String Theory Review

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Lec 1: General considerations on String Theory (String perturbation theory, D-branes, dualities, ...)

String Perturbation theory describes perturbatively the dynamics of string.

When doing calculation,
Start with free ~~answer~~ answer (which we get in free theory; where strings dont interact)

and then write formal power series which corrects free answer order by order.

$$? = A_{string} = A_{Free} + gA_1 + g^2 A_2 + \dots$$

$$\boxed{A_{QFT} = A^{QFT}_{free} + gA_1^{QFT} + g^2 A_2^{QFT} + \dots}$$

... we can also think of path integral

$$\left.\int D\phi \cdot e^{iS_{free} + igS_{interaction}}\right\} \Rightarrow$$ QFT has Non perturbative defination.

QFT has non-perturbative defination.

In string theory we dont have clue, how to give general defination of non-perturbative theory.
We only perturbative expansion + Extra info.

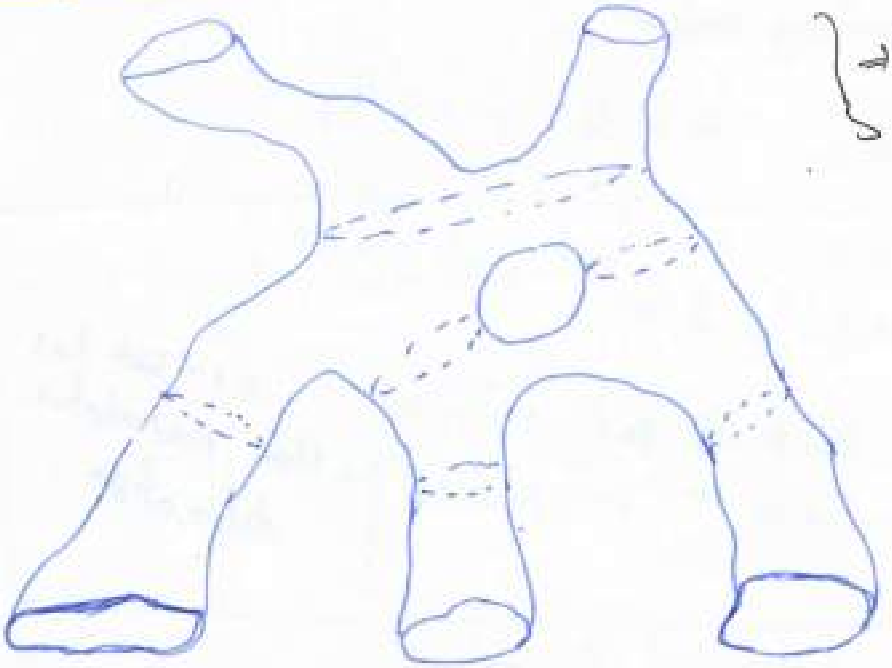
Perturbative expansion is just formal ...
can approach Astring..

$$\left.~~A_{string}~~\ A_{free} + g_s A_1 + g^2 A_2 + \dots\right\} \Rightarrow$$ This is not full defination of a theory.

QFT

Typical perturbative processes in QFT

String Theory (basic particles replaced by string)



→ We will try to make sense of this in perturbative expansion.

We can in ~~...~~ principle ~~...~~ can add interaction (where special things happens) ~~...~~ ... but till now no body knows to do it in good way.

→ And its also not needed.

The nice property of string is that they come with natural interaction ... We dont have to add interaction vertex ... We just require strings to ~~...~~ smoothly join, and then set out again.

Strings comes with a dimensionful quantity (which is analogous to mass of relativistic particle).

The tension gives us an energy scale.

l_s^{-1} $\qquad$ $l_s \Rightarrow$ string length.

This scales determines the ~~—~~ energy scale of internal vibration of modes of string.

~~various~~

A string being an extended object has lots of internal degrees of freedom. It can vibrate in various ways.

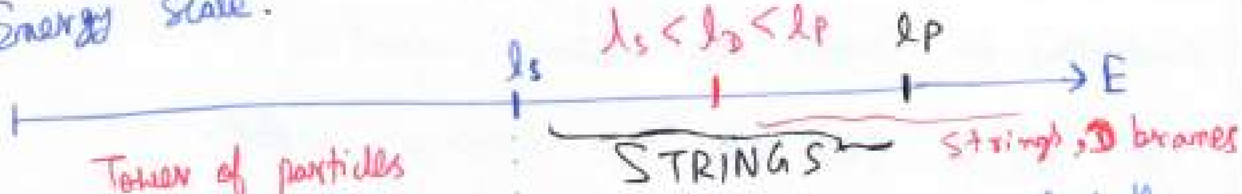
We can think of vibrational modes; as whole bunch of Harmonic Oscillators (one for each mode): And the energy of this Harmonic Oscillator is controlled by this Energy scale.

If we look at strings; at energies much lower than the string scale: very few oscillators are excited.

$\Longleftrightarrow$ The energy looks like a point like object (with small amount of vibrational object)

(This is true in general for any Extended Object)

Energy scale.



Tower of particles
(like modes)

String Theory looks like QFT
(The lowest levels of this tower, the lightest particle in this tower are actually massless : (Although there is Scale))

we also have Gravitons in these tower of particles.

(we really see strings ; And the behaviour is different from that of particles)

↳ we dont get UV divergences

(one of the basic consequence of having discrete interaction points in QFT ; is that when these interaction points comes together we get divergences)

↳ because strings are not point like and interact in diffused way ... It has much better UV behavior

String Perturbation Theory is free of UV divergences.

(U.V. divergences are cutt off by string length)

(String Theory is a way to cure perturbative problems of Quantum Gravity)

~~Gravity~~ Gravity + QFT fields breaks down at planck scale

(because near planck scale ; things diverge very badly)

We hear that there are troubles with treating Gravity as QFT

↳ There is no problem with treating gravity as an Effective low energy QFT perturbatively.

(problems start when we try to make it non-perturbative)

For each parameter we chose in our theory (String Theory); there is a massless excitations which describe's dynamical fluctions in that parameter.

↳ Its nice... It tells that whatever we are trying to define is quite unique. We dont have variety of perturbative string ~~d~~ theory. There is just one.

↳ (we might hope that this unicity keeps holding to the underlying Theory) here is the dream that Quantum Gravity is so hard, that there is probably just unique way. ~~to find the~~

Now, we might feel that everything is under-control. Here we have theory which is so rigid & unique & includes gravity (couple with spacetime; then we will have massless excitations which describes dynamical fluctuation of that parameter ⇒ Hence we get dynamical theory of gravity)

but:

The Problem is: All of the choices became dynamical: the dynamics is important

↪ The other lesson we get is that the universe could look very very different depending on the dynamical processes. The same underlying set of laws might give us completely different low energy dynamics.

(And which low energy dynamics you get; depends on the dynamics, ~~the~~ initial conditions, boundary conditions, etc.)

