Quantum Field Theory II

An advanced course in QFT

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QUANTUM FIELD THEORY II

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These notes are consequence of my self study; and are mostly inspired from Dr. Francois lectures on **QFT II**. These notes covers path integral formalism, and can be effectively regarded as a book on Advanced Quantum Field Theory II. There is also a supplementary book titled **Problems & Solutions on QFT II** written by me for the present book which contains tutorials, exercises and their solution which should be read in parallel with these notes. These notes are effectively broken up into 8 modules.

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Quantum Field Theory 2 || Module 1 || Path Integrals in Quantum Mechanics

Lec 1.1 || Path Integral for a non-relativistic Particle, Euclidean Time

— Shoaib Akhtar
16/5/2020.

1| Path and Functional Integrals in QFT.
2| Perturbation Theory. & Renormalization.
3| Non-abelian gauge Theories.

1.1 || Path Integrals in Q.M.

★ non-relativistic massive particle in a potential (no charge) 1-dimension.

mass m, position $q(t)$, t time.

Classical Theory is given by $\mathcal{L} = \frac{m}{2}\dot{q}^2(t) - V(q(t))$

$V(q)$ "smooth function" ; $\dot{q}(t) = \frac{dq(t)}{dt}$ velocity.

Equation of Motion: $\boxed{m\ddot{q} + V'(q) = 0}$

$$\text{Action} = \int_{t_1}^{t_2} dt \; \mathcal{L}(q, \dot{q})$$

Quantum Mechanics (Dirac notation)

states: $|\Psi\rangle$ $\qquad \Psi(q) = \langle q|\Psi\rangle$

position sate $|q\rangle$; $Q|q\rangle = q|q\rangle$

$\uparrow$ operator. (Position operator)

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = H \cdot |\Psi(t)\rangle \quad ; H = \frac{P^2}{2m} + V(Q)$$

$P \Rightarrow$ Momentum operator.

$t = t_1$ (initial time) ; $|\Psi_1\rangle$ initial state.

t , $|\Psi(t)\rangle = U(t, t_1)|\Psi_1\rangle$

U evolution operator ; unitary

$$\Rightarrow U(t, t_i) = U(t - t_1) = \exp\left(\frac{t - t_1}{i\hbar}H\right)$$

positition state basis.

~~$\langle q_2 | U(t) | q_1 \rangle$~~ =

$|q_2\rangle$ state at time t_2.

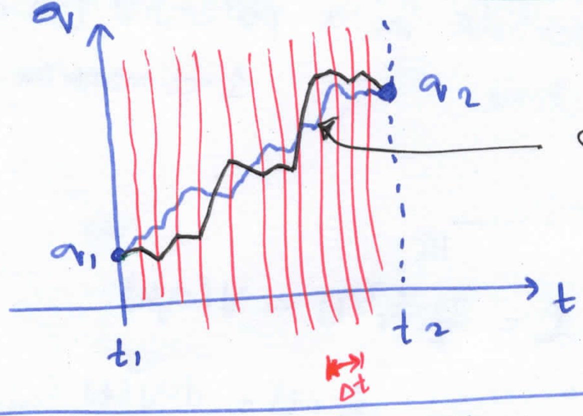
$|\psi_2\rangle$

~~(matrix element~~ $\langle q_2 | U(t_2 - t_1) | q_1 \rangle = K(q_2, t_2; q_1, t_1)$ ← Path Integral.

(matrix element)

Path Integral (feynmann)

$$K(q_2, t_2; q_1, t_1) = \int_{\substack{q(t_1)=q_1 \\ q(t_2)=q_2}} D[q(t)] \cdot e^{\frac{i}{\hbar} S[q]}$$



S[q] action of the trajectory.

quantum amplitude as sum of histories; amplitude associated with each trajectories.

Defination: use time cut off.

~~discritize the time~~

~~t~~

~~discret~~

~~discreatize time t; $t \rightarrow t_i = i \cdot \Delta t$~~

$q(t)$ ~~$\rightarrow q(t)$~~

Discreatize time t; $t \rightarrow t_i = i \cdot \Delta t$, $i = 0, \dots, N$

$q(t) \rightarrow q_i = q(t_i)$ $N = \frac{t_2 - t_1}{\Delta t}$

now the integral becomes finite dimensional integral.

$$S[q] \longrightarrow S[q_{\text{linear piecewise}}]$$

$$D[q] \longrightarrow \prod_{i=1}^{N-1} (dq_i) * \left(\frac{2 i \pi \hbar \Delta t}{m}\right)^{-N/2}$$

(measure depends on descreatization)

Continuum limit $\Delta t \to 0$ well defined.

$D q$ & $e^{i/\hbar S[q]}$ as single object are singular; not properly well defined.

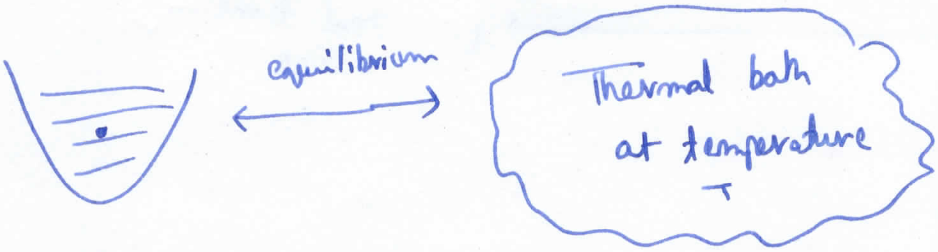
Mixed States Quantum system in a mixed state is described by density matrix $\rho \geq 0$ $\rho = \rho^\dagger$; $\rho \Rightarrow$ is an operator

(all its eigenvalues ≥ 0) & $\mathrm{Tr}[\rho] = 1$

we can so diagonilize and write.

$\rho = \sum_i p_i |i\rangle\langle i|$; $\langle i|j\rangle = \delta_{ij}$; $p_i \geq 0$; $\sum_i p_i = 1$

$p_i \Rightarrow$ " probabilities to be in the pure state $|i\rangle$ "



Interested in Stationary State.

Gibbs state ; $\rho = \frac{1}{Z} \exp(-\beta H)$; $\beta = \frac{1}{k_B T}$; $H \Rightarrow$ Hamiltonian.

$p_i = \frac{1}{Z} \cdot \exp(-\beta \underset{\uparrow \text{Energy.}}{E_i})$; $Z \Rightarrow$ Partition Function

$Z = \mathrm{Tr}[\exp(-\beta H)]$

Z of cours depends on T so; Z_T

Observable A operator $A = A^+$

$\langle A \rangle_\rho = \mathrm{Tr}[A \cdot \rho]$

In particular, $\underset{\text{Temperature}}{\langle A \rangle_T} = \frac{1}{Z_T} \mathrm{Tr}[A \cdot \underbrace{\exp(-\beta H)}]$

mathematically quite similar to evolution operator $U(t) = \exp\left(\frac{t}{i\hbar} H\right)$

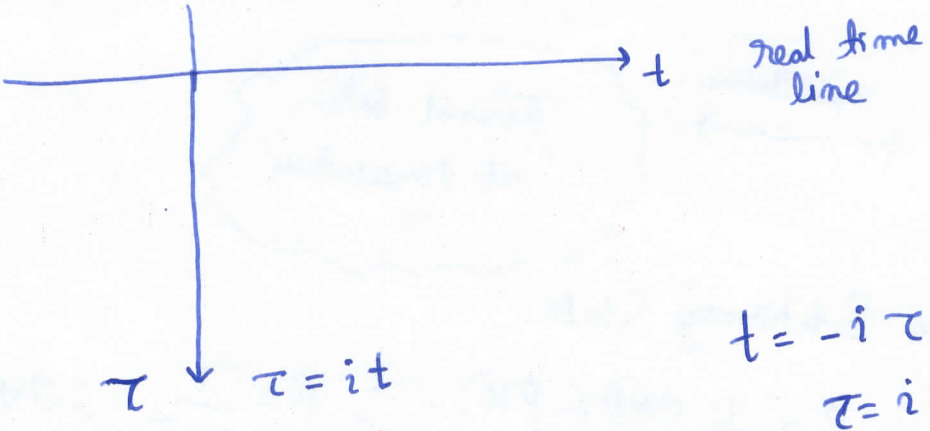
Formally,

$\exp(-\beta H) = U(\underbrace{-i\tau}_{\text{imaginary time}})$ $\Rightarrow \beta = \frac{1}{k_B T} = \frac{\tau}{\hbar}$; $\tau \geq 0$

The evolution operator U at imaginary time $-i\tau$
$\tau = \hbar\beta$ is mathematically equal to un-normalized density matrix of the system at finite temperature.

$$\beta = \frac{1}{k_B T} = \frac{\tau}{\hbar} \; ; \; \tau \geq 0$$

Temperature of the Quantum system



$t = -i\tau$ or
$\tau = it$

"Imaginary time"

↳ ~~real~~ is a real positive parameter.

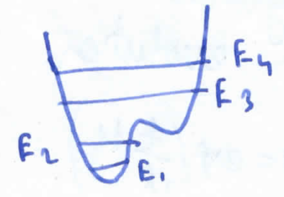
↳ We will call it Euclidean Time.

U(t) is defined for t-complex.

$H|i\rangle = E_i|i\rangle$

$E_0 < E_1 < E_2 < \dots$

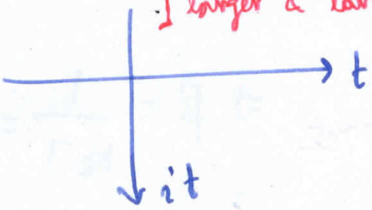
$$U(t) = \sum_i e^{\frac{t}{i\hbar}E_i} |i\rangle\langle i|$$



Im $t > 0$; $\left|e^{\frac{t}{i\hbar}E_i}\right| \to \infty$ with t

Im $t < 0$; $\left|e^{\frac{t}{i\hbar}E_i}\right| \to 0$ with t.

} larger & larger... problem expected in defining here.



Bounded Operators

An operator A is bounded if for any $|\psi\rangle$ state

$$\frac{\langle\psi| AA^{+} |\psi\rangle}{\langle\psi|\psi\rangle} \leq \|A\|^2$$

$= \sup |\text{eigenvalues } AA^{+}|$

$= |\text{diam (spectrum of } A)|^2$

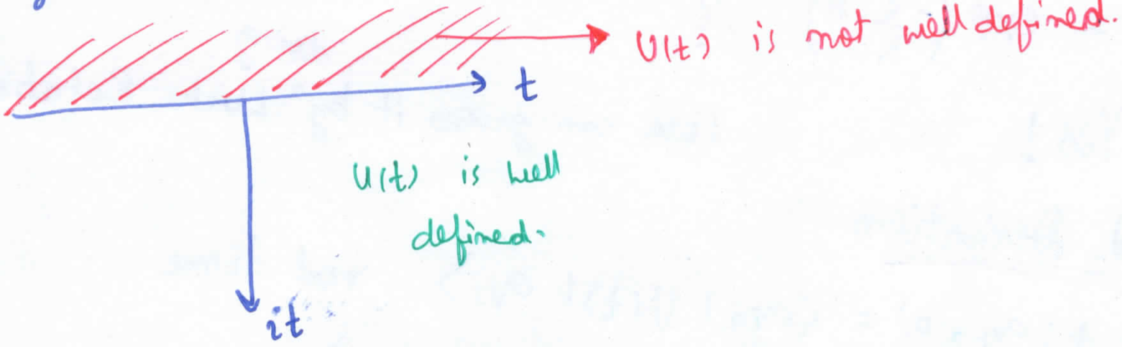
~~if dim~~

if $\dim(\mathcal{H}) = \infty$, not all operators are bounded.

(Bounded operators form a sub-algebra of algebra of all possible operators acting on the hilbert space)

Bounded operators are important to do Quantum Physics.

In general, $U(t)$ is bounded iff $\operatorname{Im} t \leq 0$



$ds^2 = -dt^2 + d\vec{x}^2$

$c=1$

$\eta_{\mu\nu} = (-,+,+,+)$
(East coast Signature)

we will use this in this book.

$(+,-,-,-)$
(West coast Signature

so; starting with Minkowski spacetime $\mathbb{M}^{1,3}$ (minkowski)

$t \to -i\tau \Rightarrow ds^2 = d\tau^2 + d\vec{x}^2$
(ordinary Euclidean metric on $\mathbb{R}^{1+3}$)

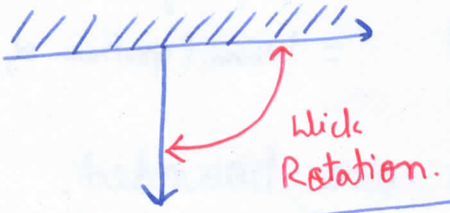
$X = (t, x_1, \ldots, x_d)$ in $\mathbb{M}^{1,d}$

$X = (\tau, x_1, \ldots, x_d)$ in $\mathbb{R}^{1+d}$

τ = another space coordinate

Poincare Group $\longrightarrow$ Euclidean Group.

The trick of going from ~~real~~ real t ~~&~~ to euclidean t is performed by Wick's Rotation.



$$\boxed{\exp(-\beta H) = U(-i\tau) \equiv U_E(\tau)}$$

" Euclidean Evolution Operator.

$U^{-1} = U^{\dagger}$ for t real.

while $U_E = U_E^{\dagger} > 0$; for τ real

H is bounded from below.

$\Rightarrow$ Unitarity.

Is there a path Integral Representation for $U_E(\tau) = \exp\left(-\frac{\tau}{\hbar}H\right)$?

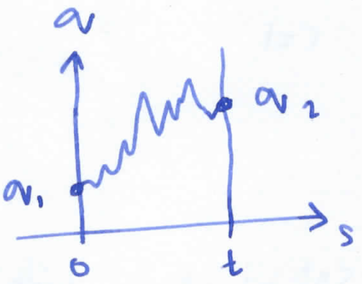
Answer// Yes! (We can guess it by doing Wick's Rotation)

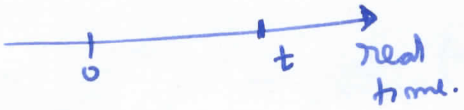
Formal Derivation

$$K(q_2, t; q_1, 0) = \langle q_2 | U(t) | q_1 \rangle \quad \text{real time}$$

$$= \int_{\substack{q(t_1)=q_1 \\ q(t_2)=q_2}} D[q] \cdot \exp\left(\frac{i}{\hbar} S[q]\right)$$

where; $S[q] = \int_0^t ds \left[\frac{m}{2}\left(\frac{dq(s)}{ds}\right)^2 - V(q(s))\right]$





Wick Rotation// $t = -i\tau$; $s = -i\sigma$; $\sigma \in [0, \tau)$

$0 < s < t$

$$iS[q] = i\int_0^t ds \left[\frac{m}{2}\left(\frac{dq}{ds}\right)^2 - V(q)\right]$$

$$iS[q] = i(-i)\int_0^\tau d\sigma\cdot\left[\frac{m}{2}\left(i\frac{dq}{d\sigma}\right)^2 - V(q)\right]$$

now .... a history of $q(\sigma)$

$$iS[q] = -\int_0^\tau d\sigma\left[\frac{m}{2}\left(\frac{dq}{d\sigma}\right)^2 + V(q)\right]$$

$$\underline{\langle q_2|U_E(\tau)|q_1\rangle} = \int_{\substack{q(0)=q_1\\ q(\tau)=q_2}} D_E[q]\exp\left(-\frac{1}{\hbar}S_E[q]\right)$$

← Positive measure probability on path (histories)

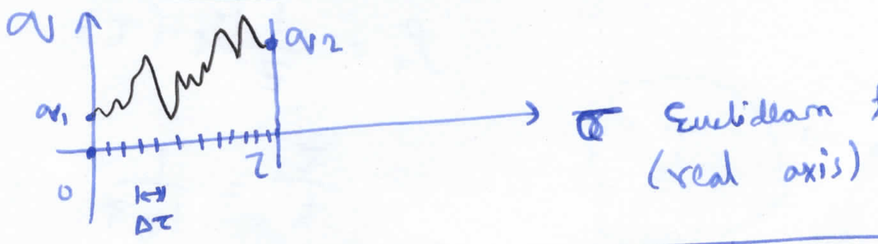
⇓ must be real number because $U_E(\tau)$ is self adjoint.

Path integral over trajectories $q(\sigma)$; $0<\sigma<\tau$ in Euclidean time

with $q(0)=q_1$; $q(\tau)=q_2$

where Euclidean Action is defined by

$$S_E[q] = \int_0^\tau d\sigma\cdot\left[\frac{m}{2}(\dot{q})^2 + V(q)\right]$$



Path integral in Euclidean time.

$$D_E(q) = \prod dq_i\left(\frac{2\pi\hbar\Delta\tau}{m}\right)^{-1/2}$$

Discretize Euclidean time by $\sigma_i = \Delta\tau\cdot i$

$i=0,\dots,N=\frac{\tau}{\Delta\tau}$

$$\sigma_i = \Delta\tau\cdot i$$

$$i=0,\dots,N=\frac{\tau}{\Delta\tau}$$

& take the limit $\Delta\tau\to 0$

Defines Probability Space.

Path with largest probability ?

q such that $S_E[q]$ is minimal.

ie; saddle point of this integral.

so; it is solution of $\boxed{m\frac{d^2q}{d\sigma^2} - V'(q) = 0}$

…… Euclidean trajectory.

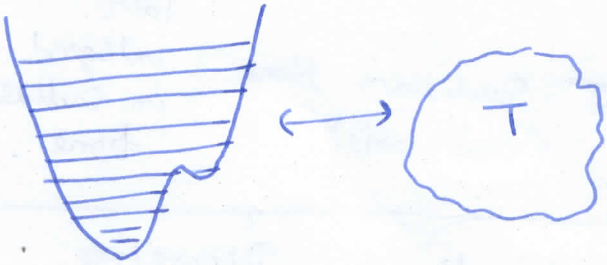
Can think of it as analytic continuation of

$m\ddot{q} + V(q) = 0$ at reale time

$\Longleftrightarrow$ $-m\frac{d^2q}{d\sigma^2} + V'(q) = 0$

Analytic continuation by Wick of Classical Trajectory gives Euclidean Trajectory.

Classical Trajectory $\xrightarrow{\text{Wick}}$ Euclidean Trajectory.



$\rho_T = \frac{1}{Z}U_E(\tau)$

$\frac{\tau}{\hbar} = \frac{1}{k_B T}$

$Z = Tr[U_E(\tau)]$: Partition Function.

$$Z = \sum_{i=0}^{\infty} \exp\left(-\frac{1}{k_B T}E_i\right)$$; $H|i\rangle = E_i|i\rangle$

Gibbs State.

$$= \int_{\mathbb{R}} dq \, \langle q| U_E(\tau)|q\rangle$$

$\langle q | q' \rangle = \delta(q-q')$; $\int dq \, |q\rangle \langle q| = \mathbb{1}$

$$\Rightarrow Z = \int \mathcal{D}_E[q] \, \exp\left[-\frac{1}{\hbar} S_E[q]\right]$$

where $q(\sigma) : 0 < \sigma < \tau$

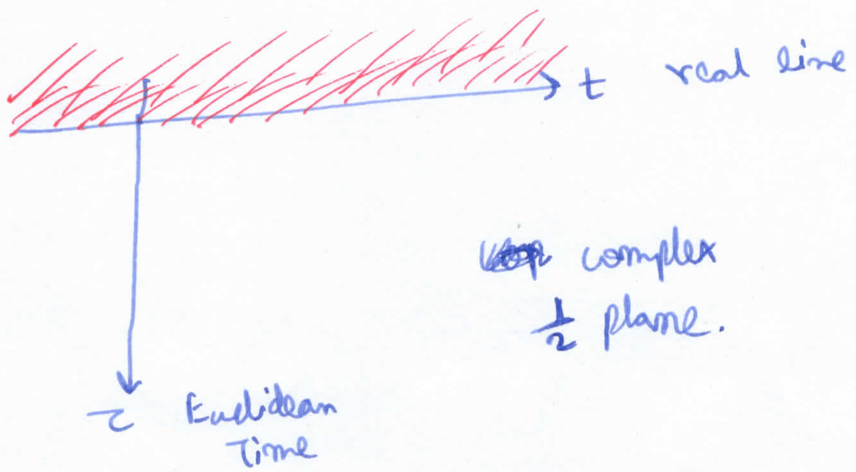
and $q(0) = q(\tau)$

So; we are considering Path Integral over periodic trajectories in Euclidean time with period τ.

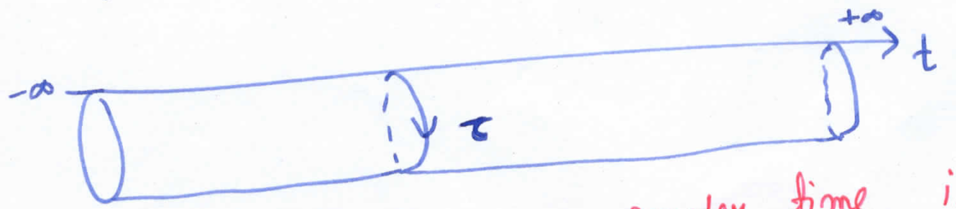
The period is related to T,

$$\text{Period} = \tau = \frac{\hbar}{k_B T}$$

So; in general; if we consider complex real time



At finite temperature, convenient to view as cylinder.



"At finite temperature, complex time is a cylinder"

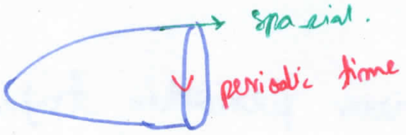
Euclidean time is periodic with period $\frac{\hbar}{k_B T}$

$t = -i\tau$

KMS states (Generalization of Gibbs matrix for ∞ dimensional Hilbert Space $\mathcal{H}$)

If you have quantum system that can be formulated in some way as living on periodic Euclidean Spacetime, then it has temperature associated ~~to~~ to it.

ex// consider QFT in a black-hole classical background.
↳ go to Euclidean Spacetime.



period ∝ mass

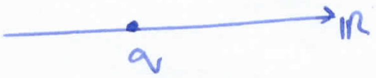
Euclidean schwarzschild.

so; Black hole has quantum temperature inversely proportional to mass.

Quantum Field Theory 2 Module 1|1 – Shoaib Akhtar 17/5/2020 (Pg 1)

Lecture 1.2 | Operators and Correlation functions in the path integral formalism, Thermal expectation values, Free Scalar Fields

1) Operators & Correlations by Path Integrals.
2) Free Field. (Introduction)



$H = \frac{P^2}{2m} + V(q)$, $U(t) = \exp\left(\frac{t}{i\hbar} H\right)$

Schrodinger Picture $|\Psi(t)\rangle \Rightarrow$ state at time t. ; $i\frac{d}{dt}|\Psi(t)\rangle_s = H|\Psi(t)\rangle_s$

Observables correspond to operators Q, P, etc.

Heisenberg Picture Time independent representation of the ~~operator~~s state vectors)

If state $|\Psi_s\rangle$ at tim time t in Schrodinger picture.

$\therefore |\Psi; t\rangle_H = U^{-1}(t)|\Psi\rangle_s$; $|\Psi(t); t\rangle_H = |\Psi_0\rangle$

where $|\Psi(t=0)\rangle_s = |\Psi_0\rangle$

Observable changes because basis changes. "states do not evolve)

If an observable is given operator A in Schrodinger picture; then it is represented at time t by $A(t) = U^{-1}(t)\, A\, U(t)$

(These two representations are physically equivalent)

↳ Operators do evolve ; $\langle A\rangle_{\Psi \text{ state at time } t} = {}_s\langle\Psi(t)|A|\Psi(t)\rangle_s$

Schrodinger picture.

$= {}_H\langle\Psi; t|\, A(t)\, |\Psi; t\rangle_H = \langle\Psi_0|A(t)|\Psi_0\rangle$

Heisenberg picture.

So; Schrodinger eqn is replaced by equation for operators.

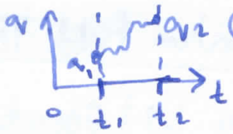
$$i\hbar \frac{\partial}{\partial t} A(t) = [A(t), H]$$

since $[H, U(t)] = 0 \Rightarrow \underline{H(t) = H}$ in Schrodinger picture. H does not evolve with time. ↳ Conservation of energy.

~~H(t) doesnot evolve~~ with time.

~~$K(q_2, t_2; q_1, 0)$~~

$$k(q_F, t; q_I, 0) = \langle q_2 | U(t) | q_1 \rangle = \int_{\substack{q(0)=q_I \\ q(t)=q_F}} D[q] \exp\left(\frac{i}{\hbar} S[q]\right)$$



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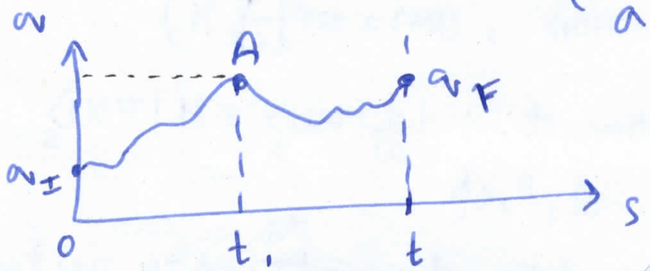
now; choose t, between 0 & t ; $0 < t_1 < t$.

Consider an observable $A = a(q)$ at time t_1

$\uparrow$ a is some function of q



$$\int_{\substack{q(0)=q_I \\ q(t)=q_F}} D[q]\, e^{\frac{i}{\hbar}S[q]} . a(q(t_1))$$

$|q, t\rangle_H = U(-t)\, |q\rangle_S$

$|q\rangle =$

$|q, t\rangle = U(-t)|q\rangle$

$=$

$$= \int_{\mathbb{R}} dq_1 \int_{\substack{q(0)=q_I \\ q(t_1)=q_2}} D[q]\, e^{\frac{i}{\hbar}S} \times a(q_1) \times \int_{\substack{q(t_1)=q_1 \\ q(t)=q_F}} D[q]\, e^{\frac{i}{\hbar}S}$$

$\rightarrow k(q_1, t_1, q_I, 0) \times a(q_1) \times k(q_F, t, q_1, t_1)$

$$= \int_{\mathbb{R}} dq_1 \int_{\substack{q(0)=q_I \\ q(t_1)=q_1}} D[q]\, e^{\frac{i}{\hbar}S} \cdot a(q_1) \int_{\substack{q(t_1)=q_1 \\ q(t)=q_F}} D[q]\, e^{\frac{i}{\hbar}S}$$

Amazing.... split S into slow modes & fast modes ....

$$= \int_{\mathbb{R}} \langle q_F | U(t-t_1) \boxed{|q_1\rangle a(q_1) \langle q_1|} U(t_1) | q_I \rangle \, dq_1$$

a(Q) represented in its diagonal basis

$$= \langle q_F | U(t-t_1) \left(\int_{\mathbb{R}} dq_1 |q_1\rangle a(q_1) \langle q_1| \right) U(t_1) | q_I \rangle$$

$\rightarrow$ This is $a(Q)$

Similar to what we did in Renormalization

$S[\phi] = S[\phi_<] + S[\phi_>]$

... Then you get product...

$$= \underbrace{\langle q_F | U(t)}_{{}_H\langle q_F, t|} \underbrace{U^{-1}(t_1)\, A\, U(t_1)}_{A(t_1)} \underbrace{|q_I\rangle}_{|q_I; 0\rangle_H}$$

$\Rightarrow \;{}_H\langle q_F ; t |\, A(t) \,| q_I ; 0 \rangle_H$; matrix element of an operator A in Heisenberg representation.

Inserting the classical quantity $a(q(t_1))$ is equivalent to inserting the operator $a(Q)(t_1)$ in the Heisenberg picture.

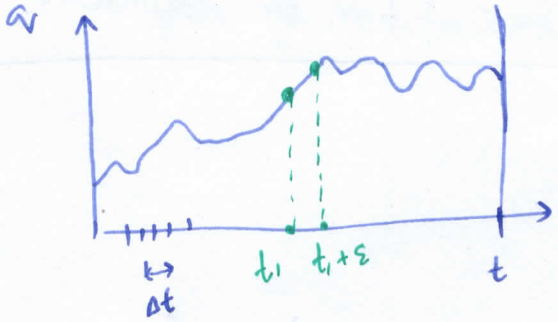
Inserting the classical quantity $a(q(t_1))$ $\Longleftrightarrow$ Insert the operator $a(Q)(t_1)$ in the Heisenberg picture

not: $K(q_F, t, q_I, 0) = {}_S\langle q_2 |\, U(t) \,| q_2 \rangle_S = {}_H\langle q_2, t | q_1, 0 \rangle_H$

$$\boxed{{}_H\langle q_F ; t \,|\, A(t_1) \,|\, q_I ; 0 \rangle_H = \int \mathcal{D}[q]\, e^{\frac{i}{\hbar} S[q]} \cdot a(q(t_1))}$$

momentum $p = m \cdot v = m \cdot \frac{dq}{dt}$ $\leftarrow$ problematic in path integral to define velocity

usually $\dot{q} \approx \pm \infty$



$$v(t_1)_\varepsilon = \frac{q(t_1+\varepsilon) - q(t_1)}{\varepsilon}$$

regularized by ε.

and then compute

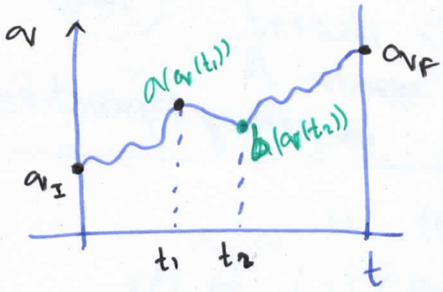
$$\lim_{\epsilon \to 0} \lim_{\Delta t \to 0} \int \mathcal{D}[q]\, e^{\frac{i}{\hbar} S[q]}\, v(t_1)_\varepsilon = {}_H\langle q_F ; t \,|\, \tfrac{1}{m} P(t_1) \,|\, q_I, 0 \rangle_H$$

$\hookrightarrow$ dont mess with order of limit. ... This limit exists.

Heisenberg picture $[Q, H] \approx P$ ~~$\frac{1}{\varepsilon}[Q(t+\varepsilon) - Q(t_1)]$~~

$$\frac{1}{\varepsilon}\big[\underbrace{Q(t+\varepsilon)}_{\Downarrow} - Q(t_1)\big] = \frac{1}{m} P(t_1) + O(|\varepsilon|)$$

$U^{-1}(\varepsilon)\, Q(t_1)\, U(\varepsilon)$

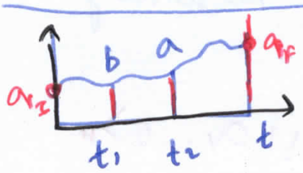


$$A(t_1) = a(Q)(t_1)$$
$$B(t_2) = b(Q)(t_2)$$

Heisenberg Picture.

$${}_H\langle q_F;t|\, B(t_2)\, A(t_1)\, |q_I;0\rangle_H \quad ; 0 < t_1 < t_2 < t$$

2 operators at
2 ≠ times



$$\langle q_F;t\,|\,A(t_1)\,B(t_2)\,|q_I,0\rangle \quad ; 0 < t_2 < t_1 < t$$

Since, $[Q, H] \neq 0$

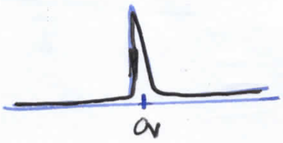
So: $[A(t_1), B(t_2)] \neq 0$ if $t_1 \neq t_2$

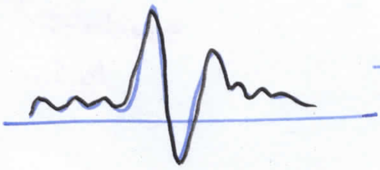
Time Ordered Products of A & B:

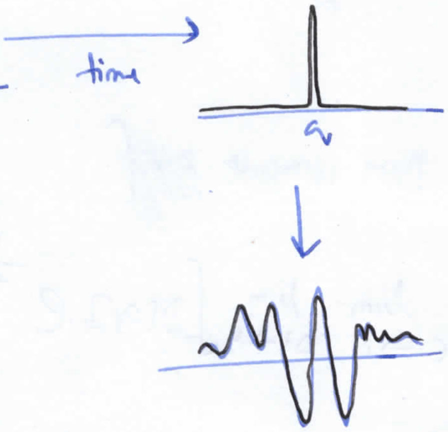
$$T[A(t_1), B(t_2)] = \begin{cases} B(t_2)\,A(t_1) & t_1 < t_2 \\ A(t_1)\,B(t_2) & t_1 > t_2 \end{cases}$$

Path Integral build automatically time ordered products of operators in Heisenberg representation by defination.

~~Go back to "Euclidean Time"~~

$|q\rangle =$


$|q,t\rangle\rangle = U(-t)|q\rangle =$




Go back to "Euclidean Time" : Correlation Functions

$U_E(\tau) = \exp\left(-\frac{\tau}{\hbar} H\right)$ well defined when $\mathrm{Re}\,\tau \geq 0$

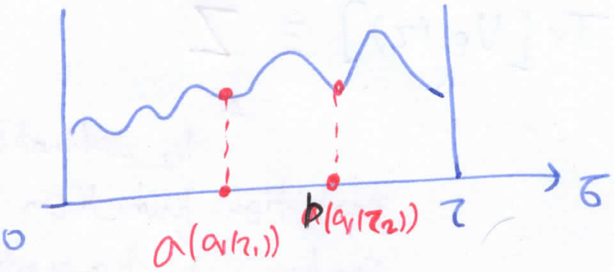
& H is bounded from below ; ie: there is a ground state $|0\rangle$

$U_E(\tau)$ is bounded operator provider H is bounded from below if $\mathrm{Re}\,\tau \geq 0$

$$\int D_E[q]\,\exp\left(-\frac{S_E[q]}{\hbar}\right) \equiv \langle q_F | U_E(\tau) | q_I\rangle$$

$q(0)=q_I$
$q(\tau)=q_F$

$$S_E[q] = \int_0^{\tau} d\sigma \left[\frac{m}{2}\dot{q}^2 + V(q)\right]$$



$$\int D_E[q]\,\exp\left(-\frac{S_E[q]}{\hbar}\right) a(q(\tau_1))\, b(q(\tau_2))$$

$q(0)=q_I$
$q(\tau)=q_F$

$$= \langle q_F | U_E(\tau-\tau_2)\, b(Q)\, U_E(\tau_2-\tau_1)\, a(Q)\, U(\tau_1) | q_I\rangle$$

if $0<\tau_1<\tau_2<\tau$

we will use same kind of notations.

$$= \langle q_F, \tau | B(\tau_2) A(\tau_1) | q_I, 0\rangle$$

Some kind of Heisenberg picture in Imaginary time … be careful.

If summing over periodic trajectories in euclidean time such that $q(\tau) = q(0)$

Then we get Trace of the operators ~~$B(\tau_2)A(\tau_1)$~~

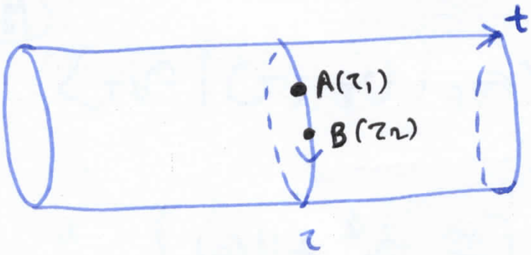
i.e.

$$\int_{\text{Periodic}} D_E[q]\, e^{-\frac{S_E[q]}{\hbar}} \cdot a(q(\tau_1))\, b(q(\tau_2))$$

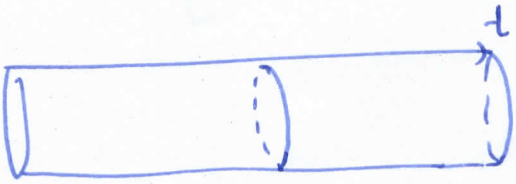
$$= \mathrm{Tr}\left[U_E(\tau-\tau_2)\cdot B\cdot U_E\cdot(\tau_2-\tau_1)\cdot A\cdot U_E(\tau_1)\right]$$

$$= \mathrm{Tr}\left[U_E(\tau_1)\, U(\tau-\tau_2)\, B\, U_E(\tau_2-\tau_1)\, A\right]$$ (use cyclicity of trace)

$$= \mathrm{Tr}\left[U_E(\tau-(\tau_2-\tau_1))\cdot B\cdot U_E(\tau_2-\tau_1)\, A\right]$$



$$= \mathrm{Tr}\left[U_E(\tau-(\tau_2-\tau_1))\, B\cdot U_E(\tau_2-\tau_1)\cdot A\right]$$

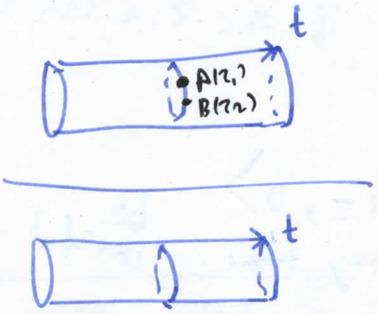


$$= \mathrm{Tr}\left[U_E(\tau)\right] \equiv Z$$

↑ by defination

partition function of system at temperature T

with $\dfrac{\tau}{\hbar} = \dfrac{1}{k_B T}$



$$= \frac{\mathrm{Tr}\left[U_E(\tau-(\tau_2-\tau_1))\, B\, U_E(\tau_2-\tau_1)\, A\right]}{\mathrm{Tr}\left[U_E(\tau)\right]}$$

by defination

$$\equiv \langle B(\tau_2)\, A(\tau_1)\rangle$$

notation for this object.

Gibbs State at temperature T

Oh Gibbs State characterized by the boltzman factor

$\left[\dfrac{\tau}{\hbar} = \dfrac{1}{k_B T} = \beta\right.$

$$\boxed{\rho_T = \frac{1}{Z} U_E(\tau)}$$

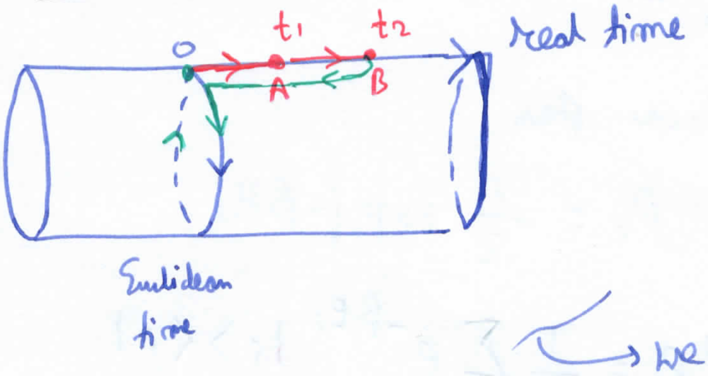
↳ $\mathrm{Tr}(\rho_\tau) = 1$

$\Rightarrow 1 = \dfrac{1}{Z}\,\mathrm{Tr}(U_E(\tau))$

$\Rightarrow Z = \mathrm{Tr}(U_E(\tau))$

Euclidean + Real Time

~~going backward means instead~~

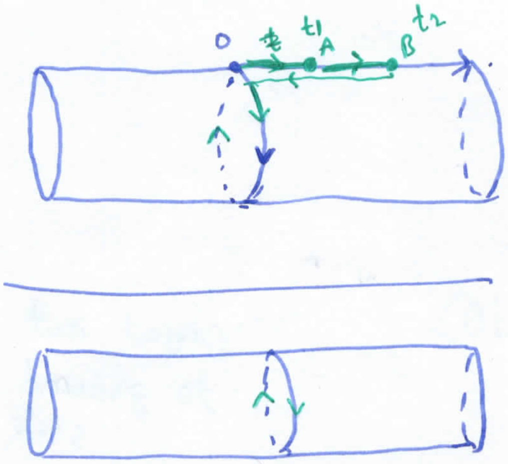


⟶ we will compute this .

⟵ : Going backward in time means instead of having path integral with phase factor $e^{\frac{i}{\hbar}S[q]}$, you take $e^{-\frac{i}{\hbar}S[q]}$ (like taking complex conjugate)

we are taking convolution, gluing path integrals is well defined object.

We want to evaluate .



$= \xi$

(lets call it ξ for now to save space. is

an object which is essential in study of quantum statistical mechanics.

