Quantum Field Theory III — Shoaib Akhtar 18/7/2020

Lecture 1 | Gloriness of CFT part 1

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Conformal Field Theory (CFT)

CFT is a QFT with extra spacetime symmetry.

- QFT is invariant under Poincaré algebra (Relativistic QFT)
  (Theory may & may not be invariant under disconnected components of Lorentz group;
  but invariant under connected components)

$$\text{CFT} = \underset{\text{(Poincare invariance)}}{\text{Poincaré algebra}} + \left(\text{some other symmetry}\right)$$

ex $L = \frac{1}{2}\dot{x}^2 - V(x)$; $E = \frac{1}{2}\dot{x}^2 + V(x)$

- We can also look for Galilean algebra invariant QFT (non-relativistic QFT)

$\left\{\begin{array}{l}\text{- Poincaré algebra (relativistic QFT)} \\ \text{- Galilean algebra (non-relativistic QFT)}\end{array}\right\} \Rightarrow$ ~~CFT~~ Conformal Symmetry

* Notion of Conformal symmetry is unique for relativistic QFT.
* " " " " .. not unique " non- " " .

"Conformal Symmetry is the largest spacetime symmetry of interacting QFT", true in $D > 2$.

Scale Transformation (Dilatations)

$\vec{x} \to \lambda \vec{x}$: $\lambda \in \mathbb{R}$

$t \to \lambda^z . t$; z: dynamical critical exponent.

- Relativistic : $(z=1)$; $x^\mu \to \lambda x^\mu$
- Non-relativistic : ex// $-\frac{1}{2}\nabla^2\psi = i\frac{\partial\psi}{\partial t}$ here $z=2$

ex// $\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 + (\nabla^2\phi)^3$; $(z=3)$

Conformal transformation

$x^\mu \to \tilde{x}^\mu(x)$ that preserves angles.

(It does not necessarily preserve distances)

ie; $ds^2 \longrightarrow e^{2\sigma(x)} ds^2$

This tells that transformation has to be isotropic; we cannot change the length element inhomogeneously.

ex// Poincaré (because it preserves metric) with $\sigma(x) \equiv 0$.

ex// Dilatations : $e^{2\sigma(x)} = \lambda^2$

ex// Special Conformal transformation.

Finite dimensional set of transformation (which preserves angles)

↳ and this will generalize Poincaré group to Conformal.

Why CFTs?

- Asymptotic large distance behavior of QFT.

① Trivial / gapped (massive)

$\langle \theta(x)\,\theta(0)\rangle \sim e^{-|x|/\xi}$ (when massive)

$\xleftrightarrow[\text{extreme I.R.}]{}$ no-local operators.

Non-trivial topological quantum field theory (TQFT)

- (don't have any local operators)

They have no local operators (because it knows only about global picture)
But can have non-local operators. (Wilson loop in gauge theory)

~~Non-local operators~~
~~→ They describe IR dynamics in~~
~~— QCD in 2+1 d~~
~~— Condensed matter.~~

Non-local operators ; describe IR dynamics in
- QCD in 2+1 d
- Condensed matter.

Wilson loop:

$$\therefore U(C) = e^{i\oint_C A_\mu \frac{dx^\mu}{d\tau}}$$

② Scale Invariant Theory

$$\underbrace{\text{Poincaré} \longrightarrow \text{Poincaré}' \times \mathbb{R}}$$ ← Dilation. ⟶ Conformal

→ Scale invariance happens here by defination; because we have removed all the scale, so there is no scale in the problem.

⇓

$$\langle \theta(x)\, \theta(0) \rangle = \frac{1}{x^{\Delta}}$$

Correlation function will have homogeneous properties under scale transformation.

(have power law)

Scaling dimensions of operator

CFTs provide an ordering in the space of QFTs.

Examples /

- $\Box\phi + \lambda\phi^3 = 0$
- $\partial_\mu F^{\mu\nu} = 0$ (Maxwell), $D_\mu F^{\mu\nu} = 0$ (Yang - Mill) g is dimensionless.
- $\mathcal{L} = \bar{\Psi}\gamma^\mu\partial_\mu\Psi + (\bar{\Psi}\Psi)^2$ ← scale invariant in $D=2$

We can think coupling ; Yukawa ; that we can write ~~g~~ $g\phi\bar{\Psi}\Psi$

(Lagrangian of Yukawa Theory is scale invariant if ~~masses~~ masses are zero)

⟹ Essentially ; if you turn of masses in all of your theories ∃ The theory you get is scale invariant.

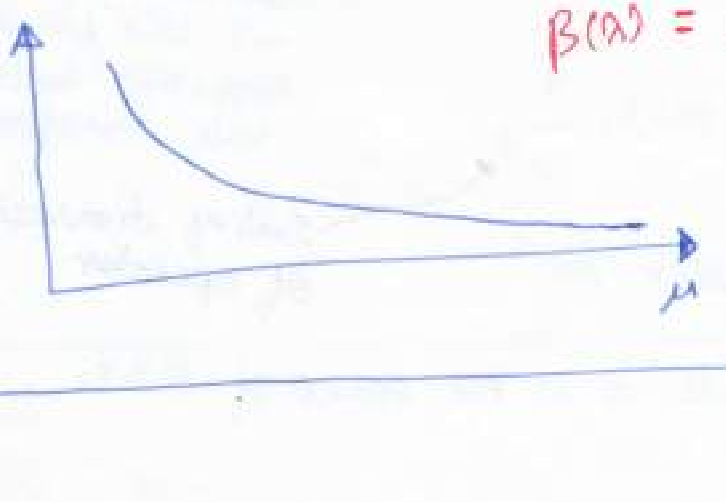
↪ These are Classical Statements.

In Q.M. ; quantum corrections (effects) change this
∴ Coupling constant of theory due to virtual particle gets dependence ~~of~~ on energy scale.

⇓

$g(\mu)$ ← coupling dependend on Energy scale.

$\beta(\lambda) = \mu \frac{d}{d\mu}\lambda(\mu) \neq 0$ (quantum Mechanically)



$\mathcal{L} = \mathcal{L}_0 + \lambda \mathcal{O}$ — Deforming our theory little bit.

$\uparrow$ classically CFT

; and ask what happens in IR limit?

i.e. $\mathcal{L} = \mathcal{L}_0 + \lambda \mathcal{O}$

• IR physics depends on the dimension of the the operator $\mathcal{O}$
$\overset{\shortparallel}{\Delta}$

D is spacetime dimension

• if $\Delta > D$: irrelevant (unimportant in IR) ex $\lambda(\Box\phi)^3$

• if $\Delta < D$: relevant (important in IR) ex ~~…~~ $\bar{\psi}\psi$

• if $\Delta = D$: marginal ; but Quantum Effects can make this …
(classically scale invariant)

- Marginally irrelevant : e.g. $\lambda\phi^4$, $\beta > 0$
- Marginally relevant : e.g. Yang-Mills, $\beta < 0$

★ The fact that low energy physics is controlled by operator with ($\Delta < D$) or ($\Delta = D$) is crucial for us to make any progress. (These set of operators is not very large.. so we can solve)

Most of the operators for a given theory fits in the subset ($\Delta > D$) ; like say ($\mathcal{O}$) million.
... Haha....

★ Δ is charge of operator $\mathcal{O}$ under dilation;

Lecture 2 | Gloriness of CFT part 2

CFTs are universal. ~~They appear to~~

① Appear as Extreme IR limit of QFT

② Any QFT can be thought of as a perturbation/deformation of CFT.

$$\mathcal{L} = \mathcal{L}_{CFT} + \sum_i \lambda_i \mathcal{O}_i$$

$\mathcal{O}_i$ ← operators.

$\mathcal{L}_{CFT}$ ↑ Usually a free field theory (with which you start)

CFTs induce an ordering in the space of QFTs

Start with a theory CFT_1 ; and deform it by a relevant operator.

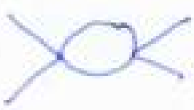
UV CFT_1 (deform it by a relevant operator)

IR CFT_2

Wilson-Fisher CFT (study of 3D Ising Model)

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \lambda\phi^4$$

CFT appears whenever $\beta(\lambda^*) = 0$: whenever there is a root of β function (because there we will have scale invariance)



$$\beta(\lambda) = |A|\lambda^2 \geq 0$$

(coupling constant grows toward UV.)