⟶ In string theory; the laws of nature which we see at low energy will just be determined by some dynamical mechanics.

String perturbation theory allows us ~~to~~ countless different ways to realize ~~[illegible]~~ semagalistic universes within it. And gives very little guidence of what non-perturbative dynamics could actually select when universe is describe by a string theory;

and to which degree is it matter of initial Condition, or matter of intrinsic property of the theory.

→ ie: Even unique underlying theory; might not gives us unique ~~~~ set of low energy laws of nature.

Understanding non-perturbative defination of string theory might help. ~~(It's still a hard~~

The two major problems in String Theory:

(i) Understanding if there is non perturbative defination.

(ii) Try to figure out what kind of universe will such a ~~universe~~ theory give; And how much the low energy universe depends on Boundary Conditions, or whatever that means.

Holography gives a non-perturbative defination of String theory in some grounds (which dont look like our universe; but look like universes of negative cosmological constant). Amusingly, we get different non ~~pertub~~ perturbative defination of theory depending on what is large scale structure of the Universe.

There is not much freedom to modify the way closed strings interact & behave.

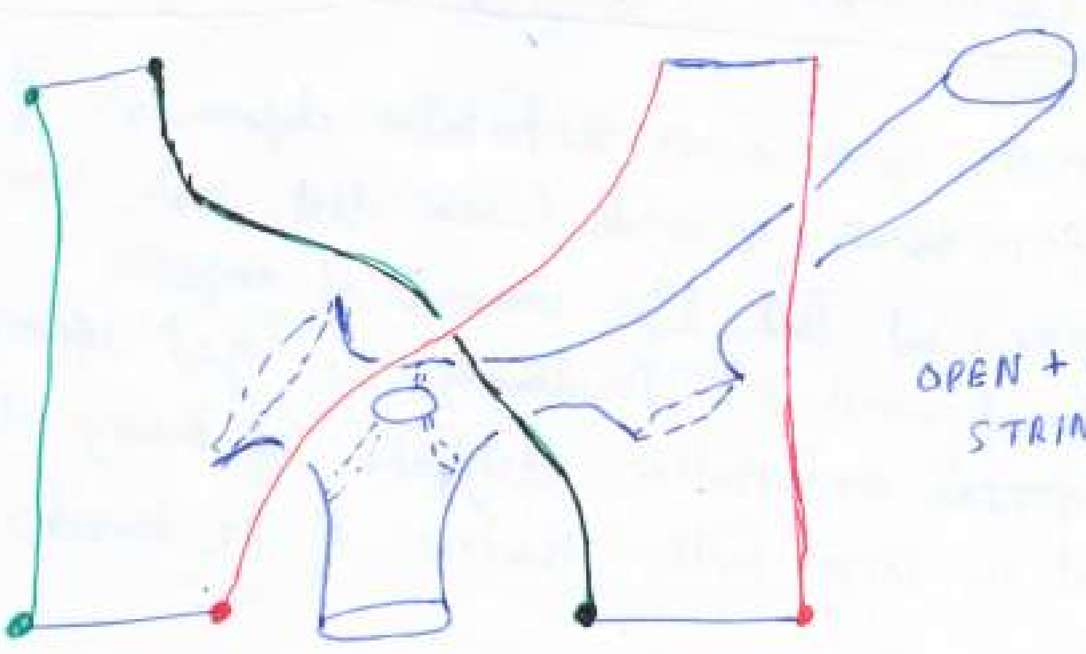
* We can ~~do~~ also discuss oriented strings.

We can also allow for open strings.

The same way we put different particles in QFT; now we get choices in String Theory.

We can put choices at the end points of strings which is analogues to choosing different ~~particles in~~ ~~QFT~~ particles in QFT.

These evolves, and boundary propagate out



OPEN + CLOSED STRINGS

- Endpoint is free in spacetime.
- Endpoint is constrained to lie in some ~~line~~ line in spacetime
- Endpoint is constrained to lie on some surface

} ex example

The colors corresponds to different rules how the end points can behave.

For every pair of such rules, we get some tower of open string modes

(String Theory makes every choice dynamical)

for example;

~~to~~ We will find a massless open string which describes fluctuations of this line

..... ~~these~~ these objects (or rule) become dynamical objects call D-brane.

Now; since these are dynamical objects; they can even appear & disappear from nothing. We can create it & destroy it by Quantum Processes.

D-Branes are unavoidable ingredients of string theory.

because

(we can find strong hints; that orignal theory of closed strings secretely had to have D-branes)

We realize that; String Theory is not exactly the theory of strings. All really what happened was: whatever was the underlying Theory, the lightest in these regime of parameters l_s^{-1} where string theory is perturbative;

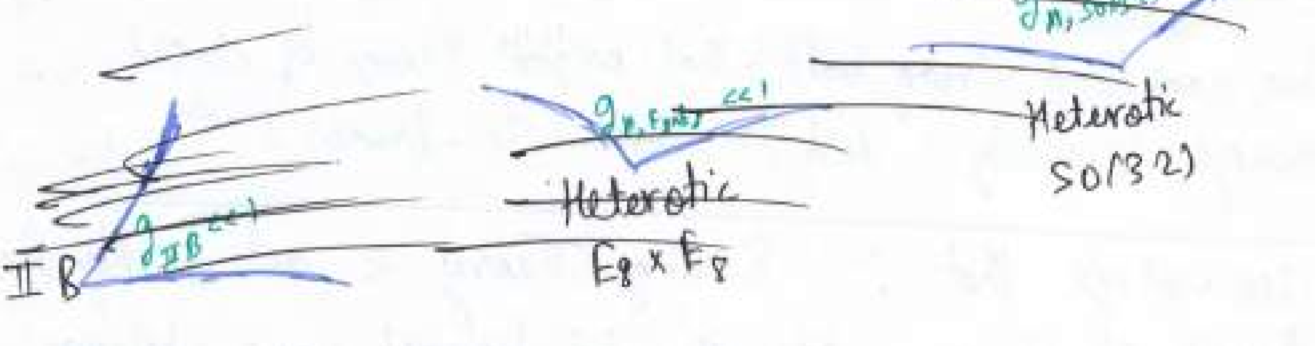
The lightest objects are strings. There are other ~~...~~ heavier objects. Therefore in some magical way, the dynamics of the lighter objects seems to be automatically producing the heavier objects. 1910