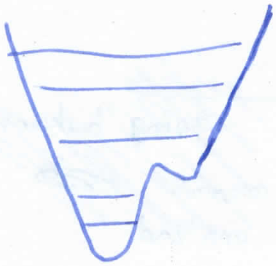
So:

$$\xi = \mathrm{Tr}\Big[\underbrace{\rho_T}_{\text{evolution operator at imaginary time}}\underbrace{\big(B(t_2)A(t_1)\big)}_{\text{Products of }\cancel{\text{operator}}\text{ operator at real time.}}\Big]$$

We obtain

$$\xi = \langle B(t_2)A(t_1)\rangle_{\text{Gibbs State}} \Big\} \Rightarrow$$

2 time correlation function of a quantum system in a Gibbs state.(which is a stationary state.)



$$H = \sum_{i=0}^{\infty} E_i |i\rangle\langle i|$$

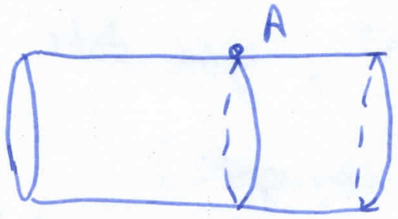
we know then

$$\rho_T = \frac{1}{Z} \exp(-\beta H)$$

$E_0 < E_1 < E_2 < \dots$

E_0 energy of ground state $|0\rangle$

$$\Rightarrow \rho_T = \frac{1}{Z} \sum_i e^{-\beta E_i} |i\rangle\langle i|$$



evaluate operator A at time 0.

$$\mathrm{Tr}(\rho_T A) = \frac{\sum_i e^{-\beta E_i} \langle i|A|i\rangle}{\sum_i e^{-\beta E_i}}$$

limit where period (Euclidean) ↗ ∞

$\beta \nearrow \infty$; $T \rightarrow 0$

so; $\mathrm{Tr}(\rho_T A) \xrightarrow[\beta \nearrow \infty]{} \langle 0|A|0\rangle$

project out to ground state

Ground state matrix element of A

or expectation value of A in the ground state.

~~ground state is equivit.~~

$\beta \rightarrow \infty$ ⟺ Projection on the ground state

in QFT $|0\rangle$ = vacuum state (state of no particle)

2) Free Scalar Field (Klein-Gordon field by Path Integral quantization)

We will be working in $\mathbb{M}^{1,D}$ $\quad d = 1 + D$ ← dimensions of space. ← dimensions of space-time

a vector is just $x = (t, \vec{x})$

will be using east-coast metric with $c = 1$; $ds^2 = -dt^2 + d\vec{x}^2$

The Classical free Scalar Field is given by the action :

$$S[\phi] = \int dt \int d^D\vec{x} \overbrace{\left[\underbrace{\underbrace{\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2}_{\text{Kinetic}} - \underbrace{\left\{\frac{1}{2}\left(\frac{\partial\phi}{\partial \vec{x}^j}\right)^2 + \frac{m^2}{2}\phi^2\right\}}_{\text{potential.}}}_{\text{Lagrangian.}} \right]}^{\text{Lagrangian Density.}}$$

$\phi(x)$: real variable

Euler Lagrange Equation ⟹ gives Klein gordon equation → gives plane waves with energy $E = \sqrt{\vec{k}^2 + m^2}$

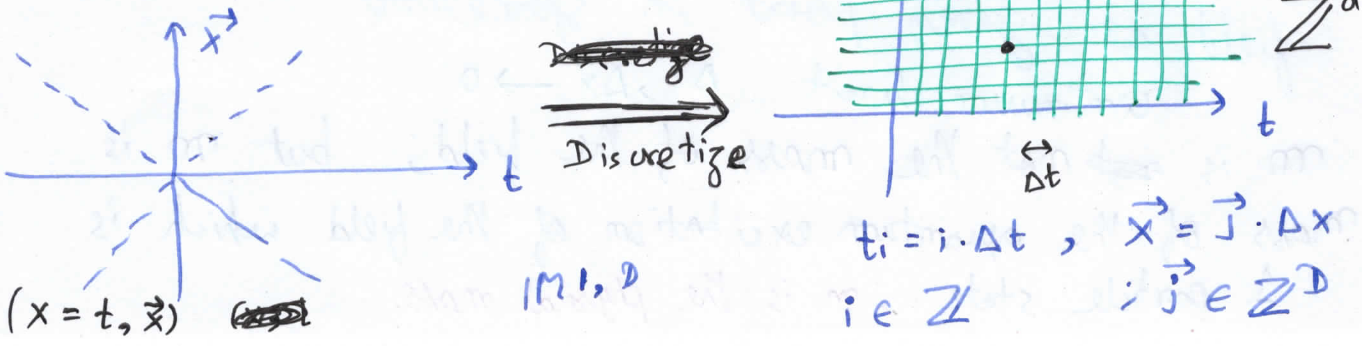
How to quantize using Path Integral? Recipie is here.

$$\int \underset{\substack{\uparrow \\ \text{Integrate over } \phi(x)}}{\mathcal{D}[\phi]} \exp\left(\frac{i}{\hbar} S[\phi]\right)$$

The measure will be formally

$$\mathcal{D}[\phi] = \prod_{x \in \mathbb{M}^{1,D}} d\phi(x) \cdot \underset{\substack{\uparrow \\ \text{some normalization factor.}}}{\mathbb{C}}$$

We can do it more carefully by discreatizing time & discreatizing space.



$t_i = i \cdot \Delta t$, $\vec{x} = \vec{j} \cdot \Delta x$

$i \in \mathbb{Z}$; $\vec{j} \in \mathbb{Z}^D$

so; Minkowski spacetime is replaced by square lattice $\mathbb{Z}^d$.

for simplicity; we can take $\Delta t = \Delta x$

Poincaree symmetry $\xrightarrow[\text{broken to}]{}$ Lattice Symmetry

$$\phi(x) \longrightarrow \phi_{\vec{I}} = \phi(x_I)$$

$$\vec{I} = (i, \vec{j}) \in \mathbb{Z}^d$$

usually we call m (the parameter) as mass because it is mass of the particle, but it does not play the same role of the mass in ~~by~~ path integral of Harmonic osillators

$S[\phi] \longrightarrow \underset{\text{discrete}}{S}[\phi]$: replace $\frac{\partial}{\partial t}, \frac{\partial}{\partial \vec{x}}$ by finite differences.

$$\int dt \int d\vec{x} = \Delta t \, \Delta x^D \sum_{\vec{I} \in \mathbb{Z}_D}$$

$(\Delta x)^{D-1}$ is a parameter which you can view as mass of elementary d.o.f of fields on lattice ... but can ~~never observe~~ never observe it.

Then, the measure is defined formally & becomes

$$D[\phi] = \prod_{\vec{I} \in \mathbb{Z}_d} \left\{ d\phi_{\vec{I}} \cdot \left[\frac{2\pi i \hbar \Delta t}{(\Delta x)^{D-1}} \right]^{-1/2} \right\}$$

~~how~~ here; the parameter m is not the mass ~~of the~~ of the field now; but $(\Delta x)^{D-1}$ is mass of the field at given point in spacetime

continuum limit $\Delta t, \Delta \vec{x} \to 0$

m is ~~not~~ not the mass of the field; but m is mass of the quantum excitation of the field which is 1 particle state ; m is the physical mass.

Quantum Field Theory 2

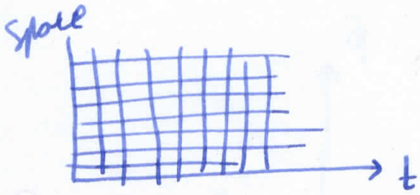
Module 2 || Functional Integrals and the Free Scalar Field

Lec 2.1 || ~~Functional Integrals~~ Propagator of the free scalar field, Correlation functions

— Shoaib Akhtar 17/5/2020

Free field (Scalar) : $\phi(x) \in \mathbb{R}$, $x = (t, \vec{x})$; $ds^2 = -dt^2 + d\vec{x}^2$

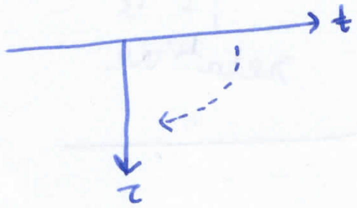
$S = \int d^dx \left[\frac{1}{2} (-\partial_\mu \phi \partial^\mu \phi) - \frac{m^2}{2} \phi^2 \right]$ $\eta_{\mu\nu} = (-1, +1, +1, +1)$

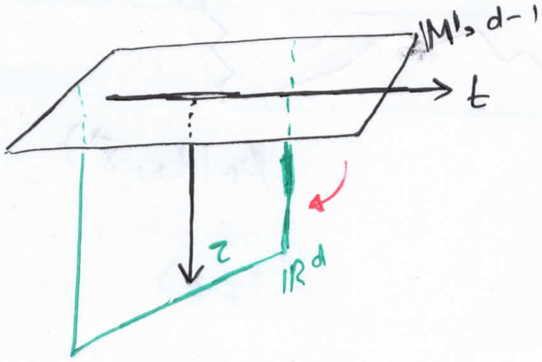


$\int D[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$ functional integral.

~~$d\phi$~~ = $D[\phi] = \prod_x d\phi(x) \cdot C$

"Euclidean Time" $t = -i\tau$





$x_E = (\tau, \vec{x})$

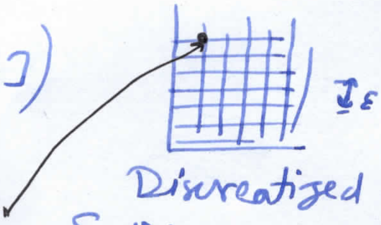
$ds^2 = d\tau^2 + d\vec{x}^2 = (dx_E)^2$ $\phi = \{\phi(x_E)\}$

$S_E[\phi] = \int d^dx_E \left[\frac{1}{2} (\partial_\mu \phi \partial_\mu \phi) + \frac{m^2}{2} \phi^2 \right]$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 & \\ & & 1 \end{pmatrix}$

Euclidean functional Integral $\int D[\phi] \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)$

$\Rightarrow \mathbb{R}^d \longrightarrow \mathbb{Z}^d$



Discreatized Euclidean Spacetim

$\phi_{\vec{i}} = \phi(\vec{x}_{\vec{i}})$

Then

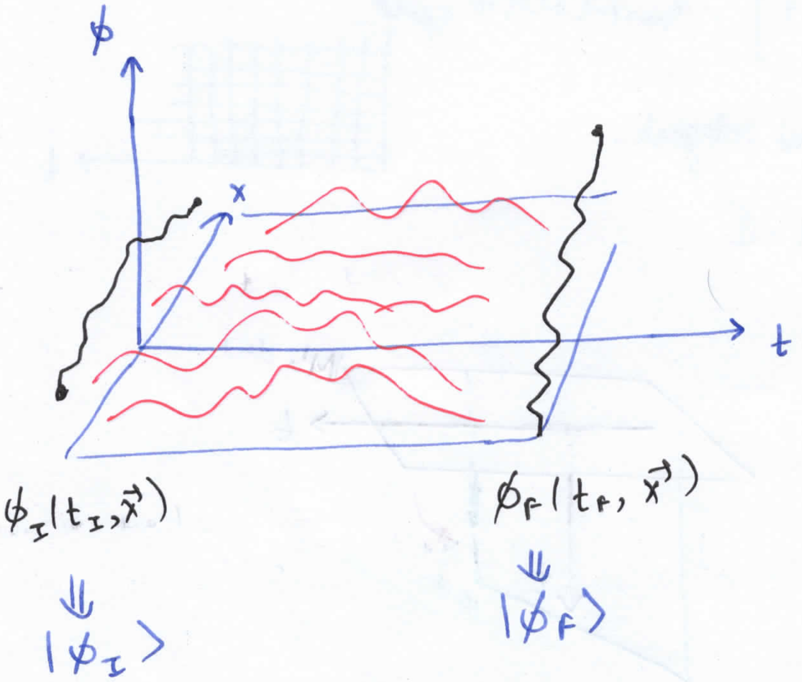
$$\boxed{\mathcal{D}[\phi] = \prod_{\vec{i} \in \mathbb{Z}^d} \left[d\phi_{\vec{i}} \cdot \left[\frac{2\pi\hbar}{\epsilon^{d-2}} \right]^{-1/2} \right]}$$

Discreatized euclidean Action $\overset{\text{lat}}{S_E}[\phi]$

$x_E = (x^0, x^1, \ldots, x^{d-1})$; $e^\mu \Rightarrow$ unit vector in direction μ. $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

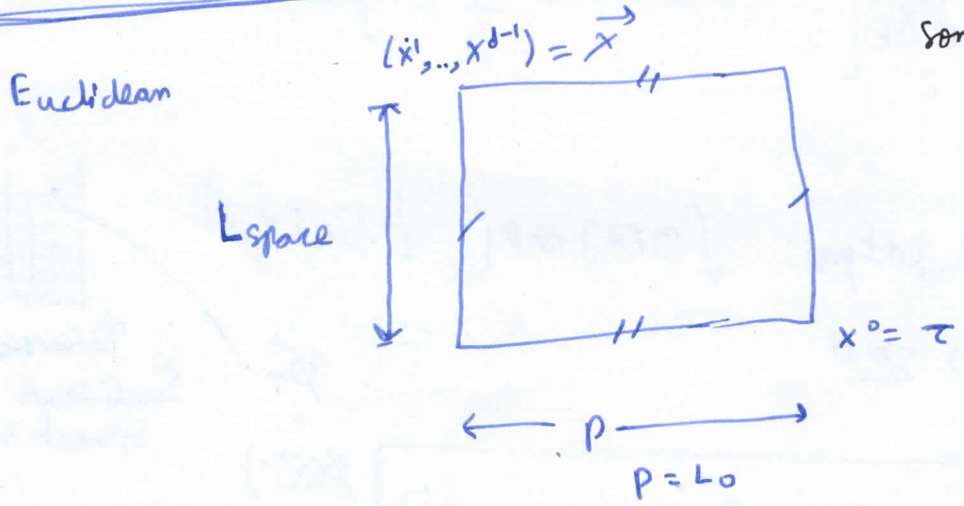
$$S_E^{\text{discretize}}[\phi] = \sum_{\vec{i}\in\mathbb{Z}^d} \epsilon^d \left[\frac{1}{2}\sum_{\mu=0}^{d-1}\left[\frac{\phi_{\vec{i}+\vec{\epsilon}_\mu} - \phi_{\vec{i}}}{\epsilon}\right]^2 + \frac{m^2}{2}(\phi_{\vec{i}})^2\right]$$

$$S_E^{\text{discr}}[\phi] = \sum_{\vec{i}\in\mathbb{Z}^d} \epsilon^d \left[\frac{1}{2}\sum_{\mu=0}^{d-1}\left\{\frac{\phi_{\vec{i}+\vec{\epsilon}_\mu} - \phi_{\vec{i}}}{\epsilon}\right\}^2 + \frac{m^2}{2}\phi_{\vec{i}}^2\right]$$



Having something at finite temperature is not lorentz invariant ... it is relative

Periodic Boundary condition. (simplest case) ~~$\phi(x^0 + L_0 \cdot m, \vec{x}$~~ $\Rightarrow$ because we get something at finite temperature.



$$\phi(x^0 + L_0 \cdot n, \vec{x} + L \cdot \vec{m}) = \phi(x^0, \vec{x})$$

Space is a Torus L_s.

Euclidean Time is periodic. L_0, QFT at finite temperature.

$$Z = \int \mathcal{D}_E[\phi_E] \exp\left(-\frac{1}{\hbar} S[\phi_E]\right)$$ Gaussian Integral

= Partition function of the Quantum Field on a torus at finite Temperature.

now if $L_0 \to \infty \Leftrightarrow$ Temp $\longrightarrow 0 \Rightarrow$ project on $|0\rangle$.

~~$S[\phi] = \sum_{ij} \phi_{\vec{i}} K_{\vec{i}\vec{j}} \phi_{\vec{j}} \xrightarrow{\epsilon\to 0}$~~

now; $X_E = X$ (drop E)

$$S[\phi] = \sum_{ij} \phi_{\vec{i}} K_{\vec{i},\vec{j}} \phi_{\vec{j}} \underset{\epsilon\to 0}{=} \frac{1}{2}\int_{\text{Torus}} d^dx\, \phi(x)\left(-\Delta + m^2\right)\phi(x)$$

(continuum limit)

$$\Delta = \sum_{\mu=0}^{d-1}\left(\frac{\partial}{\partial x^\mu}\right)^2$$ Laplace-Beltrami operator in d-dimensions

$$\int_{\text{Periodic Boundary condition}} \partial_\mu\phi\,\partial^\mu\phi = \int \phi.\left(-\partial_\mu\partial^\mu\phi\right)$$

integrate by parts (boundary terms vanishes because we are here having periodic Boundary condition)

$$Z = \mathcal{C}\cdot\left(\det\left[\frac{K}{2\pi}\right]\right)^{-1/2} \underset{\epsilon\to 0}{=} \left(\det\left[-\Delta + m^2\right]\right)^{-1/2}$$

normalization factor

discreate case

K is matrix

Differential operator

$$Z = \left(\det\left[-\Delta + m^2\right]\right)^{-1/2}$$

Correlation functions

$\phi(x) \longleftrightarrow \Phi(t, \vec{x})$

Random field variable in path integral

field operator in canonical quantization. in Heisenberg picture.

Functional Integral Methods

2 point function: (Euclidean spacetime)

$$\underbrace{\langle \phi(x_1)\phi(x_2)\rangle}_{\Downarrow} := \frac{\int D[\phi] . \exp\left(-\frac{1}{\hbar} S[\phi]\right) \phi(x_1)\phi(x_2)}{\int D[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right)}$$

Cumulant of a Gaussian Random Variable.

$$= (K^{-1})_{\vec{i}\vec{j}} = \langle \phi_{\vec{i}} . \phi_{\vec{j}} \rangle$$

$$\langle \underline{(\phi(x_1) - \langle\phi(x_1)\rangle)(\phi(x_2) - \langle\phi(x_2)\rangle)} \rangle = \langle \phi(x_1)\phi(x_2)\rangle_{\text{connected}}$$

↳ Cummulant.

~~Since gaussian here~~;

Since gaussian + Then $\phi(x) \to -\phi(x) \Rightarrow \langle \phi(x_i)\rangle = 0$

$$\langle \phi(x_1)\phi(x_2)\rangle = \left(\frac{1}{-\Delta + m^2}\right)_{x_1, x_2} = G(x_1, x_2)$$

$G(x_1,x_2)$: Kernel of the operator $(-\Delta + m^2)$ is $\left(\frac{1}{-\Delta+m^2}\right)_{x_1, x_2}$.

lets call, $G_{ij} \equiv (K^{-1})_{ij}$ ie $K \cdot G = \mathbb{1}$

$\Rightarrow \sum_j K_{ij} G_{jk} = \delta_{ik}$

as here G_{jk}

$$\boxed{K \cdot G = \mathbb{1}}$$

↳ going to continuum limit.

$(-\Delta_{x_1} + m^2)$

⟶ because applying to. left index of $G_{x_1 x_1}$

matrix element of $\mathbb{1}$ in continuum limit δ

$$(-\Delta_{x_1} + m^2)\cdot G(x_i, x_2) = \delta(x_1 - x_2) \quad \text{in } \mathbb{R}^d$$

ie; $$\boxed{(-\Delta_{x_1} + m^2)\, G(x_1, x_2) = \delta(x_1 - x_2)} \text{ in } \mathbb{R}^d \text{ or Torus.}$$

Δ_{x_1} is symmetric opperator ⇒ so; $G(x_1, x_2)$ will be symmetric

so; we can solve this . . . Symmetric Greens Function.

$G(x_1, x_2) = G(x_2, x_1)$: Symmetric

$G(x_1, x_2) = G(x_1 - x_2)$: Translational invariance.

⇒ because of these symmetry; we will just use $G(x)$ ∴ $G(x) = G(-x)$ moreover its even. function.

The simplest way is to take the Fourier Transform.

$$\hat{G}(k) = \int d^dx\, e^{-ik\cdot x}\, G(x)$$

In fourier space the equation becomes

$$(k^2 + m^2)\,\hat{G}(k) = 1$$

$$\Rightarrow \boxed{\hat{G}(k) = \frac{1}{k^2 + m^2}}$$

k is a vector in $\mathbb{R}^d$ which is reciprocal space of Euclidean space

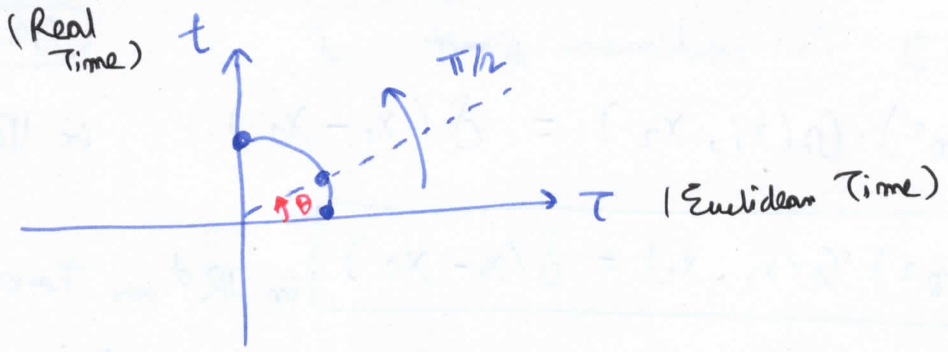
$$G(x) = \int \frac{d^dk}{(2\pi)^d}\cdot e^{ik\cdot x}\cdot \frac{1}{k^2 + m^2}$$

Euclidean 2 point function (Propagator)

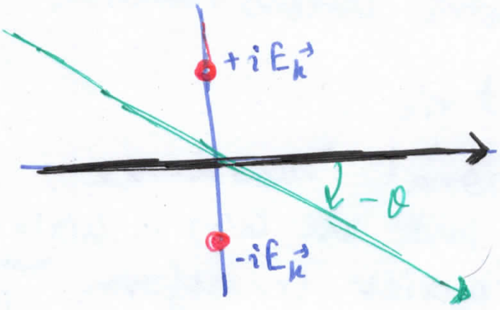
Euclidean $\xrightarrow[\text{make inverse Wicks rotation.}]{}$ Real time. $k_E = (k_0, \vec{k})$

$$x_E = (\tau, \vec{x}) \longmapsto x = (t, \vec{x}) \quad \text{with } t = -i\tau$$

$$G(x_E) = \int_{-\infty}^{+\infty} \frac{dk_0 \cdot d^{d-1}\vec{k}}{(2\pi)\,(2\pi)^{d-1}}\, e^{ik_0\cdot\tau + i\vec{k}\cdot\vec{x}}\cdot \frac{1}{k_0^2 + \vec{k}^2 + m^2}$$



Complex Time



Contour k_0 plane

now we see that if we want to make wick's rotation.

$G(x_E) \xrightarrow[\text{Rotation}]{\text{Wick}}$

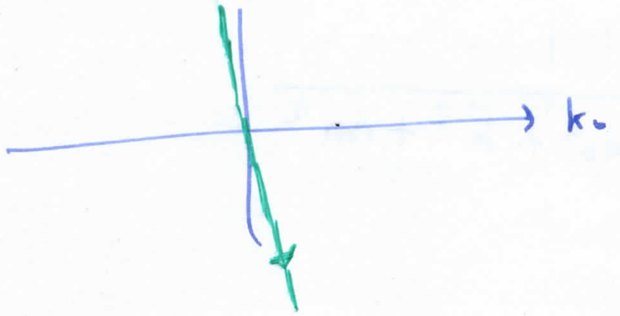
we want to keep $k_0 \cdot \tau$ a pure phase.

Rotate the contour $\int dk_0$ (allowed to do this because $\frac{1}{k_0^2+\vec{k}^2+m^2}$ analytic in k_0 as long as we dont hit singularity)

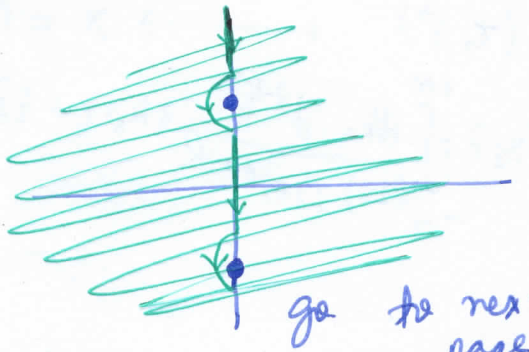
↪ but it has pole at

$$k_0 = \pm i\sqrt{\vec{k}^2+m^2}$$
$$\equiv \pm i E_{\vec k}$$

When we rotate by $\theta = \pi/2$



i.e. $\Rightarrow$



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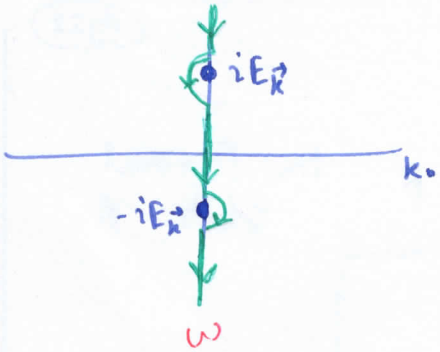


Then $\underline{k_0 = -i\omega}$ || we define this now.

$$\Rightarrow G(x) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{+\infty} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \cdot \frac{e^{i(\omega t + \vec{k}\cdot\vec{x})}}{\omega^2 - \vec{k}^2 - m^2} \quad i$$

$$\Rightarrow G(x) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{+\infty} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \cdot e^{i(\omega t + \vec{k}\cdot\vec{x})} \quad \frac{i}{\omega^2 - \vec{k}^2 - m^2 + i\varepsilon}$$

Feynman prescription for the integral

Functional Integral.

$$\equiv G_{\text{Feynman}}(t, \vec{x})$$

we expect this to be equal to metrix element of vacuum to vacuum $\langle 0| \quad |0\rangle$ of Time ordered product ~~$T[\Phi(t,\vec{x})\,\Phi(t,\vec{x}$~~

$$T[\Phi(t,\vec{x})\,\Phi(0,0)].$$

ie;

$$\boxed{G(x) = \langle 0| T[\Phi(t,\vec{x})\,\Phi(0,\vec{0})]|0\rangle}$$

So

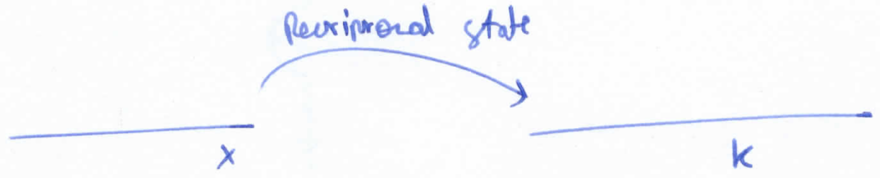
Functional Integral

||

Cannonical Quantization.

doing same calculation with periodic Euclidean Boundary Condition

$\hookrightarrow$ we get $\langle T[\phi,\phi]\rangle_\beta$ in thermal state β.



(Reciprocal state of line is line)

Reciprocal state of Torus is $\mathbb{Z}$

Properties of $G_E(x)$

$G_E(x)$ is also rotational invariance

so; actually $G_E(x) = G_E(|x|)$

so; large distance property

$|x| \nearrow \infty$ Then $G_E(x) \simeq \exp(-m|x|)$ $m>0$

(decays exponentially at large distances)

so; short distance property

$|x| \searrow 0$ ~~Then;~~

ie; $|x| << \frac{1}{m}$ Then only $|k| >> m$ counts

so; $G(x)$ behaves like

$$G(x) \simeq \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x} \cdot \frac{1}{k^2}$$ (neglect mass term)

$G(x)$ has dimension of $(\text{momentum})^{d-2}$

so: $= (\text{distance})^{2-d}$

so; $$\boxed{G(x) \approx C \cdot |x|^{2-d}}$$

so: $G(x)$ is singular at $x=0$ as long as $d \geq 2$

so; short distance singularity ⚠

in $d=4$; $G(x)$: $\frac{1}{4\pi^2}\frac{1}{|x|^2}$ quadratically divergent.

in $d=3$; $G(x)$: $\frac{1}{4\pi} \cdot \frac{1}{|x|}$ Columb potential

in $d=2$; $G(x)$: $-\frac{1}{2\pi}\log(m.|x|)$ logarithmic singularity.

larger d ; stronger is the divergence.

∴ in d=1 ; ie; time only. (we dont have fields that depend on spacial dimensions)

d=1 , means non-relativistic Quantum Mechanics.
(no divergence in d=1)

Position is not an observable in QFT; we cannot construct an operator which says my field is located at a given point.

⚠ $\left[\begin{array}{c}\text{larger } d \\ \text{stronger the} \\ \text{divergence}\end{array}\right] \Rightarrow$ problem of U.V. singularity.

(actually an important feature QFT ; related the Renormalization)

In Euclidean Space Time

$\phi(x)$ in functional integral is a random field. is very wild at short distance.

Lecture 3.1|| Wicks Theorem, Quantization of ϕ^4 theory, Feynman diagrams

— Shoaib Akhtar ; 18/5/2020.

1|| Free Scalar Field: continued.... (Euclidean)

- Wicks Theorem
- QFT ⟷ Statistical Mechanics.

$$S_E[\phi] = \int d^dx \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2} \phi^2 \right)$$

We have to compute functional integral of the form

$$\int D[\phi] \, \exp\left(-\frac{1}{\hbar} S_E[\phi]\right).(\dots \quad)$$

periodic B.C.
or
L → ∞ limit

$$\langle \phi(x_1) \phi(x_2) \rangle = \left(\frac{\hbar}{-\Delta + m^2} \right)_{x_1, x_2} \quad \cancel{= \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x} \cdot \frac{1}{k^2+m^2} \quad x = x_1 - x_2}$$

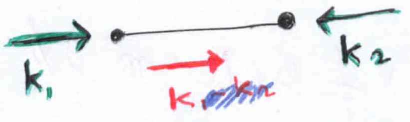
$$= \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x} \frac{1}{k^2+m^2} \quad ; \quad x = x_1 - x_2$$

Propagator

$= G(x_1, x_2)$

$$\hat{G}(k_1, k_2) = \langle \hat{\phi}(k_1) \hat{\phi}(k_2) \rangle$$
$$= (2\pi)^d \, \delta(k_1 + k_2) \cdot \frac{1}{k_1^2 + m^2}$$

} Propagator in momentum space.



$= \frac{1}{k^2+m^2}$; $\delta(k_1+k_2)$ for conservation of momentum

$$\langle \phi(x_1) \dots \phi(x_N) \rangle = 0 \quad \text{if } N \text{ is odd.}$$

$$= \sum_{\substack{\text{pairing} \\ \text{into } M \\ \text{different pairs.}}} \langle \phi_{\sigma_1} \phi_{\sigma_2} \rangle \dots \langle \phi_{\sigma_{2M-1}} \phi_{\sigma_{2M}} \rangle \quad \text{if } N = 2M \text{ (even)}$$

$$\langle \phi(x_1)\ldots\phi(x_N)\rangle = \begin{cases} 0 & \text{if } N \text{ is odd.} \\ \displaystyle\sum_{\substack{\text{pairing into} \\ \text{M different pairs}}} \langle \phi_{\sigma_1}\phi_{\sigma_2}\rangle \ldots \langle \phi_{\sigma_{2M-1}}, \phi_{\sigma_{2M}}\rangle & \text{if } N = 2M \text{ (even)} \end{cases}$$

→ Correlation between independently distributed (i d) Gaussian variables.

Generating Functional

notation

$$J\cdot\phi = \int d^dx\, J(x)\, \phi(x)$$

source term (classical function: does not fluctuate)

$$Z[\vec{J}] = \int D[\phi]\, \exp\left[-\frac{1}{\hbar}\left(S[\phi] - \vec{J}\cdot\vec{\phi}\right)\right]$$

↳ functional of a classical function.

since $S[\phi]$ can be written as quadratic form

$$S[\phi] = \frac{1}{2}\left[\phi\cdot(-\Delta+m^2)\cdot\phi\right]$$

$= \Big($ shorthand for $\displaystyle\int d^dx\, \frac{1}{2}\,\phi(-\Delta+m^2)\phi\Big)$

$$\Rightarrow Z[\vec{J}] = \left(\text{"det"}[-\Delta+m^2]\right)^{-1/2}\cdot \exp\left(+\frac{\hbar}{2}\,\vec{J}\cdot(-\Delta+m^2)^{-1}\cdot\vec{J}\right)$$

↳ because of linear term $\vec{j}\cdot\vec{\phi}$.

Short hand notation;

$$\vec{J}\cdot(-\Delta+m^2)^{-1}\cdot\vec{J} = \int d^dy_1\, d^dy_2\; J(y_1)\, J(y_2)\, G(y_1, y_2)$$

$$\boxed{\langle\phi(x_1)\dots\phi(x_N)\rangle = \hbar^N \frac{\frac{\delta}{\delta j(x_1)}\dots\frac{\delta}{\delta j(x_N)}\, Z[\vec{j}\,]}{Z[0]}\Bigg|_{\vec{j}=0}}$$

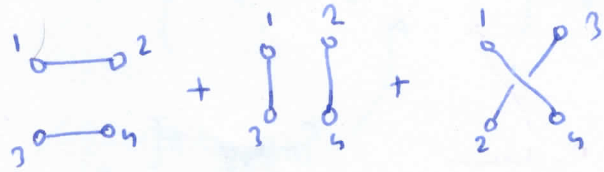
$$= \hbar^N \frac{\delta}{\delta j(x_1)}\dots\frac{\delta}{\delta j(x_N)}\cdot \exp\left(\frac{\hbar}{2}\, j\cdot G\cdot j\right)\Big|_{\vec{j}=0}$$

(we get Wicks Theorem)

---

ex

$\langle\phi(x_1)\dots\phi(x_4)\rangle =$



3 terms

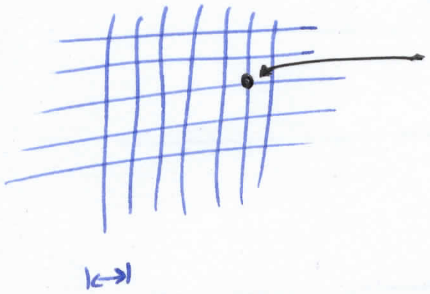
ex $\langle\phi(x_1)\dots\phi(x_6)\rangle =$ 15 terms

ex $\langle\phi(x_1)\dots\phi(x_{2M})\rangle = \dfrac{N!}{2^M M!}$ terms

---

Free Field: Euclidean d-dim

Lattice discretization $\mathbb{R}^d \to \mathbb{Z}^d$



$x_{\vec{i}} = \vec{i}\cdot\varepsilon \qquad \vec{i}\in\mathbb{Z}^d$

$\phi(x_i) = \phi_i$

ε distance

$\vec{e}_\mu$ unit vector in direction μ

$$S[\phi] \to \sum_{\substack{\vec{i}\\ \text{Lattice}}} \frac{\epsilon^{d-2}}{2}\sum_\mu \left(\phi_{\vec{i}+\vec{e}_\mu} - \phi_{\vec{i}}\right)^2 + \sum_{\vec{i}} \frac{m}{2}\epsilon^d \phi_{\vec{i}}^2$$

$$S[\phi] \to \sum_{\vec{i}} \frac{\epsilon^{d-2}}{2} \sum_{\mu} \left(\phi_{\vec{i}+\vec{e}_\mu} - \phi_{\vec{i}}\right)^2 + \sum_{\vec{i}} \frac{m}{2} \epsilon^d \phi_{\vec{i}}^2$$

Lattice

so:

$$Z = \int \prod_{\vec{i}} d\phi_{\vec{i}} \, \exp\left(-\frac{1}{\hbar} S[\phi_{\vec{i}}]\right) \Bigg\} \Rightarrow$$

Thinking of QFT as some extended statistical Mechanical system.

$$= \text{"}\sum_{\substack{\text{configuration} \\ \{\phi_{\vec{i}}\}}}\text{"} \exp\left(-\beta \, E[\{\phi_{\vec{i}}\}]\right) = \text{Partition Function}$$

Boltzman factor

$\beta \sim \frac{1}{k_\beta T}$

Energy of a configuration $\{\phi_{\vec{i}}\}$

QFT	Statistical Mechanics
Space-time x dim = 1 + (d-1)	Space (lattice) dim = d
Euclidean Time	1 dimension.
Field $\phi(\vec{x})$	local order parameter (local spin) $\phi(\vec{x}_{\vec{i}})$
Action	Energy
$\hbar$ (reduced Planks constant)	Temperature

$$S_E = \int dt \left[\frac{m}{2}\dot{q}^2 + V(q)\right]$$

$$\approx H = \frac{p^2}{2m} + V(q)$$

Hamilton Jacobi Action $S_{H.J.} = \int dt \left[p \cdot \dot{q} - H(p, q)\right]$

↪ we get $\dot{p} = -V'(q)$; $\dot{q} = p/m$

QFT	Stat. Mechanics
Quantum fluctuations (because $\hbar$)	Thermal fluctuation (because T)
Tem Q $\frac{\text{Period}}{\hbar} = \frac{1}{k_B T_Q}$ $T_Q \Rightarrow$ Temperature of quantum system. $T \propto \frac{1}{\text{size in direction of Euclidean time}}$	~~Tem~~ don't identify T_Q with T_S (temperature of Statistical System)

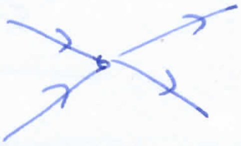
2) Interacting ϕ^4 QFT (Euclidean)

Free field : particle $\overset{m}{o}\rightarrow$

(describes particle of mass m moving freely without interacting)

ϕ^4

contact interaction.



⇒ described by ϕ^4.

$$S_E[\phi] = S_0[\phi] + \int d^4x \, \frac{g}{4!} \phi^4(x)$$

g > 0 ; coupling constant

g > 0 ⇒ repulsive interaction (stability)

(g < 0 ⇒ attracting interaction) ⇒ ie; two particles at same point gain energy ... stable .....

& since particles are bosons; you will have condensation of particles.

... can create lot of particles

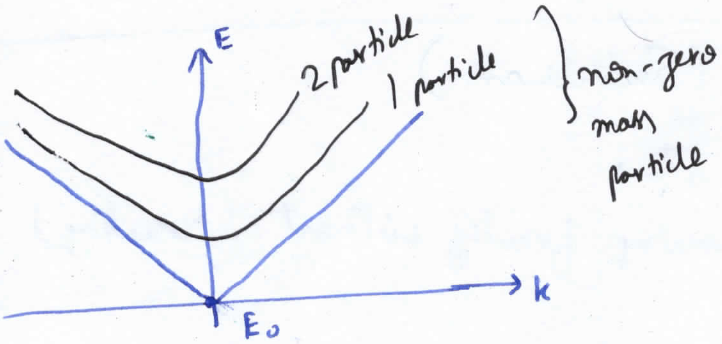
$$S_0[\phi] = \int \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2$$

Correlation Functions in perturbation theory.

$$\langle \phi(z_1)\ldots\phi(z_N)\rangle = \frac{\int D[\phi]\exp\left(-\frac{1}{\hbar}S[\phi]\right)\phi(z_1)\ldots\phi(z_N)}{\int D[\phi]\exp\left(-\frac{1}{\hbar}S[\phi]\right)}$$

We divide by this to normalize so that $\langle 1\rangle = 1$

$\Rightarrow \underbrace{\langle \ldots\ldots\rangle_0}_{\text{expectation value for free theory}}$ (replace S by S_0)



Spectrum gap

for massless particles ~~spe~~ spectrum starts from zero.

We will take the measure in ~~perturbation~~ interaction theory $D[\phi]$ to be the measure as defined for free theory $D_0[\phi]$

Expand in a series of g^k.

$$\int D_0[\phi]\exp\left(-\frac{1}{\hbar}S_0[\phi]\right)\cdot\underbrace{\exp\left(-\frac{1}{\hbar}\cdot\frac{g}{4!}\int d^dx\,\phi^4(x)\right)}\cdot\phi(z_1)\ldots\phi(z_N)$$

$$\Downarrow$$

$$= \text{---}\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-g}{4!}\right)^k\int d^dx_1\ldots d^dx_k\,\phi^4(x_1)\ldots\phi^4(x_k)\right]\text{---}$$

$$\int D[\phi]\sum_k g^k\ldots = \sum_k g^k\int D[\phi]\ldots$$

warning ⚠ Dangerous

.. this is dangerous

$$\underbrace{\int D[\phi] \sum_k g^k \dots}_{\infty \text{ Radius of convergence}} \overset{?}{=} \underbrace{\sum_k g^k \int D[\phi] \dots}_{0 \text{ Radius of convergence.}} \quad ⚠$$

...so have to use resummation methods

ex Stirling's Formula.

So by nively doing this exchange of summation & integration we get.