Imagine theory in $D = 4 - \varepsilon$ dimensions

In case; if you decrease the dimension of spacetime, the no. of relevant operators increases;
just because the dimensions of fields decreases a little bit, and therefore by taking more polynomials of the field we can construct operators which are relevant.

$$\beta(\lambda) = |A|\lambda^2 \geq 0 \qquad D=4$$

In $D = 4-\epsilon$; $\beta(\lambda) = -\epsilon\lambda + |A|\lambda^2$

Tree-level contribution

so i.e; $\beta(\lambda) = -\epsilon\lambda + |A|\lambda^2$

Now; lets look for zeroes of $\beta(\lambda)$

There are ~~two~~ solutions.

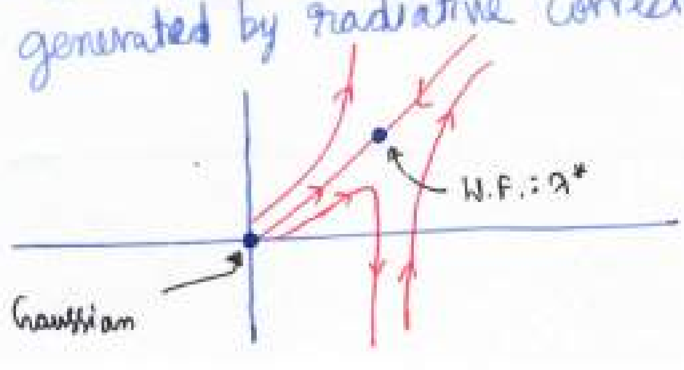
- $\lambda^* = 0$; Free field theory point
  or say Free field (Gaussian) fixed point
  because the action is quadratic.
- $\lambda^* = \frac{\epsilon}{|A|}$ W.F. fixed (CFT) : can be reliably studied in ϵ-expansion.

At both these fixed point we have CFT.

so; starting from free theory (which is a CFT) by deforming λ. we went to Wilson Fisher CFT

RG flow diagram

This theory has two couplings λ, mass.
(Even if it does not have mass classically; it will be generated by radiative corrections)



$$\beta(\lambda) = \mu \frac{d}{\mu}\lambda$$

arrow in diagram means you flow to infrared.

CFT ~~..~~ sitting at W.F. fixed point describes the universality class of the critical 3d Ising Model.

CFT's induce an ordering.
lets assume $\exists$ an R.G. flow that maps you from

$$\begin{array}{ll} \text{U.V.} & CFT_1 \\ & \big\downarrow \\ \text{I.R.} & CFT_2 \end{array}$$

$\nexists$ an R.G. flow

$$\begin{array}{c} CFT_2 \\ \big\downarrow \\ CFT_1 \end{array}$$

assuming above statement is true.

We can assign to a CFT a <u>height function</u> (c-function)

C: is an intrinsic property of CFT.

C must be monotonically decreasing under R.G. flow.

i.e., $C_{U.V.} > C_{I.R.}$ (c-Theorem)

In even dimensions: 2, 4, 6 ... c has to do with <u>conformal anomalies</u>.

Other uses of CFT

- Critical Phenomenon in statistical mechanics. (Thermal fluctuations)

Describes 2nd order phase transitions ($\xi \to \infty$)

Characterised by some non-analyticity of $Z[B, T, \ldots]$

like $T \to T_c$.

Then canonical critical exponent we can define is.

$$\xi \sim (T - T_c)^{-\nu} \qquad \chi \sim (T - T_c)^{-\gamma}$$

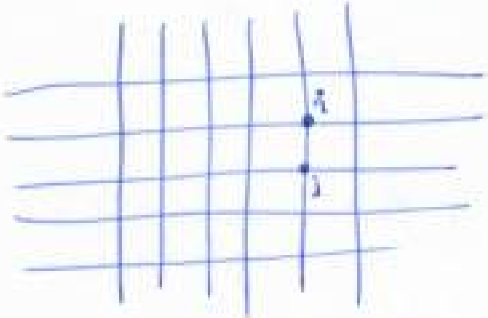
$$M \sim (T_c - T)^{\beta}$$

$$M \sim B^{\delta} \qquad \langle \sigma(x)\, \sigma(0) \rangle = \frac{1}{x^{D-2+\eta}}$$

$$C \sim (T - T_c)^{-\alpha}$$

All the exponents are determined in terms of just two, say (ν, η)

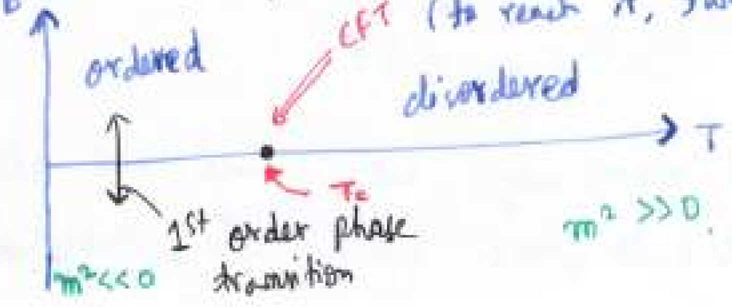
The (ν, η) has interpretation of scaling dimension of relevant operators in C.F.T.



$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \qquad (*)$$

nearest neighbours

This theory has the following phase diagram:



The theory governed by Hamiltonian (*)
has $\mathbb{Z}_2$ symmetry.

In disordered phase : $\mathbb{Z}_2$ is unbroken.
In ordered " : $\mathbb{Z}_2$ is broken.

The Wilson Fisher Lagrangian $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 + \lambda \phi^4$.
also has $\mathbb{Z}_2$ symmetry.

→ The continuum field theory description of the lattice model captures in one shot lots of physics, which require very delicate analysis which partition function of the model led you to do.

We gave the different ~~description of~~ U.V. description of theory that flow to a the Ising CFT, - (by tuning parameters we can go to Ising CFT)

Ising Model $\qquad \lambda\phi^4$

$\downarrow \qquad \downarrow$

Ising CFT

The fact that there are many ways, we could have also found other ways, which lead to same CFT. In statistical mechanics

This is called the idea of Universality.

The universality also appears in experiment.
(for example; you have very different systems like magnet & water have same critical ~~exponents~~ exponents)

Ising CFT has ∞ number of operators ~~(primary)~~ (primary)

The Ising CFT has two relevant operators which control the dynamics σ, ε.

	Δ
σ	0.5
ε	1.5

$\sigma \Rightarrow$ Spin operator. (operator which is the image of local spin)

$\varepsilon \Rightarrow$ Energy (note ε is $\mathbb{Z}_2$ even)

WF	
ϕ	σ
ϕ^2	ε

ϕ^4 is irrelevant.

ϕ^3 is a descendant of ϕ
we write $\phi \supset \phi^3$

Equation of motion $\Box \phi = \phi^3$.

So; The physics of operator ϕ^3 is governed by ϕ.

We will study the Ising CFT.

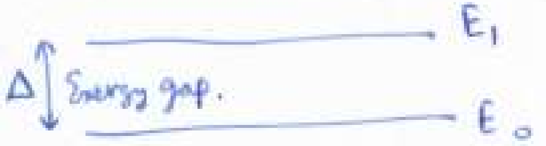
Ising CFT ~~st~~ is like a kind of object which exists intrinsically & irrespectively of U.V- defination

lets find a way to study them directly

Quantum phase transitions at $T = 0$ (driven by $\hbar$ - fluctuations)

Quantum fluctuations.

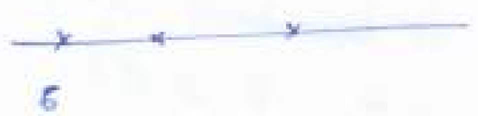
Hamiltonian.



as $\lambda \to \lambda^*$: closes the energy gap ; ie; $\Delta(\lambda^*) = 0$.

So, we have a situation in which a gapped Hamiltonian becomes gapless ; and again the theory does not have any length scale : Therefore it must be described by Conformal field Theory.

ex)



$$H = - J \sum_i \left(\sigma^z_i \, \sigma^z_{i+1} - g \, \sigma_i^x \right) \qquad (\star\star)$$

$\sigma_i^{x,y,z}$ are pauli matrices.

~~thus~~ The spectrum of the theory H $(\star\star)$ has gap ; and the gap closes as $g \to 1$

So; we get CFT as $g \to 1$.

