Knot Theory

The general mathematical theory of Knots.

SHOAIB AKHTAR

KNOT THEORY

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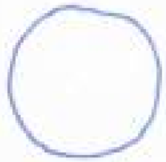
These notes are consequence of my self study; and are mostly inspired from Prof Anthony Bosman lectures. Also the book by Colin C. Adams was followed. These notes can be considered as a first course in Knot Theory; and builds up the general mathematical theory without much Physics.

Knot Theory   Shoaib Akhtar 21/7/2020   (Pg1)

Lec 1: Introduction to Knots & Invariants.

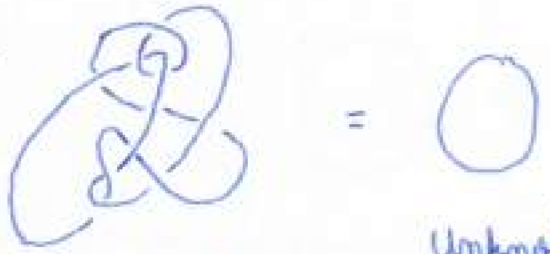
Mathematical Knots ≠ Tied Knots.
(Ends of Knots are joined together)

Knot is embedding of Circle in 3d space.



Unknot
(Trivial Knot)

* The Culprit Knot can be transformed to Unknot.



Culprit Knot = Unknot.

3_1
(Trefoil)

Crossing Number : Minimum no. of crossings given any knot diagram.
ex (here 3)

Knot index : Arbitrary index assigned to specific knot of same crossing number (to differentiate it from other knots with same crossing Number)
ex (here 1)

Ambient Isotopy : The process of deforming a knot without passing through itself.

3_1 is also known as Trefoil.
4_1 " " " " Figure Eight.

Reidemeister Moves

Move 1: "Twist"

Move 2: "Poke"

Move 3: "Slide"

Reidemeister Moves $\Longleftrightarrow$ Ambient Isotopy.

Knot Invariants

unchanging characteristics, unaffected by Reidemeister moves

Tricolorability: A knot's ability to be colored with three different colors

s.t.

1. At least two colors must be used
2. Incident crossing strands are either: all the same color or all different colours.

Under these criteriation; we see that there are two ~~kinds of~~ kinds of knots ① Tricolorable ② Non-tricolorable

Trefoil is tricolorable 

What about unknot, 

Unknot is not tricolorable:

because there is only one component to unknot; and we know that Reidemeister Moves preserve Colorability $\Rightarrow$ There is no way to color the unknot because there is only possibility of using one color, violating the rules of Tricolorability.

So:  $\neq$ 

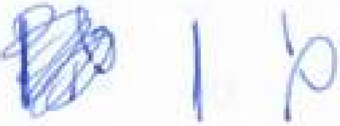
Figure Eight $\neq$ 

Trefoil

Trefoil $\Rightarrow$ Tricolorable

Figure Eight $\Rightarrow$ Non-tricolorable

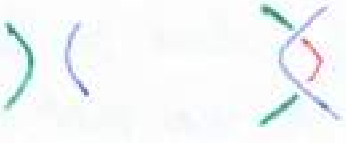
Reidemeister Moves preserves Colorability

Move 1) 

(Non-tricolorable)

 violation of rule 2

Move 2)  (Non tricolorable)

 (Tricolorable)

Knot Determinant.

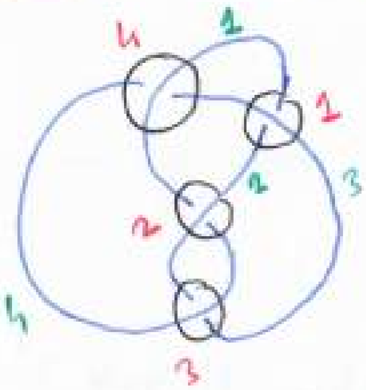
For each crossing

① Put 2 in column corresponding to <u>over crossing component</u>

② Put -1 in columns corresponding to <u>under crossing components</u>

③ Put 0 in columns corresponding to components not involved in crossing.

<u>Calculate</u> First no. each component and each crossing.



And then create $n \times n$ matrix for your knot. (where n is no. of crossing)

$$
\begin{array}{c c}
 & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \; \mapsto \text{components} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & -1 & -1 & 2 \\ 2 & 0 & -1 & -1 \end{bmatrix} \\
\Downarrow & \\
\text{crossing} &
\end{array}
$$

Once you get $n \times n$; delete a row & column in the matrix and calculate the determinant of the new matrix.

$$\begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & -1 & -1 & 2 \\ 2 & 0 & -1 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} -1 & -1 & 2 \\ -1 & 2 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\Downarrow$$

$$\det\left(\begin{bmatrix} -1 & -1 & 2 \\ -1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right) = 5$$

The prime factor of these determinants corresponds to no. of colors ~~they~~ that may be used to color using the coloring rules.

(hence, The tricolorability is just the subset of more general

p - colorability , p is prime)

Topics of Investigation

Alexander Polynomial

Surfaces & Genus

Braids

Fundamental Group

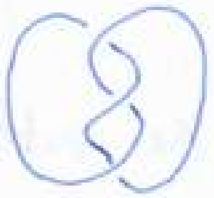
Lec 2: Coloring

Is that Knot Knotted or Not?


Unknot


Trefoil.

⇓



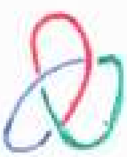
Def^n A knot is an embedding of a circle S^1 into $\mathbb{R}^3$.

We say, two knots are equivalent if they are Ambient Isotopy

Def^n A knot diagram is colorable if each arc can be colored using 3 colors s.t.:

1) Use at least 2 colors.
2) At any crossing, if 2 colors are used, three must be used.

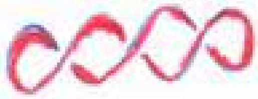
(Knot diagram is projection of knot on 2d surface; This is effectively what we are drawing on our paper)

~~Coloring~~ Ex  Trefoil is colorable.

ex  unknot, is not colorable.

(Another version of Trefoil)

This is colorable

Unknot  is Ambient Isotopy to 

(This is also not colorable)

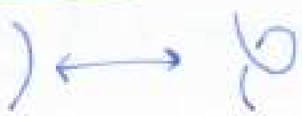
Conjecture: If a knot diagram is colorable, then all equivalent diagrams are colorable.

Likewise if a knot diagram is not colorable, then all equivalent diagrams are not colorable.

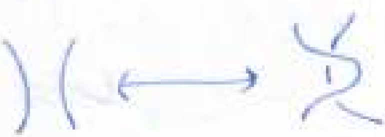
Not colorable

Not colorable

Reidemeister Moves

RI : 

(twist)

RII : 

(poke)

RIII : 

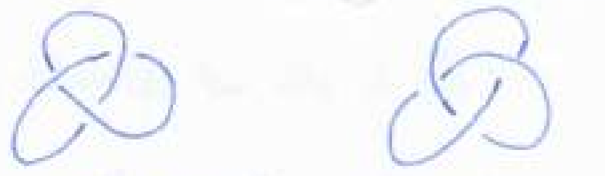
(slide)

If two diagrams are related by a sequence of R. moves

Theorem

$\Updownarrow$

They represent equivalent knots.



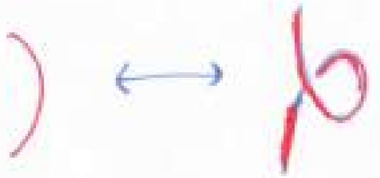
Left Handed Trefoil

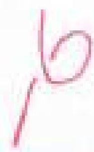
Right Handed Trefoil

Now, to prove our conjecture; we just have to show that each R moves preserves colorability.

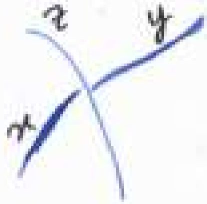
Proof of conjecture:

Show RI, RII, RIII preserve colorability. (see page 3)

eg. RI  How to check that at least two color was used.

if we assume  which is a part of a diagram to be with same color; Then there must be some other color in the other part of diagram ...... same for 

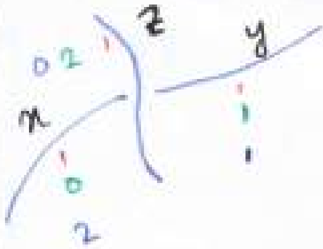
Another way to think about condition 2 of colorability



where $x, y, z \in \{0, 1, 2\}$ ⟶ Label for colors.

Condition 2 is equivalent to

$$x + y \equiv 2z \pmod 3$$



$x + y \equiv 2z \pmod 3$

$1 + 1 \equiv 2$ (differ by zero... so mod 3)

$0 + 1 \equiv 2 \times 2 = 4$ ($4 - 1 = 3$ ✓)

$2 + 1 \equiv 0$ ($3 - 0 = 3$ ✓)

We can generalize $x + y \equiv 2z \pmod 3$

We say a knot is p-colorable if : ($p \geq 3$, prime)

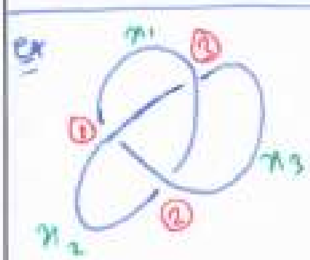
1. Use at least 2 labels from $\{0, 1, 2, \ldots, p-1\}$
2. At each crossing  : $x + y \equiv 2z \pmod p$

Goal:

Knot $\rightsquigarrow$ Matrix M
det (M)

p-colorable if and only if $p \mid \det(M)$

ex



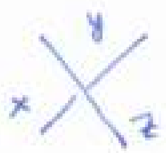
$$\begin{array}{c c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 2 & -1 & -1 \end{pmatrix} \end{array}$$

overcrossing $\equiv +2$

undercrossing $= -1$

motivation is.

Coloring condition:



$x + y \equiv 2z \pmod p$

$\Rightarrow \boxed{2z - x - y \equiv 0 \pmod p}$

Delete any 1 row & column

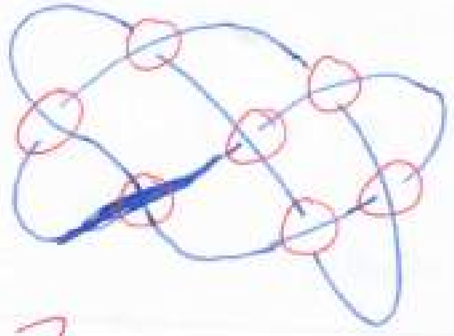
$\begin{pmatrix} -1 & 2 & \cancel{-1} \\ -1 & -1 & \cancel{2} \\ \cancel{2} & \cancel{-1} & \cancel{-1} \end{pmatrix}$; $M = \begin{pmatrix} -1 & 2 \\ -1 & -1 \end{pmatrix}$; $\det M = 1 + 2 = 3$

so: 3-colorable !

Trefoil is 3 colorable,
and not 5, 7, 11 colorable.

---

ex



7_4

$\det(7_4) = 15$

so

7_4 is 3, 5 colorable.

so: 7_4 is not equivalent to Trefoil
because 7_4 is 5 colorable
but Trefoil is not 5 colorable.

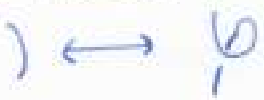
Lec 3: Alexander Polynomial.

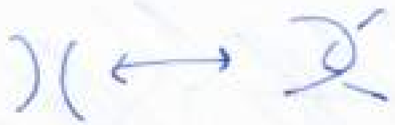
Alexander Polynomial of a Knot.

Two knots are equivalent (ambient isotopic)

$\Updownarrow$

They are related via Reidemeister Moves.

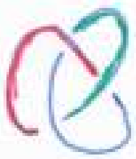
RI: 

RII: 

RIII: 

Colorable
- use ≥ 2 colors.
- At each crossing; all same or all different.

ex



Colorable!

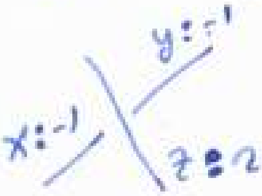
Not colorable!

Not colorable!

p-colorable:
- $\{0, 1, \dots, p-1\}$
- 

$x + y \equiv 2z \pmod p$

$2z - x - y \equiv 0 \pmod p$

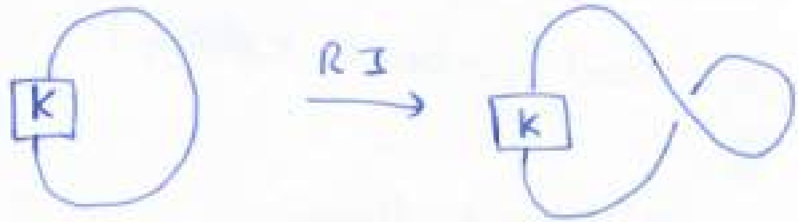


Theorem: K is p colorable $\Leftrightarrow$ $p \mid \det(M_K)$

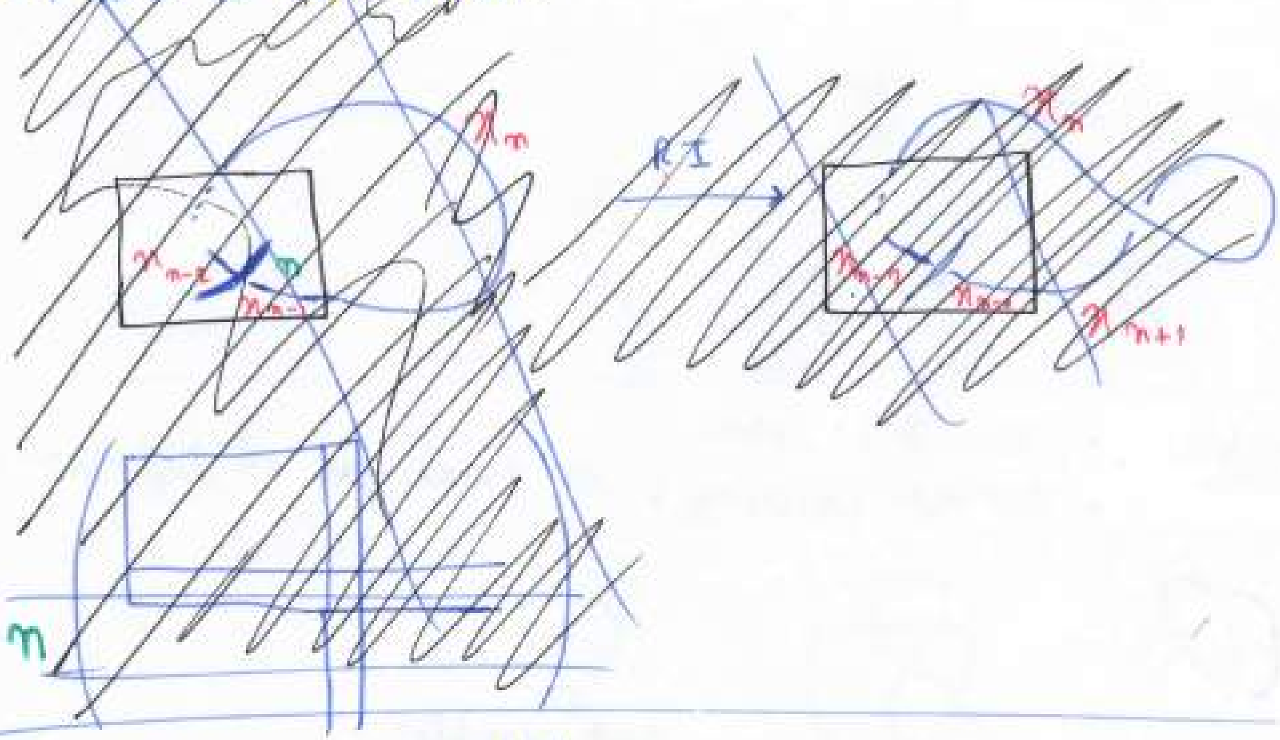
<u>Question</u> How do we know different diagrams will give same determinant?

<u>Answer</u> Just show det is preserved under RI, RII, RIII.

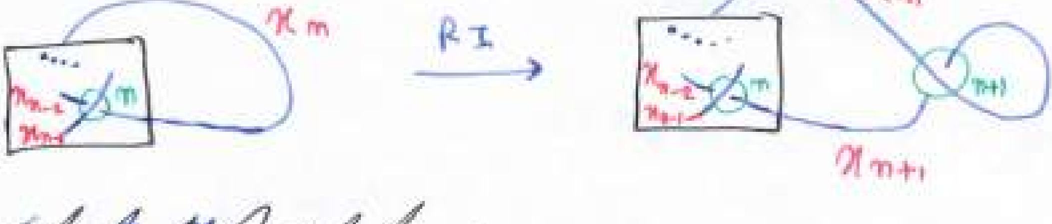
$\boxed{\text{R I}}$

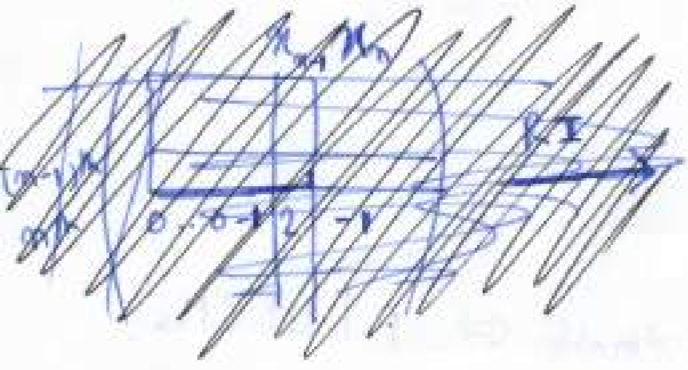


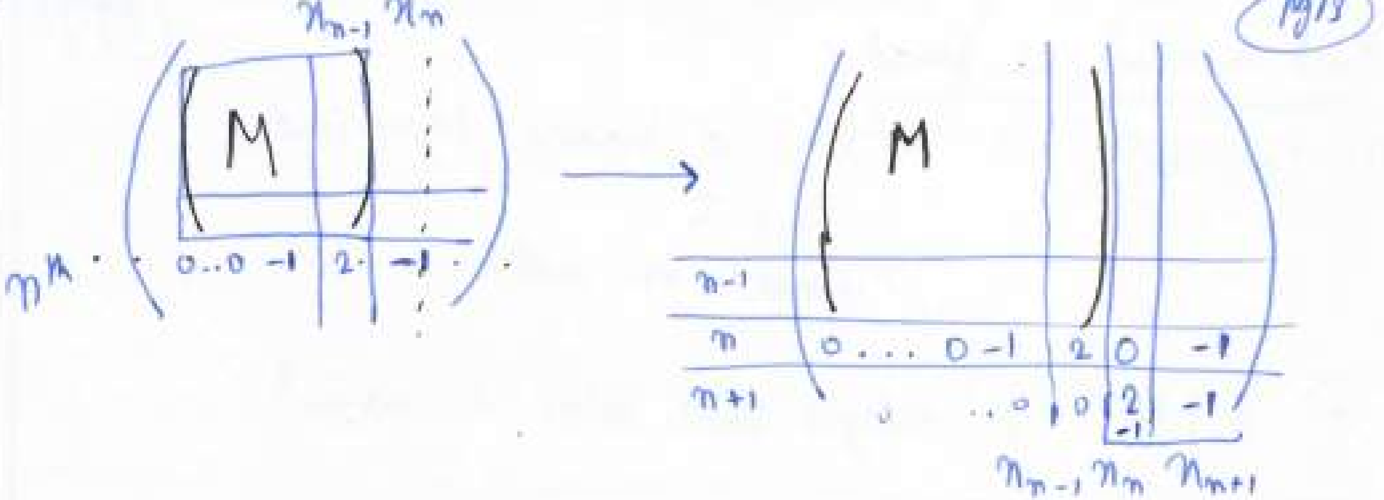
~~lets write in more detail.~~



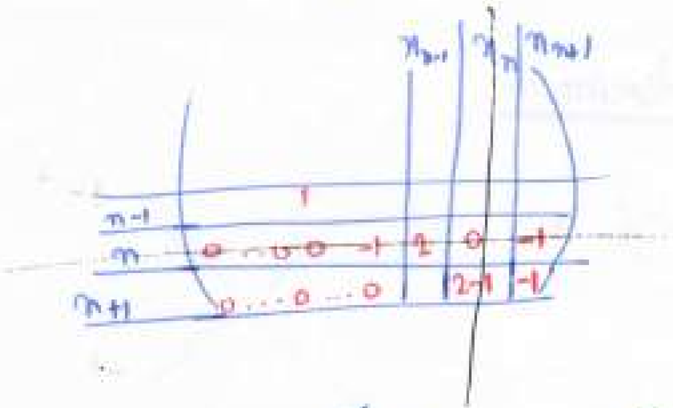
lets write in more detail







$2-1=1$



$$= \begin{pmatrix} (M) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \dots 0 & -1 \end{pmatrix} = M'$$

So; we see $\det(M) = -\det(M')$

We can check, under R I, R II, R III ; determinant can at most change by sign.

So; we can define $\det(K) = |\det(M)|$ where M is matrix obtained from any diagram K } Knot Invariant

☆ $\det(K)$ is not invariant.

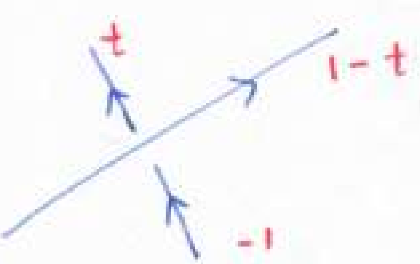
Knot invariants we found.

① Colorabity : Its a binary invariant.
0 , 1
(coloured or not)

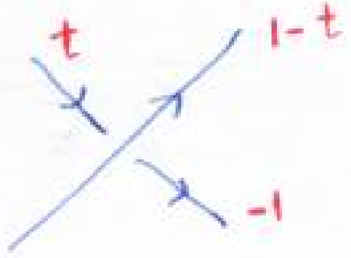
② det(K) is integer ~~value~~ value invariant.

③ We want to develop polynomial invariant of a knot.