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \left(\frac{-g}{\hbar 4!}\right)^k \underbrace{\int D_0[\phi] \exp\left(-\frac{1}{\hbar} S_0[\phi]\right) \cdot \int d^dx_1 \dots d^dx_k \, \phi^4(x_1)\dots\phi^4(x_k)\,\phi(z_1)\dots\phi(z_N)}_{Z_0 \cdot \left[\int d^dx_1\dots d^dx_k \underbrace{\langle \phi(z_1)\dots\phi(z_N)\,\phi^4(x_1)\dots\phi^4(x_k)\rangle_0}_{\hookrightarrow \text{ free field expectation value.}}\right]}$$

$$\langle \phi(z_1)\dots\phi(z_N)\rangle_{\substack{g\neq 0 \\ \text{(interacting} \\ \text{theory)}}} = \frac{\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-g}{\hbar 4!}\right)^k \int d^dx_1\dots d^dx_k \, \langle \phi(z_1)\dots\phi(z_N)\,\phi^4(x_1)\dots\phi^4(x_k)\rangle_0}{\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-g}{\hbar 4!}\right)^k \int d^dx_1\dots d^dx_k \, \langle \phi^4(x_1)\dots\phi^4(x_k)\rangle_0}$$

Diagramatic representation in terms of Feynman Diagrams & Amplitudes.

$$\langle \underbrace{\phi\dots\phi}_{N}\ \underbrace{\phi^4\dots\phi^4}_{K}\rangle_0 = \sum_{\text{pairing}} \underbrace{\langle\phi\phi\rangle_0 \dots \langle\phi\phi\rangle_0}_{\frac{N}{2}+2k \text{ propagators.}} \quad \text{well defined object}$$

$$\int d^dx_1\dots d^dx_k \left(\langle \dots \rangle_0\right) \Longleftarrow \text{do diverge at short distances.}$$

(U.V. singularities)

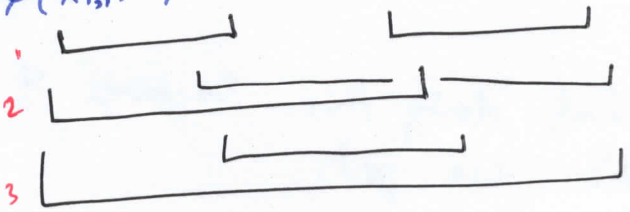
Denominator ⟵ vacuum diagrams

$N=0$, $k=0$ ⇒ 1

$k=1$ ⇒ $\frac{-g}{\hbar 4!}\int d^dx_1 \langle \phi^4(x_1)\rangle_0$

$x_1 = x_{1,1} = x_{1,2} = x_{1,3} = x_{1,4}$

$\langle \phi(x_{1,1})\, \phi(x_{1,2})\, \phi(x_{1,3})\, \phi(x_{1,4}) \rangle$



} 3 ....Wicks contraction.

$\langle \phi\phi \rangle = \hbar$ 

~~So; we set $\frac{-g}{\hbar 4!}$~~

So; we get

$$\frac{-g}{\hbar 4!}\int d^dx_1 \langle \phi^4(x_1)\rangle_0 = \frac{-g}{\hbar 4!}\hbar^2 \times 3 \times \int d^dx_1\, [G_0(0)]^2$$

$$= \frac{-g}{4!} \times 3 \times \hbar \int d^dx_1$$ 

As we see, This object is already singular, ~~if~~ because
$G(0) = \infty$ if $d \geq 2$

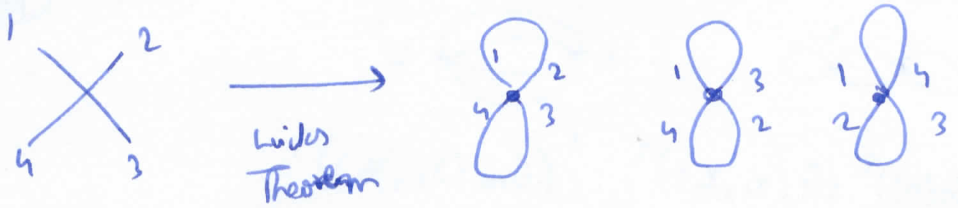
So; we have U.V. singularity (already at the simplest vacuum diagram)

$$= \boxed{\frac{-g}{8}} \times \hbar \int d^4x_1$$ 

$\boxed{\frac{1}{8}}$ ⇒ comes from symmetries of the diagram

$$= -g \times \frac{1}{8} \times \hbar \int d^dx_1$$ 

"Combinatorics comming out of the Wicks Theorem"



Same diagrams upto rellabelling of the legs of the vertex

$$\frac{3}{4!} = \frac{1}{8}$$

we normalized 8 by 4! ! because

4! ⇒ no. of possible rellabellings of the vertex.

$$\frac{1}{8} = \frac{3}{4!} = \frac{\text{\# of ways of rellabelling the graph.}}{\text{\# of possible ways of rellabelling the vertex}}$$

$$= \frac{1}{\text{\# of rellabellings that do not change diagram (labelled graph)}}$$

~~here (1→2), (3→4), (1→2)~~.

here (1→2), (3→4), [(1,2)→(3,4)]

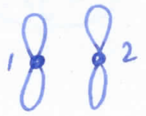
$\mathbb{Z}_2$ symmetry (the group)

8 elements = $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

$G_0(x) \simeq |x|^{2-d} = \int \frac{d^dk}{(2\pi)^d} \cdot \frac{1}{k^2+m^2} = G_0(0)$ diverge when $k \to \infty$

$N=0, k=2$: $\frac{1}{2}\left(\frac{-g}{\hbar 4!}\right)^2 \int d^dx_1\, d^dx_2 \langle \phi^4(x_1)\phi^4(x_2)\rangle_0$

get three different kind of interaction by wicks theorem



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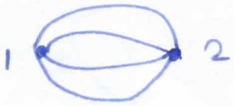


$(G(0))^4$

$=(G(0))^4$

$\frac{1}{2}\left(\frac{1}{8}\right)^2$

⇓

strong divergence

(contain ultraviolet singularity)

$(G(0))^2 (G(x_1,x_2))^2$

$=(G(0))^2 (G(x_1-x_2))^2$

$\frac{1}{16}$

⇓

divergence.

(.... contains ultraviolet singularity..)

$(G_0(x_1,x_2))^4$

$(G_0(x_1-x_2))^4$

$\frac{1}{2\cdot 4!}$

⇓

no divergence.

} Symmetry factor.

↳ can have contribution to divergence from $G(x_1-x_2)$ if x_1 is close to x_2.

(may contain ultraviolet singularity)

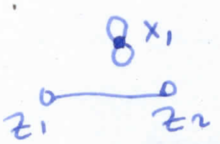
Numerator

$N=2\,;\,k=0$

$N=2\,;\,k=1$

$$\left(\frac{-g}{4!}\right)^2\int dx_1\ \langle \phi(z_1)\phi(z_1)\,\phi^4(x_1)\rangle_0$$

So; we get diagrams of the form



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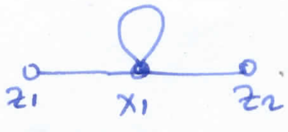


⇓

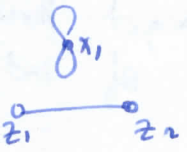
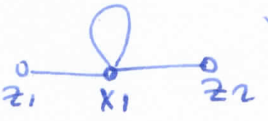
$$\int G_0(z_1-z_2)\,G_0(0)^2 dx_1$$

$\frac{1}{8}$

⇓

$$\int dx_1\ G_0(z_1-x_1)\,G_0(x_1-z_2)\,G_0(0)$$

$\frac{1}{2}$

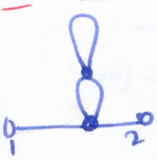
So; $-g\Big[\frac{1}{8}\Big($  $\Big)+\frac{1}{2}\Big($  $\Big)\Big]$

$N=2$, $k=2$

$$\frac{1}{2!(4!)^2}\int_{x_1}\int_{x_2}\langle\phi(z_1)\phi(z_1)\phi^4(x_1)\phi^4(x_2)\rangle_0 =$$

$\frac{1}{4}$

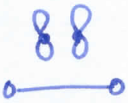
$+\ \frac{1}{4}$



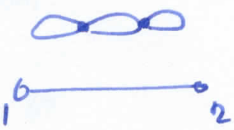
$+\ \frac{1}{6}$

$+\ \frac{1}{16}$

$+\ \frac{1}{128}$



$+\ \frac{1}{16}$



$+\ \frac{1}{48}$

Quantum Field Theory 2II   Module 3   Pg 42

Lecture 3.2| Cancellation of Vacuum diagrams, structure of perturbation theory, Generating functionals

— Shoaib Akhtar 19/5/2020.

$$\langle \phi \ldots . \phi \rangle = \sum_{k=0}^{\infty} g^k . \sum C_G \cdot I_G (z_1 \ldots z_N)$$

N external $\phi(z_i)$

Diagrams G with k internal vertices

N external vertices

G for graph.

Combinatorial factor (symmetry) ∴ comes from Wicks contraction

Amplitude (integral) for diagram G.

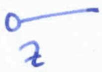
---

ex

Position Space Propagator

$G_0(y_1 - y_2)$

$\int d^d x$



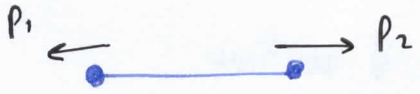
Integrate : $I_G = \int \prod_{a=1}^{N} d^d x_a \cdot \prod_{\text{lines}} G(y_\square - y_\square)$

Internal vertices

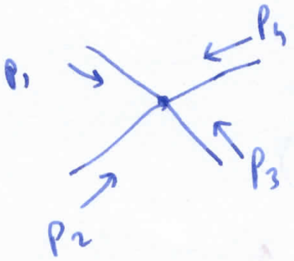
Its better to compute Feynman Diagrams in momentum representation.

Impulsion / momentum

(taking fourier transform of everything....)

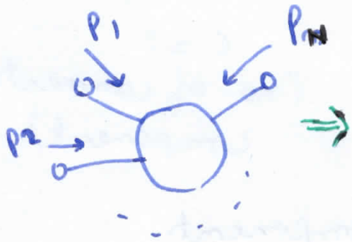


$$\hat{G}_0(p_1, p_2) = (2\pi)^d \, \delta(p_1+p_2) \cdot \frac{1}{p_1^2+m^2}$$



: $(2\pi)^d \, \delta(p_1+p_2+p_3+p_4)$

(conservation of momenta $\Leftarrow$ translation invariance)

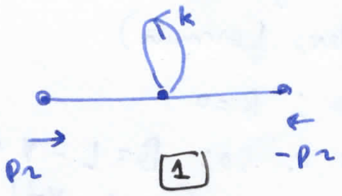


connected

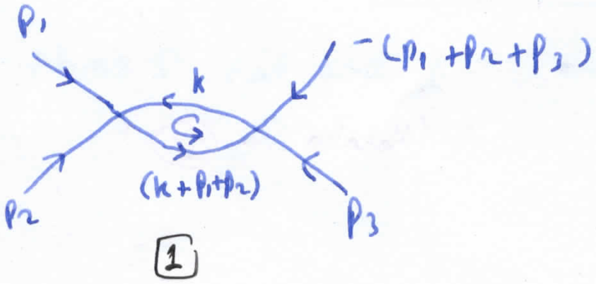
$$\Rightarrow \hat{I}_G(p_1, \ldots, p_N) = (2\pi)^d \, \delta(p_1 + .. + p_N) \times$$

$$\times \int \prod_{\substack{\text{Internal} \\ \text{momenta}}} d^d k \cdot \prod_{\text{lines}} \frac{1}{(\text{momenta})^2 + m^2}$$

ex

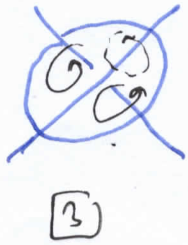


ex



How many independent internal momenta ?

ex



Theorem: (Euler) : Eulers Relation.

\# of independent momenta = \# of independent loops of diagram G

= 1^{st} Betti Number

let G be a general diagram with V vertices.

$V = V_{internal} + V_{external}$ vertices ; $V = V_{ext} + V_{int}$;

L = \# of lines , B = \# of loops.
(B for Betti)

for connected graph G : $B = L - V + 1$ ↳ because $c = 1$ (no. of connected component)

Euler Relation.

$\boxed{B = L - V + c}$ $c \Rightarrow$ \# of connected component.

---

for ϕ^4 diagram

1 line has 2 ends
(Hahaha.... ☺)

(can think of it as consequence of Eulers formula)
; line so ; $B = 0$
& $L = 1$ ∴ so; $B = L - V + 1$
$\Rightarrow 0 = 1 - V + 1$
$\Rightarrow \boxed{V = 2}$

Internal vertex has 4 "legs"

External vertex has 1 leg

so; $2L = 4V_{int} + V_{ext}$
$= 4K + N$

So; $B = 2K + \frac{N}{2} - K - N + 1$

$\Rightarrow \boxed{B = K - \frac{N}{2} + 1}$ $\boxed{B = K - \frac{N}{2} + 1}$

ħ factors in perturbation Theorem

✕ : $\left(\frac{-g}{\hbar}\right)$

— : $\hbar$ ⇒ (because $\frac{S}{\hbar}$)

Diagram G

$(-g)^k \cdot \left(\text{symmetric factor}\right) \cdot \hbar^{L-K}$

remember $B = L - (K+N) + 1$

so; G with B loops $\Longrightarrow \hbar^{B+N-1}$

Perturbation theory is an expansion in ħ ; ie; $\hbar^k \ldots$

↪ so, it is semi classical expansion.

B=0 graph; ie; tree level graphs

↪ we get theory of ħ=0 ; we get classical physics.

B=1 graph; 1 loop diagram

↪ leading order quantum correction.

$$\int D[\phi] \exp\left(-\frac{1}{\hbar}\underbrace{(S_0 + gS_{int})}\right) = \left(\sum_k g^k h^k ..\right) \hbar^{**}$$

$$S[\phi] = S_0 + g\phi^4 + g^*\phi^6 + \ldots \infty$$ ~~keep~~

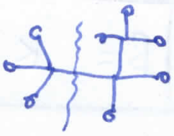
(replace by more general action.

Then; we get $\exp\left(-\frac{1}{\hbar} S[\phi]\right)$

Propagators for all times :

⇑

Correlation function



⇒ Scattering Amplitude

Generating Functional)s for the $\langle \dots \rangle$

$$Z[\vec{J}] = \int D[\phi] \exp\left(-\frac{1}{\hbar}\left(S[\phi] - J\cdot\phi\right)\right)$$

$\vec{J} = \{ J(x) , n \in \mathbb{R}^d \}$ classical field source (so we dont treat it as Random variable, but as a parameter of the theory)

$\phi = \{\phi(x)\}$

$J\cdot\phi = \int d^dx\, J(x)\, \phi(x)$

Correlation functions ⟵ $\langle 0 | T[\quad \dots \quad] | 0 \rangle$

$$\langle \phi(z_1) \dots \phi(z_N) \rangle = \hbar^N \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_N)} Z[J] / Z[0] \Big|_{\vec{J}=0}$$

$Z[J]$ ⇒ generating functional of correlation functions.

Connected Correlation Function

$$\langle \phi(z_1) \dots \phi(z_N) \rangle_{\text{connected}} = \sum_{k=0}^{\infty} g^k \sum_{\substack{G \text{ with} \\ k \text{ vertices} \\ \& N \text{ external} \\ \text{legs ; connected}}} C_G\, I_G(z_1, \dots z_N)$$

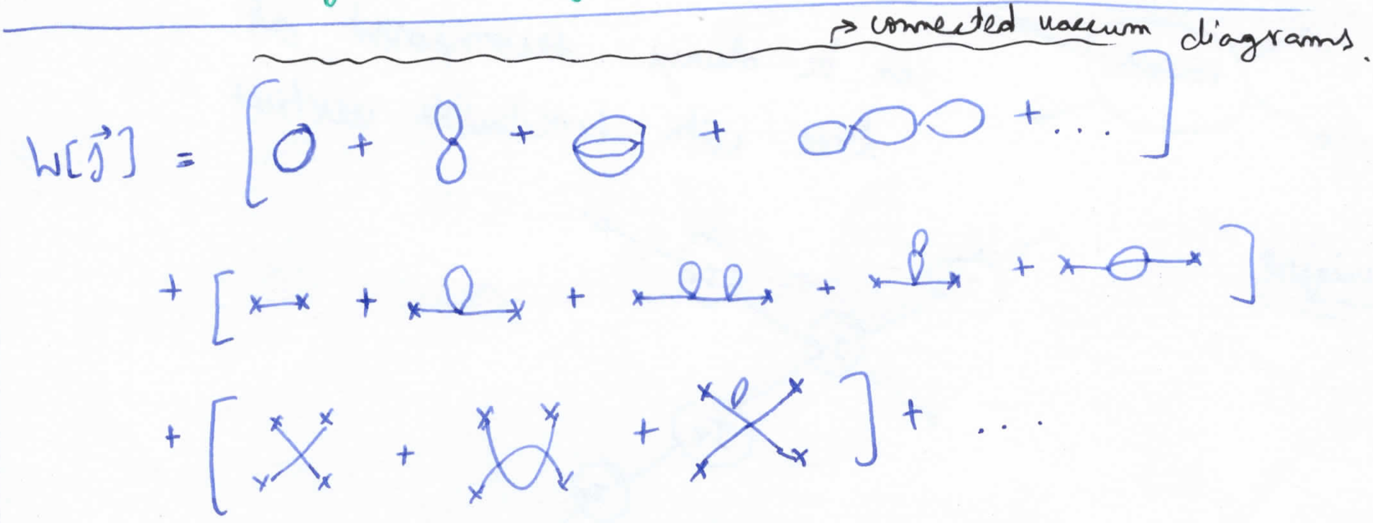
define ;

$$W[J] = \sum_{N=0}^{\infty} \frac{1}{N!} \int dz_1 \dots dz_N\, J(z_1) J(z_2) \dots J(z_N) \langle \phi \dots \phi \rangle_{\text{connected}}$$

↳ Generating Function of connected correlation functions

Theorem $W[J] = \hbar \log(Z[J])$

$W[\vec{J}] =$

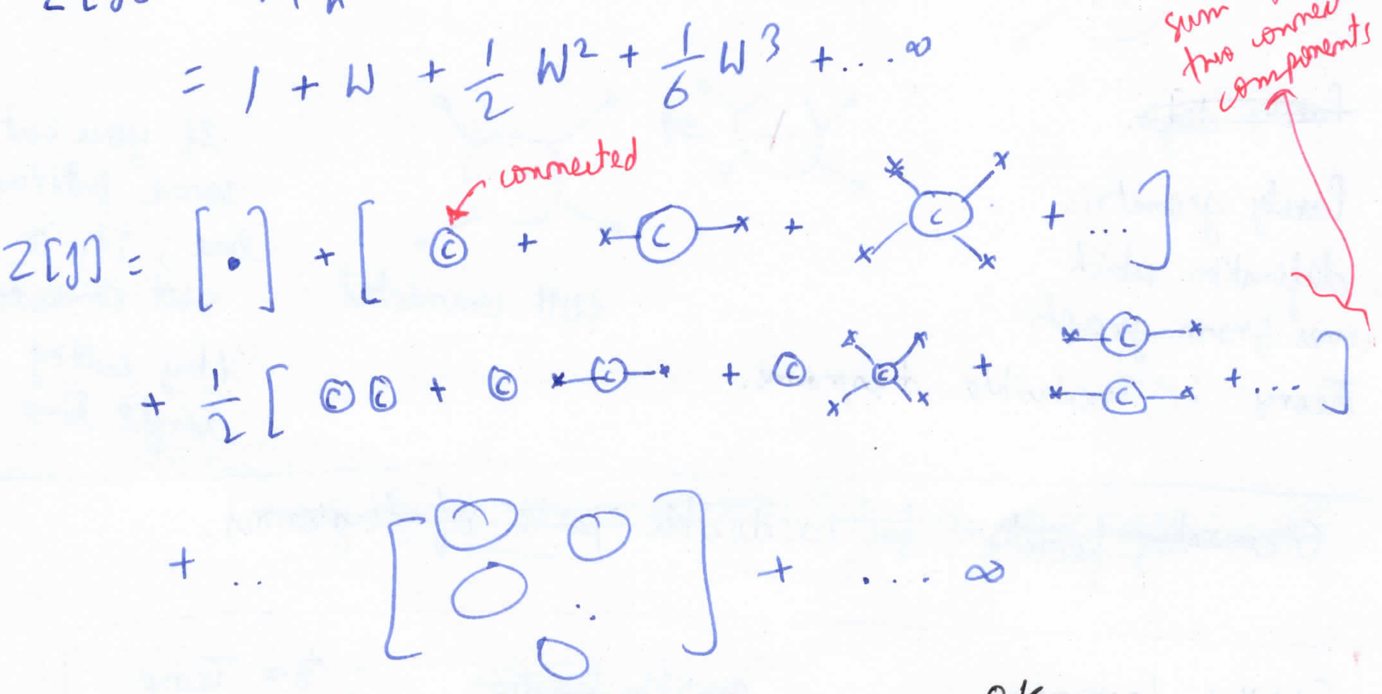


$\overset{\times}{—} = J$

$$Z[J] = \exp\left(\frac{1}{\hbar} W[J]\right)$$

$$= 1 + W + \frac{1}{2} W^2 + \frac{1}{6} W^3 + \dots \infty$$

$Z[J] =$



Combinatorics symmetric factor are OK.

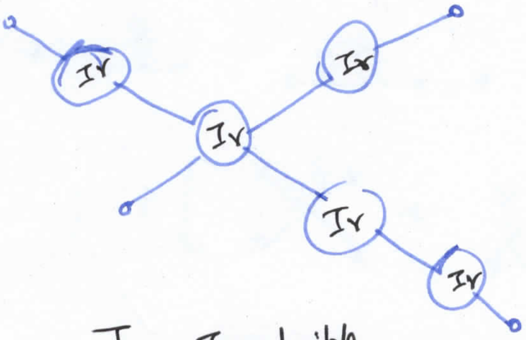
* $W[J]$ is just functional way to manupulate combinatorics of diagram.

⇒ This concept of using functions to represent combinatorial relation between objects is also an ~~inve~~ invention of Euler Leonard. ... The method of generating functions.

Irreducible Functions

can be always decomposed as tree with irreducible vertices.

unique

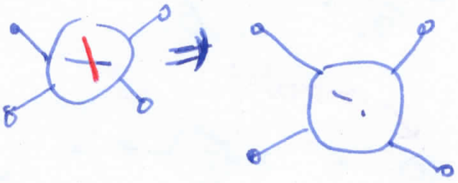


Tree Irreducible vertices

: 1-particle / line irreducible sub-diagrams

~~Purely defin~~

Purely geometric defination which comes from graph Theory : Iredueible diagrams.



still connected

If you cut some internal line ; it is still connected. (by cutting single line)

~~Generating functions for irreducible parts of diagrams.~~

Stat Mech . Lagrangian : Z partition function ; ħ = Temp

W = − Free Energy

Γ = Gibbs Potential.

$$\boxed{\langle \phi \ldots \phi \rangle_{connec} = \frac{\delta}{\delta J} \cdots \frac{\delta}{\delta J} W[J] \Big|_{J=0}}$$

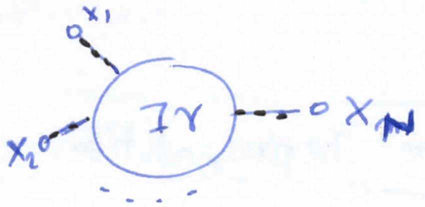
⇒ no need to divide by W[0] because we want vaccum diagrams

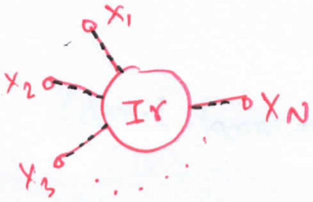
Generating functions for irreducible parts of diagrams.

we dont want to include external lines ; otherwise it will not be irreducible.

let Γ be an irreducible part.

$$I_\Gamma(x_1 \dots x_N)$$



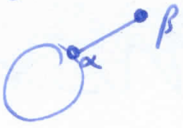


~~also~~ associate some position to external legs.

$$I_\Gamma(x_1 \dots x_N) = \int \underset{\substack{\text{Internal}\\ \text{vertices}}}{dx} \prod_{\substack{\text{Internal}\\ \text{lines}}} \underbrace{G_0}_{\text{ordinary propagator}} (x_{int} - x_{int}) \prod_{\substack{\text{external}\\ \text{lines}}} \underbrace{\delta(x_{ext} - x_{int})}$$

we dont want to include external lines because they are not part of diagram.

So, we enforce ~~so, we infore~~

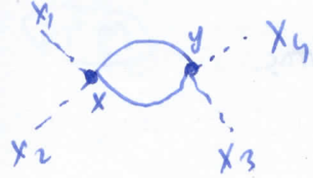


α & β are same point by introducing δ function.

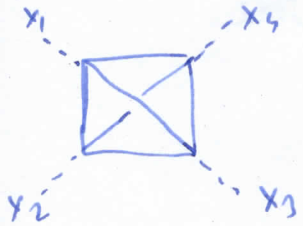
ex

$$: \int dx\, G_0(0)\, \delta(x_1 - x)\, \delta(x_2 - x)$$

$$= \delta(x_1 - x_2)\, G_0(0)$$



: $\delta(x_1-x_2)\,\delta(x_3-x_4)\left[G_0(x_1-x_3)\right]^2$.



: $G_0(x_1-x_2)\,G_0(x_1-x_3)\dots\,G(x_1-x_4)$

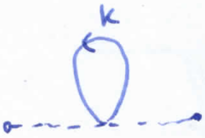
(here no delta function ~~const~~ constraint at last)

Momentum representation

$= \dfrac{1}{p^2+m^2}$ (Internal lines)

$= 1$ (External legs : Amputated legs)

→ because does not carry any momentum.



$= \int \dfrac{d^dk}{(2\pi)^d}\cdot\dfrac{1}{k^2+m^2}$

→ External legs carry no momenta.

$\dfrac{(p^2+m^2)}{1}$ ⇒ once amputated

$\dfrac{1}{(p^2+m^2)}$ ⇒ propagator

~~But Amputate single propagator. (remove left & right extremeties)~~

But,

$= (p^2+m^2)$

amputating single propagator (remove left and right extremeties) $= \dfrac{(p^2+m^2)^2}{p^2+m^2}$

so; have to amputate twice.

$\dfrac{1}{p^2+m^2}$ ⇒ propagator

$\dfrac{(p^2+m^2)^2}{1}$ ⇒ twice amputated

$$\Gamma[\phi] = \sum_{N=0}^{\infty} \frac{1}{N!} \int dz_1 \ldots dz_N \, \phi(z_1) \ldots \phi(z_N) \sum_{k=0}^{\infty} g^k \underbrace{\sum}_{\substack{\text{Irreducible} \\ \text{diagrams}}} I_{\Gamma}(z_1 \ldots z_N)$$

N amputated legs
K internal vertices

Source term for amputated functions

source term (not using the notation J . . . .. will see later why so)

$\Gamma[\phi] \Rightarrow$ generating function . . .

Theorem ~~$\Gamma[\phi] =$~~ $\Gamma[\varphi] = j_{\varphi} \varphi - W[J_{\varphi}]$

Legendre Transform of $W[J]$

$\Gamma[\phi]$ is legendre transform of $W[J]$.

We have see that one point function $\langle \phi \rangle$ is functional derivative of $W[J]$; $\langle \phi \rangle_J = \frac{\delta}{\delta J} W[J]$

functional of J

~~Keeping J~~ ~~different~~ from

Expectation value of ϕ in an external source classical field J. (ϕ is quantum field)

This is from where we start.

$$\boxed{\Gamma[\varphi] \Rightarrow \text{Effective Action of QFT (classical Quantity)}}$$

lets define $\varphi_J = \langle \phi \rangle_J$

This is a classical field; which is ~~functional~~ functional of a classical field : given by the expectation value of quantum field

$$\boxed{\varphi \Rightarrow \text{Background field (classical field)}}$$

↪ ~~classical~~ quantum fluctuations of quantum field is around this classical configuration lets assume we can ~~change vari~~ invert variables

$$\varphi_j = \langle \phi \rangle_j \quad \leftarrow - - \rightarrow \quad j_\phi$$

Invert variables.

(its possible in perturbation theorem)

φ = \varphi

ϕ = \phi

& lets say we have control on φ, not on j.

...... and then do minimization problem.

$$\Rightarrow \boxed{\Gamma[\varphi] = j_\varphi \cdot \varphi - W[j_\varphi]}$$

here j_φ as a function of φ

(not φ as function of j)

$\varphi \Rightarrow$ Background field viewed by the quantum theory.

$$W[j_\varphi] = j_\varphi \cdot \varphi - \Gamma[\varphi]$$

now: $\varphi_j = \dfrac{\delta W[j]}{\delta j}$ ie; φ is considered as function of j

so; φ_j.

$$\Updownarrow$$

$$j_\varphi = \frac{\delta \Gamma[\varphi]}{\delta \varphi}$$

Properties

① so; if W is the legendre transform of Γ

$$\underset{\substack{\downarrow \\ \text{Legendre transform}}}{(L.T.)} \circ (L.T.) = \underset{\substack{\uparrow \\ \text{Identity}}}{I_d}$$

ie; Legendre Transformation is involution

② Minimum of $\Gamma[\varphi]$: φ_0

is solution of

$$\frac{\delta \Gamma}{\delta \varphi}[\varphi_0] = 0 \Rightarrow j_{\varphi_0} = 0$$

so; find the minimum of $\Gamma[\varphi]$; you get information on vacuum of the theory $j=0$.

recall $\varphi_{j=0} = \langle \phi \rangle_{\text{vaccum}}$

i.e; $j=0$

Find the minimum of $\Gamma[\varphi] \Rightarrow \langle \phi \rangle_{\text{vaccum}}$.

Quantum Field Theory 2

[Module 4] Perturbative Renormalization & the Renormalization Group

lecture 4.1

Effective Action at one loop, Mass Renormalization in ϕ^4 Theory

— Shoaib Akhtar - 21/5/2020

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[1] Effective Action [2] Renormalization.

$\updownarrow$

1 particle irreducible - vertex diagrams (building blocks of perturbation theory)

$Z[j]=\int D[\phi]\, \exp(-\frac{1}{\hbar}[S[\phi]\} \cdot \frac{1}{\hbar} j\cdot\phi])$

$W[j]= \hbar \log Z[j]$ connected.

$\Gamma[\varphi]= j\cdot\varphi - W[j]$;

$\varphi = \langle\phi\rangle_j = \frac{\delta W[j]}{\delta j}$ Background field.

1 loop calculation (first order in $\hbar$)
(Rember; perturbation theory can be viewed as expansion in $\hbar$)

How to estimate when $\hbar \to 0$ ($\hbar$ is small) ;
dominates ϕ_c ; ϕ_c : $\frac{\delta S[\phi_c]}{\delta\phi} - j = 0$ $\rightarrow$ ϕ_c is solution of this.

~~saddle point ; $S[\phi_c]$ is minimum~~

saddle point ; $S[\phi_c] - j\cdot\phi$ is minimum

ϕ_c is a "functional" of j.

$j\cdot\phi = \int d^dx\, j(x)\, \phi(x)$

So; any general ϕ can be written as

$$\phi = \phi_c + \sqrt{\hbar}\,\tilde{\phi}$$

expand the action. $S[\phi]$

$$S[\phi] = S[\phi_c] + \frac{\hbar}{2}\tilde{\phi}\cdot S''[\phi_c]\cdot\tilde{\phi} + O(\hbar^{3/2})$$

$+j\cdot\phi - j\cdot\phi_c$

$$\tilde{\phi}\cdot S''[\phi_c]\cdot\tilde{\phi} \equiv \int d^dx_1 \int d^dx_2\, \tilde{\phi}(x_1)\tilde{\phi}(x_2)\, \underbrace{\frac{\delta^2 S[\phi]}{\delta\phi(x_1)\,\delta\phi(x_2)}}_{\text{Hessian for } S[\phi]}$$

$$S[\phi] - j\cdot\phi = S[\phi_c] - j\cdot\phi_c + \frac{\hbar}{2}\tilde{\phi}\cdot S''[\phi_c]\cdot\tilde{\phi} + O(\hbar^{3/2})$$

$$Z[j] = \exp\left(-\frac{1}{\hbar}S[\phi_c]\right)\cdot\underbrace{\int D[\tilde{\phi}]\exp\left(-\frac{1}{2}\tilde{\phi}\cdot S''[\phi_c]\cdot\tilde{\phi}\right)}_{\text{a gaussian integral.}}\overline{\left(1 + O(\hbar^{1/2})\ldots\right)}$$

~~$= \exp\left(-\frac{1}{\hbar}S[\phi_c]\right)$~~

$$W[j] = -S[\phi_c] - \frac{\hbar}{2}\operatorname{Tr}(\log(S''[\phi_c])) + j\cdot\phi_c$$

$$= -S[\phi_c] - \frac{\hbar}{2}\log(\text{"Det"}(S''[\phi_c])) + j\cdot\phi_c$$

$$\boxed{W[j] = -S[\phi_c] - \frac{\hbar}{2}\operatorname{Tr}(\log(S''[\phi_c]))}\ \boxed{+j\cdot\phi_c}$$

$$\Rightarrow \boxed{W[j] = \underbrace{-S[\phi_c] + j\cdot\phi_c}_{\substack{\text{leading term}\\ \text{"Classical contribution"}}} - \underbrace{\frac{\hbar}{2}\operatorname{Tr}(\log(S''[\phi_c]))}_{\text{quantum correction.}}}$$

take the defination $\varphi = \frac{\delta W[j]}{\delta j}$ $\qquad$ ϕ_c depends on j.

$$= \phi_c + \frac{\delta\phi_c}{\delta j}\left[\left(\underbrace{-\frac{\delta S[\phi]}{\delta\phi} + j}_{\substack{\text{by defination}\\ \text{zero ... defination}\\ \text{of } \phi_c}}\right) + \underbrace{O(\hbar)}_{\substack{\text{from}\\ \text{functional}\\ \text{derivative of}\\ \frac{\hbar}{2}\operatorname{Tr}(\log(S''\ldots)}}\right]\quad \phi = \phi_c$$

$$\varphi = \phi_c + \frac{\delta\phi_c}{\delta j}\left[\left(-\frac{\delta S[\phi]}{\delta\phi} + j\right) + O(\hbar)\right]$$

~~$\varphi = \phi + O(\hbar)$~~ $\Rightarrow$ $\varphi = \phi_c + \underbrace{O(\hbar)}_{}$ quantum correction

so, classically; Background field is equal to saddle point ϕ_c

$$\varphi = \phi_c + \frac{\delta \phi_c}{\delta J}\left[\left(-\frac{\delta S[\phi]}{\delta \phi} + j\right) + O(\hbar)\right]_{\phi = \phi_c}$$

$$\boxed{\varphi = \phi_c + O(\hbar)}$$

Background field

Classical field (saddle point)

now, if we take the definition of $\Gamma[\varphi]$

$$\Rightarrow \Gamma[\varphi] = j\cdot\varphi - W[j]$$
$$= j\cdot\varphi - j\,\phi_c + S[\phi_c] + \frac{\hbar}{2}\mathrm{Tr}(\log S''[\phi_c]) + \ldots$$

$$\Gamma[\varphi] = j\cdot\varphi - W[j]$$
$$= \underbrace{j\cdot\varphi - j\cdot\phi_c}_{} + S[\phi_c] + \frac{\hbar}{2}\mathrm{Tr}(\log S''[\phi_c]) + \ldots$$

at classical order $j\cdot\varphi - j\,\phi_c$ vanishes.

& $S[\phi_c]$ can be replaced by φ plus some quantum correction $\hbar$.

~~$S[\phi_c] = S[\varphi] + O(\hbar)$~~

$$S[\phi_c] = S[\varphi] + O(\hbar)$$

We can show

$$j\cdot\varphi - j\cdot\phi_c + S[\phi_c] = S[\varphi]$$ (no correction here)

$$\Rightarrow \boxed{\Gamma[\varphi] = S[\varphi] + \hbar\,\frac{1}{2}\,\mathrm{Tr}\left(\log\left[S''[\varphi]\right]\right) + O(\hbar^2)}$$

valid for any QFT involving scalar fields. ~~(Bosonic)~~ (Bosonic)

... can be extended to spin 1 & 2.

ϕ^4 theory $S[\phi] = \int d^4x \left(\frac{1}{2}\phi(-\Delta + m^2)\phi + \frac{g}{4!}\phi^4\right)$

periodic boundary
state.

$$0 = \frac{\delta S}{\delta \phi(x)} - J(x) = (-\Delta_x + m^2)\phi(x) + \frac{g}{6}\phi^3(x) - J(x) = 0$$

$$\boxed{(-\Delta_x + m^2)\phi_c(x) + \frac{g}{6}\phi_c^3(x) - J(x) = 0}$$

solution to this ϕ_c.

Non-linear P.D.E .... (time independent non. linear Schrodinger equation with a source)

What is Hessian

$$\frac{\delta^2 S[\phi]}{\delta\phi(x_1)\,\delta\phi(x_2)} = S''[\phi]_{x_1,x_2}$$

$$= \left(-\Delta + m^2 + \frac{g}{2}\phi^2\right)_{x_1 x_2}$$

Integral Kernel of the operator.
||
$S''[\phi]_{x_1,x_2}$

in fact;
$S''[\phi] = -\Delta + m^2 + \frac{g}{2}\phi^2$ for ϕ^4 theory.

and its a differential operator.

$S''[\phi]$ is a linear differential operator.

ψ on $\mathbb{R}^d$; we can define.

$$(S''[\phi]\cdot\psi)(x) = \left(-\Delta_x + m^2 + \frac{g}{2}\phi^2(x)\right)\cdot\psi(x)$$

$S''[\phi]$ is a linear differential operator acting on some function ; & $S''[\phi]$ depends ~~on~~ non-linearly on ϕ. ; ψ test function.
but ϕ now is parameter

$$S''[\phi]_{x_1 x_2} = \left(-\Delta_{x_1} + m^2 + \frac{g}{2}\phi^2(x_1)\right)\cdot\delta(x_1 - x_2)$$

↳ applied on integral Kernel of Identity operator

⇒ $S''[\phi]_{x_1 x_2}$ is distribution.

In perturbation theory,

$$\mathrm{Tr}\left[\log\left(-\Delta+m^2+\frac{g}{2}\varphi^2\right)\right] = \mathrm{Tr}\left(\mathrm{Log}\left[(-\Delta+m^2)\left(\mathbb{1}+\frac{g}{2}(-\Delta+m^2)^{-1}\varphi^2\right]\right)\right.$$

$$= \mathrm{Log}\left(\det\left[\underline{(-\Delta+m^2)}\left(\mathbb{1}+\frac{g}{2}(-\Delta+m^2)^{-1}\varphi^2\right)\right]\right.$$

↳ quantum contribution for effective action for free field.

$$\det(AB) = \det(A)\det(B)$$

$$= \log\left[\det(-\Delta+m^2)\det\left(\mathbb{1}+\frac{g}{2}(-\Delta+m^2)^{-1}\varphi^2\right)\right]$$

$$= \log\left(\det(-\Delta+m^2)\right) + \log\left(\det\left(\mathbb{1}+\frac{g}{2}(-\Delta+m^2)^{-1}\varphi^2\right)\right)$$

$$= \underbrace{\mathrm{Tr}\left[\mathrm{Log}(-\Delta+m^2)\right]}_{\text{contribution of free field.}} + \underbrace{\boxed{\mathrm{Tr}\left(\mathrm{Log}\left[\mathbb{1}+\frac{g}{2}(-\Delta+m^2)^{-1}\varphi^2\right]\right)}}$$

↳ lets concentrate on this ~~[illegible]~~ term... expand in g

$$= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\cdot\frac{g^k}{2^k}\cdot\mathrm{Tr}\left[\left[(-\Delta+m^2)^{-1}\varphi^2\right]^k\right]$$

$$= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\cdot\frac{g^k}{2^k}\cdot\mathrm{Tr}\left[\left((-\Delta+m^2)^{-1}\varphi^2\right)^k\right]$$

⟶ we ~~interchanged~~ interchanged summation & Trace ... should not do ... its tricky.

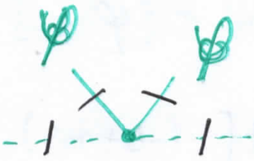
$$= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\cdot\frac{g^k}{2^k}\mathrm{Tr}\left[\left\{(-\Delta+m^2)^{-1}\varphi^2\right\}^k\right]\ \ldots$$

$\boxed{}$ will write it explicitly in terms of integral kernel.

note; $(-\Delta+m^2)^{-1}_{xy} = G_0(x-y) = x\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet y$ Propagator of free theory.

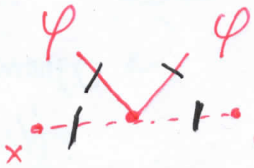
$(\varphi^2)_{xy} = \varphi^2(x)\,\delta(x-y) =$

ie;



writing it neatly.

ie; $(\varphi^2)_{xy} = \varphi^2(x)\,\delta(x-y) =$



: vertex with 2 φ attached

$\varphi(x) \to$ value of Background field at field x.

Want to compute the ħ correction term as a functional of φ background field.

$$\mathrm{Tr}\left[\{(-\Delta+m^2)^{-1}\varphi^2(x)\}^k\right] = \iint\cdots\int G_0(x_1,y_1)\cdot(\varphi^2)_{y_1,x_2}\cdot G_0(x_2,y_2)\cdot \times (\varphi^2)_{y_2 x_3}\cdots\cdots G_0(x_k,y_k)\,(\varphi^2)_{y_k\,x_{k+1}}\; dx_1\,dy_1\ldots dx_k\,dy_k$$

but $X_{k+1} = X_1$ because we are taking the trace.

→ here.... we have lot of delta function ... we can simplify it further.