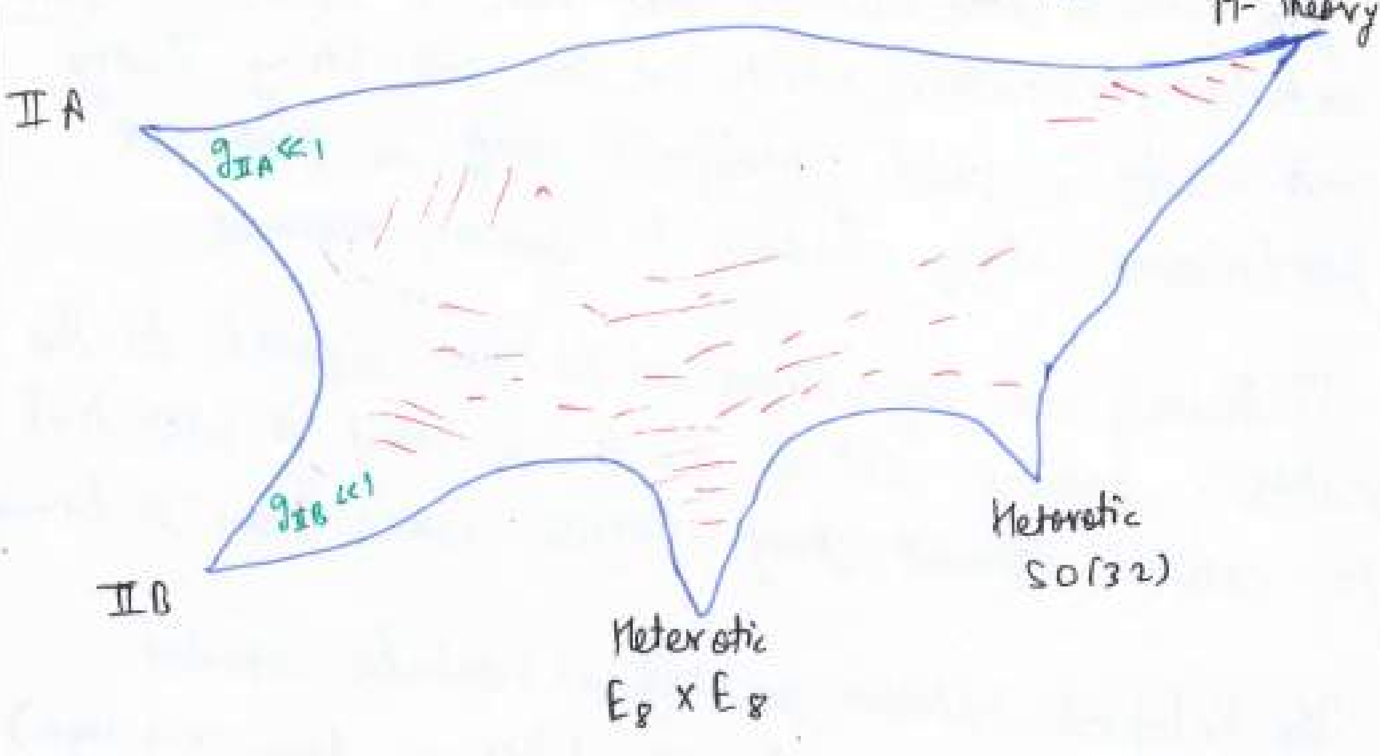
When we start defining superstring theory; we will define following types of string theory.

Each of the string theories comes with its own dynamical ~~parameter~~ parameter.

↳ It turns out that there are some quantities which can be computed exactly as the function of parameter. They just dont recieve any perturbative corrections.

If we assume for the moment that there is an underlying theory behind our perturbation theory; we can use this perturbative quantities to study the properties of the theory.





as you go inside; increase the coupling.
↳ as you go inside; it starts looking similar to other string theories.

As you start filling; mapping the space of possible configurations by looking at perturbative quantities, they join in.

And we also find completely different regions which are not String Theories, like M-Theory.

As far as these perturbative quantities are concerned, looks sensible and can be also reached from some other theories.
The space of dynamical configurations which can be probed by this perturbative quantities seems to be just connected. ⟹ This is why we talk about String Theory (rather than different types of String Theories)

This gives ~~a~~ some evidence that there is some underlying structure which we can call String Theory and study ; which manifests itself in these ~~per~~ perturbative String Theories in various regimes.

D-Branes are very important to give support to this feature ; because lot of d.o.f we need to keep track to compare different String Theories comes from D-Branes.

The Relations between ~~the~~ different weakely coupled string Theories are called Dualities. (very rich topic)

(indeed we learnt lot of dualities in QFT starting from dualities in String Theory)

~~Suppm~~ Supersymmetry it is a symmetry which relates Fermions & Bosons in our theory.

* Supersymmetry is not a necessary property of String Theory, but helps ~~in~~ a lot in studying the theory.

~~Suppme~~ Supersymmetry in general helps in studying theories because : symmetries between Bosons & Fermions sometimes allow to cancel Quantum Effects (Essentially, Bosons & Fermions gives opposite contribution when they run in loops)

$\Longleftrightarrow$ in some questions, Supersymmetry gaurantees that these contributions ~~cancel~~ cancels out.

In Supersymmetric theories we often have some degree of protection from Quantum Effects for certain quantities.

Example,

We said coupling of String Theory is dynamical.
In absence of Supersymmetry, the value of coupling is dynamical, and it will go somewhere depending on dynamics and typically does not remain small.

So; we try to define non-supersymmetric String Theory, we just have perturbative definative; if ~~you~~ our perturbative dynamics tries ~~to~~ make the coupling larger & larger: We lack consistency at ~~basic~~ 's basic level.

In supersymmetric String Theory, we have some control which allows to say that it is consistent to think about that we can couple theories. (because at least in certain configurations, we can get on to that coupling will not run, and not dynamically change)

In ~~su~~ supersymmetric String theories; the fact that there are cancellation between bosons & fermions is part of ~~the~~ the mechanism which cancels out some of the UV divergences.

Supersymmetry is useful property to put on top of String Theory. It appears naturally in ~~Strings~~ Theories which we know how to define. It is not a necessary property of String Theory

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String Theory is a theory of Gravity.

Back reaction of D-Brane $\sim \underset{\text{coupling}}{g}$

N D-Branes Back reaction $\sim Ng$

Imagine looking at a regieme where we put lot of D-branes, ie; ~~large N~~ large N

$\Longleftrightarrow$ This should give real backreaction to background.

We can study CLOSED Strings + Open strings with N D branes

$\updownarrow$ These two describe the same process or situation

or study

Closed Strings in Back reacted geometry.

(very roughly, we can think of putting together lot of D-Branes to make a Black Hole. Now we have a Black Hole geometry; and we try to study perturbative closed strings in BH ~~geomet~~ geometry)

$\longrightarrow$ There must be some equivalent between these two.

This is called <u>OPEN-CLOSED Duality</u>.

Closed + Open Strings N D-Branes $\xleftrightarrow{\text{Open-Closed Duality}}$ Closed strings in Back reacted Geometry.

Closed + Open Strings N D-Branes $\downarrow$ Decoupling limit

Closed strings in Back reacted Geometry $\downarrow$

QFT of Massless open Strings of supersymmetric variance of U(N) Gauge theory. $\xleftrightarrow{\text{HOLOGRAPHY}}$ Closed Strings in AdS ~~background~~ background.