Alexander Polynomial

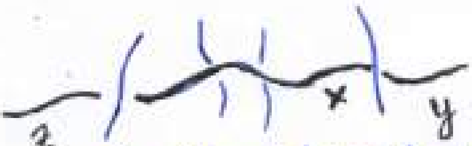


Right Handed Crossing



Left Handed Crossing.

We can assign orientation to Knots.

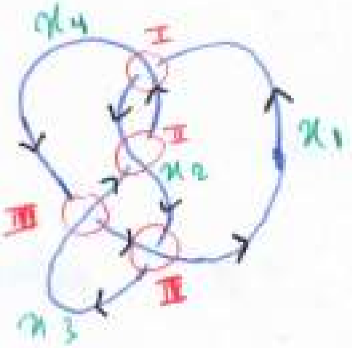


(x different from y & z)

~~An orientation can be chosen arbitrarily~~

So Choose an orientation.

ex) Figure 8



Ⓘ is right handed.
Ⓘⓘ " " "
Ⓘⓘⓘ " left "
ⒾⓋ " " "

now, build a matrix

$$\begin{array}{c} \\ \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} -1 & t & 0 & 1-t \\ 0 & 1-t & -1 & t \\ -1 & 0 & 1-t & t \\ 1-t & t & -1 & 0 \end{pmatrix} \end{array}$$

delete any one row & colum.

$$\begin{pmatrix} -1 & t & 0 & 1-t \\ 0 & 1-t & -1 & t \\ -1 & 0 & 1-t & t \\ 1-t & 1-t & -1 & 0 \end{pmatrix}$$

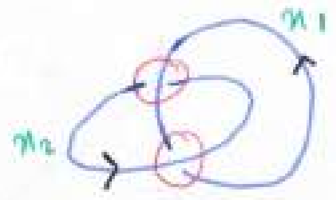
$\det(M) = -1(1-t)^2 - t(0-1) + 0 = -(1-2t+t^2)+t$

~~$\Rightarrow \det(M) = -t^2+3t-1$~~

$\Rightarrow \det(M) = -t^2+3t-1$ } We call this Alexander Polynomial.

$$\boxed{\Delta_{S_1}(t) = -t^2+3t-1}$$

We can also calculate Alexander polynomial for links.



$=: L$

$$\begin{pmatrix} t-1 & 1-t \\ 1-t & t-1 \end{pmatrix} = t-1 \equiv \Delta_L(t)$$

$t-1$ piece because same wire going down.. so; just add the two effects; $t+(-1) = t-1$

$\Delta_K(t)$ is invariant upto $\pm t^m$

$\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$

LAURENT POLYNOMIAL

S_2

$\Delta_{S_2}(t) = 2t^2 - 3t + 2$

$= 2t - 3 + 2t^{-1}$

$\boxed{\det(K) = \Delta_K(-1)}$

Note K is 2 colorable $\Rightarrow$ $\det(K)$ is odd.

What about $\Delta_K(+1)$. : we find $\underline{\Delta_K(1) = \pm 1}$.

$\Delta_K(1) = \pm 1$

K Mirror $\bar{K}$

How does mirroring impact to Alexander Polynomial.

''

Mirror

$\Delta_{\bar{K}} = \Delta_{\bar{K}}$

Left Right

$\rightarrow$ so: $\Delta_K = \Delta_{\bar{K}}$

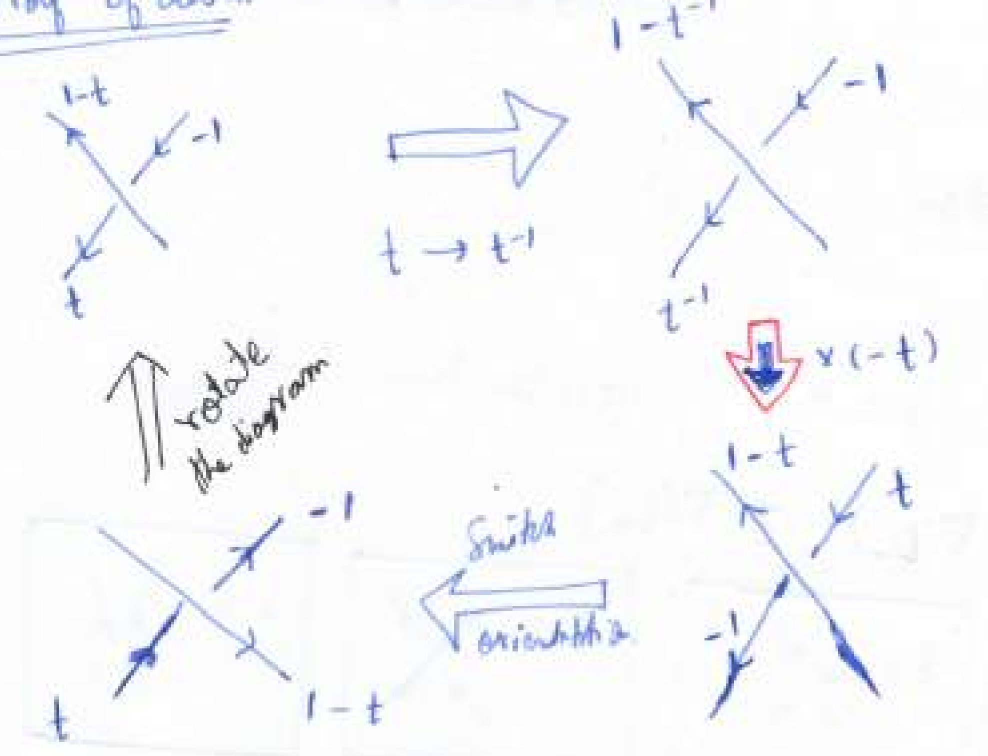
Δ_K is preserved under mirror image.

Claim: $\Delta_k(t) = \Delta_k(t^{-1})$ upto $\pm t^m$.

e.g.) $\Delta_{5_1}(t) = -t^2 + 3t - 1$

$\Delta_{5_1}(t^{-1}) = -t^{-2} + 3t^{-1} - 1 \underset{(\times t^2)}{\equiv} -1 + 3t - t^2 = \Delta_{5_1}(t)$

Proof of claim



~~A Laurent Polynomial represents the Alex~~ander

A Laurent Polynomial $f(t)$ represents the Alexander polynomial of some knot.

$\Updownarrow$

(1) $f(+1) = \pm 1$

(2) $f(t) = f(t^{-1})$ (upto $\pm t^m$)

e.g.) $f(t) = 5t^2 - 9 + 5t^{-2}$

Conway Polynomial:

~~Skein Relation~~

~~1. $\nabla(O) = 1$~~

~~2. $\nabla(OO\cdots) = 0$ for trivial links.~~

~~3.~~

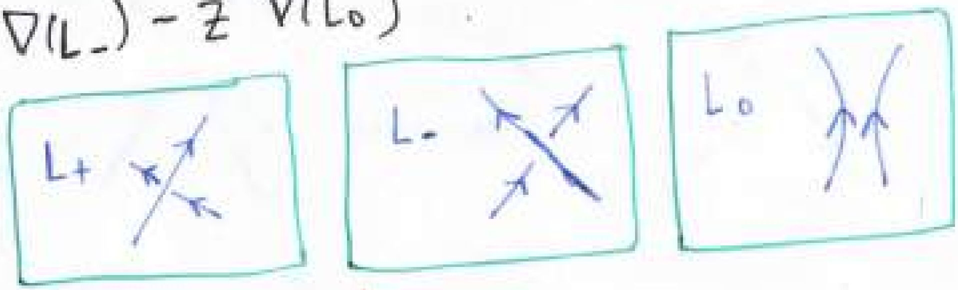
3.

Skein Relation

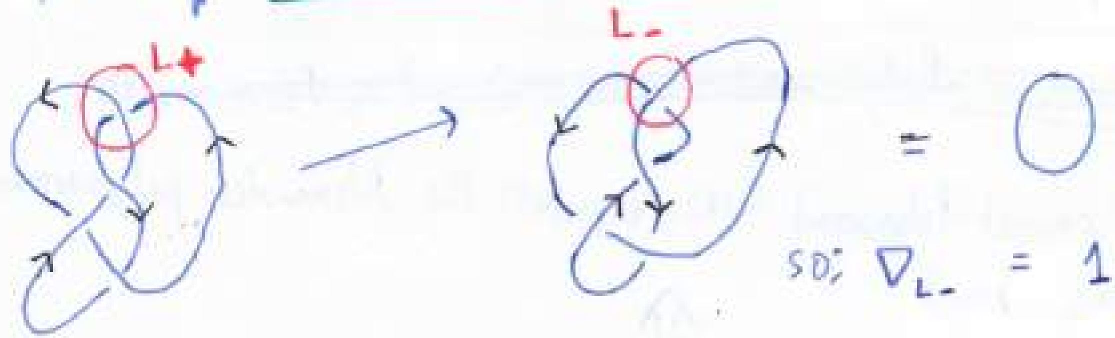
I. $\nabla(O) = 1$

II. $\nabla(OO\cdots) = 0$ (zero for trivial links)

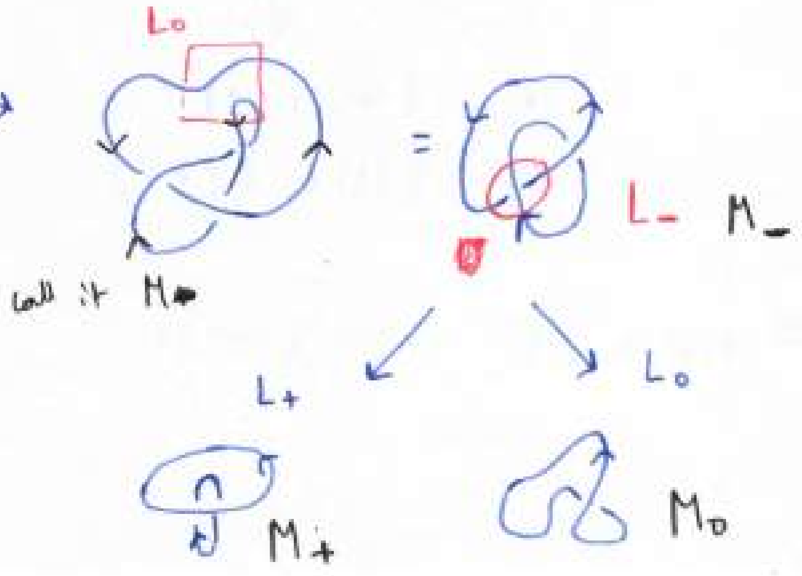
III. $\nabla(L_+) = \nabla(L_-) - z\,\nabla(L_0)$



ex)



so; $\nabla_{L_-} = 1$



so; $\nabla_{M_0} = 1$.

$\nabla_{M_+} = 0$.

~~Cosmology~~ CMB ~~Shazib Akhtar 18/7/2020.~~ (19)

$$\Rightarrow \nabla_{M_-} = \nabla_{M_+} + z\nabla_{M_0} \Rightarrow \nabla_{M_-} = z$$

so; $\nabla_{L_0} = z$.

so; $\nabla_{L_+} = \nabla_{L_-} - z\nabla_{L_0} = 1 - z \cdot z$
$= 1 - z^2$

so; $\nabla(4_1) = 1 - z^2$

4_1 is Figure Eight Knot.

$$\boxed{\Delta_K(t) = \nabla_K\left(t^{1/2} - t^{-1/2}\right)}$$

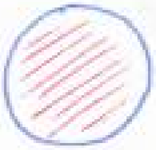
$\Delta_{4_1}(t) = 1 - (t^{1/2} - t^{-1/2})^2 = 1 - (t - 2 + t^{-1})$
$= -t + 3 - t^{-1}$

upto $(\pm t^m)$

Knot Theory

Shoaib Akhtar 21/7/2020

Lec 4: Surfaces & Genus



Unknot bounds a disk.

What kind of Surface does Trefoil bounds?

does it bound disks.

If Trefoil bounded a disk

contradiction.

Then we could move the knot inward into the disk; and ~~could~~ would get an unknot.

But we know, Trefoil is not an unknot so; it does not bound a disk.

☆ only unknot bounds a disk.

Surfaces: Compact, Orientable, without Boundary.

Non example of surface

Plane, not compact, no boundary.

Klein bottle (cant differentiate between inside & outside); not orientable,

Mobius Strip

not orientable. has boundary.

Classification : Compact, Orientable, Surfaces without boundary

faces F

edges E

Vertices V

1 

genus = 0

$\cong$

$\boxed{F - E + V = 2}$

$\cong$ equivalent upto Homeomorphism. (bi continuous map)

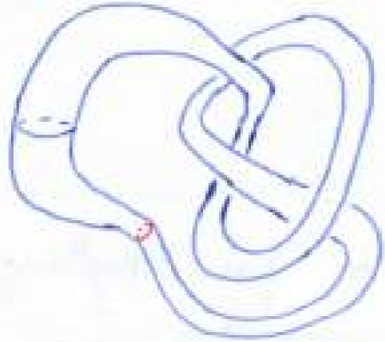
$\cong$ 

F = 6 , E = 12

V = 8

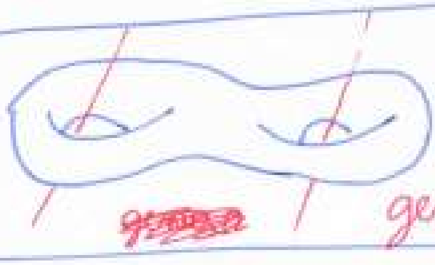
2 

$\cong$



Torus

genus = 1

3 

2-holed torus

genus = 2

4

⋮

g+1 

g-HOLED TORUS

genus = g

for sphere // genus = 0 :

proof of $F - E + V = 2$ via induction

(i) $F = 2$

we see $E = V$

$\Rightarrow F - E + V = 2$

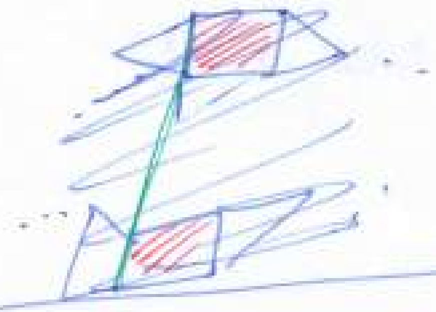
(ii) add faces

Then F go up by +1

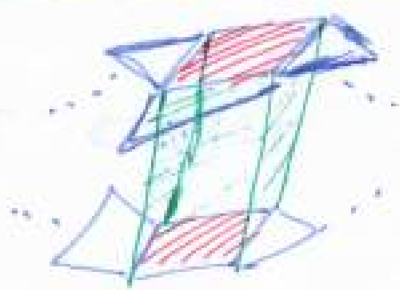
E " " " "

so; $\underset{(+1)}{F} - \underset{(+1)}{E} + V = 2$ $\quad {+1 \atop -1}$ ✓

so; $F - E + V = 2$ for anything homeomorphic to sphere.



g = 2



Remove 

& join through || ||

$\Rightarrow$ so we get hole

$\underset{(n-2)}{F} - \underset{(+n)}{E} + V = \underset{(2-2)}{0} \Rightarrow \boxed{F - E + V = 0}$

On torus.

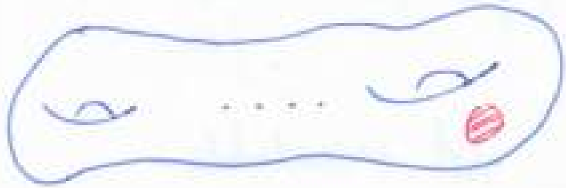
$F - E + V = 2 - 2g$ for anything homeomorphic to g-Holed Torus.

we call it χ, i.e. Euler Characteristics.

we have $\chi = 2 - 2g$ for genus g surface.

What happens when you add a boundary?

What is euler characteristic now



drill out a little face for a given boundary component.
($\Longleftrightarrow$ like taking a bite of g holed donut.)

for each boundary component we remove a face.

so; χ decreases 1

so:

$$\boxed{\begin{array}{c}\text{genus } g \text{ surface with } 1 \text{ boundary component} \\ \chi = 2 - 2g - 1\end{array}}$$

$$\boxed{\begin{array}{c}\text{Genus } g \text{ surface with } b \text{ boundary components} \\ \chi = 2 - 2g - b\end{array}}$$

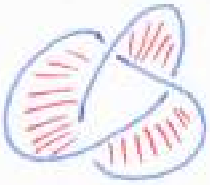
Now, lets come back to the question what kind of surface does a knot bounds.

If K is the boundary of a surface Σ,
then $\chi(\Sigma) = 2 - 2g(\Sigma) - \underline{1}$

because one boundary component

Σ is orientable compact surface.

What about Trefoil.



It is mobius band actually

give three twist & join.

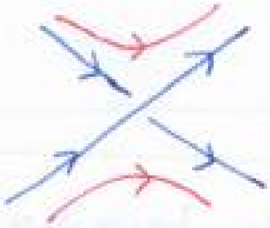
so; This Surface bounded by Trefoil is not orientable.

Can we look for orientable surface that a Knot bounds?
How to find an orientable surface:

(Seifert's Algorithm)

Take some orientation.

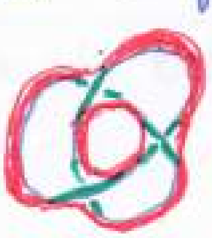
Then smooth out crossings.



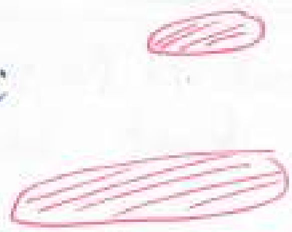
ie;

So; Smoothing out Trefoil

we get



; ie;

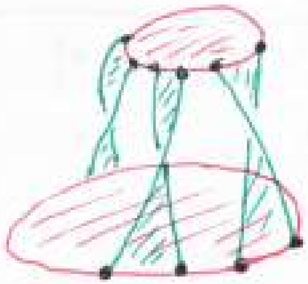


We also have crossings

Crossings are like band connecting top and bottom disk. → band with half twist.

Half Twisted Band

So, we finally get

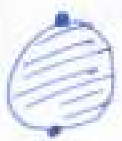


It is orientable.

What is the surface?

What is genus of this surface here.

$\chi = F - E + V$



$\chi(\text{disk}) = 1$

attaching a band χ goes down by 1.

→ for this $\chi = \underbrace{2}_{\text{2 disk}} - \underbrace{3}_{\text{3 bands}} = -1$

So: $\boxed{\chi = (\#\text{ Seifer Circles}) - (\#\text{ Crossings})}$

⇒ So, genus of the surface is **1**.

$\chi = S - C$ where

$S = \#$ of Seifert circles

$C = \#$ " crossings.

but $\chi = 2 - 2g - 1$

$\Rightarrow \quad S - C = 1 - 2g$

$$\Rightarrow \quad \boxed{g = \frac{1 + C - S}{2}}$$

where C is no. of crossings.

S is " " Seifert circles.

$$\boxed{g(\Sigma) = \frac{1 + C - S}{2}}$$

Def^n The genus of a knot :

$$g(K) := \min_{\substack{\Sigma \text{ with} \\ \text{boundary } K}} g(\Sigma) \quad , \; \Sigma \text{ with boundary } K.$$

Genus of the knot is by defination knot invariant

---

ex

$g = 0$

$g = 1$

$g = 3$

Take minimum

so;

$g(\text{unknot}) = 0.$

---

ex Trefoil (also called 3_1)

for now; we have only shown

$$g(3_1) \leq 1$$

con $g(3_1) = 0$; we know, only the unknot bounds a disk
ie; only unknot has genus $= 0$.

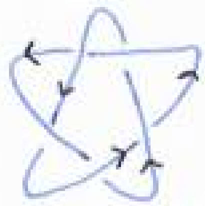
$$\boxed{\text{Only the unknot has genus} = 0}$$

So; $g(3_1)$ has to be 1.

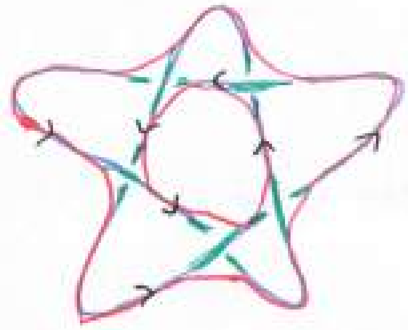
$$\Rightarrow \boxed{g(3_1) = 1}$$

---

lets consider find $g(5_1)$?

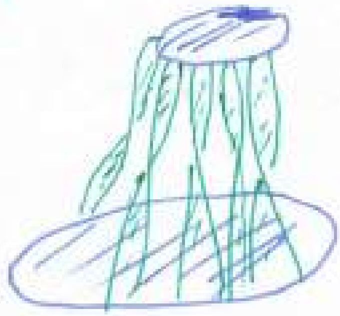


$\overset{\text{smooth out}}{\Longrightarrow}$



5_1

2 disk, attached with
5 twisted band attached.



$$g(\Sigma_1) = \frac{1 + c - s}{2} = \frac{1 + 5 - 2}{2} = 2$$

$g(5_1) \leq 2$ → This is how we get upper bound on genus.

we know $5_1 \neq \underset{\text{unknot}}{0}$

so; $g(5_1) \neq 0$ since $5_1 \neq$ unknot.