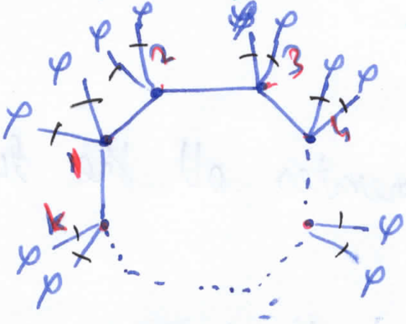
$$= \iint\cdots\int \left[G_0(x_1,y_1)\,(\varphi^2)_{y_1,x_2}\,G_0(x_2,y_2)\,(\varphi^2)_{y_2 x_3}\cdots G_0(x_k,y_k)\,(\varphi^2)_{y_k\,x_{k+1}}\,dx_1\,dy_1\ldots\ldots dx_k\,dy_k\right.$$

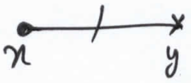
$$= \int dx_1\ldots dx_k\left[G_0(x_1,x_2)\,\varphi^2(x_2)\ldots\ldots G_0(x_k,x_1)\,\varphi^2(x_1)\right]$$

$$= \int dx_1\ldots dx_k\left[G_0(x_1,x_2)\,\varphi^2(x_2)\,G_0(x_2,x_3)\,\varphi^2(x_3)\ldots G_0(x_k,x_1)\,\varphi^2(x_1)\right]$$

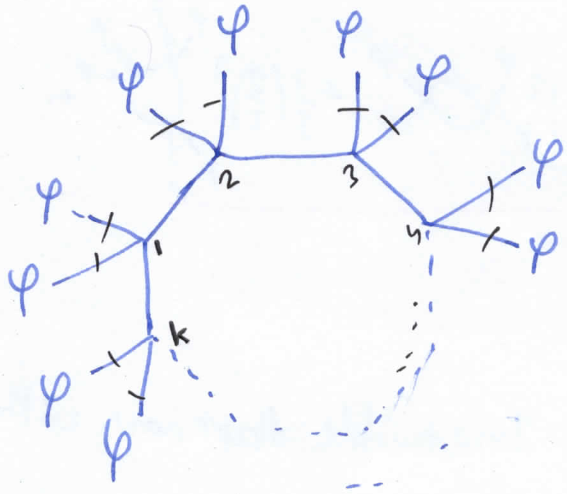
↳ Its a feynman integral ; its an integral associated to a diagram

↪ Integral of a Feynman diagram





$\varphi(y) = \varphi(x) = \int dy \; \varphi(y) \; \delta(y-x)$

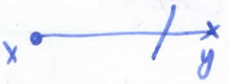


One loop diagram with 2k truncated legs.

Diagrammatic Representation

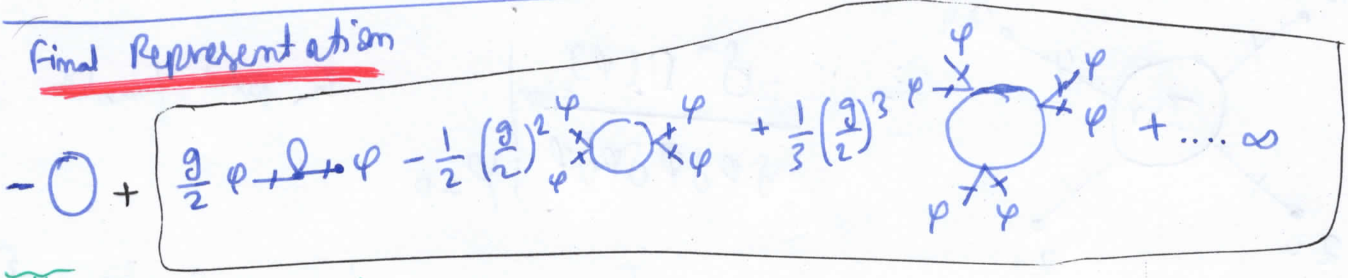
=  $= \delta(x-y)$ Truncated line

$= G_0(x-y)$ ordinary propagator



$\varphi(y) = \varphi(x) = \int dy \, \varphi(y) \, \delta(y-x)$

Final Representation



↓↓ (notation for now)
a loop with no external ~~legs~~ legs ... the free field contribution.

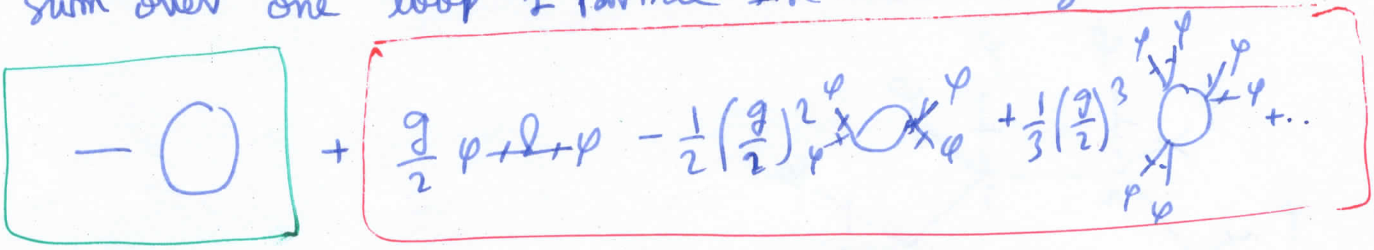
~~Inded~~ Indeed generates all one loop diagrams.

For free field we get connected diagrams with no external leg $- \bigcirc$

* we can show at ~~or~~ order $\hbar^2$, it generates all the two loop diagrams.

* at order $\hbar$, generates all the one loop diagrams.

sum over one loop 1 Particle Irreducible diagram.



$\Gamma[\varphi]$ is generating functional.

* If you want to compute what are Irreducible diagrams with two points z_1, z_2

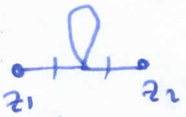
...

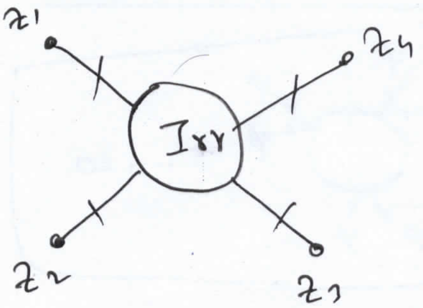
$$= \left.\frac{\delta\,\Gamma[\varphi]_{1\,loop}}{\delta\varphi(z_1)\,\delta\varphi(z_2)}\right|_{\varphi=0}$$

2 point function

because it is a generating functional.

$= \frac{\hbar}{2}g$

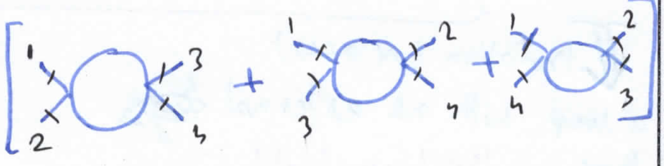


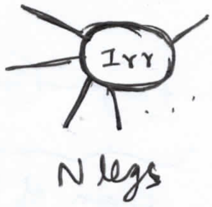


$$= \left.\frac{\delta^4\,\Gamma[\varphi]_{1\,loop}}{\delta\varphi\,\delta\varphi\,\delta\varphi\,\delta\varphi}\right|_{\varphi=0}$$

4 point function.

$= -\frac{g^2\hbar}{16}\times 8$



$$\Gamma[\varphi] = \sum_N \frac{1}{N!}\varphi^N$$ 

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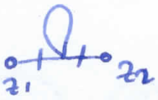
2 point Irreducible function

$$\Gamma^{(2)}(z_1,z_2) = (-\Delta+m^2)_{z_1,z_2} + \hbar\frac{g}{2}$$  $$+ O(\hbar^2)$$

: actually function of difference between two points because of translational ~~translance~~ invariance.

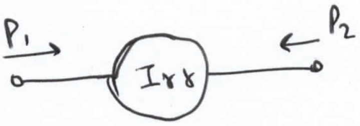
↳ lets do fourier transform

$$\hat\Gamma^{(2)}(p_1,p_2) = (2\pi)^d\,\delta(p_1+p_2)\,\hat\Gamma^{(2)}(p_1)$$

: conservation of momentum.

$$\Gamma^{(2)}(z_1,z_2) =$$ 

fourier
transform



$$\hat\Gamma^{(2)}(p_1,p_2) = (2\pi)^d\,\delta(p_1+p_2)\,\hat\Gamma^{(2)}(p_1)$$

$$\therefore\ \hat\Gamma^{(2)}(p) = p^2+m^2 + \frac{g\hbar}{2}G_0(0) \quad ;\ G_0(0) = \int\frac{d^dk}{(2\pi)^d}\cdot\frac{1}{k^2+m^2}$$

← gives terms at order $\hbar$ ← ... at $\hbar^2$.

$$\Gamma = \Gamma_{\text{classical}} + \Gamma_{1\,\text{loop}} + \Gamma_{2\,\text{loop}}$$

↑ This is just the classical action S

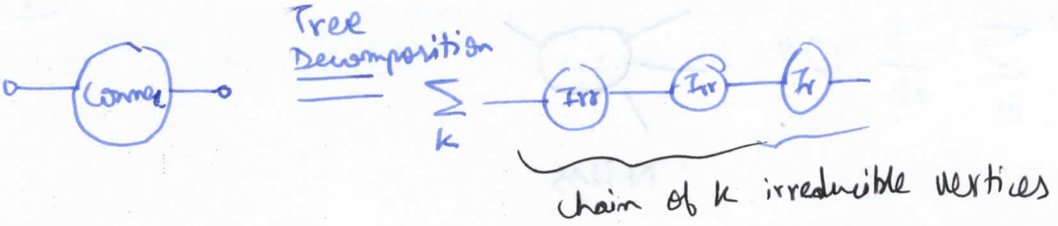
Free Theory

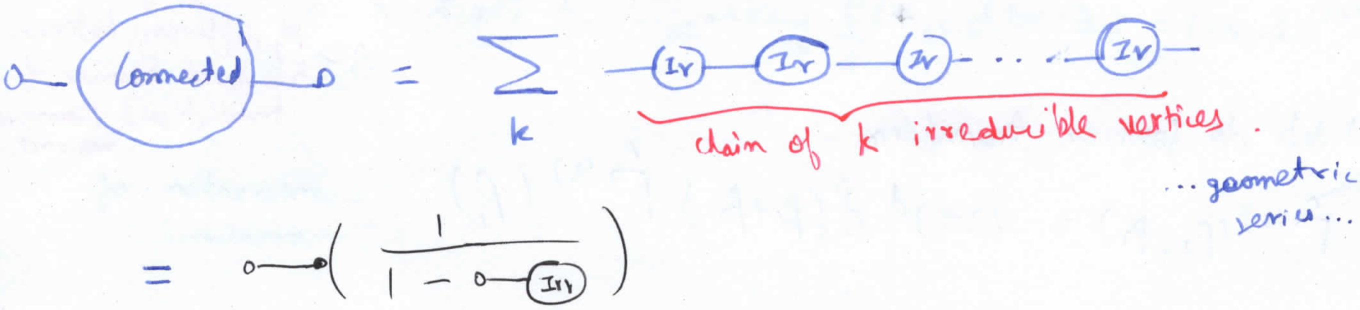
$$\hat\Gamma^{(2)}_0(p) = p^2+m^2 \ ; \quad \hat G^{(2)}_0(p) = \frac{1}{p^2+m^2}$$

$$\hat G^{(2)}(p) = \frac{1}{\hat\Gamma^{(2)}(p)}$$ ← Irreducible 2 point function.

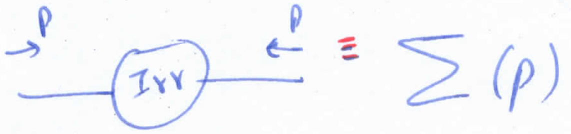
↑ connected two point function

} General Relation ... ~~true~~ true for general theory.





in p - space



(some function of p)

$$\therefore \hat{G}^{(2)}(p) = \frac{1}{p^2+m^2}\cdot\frac{1}{\left(1-\frac{1}{p^2+m^2}\Sigma(p)\right)} = \frac{1}{p^2+m^2-\Sigma(p)}$$

$$= \frac{1}{\hat{\Gamma}^{(2)}(p)}$$

$$\hat{\Gamma}^{(2)}(p) = p^2+m^2-\Sigma(p)$$ ⇒ ~~*~~ two point function in momentum space

This explains that the relation

$$\boxed{\hat{G}^{(2)}(p) = \frac{1}{\hat{\Gamma}^{(2)}(p)}}$$

is a general expression.

The diagram  has a Feynman amplitude given by the integral,

 → Amplitude of tadpole diagram $I_{\ell} = T = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2}$

U.V. divergent if $d \geq 2$ at $|k| \nearrow \infty$

(Feature of QFT)

$= G_0(x=0)$

We have seen

$G_0(x) \simeq |x|^{2-d}$ (in position space)

$\approx |k|^{d-2}$

How to deal with this problem

Modify theory to deal with short distance behavior.

Regularization Procedure

1 way) Lattice in $\mathbb{R}^d \longrightarrow \mathbb{Z}^d$

or

2nd way) Sharp momentum cut off $|k| < \Lambda$

$|\Lambda| >> m$ (so as not to change behavior of theory at physical distance)

physical scale.

3rd way) Pauli - Villars (improving cut off in smooth way)

4th way) $d \neq 4$ dimensional regularization;

but complex  ⇐ gauge theory.

5th way) The ultimate regulator l_{plank} or M_{plank}.

* l_{plank} plays the role of lattice mesh a.

~~* l_{plank} plays the role of A~~

$$T(m,\Lambda) = \int_{\substack{|k|<\Lambda \\ (\text{Ball})}} \frac{d^4k}{(2\pi)^4} \cdot \frac{1}{k^2+m^2} \qquad d=4$$

$$= \frac{1}{(4\pi)^2}\cdot\Lambda^2 - \frac{m^2}{(4\pi)^2}\log\Lambda^2 + \text{finite terms}$$

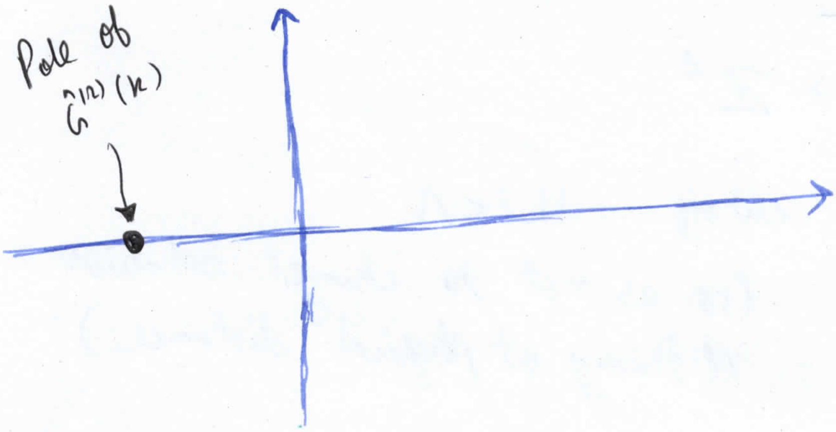
when $\Lambda \nearrow \infty$

$\uparrow$ leading quadratic divergence

$\uparrow$ sub-leading logarithmic divergence

$$G^{(2)}(x) = \int \frac{d^4x}{(2\pi)^4}\cdot e^{ik\cdot x}\,\frac{1}{\hat{\Gamma}^{(2)}(k)}$$

using $\hat{G}^{(2)}(k) = \frac{1}{\hat{\Gamma}^{(2)}(k)}$



$$|k|^2 = \vec{k}^2 \cancel{+m^2} - E^2$$
$$= \vec{k}^2 - E^2$$
$$k_E = (\pm iE, \vec{k})$$

$\hat{G}^{(2)}$ is two point function... propagator of the theory.

Euclidean.

$$\Gamma^{(2)}(k) = k^2 + m^2 + \hbar\cdot\frac{g}{2}\cdot T$$

($\Gamma^{(2)}(k)$ behaves as this)

$\uparrow$ This is divergent.

lets call $M^2 = m^2 + \hbar\cdot\frac{g}{2}T$

so: $\Gamma^{(2)}(k) = 0$ when $k^2 = -M^2$

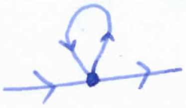
Theory has a 1 particle state with physical mass M.

↳ If free theory : 1 particle state has mass m

then in Interacting theory : 1 particle state has mass M

where $M^2 = m^2 + g\hbar \cdot \frac{1}{2} T$

$M \neq m$ quantum corrections.



↳ can think of that ; the incomming particle interacts with particles created out of vacuum fluctuations (physical languange)

- The interaction is repulsive...

all particles interact when you sit in vacuum

→ so mass increase

physical mass (M) .. what you should expect from interacting theory. > naive mass (which you expect from free theory) m

... mass is renormalized by additive correction.. by quantum effect.

---

m_0 is finite ; 1st order quantum correction

gives $M_{phys} = \infty = O(\Lambda^2)$

∴ we want a theory with finite physical mass.

ie; we want M_{phys} finite.

How can we start from free theory, to get a physical theory with finite mass.

Massless ϕ^4 theory. 1st order ~~interacti~~ in g $(\hbar)$

will have $M_{physical} = 0$

- (will have properties of conformal invariance) (scale invariance)

- simple; coupling constants are renormalized

→ quantum feature of QFT.

(Renormalization of mass is also there in QFT... but this is something which we can have in classical theory also)

↪ Renormalization of coupling constants is really the new added feature of QFT.

Lec 4.2: Renormalization of massless ϕ^4 theory at one loop, Beta function

— Shoaib Akhtar
22/5/2020

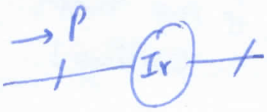
We saw, $M_{physical}$ $\neq$ m parameter in the action of the functional integral. (classical mass)

Physical mass of particle is defined as pole of two point function expressed in momentum space.

Pole in 2 point function.

Irreducible 2pt $\Gamma(p) = 0 \ni p^2 = -E^2 + \vec{k}^2 = -M^2_{phy}$

~~Eqn that defines mass of the~~



$\int D[\phi] \exp\left(-\frac{1}{\hbar} S_R[\phi]\right)$ In Euclidean space.

S_R ← Renormalized action.

$$S_R[\phi] = \int d^4x \left(\frac{A}{2} (\partial_\mu \phi)^2 + \frac{B}{2} \phi^2 + \frac{C}{4!} \phi^4 \right)$$

U.V. regulator. $|k| < \Lambda$

ϕ → renormalized field (operators) Φ (can also use this big Φ operator symbol)

↳ physical observables

Φ → operator out of which we construct physical observables

~~$\langle \phi(z_1) \phi(z_2) \ldots \phi(z_m) \rangle_R = \langle \Omega | T ($~~

So:

$$\langle \phi(z_1) \ldots \phi(z_m) \rangle_R = \langle \Omega | T(\Phi(z_1) \ldots \Phi(z_m)) | \Omega \rangle$$

Ω ↑ physical vacuum.

↳ correlation function expressed with Renormalized action.

2 point function

 $= \Gamma^{(2)}(p) = Ap^2 + B + \frac{1}{2}$  $+ \ldots \infty$

 $= \frac{1}{Ap^2+B}$;  $= C$

 $= T(m,\Lambda) = \int_{|k|<\Lambda} \frac{d^4k}{(2\pi)^4} \cdot \frac{1}{k^2+m^2} = \frac{1}{(4\pi)^2}\Lambda^2 - \frac{m^2}{(4\pi)^2}\log\left(\frac{\Lambda^2}{m^2}\right))$

Massless || Classical $\int\left[\frac{1}{2}(\partial\phi)^2 + \frac{g}{4!}\phi^4\right]d^4x$

The theory is scale invariant (since if there is no mass; there is no mass scaling)

If you have classical solution of E.O.M. $\phi_{cl}(x)$

Then $\phi_\lambda(x) = \lambda\cdot\phi_a(x)$ is also a classical solution $\lambda\in\mathbb{C}$ or $\mathbb{R}$.

In quantum theory: massless means $M_{physical} = 0$

ie; Two point function of the theory vanishes at $p^2=0$

ie; $\Gamma^{(2)}(p) = 0$ at $p^2 = 0$

$\Rightarrow B + \frac{1}{2}$  $+ \ldots = 0$.

ie; $B + \frac{1}{2}\cdot\frac{C}{A}\cdot T\left(\frac{B}{A},\Lambda\right) = 0$

condition we get

$$\boxed{B + \frac{1}{2}\cdot\frac{C}{A}T\left(\frac{B}{A},\Lambda\right) = 0}$$

so; B cannot be set to zero.
↪ You should not start will classical action in functional integral which is massles.

~~We can start from that stage, when A=1.~~
B

We can start from the stage when

$A=1$; $B = \underbrace{-\frac{\Lambda^2}{(4\pi)^2}\cdot\frac{1}{2} g_R \hbar}$ → 1 loop-counter term (ℏ correction term) ; $g_R \Rightarrow$ renormalized coupling

$C = g_R$

(∵ classically $B=0$. → Its theres to cancle the contribution of 1 loop diagram to the mass

with this choice; we see

$$M^2_{phys} = 0 + O(g_R^2)$$

→ it contains logarithmic divergence ... buts its 2 loop order.

With this notation

$$\Gamma^{(2)}(p) = p^2 + O(g_R^2)$$

↑ 2 loops.

↑ propagator of massless classical propagator.

... works only because the diagram  is independent of incomming momentum p.

... its just integral over internal momenta.

Incoming momenta comes in & flows out without coupling to internal momenta.

⟷ This is not true in general.

for instance in QED 

$\Rightarrow \log\Lambda\, \not{p}$

in fact there is logarithmic divergence

here: internal momenta gets mixed with p

The logarithm diverges in QED $\log \Lambda \cdot \not{p}$ implies that $A \neq 1$

$\rightarrow \log \Lambda \cdot \not{p}$ in QED $A \neq 1$

since $B \propto \hbar$

we can write.

$$S_0[\phi] = S_{\text{classical}}[\phi] + \hbar\, S_{(1)}[\phi] + \ldots$$

Classical part of the action

counter-term

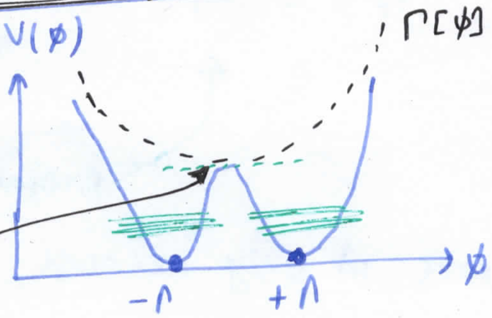
$A = 1$, $C = g_R$ is classical

$B = -\frac{\Lambda^2}{(4\pi)^2} \cdot \frac{1}{2} g_R \hbar$ is counter-term.

---

potential $g_R(\phi^4 - \Lambda^2\phi^2)$

$V(\phi) = g_R \cdot (\phi^4 - \Lambda^2\phi^2)$



effective flat potential
‖
massless

Classically we are in symmetry broken phase.

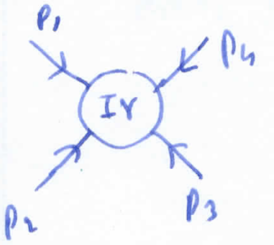
↪ Because of quantum correction (there are quantum fluctuations) so; we adjust the terms so that we have effective ~~potential~~ ~~of~~ potential of the theory $\Gamma[\phi]$

★ $\Gamma[\phi]$ is flat at its minima... (second derivative is zero; so the theory is massless)

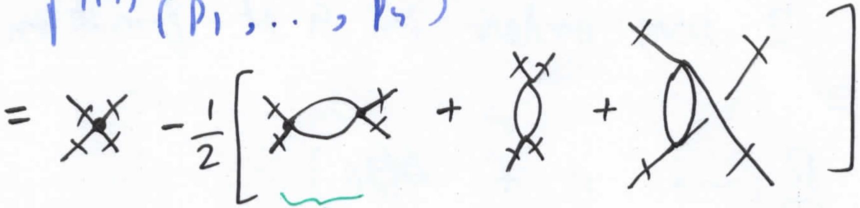
⇒ so; we ~~adjust the~~ counter term so that although classical theory looks massive with two vacuum

⟵ The quantum theory has only one vacuum and $\Gamma(\phi)$ is flat ⇒ so the theory is massless.

Is this enough? No! Consider 4pt. function.



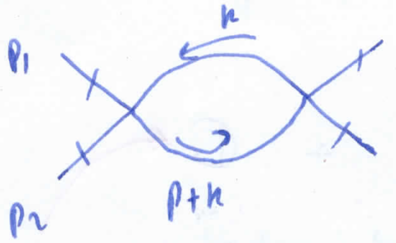
$= \Gamma^{(4)}(p_1, \ldots, p_4)$



$p_1 + p_2 + p_3 + p_4 = 0$

↳ These diagrams contains some U.V. singularities.

s, t, u channel …



$= B(p^2)$

B for Bubble

Hahaha..

$p = p_1 + p_2$

$$B(p^2) = \int_{|k|<\Lambda} \frac{d^4k}{(2\pi)^2} \cdot \frac{1}{k^2+m^2} \cdot \frac{1}{(k+p)^2+m^2}$$

$$= \frac{1}{(4\pi)^2} \log(\Lambda^2) + \text{finite terms}$$

↳ has logarithmic divergence.

(the integral behaves like ; $\displaystyle\int_{|k|<\Lambda} \frac{d^4k}{(2\pi)^4} \cdot \frac{1}{|k|^4} \quad \ldots$)

so we get.

$$\Gamma^{(4)}(p_1, \ldots, p_4) = C - \frac{1}{2} C^2 \cdot \frac{1}{A^2} \Big[B\Big((p_1+p_2)^2 ; \frac{B}{A}, \Lambda \Big) + \text{''} . + . \text{''} \Big]$$

what we must choose is indeed

$\boxed{A = 1}$ $\boxed{B = \frac{-1}{2} \cdot \frac{1}{(4\pi)^2} \cdot \Lambda^2}$ $\boxed{C = g_R + \ldots}$

Massless theory;

↳ we can treat mass = 0 in  because correction induced by mass counter term will be seen only at 2 loop order in 4 pt. function.

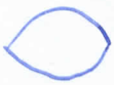
$$\frac{B}{A} \to 0 + O(g_R)$$

↳ 2 loop order in  4pt. function)

So; $B(p^2, m^2=0, \Lambda) = \frac{1}{(4\pi)^2}\cdot\log\left(\frac{\Lambda^2}{p^2}\right) + \mathbb{C}$

important to make argument of log dimensionless

constant : depends on form of regulator.

 Bubble diagram

---

Feynman rule

$$C\cdot\int dk \frac{1}{Ak^2+B} = \frac{C}{A}\int dk \frac{1}{k^2+B/A}$$

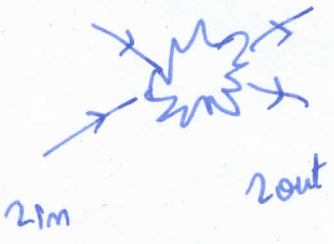
~~for vertex~~

Comes for vertex

(no rescaling of field ϕ)

How do we "define" a renormalized coupling constant in a massless (& massive) theory

ex scattering



cross section will give access to the strength of coupling constants.

(This is how coupling constant of QCD is measured in scattering experiment)

Since we are dealing with massless particles; we cannot measure interaction at rest.

At which energy you perform your experiment?

Coupling constant depends on energy

It (may) depend on the energy / momentum scale at which you are discussing physics.

In this calculation, I choose to define the coupling constant g_R as the value of 4-pt. irr. function at some reference point in momentum space.

μ : renormalization scale.

We choose symmetry point where.

$$(\bar{p}_1+\bar{p}_2)^2 = (\bar{p}_2+\bar{p}_3)^2 = (\bar{p}_1+\bar{p}_3)^2 = \mu^2$$

(specific point for Euclidean momentum)

↳ Definitely does not correspond to experiment because we are dealing with momenta with positive norm
⟹ so energy is imaginary . . . . . . but... Maths are OK

By defination, we call

$$g_R := \Gamma^{(4)}(\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4, \text{massless}, \Lambda)$$

↖ Renormalized Coupling Constant.

P has ultravoilet cut off... choose the reference point... use bar : i.e; $\bar{p}$...)

3 because of the three diagrams.

$$g_R = c - \frac{3}{2} c^2 \cdot \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right) + \cdots \infty$$

↖ choice of renormalization scale

$c \neq g_R$ (c is different from g_R; which is a coupling constant ~~a~~ measured or ~~ate~~ defined at scale μ)

With this we are done;
↳ because we want a quantum theory that makes sense at scale μ.

4 point function well defined and well behaved at $\{p_i\}$ of order μ.

In order to do this ; it is enough to adjust the coupling ~~constants~~ constants.

ie; Renormalize the coupling constant

$$C = g_R + g_R^2 \cdot \frac{3}{2} \cdot \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right)$$

With this we end with four point function of the renormalized theory.

$$\Gamma^{(4)}(p_i \;) = g_R - \frac{3}{2} g_R^2 \cdot \frac{1}{(4\pi)^2} \times \left[\log\left(\frac{\mu^2}{(p_1+p_2)^2}\right) + \log\left(\frac{\mu^2}{(p_1+p_3)^2}\right) + \log\left(\frac{\mu^2}{(p_1+p_4)^2}\right)\right] + \left[\begin{array}{l}\text{higher order terms} \\ \text{in } g_R; \\ \text{it contains } \log\Lambda \\ \text{divergences}\end{array}\right]$$

With this choice

~~A = 0~~ $A = 1$

$B = 0 + g_R \cdot \left(-\frac{1}{2} \frac{1}{(4\pi)^2} \Lambda^2\right)$

$C = g_R + g_R^2 \cdot \frac{1}{(4\pi)^2} \cdot \log\left(\frac{\Lambda^2}{\mu^2}\right)$

With this choice, 2 pt & 4 pt functions are well defined in the limit $\Lambda \to \infty$

we can check; Poincare, locality, unitary are recovered.

→ This Renormalization procedure is ~~correct~~ correct.

→ n-point functions are also finite at 1-loop.

---

g_R : renormalized coupling constant.

$m_R = 0$ (renormalized mass)

→ This is the mass which we measure & can relate it to physical mass.

(for the moment $m_R = 0$; because physical mass is ~~zero~~ zero)

The parameters B & C are called Bare parameters

Renormalized V/s **" Bare "**

$S_R[\phi] \Rightarrow$ Renormalized Action

$S_R[\phi] = \int d^4x \left(\frac{A}{2}(\partial_\mu\phi)^2 + \frac{B}{2}\phi^2 + \frac{C}{4!}\phi^4 \right)$

Bare field $\phi_B = \sqrt{A}\,\phi_R$

(we define bare field Bare field ϕ_B)

$$\boxed{\phi_B = \sqrt{A}\,\phi_R}$$

In terms of bare fields we can rewrite S as

$$S_R[\phi] = \underbrace{\int d^4x \left[\frac{1}{2}(\partial_\mu \phi_B)^2 + \frac{m_B^2}{2}\phi_B^2 + \frac{g_B}{4!}\phi_B^4 \right]}$$

→ we will call this Bare Action expressed interms of bare fields

Bare mass : $m_B^2 = B/A$

Bare coupling : $g_B = \frac{C}{A^2}$

$$S_B[\phi_B] = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi_B)^2 + \frac{m_B^2}{2}\phi_B^2 + \frac{g_B}{4!}\phi_B^4 \right]$$

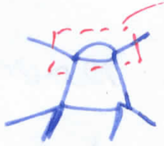
if $A \neq 1$ (if A is different from 1) ; then Field renormalization is required.

---

6 pt function

U.V. finite and it is of order $O(g^3_R)$

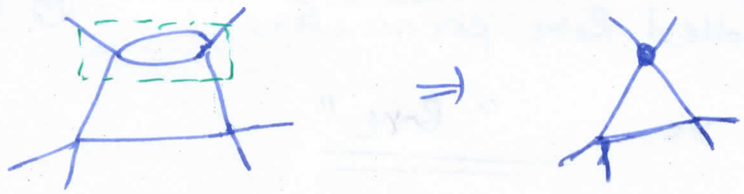
Renormalization of g_R will deal with U.V. singularities of diagram like these "



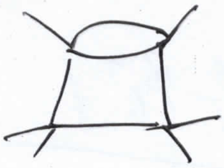
"

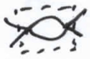
only at 2 loops

This will be giving U.V. singularity to the whole diagram ; which will be Cancelled by replacing coupling constant with g_R + counterterm ...



for higher point function ; One loop diagram are U.V. finite and renormalization will be part of the game of ~~extracting~~ substracting the U.V. divergence at two loops ... only at two loops !



This is divergent due to the sub-diagram 

~~U.V.~~ U.V. singularities can also appear because of subdiagrams like 

---

let's discuss physical significance of this Renormalization scale

Significance of Renormalization Scale μ ⟹ leads to the ~~concept~~ concept of Renormalization Group.

The 2 pt function is OK

we saw ; the 4 pt. func $\Gamma^{(4)}[\{p\}]$ depends on μ

$$\Gamma^{(4)}[\{p\}] = g_R - \frac{1}{2} g_R^2 \frac{1}{(4\pi)^2}\left[\log\left(\frac{\mu^2}{s}\right) + \log\left(\frac{\mu^2}{t}\right) + \log\left(\frac{\mu^2}{u}\right)^2\right]$$

where ; ~~$s = p_1^2 + p_2^2$~~ $s = (p_1+p_2)^2$ $t = \ldots$ $u = \ldots$ } The mandelstam variables

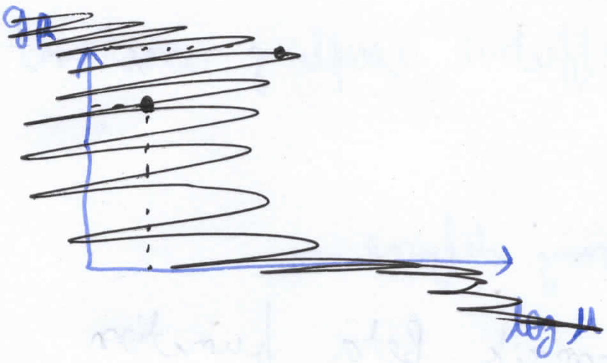
→ it seems that it depends on μ and g_R

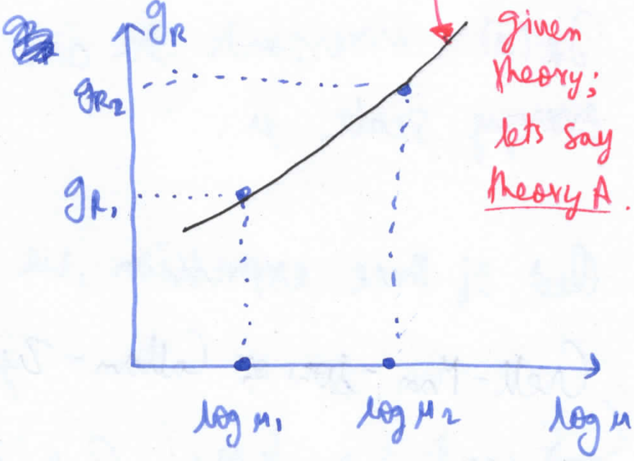
g_R is defined w.r.t. renormalized mass.

⇒ so ; for the same quantum theory (defined by its correlation functions) different $\mu \xrightarrow[\text{to}]{\text{leads}}$ different g_R

Same Quantum Theory $\quad$ different μ $\xrightarrow{\text{~~leads~~}}$ different g_R

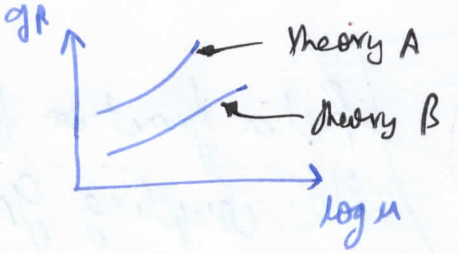
~~to~~ leads to





The curve corresponds to a given theory; lets say theory A

lets consider another system ...
you getting different curves



} These two ~~curves~~ curves dont match... so its like these theories are different

.. i.e; considering different theory with different physical property.

.. like different universe~~s~~

~~or~~ you can have ϕ^4 theory.

Theory A: can be strongly interacting theory

Theory B: can be weakly interacting theory.

If you state from a reference scale μ_0 and g_{R_0}

for same theory at scale μ ; g_R will depend on μ

ie; $g_R = g_R(\mu)$ $\quad$ (There is relation between g_R & μ)

for instance we can write; it has to satisfy.

$$\boxed{g_R(\mu) - g_R^2(\mu)\cdot\frac{3}{2}\frac{1}{(4\pi)^2}\cdot\log\left(\frac{\mu^2}{\mu_0^2}\right) = g_{R_0} + \text{(higher order terms)}}$$

↳ valid at 1 loop order.

$$\Gamma^{(4)}[\{p\}; g_R(\mu), \mu] = \Gamma^{(4)}[\{p\}; g_{R_0}, \mu_0]$$

↑ using this for any $\{p\}$.

$g_R(\mu)$ corresponds to an effective coupling constant at energy scale μ.

Out of these expression, we may define

Gell-Man-Low or Callan-Zymanzik Beta function

↳ contains how $g_R(\mu)$ depends on μ.

we start from a point & make small variation.

$$\boxed{\mu \frac{d}{d\mu} g_R(\mu)\Big|_{\substack{\mu_0,\, g_R(\mu_0)=g_{R_0} \\ \text{fixed}}} \equiv \beta_g(g_R(\mu))}$$

Beta function for the coupling g_R.

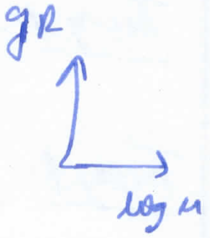
⟹ This equation is flow equation.

This is logarithmic derivative of g_R w.r.t. μ

ie; $\frac{dg(R)}{d(\log\mu)} = \mu \frac{dg_R}{d\mu}$

why we want log...

...we draw the graph



Beta function ~~too~~ concept was introduced in QED by Gellman & Low; and was made ~~p~~ system & formulated in renormalization by Callan-Zymanzik later on.

---

in our case we get

$$\left.\begin{aligned}\boxed{\beta_g(g_R) = -\frac{3}{(4\pi)^2}\cdot g_R^2} + \left(\begin{matrix}\text{2 loops}\\ \text{order}\end{matrix}\right)\end{aligned}\right\} \Rightarrow \text{flow equation}$$

Change of scale $\mu \to \mu' = S\mu$: scale transformation

multiplicative (semi)- group

(Semi because some time you cannot go down)

~~Log.~~ ⇒ β_g cannot depend explicitly on μ

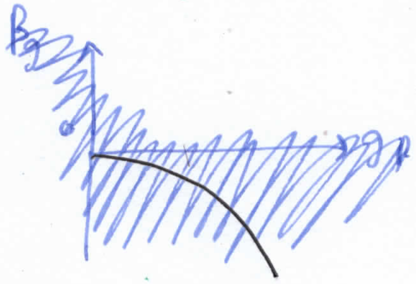
it can only ~~depen~~ depend on g_R.

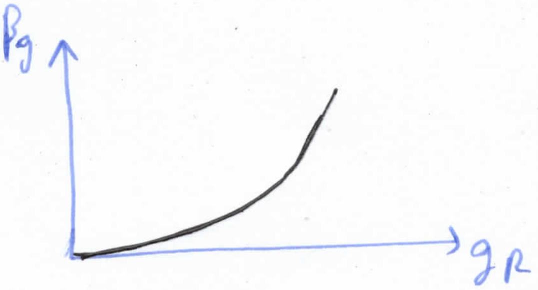
~~The coeff~~ $\beta_g(g_R) = +\frac{3}{(4\pi)^2} g_R^2$

minus sign; because you have $+(g_R)^2 \log \mu^2$ ....

here ⇒ so its related to the fact that interaction is repulsive between particles in ϕ^4 theory.

This is related to the coefficient of log Λ





If you ~~inc~~ increase the energy/momentum E;

then $g_{eff}(E)$: effective coupling constant

$g_{eff}(E)$ must obey the differential equation

$$E \frac{\partial}{\partial E} g_{eff}(E) = \beta(g_{eff}(E))$$

~~⇒ This is~~ ~~Neg~~ ~~negative~~

since $\beta(g_{eff}(E))$ is positive

⇒ E↑ $g_{eff}(E)$↑

So: ~~E increase ⇒ Geff becomes smaller~~

Since $\beta(g_{eff}(E)) \geq 0$

$$E \uparrow \quad \Rightarrow \quad g_{eff}(E) \nearrow$$

low distance $$E \searrow \quad \Rightarrow \quad g_{eff}(E) \searrow 0$$: Screening

when you have a theory where interaction becomes weaker when you go to larger distances; means you have phenomenon of screening.

Lec 4.3) Renormalization of massive ϕ^4 theory at one loop, Wilsonian Renormalization.