AdS CFT

Gravitational theory

$\int \sqrt{g}\,(R + \Lambda + \dots)$ semi-classical...

All the physics here in AdS$_{D+1}$ is described by CFT$_D$ in one dimension less.

Any question you ask about the bulk, has an answer at the boundary.

String Theory

The dynamics of string is described by 2d CFT.

$\Downarrow$ Conformal invariance

gives rise to

- Dirac Equation
- Yang - Mills equation
- Einstein equation
- ⋮
- ⋮

Lecture 3 | Conformal transformations & Conformal algebra.

In CFTs, the correlation function of any number of local operators $O_i(x)$ are completely determined from 2pt & 3pt functions

$$\langle O_i(x)\, O_j(0) \rangle \qquad \langle O_i(x_1)\, O_j(x_2)\, O_k(x_3) \rangle.$$

The index $i \in \{1, \dots, \infty\}$ for $D \geq 3$

$i \in \{1, \dots, n\}$ for. $D = 2$

← Rational CFT's.

Cookbook recipe for any problem in physics

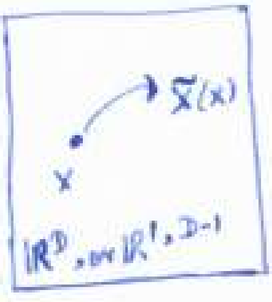
1. Kinematics.
2. Dynamics.

Kinematics of CFT

The geometry of Conformal Transformations.

Transformations

Conformal symmetry is the generalization of the notion of Poincaré Symmetry.



Conformal Transformation $\tilde{x}^\mu = \tilde{x}^\mu(x)$

that

1. Preserves angles between vectors.
2. Preserves the light-cones.

$$ds^2 \longrightarrow e^{2\omega(x)} ds^2$$

$$\eta_{\rho\sigma}\, d\tilde{x}^\rho d\tilde{x}^\sigma = e^{2\omega(x)} \eta_{\mu\nu}\, dx^\mu dx^\nu .$$

$\Updownarrow$

we want to solve for $\tilde{x}$ which obeys the following property $\eta_{\rho\sigma} \cdot \frac{\partial \tilde{x}^\rho}{\partial x^\mu} \cdot \frac{\partial \tilde{x}^\sigma}{\partial x^\nu} = e^{2\omega(x)} \eta_{\mu\nu}$

if $\omega(x) = 0$: It would be transformations which will preserve the metric (Isometries)

Isometries of flat space is Poincaré Transformation.

lets define $R^\rho{}_\mu = \frac{\partial \tilde{x}^\rho}{\partial x^\mu} e^{-\omega(x)}$

$$R^\rho{}_\mu(x) = \frac{\partial \tilde{x}^\rho}{\partial x^\mu} e^{-\omega(x)}$$

<u>what equation does $R^\rho{}_\mu(x)$ obeys?</u>

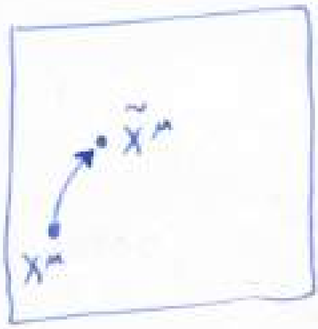
The equation in matrix form is $R^T \eta R = \eta$

which is precisely the Lorentz Transformations.

so; This means that $\frac{\partial \tilde{x}^\rho}{\partial x^\mu}$ can be thought off as local Lorentz transformations X dialatation (combined with dialatation)

Now we focus on Transformations that are connected to identity.

(ie: They map which any point to near by point.)



we linearize it

$$\boxed{\tilde{x}^\mu = x^\mu + \xi^\mu(x)}$$

(This is what it means to be close to identity)

$\xi^\mu(x)$ small

ie; $\tilde{x}^\mu = x^\mu + \xi^\mu(x)$.

We can think of $\xi^\mu(x)$ as vector fields which shifted the coordinates $\xi = \xi^\mu \partial_\mu$

exs $\xi(x^\mu) = \xi^\mu$.

---

by taking determinant ; we see that.

$$\left|\frac{\partial \tilde{x}^\rho}{\partial x^\mu}\right| \equiv \det\left(\frac{\partial \tilde{x}^\rho}{\partial x^\mu}\right)$$

ie; we have

$$\boxed{\left|\frac{\partial \tilde{x}^\rho}{\partial x^\mu}\right|^2 = e^{2D\cdot\omega(x)}}$$

---

lets go back to linearization of Conformal Transformations :

recall $\tilde{x}^\mu = x^\mu + \xi^\mu(x)$.

$$\boxed{\eta_{\sigma\rho}\left(\delta^\rho_\mu + \partial_\mu\xi^\rho\right)\left(\delta^\sigma_\nu + \partial_\nu\xi^\sigma\right) = \left(\eta_{\mu\nu} + \frac{2}{D}\eta_{\mu\nu}(\partial\xi)\right)} \quad (*)$$

Exercise: $\left|\frac{\partial \tilde{x}}{\partial x}\right| = 1 + (\partial\cdot\xi)$

we $\det M = e^{\mathrm{Tr} \log M}$

$$= e^{\mathrm{Tr}(\log(\delta^\rho_\mu + \partial_\mu \xi^\rho))}$$

$$= e^{\mathrm{Tr}(\partial\cdot\xi)} = 1 + \partial\cdot\xi$$

working in linear order in ξ.

Comparing terms order by order in equation:

$$\eta_{\rho\sigma}\cdot(\delta^\rho_\mu + \partial_\mu \xi^\rho)\cdot(\partial^\sigma_\nu + \partial_\nu \xi^\sigma) = \left(\eta_{\mu\nu} + \frac{2}{D}\eta_{\mu\nu}(\partial\cdot\xi)\right)$$

$(\xi)^0$: $\eta_{\mu\nu} = \eta_{\mu\nu}$ ✓ trivially true

$(\xi)^1$: $\boxed{\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{D}\eta_{\mu\nu}\ \partial\cdot\xi}$ $(**)$ The solution of these equations are those; whose all solutions parametrize all conformal transformation that are closed to identity.

if $\partial\cdot\xi = 0$ then equation $(**)$ will be like Killing Vector equation of minkowski space whose solutions are the Poincaré Transformation.

ie; if $\partial\cdot\xi = 0$ Then we get Killing Vectors $\Rightarrow$ Poicaré

ie; $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0$

$$\underset{\substack{\uparrow \\ \text{Parametizes} \\ \text{translations.}}}{\xi^\mu = a^\mu} + \underset{\substack{\uparrow \\ \text{Parametrizes Lorentz} \\ \text{transformations.}}}{\omega^\mu{}_\nu X^\nu} \quad : \omega_{\mu\nu} = -\omega_{\nu\mu}.$$

from $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0$ we could ~~e~~ easily derive $\boxed{\partial_\mu \partial_\nu \xi^\rho = 0}$.

Similar analysis of $\boxed{\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{D} \eta_{\mu\nu} \partial \cdot \xi}$

Conformal Killing Vector

$$\Rightarrow \boxed{\partial_\mu \partial_\nu \partial_\rho \xi^\sigma = 0}$$

So, we learn: The space of infinetisimal conformal transformations is finite dimensional; because it can be only as large as space of quadratic polynomials in coordinates; which is of course finite dimensional.

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for $D=2$; $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \eta_{\mu\nu} \partial \cdot \xi$

use Complex Coordinates: $z = x^1 + i x^2$
$\bar{z} = x^1 - i x^2$

in these coordinates; the metric of Euclidean Space is just $ds^2 = d\bar{z}\, dz$.

so; In components; $\eta_{\bar{\mu}\bar{\nu}} \propto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

so; we get $\boxed{\bar{\partial}_z \xi_{\bar{z}} = 0}$ $\Rightarrow$ The most general solution is Holomorphic functions.

ie; $\xi_{\bar{z}}(z)$; it should not be function of $\bar{z}$. pg17

Instead of $\bar{\partial}_z \xi_{\bar{z}} = 0$ its better if we write $\partial_{\bar{z}} \xi_{\bar{z}} = 0$ just for the sake of good notations.