We have a lower bound on genus, from the following relation

$$\boxed{\frac{1}{2}\,\text{span}\ \Delta_K(t) \le g(K)}$$

ex) $\Delta_{5_1}(t) = t^{2} - t + 1 - t^{-1} - t^{-2}$

$= t^{4} - t^{3} + t^{2} - t - 1\,t^{0}$

$\text{span}\,(\Delta_{5_1}(t))$ = Difference between highest & lowest coefficient

$= 2 - (-2) = 4$

or

$= 4 - 0 = 4$

so; $\text{Span}\,(\Delta_{5_1}(t)) = 4$

so; $\frac{1}{2} \cdot 4 \le g(5_1) \Rightarrow \boxed{2 \le g(5_1)}$

but we found $g(5_1) \le 2$

Hence $\boxed{g(5_1) = 2}$

---

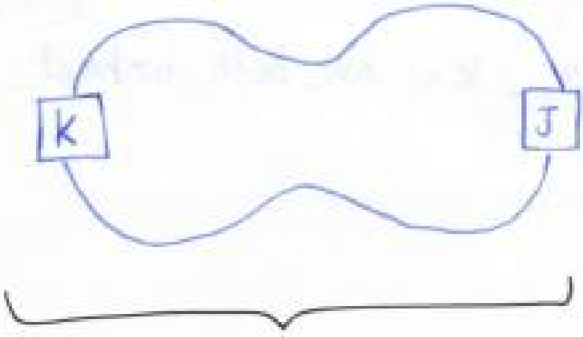
ex $\Delta_{3_1}(t) = t^{1} - t + t^{-1}$

$\text{Span}\ \Delta_{3_1}(t) = 1 - (-1) = 2 \Rightarrow \frac{1}{2} \cdot 2 \le g(3_1)$

$\Rightarrow 1 \le g(3_1)$

we showed $g(3_1) \le 1$

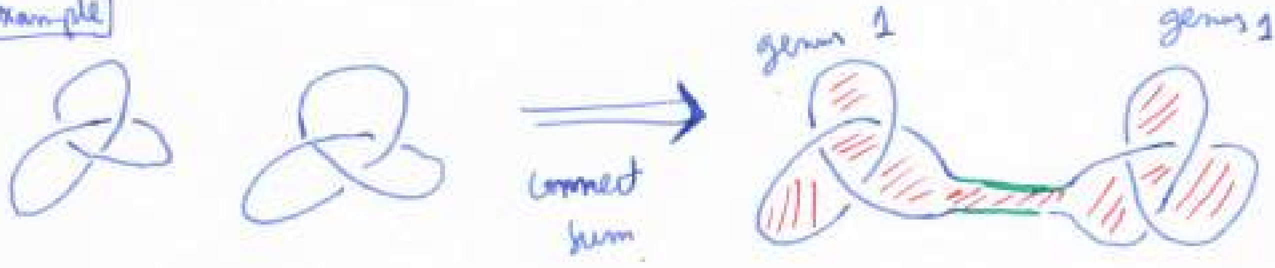
$\Rightarrow \boxed{g(3_1) = 1}$



$K \# J$

(K connect sum J)

Example



$3_1 \# 3_2$

genus 2

Question When does $K \# J = \text{unknot}$?

Proposition $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$

but, we can show ; $\boxed{g(K_1 \# K_2) = g(K_1) + g(K_2)}$

Recall $g(\text{unknot}) = 0$

$g(K) > 0 \quad K \neq \text{unknot}$

if: $g(K_1 \# K_2) = g(\text{unknot}) = 0$

$\Rightarrow g(K_1) + g(K_2) = 0$

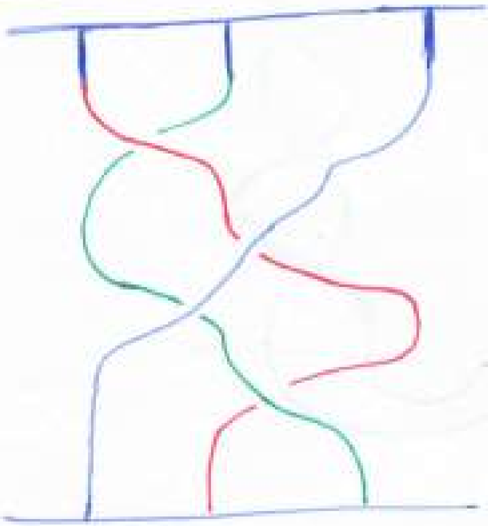
but $g(K_1) \geq 0$, $g(K_2) \geq 0$ $\Rightarrow$ K_1, K_2 both have to be unknot.

Hence we proved the following corollary:

Corollary: $K_1 \# K_2 = \text{unknot} \iff K_1, K_2$ are both unknot.

Knot Theory — Shoaib Akhtar 21/7/2020 (1971)

Lec 5: Braids

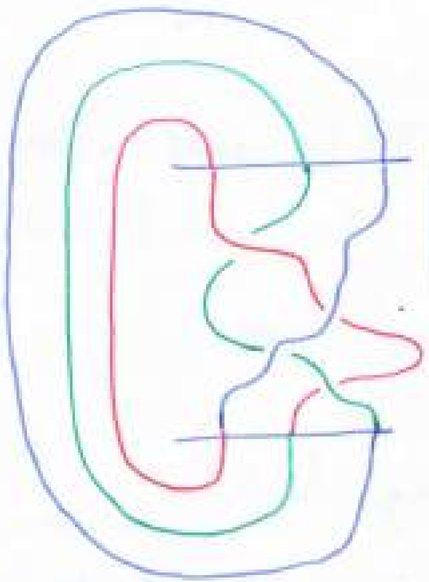


Braid

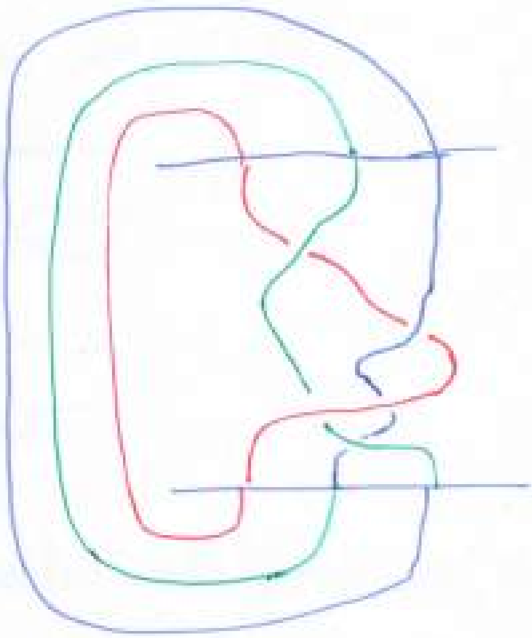
↓ close it

Knot or Link.

ie;



This is the knot we get after closing the above braid.

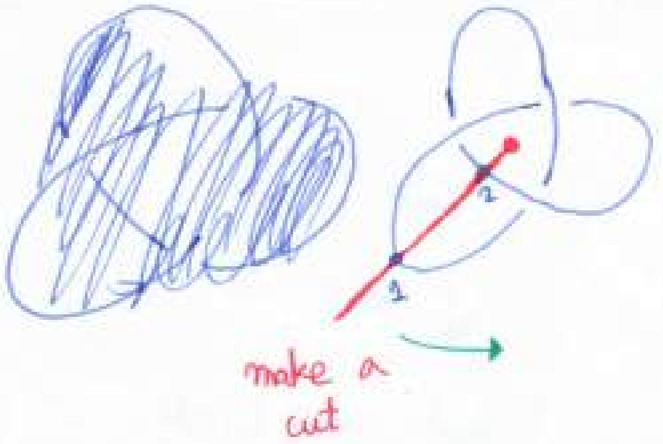


Here we dont get a knot, but a link

Can we go backward:

If we began with a knot; can we turn it into a braid.

Turn it into a braid; which closes up to this knot.



$(\sigma_1^{-1})(\sigma_1^{-1})(\sigma_1^{-1}) = \sigma_1^{-3}$

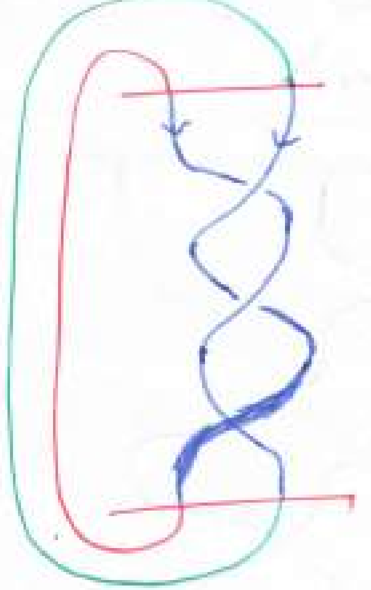
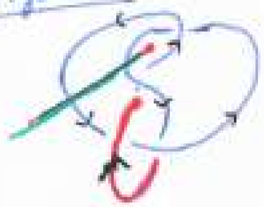


Figure 8



This is no longer well behave braid.

we dont allow this in braid..

... Its not good.

In braid, we dont allow doubling back.

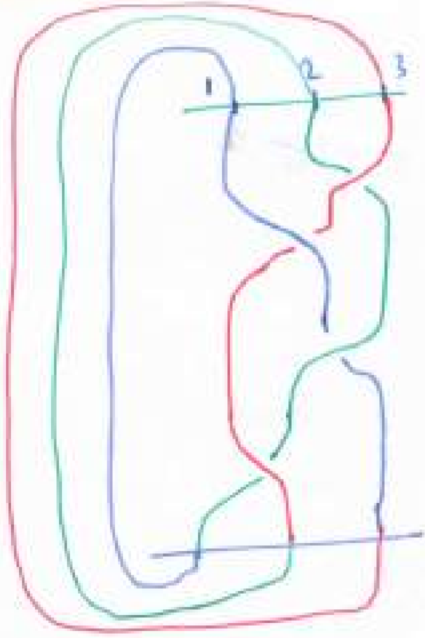
To move into a braid, we need the knot to always be counter clockwise with respect to some center.

→ There is issue with the red part.
... grap it, and pull it over the centre point.

we get

Now ; all is moving counter_clockwise (CCW)

Took overcrossing / undercrossing that is CW and pull over to make CCW.



Using Braid words we can write it as

$\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$ → This is a word that describes a braid that closes up to give figure 8 knot.

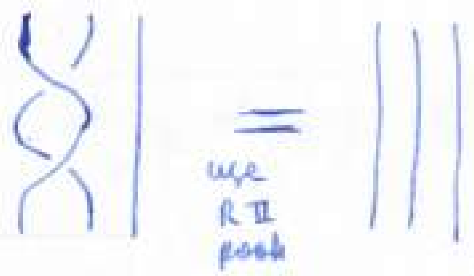
Braid words:

ALPHABET

σ_1 σ_2 σ_k

σ_1^{-1} σ_2^{-1} σ_k^{-1}

How does $\sigma_1\sigma_1^{-1}$ looks

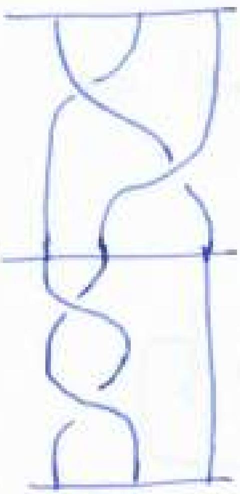


so: $\underline{\sigma_1\sigma_1^{-1} = 1}$

we call it 1 (identity)

It is not braided

We can multiply braids by stacking them.



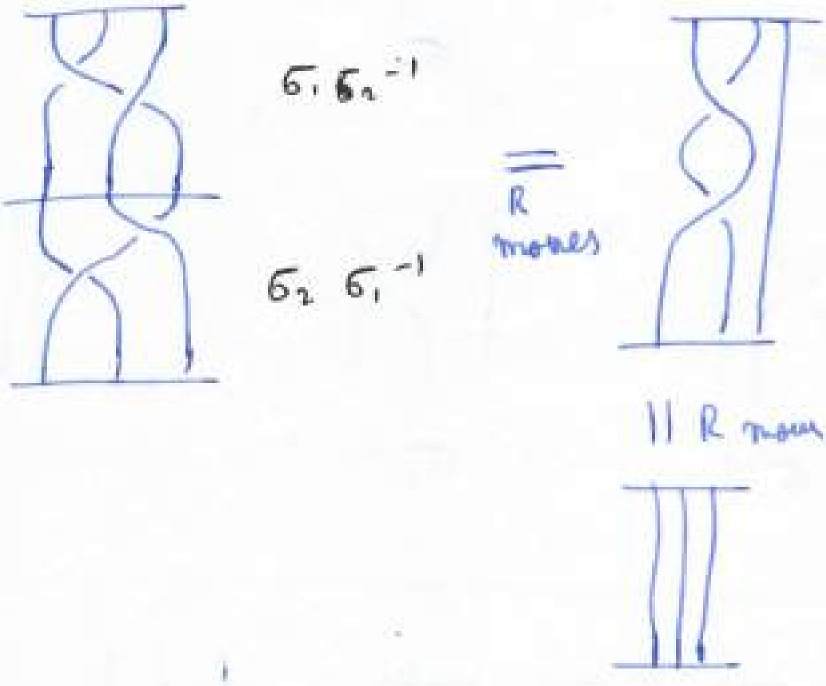
$$\left.\begin{array}{l}\left.\right\} \sigma_1 \sigma_2^{-1} \\ \left.\right\} \sigma_1^{2}\end{array}\right\} \sigma_1 \sigma_2^{-1} \sigma_1^{2}$$

ex// Inverse of $\sigma_1 \sigma_2^{-1}$ is $\sigma_2 \sigma_1^{-1}$

because when we combine them,

algebraicly we see $\sigma_1 \sigma_2^{-1} \sigma_2 \sigma_1^{-1} = \sigma_1 1 \sigma_1^{-1} = \sigma_1 \sigma_1^{-1} = 1$

lets see it



Inverse of $(\sigma_{i_1}^{k_1} \sigma_{i_2}^{k_2} \dots \sigma_{i_m}^{k_m})$ is

$(\sigma_{i_n}^{-k_n} \dots \sigma_{i_1}^{-k_1})$ $\boxed{(\sigma_{i_1}^{k_1} \sigma_{i_2}^{k_2} \dots \sigma_{i_m}^{k_m})^{-1} = \sigma_{i_m}^{-k_m} \dots \sigma_{i_1}^{-k_1}}$

Is the multiplication commutative

$\sigma_1\sigma_2$ $\sigma_2\sigma_1$

~~to~~ They both look like mirror image

But; It is associative.

So: if we take Braid with n-strings;
we see that, we have
- Identity
- Inverse
- Associative

So: They form a group.

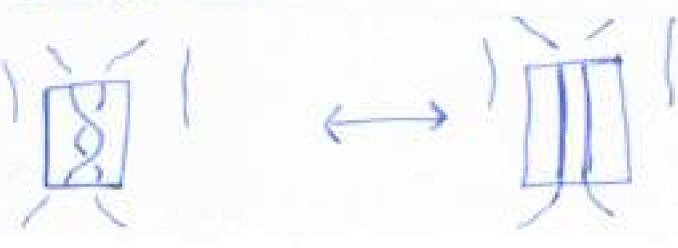
It is called <u>The Braid group with n-strings.</u>

Do Knots form a group?
- We have identity; which is unknot.
- But we dont have inverses.

so; Knots dont form a group.

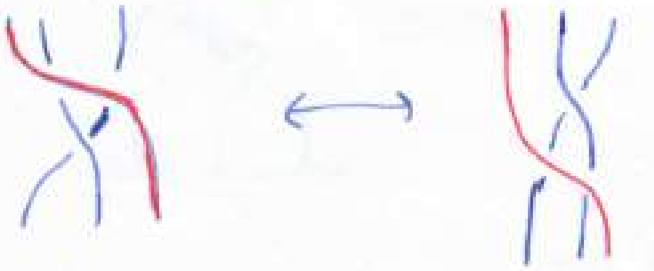
How do we know that two braids are same?

We can add or remove $\sigma_i\sigma_i^{-1}$ or $\sigma_i^{-1}\sigma_i$



This is <u>R II</u>

We can also do **R III**



$\sigma_1 \sigma_2 \sigma_1$ $\qquad\qquad$ $\sigma_2 \sigma_1 \sigma_2$

We can switch $\sigma_i \sigma_{i+1} \sigma_i \longleftrightarrow \sigma_{i+1} \sigma_i \sigma_{i+1}$

$\sigma_1 \sigma_3 \longleftrightarrow \sigma_3 \sigma_1$

We can switch $\sigma_i \sigma_j \longleftrightarrow \sigma_j \sigma_i$

when $|i-j| > 1$.

Braids are equivalent if they are represented by words that are equivalent upto a sequece of such three moves.

Theorem (Alexander, 1923) Braids are equivalent iff they are represented by words that are equivalent upto a sequence of such moves

---

~~Example~~ $\cancel{\sigma_2^{-1} \sigma_1 \sigma_3^{-1} \sigma_1 \longleftrightarrow}$

Example:

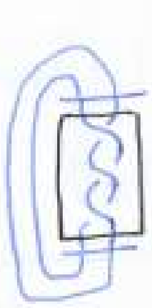
$$\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2 \longrightarrow \sigma_2^{-1}\cancel{\sigma_1\sigma_1^{-1}}\sigma_3^{-1}\sigma_2^{-1}$$

$$\downarrow$$

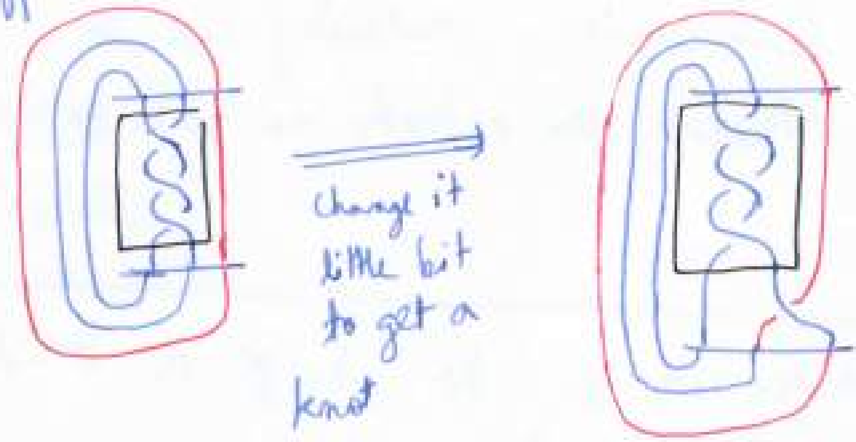
$$\sigma_2^{-1}\sigma_3^{-1}\sigma_2$$

$\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2$ and $\sigma_2^{-1}\sigma_3^{-1}\sigma_2$ are equivalent braids.

Question: Can different braids close to same knot or link?



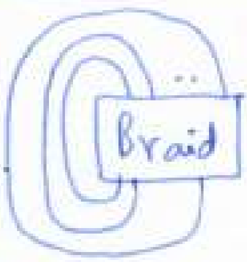
braid word w



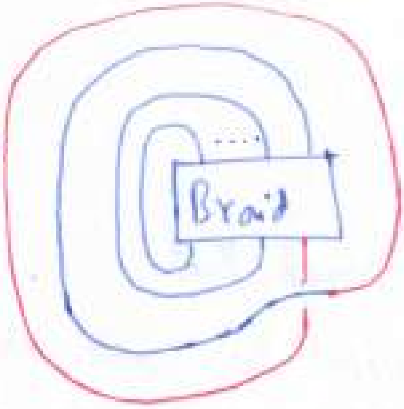
$\hookrightarrow$ Braid word for this is $w\sigma_2$

The two words w & $w\sigma_2$ represent the same knot.

$$w \xleftrightarrow[\text{Stabilization}]{} w\sigma_2$$



$\xleftrightarrow[\text{Stabilization.}]{}$



Start with n strings ; add

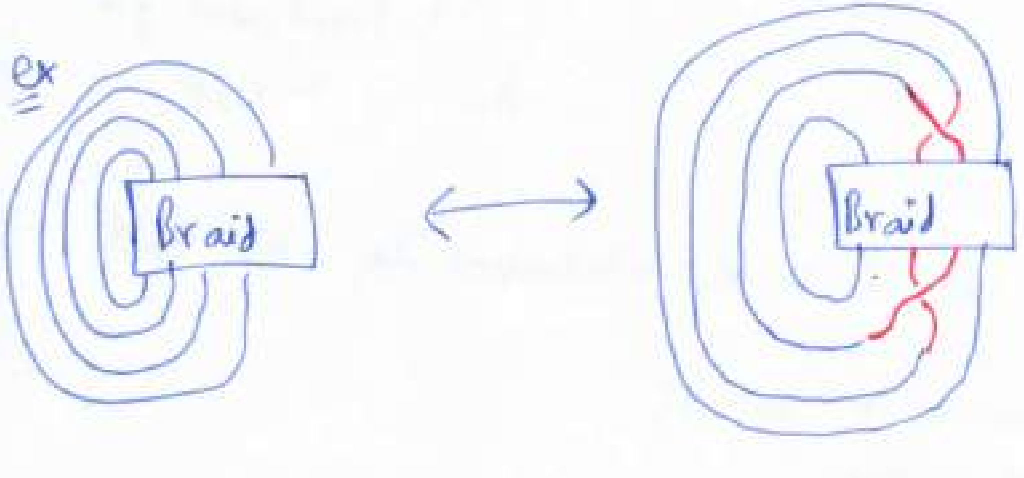
σ_n, σ_n^{-1} end

or σ_1, σ_1^{-1} beginning.

(If you add σ^1 or σ^{-1} to the begining of a braid word, you neccessarily have to change the values of the rest of the braid words , increasing each by 1.
Otherwise you actually do get entirely different links / knots.)

What else you can do to change a braid ; and still get the same knot.

ex



$$W \longleftrightarrow W \longleftrightarrow \sigma_i W \sigma_i^{-1}$$

Conjugation

Markov showed that

Two words represent the same knot / link iff they are related by stabilization , or conjugation , or the three moves that preserve the braid.

Knot / Link preserving moves (n-string braid)

- add / remove $\sigma_i \sigma_i^{-1}$
- switch $\sigma_i \sigma_j \longleftrightarrow \sigma_j \sigma_i$ when $|i-j| > 1$
- switch $\sigma_i \sigma_{i+1} \sigma_i \longleftrightarrow \sigma_{i+1} \sigma_i \sigma_{i+1}$
- add σ_n, σ_n^{-1} at end ; σ_1, σ_1^{-1} at begining
- $W \longleftrightarrow \sigma_i W \sigma_i^{-1}$

Example) $\sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_1 \rightarrow \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_3$

Braid for figure eight

$$\downarrow$$

$$\sigma_1^{-1}(\sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_3)\sigma_1$$

$$\downarrow$$

$$\sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_1 \sigma_2^{-1} \sigma_1^{2} \sigma_3$$

$1 \longrightarrow \sigma_1 \sigma_1^{-1} \longrightarrow \sigma_1 \sigma_1^{-1} \sigma_2 \rightarrow \dots$

unknot

can make complicated knot.