1. Renormalization at 1-loop.
2. Perturbative v/s Wilsonian Renormalization.

① for massless ϕ^4

g_R : Renormalized coupling. $g_B = C$ "Bare Coupling"

μ: Renormalization scale. Λ : U.V. momentum cut-off.

$$g_B = C = g_R + g_R^2 \cdot \frac{3}{2} \cdot \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right)$$

(in continuum limit set $\Lambda \to \infty$; keeping g_R fixed)

β-function (for g) : how g_R changes ~~for~~ with scale for a given theory.

$$\beta(g_R) = \mu \frac{\partial}{\partial \mu} g_R \Big|_{\text{Physics is fixed}}$$

→ can be done ~~by~~ by –

is one way:

Λ and g_B fixed $\Leftrightarrow$ $S_R[\phi]$ is fixed Λ fixed.

(The functional integral is what it is)

we have to take the continuum limit.

$$\beta(g_R) = \lim_{\substack{\Lambda\to\infty \\ g_R \text{ fixed}}} \left[\mu \cdot \frac{\partial}{\partial \mu} g_R \Big|_{g_B, \Lambda \text{ fixed}} \right]$$

If we make variation in μ : $d\mu$: it induces variation in g_r : dg_R $\qquad d\mu \to dg_R$

i.e.; $$0 = dg_R\left(1 + 2g_R \cdot \frac{3}{2}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda}{\mu}\right)^2\right) - 2\frac{d\mu}{\mu} \cdot g_R^2 \cdot \frac{3}{2}\frac{1}{(4\pi)^2}$$

$$\mu \cdot \frac{dg_R}{d\mu} = 3 \cdot g_R^2 \cdot \frac{1}{(4\pi)^2} + O(g_R^3)$$

↑ contains $\log \frac{\Lambda}{\mu}$

$$\boxed{\mu \cdot \frac{dg_R}{d\mu} = \underbrace{3 g_R^2 \cdot \frac{1}{(4\pi)^2}}_{\text{1 loop contribution}} + \underbrace{O(g_R^3)}_{\text{contribution of 2 loop}}}$$

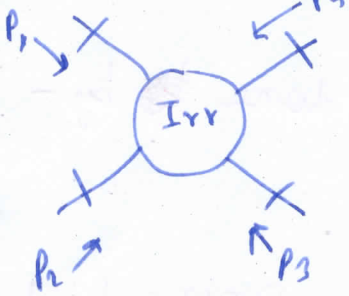
This is Beta function.

---

② • $g_R(\mu)$ find it by solving the differential equation.

... Effective coupling at scale μ.

we may consider say



$$= \Gamma^{(4)}(\{P_i\}; g_R; \mu)$$

we may ask: how does this scale with energy?

$P_i = (iE, \vec{k})$
(Euclidean spacetime)

ie; rescalling all the momenta $\quad P_i \to \lambda P_i$

λ is scaling factor.
$(\lambda \in \mathbb{R})$ (pure number)

define S as $\quad \lambda = \frac{1}{S}$ $\quad$; $S \Rightarrow$ is scaling factor is position space.

$\lambda \nearrow \infty \Leftrightarrow S \searrow 0$ small distances.

now: $\Gamma^{(4)}(\{P_i/s\}, g_R ; \mu)$

we know:

$$\Gamma^{(4)}(\{P_i\}, g_R;\mu) = g_R + \xi g_R^2 \sum_{\text{channels}} \log\left(\frac{\mu}{p}\right)^2$$

So by dimensional analysis

$$\Rightarrow \underbrace{\Gamma^{(4)}(\{P_i/s\}, g_R ; \mu)}_{\text{Classical statement}} = \underline{\Gamma^{(4)}(\{P_i\}, g_R , \mu s)}$$

now this is change in μ.

changing μ amounts to change in g_R

Quantum statement

$$= \Gamma^{(4)}(\{P_i\}, g_R(s), \mu)$$

So; renormalization theory tells us

$$\Gamma^{(4)}(\{P_i/s\}, g_R ; \mu) = \Gamma^{(4)}(\{P_i\}, g_R(s), \mu)$$

when you work out, you find $s \cdot \frac{d}{ds} g_R(s) = -\beta(g_R(s))$

$g_R(s)$ is just the effecting (or running) coupling at energies scaled by a factor s^{-1} or distances scaled by factor s.

$$\Gamma^{(4)}(\cdot, g_R, \mu) = \Gamma^{(4)}(\cdot, g_R(\mu'), \mu')$$

choosing $\mu' = \mu s$ & we ask value of $g_R(\mu')$.
we get it by $s \cdot \frac{d}{ds} g_R(s) = -\beta(g_R(s))$

---

③ The fact that $\beta(g) \neq 0$: scale invariance is analmolous broken by quantum effect.

Scale Anamoly.

Classical ϕ^4 theory in $d=4$ is scale invariant

which means $\phi(x) \to \phi_s(x) = s\,\phi(sx)$ (scale transformation)

then $S[\phi_s] = S[\phi]$. (symmetry)

$\Downarrow$ (Noether's Theorem)

current

(Rescaling around origin)

There is a current J^μ_{scale} associated to scale invariant.

$$\boxed{J^\mu_{scale} = T^\mu{}_\nu \cdot x^\nu + \phi\,\partial^\mu\phi}$$

current associated to scale anamoly; when you dialate around the origin

i.e; $x \to s\,x$

we can check;

classically $\partial_\mu J^\mu_{scale} = 0$

compute J^μ_{scale} in quantum theory.

~~locally~~ ~~Scaling~~.

recall; $T_{\mu\nu} = \partial_\nu\phi\,\partial_\nu\phi - \delta_{\mu\nu}\mathcal{L}$

$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi) + \frac{g}{4!}\phi^4$

locally scalling is some kind of translation

↪ so; intuitively its clear that J^μ_{scale} must involve $T^\mu{}_\nu$

$$J^\mu_{scale} = \underbrace{T^\mu{}_\nu x^\nu}_{\Downarrow} + \underbrace{\phi\,\partial^\mu\phi}_{\Downarrow}$$

Comes from dialation of space

comes from dialation of field.

At 1 loop in quantum theory: short distance singularity ⚠

$$\partial_\mu J^\mu_{scale} = g_R^2 \cdot \frac{3}{(4\pi)^2} \cdot \frac{\phi^4}{4!}$$

$\frac{\phi^4}{4!}$ → also proportional to the interaction.

g_R^2 ↪ one loop correction of order g_R^2.

$$\partial_\mu J^\mu_{scale} = \underbrace{g_R^2 \cdot \frac{3}{(4\pi)^2}}_{\text{one loop contribution.}} \cdot \frac{\phi^4}{4!} \neq 0$$

~~anat~~ anamoly

(symmetries broken by anamoly)

$$= \beta(g_R) \cdot \frac{\phi^4}{4!} \quad \left(\underset{\text{Related to}}{\sim} T^\mu_{\ \mu} \right)$$

here scale invariance symmetry is broken by quantum effects.

so; $\beta(g_R)$ can be viwed as anamoly

β function = scale anamoly.

$T^\mu_{\ \mu}$ is dimensionless. (trace is dimensionless)

because $\underset{\text{dimension}}{g^{\mu\nu}} \; \underset{\text{dimension}}{T_{\mu\nu}}$ } as a whole dimensionless. ~~dimensionless~~.

for more : CFT & String Theory.

---

④ Massive Theory : $M_{phys} \neq 0$.

means that in Renormalized action ....

$$S_R[\phi] = \int d^4x \left(\frac{A}{2}(\partial_\mu \phi)^2 + \frac{B}{2}\phi^2 + \frac{C}{4!}\phi^4 \right)$$

for massless theory we should

$A = 1$

$B = -\Lambda^2 - g_R \cdot \frac{1}{(4\pi)^2} \cdot \frac{1}{2}$

$C = g_R + g_R^2 \cdot \frac{3}{(4\pi)^2} \cdot \frac{1}{2} \log\left(\frac{\Lambda^2}{\mu^2}\right)$

~~for massive~~

for massive theory ; we ~~expect~~ expect.

B has an additional contribution.

$$= T = z\,\Lambda^2 + m^2 \cdot z \log \Lambda$$

Some factors.

There is logarithmic divergence proportional to mass; which you have adjust in order to have a theory which is U.V. finite for massive theory.

So, we finally find (after calculation)

$$B = \left(-\Lambda^2 \cdot g_R \cdot \frac{1}{(4\pi)^2} \cdot \frac{1}{2}\right) + m_R^2 + \left(m_R^2 \cdot g_R \cdot \frac{1}{(4\pi)^2} \cdot \frac{1}{2} \log\left(\frac{\Lambda^2}{\mu^2}\right)\right)$$

These are one loop counter-terms

$$B = \left(-\Lambda^2 \cdot g_R \cdot \frac{\hbar}{(4\pi)^2} \cdot \frac{1}{2}\right) + m_R^2 + \left(m_R^2 \cdot g_R \cdot \frac{\hbar}{(4\pi)^2} \cdot \frac{1}{2} \log\left(\frac{\Lambda^2}{\mu^2}\right)\right)$$

$$C = g_R + g_R^2 \cdot \frac{3}{(4\pi)^2} \cdot \frac{\hbar}{2} \log\left(\frac{\Lambda^2}{\mu^2}\right)$$

$$A = 1$$

This is new counter-term which we have to adjust for theory to be ~~to~~ massive & U.V. finite.

we find now

$$\Gamma^{(2)}(p) = p^2 + M^2_{phys} + O(g_R^2)$$

2 loop effects.

$$\boxed{M^2_{phys} = m_R^2 \left[1 + g_R \cdot \frac{1}{(4\pi)^2} \cdot \frac{\hbar}{2} \log\left(\frac{m_R^2}{\mu^2}\right)\right]}$$

so there is an effective renormalized for particles

⇒ but M_{phys} is fixed.

m_R ⇒ renormalized mass

g_R ⇒ " coupling.

we see;

change in μ must be re absorbed in change in g_R and m_R in order to keep the same physics.

change in μ $\longleftrightarrow$ change in g_R & m_R

physical mass M_{phys} can be observed.

★ Physical observables like M_{phys} corresponds to quantity which we can measure.

★ Renormalized parameters m_R, g_R corresponds to coordinates in the space of theory.

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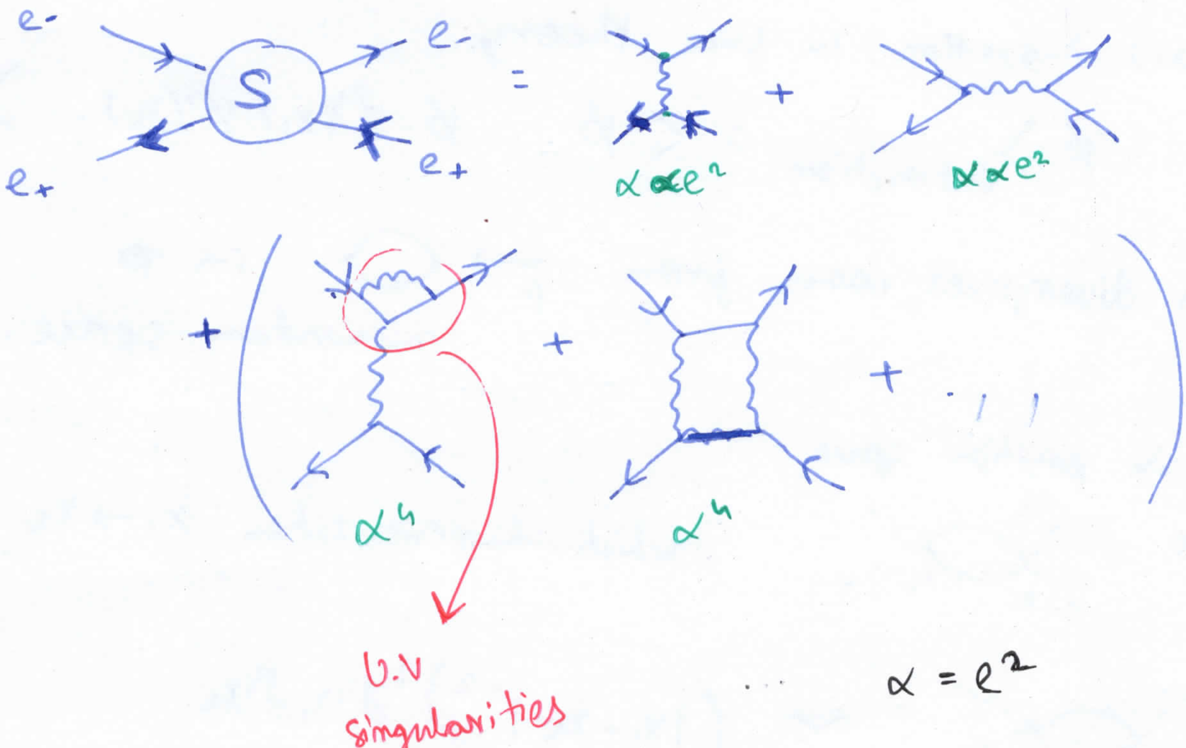
example

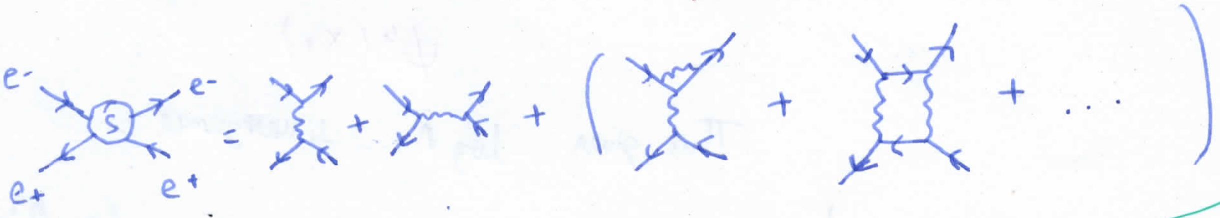


$\alpha = e^2$

..so; have to renormalize ~~electr~~ electron charge e.

α_R =

Feynman diagrams are very important. But keep in mind .. they are tools!



→ sometimes people avoid it in CFT .. there are also other tools.

Why do Renormalization works?
What if $d \neq 4$?

Short distance singularity in QFT are proportional to local operators. (consequence of QFT being "LOCAL" theories)

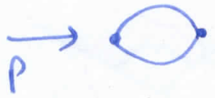
locality $\Rightarrow$ Causility, Lorentz Invariance.

no Faster than Light (FTL) effects (physical)

When you compute $\langle \phi \ldots \phi \rangle_{\text{interaction}}$ ~~in~~ correlation function in interacting theory; you reduce it to computation of $\langle \phi \ldots \phi \; \phi^4(x_1) \phi^4(x_2) \ldots \rangle_0$ with $\phi^4(x_i)$ insertion in free theory.

$$\langle \phi \ldots \phi \rangle_{\text{interaction}} = \langle \phi \ldots \phi . \phi^4(x_1) \, \phi^4(x_2) \ldots \rangle_0$$

famous divergences comes from



in momentum space.

Which in position space is

which happens when $x_1 \to x_2$

$x_1 \to x_2$

$$\propto \left(|x_1 - x_2|^{-2} \right)^2 d^4x_1 \, d^4x_2$$

when $x_1 \to x_2$ this diverges logarithmically.

$y = x_1 - x_2$

$$\approx \int \frac{d^4y}{|y|^4} \times \int d^4x_1$$

$\parallel$

$\phi^4(x_1)$

This gives $\log \Lambda$ divergence.

$$\Lambda \sim \frac{1}{(\text{minimal distance})}$$

; so when we come to this graph we see; divergence is proportional to ϕ^4 operator.

So; we are interested in

$$\phi^4(x_1)\,\phi^4(x_2) \underset{x_1\to x_2}{=}$$

this feynman diagram tells that part of products of these two operators diverges like $|x_1-x_2|^{-4}\cdot\phi^4(x_1)$

$\overset{\times}{\phi^4}\ \overset{\times}{\phi^4} =$  $+$  $+$ 

$\Downarrow$ $\mathbb{1}$ — diverge like $(|x_1-x_2|^{-2})^4$

$\mathbb{1} \Rightarrow$ something that does not depend on whats going on.

$\Downarrow$ $\phi^2;(\partial\phi)^2$ — diverge like three propagator $(|x_1-x_2|^{-2})^3$

get ~~so~~ second derivative by solving it little further.

$\Downarrow$ ϕ^4 — diverge like two propagator $(|x_1-x_2|^{-2})^2$

$$\boxed{\phi^4(x_1)\,\phi^4(x_2) \underset{x_1\to x_2}{=} |x_1-x_2|^{-8}\,\mathbb{1}(x_1) + |x_1-x_2|^{-6}\,\phi^2(x_1) + |x_1-x_2|^{-4}(\partial\phi)^2 + |x_1-x_2|^{-4}\,\phi^4(x_1)}$$

.... Operator Product Expansion. (Wilson)

(a very important feature of QFT; at short distance you can expand products of operators in terms of sum of local operators)

we see diverging coefficients like $|x_1-x_2|^{-8}$ which gives rise to U.V singularity are proportional to local operators are $\mathbb{1}(x)$ or $\phi^2(x_1)$, etc.

This is why all the U.V. singular terms reorganize themselves into local operators ... & by renormalization you can control them.

If here on RHS we had non-local operator or some wild object : theory would not be renormalized.

⇒ diverging coefficients can be computed by dimensional analysis of local operators.

$D = 4$: $\phi^4 \times \phi^4 = |y|^{-4} \phi^4$ Renormalizable.

$D<0$: $\phi^4 \times \phi^4 = |y|^{-2(D-2)} \phi^4$ Super Renormalizable.

D ⇒ space time dimensions

$D>4$: $\phi^4 \times \phi^4 =$ badly divergent ~~badly~~ ~~divergent~~ : Non-renormalizable

---

QFT Renormalization . (Gall. Man, Low, Callen-Zymanzik)

BPHZ : "Theorem" due to those mathematical physicist ; which tells renormalization works at all order.

Wilson-Renormalization Group.

Renormalization scale μ (This is the scale at which we are doing experiments & measuring things)
(we want to understand physics in this domain)

we introduce U.V. ~~co~~ cut off Λ

HEP ⇒ High Energy physics.

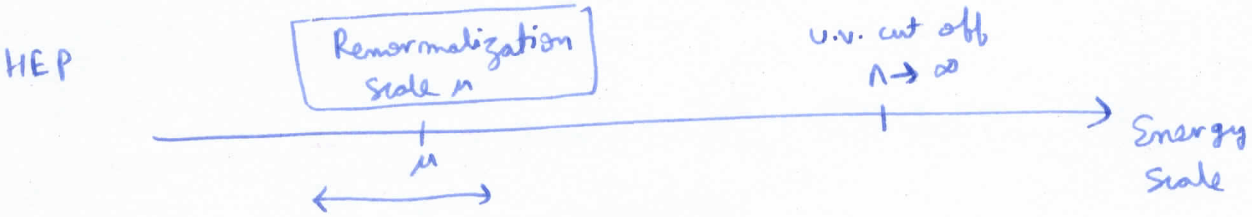
$(\Lambda >> \mu)$

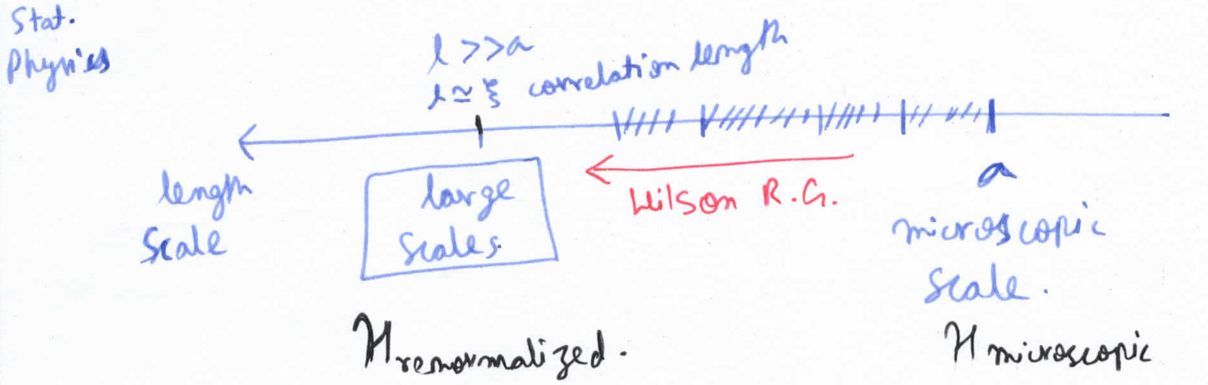
QFT ⟷ Statistical mechanics.

$\int \mathcal{D}[\phi] e^{-S[\phi]}$ for ϕ^4 theory is very similar to partition function of a model of statistical mechanics $\sum_{\sigma_i} \exp\left(\frac{-1}{T}\mathcal{H}[\sigma_i]\right)$

for ϕ^4 ; $\sum_{\sigma_i} \exp\left(\frac{-1}{T}\mathcal{H}[\sigma_i]\right)$

it is Landau - Ginsberg - Wilson Hamiltonian which describes physics ~~of~~ near critical ~~Point~~ point of Ising model.





integrate out short distance physics by steps

Wilson R.G. (Renormalization Group)

having $\xi \to \infty$ means zero renormalized mass

Wilsonian like calculation of renormalization in ϕ^4 at 1 loop; and show the connection.

We will do calculation in Local potential approximation.

↳ relies on the defination of Effective Action $\Gamma[\phi]$ at one loop. will be used.

Quantum Field Theory 2

Module 5 Renormalization Group Flow (Pg93)

Wilson Renm. &

Lecture 5.1 Wilson Renormalization

— Shoaib Akhtar 28/5/2020.

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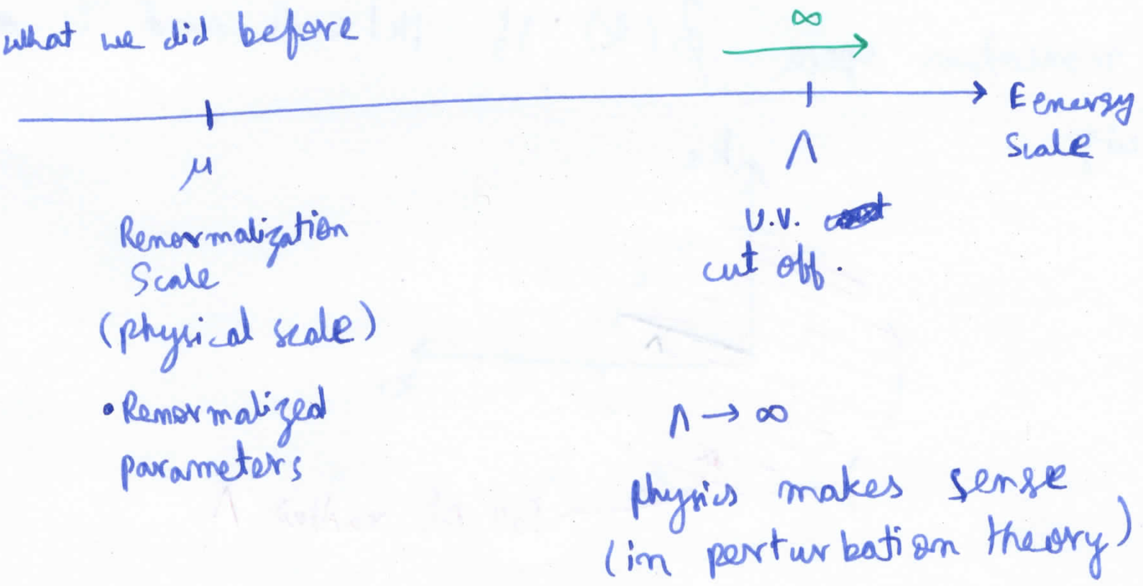
① Renormalization of ϕ^4 as viewed from Wilsonian.

② Non-perturbative issues about renormalizability of ϕ^4

This is what we did before



Start with a theory with sharp U.V. cut off Λ at this energy scale, take a QFT with an action $\mathcal{A}[\phi]$ (not renormalized Action)

$\mathcal{A}[\phi]$ is action with sharp U.V. cut off Λ

$\mathcal{A}[\phi]$ is same as $S[\phi]$ & $\mathcal{H}[\phi]$ in stat-mech lecture.

$$\mathcal{A}[\phi] = \underset{\substack{\uparrow \\ \text{previous} \\ \text{lecture}}}{S[\phi]} = \underset{\substack{\uparrow \\ \text{stat-mech} \\ \text{lecture.}}}{\mathcal{H}[\phi]}$$

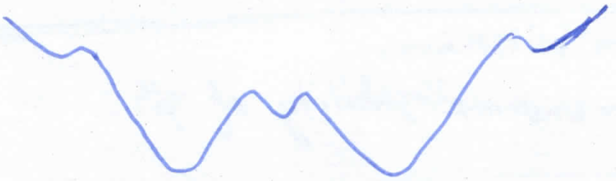
so; The theory is defined by

$$\int \mathcal{D}[\phi] \exp(-\mathcal{A}[\phi]) \qquad \text{set } \hbar = 1.$$

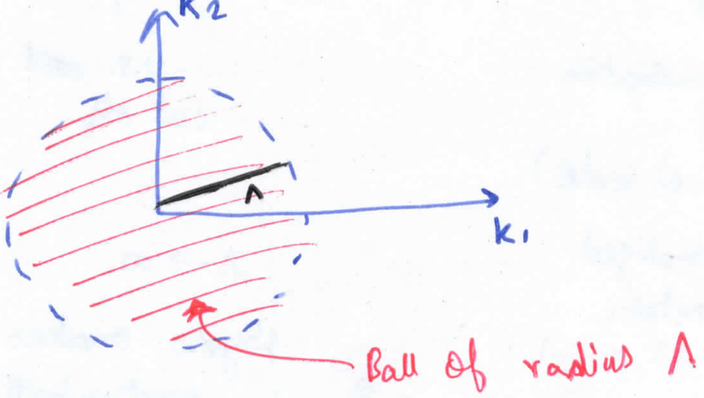
$$\mathcal{A}[\phi] = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi)^2 + V(\phi)\right]$$: general potential as an interaction term.

We can take $V(\phi)$ as

$$V(\phi) = t_0 + \frac{t_2}{2}\phi^2 + \frac{t_4}{4!}\phi^4 + \frac{t_6}{4!}\phi^6 + \dots \infty$$



In momentum space $\tilde{\phi}(k)$ if $|k| < \Lambda$ and 0 ~~otherwi~~ otherwize



From the path integral; the theory is defined by effective action.

$$\Gamma[\phi] = A[\varphi] + \frac{1}{2}\mathrm{Tr}\left[\mathrm{Log}\left(-\Delta_x + V''(\varphi)\right)\right]$$

Quantum

Classical Action

Effective action at one loop

(setting $\hbar = 1$)

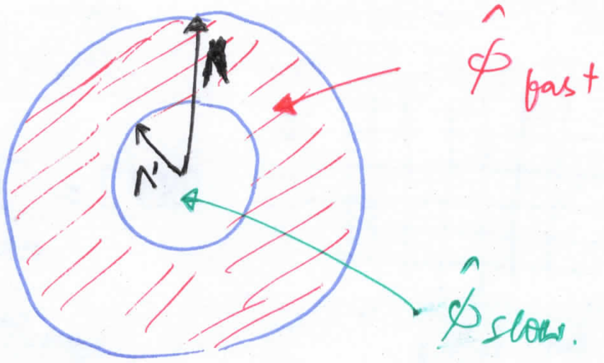
(we insist that we neglect other terms)

*Note] All the physics of the Quantum System is contained in Effective Action... because out of it you can reconstruct correlation functions of the theory... ...so contains the Quantum physics.

Wilsonian Framework

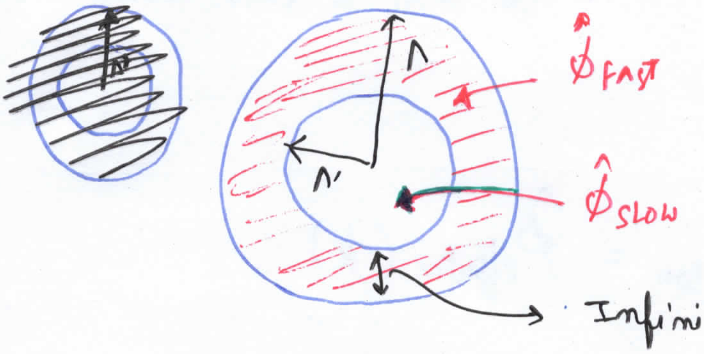
seperate ϕ into ϕ fast & ϕ slow.

$$\phi = \phi_{fast} + \phi_{slow}.$$



ϕ_{fast} comes from momentum shell k between Λ and $\Lambda/$(some scaling factor)

ie; Momentum shell $\Lambda' = \frac{\Lambda}{s}$; $\Lambda' < |k| < \Lambda$



scaling factor $\underline{S > 1}$

$S = 1 + \varepsilon$

Infinitely small shell of width $\varepsilon . \Lambda$.

Renormalization Procedure

$$Z = \int D[\phi] \exp(-A[\phi])$$

decompose Z as

step 1

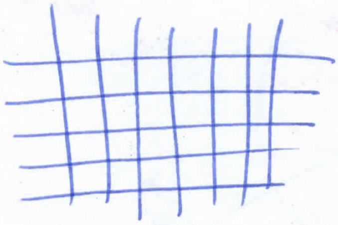
$$Z = \int D[\phi_{slow}] \cdot \underbrace{\int D[\phi_{FAST}] \cdot \exp\left(-A[\phi_{fast} + \phi_{slow}]\right)}$$

$$= \int D[\phi_{slow}] \cdot \exp\left(-A_{effective}[\phi_{slow}]\right)$$

- Integration over the fast mode — "Block-spin transformation" in stat-mech if you are on lattice.

- "Decimation procedure"

$$A[\phi] \longrightarrow A_{effective}[\phi_{slow}]$$



on lattice

$$D[\phi] = \prod_{\text{sites}} d\phi(x) \quad *$$

$$= \prod_{\substack{\text{Fourier} \\ \text{modes}}} d\hat{\phi}(k) \quad *$$

↳ some normalization factor.

$\phi(x) \longrightarrow \hat{\phi}(k)$ linear.

∴ so there is some Jacobbian.

$$A \to A_{effective}$$

$\hat{\phi}(k)$

$|k| < \Lambda$

$$\hat{\phi}_{slow} = \hat{\phi}_{effective}(k)$$

$$|k| < \frac{\Lambda}{(1+\varepsilon)} = \Lambda(1-\varepsilon)$$

So.. they dont live in same space of function.

Step 2: Rescale (to compare A & $A_{effective}$)

- positions x i.e; momenta k

  $x \to x'$ $k \to k'$ so that, $|k'| < \Lambda$

  ~~so that $|k'| < \Lambda$~~

- Field $\phi_{effe} \longrightarrow \phi'$ when you make decimation procedure, you average fields ......

  i.e; $\phi_{eff}(x) \longrightarrow \phi'(x')$ rescale by some factor

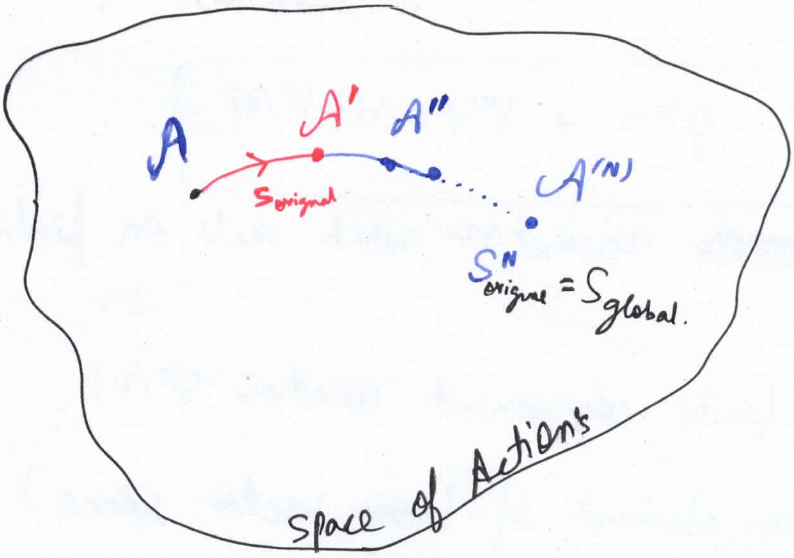
Part of rescaling field has choices .. but their is one convinient choice.

Rescale .... so that

$$\boxed{A_{\text{eff}}[\phi_{\text{eff}}] = A'[\phi']}$$ its an identity.

and we will call ϕ' to be "Renormalized field variable" & A' "renormalized action"

and of course $D[\phi_{\text{eff}}] = D'[\phi']$ (trivial)



$\nearrow_s$: performing Renormalization group action.

Step 3] Iterate.

rescaling by a factor S^N gives $A^{(N)}[\phi^{(N)}]$

---

if $s_{\text{original}} = [1+\epsilon]$

if you take N large ; $\boxed{N = \frac{s}{\epsilon}}$

Then $S^N_{\text{original}} = (1+\epsilon)^{s/\epsilon} = \exp(s) = S_{\text{global}}$.

Renormalization group can be viewed as some kind of finite difference approximation scheme for performing a direct path integral calculation.

- Local Potential (effective) approximation.

- $V(\phi)$ is the local potential. [illegible]
  no higher derivative terms.

- $$\text{Tr}[\text{Log}(-\Delta + V''(\phi))] = \int d^dx \, \langle x| \underbrace{\text{Log}(-\Delta + V''(\phi))}_{\text{after taking log it } \cancel{\text{doe}} \text{ becomes highly non-local.}} |x\rangle$$

  ... non-local operator.

$$\boxed{[-\Delta + V''(\phi)]\,\psi(x) = -\Delta_x \psi(x) + V''(\phi(x))\cdot\psi(x)}$$

$[-\Delta + V''(\phi)]$ → an ~~appres~~ operator which acts on fields.

$$\psi = \begin{pmatrix} \psi(x_1) \\ \psi(x_2) \\ \vdots \end{pmatrix}$$ an infinite component vector $\psi(x)$.

(an element of linear vector space)

x labels the components of the vector ψ.

Taking trace means, taking sum over diagonal elements.

bracket Notation of dirac. $\psi(x) = \langle x|\psi\rangle$

$|x\rangle$ basis

$$\text{Tr}(O) = \sum_x \langle x|O|x\rangle$$

sum over $\sum_x$ x means integral $\int dx$

$$\Rightarrow \text{Tr}(O) = \int dx \, \langle x|O|x\rangle$$

Approximation ; justified in 1 loop calculation

$$\langle x_0 | \text{Log}(-\Delta + V''(\phi)) | x_0 \rangle \simeq \langle x | \text{Log}(-\Delta + V''(\phi(x_0))) | x_0 \rangle$$

function (this depends on x)

constant function

The difference between these comes in higher loop orders . .

. . . its very well justified in loop calculation .

now ; $V''(\phi(x))$ is a number

→ it plays the role of mass .

now ; $(-\Delta + V''(\phi(x)))$ is a differential operator which is invariant by ~~translat~~ translation because $V''(\phi(x))$ is just a number

so ; its eigen vectors are plane waves .

$[-\Delta + V''(\phi(x_0))]$ eigenvectors are plane waves $\exp(-i k . x) = \psi_k(x)$

and the eigenvalue is $k^2 + V''(\phi(x_0))$

from this we can check very easily that ;

$$\langle x_0 | \text{Log}(-\Delta + V''(\phi(x)) | x_0 \rangle = \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^d} \log(k^2 + V''(\phi(x_0)))$$

Renormalization procedure at step 1

$\mathcal{A}[\phi] \longrightarrow \Gamma[\phi]$ if you integrate over all the modes ~~$0<|k|<\Lambda$~~ $0<|k|<\Lambda$

now; if we integrate over momentum shell $\Lambda(1-\varepsilon)<|k|<\Lambda$

$$\mathcal{A}[\phi] \longrightarrow \mathcal{A}_{eff}[\phi_{eff}]$$

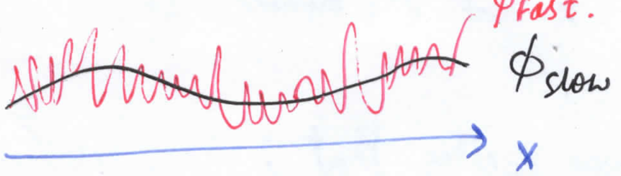
$$= \mathcal{A}[\phi_{eff}] + \frac{1}{2}\int d^dx \int_{\Lambda(1-\varepsilon)<|k|<\Lambda} \frac{d^dk}{(2\pi)^2} \log\left(\frac{k^2+V''(\phi(x))}{\Lambda^2}\right)$$

ie; $\mathcal{A}_{eff}[\phi_{eff}] = \mathcal{A}[\phi_{eff}] + \frac{1}{2}\int d^dx \int_{\Lambda(1-\varepsilon)<|k|<\Lambda} \frac{d^dk}{(2\pi)^2} \log\left(\frac{k^2+V''(\phi(x))}{\Lambda^2}\right)$

explicit formula for $\mathcal{A}_{eff}$.

The approximation is quite justified when you integrate over thin shell.

Integrate over $\hat{\phi}_{fast}(k)$; & keep $\hat{\phi}_{slow}(k)$ fixed.



Another derivation

$\Gamma[\phi] = \mathcal{A}[\phi] + \frac{1}{2}\mathrm{Tr}[\log(-\Delta + V''(\phi))]$

Trace is estimated with cut off Λ

by performing step 1; we get $\mathcal{A}_{eff}[\phi] + \frac{1}{2}\mathrm{Tr}_{\Lambda'}[\log(-\Delta + V''_{eff}[\phi])]$

~~$\Gamma[\varphi] = A[\varphi] + \frac{1}{2}\mathrm{Tr}_{\Lambda}\left(\log[-\Delta + V''(\varphi)]\right)$~~

$$\Gamma[\varphi] = A[\varphi] + \frac{1}{2}\mathrm{Tr}_{\Lambda}\left(\log[-\Delta + V''(\phi)]\right)$$

|| we want this to be equal because the final result is same.

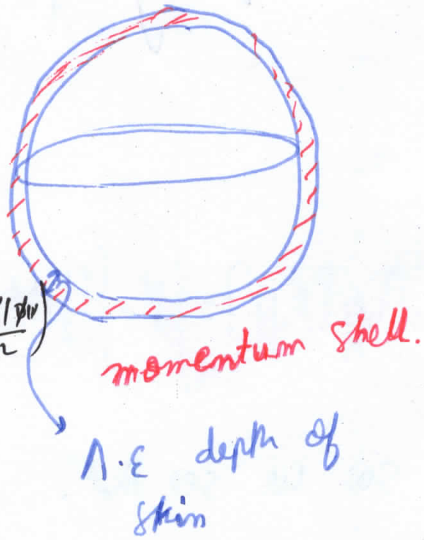
$$\Gamma[\varphi] = A_{\text{eff}}[\varphi] + \frac{1}{2}\mathrm{Tr}_{\Lambda'}\left(\log[-\Delta + V''_{\text{eff}}(\varphi)]\right)$$

$$A_{\text{eff}}[\phi_{\text{eff}}] = A[\phi_{\text{eff}}] + \boxed{\frac{1}{2}\int d^dx \int_{\Lambda(1-\varepsilon)<|k|<\Lambda} \frac{d^dk}{(2\pi)^2}\log\left(\frac{k^2+V''(\phi(x))}{\Lambda^2}\right)}$$

Since integrating over momentum shell; we can just take k to be ~~Λ~~ Λ (if ε is small)

$$\frac{1}{2}\int d^dx\int\frac{d^dk}{(2\pi)^2}\log\left(\frac{k^2+V''(\phi(x))}{\Lambda^2}\right)$$ [redacted]

$$= \frac{1}{2}\int d^dx \cdot \frac{\Lambda^d\cdot\epsilon}{(2\pi)^d}\cdot \mathrm{Vol}\begin{pmatrix}\text{unit sphere}\\ \text{dim. } d-1\end{pmatrix}\times\log\left(1+\frac{V''(\phi(x))}{\Lambda^2}\right)$$



so; volume of shell

$= \Lambda\epsilon \times (\text{area of sphere})$

$= \Lambda\cdot\epsilon \times (\Lambda^{d-1}\,\text{vol (unit sphere)})$

$= \Lambda^d\cdot\epsilon\cdot\text{Vol}(\text{unit sphere})$

$$\cancel{\boxed{\quad} = \frac{1}{2}\int d^dx \int \frac{d^dk}{(2\pi)^d}\,\Lambda^d\cdot\epsilon}$$

$$\boxed{\quad} = \frac{1}{2}\int d^dx \frac{\cancel{d^dk}}{(2\pi)^d}\cdot\Lambda^d\cdot\epsilon\cdot \text{Volume}\begin{pmatrix}\text{unit sphere of}\\ \text{dimension } d-1\end{pmatrix} \times \log\left(1+\Lambda^{-2}V''(\phi(x))\right)$$

because of this ϵ we see that difference between A & A_{eff} is ~~is~~ of the order ϵ.