So, the most general infinitisimal conformal transformation is parametrized by the following vector fields. **(D > 2)**

$$\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu b\cdot x$$

P_μ (Translation)

$M_{\mu\nu}$ (Lorentz Transformation)

D (Dilatations)

K_μ (Special Conformal Transformations)

When you have transformation, we can associate generator to transformation, that has the same number of indices as the parameter of the transformation.

How many transformations do we get from:

a^μ : D

$\omega^\mu{}_\nu$: $\frac{D(D-1)}{2}$

D : 1

b^μ : $\frac{(D+1)(D+2)}{2}$

"Usually transformations that we can do simultaneously generate a group."

So, the set of transformation here will also generate a group : group which has as many parameters as those of transformations.

- Transformations form a group.

We can associate to group of transformations close to the identity a Lie Algebra.

$$g(\alpha)\, g(\bar{\alpha}) = g\left(f(\alpha, \bar{\alpha})\right)$$

: $g(\alpha)$ are group elements.

α^s are constant parameters

if we look at transformations close to identity :

① $g(\alpha) = 1 + i\alpha^a t_a + \frac{1}{2}\alpha^a \alpha^b t_{ab} + \ldots$

② $f(\alpha, 0) = \alpha$

③ $f(0, \bar{\alpha}) = \bar{\alpha}$

Expanding at the quadratic order we find that t_a and t_b obeys the following equation (which is essentially the defination of Lie Algebra)

$$\boxed{[t_a, t_b] = i f_{ab}{}^{c} t_c}$$

and $f_{ab}{}^{c}$ satisfy Jacobi Identity:

Continuous group of transformations close to identity generate a Lie Algebra.

The Lie Algebra, in a sense contains all the information which are close to identity.

Determine Lie Algebra generated by Conformal Transformation.

$$[\xi_1, \xi_2] = \xi_3$$ Lie bracket between vector fields

$$\xi_1 = \xi^\mu_1 \partial_\mu \qquad \xi_2 = \xi^\mu_2 \partial_\mu$$

associate $\xi_{P^\mu} = a^\mu \partial_\mu$: we can associate to each parameter of transformation a vector field.

$$\xi_{M^{\mu\nu}} = \omega^\mu{}_\nu x^\nu \partial_\mu$$

$\hookrightarrow$ parametrizes direction of transformation.

$$\xi_D = \lambda x^\mu \partial_\mu$$

$$\xi_{K^\mu} = (b^\mu x^2 - 2x^\mu b\cdot x)\,\partial_\mu \quad \text{(check)}$$

To compute Lie Algebra; compute the Lie Bracket of the above vector fields.

So, we get the following commutation relations.

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\,\eta^{\mu\rho} M^{\nu\sigma} + \ldots$$ $\hookrightarrow$ fixed by anti-symmetry of $M^{\mu\nu}$.

$$[M^{\mu\nu}, P^\rho] = i\,\eta^{\mu\rho} P^\nu + \ldots$$ $\hookrightarrow$ terms : get by antisymmetrizing μ & ν

Now, Translations commute: $[P^{\mu}, P^{\nu}] = 0$

$[M^{\mu\nu}, K^{\rho}] = i\eta^{\mu\rho}K^{\nu} + \dots$

K^{ρ} has an index; this means that it transforms as a vector to the Lorentz Group.

So: It should transform the same way as P^{ρ} does.

from the point of ~~of~~ Lorentz Group, P^{ρ} & K^{ρ} are just Lorentz vectors.

So: $[M^{\mu\nu}, K^{\rho}] = i\eta^{\mu\rho}K^{\nu} + \dots$

Same as for P^{ρ}.

The hero of conformal field Theory is D.

The dilation operator plays the role in CFT to what the Hamiltonian plays the role in non-relativistic Q.M.

The eigenvalues of D tells us the critical exponent of the theory.

$$[D, P_{\mu}] = -iP_{\mu}$$

$$[D, K_{\mu}] = iK_{\mu}$$

This is like harmonic oscillator a & a^{+}.

P_{μ} in the right convention will be the raising operator; so if you have an operator & you act P_{μ}: it will increase the dimension by 1 (fits your intuition: P_{μ} is derivative & so it has dimension 1)

K_μ will lower the dimension.

$$[P^\mu, K^\nu] = -2i\,(M^{\mu\nu} + \eta^{\mu\nu} D)$$

$[D, M^{\mu\nu}] = 0$ (D commutes with $M^{\mu\nu}$, because D is lorentz scalar; & lorentz scalar commutes with lorentz transformations by defination)

$$\boxed{[M^{\mu\nu}, D] = 0}$$

$\mu \in \{0, \ldots, D-1\}$.

$M = \mu, D, D+1$ (Made our matrix bigger by two entries)

And now, lets make the following defination.

$$L_{MN} = \begin{cases} L_{\mu\nu} = \hat{M}_{\mu\nu} \\ L_{D,D+1} = \hat{D} \\ L_{\mu D} = \frac{1}{2}(\hat{P}_\mu + \hat{K}_\mu) \\ L_{\mu, D+1} = \frac{1}{2}(\hat{P}_\mu - \hat{K}_\mu) \end{cases}$$

With these definations, L_{MN} obeys the following Lie Algebra

$$[L_{MN}, L_{PQ}] = i\,\eta^{MP} L^{NQ} + \ldots$$

where $\eta_{MN} = (\underbrace{- + + + \ldots +}_{\text{lorentz signature}} \; \overset{\to D}{+} \; \overset{\to D+1}{-})$

↑ metric in Higher dimensional space

$[L_{MN}, L_{PQ}] = i\eta^{MP} L^{NQ} + \ldots\ldots$

↳ Lorentz Algebra ....

This Lie Algebra is ~~call~~ called $SO(2, D)$

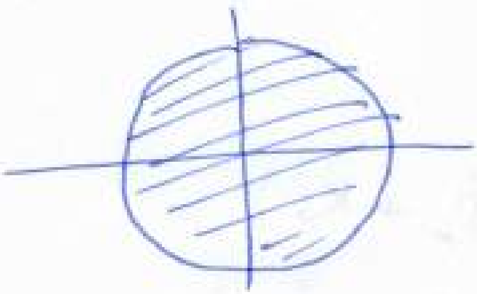
$\boxed{\text{c.f. Lorentz } SO(1, D-1)}$

~~co~~ c.f. = ... compare with.

★ The conformal Algebra in D dimensions is Lorentz Algebra $SO(2, D)$!

Discrete Conformal Transformations

Inversion $I : \; x^\mu \longrightarrow \dfrac{x^\mu}{x^2}$.



changes the inside & outside the unit ball

I is not close to ~~to~~ identity.

1) $\mathbb{Z}_2$ transformation (if act twice, we get back the orignal thing)

2) Changes the orientation of spacetime.

Conformal transformations can be obtained by combining Poincaré + I.

$$\left.\begin{array}{l} I P_\mu I = K_\mu \\ I M_{\mu\nu} I = M_{\mu\nu} \end{array}\right\} \rightarrow$$ starting with this we can get everything..

$$I K_\mu I = P_\mu$$

$$I D I = -D$$

I

Lec 4 : Energy momentum tensor, Primary Operators, Correlation functions

Infinitisimal (∞'al) conformal transformations implemented by vector fields.

$$\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu b\cdot x$$

P_μ $\quad M_{\mu\nu}$ $\quad D$ $\quad K_\mu$

$$\rightarrow \begin{cases} SO(2,D) & \mathbb{R}^{1,D-1} \\ SO(1,D) & \mathbb{R}^D \end{cases}$$

Conformal transformations generated by Poincaré $\times$ **I**

$I: x^\mu \longrightarrow \frac{x^\mu}{x^2}$ (Inversion) $\quad$ e.g. $\; I P_\mu I = K_\mu$.

(CFT's ~ Anti-de-Sitter space) Isometries of AdS_{D+1} is $SO(2,D)$

So, we see ; At the level of kinematics only that, The Hilbert space of Quantum ~~Gravity~~ in AdS_{D+1} will have to furnish the representation of $SO(2,D)$ which is precisely the same symmetry of CFT in one dimension less, i.e; CFT_D

$$CFT_D \sim AdS_{D+1}$$

$$x^\mu \xrightarrow{I} \frac{x^\mu}{x^2} \xrightarrow{P_\mu} \frac{x^\mu}{x^2} + b^\mu \xrightarrow{I} \frac{\frac{x^\mu}{x^2}+b^\mu}{\left(\frac{x^\mu}{x^2}+b^\mu\right)^2} = \frac{x^\mu + b^\mu x^2}{(1+b^\mu x_\mu)^2}$$

$P_\mu I$ $\qquad$ $I P_\mu I$

This is finite transformation.

for ∞'sal, we get precisely K_μ.