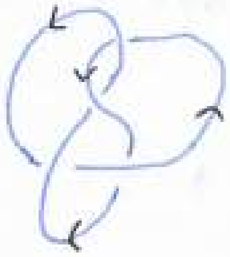
Minimum number of strings needed in a braid to represent K ; This no. is called Braid index of K.

ex) $br(\text{figure-eight}) \leq 3$. ; we can show $br(\text{figure-eight}) = 3$

Theorem

$br(K)$ = minimum number of S-circles needed for a diagram of K.

ex



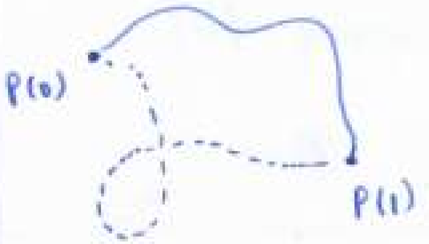
$\Rightarrow$

Lec 6: Fundamental Group.

A <u>path</u> $p : [0,1] \longrightarrow X$ (topological space)

has endpoints $p(0)$ & $p(1)$.

Called a <u>loop</u> if $p(0) = p(1)$.

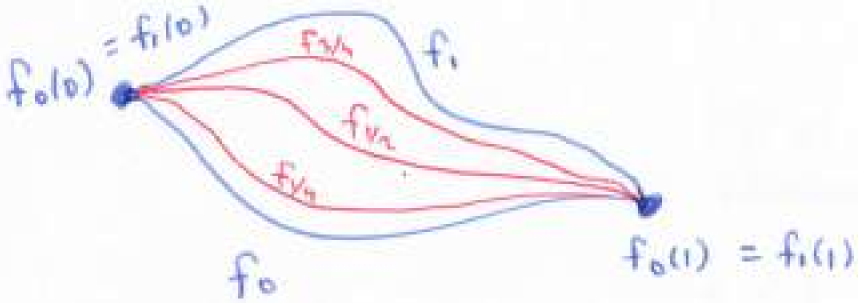


A <u>homotopy</u> between paths f_0, f_1 with same endpoints is a continuous map.

$F(s,t) : [0,1] \times [0,1] \longrightarrow X$

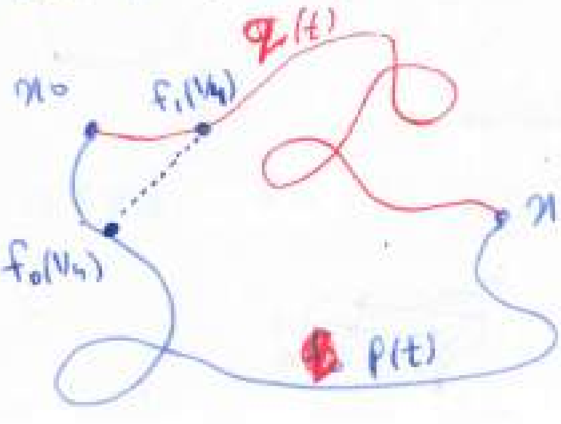
where $F(s,t) = f_t(s)$

Such that $f_t(0) = f_0(0)$, $f_t(1) = f_1(1)$ $\quad 0 \leq t \leq 1$



<u>Example:</u>

Any two paths in $\mathbb{R}^2$ with endpoints x_0, x_1 are homotopic.



let $p(t), q(t)$ be paths from x_0 to x_1

ie; $p(0) = q(0) = x_0$

$p(1) = q(1) = x_1$

Define $f_t(s) = (1-t)p(s) + t\,q(s)$

Note at $t=0$; $f_0(s) = p(s)$

$t=1$; $f_1(s) = q(s)$.

& f_t continuously as t varies.

f_t is like family of equation which is moving $p(s)$ path to $q(s)$ path.

<u>We can multiply paths:</u>

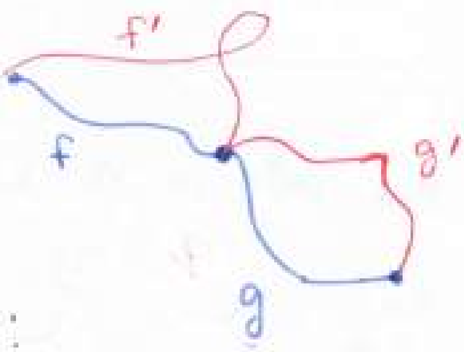
$f, g : [0,1] \longrightarrow X$ if $f(1) = g(0)$

~~$f \cdot g = \{ f(2s) \quad 0 \le s \le \frac{1}{2}$~~

$$f \cdot g = \begin{cases} f(2s) & : 0 \le s \le \frac{1}{2} \\ g(2s-1) & : \frac{1}{2} \le s \le 1 \end{cases}$$

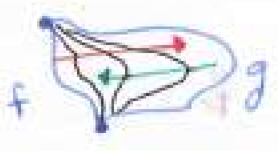
<u>Note</u> || If $\underset{(\text{Homotopic})}{f \simeq f'}$, $g \simeq g'$

Then $f \cdot g \simeq f' \cdot g'$



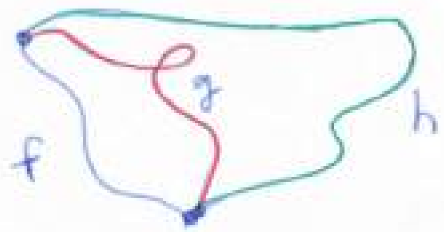
Homotopy is an equivalence relation:

- Reflexive : $f \simeq f$; define $f_t = f \quad 0 \le t \le 1$
- Symmetric : $f \simeq g \Rightarrow g \simeq f$



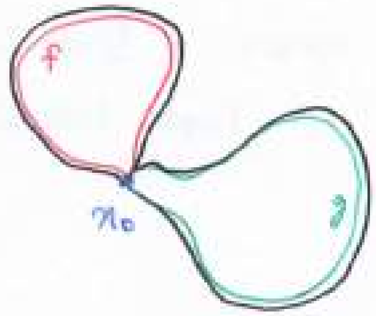
$f_t \qquad f_{1-t}$

• Transitive If $f \simeq g$, $g \simeq h$
Then $f \simeq h$.



Fixing some base point $x_0 \in X$, the set of loops based at x_0 with the operation • is a group.

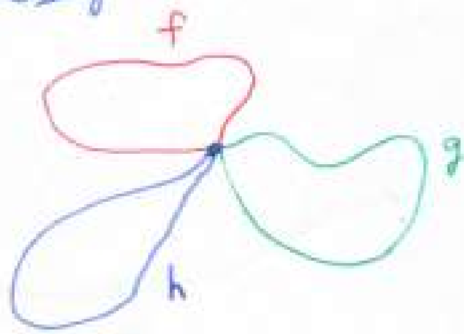
Group operation •



$f \cdot g$

* Associative $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$

~~• Identity~~



$$\underbrace{(f \cdot g)}_{1/2} \cdot \underbrace{h}_{1/2} \simeq \underbrace{f}_{1/2} \cdot \underbrace{(g \cdot h)}_{1/2}$$

times
s $f - 1/4$ $g \sim 1/4$ $g \sim 1/4$ $h - 1/4$

we travel the same; but at different rates.

... homotopy will adjust the speed.

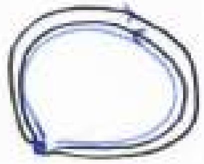
* Identity : c constant ; $c(s) = x_0$

$f \cdot c \simeq f$

* Inverse we denote inverse of f by $\bar{f}$.

$$\bar{f}(s) = f(1-s)$$

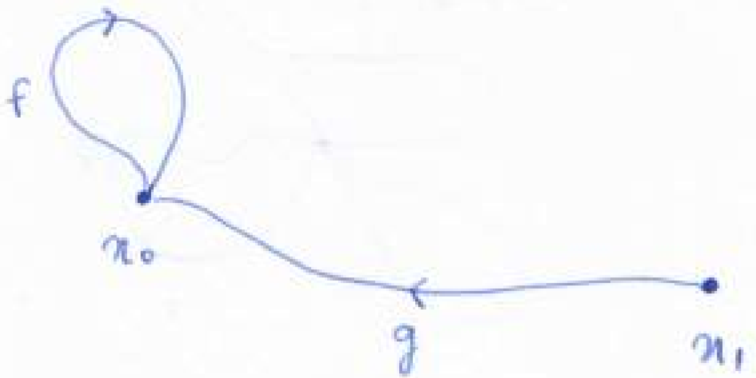
$$f \cdot \bar{f} \simeq c$$



does not move at all.

The group of loops in X based at x_0 is denoted by $\pi_1(X, x_0)$ and is called the fundamental group of X.

group of loops in X (upto homotopy)



$$\pi_1(X, x_0) \underset{\text{isomorphic}}{\cong} \pi_1(X, x_1)$$

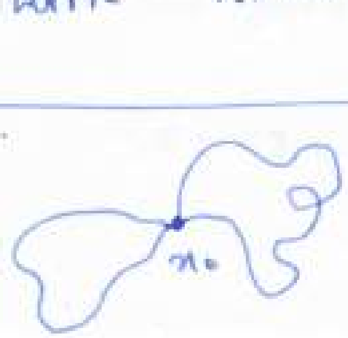
$$f \longleftrightarrow g \cdot f \cdot \bar{g}$$

as long as X is path connected.

So, can just write $\pi_1(X)$; (dont worry about base point)

Examples

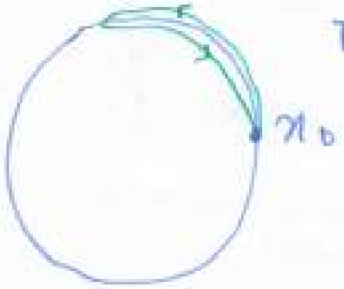
① $\pi_1(\mathbb{R}^n)$



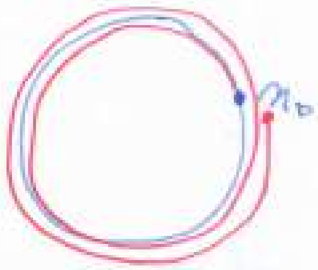
all loops are homotopic. so; $\pi_1(\mathbb{R}^n)$ has only one element.

$\pi_1(\mathbb{R}^n) \cong 1$ Trivial group.

② The Circle (S^1).



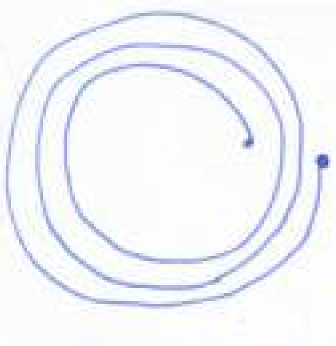
This is equivalent to constant path.



going around twice ; it can't be homotopic to staying constant.

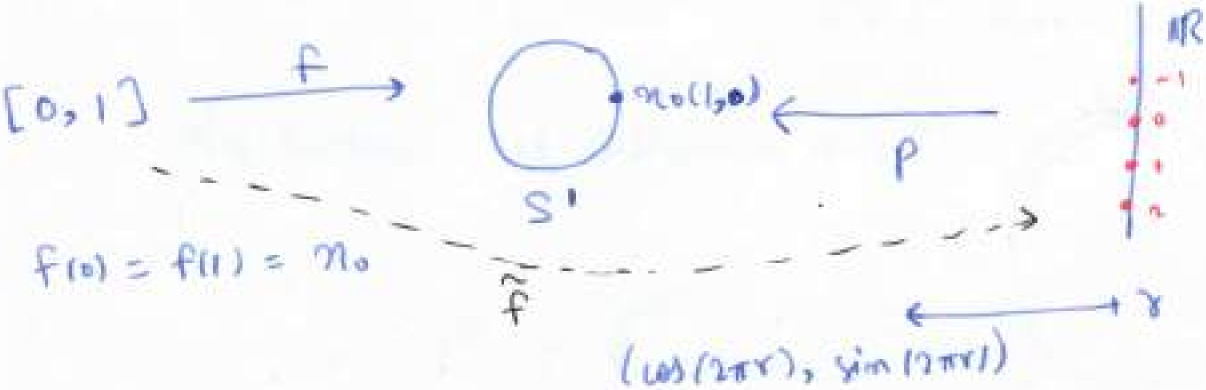
These all paths are living on the circle ... for the sake of clarity we are drawing them outside S^1.

we can have



we can also have negative winding number ; going in opposite direction

~~[illegible]~~ So; somehow the ~~[illegible]~~ paths here seem to correspond to integers..



we see that $n \in \mathbb{Z}$, $p(n) = x_0$

given any path f; ~~[illegible]~~

There exists a unique lift. $\tilde{f}$ of f

such that $\tilde{f}(0)$

Any two paths from 0 to $n \in \mathbb{Z}$ in $\mathbb{R}$ are homotopic.

Each $n \in \mathbb{Z}$ corresponds to a unique path up ~~to~~ to. homotopy.

so; $\underline{\pi_1(S^1) \cong \mathbb{Z}}$.

ex

Brouwer's Fixed Point Theorem

Any continuous map $h: D^2 \longrightarrow D^2$ has a fixed point. $h(x) = x$ for some $x \in D^2$

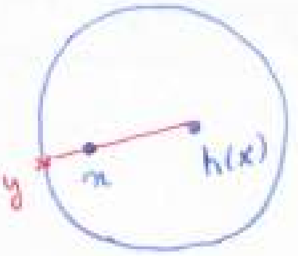
D^2 is disk.

ie ~~[illegible]~~

proof suppose not..

given any x we get $h(x)$

$h(x) \neq x$ for all x.



Define the map $r: D^2 \longrightarrow S^1$

$x \longmapsto y$

(ray map...)

Note: r is continuous

let f be a loop in S^1

Then f is a loop in D^2

and in the disk; $f \simeq c$

ie: ~~There is~~ Then, there is some f_t s.t. $f_0 = f$ and $f_1 = c$.

Then $r \circ f_t$ is a homotopy in S^1 from f to c.

Hence $\pi_1(S^1) = 1$ # contradiction

In Calculus;

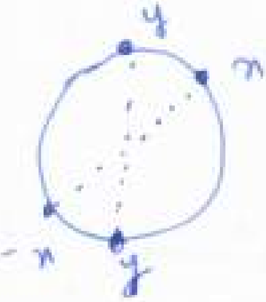
Any continuous map $h: S^1 \longrightarrow \mathbb{R}$ has some antipodal points $x, -x$ such that $h(x) = h(-x)$.

$f(x) = h(x) - h(-x)$

If $f(x) = 0$; Then done.

If $f(x) > 0$, Then $f(-x) = -f(x) < 0$

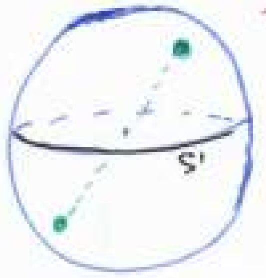
By IVT, there exists y s.t. $f(y) = 0$.



We can generalize this

Borsuk - Ulam Theorem

If $f : S^2 \to \mathbb{R}^2$ continuous;
then there exists antipodal $x, -x$
such that $f(x) = f(-x)$.



Earth

Temperature, Humidity.

There exists a point on Earth s.t.; The diametrically opposite point has exactly same temperature and Humidity.

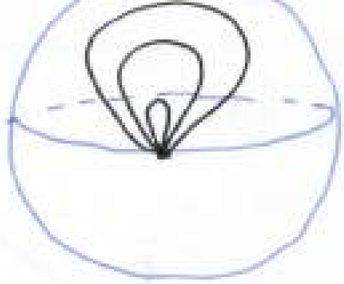
Proof: Suppose not.
Then, define $h(x) = \dfrac{f(x) - f(-x)}{|f(x) - f(-x)|}$ unit vector

so; $h : S^2 \longrightarrow S^1$

also note; $h(-x) = -h(x)$

Any loop on the equator is homotopic to constant loop in S^2.

via some homotopy g_t.

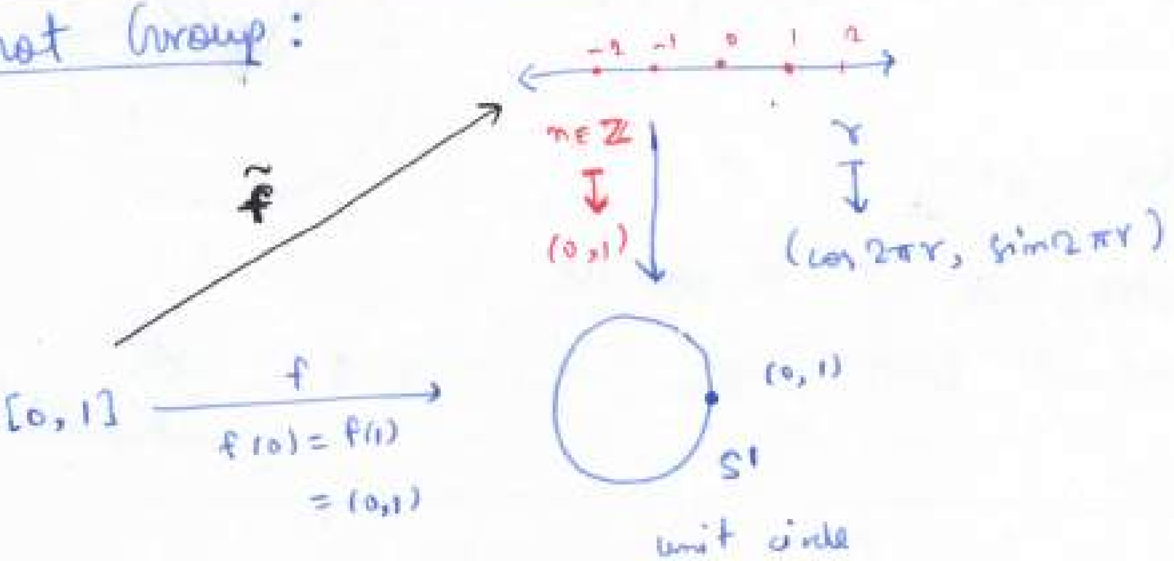


But, then $h \circ g_t$ is a homotopy in S^1 to the constant loop. Thus $\pi_1(S^1) \cong 1$ $\#$ contradiction.

Lec 7: The Knot Group.

Knot Group:

Recall



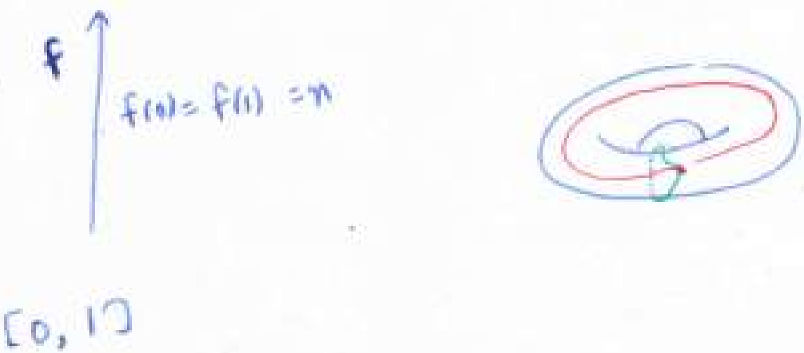
We can uniquely define a lift $\tilde{f}$

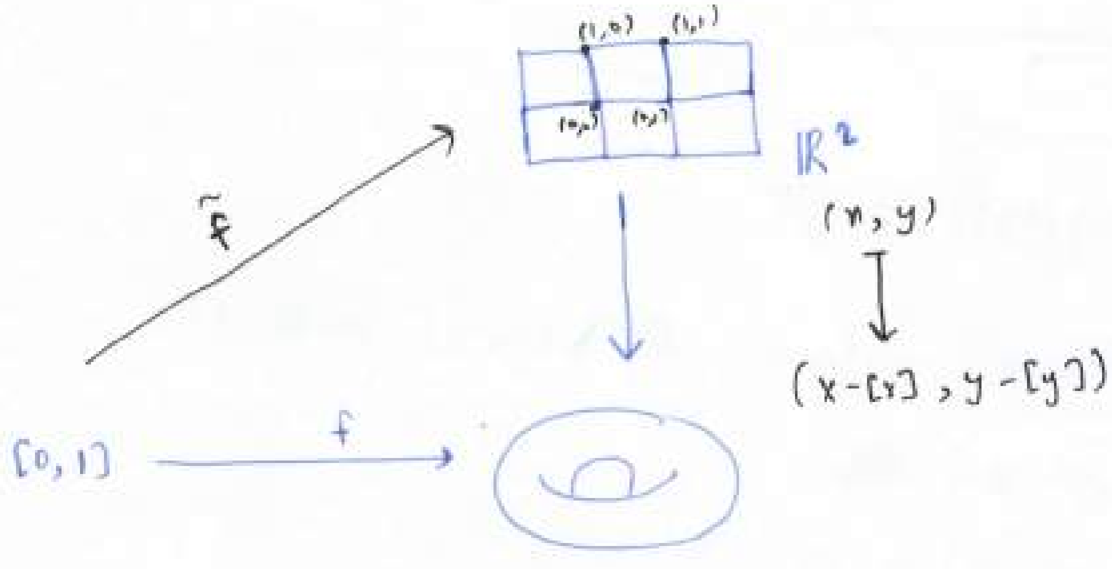
s.t. $r \circ \tilde{f} = f$

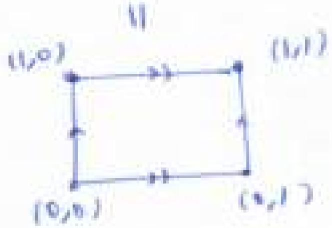
and $\tilde{f}(0) = 0$

Every lift of a path ends at some $n \in \mathbb{Z}$.