So; we get

$$A_{eff}[\phi_{eff}] = \int d^dx \left\{\left[\frac{1}{2}(\partial\phi_{eff})^2 + V(\phi_{eff})\right] + \epsilon\,\frac{1}{2}\cdot\frac{\Lambda^d}{(2\pi)^d}\cdot\left(\frac{2\,\pi^{d/2}}{\Gamma(\frac{d}{2})\cdot\cancel{\quad}}\right)\log(1+\Lambda^{-2}V'')\right\}$$

ie;

$$A_{eff}[\phi_{eff}] = \int d^dx \left\{\left[\frac{1}{2}(\partial\phi_{eff})^2 + V(\phi_{eff})\right] + \epsilon\cdot\frac{1}{2}\cdot\frac{\Lambda^d}{(2\pi)^d}\cdot\left(\frac{2\cdot\pi^{d/2}}{\Gamma(d/2)}\right)\cdot\log\left(1+\Lambda^{-2}V''(\phi_{eff}(x))\right)\right\}$$

non linear but local in x.

So; we see that,

in this scheme; going from renormalization amounts to change the potential.

$$V[\phi] \rightarrow V_{eff}[\phi_{eff}] = V(\phi_{eff}) + \epsilon\cdot\frac{\log\left(1+\Lambda^{-2}V''(\phi_{eff})\right)]}{(4\pi)^{d/2}\cdot\Gamma(d/2)} \qquad (*)$$

original effective potential

contribution of order ϵ.

Only the local potential is renormalized

$$V[\phi] \rightarrow V_{eff}[\phi_{eff}] = V[\phi_{eff}] + \epsilon \cdot \Lambda^{d} \cdot \frac{1}{(4\pi)^{d/2}\,\Gamma(d/2)} \log\left(1+\Lambda^{-2} \cdot V''(\phi_{eff})\right)$$

The result of step 1 ; is only to renormalize the local potential.

Step 2 $x \rightarrow x' = \frac{x}{1+\epsilon}$; $k \rightarrow k' = k(1+\epsilon)$

so that cut off becomes Λ again.

The measure changes; the $\partial\phi_{eff}$ derivate changes.

ie, $\phi_{eff} \rightarrow \phi'(x') = \phi_{eff}(x) \cdot (1+\epsilon)^{\Delta\phi}$

$\Delta\phi$ is called scaling dimension of field.

easy to check $\underbrace{\Delta_\phi = \frac{d-2}{2}}$ so that

$\int d^dx \, \frac{1}{2} (\partial\phi_{eff})^2$ does not change.

Scaling dimension of ϕ

i.e, $\int d^dx \, (\partial\phi_{eff})^2 = \int d^dx' \, (\partial\phi')^2$

$d^dx \cdot s^{-d}$ $\qquad$ $V_{Ren}(\phi') = s^d \cdot V_{eff}(\phi..)$

$$\int d^dx \, V_{eff}(\phi_{eff}) = \int d^dx' \cdot V_{Ren.}(\phi')$$

Final Result: in terms of Renormalized quantities,

after, $N = \frac{\log s}{\epsilon}$ iteration . Rescaling by a global factor s

ie, doing Renormalization by global factor S.

$$\underset{\substack{\text{Initial}\\\text{potential}}}{V(\phi)} \xrightarrow{s} \underset{\substack{\text{Renormalized}\\\text{potential}}}{V_s(\phi_s)}$$

equation (*) gives us differential equation for our local potential.

$$s \cdot \frac{\partial}{\partial s} V_s(\phi) = \overbrace{d \cdot V_s(\phi) - \left(\frac{d-2}{2}\right) \cdot \phi \cdot \frac{\partial}{\partial \phi} V_s(\phi)}^{\text{These are classical terms}} + A \log(1 + \ldots)$$

Comes from rescaling of x in step 2.

comes from rescaling of ϕ.

Set $\Lambda = 1$

A is some constant.

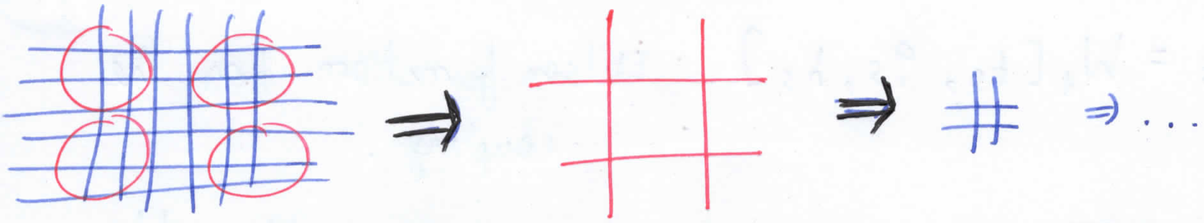
$$\boxed{\begin{aligned} s \cdot \frac{\partial}{\partial s} V_s(\phi) &= \overbrace{\underset{\uparrow \text{ dimension of } V(\phi)!!!}{d} \cdot V_s(\phi) - \left(\frac{d-2}{2}\right) \phi \cdot \frac{\partial}{\partial \phi} V_s(\phi)}^{\text{classical scaling}} + \overbrace{A \log\left(1 + \frac{\partial^2}{\partial \phi^2} \cdot V_s(\phi)\right)}^{\text{one loop contribution.}} \\ A &= \frac{1}{(4\pi)^{d/2} \cdot \Gamma(d/2)} \end{aligned}}$$

~~non-linear partial differential~~ equation

Renormalization group flow equation in space of potentials $V(\phi)$

(Non-linear P.D.E. (w.r.t. ϕ) in the space of potentials)

This flow is richer than what we did in previous lectures & stat-mech course.



$k \rightarrow \frac{k}{\Lambda} = q$ (can work in dimensionless quantities)

$x \rightarrow x\Lambda = y$

$\phi \rightarrow \phi \cdot \Lambda^{\frac{d-2}{2}} = \psi$ $\Rightarrow$ These all are dimensionless.

↳ express in units where $\Lambda = 1$.

example.

$$V(\phi) = \frac{t}{2}\phi^2 + \frac{g}{4!}\phi^4 + \frac{h}{6!}\phi^6$$

$$V''(\phi) = t + \frac{g}{2}\phi^2 + \frac{h}{4!}\phi^4$$

$$\log(1+V''(\phi)) = \log\left(1+t+\frac{g}{2}\phi^2 + \frac{h}{4!}\phi^4\right)$$

$$= \log(1+t) + \log\left(1 + \frac{1}{2}\left(\frac{g}{1+t}\right)\phi^2 + \frac{1}{4!}\left(\frac{h}{1+t}\right)\phi^4\right)$$

$$= \log(1+t) + \underbrace{\left[\frac{g}{2(1+t)}\phi^2 + \frac{1}{4!}\left(\frac{h}{1+t}\right)\phi^4\right]}_{\text{first order.}} - \frac{1}{2}\left[\frac{g}{2(1+t)}\phi^2 + \frac{1}{4!}\left(\frac{h}{1+t}\right)\phi^4\right]^2$$

$$+ \frac{1}{3}\left[\frac{g}{2(1+t)}\phi^2 + \frac{1}{4!}\left(\frac{h}{1+t}\right)\phi^4\right]^3 + \dots$$

Truncating at order ϕ^6;

we get set of non-linear flow equations for the coupling.

$$V_s(\phi) = \frac{t_s}{2}\phi^2 + \frac{g_s}{4!}\phi^4 + \frac{h_s}{6!}\phi^6$$

; t_s, g_s, h_s renormalized couplings.

$$s \cdot \frac{d}{ds} t_s = W_t[t_s, g_s, h_s]$$

Wilson function for the coupling.

$$s \cdot \frac{d}{ds} g_s = W_g[t_s, g_s, h_s]$$

W_t, W_g, W_h.

$$s \cdot \frac{d}{ds} h_s = W_h[t_s, g_s, h_s]$$

These functions are just polynomials.

This W, Wilson Renormalization Group (RG) function is just "$-\beta$" function in perturbation theory.

$\boxed{\text{In } d=4}$

$$\boxed{\begin{aligned} W_t &= 2t + A g + \dots \\ W_g &= -3 A g^2 + \dots \end{aligned}}$$

In perturbation theory in 4 dimensions we found

$$\beta_g(g) = +\frac{3}{(4\pi)^2} g^2$$

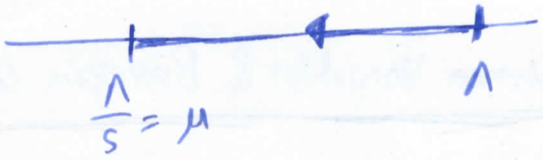
(Related by minus sign)

Action $A[\phi]$ is like bare action with which you start in begining.

$A[\phi]$ depends on bare parameters t, g, h.

$\downarrow$ R.G.

$A_s[\phi_s]$ can be expressed in terms of renormalized parameters t_s, g_s, h_s.



in $D=4$:

$\mu^2 t_s = m_R^2$

$g_s = g_R$

dimensionless

~~t_s is dimensionless~~

$\Lambda^2 t = m_B^2$

$g = g_B$

The parameters are always dimensionless because they are always expressed in terms of units of scale along each step

ex in $D=4$

$\mu^2 t_s = m_R^2$

$g_s = g_R$

renormalized parameter

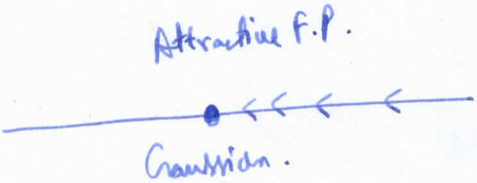
$\Lambda^2 t = m_B^2$

$g = g_B$

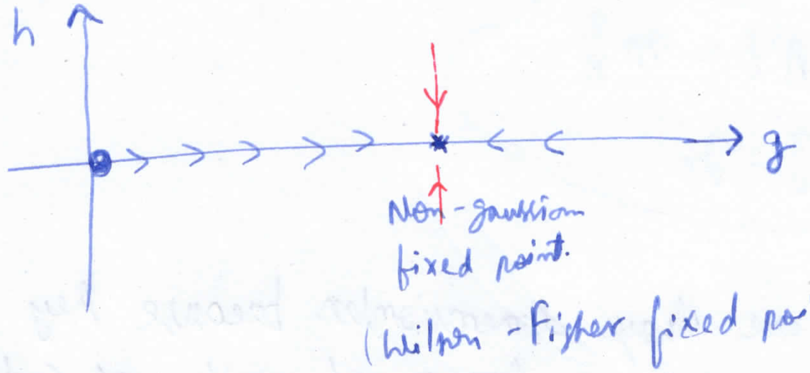
Bare parameters.

Lecture 5.2 | Renormalization Group Flow, Grassmann Variables & Berezin Calculus

The concept of relevance or irrelevance is relative to the fixed point you consider (perturbatively relevant or irrelevant)



D < 4



The Problem with ϕ^4 Asymptotic " slavery"

as a function of length scale, if we perform ~~rescalling by~~ rescalling by factor of S.

length scale - rescaling by S

running coupling $g(s)$.

$$s \cdot \frac{d}{ds} g(s) = - A \; g^2(s) \quad .$$

(result of one loop calculation)

We can very easily solve this

$$A = \frac{3}{(4\pi)^2} > 0$$

Initial $S = 1$, $g = g_0 > 0$.

$$g(s) = \frac{g_0}{1 + A \cdot g_0 \cdot \log S}$$

Infrared Regime ; IR $s \nearrow \infty$ $g(s) \searrow 0$

$x \to sx$

U.V. $s \searrow 0$ $g(s) \nearrow \infty$ at a finite rescaling factor because there is a pole there.

ie; when ; $s = \exp\left(\frac{-1}{A g_0}\right)$

$\hookrightarrow$ at this point $g\left(s = \exp\left(\frac{-1}{A g_0}\right)\right) = \infty$

<u>This is problem</u>

because if start with given g_R at given scale μ.

and you ask what is g_B at scale Λ

$\therefore$ There is then, maximal energy scale where you cant trust the ~~g~~ theory any more.

g_R, μ

$\downarrow$?

g_B, Λ $\quad \Lambda_c = \mu \exp\left(-\frac{1}{A g_R}\right)$ $\leftarrow \Lambda_{critical}$ I cannot trust my calculation.

$\Longleftrightarrow$ Perturbation theory is not really consistent.

(occurs of course because $A > 0$)

This problem occurs for QED ; ϕ^4 .

$\Longrightarrow$ The problem was raised by Landau. <u>1960</u>

occurs for $\phi^4_{d=4}$, QED , Higgs sector in Standard model ☹

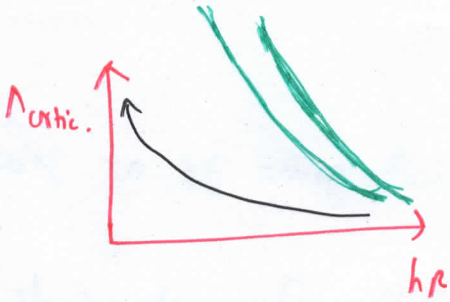
Not a problem for Yang-Mills theories, Gauge theory ☺

Not really problematic for QED ; because the energy scale at which we have problem is very large.

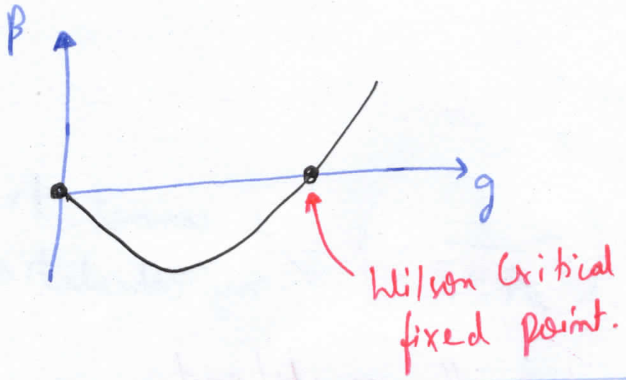
$$E_{problem} \approx \exp(.137) \underset{\text{much more larger}}{>>>} \text{(Plank energy scale)}$$

ϕ^6 ; $h_R = 0$

Problem $h_B = \infty$ at $\Lambda_{criti.} \approx (h_R)^{-2}$



D = 3 ; ϕ^4 ; $s \cdot \frac{dg}{ds} = g - A g^2$



Fermionic Path Integrals (Berezin Calculus)

- Dirac particles ; spin $\frac{1}{2}$ ↔ Fermi-dirac statistics.
- Non-Abelian gauge theory → Fadev-Popof Ghost fields.

Bosonic $[\Phi(x), \Phi(y)] = 0$ if $(x-y)^2 > 0$ spacelike.

Fermions $\{\Psi(x), \Psi(y)\} = 0$ if $(x-y)^2 > 0$ spacelike.

For Causality.

ordinary numbers dont anti-commute.

$\{A,B\} = AB+BA$.

how can we realize $\int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$

anticommutes instead of commuting.

Calculational Rules. with "stochastic" process variables.

What is Algebra behind fermions?

Grassmann Algebras/(Exterior Algebras) over the complex field $\mathbb{C}$.

(we can define Grassmanian algebra over real field $\mathbb{R}$; but since most fermions are charged, so its better to consider over complex field $\mathbb{C}$)

G_N associative algebra over $\mathbb{C}$

means it is a vector space.

N is integer >0 (related to dimension of algebra; & is related to no. of generators)

we have ★ multiplication by some complex number.

★ Addition. $+$

★ Products. $\cdot$

(we cannot represent G_N as algebra of matrices)

G_N has 2N generators. θ_i, $\bar{\theta}_i$ $i=1,\dots,N$

Algebraic objects.

They are going to be anti-commuting generators

$$\theta_i\theta_j + \theta_j\theta_i = 0 \quad \& \quad \theta_i\bar{\theta}_j + \bar{\theta}_j\theta_i = 0.$$

$$\bar{\theta}_i\bar{\theta}_j + \bar{\theta}_j\bar{\theta}_i = 0$$

These generators are nilpotent: $\bar{\theta}_i^2 = 0$ & $\theta_i^2 = 0$.

$\theta_i^2 = 0 = \bar{\theta}_j^2$ nil ponent.

General Element g of the algebra G_N is of the form.

$$g = \sum_{k=0}^{N}\sum_{H=0}^{N}\underbrace{\sum_{i_1<i_2<\dots<i_k}}_{I}\underbrace{\sum_{j_1<j_2<\dots<j_H}}_{J} C_{IJ}\,\theta_{i_1}\theta_{i_2}\dots\theta_{i_k}\bar{\theta}_{j_1}\bar{\theta}_{j_2}\dots\bar{\theta}_{j_H}$$

C_{IJ} is a complex number.

$\Downarrow$

The number of linear combination we can build out of this will give us the dimension of the algebra.

obviously, $1 \in G_N$

~~we can check that~~

we can check that $\boxed{\dim_{\text{on } \mathbb{C}} (G_N) = 2^{2N}}$

if N=1 $\quad g = a + b\theta + c\bar{\theta} + d\theta\bar{\theta}$

↑ generic element in G_1.

$a, b, c, d \in \mathbb{C}$. complex coefficients.

$$g_1 \cdot g_2 = (a_1 + b_1\theta + c_1\bar{\theta} + d_1\theta\bar{\theta})\cdot(a_2 + b_2\theta + c_2\bar{\theta} + d_2\theta\bar{\theta})$$

$$= a_1a_2 + (a_1b_2 + b_1a_2)\theta + (a_1c_2 + c_1a_2)\bar{\theta} + (a_1d_2 + d_1a_2 + b_1c_2 + c_1b_2)\theta\bar{\theta}$$

$$= a_1a_2 + (a_1b_2 + b_1a_2)\theta + (a_1c_2 + c_1a_2)\bar{\theta} + (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)\theta\bar{\theta}$$

more complicated examples .....

in general $g_1 g_2 \neq g_2 g_1$; but can have commuting numbers.

Conjugation operation (analogue of complex conjugate).

G_N is non-commuting algebra; actually mixture of commuting & non-commuting numbers.

if K and H fixed, then we can check that when you (K+H) multiply two numbers

These d_1, d_2 are gradation of grassman numbers being multiplied.

(K+H) is called gradation of the number.

or gradation degree

d_1 \ d_2	even	odd
even	commute	commute
odd	commute	anti-commute

$*$ operation (conjugation operation)

c complex , $c^* = \bar{c}$ (ordinar complex conjugation) for complex numbers.

θ_i , $\theta_i^* = \bar{\theta}_i$

$\bar{\theta}_i$, $\bar{\theta}_i^* = \theta_i$

} very much like conjugation of complex numbers.

"good basis for my algebra".

$$(g_1 \cdot g_2)^* = g_2^* \cdot g_1^*$$

---

with this rules ; example N=1 continued...

$$g^* = \bar{a} + \bar{c}\,\theta + \bar{b}\,\bar{\theta} + \bar{d}\,\theta\bar{\theta}$$

View $g \in G_N$ as "functions" of the anti-commuting generator θ_i, $\bar{\theta}_i$."

"non-commutative space".

Can we define Derivation w.r.t. θ_i or $\bar{\theta}_i$?

& notion of Integration w.r.t. θ_i or $\bar{\theta}_i$?

Derivation w.r.t. θ_i & $\bar{\theta}_j$.

$$\frac{\partial}{\partial \theta_i}\left(\theta \ldots \theta_i \, \theta \;\bar{\theta} \ldots \bar{\theta}\right) = (-1)^{*} \theta_1 \ldots \theta \, \bar{\theta} \ldots \bar{\theta}$$

* → some number...

θ_i removed.

first move θ_i to left using anti-commutation relation..

... & then remove it by using the fact that $\frac{\partial}{\partial \theta_i}(\theta_i) = 1$. & $\frac{\partial}{\partial \theta_i}(\theta_j) = 0$.

if there is no θ_i in $\theta \ldots \theta \bar{\theta} \ldots \bar{\theta}$

then $\frac{\partial}{\partial \theta_i}(\_\_\_\_) = 0$.

$$\boxed{\frac{\partial \theta_j}{\partial \theta_i} = \delta_{ij}}$$

$$\boxed{\frac{\partial \bar{\theta}_j}{\partial \bar{\theta}_i} = \delta_{ij}} \quad \boxed{\frac{\partial \bar{\theta}_j}{\partial \theta_i} = 0} \quad \boxed{\frac{\partial \theta_j}{\partial \bar{\theta}_i} = 0}$$

$$\boxed{\frac{\partial \theta_j}{\partial \theta_i} = \delta_{ij} = \frac{\partial \bar{\theta}_j}{\partial \bar{\theta}_i}} \quad \boxed{\frac{\partial \bar{\theta}_j}{\partial \theta_i} = 0 = \frac{\partial \theta_j}{\partial \bar{\theta}_i}} \to \text{defination}$$

with this we can check ... Derivatives anti-commute

anti-commute. → property.

$\frac{\partial}{\partial \bar{\theta}}$ first bring to left .. & then remove it.

Example

$$\frac{\partial}{\partial\theta}(a + b\theta + c\bar{\theta} + d\theta\bar{\theta}) = b + d\bar{\theta}$$

$$\frac{\partial}{\partial\bar{\theta}}(\underline{\qquad\qquad}) = c - d\theta$$

Berezin Integration.

define $\int d\theta_i$ such that $\underline{\int d\theta_i \frac{\partial}{\partial\theta_i}} = 0$

↳ usually we get boundary terms;

define $\int d\theta_i$ s.t. $\int d\theta_i \frac{\partial}{\partial\theta_i} = 0$

but here we are in space of non-commutative functions with no boundary ; so zero.

"Space of anti-commuting numbers have no boundary"

since Derivatives anti-commute ; we can take as a defination

$$\int d\theta_i = \frac{\partial}{\partial\theta_i}$$

Integral operation is Same as derivative.. . . .

$$\int d\bar{\theta}_i = \frac{\partial}{\partial\bar{\theta}_i}$$

Anti-commutation.

The operation of integration here is algebraicly same as the operation of differentiation here.

Integration = Derivation.

— Shoaib Akhtar
28/5/2020

Lecture 6.1 // Fermionic Path Integrals

① Functional Integral Quantization of (Dirac) Fermions.

Grassman (Exterior) Algebra : introduced by Berizen.

G_N algebra over $\mathbb{C}$, has dim $(G_N) = 2^{2N}$

$2N \Rightarrow$ no. of generator of G_N.

$\theta_i, \bar{\theta}_i$; $i = 1, 2, .. N$ anticommuting numbers. (algebraic objects)

Basis $\{1, \theta_i, \bar{\theta}_i, \theta_i\bar{\theta}_j, \theta_i\theta_j, \bar{\theta}_i\bar{\theta}_j, \ldots\}$

+ , × , conjugation.

Derivation : anti-commutes $\frac{\partial\theta_j}{\partial\theta_i} = \delta_{ij} = \frac{\delta\bar{\theta}_j}{\delta\bar{\theta}_i}$; $\frac{\partial\theta}{\partial\bar{\theta}} = 0 = \frac{\partial\bar{\theta}}{\partial\theta}$

Integration: $\frac{\partial}{\partial\theta_i} = \int d\theta_i$ → same as derivation; because integrating means ; integrating over one of those variable and that disappears from result of integration.

... and derivation also does that.

... & since the elements of algebra can contain only one θ_i ; integrating over one θ_i or removing θ_i is same thing.

$\int d\theta_i$ integration also anti-commutes

"Gaussian Integral"

$A = A^\dagger$ NxN matrix (self adjoint) $A = \{A_{ij} ; i,j = 1,..,N\}$ ∴ $A_{ij} = \bar{A}_{ji}$

$$\exp(-\bar{\theta}_i A_{ij} \theta_j) \equiv \exp(-\bar{\theta} \cdot A \cdot \theta) = e$$

$$e \in G_N \quad : \; g = \sum_0^\infty \frac{1}{k!} (-\bar{\theta} \cdot A \cdot \theta)^k$$

$$= \sum_{k=0}^{N} \underbrace{\theta\theta \cdots \theta}_{k} \cdot \underbrace{\bar{\theta} \cdots \bar{\theta}}_{k} \; \underbrace{AA \cdots A}_{k}$$

This's defination is independent of A being hermitian

polynomial.

N=1 Then

$$\exp(-\bar{\theta} A \theta) = 1 + A\theta\bar{\theta}$$

$$\int \prod_{i=1}^{N} d\bar{\theta}_i \cdot d\theta_i \; \exp(-\bar{\theta} \cdot A \cdot \theta) = \det[A]$$

Gaussian Integral. ⟹ determinant.

N=1 $\int d\bar{\theta} d\theta \exp(-\bar{\theta} a \theta) = a$

N=2 $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$; $\exp(-\bar{\theta} \cdot A \cdot \theta) = \ldots + \ldots + (\;)\, \theta_1\theta_2\bar{\theta}_1\bar{\theta}_2$

$$= \ldots + \ldots + (A_{11}A_{22} - A_{12}A_{21})\, \theta_1\theta_2\bar{\theta}_1\bar{\theta}_2$$

$$\Rightarrow \int --- = A_{11}A_{22} - A_{12}A_{21} = \det[A]$$

Gaussian integral for N=2

The analogue for commuting numbers;

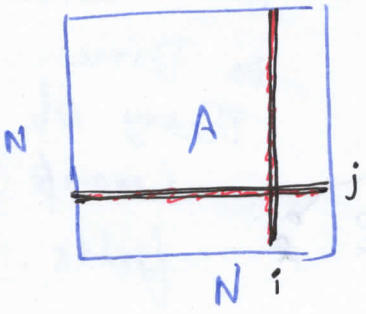
z_i, $\bar{z}_i$ complex numbers

$$\int \prod_i d\bar{z}_i dz_i \exp(-\bar{z}_i A_{ij} z_i) = [\det A]^{-1}$$

"Correlators" ; ~~cor~~ cumulants of this distribution...

$$\langle \theta_i \bar{\theta}_j \rangle := \frac{\int \prod_k d\bar{\theta}_k d\theta_k \exp(-\bar{\theta}\cdot A\cdot\theta)\, \theta_i \bar{\theta}_j}{\int \prod_k d\bar{\theta}_k d\theta_k \exp(-\bar{\theta}\cdot A \cdot \theta)}$$

$$= \frac{\det[\text{Minor}_{ij}(A)]}{\det[A]} = (A^{-1})_{ij}$$



$$\Rightarrow \langle \theta_i \bar{\theta}_j \rangle = \frac{\det[\text{Minor}_{ij}[A]]}{\det[A]} = (A^{-1})_{ij}$$

$$\boxed{\langle \theta_i \bar{\theta}_j \rangle = (A^{-1})_{ij}} \Rightarrow \boxed{\langle \bar{\theta}_i \theta_j \rangle = -(A^{-1})_{ji}}$$

~~$\langle \bar{\theta}_i \theta_j \rangle =$~~

$= \langle \theta_i \bar{\theta}_j \rangle$

line flows from θ to $\bar{\theta}$

$\langle \bar{\theta}_i \theta_j \rangle = -(A^{-1})_{ji} =$

Two-point function.

4 point function

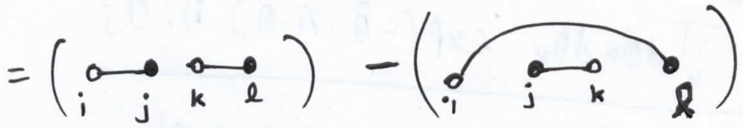
2θs & $2\bar{\theta}$s ; otherwise it is ~~zero~~ zero.

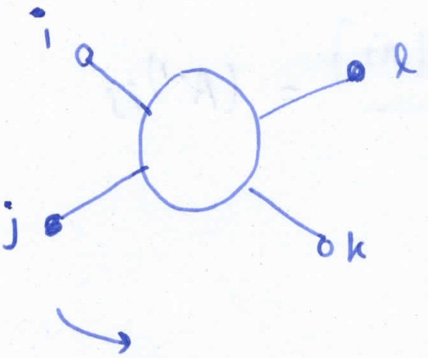
$$\langle \theta_i \bar{\theta}_j \theta_k \bar{\theta}_l \rangle = \underbrace{(A^{-1})_{ij} \cdot (A^{-1})_{kl}}_{\text{contribution from } \underset{\sqcup}{\theta_i \bar{\theta}_j}\ \underset{\sqcup}{\theta_k \bar{\theta}_l}} - \underbrace{(A^{-1})_{il} (A^{-1})_{kj}}_{\text{comes from } \theta_i \bar{\theta}_j \theta_k \bar{\theta}_l}$$

Looks like Wicks theorem.

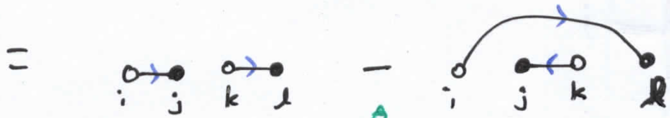
minus 1 comes from anti-commutation

$$\langle \theta_i \bar{\theta}_j \theta_k \bar{\theta}_l \rangle = (A^{-1})_{ij} (A^{-1})_{kl} - (A^{-1})_{il} (A^{-1})_{kj}$$





$$= \langle \theta_i \bar{\theta}_j \theta_k \bar{\theta}_l \rangle$$



Minus one is associated to signature of permutation

Wicks Theorem in Dirac Theory of fermi fields.

~~forget about arrow rules.~~

Relation with Quantum Mechanics (non-relativistic Fermions)

θ & $\bar{\theta}$ are associated to creation & annihilation operator.

$$\theta, \bar{\theta} \longleftrightarrow a, a^{\dagger} \text{ operators}$$

using formalism of, Fermionic Coherent states representation.

Dirac Field

4-dimensions.

$\psi = (\psi_a) \quad a = 1, .., 4 = 2^{[d/2]}$: Dirac indices

γ^{μ} : 4×4 matrices (Dirac) ; $\{\gamma^{\mu}, \gamma^{\nu}\} = -2h^{\mu\nu}$

$h^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ East coast metric

~~Dirac~~

Dirac "Action"

$$S[\bar{\psi}, \psi] = \int_{\mathbb{M}^{1,3}} d^4x \left\{ (\bar{\psi}(x))(i \not{\partial}_x - m)\psi(x) \right\}$$

$$\not{\partial} = \gamma^\mu \frac{\partial}{\partial x^\mu}$$

Classical "E.O.M."

$$\frac{\delta S}{\delta \bar{\psi}(x)} = 0 \Rightarrow \boxed{(i\not{\partial} - m)\psi(x) = 0}$$

Dirac Equation.

2nd Quantization Fermi-Dirac statistics ⇒ unitary QFT. (positive normed states)

Functional Integral

$\psi(x)$, $\bar{\psi}(x)$ need to anti-commute.

Big Grassmann Algebra

whose generators are $\psi = \{\psi^a\}$. field of anti-commuting numbers associated at each point in space.

$$\theta = \{\theta_i\} \longrightarrow \psi = \{\psi^a(x) ; x \in \mathbb{M}^{1,3} ; a = 1,..4 \text{ dirac indices}\}$$

infinite no. of generators labelled by position in space x & dirac indices a.

$$\bar{\theta} = \{\bar{\theta}_i\} \longrightarrow \bar{\psi} = \{\bar{\psi}^a(x) ; x \in \mathbb{M}^{1,3}, a = 1,..,4\}$$

not exactly the dirac. adjoint.

Dirac Action:

$$\int d^4x\,[\bar{\Psi}(x)\,(i\not{\partial}-m)\,\Psi(x)] = \int d^4x\,\bar{\Psi}^{a}(x)\left[i\gamma^{\mu}_{ab}\frac{\partial}{\partial x^{\mu}} - m\,\delta_{ab}\right]\Psi^{b}(x)$$

$$= S[\bar{\Psi},\Psi]$$

Fermionic Path Integral

$$Z = \int D[\bar{\Psi},\Psi]\ \exp\left(i\,S_{Dirac}[\bar{\Psi},\Psi]\right)$$

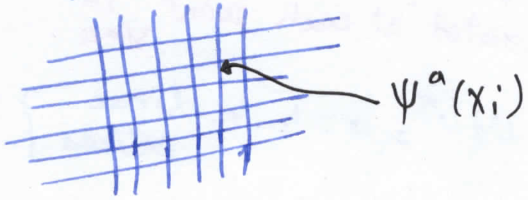
The measure $D[\bar{\Psi},\Psi]$ means

$$D[\bar{\Psi},\Psi] = \prod_{x}\prod_{a} d\bar{\Psi}^{a}(x)\,d\Psi^{a}(x)$$

$$\dim(G_{Dirac}) = \infty.$$

$$(i.e.\ 2^{\infty})$$

If you discreatize spacetime, you have $\Psi^a(x_i)$ at each site



Better now,
but there is phenomemam
called "Fermion Doubling".
if naive discretization.

$$Z = \int D[\bar{\Psi},\Psi]\exp\left(i\,S_{Dirac}[\bar{\Psi},\Psi]\right) \propto \text{"Det"}[\,i\not{\partial} - m\,]$$

Correlation Function

$$\langle \Psi^{a}(x)\,\bar{\Psi}^{b}(y)\rangle = \int\frac{d^4p}{(2\pi)^4}\cdot\underbrace{\left(\frac{i}{-\not{p}-m-i\varepsilon_{+}}\right)}_{}\cdot e^{ip\cdot(x-y)} = \underbrace{G_F(x-y)}_{}$$

Feynmann Propagator for Dirac Field.

4x4 matrix in dirac indices

$$= \langle 0|\,T[\Psi(x)\bar{\Psi}(Y)]\,|0\rangle$$

field operators

Time ordered product for fermions

$G_F(x-y) = \langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle =$

$$T[\psi(x)\bar{\psi}(y)] = \begin{cases} \psi(x)\,\bar{\psi}(y) & \text{if} \\ -\bar{\psi}(y)\psi(x) & \text{if} \end{cases}$$

finally from this we get; Correct Wicks Theorem for Dirac Field. with correct "-1" singn..

Last Remark (notational problem) (not real problem... it's a feature)

Grassmann "variable" $\psi(x) \xrightarrow[\text{field operator in canonical formalism}]{\text{associated to}} \Psi(x)$ Field operator

Grassman Conjugate $\psi^*(x) \longrightarrow \Psi^\dagger(x)$ Hermitian conjugate of ~~the~~ the field operator.

$$\psi^\dagger = \bar{\psi}\cdot\gamma_0$$

so; $\bar{\psi}(x) \longrightarrow \bar{\Psi}(x)$

$$\boxed{\psi^* = \bar{\psi}\gamma_0}$$

If one uses ψ^* instead of $\bar{\psi}$; then Dirac action is not explicitly Lorentz Invariant.

Non-Abelian Gauge Theories

Non-Abelian Symmetries in QFT

* Group theory * Representation Theory of groups.

SU(2) = G group (simplest Lie group which is not U(1) group)

In QED : 1 type of charge (say electric charge)
$\Longleftrightarrow$ so 1 type of conserved current.

$\Longleftrightarrow$

current - current interaction mediated by photon, which is ~~natural~~ ~~vect~~ neutral vector particle of spin 1.

What happens when you have several types of charge?

↳ here interation is not just mediated by neutral vector ~~the~~ particle; but ~~ste~~ also through charged vector particles which are Gauge Bosons.

---

have Global Symmetry; ~~and~~ continuous SU(2) Group

; then fields (namely particles of the theory) belongs to some irreducible ~~rep~~ representation of group G. (unitary or anti-unitary)

A ~~group of Sym~~ A group of symmetry is a group that acts on Hilbert space of your theory; but commutes with Hamiltonian ⇒ so does not change the dynamics.

i.e. continuous, global symmetry ; group G then fields (particles) belongs to some irreducible representation of group G (unitary or ~~anti-unitary~~)

since SU(2) is continuous; infact anti-unitary is not allowed.

* when G is continuous; then only unitary representations are associated.

Note!!

Dirac index "a", is the index of irreducible representation of Lorentz group of spin $1/2$.

Scalar field ϕ ; spin 0.

---

cohice;

R = Fundamental Representation F

$\dim_{\mathbb{C}}(F) = 2$ $\qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ 2 different scalar fields complex

(complex field because they carry charge)

(because they belong to fundamental representation of SU(2); which is group of unitary transformation)

Global Transformation:

$g \in SU(2)$ 2x2 complex matrix with $g \cdot g^{\dagger} = 1$
(unitary matrix)

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$$

Action of group $\phi(x) \xrightarrow{g} g \cdot \phi(x)$

(its a transformation which mixes ϕ_1 & ϕ_2; so it changes the State of particle)

Conjugate ϕ , $\bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2)$ $\quad \bar{\ } =$ complex conjugate.

$\bar{\phi} \xrightarrow[g]{} \bar{\phi} \cdot g^{\dagger}$ (by definition)

...

- Since $SU(2)$ has 3 generators, $\dim_{\mathbb{R}}(SU(2)) = 3$

we expect there are three conserved currents.

$$J_{\mu}^{1,2,3} \;, \quad \partial_\mu J^{\mu}_{1,2,3} = 0 \quad \text{independently.}$$

- $SU(2)$ invariant action (renormalizable)

$\xrightarrow{}$ which corresponds to renormalizable theory

$$S_E[\Phi] = \int d^4x \left[\frac{1}{2}(\partial_\mu \bar{\Phi} \cdot \partial^\mu \phi) + \frac{m^2}{2}\bar{\Phi}\cdot\phi + \frac{g}{8}(\bar{\Phi}\cdot\phi)^2 \right]$$

[redacted]

This is only action which is renormalizable in 4-dimensions (contain upto ϕ^4 term) which is invariant under global action of the group $SU(2)$.

because $\bar{\phi}\cdot\phi$ is invariant ...

global transformation ; so g is independent of x

$\Rightarrow (\partial_\mu \bar{\Phi} \cdot \partial^\mu \phi)$ is also invariant.

★ Non-abelilian version of photons $\Rightarrow$ gauge bosons.

Lecture 7.1) Non-Abelian Gauge theory, Coupling non-abelian gauge fields to matter.

Non-Abelian gauge Theories : $G = SU(2)$ example.

2x2 complex matrix g : $g g^{\dagger} = 1$

$\det g = 1$

3 dimensional (real) group.

The group has Lie Algebra $\mathfrak{g}$

$$\mathcal{G} = su(2)$$

; $g = \exp(i\alpha)$

where α is traceless, anti-hermitian matrix.

This means, $g = 1 + i\alpha - \frac{1}{2}\alpha^2 + \ldots \infty$

t_a : basis of g.

$\alpha = \alpha^a t_a$; α^a real numbers.

$$g = 1 + i\alpha \qquad h = 1 + i\beta$$

; $g h g^{-1} h^{-1} = 1 - [\alpha, \beta]_{Lie} + \ldots$

We know properties of Lie brackets, by writing Lie brackets of generator's.

$$[t_a, t_b]_{Lie} = i f^{c}_{ab} t_c$$

$\uparrow$ structure constant.

for su(2),
we have well known basis

$t_a = \frac{1}{2}\sigma_a$; where σ_a, $a=1,2,3$ are the three pauli matrixes

$\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}$; $\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$; $\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

or σ_1 ; or σ_2 ; or σ_3

In this basis ; the structure constant is

$$f^{c}_{ab} = \epsilon^{abc}$$

and hence the lie bracket becomes ordinary commutator

$$[\alpha, \beta]_{Lie} = [\ ,\ ]_{commutator}.$$

★ QFT with a global SU(2) symmetry

3 currents (three kind of charge, labelled by three generators)

$J_\mu^{\ a}$; $a=1,2,3$

ex Spin 0 Field in the fundamental representation.

I have a vector field $\phi = \begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix} = \phi^i$; $i=1,2$ group indices of the fundamental representation.

(has two components

ϕ^i is complex field.

2 complex field = 4 real fields.

ex Spin ½ field in the fundamental representation

$\psi = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}$ ← Dirac field.

each field ψ^i ; $i=1,2$; carries dirac indices α

ie; $\psi = (\psi_\alpha^{\ i})$

$i \Rightarrow$ group indices of the fundamental representation

$\alpha \Rightarrow$ Dirac indices

$$S[\phi] = \int d^4x \ \frac{1}{2}\partial_\mu \bar{\phi}\cdot \partial_\mu \phi + \frac{m^2}{2}\bar{\phi}\phi + \frac{g}{8}(\bar{\phi}\cdot\phi)^2$$

$$\bar{\phi} = (\bar{\phi}^1, \bar{\phi}^2)$$

global ~~gauge~~ invariance ; $\phi \to g\phi$, $\bar{\phi} \to \bar{\phi} g^+$

$$S[\psi] = \int d^4x \ \bar{\psi}(i\not{\partial} - m)\psi \quad ; \quad \bar{\psi} = (\bar{\psi}^1, \bar{\psi}^2)$$

$= \int d^4x [\bar{\psi}^\alpha_i (i\gamma^\mu_{\alpha\beta}\partial_\mu - m\delta_{\alpha\beta})\psi_i]$

global invariance ; $\psi \to g\psi$; $\bar{\psi} \to \bar{\psi}\cdot g^+$

Each of the two Dirac field obey their own Dirac equation... dont mix here.