Quantum Field Theory

Operator that appears in any QFT.

In any Poincaré invariant QFT, $\exists$ a local operator that implement Poincaré transformations.

$T_{\mu\nu}$ Energy Momentum Tensor.

Any local QFT has Energy Momentum Tensor $T_{\mu\nu}$
Any poincaré invariant QFT has Symmetric Energy Momentum Tensor ; $T_{\mu\nu} = T_{\nu\mu}$

1) $T_{\mu\nu} = T_{\nu\mu}$: Symmetric

2) Conserved $\partial^\mu T_{\mu\nu} = 0$ (Noether's Theorem)

$T_{\mu\nu}$ is the Noether Current associated to translations.

$$\delta S = \int d^D x \ \frac{1}{2} T_{\mu\nu} \, \delta h^{\mu\nu}$$ An arbitrary metric variation.

$$= \int d^D x \ \frac{1}{2} T_{\mu\nu} \left(\partial^\mu \xi^\nu + \partial^\nu \xi^\mu \right)$$ Transformation of metric under infinitesimal diffeomorphism.

true for arbitrary diffeomorphism

$$\sim \int d^D x \cdot \frac{1}{D} \cdot T_{\mu\nu} \left(\eta^{\mu\nu} \partial \cdot \xi \right)$$ use Conformal Killing Vector equation. $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{D} \eta_{\mu\nu} \partial \cdot \xi$.

$$= \int d^D x \ \left(T_{\mu\nu} \eta^{\mu\nu} \right) \partial \cdot \xi$$

Trace of the Energy Momentum Tensor. $T^\mu{}_\mu$

True for diffeomorphism which implement Conformal Transformation for Conformal $\delta S \sim \int d^D x \, (T_\mu{}^\mu) \, \partial \cdot \xi$

What should $T_\mu{}^\mu$ obey?

☆ Dilation Symmetry $\xi^\mu = \lambda x^\mu$ $\therefore \partial \cdot \xi = \lambda =$ constant

If conformal transformation is symmetry of the theory;

Then action; more precisely the partition function is invariant under Conformal transformation.

so; $\delta S \sim \int d^D x \, (T^\mu{}_\mu) \, \partial \cdot \xi$ has to vanish.

$\delta S = 0$: What is most general solution.

It is $T^\mu{}_\mu = \partial_\mu L^\mu$

Then $\delta S = 0$

so; Conformal Invariant Theory has the constraint : $\underline{T^\mu{}_\mu = \partial_\mu L^\mu}$

(i.e; Trace of Energy Momentum Tensor is total derivative)

☆ Special Conformal Transformation

$\partial \cdot \xi \sim x$

So, now ; $T^\mu{}_\mu = \partial_\mu \partial_\nu L^{\mu\nu}$

(second derivative; because one of the derivative can be used to kill x comming from $\partial \cdot \xi$, In exercises we showed

$$\boxed{T^{\mu}{}_{\mu} = \partial_\mu \partial_\nu L^{\mu\nu}} \Longleftrightarrow \boxed{\hat{T}^{\mu}_{\mu} = 0}$$

Noether : continuous symmetry results in conserved current $\partial^\mu J_\mu = 0$

But there is no the current.

We can always modify the current, and define new current which will still be conserved.

We can construct $\hat{J}^\mu$ which is also conserved $\partial^\mu \hat{J}_\mu = 0$

$Q = \int_\Sigma dx\, J_0$: Charge Associated to current.

$$\boxed{\hat{Q} = \int_\Sigma dx\, \hat{J}_0 = Q}$$

$$\hat{J}_\mu = J_\mu + \partial^\nu M_{\mu\nu}$$

Improvement transformations

for $\hat{J}_\mu$ has to be conserved:

$M_{\mu\nu}$ has to be antisymmetric

Then $\hat{J}_\mu$ will be automatically be conserved by virtue of J_μ being conserved.

Operators in CFT

Operators are labelled
- Lorentz Transformation (spin)
- Dilatation operator (Scaling dimension) Δ

Operators in CFT are

1) Primary : (transforms as Tensors)

2) Descendants : (all properties are fixed by primaries)

We care about 1) Primary now because of its importance.

- P_μ is a raising operator for D. eigenvalue : +1
- K_μ is a lowering operator for D. eigenvalue : -1.

$\mathcal{O}_\Delta$: writting operator $\mathcal{O}$ with subscript Δ, to manifestly show its scaling dimension in the notations.

We can also write index for Lorentz Quantum Number.

$\mathcal{O}_{A,\Delta}$

Δ → measuring dimension.

$$\hat{D}\,(P_\mu \mathcal{O}_{A,D}) = (\Delta+1) P_\mu \mathcal{O}_{A,\Delta}$$

$$\hat{D}\,(K_\mu \mathcal{O}_{A,\Delta}) = (\Delta-1) K_\mu \mathcal{O}_{A,\Delta}$$

P_μ is raising operator : $\hat{D}(P_\mu \mathcal{O}_{A,\Delta}) = (\Delta+1)\, P_\mu \mathcal{O}_{A,\Delta}$

K_μ is lowering " : $\hat{D}(K_\mu \mathcal{O}_{A,\Delta}) = (\Delta-1)\, K_\mu \mathcal{O}_{A,\Delta}$

<u>What is Primary?</u>

Primary operators are those which are annihilated by K_μ.

If $\mathcal{O}$ is Primary Operator,

then $K_\mu \mathcal{O} = 0$

Conformal Representation

$K_\mu \mathcal{O} = 0$: $\mathcal{O}$ primary

$$\left.\begin{array}{l} P_\mu \mathcal{O} \\ P_{\mu_1} P_{\mu_2} \mathcal{O} \\ \vdots \end{array}\right\} \text{Descendants.}$$

Primary Operators

do a transformation $x^\mu \longrightarrow \tilde{x}^\mu = (gx)^\mu$ conformal transformation.

$$\mathcal{O}_A(x) \longrightarrow \tilde{\mathcal{O}}_A(x) = \left|\frac{\partial x}{\partial \tilde{x}}\right|^{\Delta/D} L_A{}^B(R)\, \mathcal{O}_B(g^{-1}x)$$

(not writing Δ explicitly for simplicity)

"spin part of transformation which acts on internal d.o.f."

" part which acts on spacetime of the field, call it Orbital Part"

$$R^\mu{}_\nu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \cdot e^{-\omega} \quad ; \; R^T \eta R = \eta$$

Usually, the transformation of spin & orbital part is in opposite direction.

$$\tilde{\mathcal{O}}_A(x) = \left|\frac{\partial x}{\partial \tilde{x}}\right|^{\Delta/D} \cdot L_A{}^B(R) \cdot \mathcal{O}_B(g^{-1}x)$$

$\hookrightarrow$ we can have density ...

if we do $g_1 g_2$

then $L(g_1 g_2) = L(g_1)\, L(g_2)$.

This is just the statement that,

L is representation of the group.

<u>Scale Transformation</u>

$x^\mu \to \lambda x^\mu$; Scalar operator (for simplicity) (has no spin index)

i.e. $x^\mu \to \tilde{x}^\mu = \lambda x^\mu$

Then $\underline{\tilde{\mathcal{O}}(x) = \lambda^{-\Delta}\, \mathcal{O}(\lambda^{-1} x)}$

Here we see the thing for which we always had intuition about ; is that ; These are operators in the theory that transforms homogeneously with some weight under scale transformation.

The operator is scaled by some factor which depends on dimension of operator.

Now, look at formula for infinitesimal transformation.

$\tilde{x}^\mu = x^\mu + \xi^\mu$

$$\delta \mathcal{O}_A(x) = \tilde{\mathcal{O}}_A(x) - \mathcal{O}_A(x)$$

$$= -\xi^\mu \partial_\mu \mathcal{O}_A + \frac{i}{2}\Omega_{\mu\nu}(x)\, (M^{\mu\nu}_{r})_A{}^B\, \mathcal{O}_B(x) - \Delta w(x)\, \mathcal{O}_A(x)$$

"orbital" "Lorentz spin" "dilatation"

$$\begin{cases} \Omega_{\mu\nu} = \omega_{\mu\nu} - 2(x_\mu b_\nu - x_\nu b_\mu) \\ \omega(x) = \lambda - 2\, b\cdot x \end{cases}$$

correction due to Special conformal transformation.