and we get: $\pi_1(S^1) \cong \mathbb{Z}$



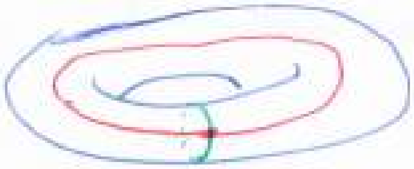




$(x,y) : 0 \le x < 1$
$0 \le y < 1$

$\tilde{f}$ s.t. $\pi \circ \tilde{f} = f$,
and $\tilde{f}(0,0) = (0,0)$

$\tilde{f}^{-1}(0,0) = (m,n) \quad ; m,n \in \mathbb{Z}$



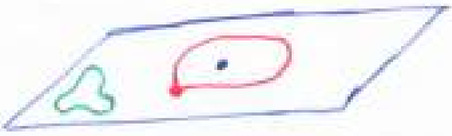
So, we get
$\pi_1(\text{Torus}) \cong \mathbb{Z} \times \mathbb{Z}$
$= \{(x,y) : x, y \in \mathbb{Z}\}$

We saw ;
$\pi_1(\mathbb{R}^n) = 0$ Trivial group (with 1 element)

any two paths $p(s)$, $q(s)$ related by a homotopy : $f_t(s) = (1-t)\,p(s) + t\,q(s)$

$f_0 = p$
$f_1 = q$

What about $\mathbb{R}^2 \setminus (0,0)$



$$\pi_1(\mathbb{R}^2 \setminus (0,0)) \cong \mathbb{Z}$$

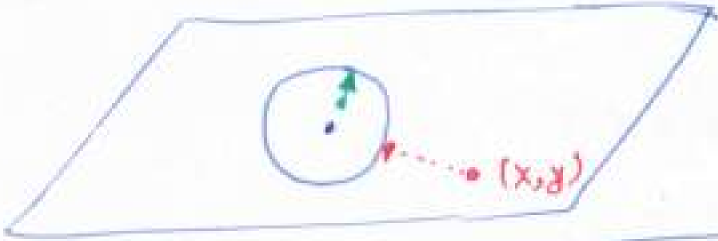
We can continuously deform $\mathbb{R}^2 \setminus (0,0)$ onto S^1:

$$f_t(x,y)$$

$$f_0(x,y) = (x,y)$$

$$f_1(x,y) = \frac{(x,y)}{\sqrt{x^2+y^2}}$$

Then $$f_t(x,y) = \frac{(x,y)}{(1-t) + t\sqrt{x^2+y^2}}$$ Deformation Retraction.



for $(x,y) \in S^1$
$\sqrt{x^2+y^2} = 1$; hence
$f_t(x,y)|_{S^1} = \frac{(x,y)}{1-t+t} = (x,y)$

A Deformation Retraction is a family $f_t : X \longrightarrow A$ continuous in t
where A subspace of X

s.t. $f_0 = \mathbb{1}$

$f_1(X) = A$

$f_t|_A = \mathbb{1}$ for all $0 \le t \le 1$

Proposition If there exists a deformation retraction $X \to A$ then $\pi_1(X) \cong \pi_1(A)$

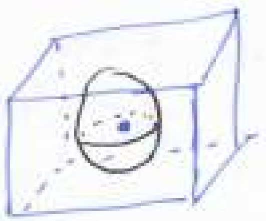
so; $\pi_1(\mathbb{R}^2\setminus(0,0)) \cong \pi_1(S^1) \cong \mathbb{Z}$

$X =$ 

can deformately retract to circle

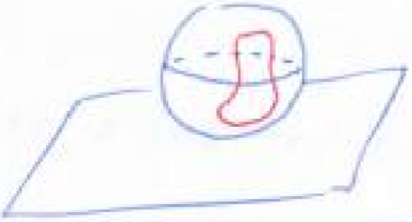
so; $\pi_1(X) \cong \pi_1(S^1) \cong \mathbb{Z}$.

$\mathbb{R}^3\setminus(0,0,0)$ deforms onto S^2.



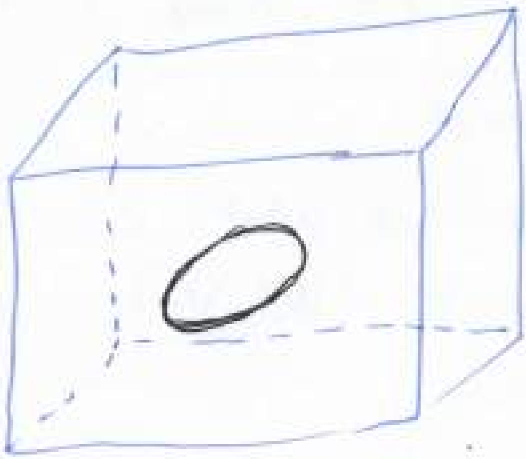
$$(x,y,z) \longmapsto \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$$

Hence; $\pi_1(\mathbb{R}^3\setminus(0,0,0)) \cong \pi_1(S^2) \cong 0$



$\Rightarrow \boxed{\pi_1(\mathbb{R}^3\setminus(0,0,0)) \cong 0}$

$\mathbb{R}^3\setminus S^1$



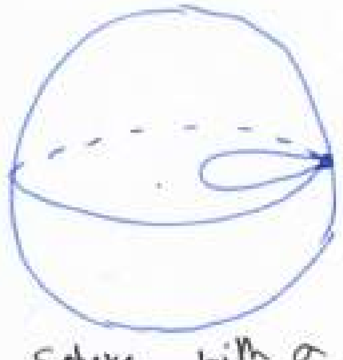
$\mathbb{R}^3\setminus S^1$

$\downarrow$ deformation retract to

sphere with a line going throuing it.

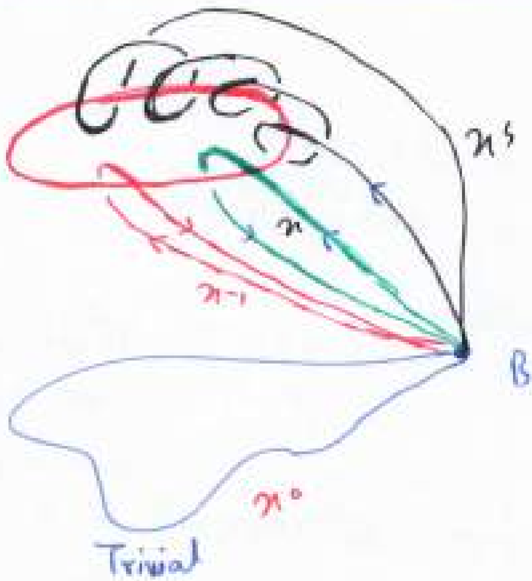
$\downarrow$ can continuously deform this

$\longleftarrow$



Sphere with a loop.

so; $\pi_1(\mathbb{R}^3 \setminus S) \cong \pi_1(S^1) \cong \mathbb{Z}$



$\pi_1(\mathbb{R}^3 \setminus S^1) = \langle x \rangle$
free group generated by x.

$\cancel{=\{x^2=x, x,}$

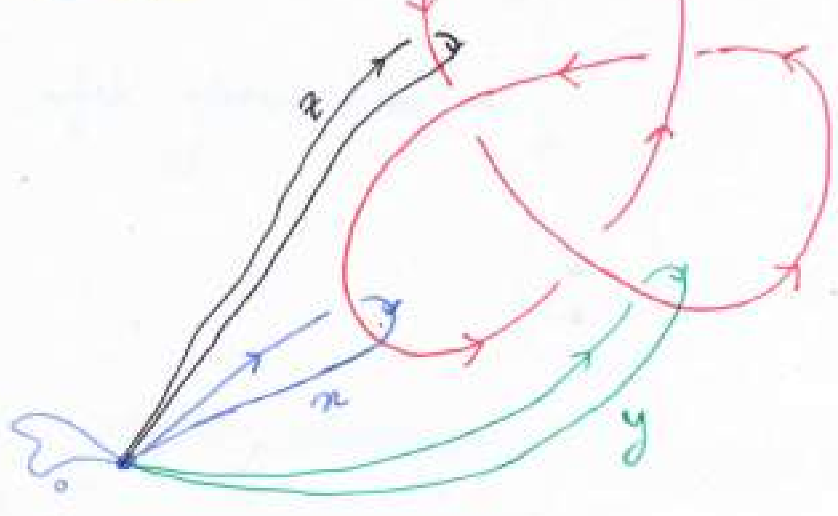
$$\langle x \rangle = \{\ldots x^{-2}, x^{-1}, x^0 = 1, x, x^2, \ldots\}$$

$$\cong \mathbb{Z}$$

x is called the generator of the group.

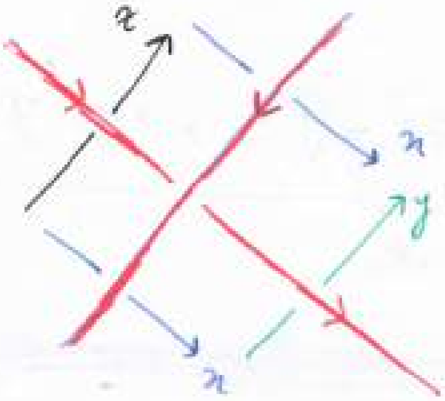
What if we began with some knot,
Say Trefoil : And ask what is $\pi_1(\mathbb{R}^3 \setminus K)$

Ex Trefoil



Any general loop in $\mathbb{R}^3 \setminus (\text{Trefoil})$ will be combinations of x, y, z.

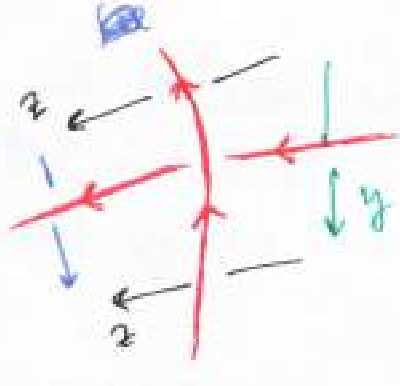
Zoom in to a crossing.



we see that

$$zx = xy$$

going to other crossing; we see



$$zx = yz.$$

finally;
looking at the last crossing.

$$xy = yz.$$

The Knot Group for knot K is $\pi_1(\mathbb{R}^3 \setminus K)$ (1956)

for Trefoil;

~~π_1~~

$\pi_1(\mathbb{R}^3 \setminus \text{Trefoil}) \cong \langle x, y, z : zx = xy,\ zx = yz,\ xy = yz \rangle$

~~$\langle x, y, z : zx = y$~~

$(*)$ $\langle \underbrace{x, y, z}_{\text{generators}} : \underbrace{zx = xy,\ zx = yz,\ xy = yz}_{\text{relations}} \rangle$

~~You can add / remove generators if they are defined~~

You can add / remove generators / relations if they are defined in terms of the other generators / relations.

Notice that in $(*)$ you can remove one of the relation; say $xy = yz$ (it imediately follows from $zx = xy$ & $zx = yz$)

from first one: $zx = xy \Rightarrow z = xyx^{-1}$

since $z = xyx^{-1}$

so we can remove out z;

and as we remove z; we find that we no more need $zx = xy$.

now look at $zx = yz \Longrightarrow (xyx^{-1})x = y(xyx^{-1})$

$\Leftrightarrow xy = yxyx^{-1}$

$\Leftrightarrow xyx = yxy.$

Hence we see that

$\langle x, y, z : zx = xy,\ zx = yz,\ xy = yz \rangle$

$= \langle x, y : xyx = yxy \rangle$

This method works for any knot!

---

example) find $\pi_1(\mathbb{R}^3 \setminus (\text{figure eight knot}))$

$\pi_1(\mathbb{R}^3 \setminus K)$ (sometime we just write $\pi_1(K)$ instead of $\pi_1(\mathbb{R}^3 \setminus K)$ for the matter of condensed notation)

$\pi_1(\mathbb{R}^3 \setminus K) \cong \langle x, \cancel{y}, z, \cancel{w} : wy = yx,\ \cancel{yw = wz},\ \cancel{xz = wx},\ zx = yz \rangle$

$y = wzw^{-1}$ so; remove y & $yw = wz$.

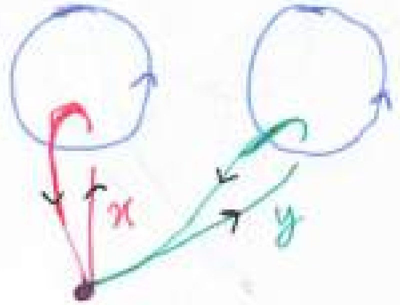
$w = xzx^{-1}$ so; remove w & $xz = wz$.

Example) Trivial link with two components

L : 

What is $\pi_1(\mathbb{R}^3 \setminus L) \cong \langle x, y \rangle$

(no relations)



~~it is called free~~

~~group~~ $\langle x, y \rangle$ is called free group of rank 2.

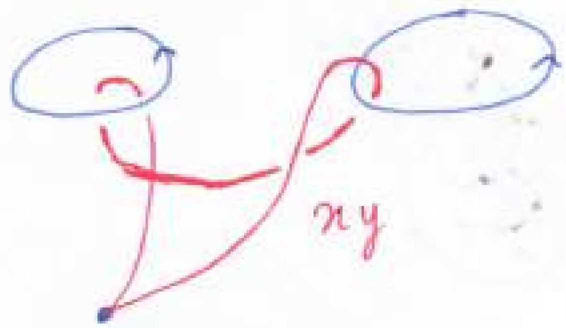
$\pi_1(\mathbb{R}^3 \setminus L) \cong \langle x, y \rangle$ $\underbrace{\text{free group}}_{\text{because no relation}}$ of $\underbrace{\text{rank } 2}_{\text{because generated by 2 generators.}}$

not abelian.

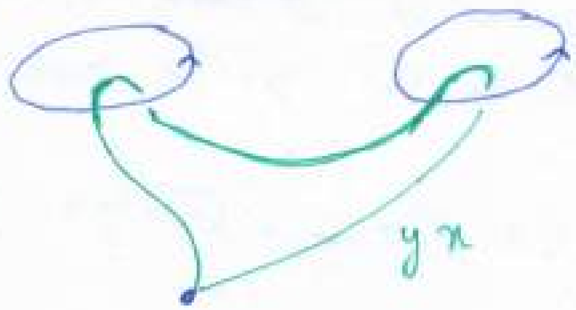
Notice that in the free group $\langle xy \rangle$

$$xy \neq yx$$

(i) xy



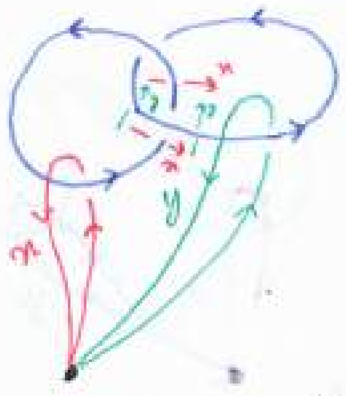
(ii) yx



We can convince ourselves that ; xy & yz are ~~funda~~ fundamentally different things.

Ex]

$L =$



here we have relationship at crossing.

We get $xy = yx$

$\pi_1(\mathbb{R}^3 \setminus L) \cong \langle x, y : xy = yz \rangle$

free abelian group of rank 2

In this group.
e.g.: we can simply a work like $xy^2x^{-1}x^3$.

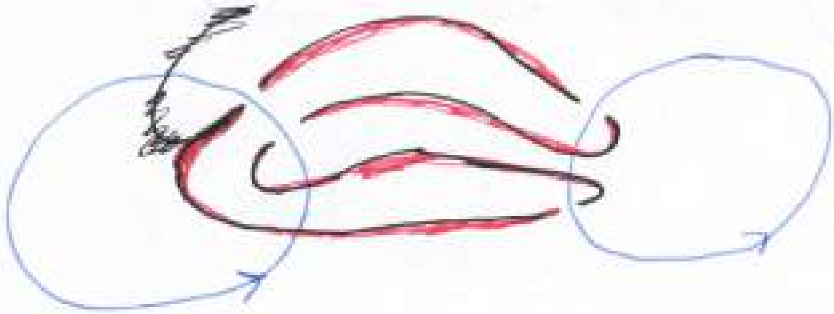
ie; $xy^2x^{-1}y^3 = xx^{-1}y^2y^3 = 1\cdot y^5 = y^5$.

---

ex1 free link 

Consider the word $xyx^{-1}y^{-1}$

(note since the group is not abelian:
$xyx^{-1}y^{-1} \neq 1$)



so; we get

so; ~~the~~
These cant be detached.

Borromean rings

---

but here; we link two loops as

Then the group becoms abelian.

& we can get 

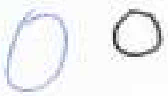
Lec 8: Linking Number

We want to develop a theory; of how linked are links.

We want to keep the intuition that

 linked once. linking ~~0~~ 1

 linked twice. Linging 2

 linking 0

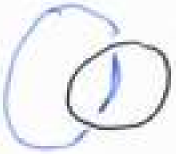
Method ~~(B)~~

Sum up  no. of times black passes over blue.

but there is a problem.

 ; according to this method; This has linking ~~too~~ 2.

→ has ~~it~~ linking 0.

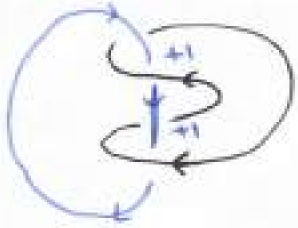
but  = 

So; we see that somehow we have to keep track of orientation

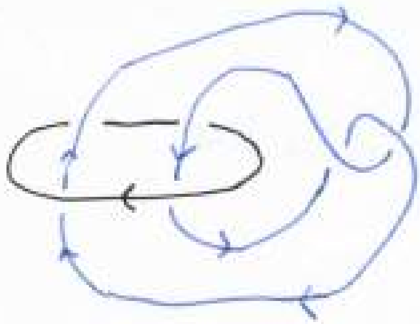
So the better method we develop is.

+1 Right Hand −1 Left Hand.

linking 1



linking 2

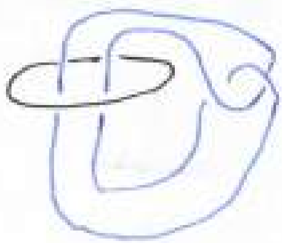


How linked are blue & black components.

has linking zero.

Linking 0

This has linking zero;
but we cannot deform it to 

So; the method we had develop to talk about linking no.
does not distinguishe  and 

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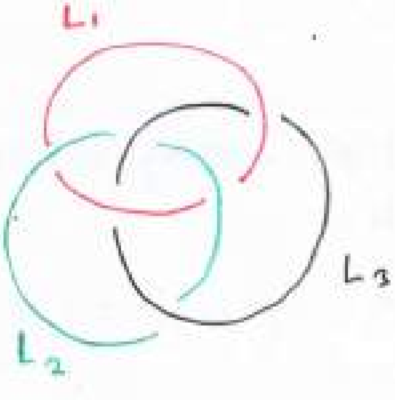
ex Borromean rings.



L_1, L_2, L_3 are linking components.

lets denote "linking number between L_i & L_j"

by $lk(L_i, L_j)$

so; ~~lk~~ $lk(L_1, L_2) = 0$

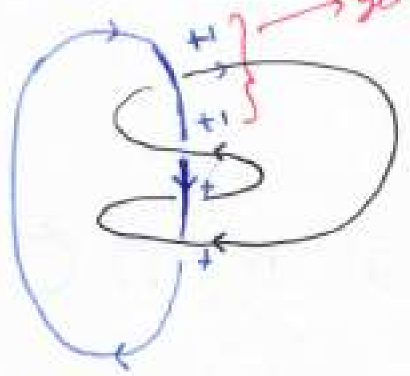
$lk(L_2, L_3) = 0$

$lk(L_1, L_3) = 0$

so; we see that; even when we have linking zero between components ⇒ we can still have higher order linking.

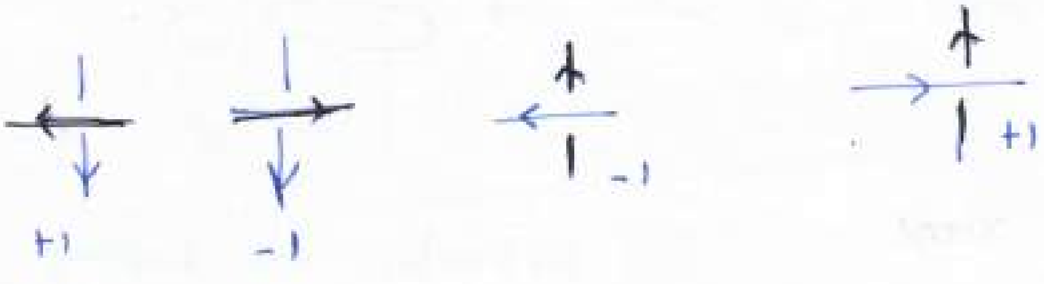
Now method — Method (2)

we can also look at when Black goes under blue.

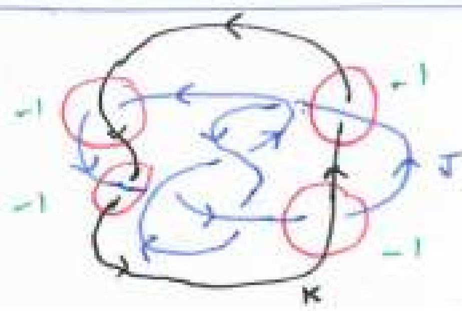


so; $\frac{(+1)+1+1+1}{2} = 2$

Sum up all crossings.



Then devide total by 2.



look at all crossing between two links.

$lk(K, J) = \frac{1}{2}(-4) = -2$.

Proposition

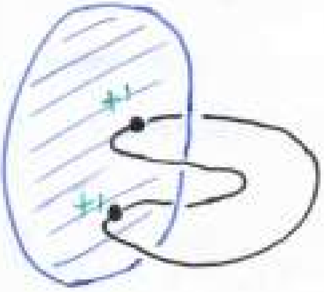
Change orientation of K to −K

so: $\operatorname{lk}(-K, J) = -\operatorname{lk}(K, J)$

$\operatorname{lk}(-K, -J) = \operatorname{lk}(K, J)$

linking number is defined upto sign depending on choice of orientation.