By applying Noether's theorem ; we have some currents.

$$J_\mu{}^a = \frac{1}{2}(\bar{\phi}\cdot t_a \cdot \partial_\mu\phi - \partial_\mu\bar{\phi}\cdot t_a\cdot\phi)$$ for scalar field.

$$J_\mu{}^a = \bar{\psi}\cdot\gamma^\mu t_a \psi$$ for Dirac field.

belongs to Lie Algebra.

3 currents in each case.

; & we can check that they are conserved as a consequence of equation of motion.

These where two simple examples of matter field

Now, lets introduce Gauge Fields.

(The analogue of photon for this symmetry in QED)

~~ie; to find~~ i.e; to introduce current - current interaction (J - J interaction)

for SU(2) we expect we have, 3 vector potentials (real) A^a_μ ; $a = 1, 2, 3$.

(since we have three currents)

QED like

U(1) charge (have one charge)

$J^\mu \to A_\mu$ vector potential.

(have one current)

3 real vector potential A^a_μ

$\swarrow$

~~belong~~ belong to multiplet $A_\mu(x) = A^a_\mu(x)\, t_a \in su(2)$

for each x & each μ ; $A_\mu(x)$ is an element of lie algebra

$A_\mu(x)$ 2x2 hermitian matrix (because of t_a, i.e; σ_a)

$$= \frac{1}{2}\begin{pmatrix} A^3_\mu & A^1_\mu - i A^2_\mu \\ A^1_\mu + i A^2_\mu & -A^3_\mu \end{pmatrix}$$

Then we can write $A = A_\mu dx^\mu$ $\leftarrow$ one form with value in SU(2)

Covariant derivative

Field strength $F_{\mu\nu} \rightarrow F, B$

Yang-Mills Action (generalization of Maxwell action for QED)

The lie algebra of SU(2), i.e. su(2) ; is the adjoint representation of SU(2).

If you have $g \in SU(2)$; and R(g) in adjoing representation.

$$\underset{\text{group } G}{g \in SU(2)} \longrightarrow R(g)$$ action of g in the adjoint ~~represent~~ representation.

$$R(g)\cdot\alpha = g\cdot\alpha\cdot g^{-1}.$$

where α is 2x2 matrix

if $\alpha \in su(2)$ ~~tan~~ lie; if α is an element of Lie Algebra)
(Lie Algebra) $\mathcal{G}$

$R(\alpha)\cdot\beta = i\,[\alpha,\beta]_{Lie}$; $\beta \in \mathcal{G}$ (β also belongs to Lie Algebra)

We want to promote global symmetry G

$$\downarrow$$

local gauge symmetry.

$\phi(x) \longrightarrow g(x)\cdot\phi(x)$ (local gauge transformation)

we need to construct covariant derivative ; i.e; derivative which are invariant under local gauge transformation.

have 3 different type of gauge transformation (because have 3 generators)

Infinetisimal gauge transformation.

$$g(x) = 1 + i\,\alpha(x) \quad ; \quad \alpha(x) = \alpha^a(x)\,t_a \qquad a=1,2,3.$$

Then, the fields $\phi(x)$ tranforms as.

$$\phi(x) \longrightarrow \phi(x) + i\,\alpha(x)\,\phi(x)$$
$$= \phi(x) + i\,\alpha^a(x)\,t_a\,\phi(x)$$

(Infinetisimal gauge transformation)

☆ Covariant Derivative : D_μ Such that;

$D_\mu\phi$ transforms under gauge transformation as ϕ.

i.e; $D_\mu\phi \longrightarrow g(x)\,D_\mu\phi$

or writting the infinetisimal version.

$$D_\mu\phi \longrightarrow D_\mu\phi + i\,\alpha(x)\,D_\mu\phi$$

Ordinary derivatives dont transform like this ; since here $\alpha(x)$ depends on X.

$$D_\mu \phi = \partial_\mu \phi - i A_\mu^a t_a \phi$$

Generator of Lie Algebra.

$$\left(= \partial_\mu \phi(x) - \frac{i}{2} A_\mu^a(x) \sigma_a \phi(x) \quad \text{for } SU(2) \right)$$

ie; $D_\mu \phi(x) = \partial_\mu \phi(x) - \frac{i}{2} A_\mu^a(x) \sigma_a \cdot \phi(x)$

COVARIAN DERIVATIVE. (defination)

$$\boxed{D_\mu \phi(x) = \partial_\mu \phi(x) - i A_\mu^a(x) t_a \phi(x)}$$ ⇒ Covariant derivative acting in Fundamental Representation.

How Gauge transformation acts on the gauge field.

we want that, under ~~gau~~ global gauge transformation,

$A_\mu(x) \longrightarrow g(x) A_\mu(x) g(x)^{-1} +$ because A_μ belongs to adjoint representation of the group.

but we know $A_\mu(x)$ gauge potential is not invariant ; it transforms in non-covariant way
∵ because you change the gauge ; you change the gauge potential.

~~$A_\mu(x) \to g(x) A_\mu(x) g(x)^{-1} + i g(x) \partial_\mu [$~~

$$\boxed{A_\mu(x) \longrightarrow \underbrace{g(x) A_\mu(x) g(x)^{-1}}_{\text{This is just } A_\mu \text{ for maxwell theory.}} + \underbrace{i g(x) \partial_\mu [g(x)^{-1}]}_{\text{effect of gauge transformation on } A_\mu.}}$$

∴ but here they dont commute; so it is not trivial. (since group of symmetry is non abelian ☆g not commutes with A.

Number of currents = dimension of group.

(not ~~on~~ dimension of representation we act on)

for $SU(3) \rightarrow 8$ currents.

Short hand notation.

$$A_\mu(x) \longrightarrow g \cdot (A_\mu + i\partial_\mu) \cdot g^{-1}$$

Infimetisimal transformation; where $g = 1 + i\alpha(x)$

$$A_\mu(x) \longrightarrow A_\mu(x) + D_\mu \alpha(x)$$

where; $D_\mu \alpha(x) = \partial_\mu \alpha(x) - i[A_\mu, \alpha]$

$$\boxed{\begin{aligned} &A_\mu(x) \longrightarrow A_\mu(x) + D_\mu\alpha(x) \\ &D_\mu\alpha(x) = \partial_\mu\alpha(x) - i\underbrace{[A_\mu, \alpha]}_{\text{commutator}} \end{aligned}}$$

Covariant Derivative acting in Adjoint Representation.

$$D_\mu \phi_{\text{Fundamental}} = \partial_\mu \phi_{\text{Fundamental}} - iA_\mu \cdot \phi_{\text{Fundamental}}.$$

(A_μ acting on fundamental representation element; so just multiplies)

$$D_\mu \alpha_{\text{Adj}} = \partial_\mu \alpha_{\text{Adj}} - i[A_\mu, \alpha_{\text{Adj}}]$$

A_μ acting on an element of Lie Algebra ∴ so have commutation.

Effectively same defination; just had to take into account how element of Lie Algebra act on representations.

Infact, mathematically its the same defination.

Summary//

$$\boxed{\begin{aligned} D_\mu \phi_{\text{Fund.}} &= \partial_\mu \phi_{\text{Fund.}} - i A_\mu \cdot \phi_{\text{Fund.}} \\ D_\mu \alpha_{\text{Adj}} &= \partial_\mu \alpha_{\text{Adj}} - i [A_\mu , \alpha_{\text{Adj}}] \end{aligned}}$$

With this defination covariant derivative is covariant

D_μ is covariant.

Once we have D_μ ; we can build analogue of of Field Strength (like "E-M" tensor)

$$F_{\mu\nu} = i[D_\mu , D_\mu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu , A_\nu]$$

This is absent in maxwell theory ; but here we have this not trivial term; because $SU(2)$ is non-Abelian.

remember ; A_μ is element of Lie Algebra

$\Rightarrow$ $[A_\mu, A_\nu]$ belongs to Lie Algera

$\Rightarrow$ $F_{\mu\nu}$ is an element of Lie Algebra. (it is 2x2 matrix)

$$\boxed{F_{\mu\nu} = F_{\mu\nu}{}^{a} \, t_a}$$

decomposing into components.

$F_{\mu\nu}^{a}$ is Real field strength. $a = 1, 2, 3$.

$F_{\mu\nu}$ is an element of Lie Algebra.

In terms of component, we have -

↳ next page.

$$F_{\mu\nu}^{\ a} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^a_{bc} A^b_\mu A^c_\nu$$

structure constant.

we see the field strength is non linear in gauge potential.

If you want to interpret $F_{\mu\nu}^a$ as usual Electric & Magnetic field ;

i.e.; $F^a_{\mu\nu} \leftarrow \vec{E}^a , \vec{B}^a$ (we have 3 ~~types~~ different Electric & Magnetic fields)

$a = 1, 2, 3$

$$A_\mu^a \longleftarrow (V^a , \vec{A}^a_\mu)$$

$a = 1, 2, 3$

now; $\vec{E}^1 = \partial_t \vec{A}^1 - \vec{\nabla} V^1 + V^2 \vec{A}^3 - V^3 \vec{A}^2$

have this extra non-linear terms involving potential & vector potential of the two other components

$$\vec{B}^1 = \vec{\nabla} \times \vec{A}^1 + \vec{A}^2 \times \vec{A}^3$$

so; we have non-linear terms (which are absent in Maxwell theory)

Since Covariant derivative is covariant

$\Downarrow$

Field strength is also covariant

(so it transforms covariantly)

$F_{\mu\nu}$ transforms covariantly

so: if you do gauge transformation. ($F_{\mu\nu}$ belongs to adjoint representation)

$$F_{\mu\nu}(x) \longrightarrow g(x) \cdot F_{\mu\nu}(x) \cdot g^{-1}(x)$$

local gauge transformation.

For infinitisimal gauge transformation.

$$F_{\mu\nu}(x) \longrightarrow F_{\mu\nu}(x) - i\,[F_{\mu\nu}(x), \alpha(x)]$$

---

of course ; $F_{\mu\nu} = -F_{\nu\mu}$. Hehe...  ..

Now, we can build , YANG-MILLS ACTION for gauge field (analogue of Maxwell Action for U(1) gauge field)

$$S_{YM}[A_\mu] = -\frac{1}{2g^2_{Y.M.}}\int d^4x \, \text{Tr}\,[F^{\mu\nu}F_{\mu\nu}]$$

(Gauge field action) looks like maxwell...

g is analogue of charge

$g_{Y.M.} \Rightarrow$ coupling constant of Yang Mill Theory... charge of particles.

$F_{\mu\nu}F^{\mu\nu}$ is 2x2 matrix $\Rightarrow$ ~~so natural~~ so natural choice is to take trace .

$\frac{1}{g^2_{Y.M.}}$ is analogue of $\frac{1}{e^2}$ in QED.

Tr = Trace in Lie Algebra. (for SU(2) only one trace ... its fine)

$$S_{YM}[A_\mu] = -\frac{1}{2g^2_{Y.M.}}\int d^4x \, \text{Tr}[F_{\mu\nu}F^{\mu\nu}]$$

$\mathrm{Tr}(t_a t_b) = \frac{1}{2}\delta_{ab}$ properties from Pauli matrices. (group theory)

$$\Rightarrow \boxed{S_{YM}[A_\mu] = -\frac{1}{4g^2_{Y.M.}}\int d^4x\, F^a_{\mu\nu}F^{\mu\nu}_a}$$

Yang-Mills action in terms of components.

↳ It is invariant under local gauge transformation.

$$S_{YM}[A_\mu] \longrightarrow S_{YM}[\cancel{g}\cdot A'_\mu \cdot \cancel{g^{-1}}] = \frac{-1}{2g^2_{YM}}\int d^4x\, \mathrm{Tr}\left(g\cdot F_{\mu\nu}\cdot g^{-1}\cdot g F^{\mu\nu} g^{-1}\right)$$

This is A_μ'

$$= \frac{-1}{2g^2_{YM}}\int d^4x\, \mathrm{Tr}\left(g\cdot F_{\mu\nu}F^{\mu\nu}g^{-1}\right)$$

So, we have classical theory which is gauge invariant.

use cyclic permutation property of Trace

$$= \frac{-1}{2g^2_{YM}}\int d^4x\, \mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

$A_\mu \to A_\mu' = g(A_\mu + i\partial_\mu)g^{-1}$

$\Rightarrow F_{\mu\nu} \to g F_{\mu\nu} g^{-1}$

$$= S_{YM}[A_\mu]$$

Physics is gauge invariant.

Since $F_{\mu\nu}$ contains non-linear terms $\Rightarrow$ This is much complicated than maxwell theory.

Lagrangian Density (Classical)

Classical Theory of Non-Abelian gauge field.

comes from the fact that you had structure constant.

$$\mathcal{L}_{YM} = -\frac{1}{4g^2_{YM}}\Big[(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \frac{1}{2}\epsilon_{abc}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)A^\mu_b A^\nu_c$$
$$+ (A_\mu^a A_\nu^b A^\mu_a A^\nu_b - A^a_\mu A_\nu^b A^\mu_b A^\nu_a)\Big]$$

↪ comes from the fact that you have two structure constant.

$$\left.\begin{aligned} f^a{}_{bc} &= \epsilon_{abc} \\ f^e{}_{ab} f^e{}_{cd} &= \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \end{aligned}\right\} \text{for } SU(2)$$

will be more complicated for SU(3), or say another gauge group

---

Coupling to Matter (Dirac Field case)

Dirac Field - Fundamental Representation.

$$S_{Dirac}[\bar{\psi}, \psi] = \int d^4x \, \bar{\psi}(i\cancel{\partial} - m)\psi$$

remember ψ is a two vector

If we want this action to be invariant under local gauge transformation.

$$\psi(x) \longrightarrow \underset{SU(2)}{g(x)}\,\psi(x)$$

↖ Fundamental Representation

so; have to replace ∂ by D

ie; $\cancel{\partial} \longrightarrow \cancel{D}$

ψ carries group indices i and dirac indices μ.

Since $\cancel{D}$ transforms covariantly as an element of adjoint representation; it absorbs $g, .. g^{-1}$.

$$\boxed{S_{Dirac}[\bar{\psi}, \psi] = \int d^4x \, \bar{\psi}(i\cancel{D} - m)\psi}$$

Invariant under local gauge transformation.

$$\mathcal{L} = \bar{\Psi}_i^{\alpha}(x)\Big\{ i\,\gamma^{\mu}_{\alpha\beta}\Big[\delta_{ij}\partial_\mu - i\,A^{a}_{\mu}(x)\cdot\frac{1}{2}(\sigma_a)_{ij}\Big] - m\,\delta_{ij}\,\delta_{\alpha\beta}\Big]\Psi^{\beta}_{j}(x)\Big)$$

$$\boxed{\mathcal{L}_{Dirac} = \bar{\Psi}^{\alpha}_{i}(x)\Big[i\gamma^{\mu}_{\alpha\beta}\Big[\delta_{ij}\partial_\mu - iA^{a}_{\mu}(x)\cdot\frac{1}{2}(\sigma_a)_{ij}\Big] - m\,\delta_{ij}\,\delta_{\alpha\beta}\Big]\Psi^{\beta}_{j}(x)}$$

g_{YM} is charge of elementary particle

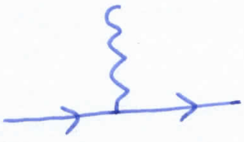
∴ if we redefine $A \longrightarrow g_{Y.M}\tilde{A}$

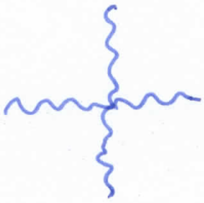
Then;

$$\mathcal{L}_{Y.M.} = (\partial\tilde{A})^2 + g_{YM}(\partial\tilde{A}\cdot\tilde{A}\cdot\tilde{A}) + g^2_{Y.M.}\cdot\tilde{A}\tilde{A}\tilde{A}\tilde{A}$$

$$\mathcal{L}_{Dirac} = \bar{\Psi}\,(i\partial\!\!\!/ - m)\Psi + g_{YM}\tilde{A}\bar{\Psi}\Psi$$

g_{YM} ⇒ charge of the Dirac Field.
but it is also charge of gauge vector boson





Gauge fields carry charges; so they interact.

Conserved current for Dirac fields coupled to gauge field in SU(2) theory

$$J^\mu_a = \bar\Psi \gamma^\mu t_a \Psi + \frac{1}{g^2_{YM}} \epsilon_{abc} F^b_{\mu\nu} A^c_\nu$$

This term was ~~absent in Ma~~ absent in maxwell theory ; because Maxwell theory is abelian and so the structure constant is zero there.
Here it is non-zero ; $\sim \epsilon_{abc}$.

is conserved by cancellation of currents ~~carry~~ carried by fermions ~~&~~ with ~~for gauge bosons~~ currents carried by gauge bosons; fermions & bosons can exchange charges (because there are 3 different type of charges)

$\partial_\mu J^\mu_a = 0$; $a = 1, 2, 3$.

Lecture 8.1] Currents, Quantization of non-abelian gauge theory

— Shoaib Akhtar 29/5/2020

SU(2) non-abelian Gauge Theory.

Gauge Theory
↳ Theory of vector particles.

- Currents
- Quantization and Gauge Fixing.

Gauge Field (Potential) : $A_\mu = \{ A^a_\mu(x) ; a = 1, 2, 3 \}$

"a" is associated to three generators of SU(2)

$A_\mu \in su(2)$; ie; A_μ lives in Lie Algebra, which is adjoint representation of SU(2)

If we take generators $t_a = \frac{1}{2}\sigma_a$ 2×2 Pauli ~~Mat~~ Matrices.

Then $A_\mu = A_\mu^a t_a = \begin{pmatrix} A_\mu^3 & A'_\mu - i A_\mu^2 \\ A'_\mu + i A_\mu^2 & -A_\mu^3 \end{pmatrix}$ 2×2 Hermitian Matrix

The effect of gauge transformation :

if $g \in SU(2)$ ie; 2×2 unitary matrix

$$A_\mu \longrightarrow g \cdot (A_\mu + i\partial_\mu) g^{-1} \equiv A_\mu^{(g)}$$

Infinitesimal ; $g = 1 + i\alpha + ..$

$$A_\mu \longrightarrow A_\mu + D_\mu \alpha$$

where ; $D_\mu \alpha = \partial_\mu \alpha - i [A_\mu, \alpha]$ Covariant derivative

(acting on adjoint representation)

Field Strength

$$F_{\mu\nu} = F_{\mu\nu}^{a}\, t_a = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

Covariant object. i.e; $F_{\mu\nu} \longrightarrow g \cdot F_{\mu\nu} \cdot g^{-1}$

Y.M. Action (Gauge Invariant)

No matter field.

$$\underset{\text{Y.M.}}{S}[A_\mu] = \frac{-1}{2g^2_{\text{Y.M.}}} \cdot \int d^4x \; \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]$$

Field associated to non-abelian gauge group.

- Gauge particles (particles which come out of gauge fields) carry charges (different from U(1) maxwell theory)

[redacted] $J_a^{\mu} = \epsilon_{abc} F^{b}_{\mu\nu} A^{c}_{\mu}$

; $J^\mu = J^\mu_a t_a = [F_{\mu\nu}, A_\nu]$

- Equations (classical) for field Strength. for $\vec{E}^a, \vec{B}^a$, $a = 1,2,3$

Two pairs of equation

$D_\mu F^{\mu\nu} = 0$ Equation of Motion

$D_\mu \tilde{F}^{\mu\nu} = 0$ Bianchi Identity.

(usual consistency equation)

↳ comes from the fact that $F^{\mu\nu}$ is ~~obta~~ obtained from vector potential A_μ

where $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

in $d = 4$

non-linear P.D.E.

(classical solutions are not just plane waves)

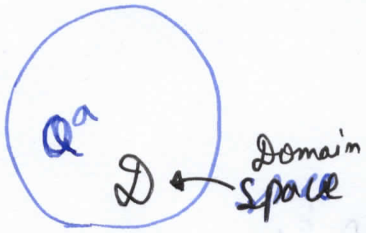
$J^\mu \propto \partial_\nu F^{\mu\nu}$ using equation of motion

from this point of view; it is very ~~each~~ easy that $\partial_\mu J^\mu = 0$

∴ hence conserved current.

★ J^μ is not gauge covariant!
(not real problem; because you have some freedom in defining current J^μ)

Charge



$$g(x) = \begin{cases} \overset{\text{a constant}}{g} & \text{out side of } \mathcal{D} \\ \text{non-constant} & \text{inside of } \mathcal{D} \end{cases}$$

with this; Charge Q transforms as

$$Q \to g \cdot Q \cdot g^{-1}$$

where g is the one which happens outside

⇒ so whatever happens inside; it does not affect the defination of charge

So; Total Charge rotates (because you have non-abelian symmetry)

↪ so, charge is defined in global way.

Quantize by Path Integral

propagator of gauge field is obtained by "Linearized Yang Mills Action"

↪ ie; keeping only linear terms in $F_{\mu\nu}$

$$S^{\text{linear}}_{YM} = \frac{-1}{4g^2_{YM}} \int_{\mathbb{M}^{1,3}} d^4x \, (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)$$

Same as Maxwell; instead of 1 photon; we [illegible] here are three types of photons.

$$S^{\text{linear}}_{\text{Y.M.}} = \frac{-1}{4 g^2_{\text{Y.M.}}} \int_{\mathbb{M}^{1,3}} d^4x \, (\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a)(\partial^\mu A^\nu{}_a - \partial^\nu A^\mu{}_a)$$

We can write;

$$S^{\text{linear}}_{\text{Y,M.}} = \int d^4x \; A^a_\mu \, K^{\mu\nu}_{ab} \, A^b_\mu .$$

Kernel operator.
(second order linear differential operator)

The propagator; carry indices $\underset{\mu}{a}$ $\underset{\nu}{b}$

ie;

$$\underset{\mu}{\overset{a}{\sim\sim\sim}}\overset{b}{\underset{\nu}{}} \simeq \langle 0 | A^a_\mu \cdot A^b_\nu | 0 \rangle_{\text{linearized}} = (K^{-1})^{ab}_{\mu\nu}$$

because of gauge invariance; K has zero modes.

At, linear order; gauge transformation is

$$A_\mu{}^a(x) \longrightarrow A^a_\mu(x) + \partial_\mu \alpha^a(x)$$

(action does not change)

↪ so zero modes.

Momentum Space $K^{\mu\nu}_{ab} = \delta_{ab}(k^\mu k^\nu - k^2 h^{\mu\nu})$

$$\Rightarrow k_\mu K^{\mu\nu}_{ab} = 0$$

so, (This is zero mode)

↪ zero modes are just total derivatives of fields that are linear in k...

so; you cannot invert k

for U(1), gauge fix.
then compute.

Gauge fixing amounts in general to change the propagator.
ie; to project out the propagator ; to project out the zero modes.

SU(2) : Gauge Fixing, ~~then correct~~

$\downarrow$ then

Construct Correct Path Integral

$\downarrow$ then

Compute.

Gauge Fixing ; choosing representative of field in gauge potential. A_μ.

ex Axial gauge, landau gauge, Lorentz gauge, Columb gauge.

lets write $A_\mu = (A_0, \vec{A})$

- Columb Gauge : $\vec{\nabla} \cdot \vec{A}_a = 0$
- Axial Gauge : $A_0^a = 0$ $\qquad a = 1, 2, 3.$
- Lorentz Gauge -landau Gauge : $\partial_\mu A^\mu_a = 0$

ie; Lorentz - Landau gauge : $\partial_\mu A^\mu_a = 0$.

Principle we want to define functional integral of the form

$$\int D[\phi] \exp(i S_{Y.M.}[A])$$

Principle of Functional Integral

$$A = \{ A^a_\mu(x) \ ; x \in \mathbb{M} \ , \underset{\substack{\uparrow \\ \text{4-dimensions}}}{\mu = 0,1,2,3} \ , \underset{\substack{\uparrow \\ SU(2)}}{a = 1, 2, 3} \}$$

gauge potential configuration.
(regularity condition . . .)

$$D[A] = \prod_{x} \prod_{\mu} \prod_{a} dA_{\mu}^{a}(x)$$ (local measure)

There is consistent defination of Gauge Field on a Lattice. invented by K. Wilson ≈ 1974 (makes discreatization which is perfectly well defined)

---

let us denote $\mathcal{A} = \{A\}$ space of gauge potential configurations.

Big Gauge Group $\mathcal{G} = \{g\}$ space of all local gauge transformation

where $g = \{g(x) \in SU(2) \; ; \; x \in \mathbb{M}\}$ a local gauge transformation; it is 2x2 unitary matrix.

~~$g(x) = \{g(x)_{ij} : i,j=1,2 , g_{ij} \cdot \bar{g}_{jk} = \delta_{ik} , \det(g) = 1\}$~~

$$g(x) = \{ g(x)_{ij} : i,j = 1,2 \; ; \; g_{ij} \cdot \bar{g}_{jk} = \delta_{ik} \; ; \; \det(g) = 1 \}$$

Gauge Transformation $A \longrightarrow g A g^{-1} + i g \cdot \partial \cdot g^{-1} \equiv A_g$

$$\mathcal{G} = \bigotimes_{x \in \mathbb{M}} G_x$$ ($G = SU(2)$ gauge group)

(collection of gauge group; associated to each point of space-time .. just have direct product)

need to define measure on big gauge group $\mathcal{G}$. will be product of measure ~~of~~ on each little group.

measure $\mathcal{G}$: $D[g] = \prod_{x \in \mathbb{M}} d_{Har}(g(x))$

Taking Harr measure on SU(2)

$d_{Haar}(g)$: The Haar Measure on SU(2)
(which is invariant measure)

left (and right) invariant measure.

~~tks~~; $d_{Haar}(g_0 \cdot g) = d_{Haar}(g) = d_{Haar}(g \cdot g_0)$

for g_0 fixed.

→ invariant under action of groups onto ~~tsd~~ itself.

an element of SU(2) can always be written in the form

$$g = \begin{pmatrix} \cos\theta \cdot e^{i\phi} & -\sin\theta \cdot e^{-i\chi} \\ \sin\theta \cdot e^{i\chi} & \cos\theta \cdot e^{-i\phi} \end{pmatrix}$$

$\theta, \phi, \chi \rightarrow 3$ coordinates.

~~Haa~~ Haar Measure is just the measure

$$d_{Haar}(g) = \sin(2\theta) \cdot d\theta \cdot d\phi \cdot d\chi .$$

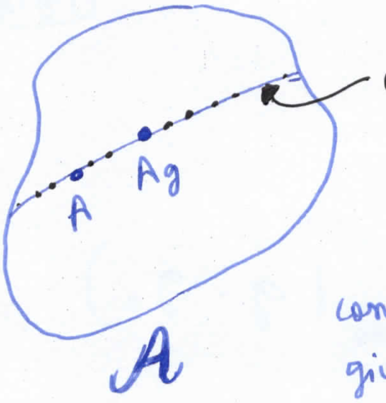
SU(2) more of less SO(3) ; $SU(2) \underset{locally}{\approx} SO(3)$ locally same group.

we want local measure D[A] to satisfy locality for Quantum Theory.

we can check ; $D[A] = \prod_{x} \prod_{\mu} \prod_{a} dA^a_\mu(x)$ is invariant under local gauge transformation.

(local rotation & local translation ≃ gauge transformation)

i.e; $\boxed{D[A] = \prod_{x} \prod_{\mu} \prod_{a} dA^a_\mu(x) \text{ is invariant under } \mathcal{G}}$

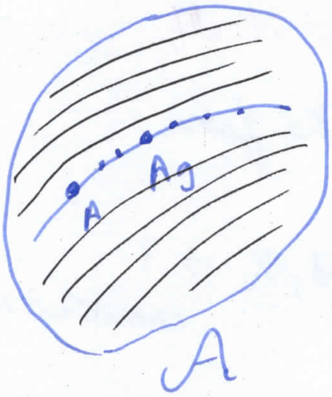


orbit of g (all points on this orbit ; i.e; all gauge configuration on a given orbit are gauge equivalent to A°, so they are physically equivalent) * Physically equivalent A's

* S_{YM} is constant

Big gauge group $\mathcal{G}$ acts on $\mathcal{A}$.

$\mathcal{G}$ acts on $\mathcal{A}$



We have different orbits associated to physically inequivalent configuration.

Space of physically inequivalent configurations $\mathcal{C} = \mathcal{A}/\mathcal{G}$

i.e; $\mathcal{C} = \mathcal{A}/\mathcal{G}$: its a quotient space

= space of orbits.

In perturbation theory;

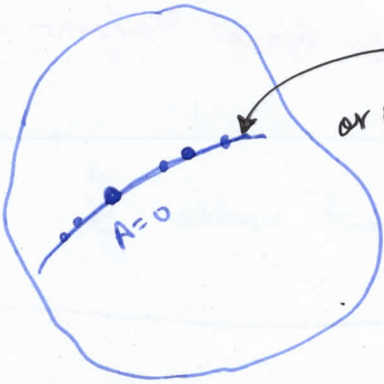
we start with classical vacuum $A = 0$ (minimize the action)

$\Updownarrow$

pure gauge configuration.

$A = \partial\alpha$

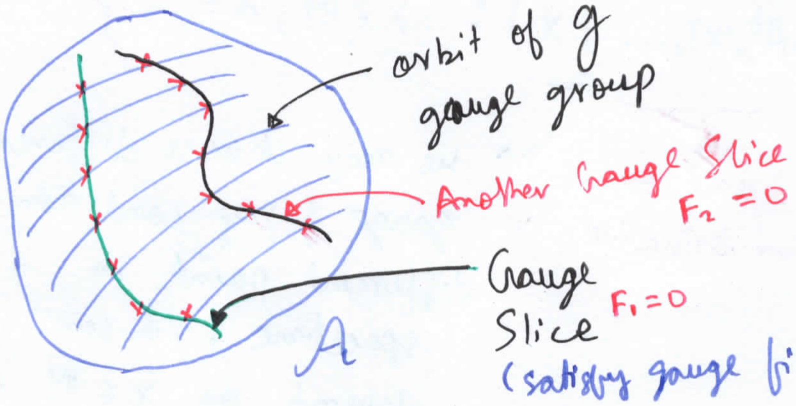
(are physically equivalent)



orbit of pure gauge or classical configuration which leads to vaccum configuration $\vec{E} = \vec{B} = 0$.

i.e; $F_{\mu\nu} = 0$.

Gauge fixing



In each orbit choose a representative

↪ This is gauge fixing procedure.

~~lets~~ lets hope; that in each gauge orbit, there is only one gauge configuration which satisfies $\partial^\mu A_\mu = 0$; then we pick it.

(or assume in each gauge orbit; there is only one which satisfies axial gauge)

We will replace the Big Integral, by an integral over the gauge slice.

★ We can take another gauge slice.

★ For the calculation ~~to~~ to make sense; the calculation ~~result~~ of integrating over different gauge slice should give same result.

(otherwise; result will be inconsistent)

"Gauge fix; in a gauge invariant way"

Gauge fixing procedure

define Gauge fixing functional $F[A] = 0$

↪ its a sketchy notation

★ it is data of set of gauge fixing condition (for each component of field)

~~$F[A] = \{F(A^b_\mu(x), \partial_\nu A^b_\mu(x), \dots)$~~

$$F[A] = \{ F(A^b_\mu(x), \partial_\nu A^b_\mu(x), \dots ; x) : x \in \mathbb{M}, a = 1,2,3 \}$$

may depend on higher derivative

we may choose different gauge fixing condition at different point in spacetime. So can depend on $x \in \mathbb{M}$ in general.

F application $A \longrightarrow \mathcal{G}_{\Pi} = \prod_{x \in \mathbb{M}} g_x$

Big Lie Algebra

i.e; Lie Algebra of the big gauge group $\mathcal{G}$.

g_x : Lie algebra of group G at poit x ie; G_x

ie; $g_x \in G_x$.

Assume that;

① if you take $F[A] = 0$; and $A \to A_g$ where g is infinitesimal gauge transformation $g = 1 + i\alpha$ (infinetisimal gauge transformation) ; we expect $F[A_g] \neq 0$ (ie; gauge slice is transverse to gauge orbits)

② $F[A] = 0$; for any $g \in \mathcal{G}$ $F[A_g] \neq 0$; ie; one gauge fixed configuration in each orbit. (we expect; gauge slice does not come back, things like that...)

we dont expect to have ⇒ this

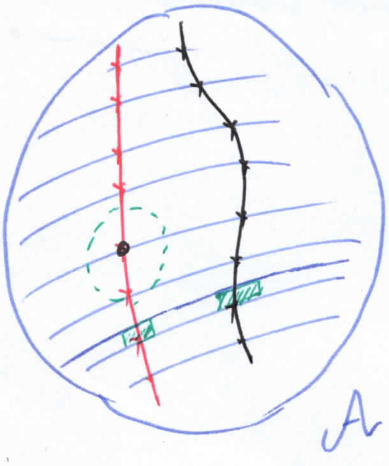
because then we would be doing some over counting.

In general, the conditions ① & ② are not true especially for landau ~~gauge~~ lorentz gauge.

This phenomenon is called Gribov - Copies Problem

→ related to Topology ... (not completely solved problem... ...open problem)

So; ~~to~~ conditions ① & ② are highly non-trivial.



◌ ⇒ study in vicinity... i.e; small fluctuation around classical vacuum (classical solution)

↪ one can show that Gribov - Copies are at finite distance; not infinetisimal distances. (so its not a problem in perturbation theory)

↪ so we will ignore this problem

★ Gribov - Copies problem, is not a problem in perturbation theory.

~~New~~ $D[A_g] = D[A]$ measure is gauge invariant.

If you integrate ~~[illegible]~~ naively on a gauge slice by taking measure induced on ~~[illegible]~~ gauge slice ; on other slice it will different ; there will be some Jacobian.

but we want $D[A_g] = D[A]$

How to write this as a measure $D[A : F[A] = 0]$ ↑ gauge fixing.

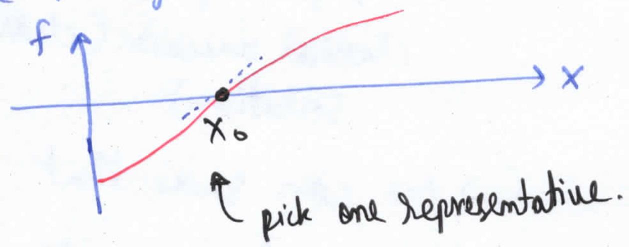
(here ; this problem is different for U(1) & non-abelian gauge theories : thats where it becomes interesting)

☆ $\underbrace{\text{Feynmann , De Witt.}}_{\text{from West}}$; $\underbrace{\text{Fadder , Popov}}_{\text{from East}}$.

---

1 dimensional example (configuration space A will be one dimensional)

replace A by ~~[illegible]~~ real variable x , which lives on the line $\longrightarrow x$



Some function $f(x_0) = 0$
"gauge fixing".

Now, I want to pick point where $f(x_0) = 0$.
~~"Pick~~ "Picking a point means delta function"

$$\int dx \, \delta(x - x_0) = 1 = \int dx \, \delta(f(x)) \cdot |f'(x)|$$

↖ derivative of f.

Jacobian of change of variable $x \to f(x)$

$$1 = \int dx \, \delta(f(x)) \cdot |f'(x)|$$

Gauge fixing

take absolute value ; if the function was negative , we still would want to pick it with weight 1

$$g(x_0) = \int dx\, g(x) \cdot \delta(x-x_0) = \int dx\, \delta(f(x))\, g(x)\, |f'(x)|$$

---

in n dimensions ; $X = (x^1, \dots, x^n)$ in $\mathbb{R}^n$

n conditions ; $F = (f^1, \dots, f^n)$

$$f^{(a)} = f(x^1, \dots, x^n)$$

given X_0 ; $F(x) = 0\Big|_{X_0}$; $\underbrace{f^1 = f^2 = \dots = f^n = 0}$

→ n gauge fixing condition ; one for each component.

we want to write

$$\int d^n x\, \delta^{(n)}(x - x_0) = 1 \Rightarrow 1 = \int d^n(x) \cdot \delta^{(n)}(F(x)) \left|\det\left(\frac{\partial F}{\partial x}\right)\right|$$

↑ Jacobian matrix of change of variable.

$$\delta^{(n)}(F(x)) = \delta(f^1)\,\delta(f^2) \dots \delta(f^{(n)})$$

$$\left(\frac{\partial F}{\partial x}\right)_{ab} = \frac{\partial f^a}{\partial x^b}$$

Lec 8.2 | Gauge Fixing, Ghosts

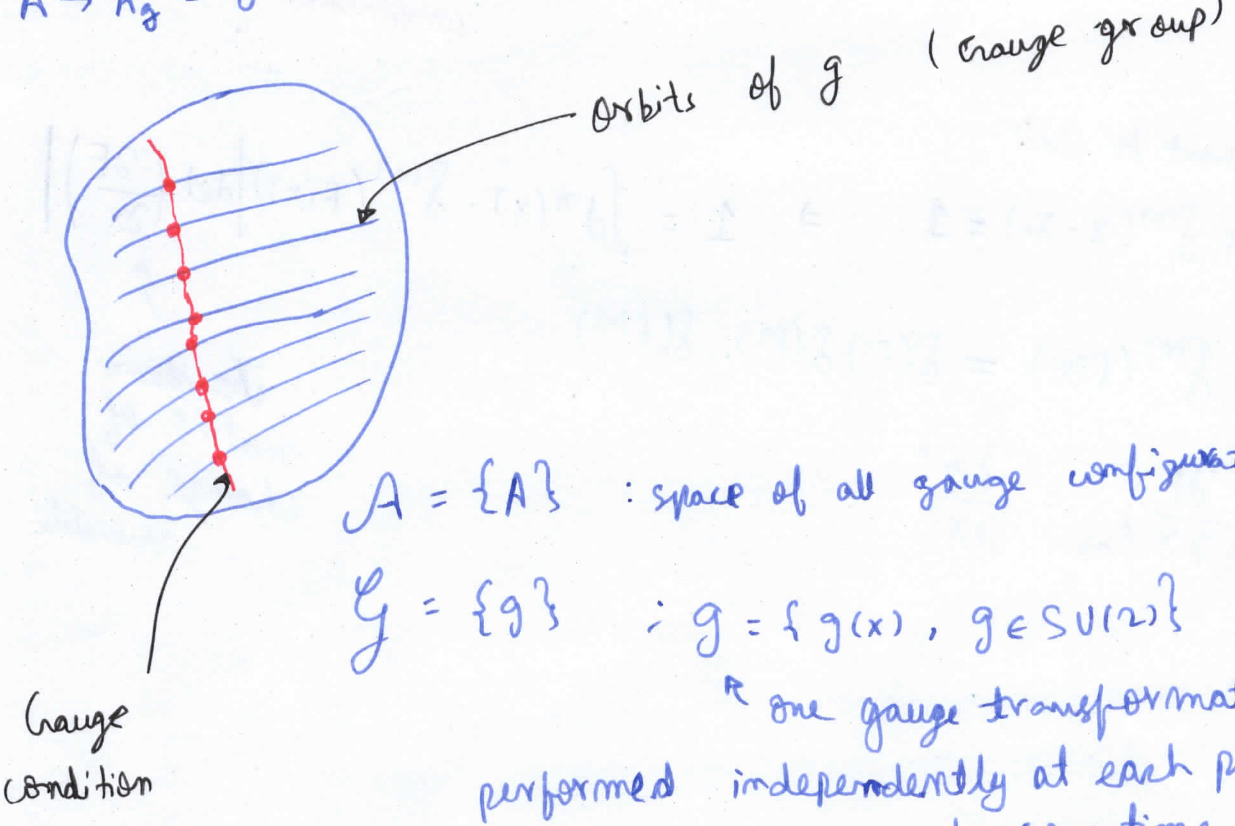
Gauge Fixing (continued): Fadeev - Popov "Ghosts".

SU(2) Yang Mills $A = \{A^a_\mu(x) \, ; a = 1, 2, 3\}$

$$\int \mathcal{D}[A] \exp(i S_{YM}[A])$$

both the measure $\mathcal{D}[A]$ and $S_{YM}[A]$ are Gauge Invariant.

$$A \to A_g = g(A + i\partial)g^{-1}$$



$\mathcal{A} = \{A\}$: space of all gauge configuration

$\mathcal{G} = \{g\}$; $g = \{g(x), \, g \in SU(2)\}$

↑ one gauge transformation performed independently at each point of spacetime

Gauge Fixing Condition: $\mathbf{F}[A] = \{F^a[A, x] \, ; \, x \in \mathbb{M}, a = 1, 2, 3\}$

enforce $F[A] = 0$ ↑ local functional.