$(M^{\mu\nu}{}_{r})_{A}{}^{B}$

r is for representation of Lorentz.

$$\delta\mathcal{O}_A(x) = -\xi^\mu \partial_\mu \mathcal{O}_A + \frac{i}{2}\Omega_{\mu\nu}(x)(M^{\mu\nu}{}_{r})_{A}{}^{B}\mathcal{O}_B(x) - \Delta\cdot\omega(x)\cdot\mathcal{O}_A(x)$$

$[\xi_{(1)}, \xi_{(2)}] = \xi_{(3)}$. This determines Lie Algebra.

At level of operator we can take

$$[\underbrace{\delta_{\xi_{(1)}}}_{\substack{\text{Transformation}\\ \text{parametrized}\\ \text{by } \xi_{(1)}}}, \underbrace{\delta_{\xi_{(2)}}}_{\substack{\text{Transformation}\\ \text{parametrized}\\ \text{by } \xi_{(2)}}}]\mathcal{O}_A = \delta_{\xi_{(3)}}\mathcal{O}_A$$

This is just the statement that, this infinitesimal transformation represents the infinitesimal Lorentz Conformal Algebra.

Transformation of Descendant

e.g. $\partial_\mu \mathcal{O}_A$

—— $\mathcal{O}_A$

—— $\partial_\mu \mathcal{O}_A$

$\mathcal{O}_A$ generates $\partial_\mu \mathcal{O}_A$

— $\mathcal{O}_A$

— $\partial_\mu \mathcal{O}_A$

— $\partial_\mu \partial_\nu \mathcal{O}_A$

When you look at transformation properties of Descendants; it can mix Level.

e.g. $\delta(\partial_\mu \mathcal{O}_A) \supset \underbrace{b_\mu . \mathcal{O}_A}$

↳ we also get this term.

So, The conformal transformation of $\partial_\mu \mathcal{O}_A$ is not just proportional to $\partial_\mu \mathcal{O}_A$; but also to the guy at top; i.e; $\mathcal{O}_A$.

~~So, they are not tensors ...~~

We can ask; What are constraints of conformal invariance of correlators of primaries.

$\boxed{\text{e.g.}}$ $\langle \mathcal{O}_A(x) \mathcal{O}_B(y) \rangle$

Two point functions.

$$\langle \mathcal{O}_A(x) \mathcal{O}_B(y) \rangle = f(|x-y|)$$

Some function of $|x-y|$.

convince yourself that $\langle \mathcal{O}_A(x) \rangle = 0$ unless $\mathcal{O}_A = \mathbb{1}$

for Lorentz Scalars

Now, we have to ask what constraint D imposes on f.

We know under dilation things transform like

know $\tilde{\mathcal{O}}(x) = \lambda^{-\Delta} \mathcal{O}(\lambda^{-1} x)$

This tells us that f has to be homogeneous function of $|x-y|$ with weight that will have scaling dimension of $\mathcal{O}_A$ & $\mathcal{O}_B$.

Dilatations tells us that

$$f = \frac{1}{|x-y|^{\Delta_A + \Delta_B}}$$

$$\boxed{f(|x-y|) = \frac{1}{|x-y|^{\Delta_A+\Delta_B}}}$$

we would get $\lambda^{-(\Delta_A+\Delta_B)}$ on LHS.

Now we can ask; what special conformal or inversion gives you.

- Conformal Invariance

Special Conformal (I) tells us that

$$\boxed{\langle \mathcal{O}_A(x)\, \mathcal{O}_B(y)\rangle = \frac{\delta_{\Delta_A, \Delta_B}}{|x-y|^{2\Delta_A}}}$$

for Lorentz Scalars.

$\delta\langle \mathcal{O}_A\, \mathcal{O}_B\rangle = 0$ under symmetry.

$\langle \delta\mathcal{O}_A\, \mathcal{O}_B\rangle + \langle \mathcal{O}_A\, \delta\mathcal{O}_B\rangle = 0$

This gives differential equation for f which has unique solution.

<u>example)</u> we can take δ_D

ie; $\langle \delta_D \mathcal{O}_A \mathcal{O}_B\rangle + \langle \mathcal{O}_A\, \delta_D \mathcal{O}_B\rangle = 0$

and will see that the solution

is $f = \dfrac{1}{|x-y|^{\Delta_A+\Delta_B}}$

So, we see that,

The two point function has one very important data of CFT: The collection of all Δ_A's

This is an infinite list $\{\Delta_A\}$

The scaling dimensions of operators $\{\Delta_A\}$ in CFT.

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3 point function

$$\langle \mathcal{O}_A(x_1)\, \mathcal{O}_B(x_2)\, \mathcal{O}_C(x_3) \rangle = \frac{C_{ABC}}{(x_{12}^2)^{\frac{1}{2}(\Delta_3-\Delta_1-\Delta_2)} \cdot (x_{13}^2)^{\frac{1}{2}(\Delta_2-\Delta_1-\Delta_3)} \cdot (x_{23}^2)^{\frac{1}{2}(\Delta_1 \dots)}}$$

completely fixed

writing it clearly.

$$\langle \mathcal{O}_A(x_1)\, \mathcal{O}_B(x_2)\, \mathcal{O}_C(x_3) \rangle = \frac{C_{ABC}}{(x_{12}^2)^{\frac{1}{2}(\Delta_3-\Delta_1-\Delta_2)} (x_{13}^2)^{\frac{1}{2}(\Delta_2-\Delta_1-\Delta_3)} \cdot (x_{23}^2)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_3)}}$$

- The spacetime dependence is completely fixed.
- The powers depend on dimension.

☆ There is new data, C_{ABC}.

Data of CFT

- $\{\Delta_A\}$ 2pt.
- $\{C_{ABC}\}$ 3pt

$\Rightarrow$ This data completely fixes all the correlation functions.

$(x_1 \quad x_2 \quad x_3)$

$\downarrow$

$(x_1' \quad x_2' \quad \infty$ can send it to infinity)

(at with symmetry: can bring x_3 to origin; and then we do inversion: it then goes to infinity)

so: $(x_1 \,,\, x_2 \,,\, x_3)$

$\downarrow$

$(x_1' \,,\, x_2' \,,\, \infty)$

$\downarrow$ finite translation

$(0 \,,\, x_2'' \,,\, \infty)$

finite translation will not move ∞ to finite point.

so; $(0 \,,\, x_2'' \,,\, \infty)$ is a D dimensional vector; has D components.

now; we want to do transformations which can bring x_2'' to place we like; we have to do it in a way so that we dont touch 0 & ∞.

we rotate.

Rotation M_{1j} $\quad j = 2, \ldots, D$

we can bring x_2'' to the canonical for

$x''_2 \rightarrow (a, 0, 0, \ldots, 0)$

Now we have succeded by collection of transformations to bring (x_1, x_2, x_3) three arbitrary points to the point of type $(0, (a, 0, .., 0), \infty)$

We can do Dilatation, and can canonically bring it to $(0, (1, 0, .., 0), \infty)$

This tells us that conformal invariance completely fixes the form of three point function because ~~thou~~ we can always bring any three points to a canonical form $(0, (1, 0, .., 0), \infty)$.

So, whatever you write for 3pt, is completely fixed by symmetry.

$\hookrightarrow$ This will stop being true at 4pt functions.

QED & Renormalization Shoaib Akhtar 31/7/2020 (Pg1)

Lec 1:

Topics: ① One particle irreducible (1PI) diagrams
② Superficial degree of divergence.
③ Dimensional regularization of QED.
④ Renormalization using counter terms with minimal substraction prescription.
⑤ Application of β function of QED.
⑥ QCD β-function.