⇒ doing perturbative expansion we get that closed strings in AdS.

This is defined perturbatively... so definition from this equivalence becomes

Holography has derived from open-closed duality, but holds on its own.

There are variety of QFT which has chance of having Holographic dual. Each of those Holographic QFT's will give us some theory of Quantum Gravity in negatively curved spacetime.

<u>Open Question</u>: Can we really define a theory of Quantum Gravity independent of large structure of Universe.

Lec 2: String perturbation theory : Nambu - Goto action, Polyakov action, Gauge fixing.

lets call this worldsheet Σ

QFT

we get graph, world line interacts

Perturbative process in QFT

In QFT we know how to compute contribution of a diagram Γ for scattering amplitudes : A_Γ^{QFT}

We also compute contribution to scattering amplitude due to the surface Σ : A_Σ^{STRING}

$$A_\Sigma = \int_{\mu[\Sigma]} d\mu \int_{X:\Sigma \to \mathbb{R}^{d-1,1}} \mathcal{D}X \, e^{-S_{STRING}(X,\mu)}$$

Integral over extra structures (It is finite dimensional integral)

We have some extra data μ which we have to ~~stud~~ specify. It somehow incodes the intrinsic geometry of this surface

In order to motivate this expression, we will write completely analogous expression for QFT.

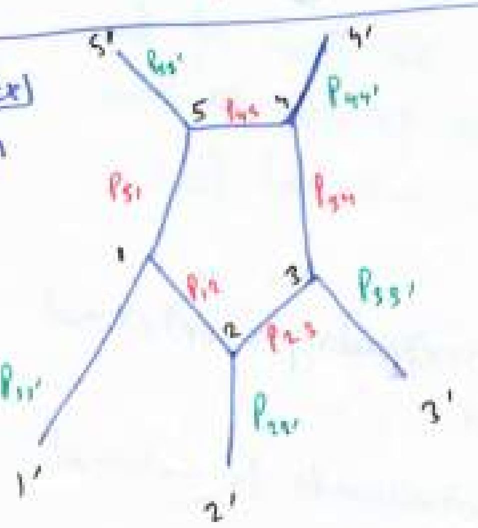
In a scalar QFT,
we can re-write our standard Feynman diagram amplitude as

$$A_\Gamma = \int dl \int_{x:\Gamma \to \mathbb{R}^{d-1,1}} Dx \; e^{S_{\text{Particle}}(x,l)}$$

$\int dl$ → finite dimensional integral

l → some parameter (like length of world lines)

Ex)
Γ



Schematically, if the integral is over internal momenta of bunch of δ-functions for vertices, and propagators

* here b is running indices ; over external points like 5', 4', 3', 2', 1' (here)

$$A_\Gamma = \int \prod_{(a,b)\,\in\,\text{Internal line}} dP_{ab} \prod_{a\,\in\,\text{Internal Vertices}} \delta\Big(\sum_b P_{ab}\Big) \prod_{(a,b)\,\in\,\text{Internal line}} \frac{1}{P_{ab}^2 + m^2}$$

upto factors of coupling constants. ~~upto coupling~~

Step 1)

$$\frac{1}{P_{ab}^2 + m^2} = \int_0^\infty dl_{ab} \cdot e^{-l_{ab} \cdot P_{ab}^2 - l_{ab} m^2}$$

Take propagators and rewrite them in proper time formalism.

Can also take δ functions & rewrite them as integral over spacetime of, position's of interaction points

$$\cancel{\delta\left(\sum_b P_{ab}\right) = \int dx} \quad \delta\left(\sum_b P_{ab}\right) = \int dx_a^d \cdot e^{i x_a \cdot \sum_b P_{ab}}$$

(Ignoring factors of 2π)

Then, we have

$$A_\Gamma = \prod_{a \in \text{Internal}} \int dx_a^D \cdot e^{i x_a \sum_{b \in \text{External}} P_{ab}} \cdot \prod_{(a,b) \in \text{Internal}} \int_0^\infty dl_{ab} \cdot e^{-\frac{(x_a - x_b)^2}{4 l_{ab}} - l_{ab} \cdot m^2}$$

We can think of this as integral over proper ~~tim~~ time lengths over internal legs

$I \Rightarrow$ Set corresponds to internal Indices

$E \Rightarrow$ Set corresponds to internal Indices.

$(I, I) \Rightarrow$ Set of internal links.

Then we can write.

$$A_\Gamma = \prod_{a \in I} \underbrace{\int dx_a^D}_{\text{integral over position of vertices}} \cdot e^{i x_a \sum_{b \in E} P_{ab}} \cdot \prod_{(a,b) \in (I,I)} \underbrace{\int_0^\infty dl_{ab}}_{\text{integral over proper time lengths of internal lines.}} \cdot e^{-\frac{(x_a - x_b)^2}{4 l_{ab}} - l_{ab} \cdot m^2}$$

define $G(x_a, b) = \int_0^\infty dl_{ab}\, e^{-\frac{(x_a - x_b)^2}{4 l_{ab}} - l_{ab} m^2}$ (pg 19)

We want to write this part as a path integral over different positions of each individual point of the edges.

lets write down action for relativistic particle.

$$S[\underline{x}] = m\int ds = m\int_0^1 \sqrt{\frac{dx_a}{du}\frac{dx_a}{du}}\, du$$

choose some parameter

$$x(u) : [0,1] \longrightarrow \mathbb{R}^{d-1,1}$$

Trajectory as a map from interval to Spacetime.

The choice of the parametrization u is completely arbitrary. We can see it from the fact that action has gauge invariance.

$u = f(v)$, Action does not change.

Quantising it

... path integral over all trajectories.

(to do this, we have to devise a way for gauge symmetry ... otherwise we will overcount trajectories a lot)

$u = f(v) \in \mathrm{DIFF}$

(Diffeomorphism)

$\frac{DX(u)}{\text{Diff}}$ ($DX(u)$ modulo Diffeomorphism)

$$\int_{X(0)=X_a}^{X(1)=X_b} \frac{DX(u)}{\text{Diff}} \cdot e^{-m\int_0^1 \sqrt{\cdots}\, du}$$

$X(u) : [0,1] \longrightarrow \mathbb{R}^{d-1,1}$

→ This path integral is hard to do because the action is not quadratic ... its square root.

$u = f(v) \in \text{Diff}$

→ So,

We introduce new degree of freedom, and write the following action.

$$S[x,e] = \frac{1}{2}\int_0^1 [e^{-1}\dot{x}^2 + m^2 e]\, du$$

This action is still diffeomorphic invariant as long as this e behaves properly, i.e; $e(v)\, dv = e(u')\, du'$

Under diffeomorphism,

$$\boxed{\begin{aligned} X(u) &\longrightarrow X(f(u)) \\ e(u) &\longrightarrow e(f(u))\frac{df}{du} \end{aligned}}$$

This is our gauge transformations now.