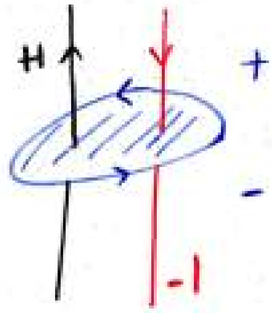
Method 3 Seifert Surfaces Perspective



passes twice through the surface.

let K bound some Seifert (S.) surface Σ

Then $\operatorname{lk}(K, J) \overset{\text{signed}}{=}$ count times J passes through $\Sigma(K)$

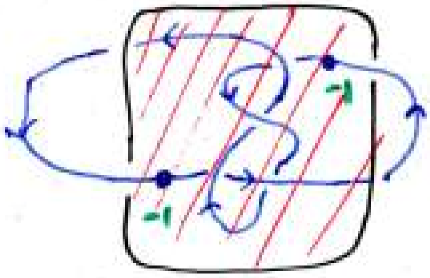


orient counter clockwise about the surface.

Since the surface Σ is orientable ; it has ~~top(+)~~ top (+) and bottom (-).

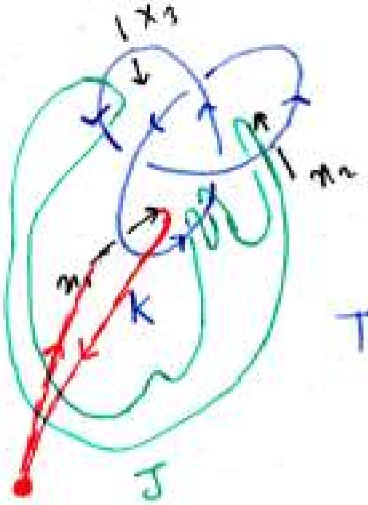
$lk(K, J)$ = Signed count times J passes through $\Sigma(K)$

ex



so; $lk(K, J) = \underline{\underline{-2}}$.

Method 4 || Fundamental Group Perspective



$\pi_1(\mathbb{R}^3 \setminus K)$ is represented by $\langle x_1, x_2, x_3$ with some relation $\rangle$

~~so π_1~~

$$\pi_1(\mathbb{R}^3 \setminus K) = \langle x_1, x_2, x_3 : r_1, r_2, r_3 \rangle$$

r_1, r_2, r_3 are relations.

can represent J by a word ; which is an element of the fundamental group.

here: $J = x_1^2 x_2 x_3^{-1} \in \pi_1(\mathbb{R}^3 \setminus K)$

linked twice (link +2)

linked one more time (link +1)

unlinked one time (link -1)

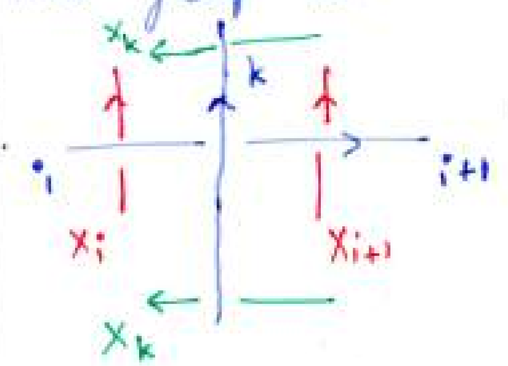
total linking $2 + 1 + (-1) = 2$.

$lk(K, J)$ = sum of the exponents in the word representing J in $\pi_1(\mathbb{R}^3 \setminus K)$

We can convince ourselves : $lk(K,J) = lk(J,K)$.

$$lk(K,J) = lk(J,K)$$

Knot group has relations:



relation $x_k x_i = x_{i+1} x_k$

(These relations are interesting ; because in general the group is non-abelian.)

so; $\pi_1(\mathbb{R}^3 \setminus K)$ is generally not abelian.

let everything commute

$\text{ABEL}(\pi_1(\mathbb{R}^3 \setminus K))$ called Abelianization of $\pi_1(\mathbb{R}^3 \setminus K)$

$x_k x_i = x_{i+1} x_k$

Abelianizing

$x_k x_i = x_k x_{i+1}$

$\Rightarrow x_i = x_{i+1}$

~~So, the gen.~~

So; The original group which looked like $\langle x_1, x_2, \ldots, x_n : r_1, \ldots, r_m \rangle$

$$\Big\downarrow \text{Abelianize}$$

$$\langle x_1, \ldots, x_n : x_1 = x_2, x_2 = x_3, \ldots \rangle$$
$$= \langle x \rangle \cong \mathbb{Z}$$

What exactly ~~did~~ we are doing when calculating lk?

We are actually abelianizing the group.

$$\pi_1(\mathbb{R}^3 \setminus K) \rightsquigarrow \text{ABEL}(\pi_1(\mathbb{R}^3 \setminus K))$$
$$x_1^2 x_2 x_3^{-1} \longmapsto x^2 x x^{-1} = x^2$$
$$\cong$$
$$2 \in \mathbb{Z}$$

linking number.

So; for now: we have seen four perspective of linking Numbers.

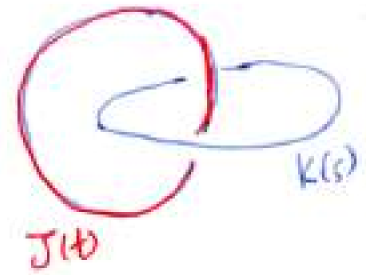
1. Combinatorial.
2. Count all crossing, but divide by 2
3. S. surface perspective

$\Bigg\}$ All equivalent

4. Group Theoretic perspective. (where we look at one component as being a word in fundamental group of the compliment of the other component: And then you see abelianisation: to where that word is sent to)

5 5th perspective   Gauss ; using calculus.

Starting thinking component knot K as curves,



$K(s) = (x(s), y(s), z(s))$

Think of component knots to be curves parametrized by some parameter s.

where $s_0 \le s \le s_1$

here: $K(s) = (\cos s, \sin s, 0)$; $0 \le s \le 2\pi$

$J(t) = (\bar{x}(t), \bar{y}(t), \bar{z}(t))$; $t_0 \le t \le t_1$

$J(t) = (0, \sin t - 1, \cos t)$; $0 \le t \le 2\pi$.

Gauss Integral

$$lk(K,J) = \iint \frac{(\bar{x}-x)(y'\bar{z}' - z'\bar{y}') + (\bar{y}-y)(z'\bar{x}' - x'\bar{z}') + (\bar{z}-z)(x'\bar{y}' - y'\bar{x}')}{4\pi\left((\bar{x}-x)^2 + (\bar{y}-y)^2 + (\bar{z}-z)^2\right)^{3/2}}\, ds\, dt$$

example here;

$$lk(K,J) = \int_0^{2\pi}\int_0^{2\pi} \frac{1}{4\pi} \frac{(-\cos s)(-\cos s \sin t - 0) + (\sin t - 1 - \sin s)(0 - \sin t \sin s) + \cos t(-\sin t \cos t - 0)}{(\cos^2 s + \cos^2 t + (\sin t - 1 - \sin s)^2)^{3/2}}\, ds\, dt$$

$= 1$

Knot Theory

Shoaib Akhtar 17/8/2020. (1968)

Lec 9: Local Moves on Links.

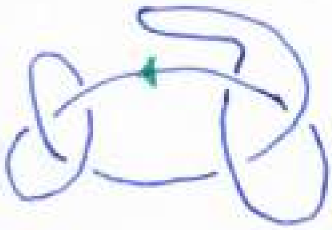
Local Moves

(i)  This means that knot can pass through itself.

Crossing Change

Proposition Any knot can be unknotted via crossing changes !

Proof eg

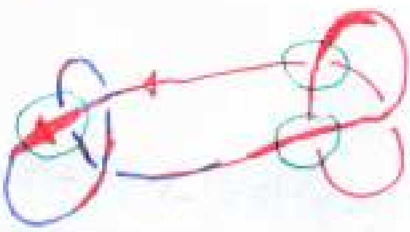


Connect sum of two trifoil

Start somewhere and traverse the knot.

As you travel, change crossings into overcrossings. unless you already passed.

⇓

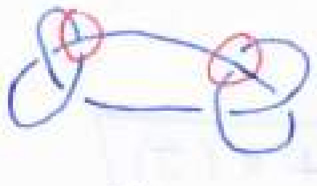


ie; always going down ... so don't get ~~un~~ knotted.

Question Given a knot what is the minimal number of crossing changes needed to turn it into unknot?

Call this $U(K)$, The ~~Unknotting~~ unknotting number of K

ex



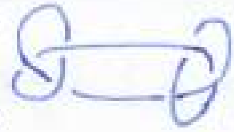
$= K$

can we unknot with fewer than 3 uncrossing.

⇓

So possible with two.

Is it possible with just one.

So $u(K) \leq 2$ ie: U() ≤ 2

Warning:
The uncrossing number is not preserved across diagrams of same knot.

Given any knot K, with $n \in \mathbb{N}$, there exists a diagram for K that requires at least n crossing changes to unknot.

ex) u() = 1

But there is some diagram for Trefoil that needs 500 crossing changes

Moral: $u(K)$ hard to calculate.

= connect sum of two trefoils.

What can be relation between.

$u(K \# J)$ $\quad$ $u(K)$ $\quad$ $u(J)$

We can show $\boxed{u(K \# J) \leq u(K) + u(J)}$

Proof) One way to unknot $K \# J$, is just unknot K, then J.

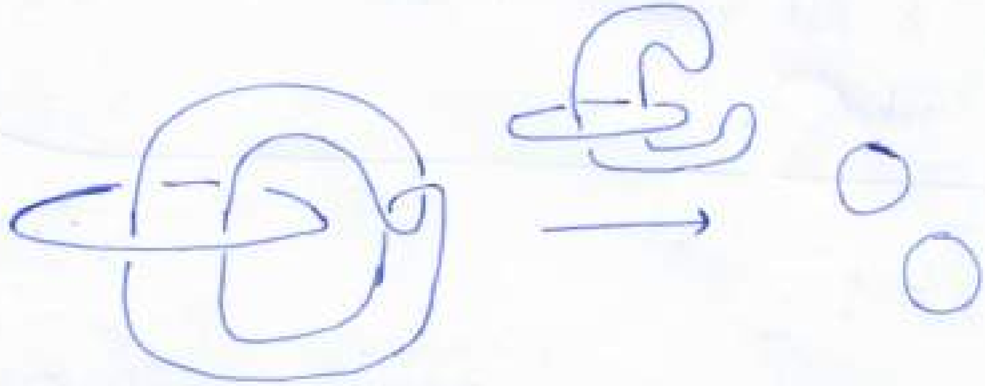
Open Conjecture $\quad u(K \# J) = u(K) + u(J)$

Recall) $g(K \# J) = g(K) + g(J)$ this is proved.

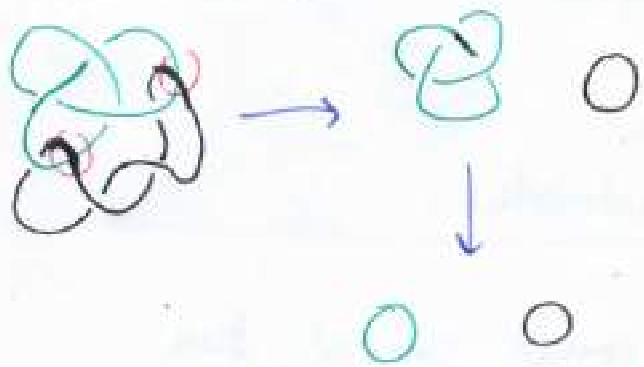
Similarly; we can define Unlinking Number.

$u(L)$ = min. crossing changes needed to obtain trivial link.

ex)



ex)



1st split the link

2nd unknot the components

Warning: This is not optimal.

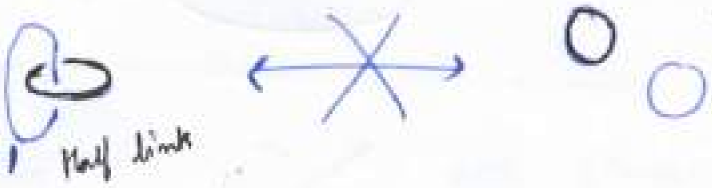
(ii)

Same component.

Link Homotopy

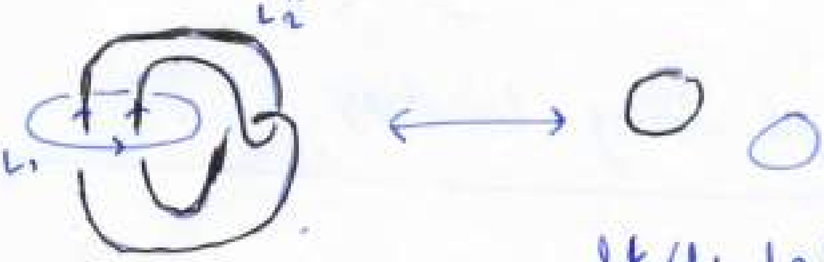
(components of a link can pass through themselves but not each other.)

ex



$lk(k_1, k_2) = 1$ Hopf link

ex

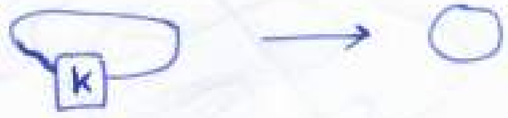


Whitehead link

$lk(L_1, L_2) = 0$

$lk(L_1, L_2) = 0$

ex Any Knot is homotopic to Unknot



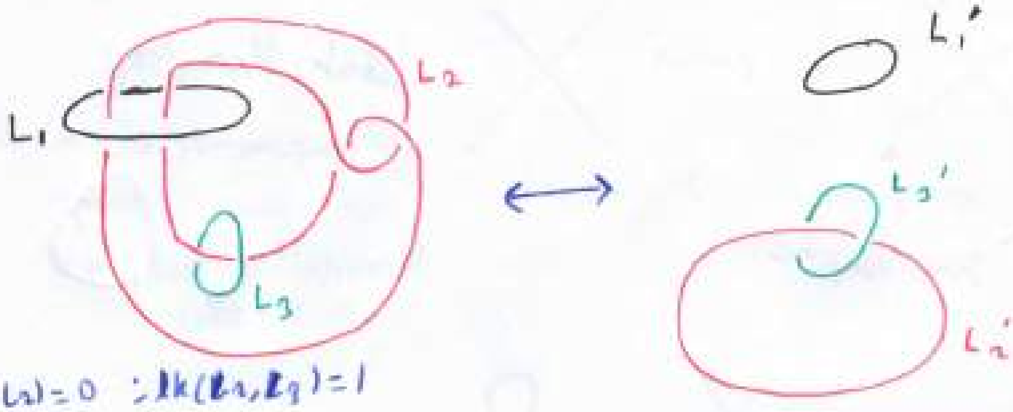
K be any knot

L is link homotopic to L'

$\Downarrow$ $\forall i,j$

$$lk(L_i, L_j) = lk(L_i', L_j')$$

* converse is true for 2-component link.
* for n-component link $(n > 2)$; we also need information about higher order linking numbers.



$lk(L_1, L_2) = 0$; $lk(L_2, L_3) = 1$

---

lk is counted by counting over  ; and these are unchanged during homotopy

---

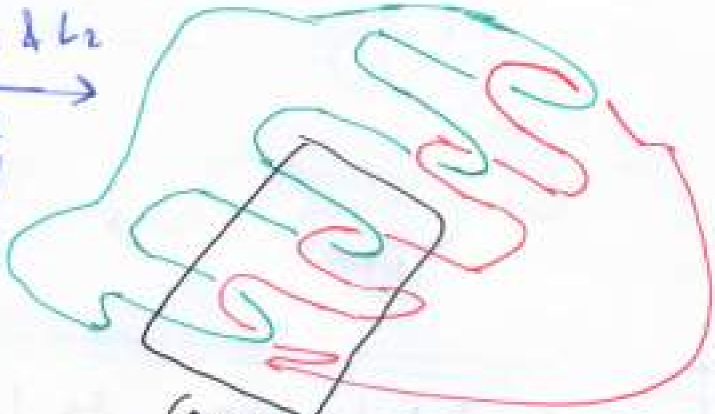
ex) <u>2-component links</u>

$L = L_1 \sqcup L_2 \qquad lk(L_1, L_2) = n$

$\qquad\qquad\qquad\qquad\qquad \parallel$

$L' = L_1' \sqcup L_2 \qquad lk(L_1', L_2')$

proof $L = L_1 \cup L_2 \xrightarrow[\text{and move into Standard Position}]{\text{unknot } L_1 \,\&\, L_2}$



canonical copy with linking number 2.

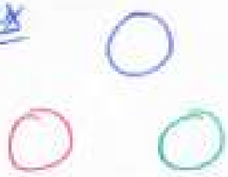
ll ex

This is linked some number of times all in same direction.

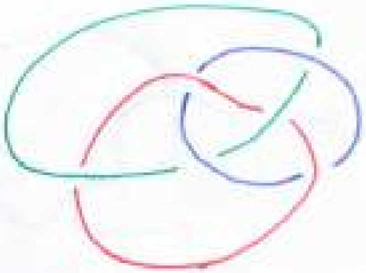
linked n times.

3 - component links

ex



$lk(L_1, L_2)$
$= lk(L_2, L_3)$
$= lk(L_3, L_1) = 0$



Counterexample.

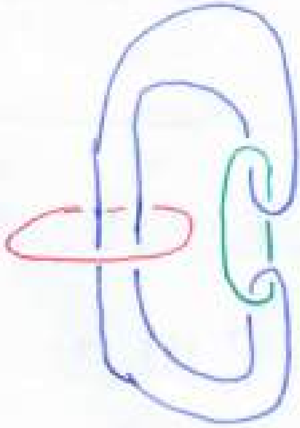
~~$lk(L_i, L_j) =$~~
$lk(L_i, L_j) = 0 \quad ; i \neq j$

↳ But these are not homotopic

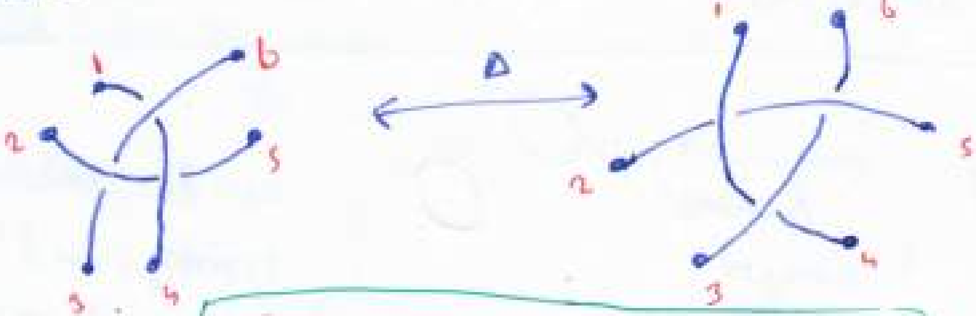
" For 3 component link, linking number does not classify link homotopy "

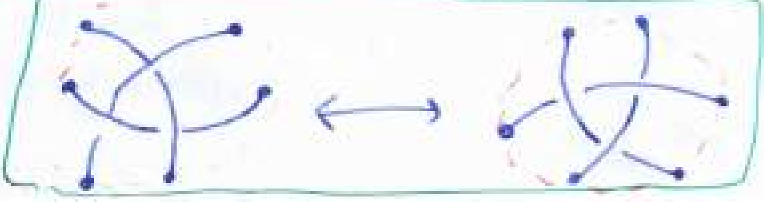
Redrawing ~~Brom~~ Borromean link

ie:

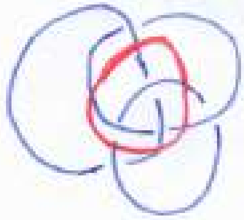


(iii) Delta - moves

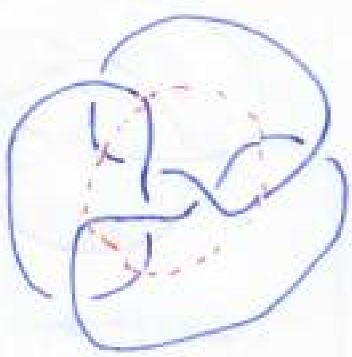




ex Borromean links



$\xleftrightarrow{\Delta}$



○ is the circle inside which we apply Δ move

||

Borromean Rings $\xleftrightarrow{\Delta}$ Trivial link.