Example:

Lorentz Landau : $F^a[A, x] = \partial_\mu A^\mu_a(x) = 0$

Feynmann Gauge (variant of Landau Lorentz-gauge) : $F^a[A, x] = \partial_\mu A^\mu_a(x) - \underset{\text{some function.}}{\epsilon_a(x)} = 0$

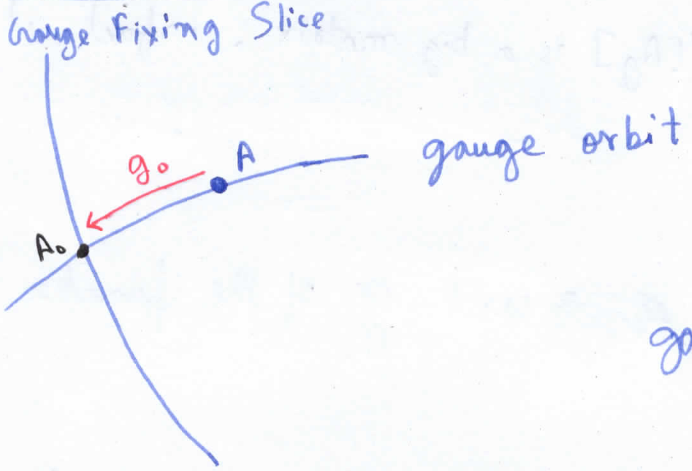
Baground Gauge (variant of Feynmann gauge)

choose a background gauge field $\bar{A}_a^{\mu}(x)$: fixed gauge potential.

$$F^{\mu}[A,x] = \bar{D}_{\mu} A^{\mu}_{a}(x) = 0$$

where $\bar{D}_{\mu}$: covariant derivative in $\bar{A}$

ie; $\bar{D}_{\mu} \# = \partial_{\mu} \# - i[\bar{A}_{\mu}, \#]$



∃ unique A_0 s.t. $F[A_0]=0$
on each
gauge
orbit

for any A, there is a (unique) gauge transformation g_0 which sends A to A_0

ie; $A_{g_0} = A_0$

such that $F[A_{g_0}] = 0$

These g_0's depend on A and on F.

lets denote $g_0 = g_F[A]$ (which means that it depends on gauge configuration & choice of gauge fixing)

$$1 = \int_{\mathcal{G}} \mathcal{D}_{Haar}[g] \cdot \delta[g, g_F[A]]$$

Dirac delta function

Integral over gauge group

Since $g_F[A]$ is determined by $F[A_{g_0}] = 0$

then; we can rewrite it as

$$1 = \int_{\mathcal{G}} D_{Haar}[g]\; \delta[g, g_F[A]] = \int_{\mathcal{G}} D[g] \cdot \delta[F[A_g]] \cdot |\det(F'[A_g])|$$

$$1 = \int_{\mathcal{G}} D_{Haar}[g] \cdot \delta[g, g_F[A]] = \int_{\mathcal{G}} D[g] \cdot \delta[F[A_g]] \cdot |\det(F'[A_g])|$$

~~$F[A_{g_0}] = 0$ infinite sets of gauge conditions~~

What is $F'[A_g]$ $F'[A_g]$ is a big matrix ; infact it will be Kernel of an operator.

What is the operator.

ie; we are making variation w.r.t. g of the function $F[A_g]$

What is variation of g ?

We have seen : $g \to g' = g + \delta g = g(1 + i\alpha)$; $\alpha \Rightarrow$ infinitisimal generator of the lie algebra

so; $\alpha(x) = \alpha^b(x)\, t_b$; $t_b = \frac{1}{2}\sigma_b$; $b = 1, 2, 3.$

$\uparrow$ basis of the lie algebra

$F'[A_g]^b_a$ will depend on, x and y.

$\uparrow$ 3x3 matrix.

x ← gauge condition taken at poin x. $F[A, x] = 0$

y ← variation at point y

$$\boxed{F'[A_g]^b_a(x, y) = \frac{F^a[A_{g(1+i\alpha)}; x] - F^a[A_g; x]}{\delta\alpha^b(y)}}$$

↑ differential expression

* gauge transformation at point y.
* And see the effect at gauge fixing term at point x

we can also write it as

$$F'[A_g]^a{}_b(x,y) = \left.\frac{\partial F^a[A_{g(1+i\alpha)};x]}{\partial \alpha^b(x)}\right|_{\alpha=0}$$

---

det(F'[A_g]) means take determinant of 3×3 matrix a, b
& of Kernel operator x, y

$\Psi = \{\Psi^a(x) : x \in M, a = 1,2,3\}$ functions in Lie Algebra

$F'[A_g]$ is an operator which sends $\Psi \to \tilde{\Psi}$ =

~~ie: $\Psi \to \tilde{\Psi} = F[A_g]\cdot\Psi$~~

$$\Psi \to \tilde{\Psi} = F[A_g]\Psi$$

Its a linear operator.

$$\tilde{\Psi}^a(x) = \int dy \; F'[A_g]^a{}_b(x,y) \cdot \Psi^b(y)$$

Its a convolution in space ; and acts as ordinary matrix in Lie Algebra.

with this defination we ; that $F'[A_g]$ is a linear operator acting on space of functions of Lie Algebra and gives an element of space of functions of Lie Algebra.

$$Z = \int D_{Haar}[g] \cdot \delta F[A_g] \, |\det[F'[A_g]]| \times \int D[A] \cdot \exp(i S_{Y.M.}[A])$$

(here I have actually inverted $\int D_{Haar}[g]$ & $\int D[A]$

What we should do is following.

$$Z = \int D[A] \left(\int D_{Haar}[g] \cdot \delta F[A_g] \, |\det[F'[A_g]]| \right) \int D[A] \exp(i S_{Y.M.}[A])$$

Insert the identity inside

.... now exchange the integral ; $\int D[g]$ & $\int D[A]$

$$Z=\int_{\mathcal{G}} D[g]\int_{A} D[A]\cdot\delta[F[A_g]]\cdot|\det[F'[A_g]]|\cdot\exp(iS_{Y.M.}[A])$$

we know; $D[A]$ & $S_{Y.M.}[A]$ are gauge invariant.

So; we can make change of variable inside $\int D[A]\ldots$

$A \rightarrow A_g$ because A_g is just dummy variable

using gauge invariance $D[A]=D[A_g]$; and $S_{Y.M.}[A]=S_{Y.M.}[A_g]$

$$Z=\int_{\mathcal{G}} D[g]\int_{A} D[A_g]\cdot\delta[F[A_g]]\,|\det[F'[A_g]]|\cdot\exp(i\,S_{Y.M.}[A_g])$$

$$=\int_{\mathcal{G}} D[g]\times\int_{A} D[\tilde{A}]\cdot\delta[F[\tilde{A}]]\times|\det[F'[\tilde{A}]]|\times\exp(i\,S_{Y.M.}[\tilde{A}])$$

$\tilde{A}=A_g$ (redefine ; and integrate ... dummy variable)

This factors out ie; $\int_{\mathcal{G}} D[g]$ and is just volume of gauge group.

(so its an infinite factor)

its like $(\text{finite number})^{\infty}$ because you have $SU(2)$ at each $x \in \mathbb{M}$.

It is independent of action... ... you have no dynamics.

This is just normalization factor.

$|\det[F'[\tilde{A}]]|$ = Fadeev Popov determinant.

$$F'[A]^a_{\ b}(x,y)=\left.\frac{\partial F^a[A_{1+i\alpha}\,;x]}{\partial\alpha^b(y)}\right|_{\alpha=0}$$

now ; F : Feynmann Gauge (working in Feynmann gauge)

what is $(A_{1+i\alpha})^a_\mu(x) = ?$

we know;

for SU(2)

$$(A_{1+i\alpha})^a_\mu(x) = A^a_\mu(x) + D_\mu \alpha^a(x)$$

$$= A^a_\mu(x) + \partial_\mu \alpha^a(x) + \epsilon_{abc} A_\mu{}^b(x)\, \alpha^c(x)$$

so; the gauge condition.

$$F[A_{1+i\alpha}; x]^a = \partial_\mu A^\mu{}_a(x) + \underbrace{\partial_\mu D^\mu \alpha_a(x)}_{\partial_\mu \partial^\mu \alpha^a(x) + \partial_\mu[\epsilon_{abc} A^b_\mu(x)\, \alpha^c(x)]} + \epsilon^a(x)$$

↑ this does not vary if you make variations in α.

so;

$$F'[A]^a_b(x,y) = \left.\frac{\partial F^a[A_{1+i\alpha}; x]}{\partial \alpha^b(y)}\right|_{\alpha=0}$$

⇒

$$\boxed{F'[A]^a_b(x,y) = \delta^a_b \cdot \Delta + \epsilon_{acb}\, \partial_\mu A^\mu_c + \epsilon_{acb} A^c_\mu \overrightarrow{\partial_\mu}}$$

$F'[A]^a_b(x,y)$ is a differential operator.

contains oridinary laplacian Δ times identity matrix and other two terms.

→ differential operator & depends on A

$$\boxed{\begin{aligned} F'[A]\cdot \psi^a(x) &= \Delta_x \psi^a(x) + \epsilon_{acb} (\partial_\mu A^\mu_c(x)) \cdot \psi^b(x) \\ &+ \epsilon_{acb} \cdot A_\mu{}^c(x)\, (\partial_\mu \psi^b(x)) \end{aligned}}$$

depends on y if you write.

$\Delta_x \delta(x-y)$

$\partial_\mu A^\mu_c(x) \cdot \delta(x-y)$

$\partial_\mu \delta(x-y)$

similar to

$\Delta_x \to$ Kernel $\Delta_x \cdot \delta(x-y)$

F'[A] will introduce new contribution to action.

~~(to compensate naive~~ (fixing on gauge induce new contribution to action : This is needed in order to compensate naive contribution of gauge fixing so that the integral is indeed gauge invariant)

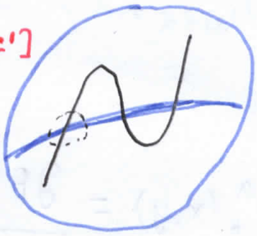
If you dont take |det F'[A]| into account ; you will have result which depends on gauge fixing condition.

How to compute the Fadeev-Popov Determinat ?

first remark : |det[F']| = det [F'] or -|det[F']| = det[-F']

valid only if det(F') ≥ 0

its an strong assumption.



It is related to the fact that you ~~do~~ are working in vicinity of classical vacuum: This det(F') basically measures slope of gauge slice with respect to orbit.

"The sign problem is related to the Gribov problem"

we can forget about these ; because this problem is irrelavant around A = 0 which is the classical vacuum.

now; if A = 0 : Then F' upto a constant is − Δ ie; F' ≃ − Δ (minus laplacians)

F' > 0

ie; F' is a positive operator

(because − Δ > 0 , i.e; minus laplacian is positive operator because eigen values are k² ; which is positive) pg161

Introduce additional anti-commuting fields to write det [F']

The trick is due to Fadeev & Popov

i.e; write this determinant as a path integral over these fields $\bar{c}$ and c ; i.e; over grassmannian anti-commuting fields.

$$\boxed{\det [F'] = \int D[\bar{c}, c] \cdot \exp\left(i\, \bar{c} \cdot F' \cdot c \right)}$$

we need to have that c is collection of three anti commuting Gauge Field.

$$C = \{ C^{a}(x) \ : \ x \in M, \ a = 1, 2, 3 \}$$
$$\bar{C} = \{ \bar{c}_{a}(x) \ : \ x \in M, \ a = 1, 2, 3 \}$$

⇒ They dont have Lorentz or dirac indices.

⇒ so They are Scalar.

Grassmann (anti-commuting variables)

fields on Spacetime

Quantum numbers belonging to Lie Algebra (as gauge potential A^{a}_{μ})

Scalar; in the sense they are spin 0 field.

They obey Fermi-Dirac Statistics (because they are Grassmann)

They violate spin-statistics theorem.

$$\det [iF'] = (i)^{\#} \det [F']$$ some number.

It turns out that the fields c & $\bar{c}$ violate unitarity.

It " " gauge invariance implies that when you take product ~~o~~ physical state. |physical⟩ with no ghost ~~o~~ it is positive

~~i.e; Gauge Invariance ⇒ ⟨ posit~~

i.e; Gauge Invariance ⇒ $\langle \text{physical} | \text{physical} \rangle \geq 0$

We can write an action for the ghost; which involves gauge field, c & $\bar{c}$.

$S_{ghost}[A, c, \bar{c}]$

$$S_{ghost}[A, c, \bar{c}] = \int d^4x \left[\bar{c}_a(x) \cdot \delta^a_b (-\Delta) \cdot c^b(x) + \epsilon_{abc} (\partial^\mu \bar{c}_a(x)) A^b_\mu(x) c^c(x) \right]$$

"Action term for the ghost"

obtained by doing a partial integration by parts; so that derivatives act on ($\bar{c}_a(x)$) rather than $A^b_\mu(x)\, c^c(x)$

This terms says that Ghosts are coupled to gauge potential; so they interact with gauge potential.

Once we do that; we get following form for path integral.

$$\int D[A] \int D[\bar{c}, c] \cdot \delta[F[A]] \cdot \exp\left[i \left(S_{YM}[A] + S_{Ghost}[A, c, \bar{c}] \right)\right]$$

lets work with Feynmann Gauge.

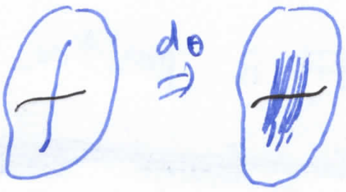
we will have $F_\epsilon[A]$

ie; $\delta[F[A]]$ will be $F_\epsilon = \partial_\mu A^\mu - \epsilon$

if we work correctly $\delta[F_\epsilon[A]]$ is independent of ϵ.

so; Instead of choosing one choice of gauge; we can average over different one (Its a trick)

ie; instead of integrating over slice; we make an average over F_ϵ



so; average over ϵ with gaussian weight.

An average of the form

$$\int D[\epsilon]\cdot\exp\left[\frac{-1}{2\xi}\int d^4x\,\epsilon^a(x)\,\epsilon_a(x)\right]\cdot\delta[F_\epsilon[A]]$$

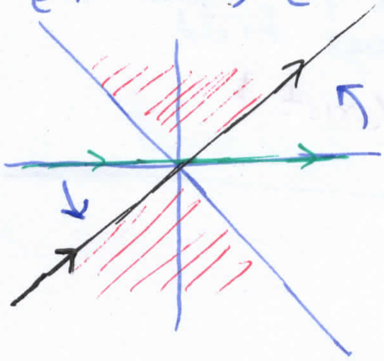
* A choice of gauge fixing procedure

ξ : some parameter.

$\delta[F_\epsilon[A]]$ is replaced by $\int D[\xi]\exp\left(-\frac{1}{2\xi}\int d^4x\,\epsilon^a(x)\,\epsilon_a(x)\right)\cdot\delta[F_\epsilon[A]]$

(gaussian average of delta function)

$e^{-x^2} \longrightarrow e^{-ix^2}$



$\epsilon(x)$ is a real variable.

Its a trick to get simple perturbation theory.

End result of this particular choice of gauge fixing condition is that; now you can forget about ϵ; i.e: integrate over ϵ. $\int D[\epsilon]$

Final Result for Path Integral

$$\int D[A]\int D[\bar{c},c]\cdot\exp\left[i\left(S_{Y.M.}[A]+S_{Ghost}[A,c,\bar{c}]+S_\xi[A]\right)\right]$$

replaced ϵ by $\partial_\mu A^\mu$

∵ comes from Gauge Fixing of Feynman.

where

$$S_\xi[A]=-\frac{1}{2\xi}\int d^4x\,\partial_\mu A^\mu_a(x)\cdot\partial_\nu A^\nu_a(x)$$

$$S_{Y.M.}[A]=\frac{-1}{4g^2_{Y.M.}}\int d^4x\cdot F^a_{\mu\nu}(x)\,F^{\mu\nu}_a(x)$$

where $F^a_{\mu\nu}(x)=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+\epsilon_{abc}A^b_\mu A^c_\nu$

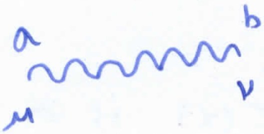
Feynman Diagrammatic Rules & Perturbative expansion.

Note!!

$$S_{linear} = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \frac{1}{\xi}(\partial_\mu A_\nu)^2$$

now this has no zero modes ; thanks to Feynmann term.

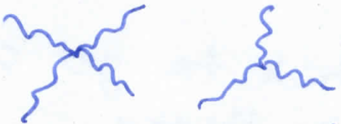
so; The propagator of gauge field ~~~ exists as in QED



= Propagator of Gauge field.
(exist)

---

we have seen



gauge field vertices

but now, we have ghosts (have ghost propagator)

remember ghost is a Fermion ; so it carries an arrow like Dirac Fermion. The ghost carry quantum numbers of gauge field; so it is charged , it carries indices a, b.

so, the propagator of ghost is

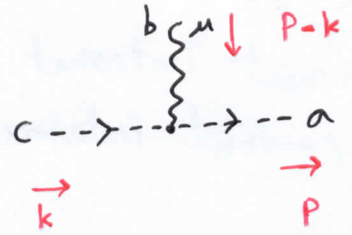
ghosts : are spin 0 massless - charged

Since ghosts have charge ; they interact with gauge field.

$$a \text{ - - - → - - - - } b = \delta_{ab} \cdot \frac{-i}{p^2 - i\epsilon_+}$$

(lets use dashed line for ghost ; because they are ghost ; & are not really there)

Since ghost interact; there is a vertex for ghost.

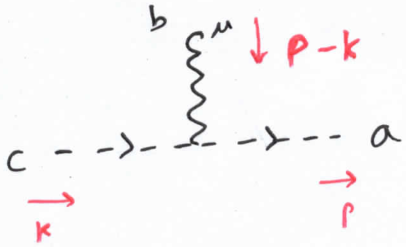


Since gauge field dont have spin; so Ghosts interact with gauge field through momentum (ie; interaction has to involve coupling of spin momentum of the ghost)

In fact you have momentum spin; comes in with $\vec{k}$ & goes out with $\vec{p}$

Then interaction is of the form $-g\cdot\epsilon_{abc}\,p^{\mu}$

so; The interaction vertex. (for having $\bar{c}$, c, and A interaction)



$$= (-g\cdot\epsilon_{abc}\,p^{\mu})$$

p is momentum of the ghost

U(1): Abelian group; there are ghosts but they are not coupled to gauge field A.

ie; det [F′] is independent of A.

↪ which means we can factor out |det [F′]|

so; it cancels out by denominator.

$$\text{ie; } \frac{\int \mathcal{D}\, e \cdots}{\int \mathcal{D}\, e}$$

i.e; in path integral there is internal loop of ghosts which interacts with nobody; and they dont interact with themselves either (because ghosts would only interact with gauge field) i.e; They are like; neutral dirac fermions with zero. They cannot interact with gauge potential because they are neutral; and they cannot interact with anybody. It turns out that they cannot interact with matter either.

lec 8.3 | Yang-Mills perturbative expansion, Renormalization of Yang-Mills theory

SU(2) Gauge Theory : $A = \{ A_\mu^a : a = 1,2,3 \}$ → $a \Rightarrow$ group indices in adjoint representation.

Dirac Field : $\Psi = \{ \Psi^i_\alpha : i = 1,2 \; ; \; \alpha = 1,2,3,4 \}$

$\bar{\Psi}$

i ← group indices in fundamental representation

α ← Dirac indices.

If we quantize the Gauge Theory in feynmann gauge ; we have to add Ghosts fields (which are way to represent gauge fixing condition)

$$C = \{ C^a : a = 1,2,3 \}$$
$$\bar{C} = \{ \bar{C}_a : a = 1,2,3 \}$$

Ghost field
Grassmanian but spin 0.

with this feynman gauge. $\partial_\mu A^\mu_a = \epsilon_a$; average over ϵ_a.

$$S_{YM} = \frac{-1}{2g^2_{Y.M.}} \int d^4x \, F^a_{\mu\nu} F^{\mu\nu}_a$$

$$S_{\substack{\text{gauge} \\ \text{fixing}}} = \frac{-1}{2\xi} \int d^4x \, (\partial_\mu A^\mu_a)(\partial_\nu A^\nu_a)$$

$$S_{Ghost} = \int d^4x . [\bar{C}_a \delta^a_b (-\Delta) C^b + \epsilon_{abc} \, \partial^\mu \bar{C}_a \, A^b_\mu . C^c]$$
$$= \int d^4x \; \bar{C} . \partial_\mu D^\mu . C$$

$$S_{Dirac} = \int d^4x \, \bar{\Psi} \, (i \not{D} - m) \Psi$$

$\Rightarrow$

$$S_{Dirac} = \int d^4x \, \bar{\Psi}^\alpha_i [(i\gamma^\mu)^\beta_\alpha \delta^i_j [\partial_\mu - i \frac{(\sigma a)}{2}^i_j . \delta^\beta_\alpha . A^a_\mu(x)] - m \delta^\beta_\alpha \delta^i_j] \Psi^\beta_j$$

~~$\int D[A] \int D[c,\bar{c}]\, D[\bar{\Psi},\psi] \exp\Big[i\Big(S_{YM}[A] + S_{G.Fixing}[A] + S_{Ghost}[A,c,\bar{c}] + S$~~

g is coupling constant ; g = charge of gause bosons & fermions (similar to e in QED)

it is also coupling.

$$Z = \int D[A] \int D[c,\bar{c}] \int D[\bar{\Psi},\Psi] \exp\Big[i\Big(S_{YM}[A] + S_{G.\,Fixing}[A] + S_{Ghosts}[A,c,\bar{c}] + S_{Dirac}[\Psi,\bar{\Psi},A]\Big)\Big]$$

$$\langle\Omega|T(\text{operators})|\Omega\rangle = \frac{\int D[\text{fields}]\cdot\exp(i\ \text{Action})\cdot(\text{operators})}{Z}$$

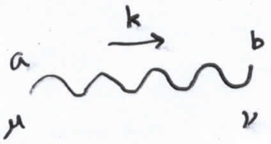
ie;
$$\boxed{\langle\Omega|T(\text{operators})|\Omega\rangle = \frac{\int D[\text{fields}]\exp(i\,\text{Action})\cdot(\text{operators})}{Z}}$$

Perturbation theory (expansion in coupling constants ; g^2 $\updownarrow$ $\hbar$)

$A^a_\mu \to g A^a_\mu$ (linearized term of order 1)
(redefination of A_μ)

gauge transformation will now depend on g.

Gauge Field Propagator



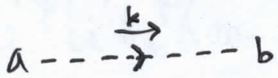
$$= G^{ab}_{\mu\nu}(k) \simeq \langle A^a_\mu(x) A^b_\nu(x)\rangle_0 = \delta^{ab}\cdot\frac{-i}{k^2 - i\epsilon_+}\cdot\left(h_{\mu\nu} + (\xi-1)\frac{k_\mu k_\nu}{k^2}\right)$$

$h_{\mu\nu} = \text{diag}(-1,+1,+1,+1)$ (spacetime minkowski metric)

ξ is a free parameter (Gauge Fixing)

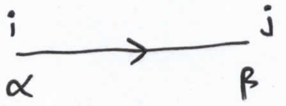
where $k^2 = -E^2 + \vec{k}^2$
$(-,+,+,+)$

Ghost Propagator



$$= \langle c^a(x)\,\bar{c}^b(x)\rangle_0 = \delta ab\cdot\frac{-i}{k^2 - i\epsilon_+}$$

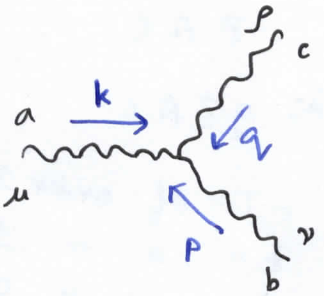
Dirac Field Propagator



$$= \langle\Psi^i_\alpha(x)\,\bar{\Psi}^j_\beta(x)\rangle_0 = \delta^{ij}\times(\text{Dirac Propagator})$$

as in QED.

Interaction Vertices



Three gauge fields
∴ comes from

$$\frac{g^3}{g^2} \cdot \partial A \cdot A \cdot A$$ in Yang Mill's Action

by rescaling

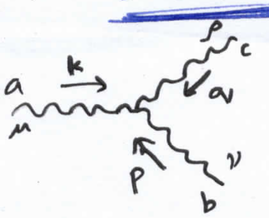
This we had initially

$a, b, c = 1, 2, 3$ gauge indices
$\mu, \nu, \rho = 1, 2, 3, 4$ Lorentz indices
k, p, q : 4-momenta
∴ and of course we have delta function for conservation of momentum

$$k + p + q = 0$$

so; it is proportional to g
ie; an interaction term proportional to charge of the particle

ie; 3 vertex interaction (3 Gauge boson interaction)



$$= g\, \epsilon_{abc} \left(h^{\mu\nu}(k-p)^{\rho} + h^{\nu\rho}(p-q)^{\mu} + h^{\rho\mu}(q-k)^{\nu} \right)$$

↖ SU(2)

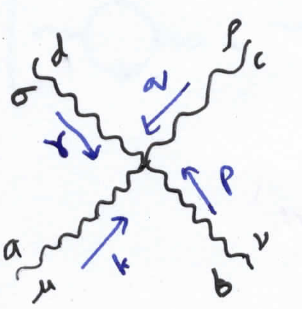
polarization - momentum coupling

---

4 vertex interaction
(4 Gauge boson interaction)

$A \cdot A \cdot A \cdot A$ coming from this $\frac{g^4}{g^2} A \cdot A \cdot A \cdot A$

so; it is of order g^2.



$$= g^2(-i) \left[\begin{array}{l} \epsilon_{abe}\epsilon_{cde}\left(h^{\mu\rho}h^{\nu\sigma} - h^{\mu\sigma}h^{\nu\sigma}\right) \\ + \epsilon_{ace}\epsilon_{bde}\left(h^{\mu\nu}h^{\rho\sigma} - h^{\mu\sigma}h^{\nu\rho}\right) \\ + \epsilon_{ade}\epsilon_{bce}\left(h^{\mu\nu}h^{\rho\sigma} - h^{\mu\sigma}h^{\nu\sigma}\right) \end{array} \right]$$

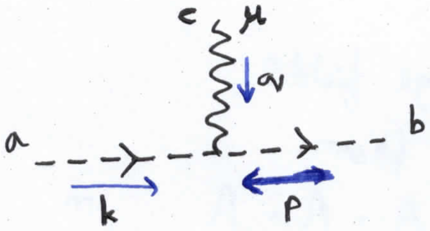
⚠ warning; might be small typo

$$\boxed{\epsilon_{ace}\epsilon_{bde} = \delta_{ab}\delta_{cd} - \delta_{ab}\delta_{bc} \quad \text{for } SU(2)}$$

⇓ look Peskin Shroder book

$a, b, c, d = 1, 2, 3$
$\mu, \nu, \rho, \sigma = 1, 2, 3, 4$
k, p, q, r, s : 4-momenta
$k + p + q + r = 0$

Ghost - Gauge boson interaction



$= g \cdot (-1) \cdot i \cdot p_\mu \cdot \epsilon^{abc}$

comes because of
$\bar{c} A c$
ie; $g \bar{c} A c$

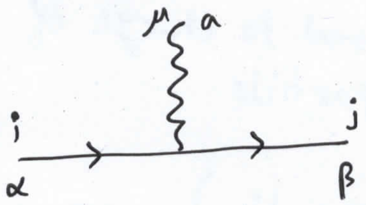
c is of order 1
$\bar{c}$ " " " 1
A " " " g

So; This term is of order g.

$k + p + q = 0$

Dirac Interaction Vertex

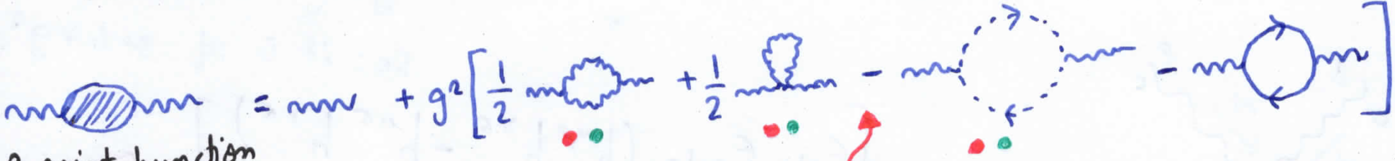
comes from the $\bar{\psi} A \psi$ term
Dirac - gauge boson interaction



$= g \cdot (\gamma^\mu)_{\alpha\beta} \cdot (\sigma^a)_{ij}$

Charges : here we mean SU(2) charges

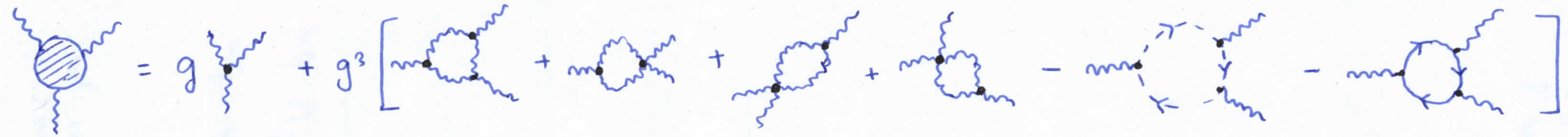
Irreducible (vertex) functions : 1-loop diagrams

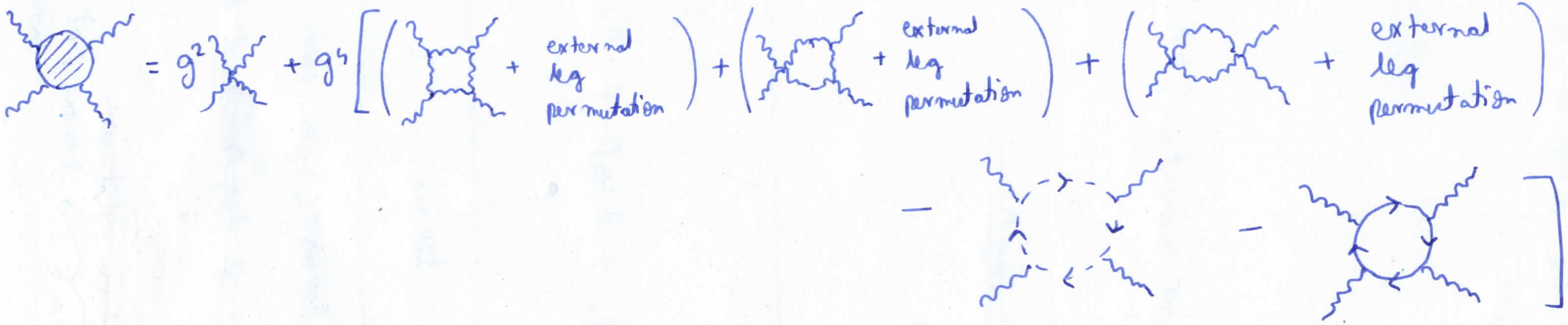


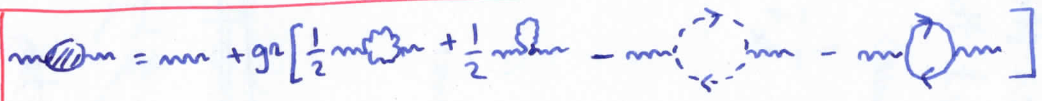
2 point function for Gauge field.

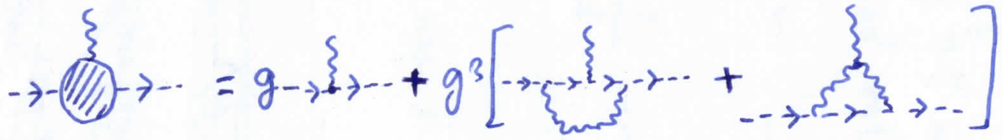
minus sign comes from closed loop (of fermions) ∵ Wicks Theorem for fermions

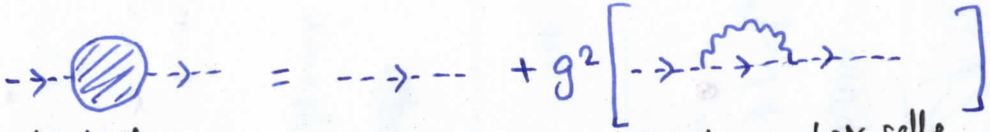
•• } ⇒ divergence contribution on Pg 162.











ghost two point function

(analogue for selfe energy diagram in QED; but now its a ghost)

Fermions

U.V. divergences and Renormalization

Non-abelian gauge theory has short distance singularity; which simply comes from the fact that the diagram of ghost & gauge field propagator behaves like..

 $\sim \frac{1}{k^4}$

 $\sim \frac{1}{k}$

if you consider that if you have some momenta ; less than Λ

ie; $|k| < \Lambda$

~~Then you have $\Lambda^2 + p^2 \log\Lambda$~~

~~Then you have divergence like,~~

~~$\delta^{ab}[\Lambda^2 k_{\mu\nu}$ $\delta^{ab}[\Lambda^2 h_{\mu\nu} + \log$~~

~~$\delta^{ab}[$~~ $\Lambda^2 h_{\mu\nu} +$ for 

Then you have divergence like

$$\delta^{ab}[\#\Lambda^2 h_{\mu\nu} + \#\log\Lambda\cdot p_\mu p_\nu + \#\log\Lambda\cdot p^2\cdot h_{\mu\nu}]$$

For  we have $\log\Lambda$ terms

 has Λ terms ; with som $\log\Lambda\cdot p^\mu$ terms. ~~$= \#\log\Lambda + \#\log\Lambda\cdot p^\mu$~~

~~term that are~~

ie:

 $= \overbrace{\#\,\Lambda}^{\text{terms that are dangerous}} + \#\,\overbrace{\log\Lambda\cdot p^\mu} \rightarrow$ terms that are less dangerous.

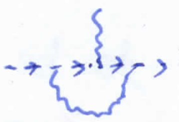
 is very dangerous ; and we dont like it at all.

because  would mean that (ie; taking  into accound) taking ~~out~~ into account quantum correction terms in action which will be linear in A_μ. Which is clearly not gauge invariant.

In QED  is trivially zero ; $\int_{-\infty}^{+\infty} d^3k \frac{k^{*\,\text{odd}}}{k^2+m^2} = 0$

These are also zero in gauge theory for the same above reason.

 contains $\# p \log p$

 contains

$\# \Lambda + \# p \cdot \log \Lambda$

$\log \Lambda$ divergences as in ϕ^4 theory of QED : because $[g]$ dimension $= 0$

dangerous Λ or Λ^2 divergences

If you think of original action $\frac{-1}{4g^2}\int d^4x\, F^a_{\mu\nu} F^{\mu\nu}_a$

$$F_{\mu\nu}^{\ a} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i\,\epsilon_{abc} A^b_\mu A^c_\nu$$

we see that for this to be dimensionally consistent

$[A] = [\partial_\mu] = (\text{length})^{-1} = (\text{mass})^1$

and so; $[F] = (\text{length})^{-2} = (\text{mass})^2$

$\Rightarrow [g]$ dimension $= (\text{length})^0 = (\text{mass})^0$

$$\int_{|k|<\Lambda} d^4k \cdot \frac{1}{k^2} \cdot \frac{1}{k^2} = \log\Lambda$$

dimension of ghost field
$[C] = (\text{legth})^{-1} = (\text{mass})^{1}$

Y.M. theory in $d=4$ have scale invariance classically.

~~$A_\mu^a(x) \to \lambda \cdot A_\mu^a(\lambda x)$~~ $A_\mu^a(x) \to \lambda \cdot A_\mu^a(\lambda x) \equiv A_\lambda$

$\lambda \Rightarrow$ global factor

then $S_{YM}[A_\lambda] = S_{YM}[A]$

so; if A is solution of classical equation of motion; then A_λ is also.

☆ also have Conformal Invariance

☆ Are Gauge Theories Renormalizable? (This problem puzzled physicists ~~[illegible]~~ for around 10 years)

- you need to have no Λ^2 and Λ divergences.
- " " " " relations between $\log\Lambda$ divergences.

~~$\Rightarrow$ gauge invariance $p_\mu\left(\overset{p}{\sim\sim\oslash\sim\sim}\right)^{\mu\nu} \neq 0$~~ ~~ne~~ new

$\Rightarrow$ gauge invariance ; we want $p_\mu\left(\overset{p}{\sim\sim\oslash\sim\sim}\right)^{\mu\nu}$ to be zero; i.e; $p_\mu\left(\overset{p}{\sim\sim\oslash\sim\sim}\right)^{\mu\nu} = 0$ otherwise, unphysical polarization states propagates as in QED.

If the coefficient of $\log\Lambda$ ....

$A_1 \log\Lambda \cdot (\partial A \cdot \partial A) + A_2 \log\Lambda \cdot (\partial A \cdot A \cdot A) + A_3 \log\Lambda \cdot (A \cdot A \cdot A \cdot A)$

coefficient of $\log\Lambda$; A_1 & A_2 & A_3 are not related in a fixed way : then it means that

$\Rightarrow$ counterterms which you introduce are not gauge invariant.

Renormalize the theory in a gauge invariant way.

- possible at 1 loop : 't Hooft 1972, (~~[illegible]~~) ↳ using a trick called dimension regularization

~~Proof~~ Gross - Wilczek, Politzer.

- proof that this is possible to do at all orders by Lee - Zinn Justin around 1973 or 1974

- Algebraic formalism BRST formalism. ⇐ CFT & String Theory.

Thanks to gauge invariance of the bare theory (the orignal theory); you can renormalize the theory

no Λ^2 and Λ divergences.

& log Λ divergences are consistent.

The theory is Renormalizable & has one ~~couplin~~ coupling constant $\alpha_{YM} = g^2$

you can define β function for $\alpha_{Y.M.}$ in the same way we did for ϕ^4.

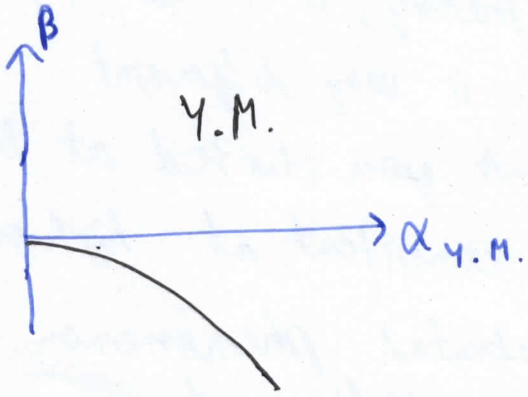
The result for Y.M. theory.

a constant which depends on group.

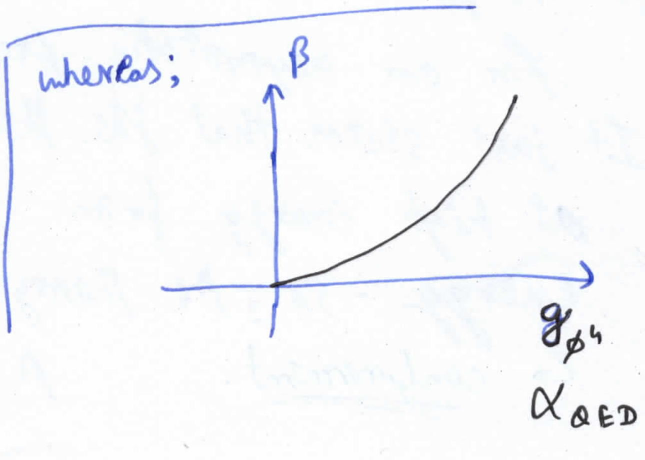
$$\beta(\alpha_{Y.M.}) = \alpha^2_{Y.M.} \cdot \frac{1}{(4\pi)^2} \cdot \left(-\frac{11}{3} C_2(G)\right)$$

$$\boxed{f^{acd} f^{bcd} = C_2(G)\,\delta^{ab}}$$ Casimier Representation.

for SO(N) ; this is equal to N ;) $C_2(G) = N$



whereas;



★ This ~~mo~~ means that Y.M. ~~theories ar~~ theories are Asymptotically free in the U.V. regime.

The running coupling constant ; $g_{eff}(E)$

$g_{eff}(E) \searrow 0$ as $\frac{1}{\log(E)}$ when energy $E \uparrow \infty$

which means that the theory is weakly coupled at high energy • (which is very nice : it explained experiment ; ie; inside hadrons the quarks may be considered as free particles interacting weakely : The gas of free quarks. So; it was justification for Parton Model)

• Also ; it says that at low energy ; the theory is strongly coupled .... which is just a justification of the fact that gauge theories leads to very compliated strongly interacting physics at low energies.

The theory which is strongly interacting at high energy is problematic ; because you assume that the theory was weakely interacting at high energy in order to define renormalizability.

If the theory is strongly interacting at high energy; it is problem with the formalism.

For an asymptotic free theory ; it is not a problem. It just states that the theory is very different at high energy from what you started at low energy. So, the theory is consistent at high energy

↪ confinement. A celebrated phenomenon of confinement.

You cannot observe gauge bosons; because they are confined; or you ~~ca~~ cannot observe dirac fermions or chiral fermions (because they are confined). ⇒ It's just a non-trivial reflection of the fact that the theory is strongly interacting.

2→ You need to use new tools. Perturbation theory is not enough to understand the physics of gauge theories at low energies.

Theory is weakely coupled at high energy.

It is nice.

- Parton Model
- Strongly interacting theory at low energies.
- Confinement.

THE ART OF INFINITIES

THANK YOU

Find the problem involving infinity and kill it to recast your theory in a beautiful piece of art
Shoaib Akhtar