Note; this ~~actual~~ action is quadratic in x, but not polynomial in e.

We can gauge fix e away

We can pick f such that $\frac{df(u)}{du} = \frac{1}{e}$

1 d.o.f in e
1 d.o.f in gauge transformation
→ So we can get rid of most of e.

We cannot get rid of **all** of e,
in the sense that $\int_0^1 e(u)du$ is gauge invariant

~~so~~ lets call it $L = \int_0^1 e(u)\, du$

We can always pick a diffeomorphism that allows to get rid of $e(u)$ almost completely except for this constant mode.

We can always Gauge Fix ; e to be a constant ~~a~~
i.e. $e = L$ (constant)

And so the Gauge fixed action is now very simple

$$S^{\text{Gauge-Fixed}} = \frac{1}{2}\int_0^1 \left[\frac{\dot{x}^2}{L} + m^2 L \right] du$$

This still depends on L, but it is constant not a function of u.

So, replace $\displaystyle\int_{x(0)=x_a}^{x(1)=x_b} \frac{Dx(u)}{\text{Diff}}\, e^{-m\int_0^1 \sqrt{\dot{x}^2}\,du}$

by $\displaystyle\int_0^\infty dl \int Dx \cdot e^{\int_0^1 \frac{1}{2}\left[\frac{\dot{x}^2}{L} + m^2 L\right] du}$

$$\int_{x(0)=x_a}^{x(1)=x_b} \frac{Dx(u)}{\text{Diff}}\, e^{-m\int_0^1 \sqrt{\dot{x}^2}\,du} \longrightarrow \int_0^\infty dL \int Dx \cdot e^{\frac{1}{2}\int_0^1 \left[\frac{\dot{x}^2}{L} + m^2 L\right] du}$$

This is more like a defination (*)

$$\int \frac{Dx\, De}{\text{Diff}} \cdot e^{-S[x,e]}$$

gauge fixed.

(*) we just dont know how to make sense of $e^{-m\int_0^1 \sqrt{\dot{x}^2}\,du}$

$$\int_0^\infty dL \underbrace{\int_{x(0)=x_a}^{x(1)=x_b} Dx \cdot e^{\frac{1}{2}\int_0^1 \left[\dot{x}^2/L + m^2 L\right] du}}_{\text{This is exactly } e^{-\frac{(x_a-x_b)^2}{4 l_{ab}} - l_{ab} m^2}}$$

So:
$$G(x_a, b) = \int_0^\infty dl_{ab} \cdot \int_{x(0)=x_a}^{x(1)=x_b} Dx \cdot e^{-\frac{1}{2}\int \left[\frac{\dot{x}^2}{l_{ab}} + m^2 l_{ab}\right] du}$$

$\rightsquigarrow$ This justifies our expression for string.

Action for String

$$S[X] = T \iint_{\Sigma} \sqrt{\det_{(a,b)}\left(\frac{dX^\mu}{dx^a}\frac{dX_\mu}{dx^b}\right)}\, du^1 du^2$$

Nambu-Gotta Action

a constant analogous to mass for point particles.

pick up local parametrization of world sheet & integrate over.

Choose the local parametrization to be u^1, and u^2

i.e. then $X = X(u^1, u^2)$

This $S[X]$ is again diffeomorphism invariant.

$$u^1 \to f^1(u^1, u^2)$$
$$u^2 \to f^2(u^1, u^2)$$

$\longrightarrow$ These are exactly what we need to ~~join~~ join different patches on the surface

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We want to try to make sense of

$$\int \frac{DX}{\text{DIFF}} \cdot e^{-S[X]}$$

again, we have some issue ; because Action is not quadratic.

$X(u^a)$, $h_{ab}(u^a)$ $\longrightarrow$ add these new degrees of freedom : metric on the word sheet.

Then, we can write the Polyakov Action

$X(u^a)$

$h_{ab}(u^a)$

$$S[X, h_{ab}] = \frac{T}{2}\int \sqrt{h}\cdot h^{ab}\frac{\partial X^\mu}{\partial u^a}\frac{\partial X_\mu}{\partial u^b}\cdot du^1 du^2$$

If we integrate away the metric in the Polyakov Action, we get back Nambu Goto action.

↳ Equation of motion for metric fix it to be pull back of the spacetime metric, induced due to embedding of surface in spacetime.

As an analogy with particle case;
we should now try to Gauge fix the metric away using diffeomorphism symmetric
(because the action is quadratic in the X, (the map);
but not quadratic in metric)

A metric in 2d has 3 components. }
Diffeomorphism has 2 components }
$\Rightarrow$ So, we cannot use diffeomorphism alone to make our metric trivial (or say flat)

↳ Its quite clear, 2d metric can have curvature.
Curvature is diffeomorphism invariant notion.
So, we clearly cant get rid of curvature by coordinate transformations.

The best we can do using diffeomorphism ~~...~~ is to map our metric to be locally conformally flat. ie; $\underbrace{h_{ab} = e^{\phi}\delta_{ab}}$,

↳ This can only be done locally (not globally)

Something nice happens.

If we take the action, and plug in something proportional to flat metric, say $e^{\phi}\delta_{ab}$

Then $S[X, e^{\phi}\delta_{ab}] = \frac{T}{2}\int \frac{\partial X^{\mu}}{\partial u^{a}} \frac{\partial X_{\mu}}{\partial u_{a}}\, du^1 du^2$

↳ it does not depend on conformal factor.

$\boxed{\text{... we will reduce the problem to free problem... where we only need to solve bunch of laplacians}}$

This tell us that;

we were trying to study $\int \frac{DX\, Dh}{Diff}$ but it turns out that the action is actually independent of one extra degree of freedom.

So; The Action is invariant under another transformations, called. Weyl Transformation

: $h_{ab} \longrightarrow e^{\lambda} \cdot h_{ab}$

So, Polyakov Action is invariant under

- Diffeomorphism
- Weyl Symmetry.

So; we should better do this $\int \frac{DX\, Dh}{Diff \cdot Weyl}\, e^{-S[X, h]}$

This is good defination.

Once we have Weyl Symmetry, along with diffeomorphism we can choose the metric to be exactly flat.

Diff x Weyl ...(fix λ to get rid of ϕ in $e^{\phi}\delta_{ab}$)

When we gauge fix, we need to remember constraints.
We have to impose equation of motion for metric.
We only need to impose them as sort of initial condition; and they remain true during evolution (but they do need to be imposed)

Even classically, we still need to impose

$$\frac{\delta S}{\delta h_{ab}} = 0 \quad : \quad \text{CONSTRAINT}$$

proportional to stress energy tensor.

$$\frac{\partial X^\mu}{\partial u^a}\frac{\partial X_\mu}{\partial u_b} - \frac{\delta_{ab}}{2}\left(\frac{\partial X^\mu}{\partial u^c}\frac{\partial X_\mu}{\partial u_c}\right) = 0 .$$

Now; we have to find analogue of lengths l_{ab}.
Which gauge invariant information did we forget when we said that we can get rid of metric completely.