Theorem $L = L_1 \sqcup L_2 \sqcup \dots \sqcup L_m$ is delta equivalent to $L' = L_1' \sqcup L_2' \sqcup \dots \sqcup L_m'$

$$\Updownarrow$$

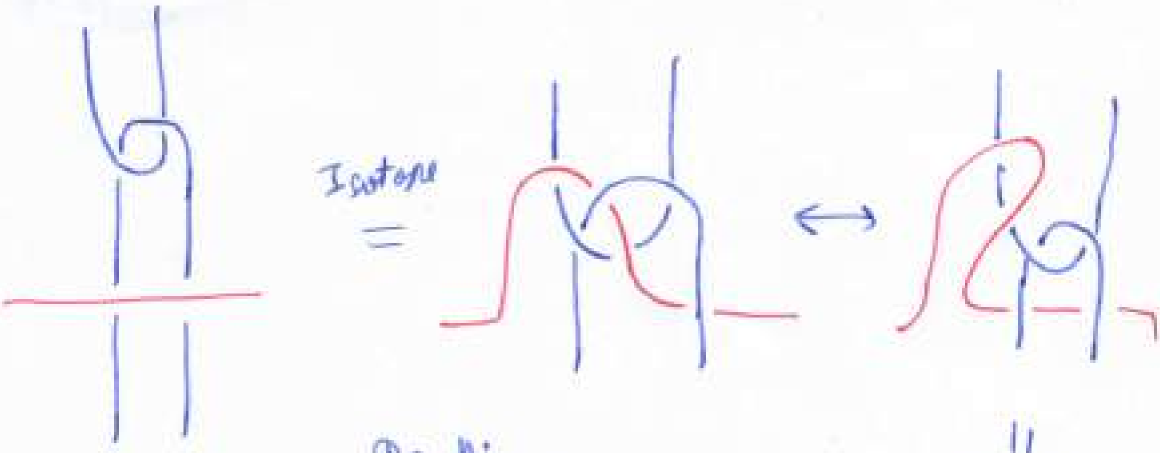
$$lk(L_i, L_j) = lk(L_i', L_j') \ \forall\ i,j$$

ex

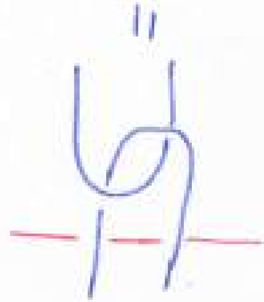
$\longleftrightarrow$ Δ-moves & ambient isotopy

we do ambient Isotopy; and then we get an opportunity to use Δ move

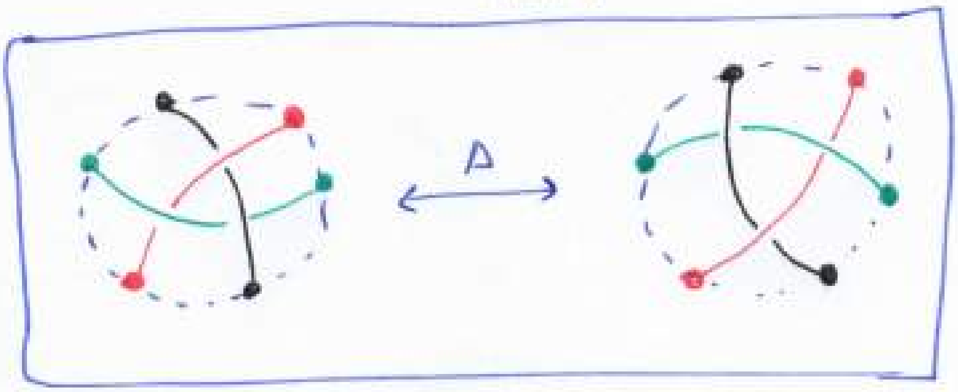
↳ Initially in this form, cant use Δ move.



Do this
to get delta.
(so that you could
perform delta
move)



so:



Lec 10: Jones Polynomial.

The Jones Polynomial

V : oriented link $\longrightarrow \mathbb{Z}[t^{-1/2}, t^{1/2}]$

Satisfies

- $V(\bigcirc) = 1$ ie, $V(\text{unknot}) = 1$
- $t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0$

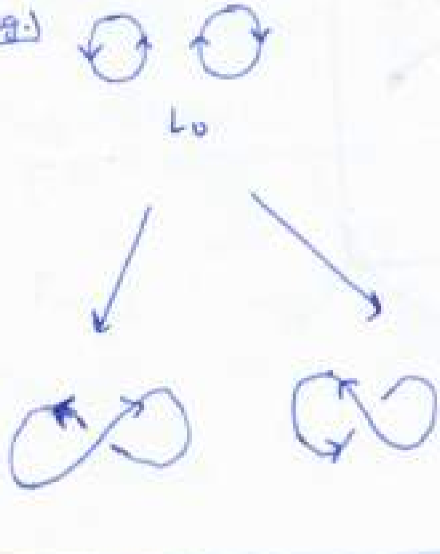
L_+ L_- L_0

Note:

① Well defined for all knots / links.

② Link invariant. (up to choice of orientation)

E.g.)



L_0

Then

$$t^{-1}\underbrace{(1)}_{V(L_+)} - t\underbrace{(1)}_{V(L_-)} + (t^{-1/2} - t^{1/2})V(OO) = 0$$

$$\Rightarrow V(OO) = -\frac{t^{-1} - t}{t^{-1/2} - t^{1/2}}$$

$$\boxed{V(OO) = -(t^{-1/2} + t^{1/2})}$$

In general. $L \sqcup O$

L L_0' L_+' L_-'

$$\rightarrow \boxed{V(L \sqcup O) = -(t^{-1/2} + t^{1/2})V(L)}$$

In particular;

$$V(\underbrace{O \ O \ O \dots \ O}_{m \text{ component}}) = (-1)^{m-1} \cdot (t^{-1/2} + t^{1/2})^{m-1}$$

$\cancel{V_L(t=1) = V}$

$$\boxed{V_{L_+}(t=1) = V_{L_-}(t=1)}$$

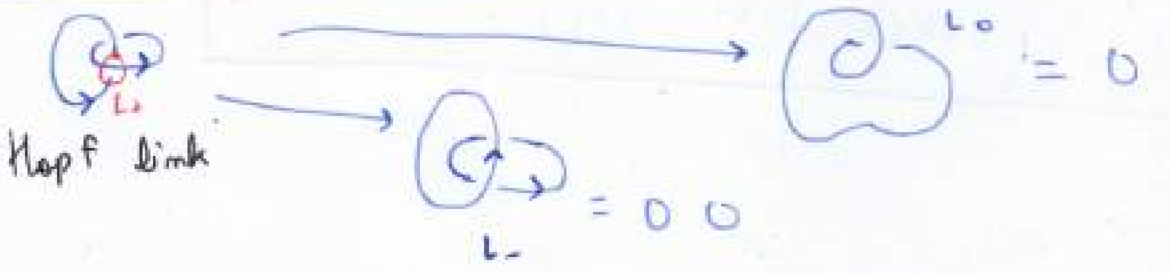
This says that as we change crossing; value of Jones polynomial does not change at $t=1$

So: we can make
The changes in crossing to make it trivial link.

$$V_L(1) = \underset{\substack{\text{link} \\ \text{(has same no.} \\ \text{of component as } L)}}{V_{\text{trivial}}}(1) = (-2)^{m-1}$$

where m is no. of component of link L

Then $V_L(1) = (-2)^{m-1}$



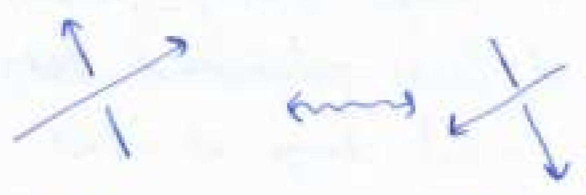
$$t^{-1}V(L_+) - t\,V(L_-) + (t^{-1/2} - t^{1/2})\,V(L_0) = 0$$

$$\Rightarrow t^{-1}\cdot V(L_+) + t\,(t^{-1/2} + t^{1/2}) + (t^{-1/2} - t^{1/2}) = 0$$

$$\Rightarrow V(L_+) = -t^{5/2} - t^{1/2}$$

$$V(L_+)\Big|_{t=1} = -2 \quad ✓$$

Reversing orientation on all components preserves Jones Polynomial.



RH $\qquad$ RH

Then $V(t) = -t^{5/2} - t^{1/2}$

Then

$$-t\,V - t^{-1}(t^{-1/2} + t^{1/2}) + (t^{-1/2} - t^{1/2}) = 0$$

$$\Rightarrow -tV = t^{-3/2} + t^{1/2}$$

$$\Rightarrow \boxed{V = -t^{-5/2} - t^{-1/2}}$$

t^3

Change orientation on L_i of L to get L',

$$\boxed{V(L') = t^{3\,lk(L_i,\,L-L_i)}\,V(L)}$$

Jones Polynomial under Mirror image

$-t^4 + t^3 + t$

calculate this

$-t^{-4} + t^{-3} + t^{-1}$

easily follows

L_- ⟷ L_+

L_0 L_0

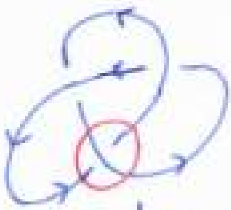
$\bar{L}$ is mirror of L.

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0$$

$$t^{-1}V(\bar{L}_-) - tV(\bar{L}_+) \quad (t^{-1/2} - t^{1/2})V(\bar{L}_0) = 0$$

$$\boxed{V_{\bar{L}}(t) = V_L(t^{-1})}$$ where $\bar{L}$ is mirror of L.

use this.



L_+

$= 0$

L_-

$V(L_-) = 1$

L_0

$V(L_0) = -t^{5/2} - t^{1/2}$

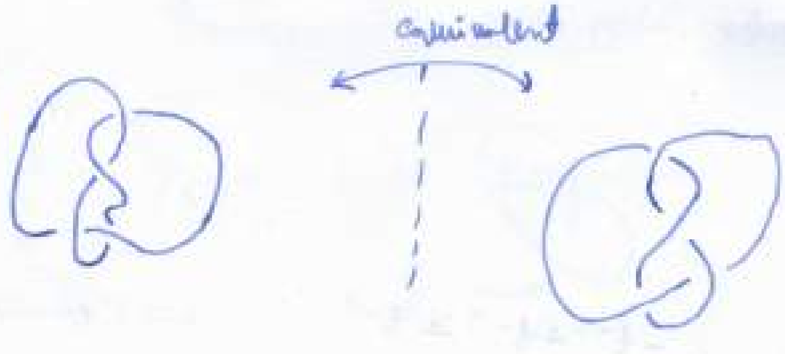
$$\Rightarrow t^{-1}V(L_+) - t + (t^{-1/2} - t^{1/2})(-1)(t^{5/2} + t^{1/2}) = 0$$

$$\Rightarrow V(L_+) = t^2 + (t^{-1/2} - t^{1/2})(t^{5/2} + t^{1/2})t$$

$$\Rightarrow V(L_+) = t^2 + (t^2 + 1 - t^3 - t)t$$

$$= -t^4 + t^3 + t$$

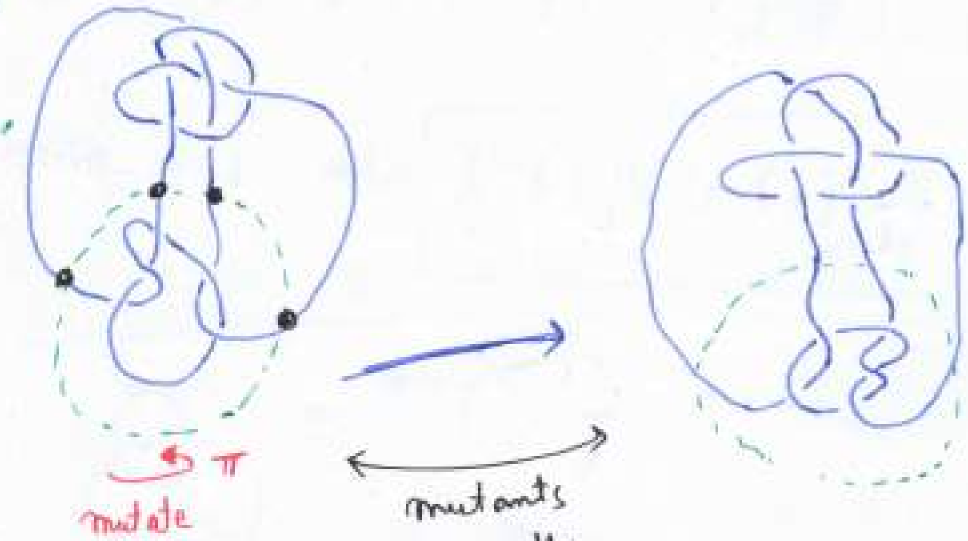
Example



$$K = J \implies V(K) = V(J)$$

Is the converse true? No, its not true.

example



mutants have the same Jones polynomial. $V(t)$!

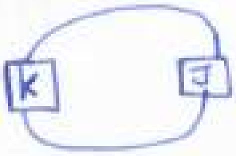
Mutants are different: they have different fundamental group $\pi_1(K)$, different genus.

(but same $V(t)$)

$K =$ unknot $\bigcirc$ $\implies V(K) = 1$

Is the converse true?

Open conjecture: If $V(K) = 1 \implies K =$ unknot



$V(K \# J)$ $\xrightarrow{\text{resolve crossing of } V}$ $V(K)$

So: $$\boxed{V(K \# J) = V(K)\,V(J)}$$

HOMFLY Polynomial. $P(\alpha, z)$. Polynomial of two variables.

defined by

- $P(O) = 1$
- $\alpha P(\nwarrow\!\!\!\!\nearrow) - \alpha^{-1} P(\nwarrow\!\!\!\!\nearrow) = z\, P(\uparrow\uparrow)$

Generalizes Jones & Alexander polynomial as follows:

$$\Delta(t) = P(\alpha = 1,\ z = t^{1/2} - t^{-1/2})$$

$$V(t) = P(\alpha = t^{-1},\ z = t^{1/2} - t^{-1/2})$$

Lec 11: Slice and Concordance

So far ... knots in $\mathbb{R}^3$ equivalent up to ambient isotopy.

Invariants:

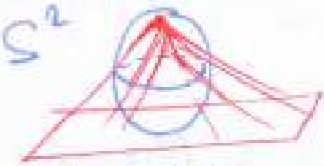
- Colorability.
- Determinant.
- Alexander Polynomial.
- Jones Polynomial.
- Genus.

Generalizing Unknot

Usually we think of knots / links in ~~$\mathbb{R}^3$~~

but now think in S^3.

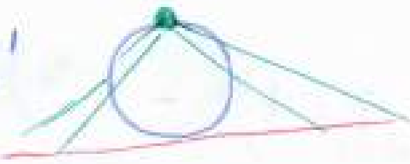
S^2



2-dim sphere

$\mathbb{R}^2$

Similarly S^1



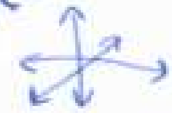
1-dim sphere

$\mathbb{R}$

Similarly S^3 ~~dim 3 sphere~~

3 dim sphere.

$\mathbb{R}^3$



S^1 bounds disk  B^2

S^2 bounds ball  B^3

S^3 bounds B^4 (4-dimensional ball)

Knot $K \subset S^3$

(ie; Knot K inside S^3)

Then think of the knot K living in 3 dimensional space; but that 3-dim space is on surface of the 4-dimensional ball.

K is the unknot $\Longleftrightarrow$ K bounds disk in S^3

~~generalizing this;~~

generalizing the above notion:

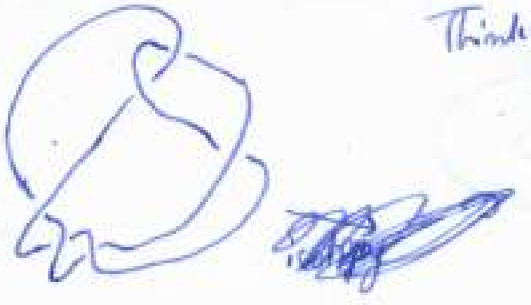
Defn] $K \subset S^3$ is Slice $\Longleftrightarrow$ K bounds smooth disk in B^4.

Warning] every knot bounds a cone in B^4.

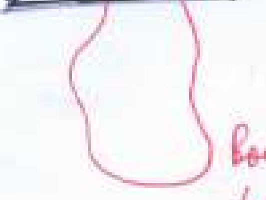
disk

Singularity.

6, :



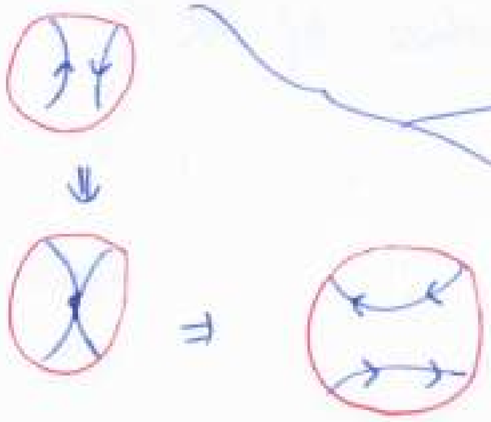
Think as



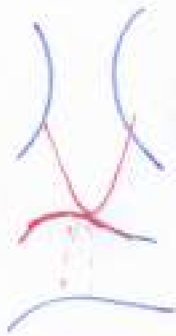
S_3

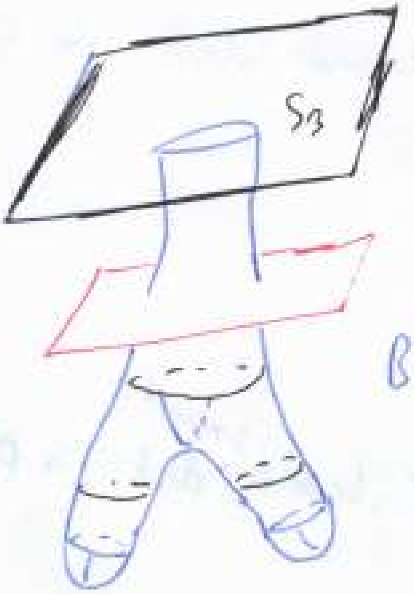
Bounds some ball in B_4

Isotopy



This corresponds to comming to a saddle





B^4

Moving down the saddle splits it in two components.

Saddle

After doing this move we get

isotopy

isotope

cap off

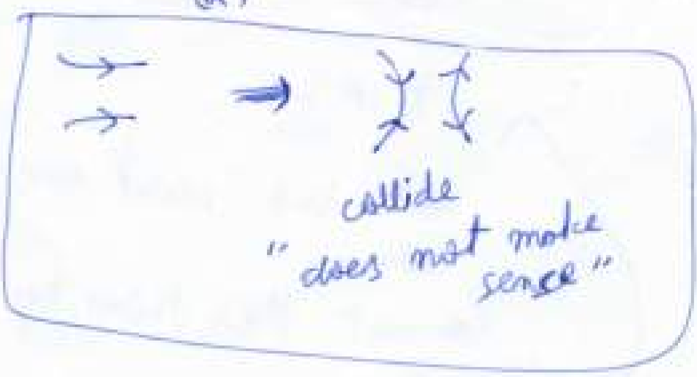
Slice Disk.

8_8

S^3

(*) because of "(*)" we first make the following isotopy move

(*)

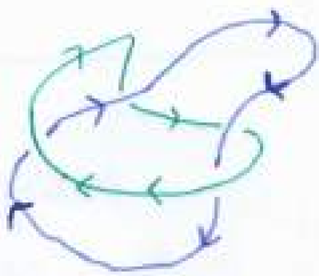


collide

"does not make sense"

passing through saddle.

isotopy



S^3

Slice Disk

Q] Is every knot (smoothly) slice?

A] No: Trefoil is not!

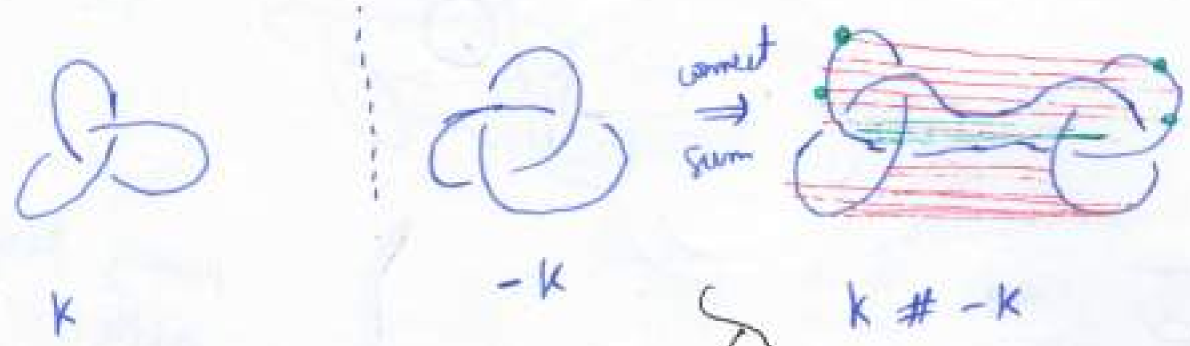
Def^n] We say knots K, J are <u>concordant</u> if

$\underbrace{K \# -J}$ is slice. $(K \simeq J)$ → Mirror image of J.

Concordance is equivalence relation.

- <u>Reflective</u> $K \simeq K$

  ie; $K \# -K$ is slice.



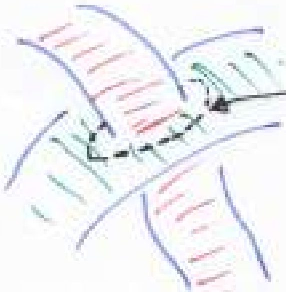
note each point has twin.
Connect the twins by line

This is a disk which passing through it self couple of times.
The knotting causes the disk to pass through itself

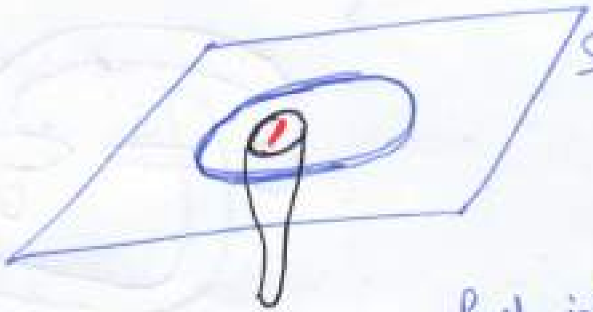
So: for $K \# -K$

disk crossing over itself couples of times.

disk crossing through itself in S^3.



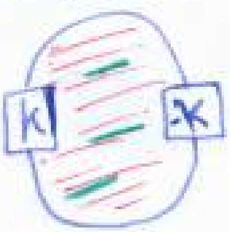
Take a neighbourhood where it crosses through itself; and push that region down to 4th dimension.



S^3

Push into B^4 to prevent it from crossing through itself.

For any knot K



— ⇒ green are the region where it cross through itselves.

⟺ We can push these regions to B^4 to prevent it from crossing ~~itselfs.~~ itselves.

We call a knot <u>Ribbon</u> if it bounds a disk that crosses through itself only in arcs thusly:

In particular $K \# -K$ is ribbon.