1PI or Proper Vertices (These are sub class of n ~~pt~~ point functions)

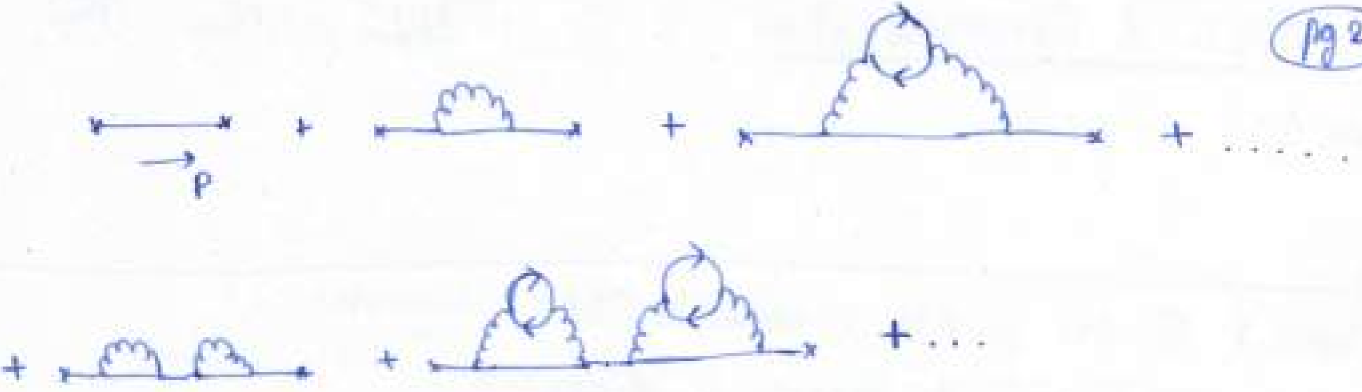
$\Gamma_c^n (x_1, \dots, x_n)$

If we have feynman diagram representation of these n-point functions; then there are sub class of these n point functions in which if we cut an internal line (an internal feynman propagator); Then the diagram does not fall apart in two parts: These are 1PI diagrams.

example  1PI

Not example 

cutting like this; Then diagram falls into two pieces.



+ .....

This is summation of all the 1 P I at each order

Define $\frac{\Sigma(p)}{i} = S_{F_0}^{-1}(p)$  $S_{F_0}^{-1}(p)$

inverse of free propagator (have applied to the series)

F corresponds to Feynman propagator.

$S_F^0(p)$: is free Feynman propagator.

Can represent $\frac{\Sigma(p)}{i}$ as 

without any external legs attached to it.

$\frac{\Sigma(p)}{i} =$ 

external legs removed

Then ;

This is like giving the legs back

$$S_F(p) = S_F^0(p) + S_F^0(p)\frac{\Sigma(p)}{i}S_F^0(p) + S_F^0(p)\frac{\Sigma(p)}{i}S_F^0(p)\frac{\Sigma(p)}{i}S_F^0(p) + \dots$$

note;

$S_F(p) =$ 

Note that $S_F(p)$ is formally a geometric series

(may not be in true sense because $\Sigma(p)$ are divergent)

and

$\hookrightarrow$ There will be divergent when we regularize them.

Then, we get

$$S_F(p) = S_F^{\circ}(p)\left[1 - \frac{\Sigma(p)}{i} S_F^{\circ}(p)\right]^{-1}$$

Now we plug the value of free propagator.

Exact Propagator

$$S_F(p) = \frac{i}{\not{p} - m - \Sigma(p) + i0^+}$$

The ϵ prescription is written as 0^+

(because the symbol ϵ will be used later for something else)

we know that, the free propagator

$$S_F^{\circ}(p) = \frac{i}{\not{p} - m + i0^+}$$

$$= \frac{i(\not{p} + m)}{(p^2 - m^2 + i0^+)}$$

The Exact Propagator

$$S_F(p) = \frac{i}{\not{p} - m - \Sigma(p) + i0^+}$$

m is the parameter which entered our lagrangian ; but according to Callen Lehman representation ; we know that pole of the propagator is at the physical mass.

So, the physical mass is given by.

$$m_{physical} = m + \Sigma(p)$$

The analysis motivates to call $\Sigma(p)$ as Electron Self Energy.

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2 point 1PI : $\Gamma^{(2)}(p)$ in the momentum space

Is the inverse of the 2 point connected Green Function (which is also the Propagator)

ie; $S_F(p)\,\Gamma^{(2)}(p) = i$

$$\Rightarrow \boxed{\Gamma^{(2)}(p) = \not{p} - m - \Sigma(p)}$$

$G_c^{n}(x_1, \dots, x_n)$ was generated by W.

Effective Action: generates the n points 1PI diagrams.

$\Gamma[\varphi] = W[J] + \int dx\, J(x)\varphi(x)$ (Switching to Scalar Theory)

(here dx stands for d^4x)

$$\Gamma^{(2)}(x,y) = \frac{\delta^2 \Gamma[\varphi]}{\delta\varphi(x)\,\delta\varphi(y)}$$

Now we will show that $\Gamma^{(2)}(p)$ is fourier transform of $\Gamma^{(2)}(x,y)$

$$\varphi(x) = \frac{\delta W[J]}{\delta J(x)}$$

$$J(x) = \frac{\delta \Gamma[\varphi]}{\delta\varphi(x)}$$

$$G_c^{(2)}(x,y) = G_c^{(2)}(x-y) = -i\,\frac{\delta^2 W[J]}{\delta J(x)\,\delta J(y)}$$

Connected 2pt function

because of Poincare invariance; its a function of x-y.

$$= -i\,\frac{\delta \varphi(x)}{\delta J(y)}$$

$$\Gamma^{(2)}(x-y) = \frac{\delta^2 \Gamma[\varphi]}{\delta\varphi(x)\,\delta\varphi(y)} = \frac{\delta J(y)}{\delta\varphi(x)}$$

$$\int G_c^{(2)}(x-y)\,\Gamma^{(2)}(y-z)\,dy = -i\,\delta(x-z)$$

This is the real space version of $S_F(p)\,\Gamma^{(2)}(p) = i$

$$\frac{\delta}{\delta J(x)} = \int dy\,\frac{\delta\varphi(y)}{\delta J(x)}\,\frac{\delta}{\delta\varphi(y)} = i\int dy\,G_c^{(2)}(x-y)\,\frac{\delta}{\delta\varphi(y)}$$

$$\Rightarrow \boxed{\frac{\delta}{\delta J(x)} = i\int dy\,G_c^{(2)}(x-y)\,\frac{\delta}{\delta\varphi(y)}}$$

$$\Rightarrow \int G_c^{(2)}(x-y)\,\Gamma^{(2)}(y-z) = -i\,\delta(x-z)$$

do the functional derivative.

$$\int \frac{\delta^3 W[J]}{\delta J(x')\,\delta J(x)\,\delta J(y)}\,\Gamma^{(2)}(y-z)\,dy + \int G_c^{(2)}(x-y)\cdot G_c^{(2)}(x'-y')\,\frac{\delta^3\Gamma[\varphi]}{\delta\varphi(y')\,\delta\varphi(y)\,\delta\varphi(z)} = 0$$

ie: $$\int \frac{\delta^3 W[J]}{\delta J(x')\,\delta J(x)\,\delta J(y)}\cdot\Gamma^{(2)}(y-z)\,dy + \int G_c^{(2)}(x-y)\cdot G_c^{(2)}(x'-y')\cdot\frac{\delta^3\Gamma[\varphi]}{\delta\varphi(y')\,\delta\varphi(y)\,\delta\varphi(z)} = 0$$

Multiply by $\int G_c^{(2)}(x'-z)\,dz$

Then we get

$$-i\int \frac{\delta^3 W[J]}{\delta J(x')\,\delta J(x)\,\delta J(y)}\cdot\delta(y-y'')\,dy = -i\int dy\,dy'\,dz\cdot G_c^{(2)}(x-y)\,G_c^{(2)}(x''-y')\cdot G_c^{(2)}(x'-z)\cdot\frac{\delta^3\Gamma[\varphi]}{\delta\varphi(y)\,\delta\varphi(y')\,\delta\varphi(z)}$$

$$\Rightarrow -i\int \frac{\delta^3 W[J]}{\delta J(x')\,\delta J(x)\,\delta J(y)}\cdot\delta(y-y'')\,dy = -i\int dy\,dy'\,dz\, G_c^{(2)}(x-y)\,G_c^{(2)}(x'-y')\,G_c^{(2)}(x''-z)\cdot\frac{\delta^3\Gamma[\varphi]}{\delta\varphi(y)\,\delta\varphi(y')\,\delta\varphi(z)}$$

$$\Rightarrow \boxed{\frac{\delta^3 W[J]}{\delta J(x)\,\delta J(x')\,\delta J(x'')} = \int dy\,dy'\,dy''\cdot G_c^{(2)}(x-y)\,G_c^{(2)}(x'-y')\,G_c^{(2)}(x''-y'')\cdot\frac{\delta^3\Gamma[\varphi]}{\delta\varphi(y)\,\delta\varphi(y')\,\delta\varphi(y'')}}$$

Graphically we can represent it as follows.