Intuitively we know that we have to preserve angles. So, our gauge invariant information must know something about angles on the surface

$\longleftrightarrow$ To do better, we need to characterize which sort of gauge freedom we have left after we do the transformations.

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locally we can surely say that our metric is flat δ_{ab}

$u_1' = u_1'(u_1, u_2)$
$u_2' = u_2'(u_1, u_2)$ } doing this transformation our flat metric is mapped to

$$\delta_{ab} \longmapsto \delta_{ab} \cdot \frac{\partial u^c}{\partial u'_a} \cdot \frac{\partial u^d}{\partial u'_b}$$

$$= e^{\lambda} \cdot \delta_{ab}$$

we want this to be proportional to flat metric.

→ We need to characterize what are coordinate transformations which map a flat metric to the conformally flat metric

⟹ These are called Conformal Killing Vectors.

In dimension ≥ 2, these are rare.

In dimension $= 2$, ~~they~~ there are lot of them.

Use complex coordinates to find these transformations.

$z = u_1 + i u_2$

Then our flat metric looks $ds^2 = dz\, d\bar{z}$

∴ now doing holomorphic coordinate transformation

$$z = z(z')$$

Then $ds^2 = dz\,d\bar{z} = \frac{dz}{dz'} \cdot \frac{d\bar{z}}{d\bar{z}'} \cdot dz'\,d\bar{z}'$

$$= \underbrace{\left|\frac{dz}{dz'}\right|^2}_{\text{real number.}} dz'\,d\bar{z}'$$

So; The transformations $z = z(z')$ definitely map a flat metric to conformally flat metric

$\hookrightarrow$ And it turns out; this is all ~~there~~ that there is.

So; if we replace two real coordinates with one complex coordinates, Then we discover that our coordinate transformations must be holomorphic $z' = z'(z)$

By doing local gauge fixing, and comparing our different local gauge fixing we have ended up covering up our surface with an atlas of complex coordinates with complex coordinate transformations ... This has complex structure ... Its a complex manifold of complex dimension 1.

The complex structure on the surface is ~~a~~ the gauge invariant quantity. This is what telling us that the surface is ~~a~~ not just flat space. It is sort of information that is left even after we gauge fix as much as we could.

It turns out that ~~every~~ given surface can be endowed with complex structure in many different ways. There is finite dimensional space of complex structure.

Conclusion ;

The analogue of integral over length there ; becomes integral over space of complex structure.

$$A_\Sigma = \int_{\mu(\Sigma)} d\mu \int_{X : \Sigma \to \mathbb{R}^{d-1,1}} DX \cdot e^{-S_{String}(X, \mu)} \qquad (*)$$

$\mu(\Sigma)$ = Space of complex structures on Σ.

We can have surfaces with m tubes, and g handles.

$\Sigma_{m,g}$

There is moduli space of such surfaces $\mathcal{M}_{m,g}$ which is actually a connected complex manifold.

To compute $(*)$ in ~~Σ~~ to find scattering amplitude in string Theory,

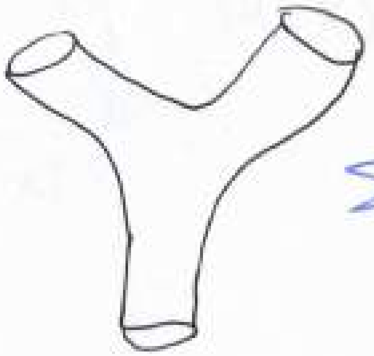
- We have to learn how to do path integrals over fields that lives on 2d surface with finite topology.
- Then, find a way to integrate over space of complex structures to get the final answer.

In QFT, simplest scattering we can write is

one line comes in,
and two goes out

In string theory

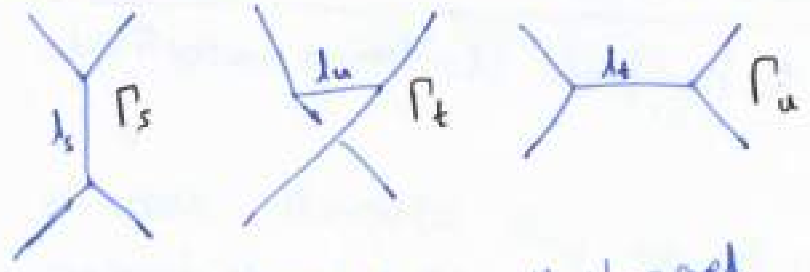
~~two~~ one tube comes in,
and two tubes goes out



$\Sigma_{3,0}$ Surface with three external legs and no handles attached.

This has just one complex structure;

$\mathcal{M}_{3,0} = \text{POINT}$ (analogous to the fact that, we dont have any internal legs in )



In QFT, we had one real d.o.f ... The internal leg to this edge.

On the other hand, we have only one topology for $\Sigma_{4,0}$

$\Sigma_{4,0}$.

We have three graphs $\Gamma_s, \Gamma_t, \Gamma_u$
and we can adopt there moduli spaces.

Moduli spaces ... the space of l over which we were integrating over

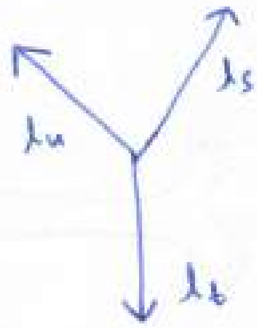
$$\Gamma_s \;,\; \Gamma_t \;,\; \Gamma_u$$

$$\mathcal{M}_s \quad \mathcal{M}_t \quad \mathcal{M}_u$$

Each of the moduli spaces looks like a line parametrized by the length of the corresponding edge.

The three graphs meet together, as the length goes to zero.

The moduli spaces looks kind of this.



We attach the three lines

because morally speaking when lines goes to zero, we just get

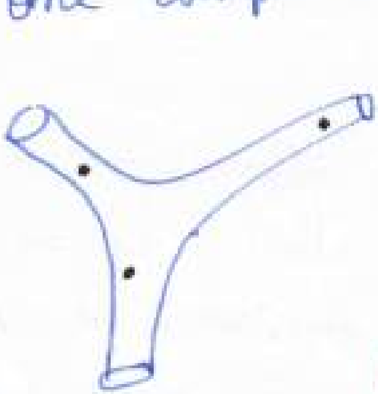
this graph.

---

lets discuss, moduli space of complex structure of the surface $\Sigma_{4,0}$ of $\Sigma_{4,0}$, i.e. $\mathcal{M}_{4,0}$

It has to be one complex dimensional space.