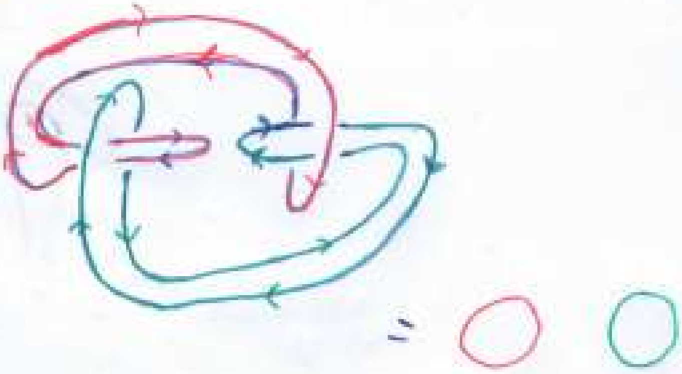
e.g.

6_1

$\boxed{\text{Lemma}\quad \text{Ribbon} \Rightarrow \text{Slice}}$

passing through itself looks

passing through saddle.



One more representation of 6_1 is

$\boxed{\text{Slice} \Longrightarrow \text{Ribbon} \quad : \text{Open Conjecture!}}$

• Symmetric: $K \simeq J \Rightarrow J \simeq K$

need to show: if $K \# -J$ slice $\Rightarrow$ $J \# -K$ is slice

$-J \# K = K \# -J = -(K \# -J)$

Now; we have to show ; given a knot is slice, its mirror is slice. Then we are done

To be a slice it has to bound a disk.

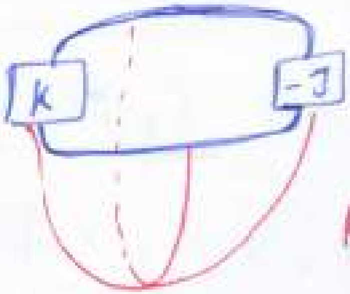
$\hookrightarrow$ if original one bounds a disk ; then its mirror image also bounds a disk.

Hence if $\boxed{K \text{ is slice} \Rightarrow -K \text{ is slice}}$

- Transitive) $K_1 \simeq K_2,\ K_2 \simeq K_3 \Rightarrow K_1 \simeq K_3$

To prove this ; we give the equivalent defination for Concordance.

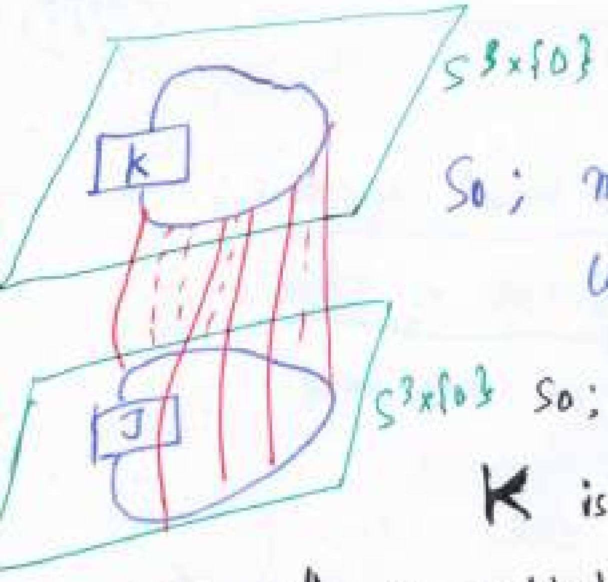
Defn of Concordance) $K \# J$ is slice means.



Bounds disk in B^4.

pull J to the bottom; and flip J over (flipping will undo the mirror.

$\hookrightarrow$ we can seal this gap (look up in next page)



So; now we talk about K ~~#~~ & J connected by Cylinder.

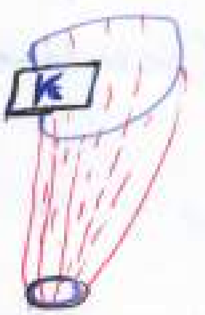
So;

K is concordant to J if there exists smooth a cylinder in $S^3 \times [0,1]$ (cylinder) with boundary $K \subset S^3 \times \{0\}$ and $J \subset S^3 \times \{1\}$

↪ from this defination ; we see that $K \simeq K$.

$K_1 \simeq K_2 , K_2 \simeq K_3 \Rightarrow K_1 \simeq K_3$

Slice Knots are concordant to the unknot.



6_1 is slice , so 6_1 is concordant to Unknot.

but

$\Delta_{6_1}(t) = -2t + 5 - 2t^{-1}$

$\Delta_{\text{unknot}}(t) = 1$

Alexander Polynomial.

$V_{6_1}(t) = t^2 - t + 2 - 2t^{-1} + t^{-2} - t^{-3} + t^{-4}$

$V_{\text{unknot}}(t) = 1$

Jones Polynomial

(Concordance dont preserve Alexander & Jones Polynomial)

Determinat (it is also not preserved under concordance)

$\det(6_1) = 9$

$\det(0) \neq 9$

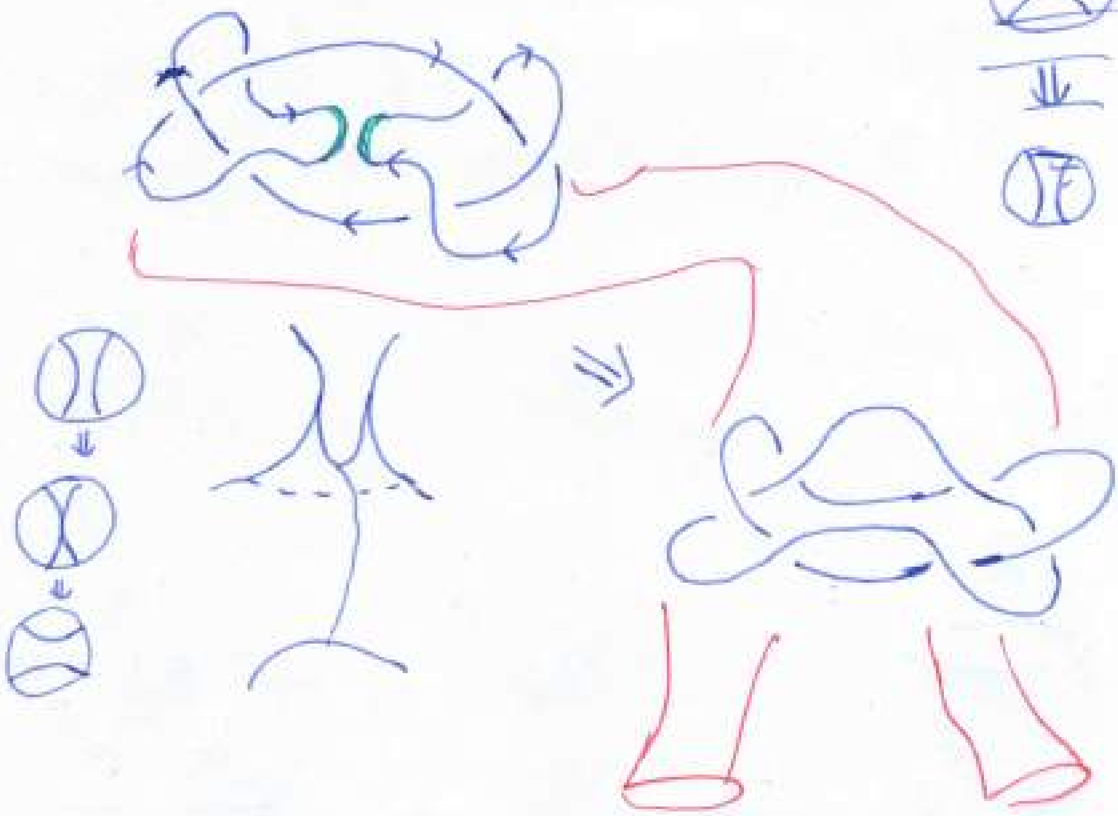
Are there invariants that are preserved under concordance ?

Lec 12] Concordance Group

Concordance:

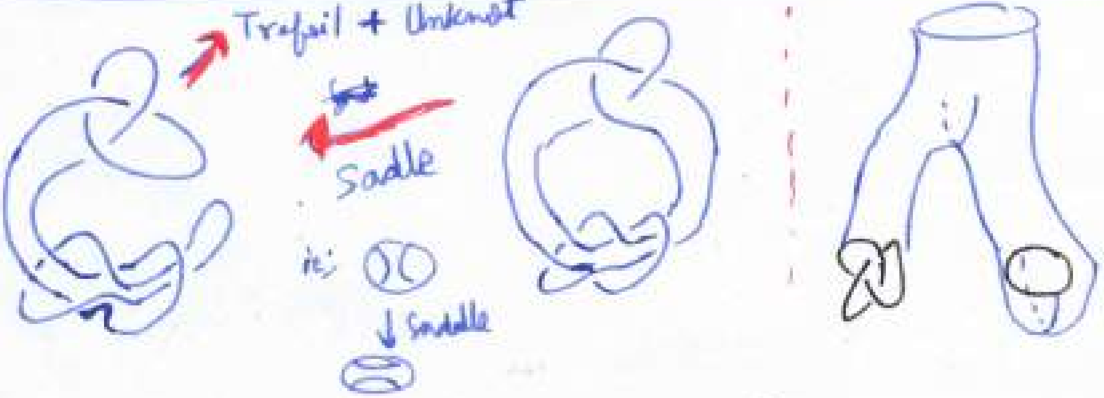
ex]

$K \# -K$

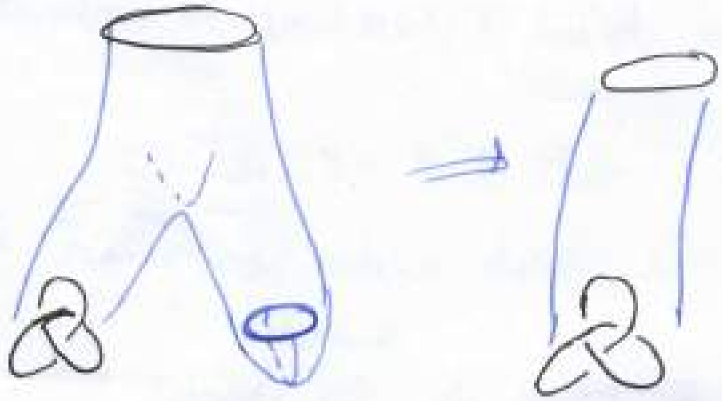


$\boxed{\text{Slice} \Longleftrightarrow \text{Concordance to Unknot}}$

eg



So, The original one is concordant to unknot



so; concordant to trefoil.

Is trefoil concordant to unknot?

Problem:

Our invariants: Alexander Polynomial,
Jones ",
Det, Genus, Knot Group are
not invariants of Concordance.

Question

- Why should we care about concordance?
- Are all knots slice? If not, how many are there upto concordance?
- Can we find invariants of concordance.

Recall: Genus of knot : $g(K)$

we showed $g(K \# J) = g(K) + g(J)$

$g(K) = 0 \iff K = \text{unknot}$

$\boxed{g(K) \geq 1 \quad , K \neq \text{unknot}}$

cannot combine knotted knots to get unknot!

Knots dont have inverses!

Thinking up to Concordance:

- Slice knots are trivial (concordant to unknot)
- For any knot K, note $K \# -K$ is slice.

(so knots under concordance have inverses)

(Mirror of the knot is its inverse)

•

~~So, knots form a group under concordance~~

The set of knots up to concordance forms a group.

Denote $\mathcal{C}$, knot concordance group.

- Slice knot is trivial element; order 1.
- Figure Eight (4_1) : $-4_1 = 4_1$

so; $4_1 \# 4_1 = 4_1 \# -4_1$ = slice.

so; Order$(4_1) \leq 2$

can show Order$(4_1) = 2$

Amphichiral knots are the knots K s.t. $K = -K$

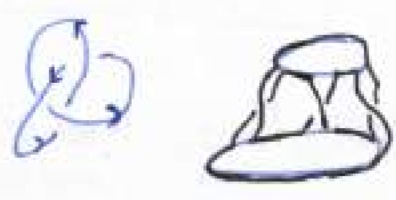
→ They are order ≤ 2.

- Knots of other orders?

Seifert Surfaces Revisited:

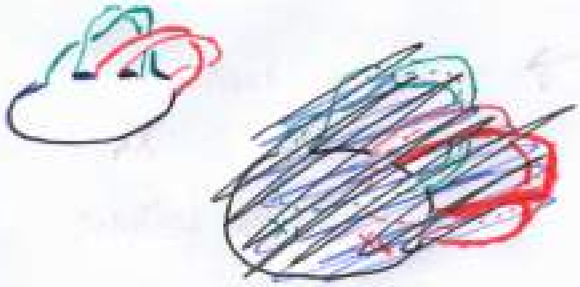
Any knots bounds orientable surface that consists of disk & bands.

Eg.



We can always reduce to 1 disk
(by using some moves ; say sliding bands)

In this case for trefoil,
we can reduce to



Eg)

(It is definitely a knot check easily)

for each band we have a circle.

We define Seifert Matrix V

(by thinking about how the circles interact with each other)

eg) of Trefoil

$$V = \begin{array}{c|cc} & x_1 & x_2 \\ \hline x_1^+ & 1 & 1 \\ x_2^+ & 0 & 1 \end{array}$$

$\hookrightarrow V + V^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

The matrix V is defined by linking numbers of x_i with the push off of x_j i.e. x_j^+

$lk(x_i, x_j^+)$

Meaning of Push-off

since the surface is orientable; we can call one side of the disk to be positive side

eg.

x_i^+ is the x_i loop pushed off the surface along positive side of the surface

Eg.

$$V = \begin{array}{c} \\ x_1^+ \\ x_2^+ \\ x_3^+ \\ x_4^+ \end{array} \begin{array}{c} \begin{array}{cccc} x^1 & x^2 & x^3 & x^4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Since x_1 does not touch x_3 & x_4 this trivially follows.

$$\hookrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

We do a very tiny push off (say by an ϵ amount, where ϵ is infinitesimal)

→ These matrices are not very interesting ; (these are not symmetric)

↪ So, we make a symmetric matrix out of it $V + V^T$

Symmetric matrices have all eigen values real.

↪ Hence ; diagonalizable.

$$D(\text{Trefoil}) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$D(K) = \begin{pmatrix} 1-\sqrt{2} & 0 & 0 & 0 \\ 0 & 1-\sqrt{2} & 0 & 0 \\ 0 & 0 & 1+\sqrt{2} & 0 \\ 0 & 0 & 0 & 1+\sqrt{2} \end{pmatrix}$$

Signature of matrix stays the same under change of basis.

$$\boxed{\text{Signature of Matrix} = (\#\text{ of +ve eigenvalues}) - (\#\text{ of -ve eigenvalues})}$$

→ call this the signature of the knot

$\sigma(K)$

$\sigma(\text{Trefoil}) = 2$

$\sigma(k) = 0$

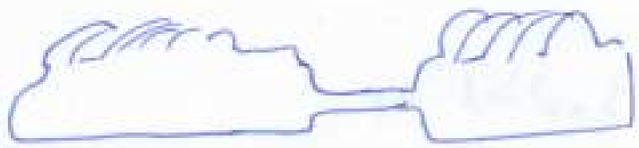
$$\boxed{K \rightsquigarrow \text{S. Surface} \rightsquigarrow \text{Seifert Matrix} \rightsquigarrow \sigma(K)}$$

Roadmap of calculation of $\sigma(k)$

Properties of Signature:

$\sigma(k)$ does not depends on S. Surface (This is important for $\sigma(k)$ to be well defined)

- $\sigma(K\#J) = \sigma(k) + \sigma(J)$



K J

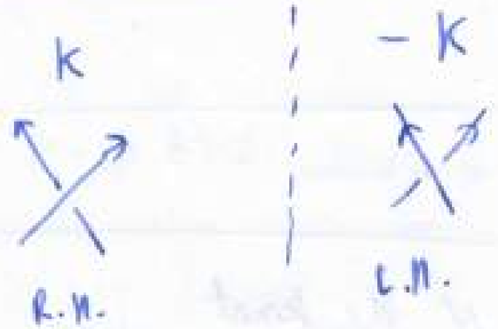
$$V_{K\#J} = \begin{pmatrix} V_K & 0 \\ 0 & V_J \end{pmatrix}$$

$$\Rightarrow \quad V_{K\#J} + V_{K\#J}^T = \begin{pmatrix} V_K + V_K^T & 0 \\ 0 & V_J + V_J^T \end{pmatrix}$$

$$\xrightarrow{\text{Diagonalize}} D_{K\#J} = \begin{pmatrix} D_K & 0 \\ 0 & D_J \end{pmatrix}$$

$\Rightarrow \sigma(K\#J) = \sigma(K) + \sigma(J)$

- Under mirror image !



V_k : $V_{-k} = -V_k$ (everything remains same; but crosses backwards)

D_k $\Downarrow$ $D_{-k} = -D_k$

$\Rightarrow$ $\sigma(k)$ $\quad$ $\sigma(-k) = -\sigma(k)$

$$\boxed{\sigma(k) = -\sigma(-k)}$$

negative number -1 i.e. $-1 \in \mathbb{R}$.

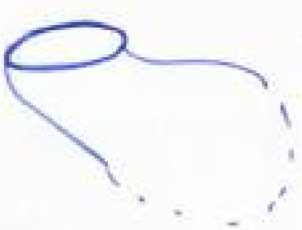
mirror image

eg)

Surface bounded by Trefoil.

A trefoil is circle ~~knotted~~ knotted up.

Start with



Then generate the surface.

Recall, Trefoil bounds Genus 1 surface.

so;

knot this up to get surface bounded by Trefoil

K is genus 2 surface

so;

$\Rightarrow$ knot it up to get surface bounded by K.

What happens when K is slice.

If K is slice, then

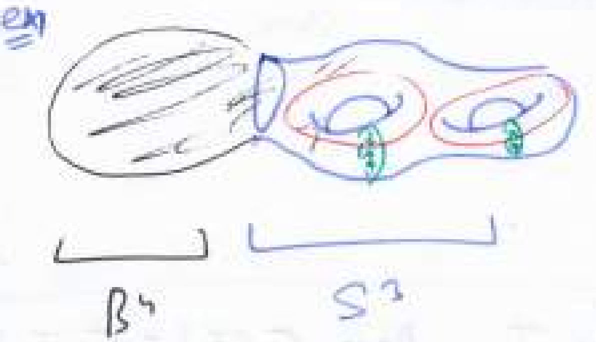
Slice disk ⊔ Seifert Surface

bounds 3-manifolds

Half Lives,
Half Dies
(Result from topology)

(because the Slice disk closes up the hole

eg



If we choose our surface wisely,

zero (null) because half of the circle dies.

Then our V will look like $V = \underbrace{\begin{array}{c} g\{ \\ \\ \end{array}\left(\begin{array}{c|c} 0 & A \\ \hline B & C \end{array}\right)}_{2g}\Big\}_{2g}$

We have a $g \times g$ block of $2g \times 2g$ which vanishes.

With some algebra,

we can show $\sigma(K) = 0$ (when K is Slice)

(given a slice knot; signature vanishes)

$\boxed{\text{Lemma: } K \text{ slice } \Rightarrow \sigma(K) = 0}$

Corollary: ~~Suppose J has S. surfaces Σ, Σ_2 then $\sigma_{\Sigma_1}(J) = \sigma_{\Sigma_2}(J)$~~

Corrollary) Suppose J has S. Surface F_1, F_2

Then $\sigma_{F_1}(J) = \sigma_{F_2}(J)$

proof) $\sigma_{F_1}(J) - \sigma_{F_2}(J) = \sigma_{F_1}(J) + \sigma_{F_2}(-J)$

$= \sigma_{F_1 \# F_2}(J \# -J)$

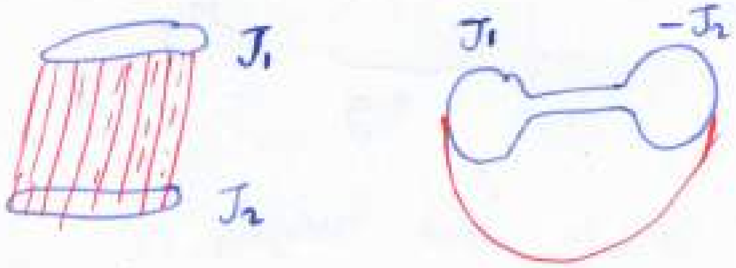
$= \sigma_{F_1 \# F_2}(\underset{\text{slice}}{J\#-J \text{ is a}})$

$= 0$

$\Rightarrow \boxed{\sigma_{F_1}(J) = \sigma_{F_2}(J)}$

Corrollary) Suppose J_1 concordante to J_2, then $\sigma(J_1) = \sigma(J_2)$

proof) J_1 conc. $J_2 \iff J_1 \# -J_2$ slice.



We know $0 = \sigma(J_1 \# -J_2) = \sigma(J_1) + \sigma(-J_2)$

$= \sigma(J_1) - \sigma(J_2)$

$\Rightarrow \boxed{\sigma(J_1) = \sigma(J_2)}$

Hence proved: Signature is invariant of Concordance

We found $\sigma(\text{Trefoil}) = 2 \Rightarrow$ Trefoil is not slice!

$$\boxed{\sigma(\#_n \text{ trefoil}) = \sum_n \sigma(\text{trefoil}) = 2n}$$

so: atleast by $\#_n$ trefoil, by varying n, we can generate infinite no. of elements out of $\mathcal{C}$.

In fact, $\mathcal{C} \xrightarrow[\text{surjective}]{} \mathbb{Z}^{\infty} \oplus \mathbb{Z}_2^{\infty} \oplus \mathbb{Z}_4^{\infty}$

K slice $\Rightarrow \sigma(K) = 0$

Is the converse true?

Ans No; we can find an example such that $\sigma(J) = 0$ & J is not slice.

TIE YOUR SHOE LACES

THANK YOU

Embedding of a one dimensional curve into ambient spaces can give rise to different mathematical structures.
Shoaib Akhtar