- L.H.S. is 3 point connected greens function.
- RHS is something with three 2 point greens function attached with leg

We can invert the equation by multiplying with $\Gamma^{(2)}$ & integrate over arguments,

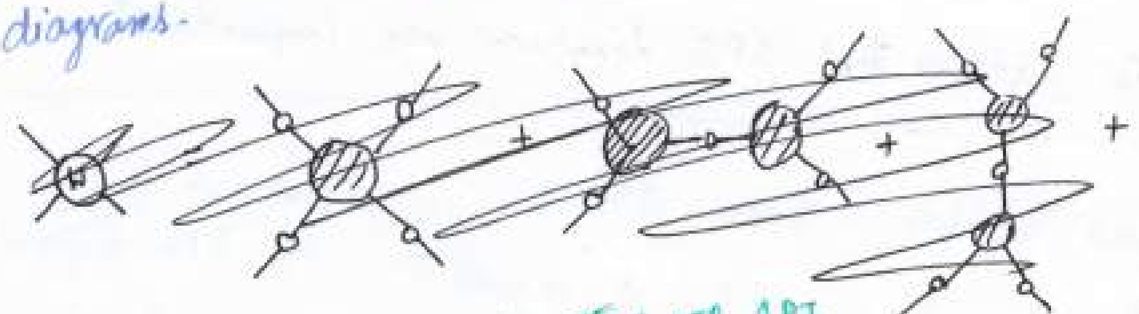
we get

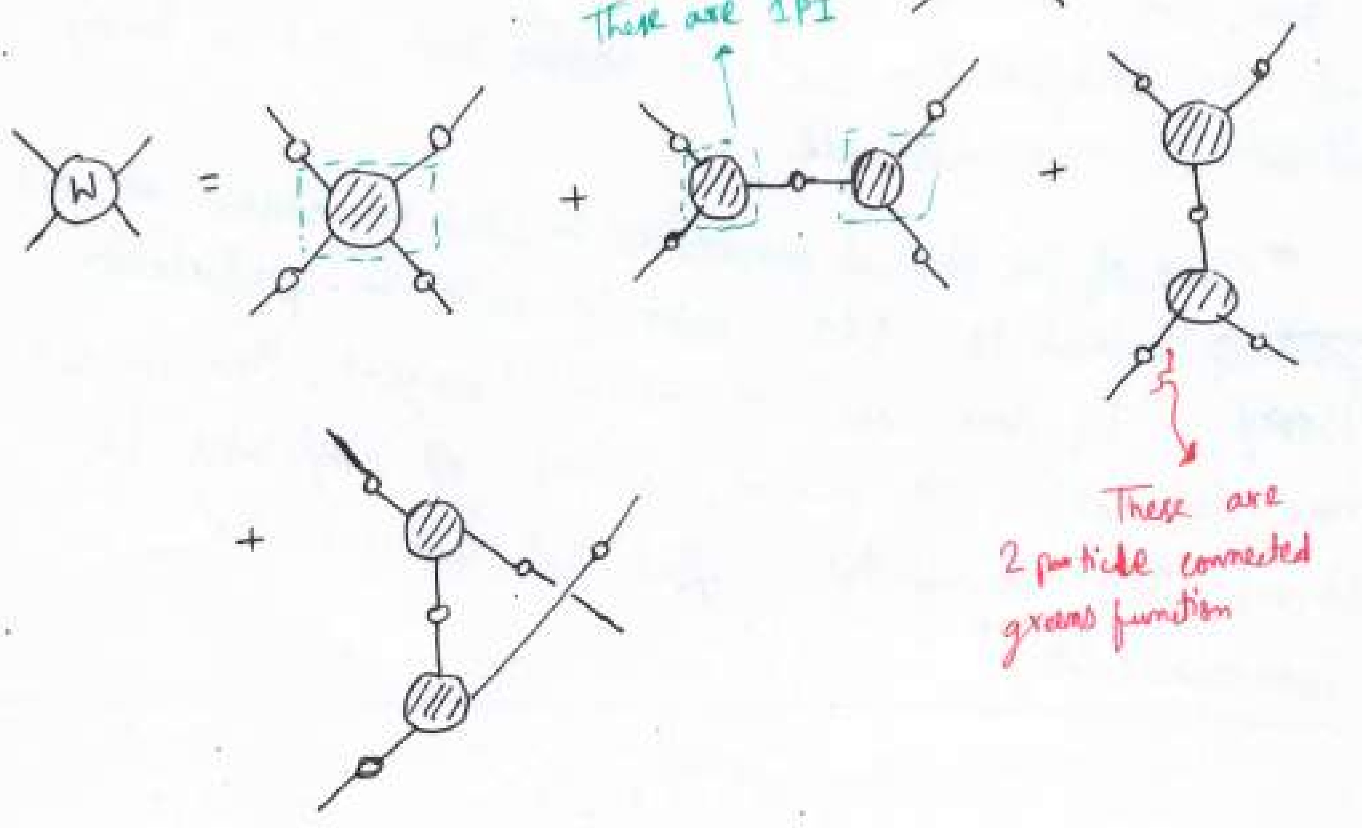
$$\frac{\delta^3\Gamma[\varphi]}{\delta\varphi(x_1)\,\delta\varphi(x_2)\,\delta\varphi(x_3)} = -\int dy_1\,dy_2\,dy_3\,\Gamma^{(2)}(x_1-y_1)\,\Gamma^{(2)}(x_2-y_2)\,\Gamma^{(2)}(x_3-y_3)\cdot G_c^{(3)}(y_1,y_2,y_3)$$

$$\boxed{\frac{\delta^3\Gamma[\varphi]}{\delta\varphi(x_1)\,\delta\varphi(x_2)\,\delta\varphi(x_3)} = -\int dy_1\,dy_2\,dy_3\,\Gamma^{(2)}(x_1-y_1)\,\Gamma^{(2)}(x_2-y_2)\,\Gamma^{(2)}(x_3-y_3)\cdot G_c^{(3)}(y_1,y_2,y_3)}$$

These algebra convinces that:
" The connected greens function & the 3 point 1PI diagram are the same object"

We can take derivative of this equation; And then can express 4 particle connected greens function, In terms of 1PI diagrams.





These analysis suggests that,:

"Whatever Correlation function we look at, we can always organize it in terms of 1PI of that order & lower order"

When we have divergences in QFT; They will be concentrated on some of these 1PI diagrams.

→ If ~~yo~~ we can fix the divergences, or regularize or renormalizes the divergences of 1PI that will ensure that our Connected Greens functions are not ~~stan~~ diverging

→ This justifies that 1PI diagrams are important

If we find that; for a given QFT, ~~(there are ∞ no. of 1PI diagrams~~

there are ∞ no. of 1PI diagrams which are divergent then we can ensure that such a theory will not be renormalizable.

" Some of the physical parameters in which we absorb ∞ are ~~[illegible]~~ related to 1PI order by order in perturbation theory. If there are ∞ no. of divergent, then we will have to redefine ∞ no. of scattering ~~[illegible]~~ amplitudes to absorb these ~~∞~~ infinities. And that theory is not renormalizable"

Divergent diagrams ~~in QED~~

We can forget higher loops, and do only tree level diagrams. But, we dont get good agreement experiments.

→ The differences between tree level prediction & observation is due to higher loops.

We organize the discussion on Divergent Diagrams according to a quantity called Superficial degree of diagrams.

D ⇒ Superficial degree of diagrams.

→ Its called superficial because lot of time ~~diverges~~ the kind of divergences with D will predict is not actually the divergence we get when we examine the diagrams very carefully

→ But, it is a quick way to see whether the theory is renormalizable or not in Perturbation theory.

In d dimensions;

If there are L loops in a diagram [it will give d powers of momentum integral (because each momentum integral will have $\int dp^{\mu}$)] → So, if we have L loops, then L independent momentum integrals to go

~~D = dL~~ $d \cdot L$

→ The more momentum integrals we have; the more divergent our diagram will be.