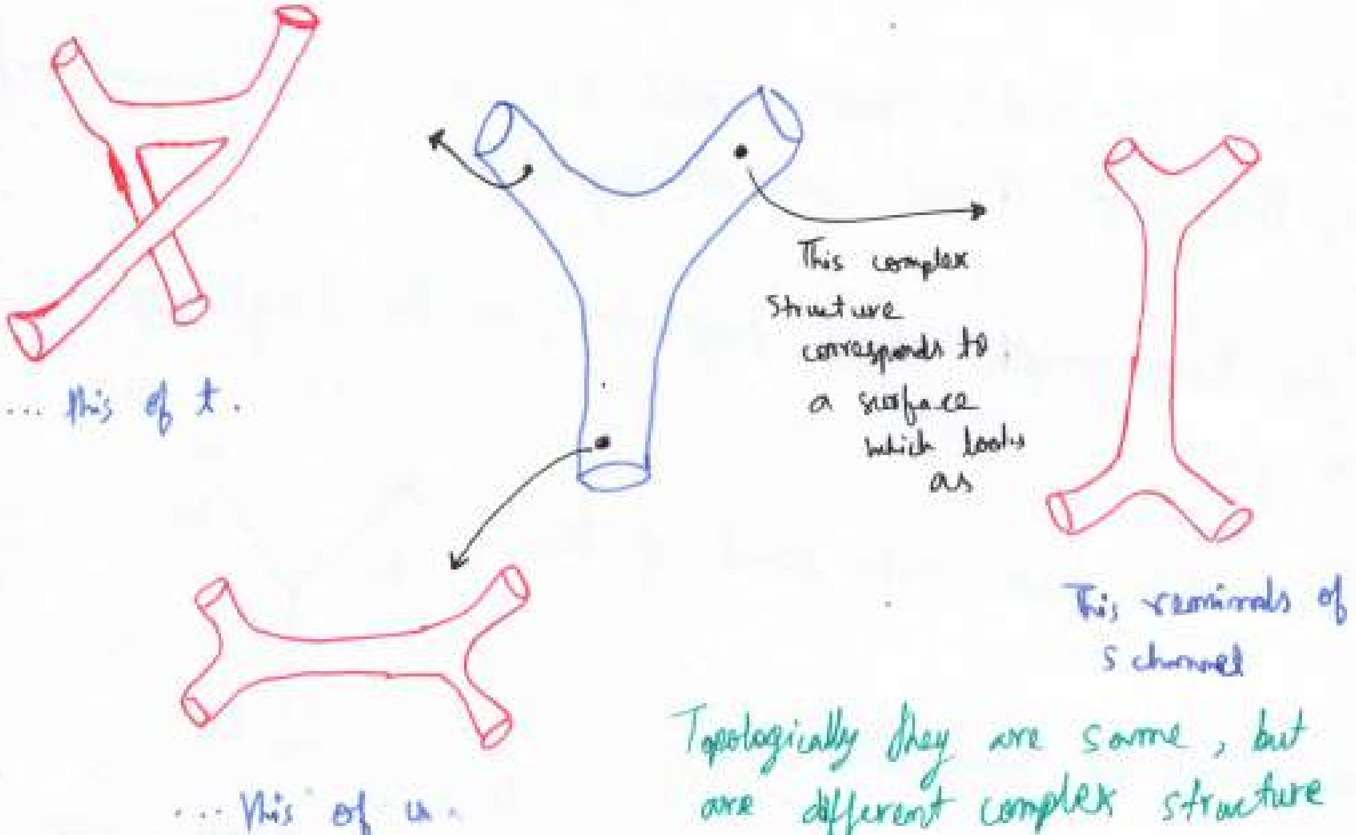
It roughly looks like



dont confuse this with world sheet.
We are not describing world sheet, but the space of complex structures.

Each point in this space describes some structure on the manifold.

$\mathcal{M}_{4,0}$



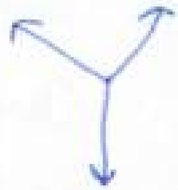
Topologically they are same, but are different complex structure

This is the parttern which repeats itself for general surfaces

☆ Complex Structure is a gauge fixed metric. So in particular to give a complex structure; we can just give metric

There can be many different metric's which gives us same complex structure. But if we need to just give a description of complex structure; we can just draw pictures. Sort of Derive the metric from picture; and then gauge fix it of complex structure.

Note) The moduli space of surface for a String Theory process (or complex structures) kind of looks like a complexified version ~~of complexified version~~ of the moduli space of lengths for Feynman ~~Dig~~ Diagrams.



Except it does not have nasty points like the intersection point. $M_{4,0}$ is nice & smooth.

M_Σ in general is very complicated space. But it has patches where the surface kind of looks like a Feynman diagram somewhat thickened out.

These patches are sort of responsible for the fact that String Theory at low energy behaves like QFT.

The fact that $M_{4,0}$ (or say M_Σ) : everything is nice & smooth is part to the reason that String Theory has no UV divergences.

(U.V. divergences in Feynman Diagram corresponds to situation where some of the interaction points collide)

$\Longleftrightarrow$ There is no such singular point in String Theory.

~~Sym~~ ~~Classical~~

Classical symmetries may not survive quantization.

It survives for sure if we can discretize or regularize our measure in a way that preserves the symmetry.

But, if we cannot; There there might be problem.

In String Theory;

The regularized measure breaks the symmetry (mostly the Weyl) and then we try to see, "if we can restore it."

(length is diffeomorphic notion, but not weyl-invariant notion)

Our regularized measure might change after ~~we~~ we do symmetry transformation. We then need to check whether we can add some counter-terms to action which will cancel this change.

We have <u>anamolies</u>, when ~~there~~ ~~is~~ more things can go wrong than things which we can use to correct.

For weyl symmetry in 2 dimension; there is one number which can go wrong, & which cannot be corrected by counter-term. This number is called the Central Charge.

~~(Each boson)~~ $\boxed{\text{Each bosonic field has central charge 1}}$

CENTRAL CHARGE C — X is maps

$C_X = 1 \times d$ — no. of X, which is dimension of spacetime

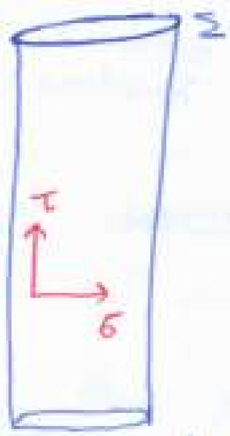
$C_{h_{ab}} = -26$ (metrics contribute this. ; Painful calculation)

So; Total central charge $C = C_X + C_{h_{ab}} = d - 26$

So; Our Bosonic theory can make sense ~~for~~ only in 26 dimensional spacetime (Because $C \neq 0$ gives anomaly)

→ ~~(Repeat)~~ (Repeating calculation for superstrings; the ~~dimen~~ dimension of spacetime comes out to be 10)

Lec 3: The two-dimensional free boson : Fock - space, symmetries, stress energy tensor, Virasoro algebra.



Single closed string propogating freely in spacetime.

This analysis will allow us to find spectrum of string states; what sort of excitations we find in String Theory.

$h_{ab} = \delta_{ab}$

We put flat coordinates on Σ in order to do calculation.

$\sigma \equiv \sigma + 2\pi L$ (identify ~~sigma~~ σ periodically

length scale ... we know that theory is weyl invariant; so ~~8~~ length scale should not really matter

... will set it to 1 later.

We will use euclidean ~~metric~~ ~~time~~ metric.

τ is euclidean time

(Why this? There was natural metric we could think of putting on Σ; as the one inherited from spacetime, which should be lorentzian)

(do it in analogy with what we do in QFT)

The action describing motion of string in flat metric looks like

$$S[X^\mu, \delta_{ab}] = \frac{T}{2} \iint \underbrace{\frac{\partial X^\mu}{\partial u^a} \frac{\partial X_\mu}{\partial u_a}}_{} du^1 du^2$$

→ This is sum of $d+1$ terms ($d+1 \Rightarrow$ dimension of spacetime)

where $u^1 = \tau$, $u^2 = \sigma$

$X^\mu = X^0, X^1, \ldots, X^d$.

we can focus on single ~~field scalar field, i.e.,~~ coordinate; which behaves like scalar field in 2d.

just call it X (and later we will take into account the no. of copies we need)

Recall $\tau = it$ (Relation between Euclidean & Lorentzian time)

$\therefore$ useful to define $\underline{s = \tau - i\sigma}$ (complex coordinates)

Then the action for single scalar can be written

$$\boxed{S[X, \delta_{ab}] \propto T \int \partial_s X \, \partial_{\bar{s}} X \, ds \, d\bar{s}}$$

take: just add up $S[X^\mu, \delta_{ab}] \propto T \int \partial_s X^\mu \partial_{\bar{s}} \partial X_\mu \, ds \, d\bar{s}$

(we treat X^μ as euclidean coordinate in spacetime, and then wick rotate)

Now working with $g^{\mu\nu} = \eta^{\mu\nu}$.

we have to be careful with $S[X^0, \delta_{ab}]$, it will have negative sign; and represents unpleasants things about commutation relation & unitarity (Analogous to quantization is SD in Electromagnetism. Careful with A^0 component)

$S[X] \propto T \int \partial_s X \, \partial_{\bar{s}} X$

We have seen canonical quantization of free scalars.
We expand scalar field into fourier modes at some fixed time.
We look at conjugate momentum, which is something like $\Pi(\sigma, 0) = \partial_\tau (X(\sigma, 0))$
and then impose canonical commutation relations.

$$X(\sigma, 0) = \sum x_m \, e^{im\sigma/L}$$

$$\Pi(\sigma,0) = \partial_\tau X(\sigma,0) = \sum p_m \cdot e^{im\sigma/L}$$

We call it field decomposed into towers of harmonic oscillators (except for zero mode, i.e. $m=0$)

Decompose field $X(\sigma, \tau)$ into modes

$$X(\sigma, \tau) = x - \frac{2i}{L} \cdot p \cdot \tau + \sum_{n \neq 0} \left[\frac{i}{n} a_n \, e^{i\frac{n}{L} \cdot (\sigma + i\tau)} - \frac{i}{n} \bar{a}_n \cdot e^{-\frac{in}{L}(\sigma - i\tau)} \right]$$

This a_n, $\bar{a}_n$ are modes of harmonic oscillators.
Commutation relations looks like.

$$[a_n, a_m] = n \, \delta_{n+m,0}$$

$$[\bar{a}_n, \bar{a}_m] = n \cdot \delta_{n+m,0}$$

$$[a_m, \bar{a}_n] = 0$$

This says that a_n, $\bar{a}_n$ are truely independent sets of oscillators.

Depending on whether n is positive or negative; they represent creation & annhilation mode of harmonic oscillator.

Over here; also have x, p

$$[x, p] = i$$

$\to$ ~~x p~~ x and p (these modes satisfy just the standard commutation relation of particle). They describe the centre of mass of string.

a_m, $\bar{a}_m$ describes the vibrational modes of string.

To check these;
we can compute equal time commutator

$$[X(\sigma, 0), \underbrace{i\partial_\tau X(\sigma', 0)}] = 4\pi i\,\delta(\sigma - \sigma')$$

$\to$ Conjugate momentum to the field.

FOCK SPACE for a_m, $\bar{a}_m$.

for zero modes; we have same Hilbert space as for a free particle on the real line.
The fock space will include $|p\rangle$ eigenvalues of centre of mass momentum.

Then turn on harmonic oscillators.

$a_{-2}|p\rangle$, $a_{-1}^2|p\rangle$, $a_{-1}\bar{a}_{-1}|p\rangle$, $\bar{a}_{-1}^2|p\rangle$, $\bar{a}_{-2}|p\rangle$,

... etc.

We can also write, dual states

$\langle p|$

These are δ function normalized states $\langle p|p'\rangle = \delta(p-p')$

These will be annihilated by a_n, $\bar{a}_m$ for $n, m < 0$

$$\langle p| a_m = 0 \quad , \langle p| \bar{a}_m = 0 .$$

---

ex: $\langle p| a_1\, a_{-1} |p'\rangle = \langle p|p'\rangle + \underbrace{\langle p| a_{-1} a_1 |p'\rangle}_{0}$
$= \delta(p-p')$

Taking holomorphic derivative of X, (is holomorphic function)

$$\partial_s X = \frac{1}{L} p + \sum_{n\neq 0} \frac{1}{L} a_n \cdot e^{-\frac{n}{L} s}$$

→ This comes from the fact that equations of motion is just $\partial_s \partial_{\bar{s}} X = 0$

so; This mean $\partial_{\bar{s}} X$ is anti-holomorphic.

& $\partial_s X$ " holomorphic

(we note that, X was just a sum of holomorphic & anti-holomorphic function)

Similarly: $\partial_{\bar{s}} X = \frac{1}{L} p - \sum_{n\neq 0} \frac{1}{L} \bar{a}_n e^{-\frac{n}{L}\bar{s}}$

These expressions are convinient to define.

$a_0 \equiv p$, $\bar{a}_0 = -p$; gives more uniform formula.

A free scalar has a simple symmetry; we can translate scalar by some amount to and the action is invariant.

In this 2 dimensional theory; there is associated 2 dimensional current with the symmetry $X \to X + \epsilon$

The current has two components $(\partial_s X, \partial_{\bar{s}} X)$

The individual components are individually conserved. (This happens in various 2d conformal theories; due underlying larger symmetry)

~~The current conservation states~~

Actually the theory has larger symmetry.

$$X \to X + f(s) + g(\bar{s})$$

This does not change the action.

The current for this symmetry looks like

$$(f(s)\, \partial_s X \;,\; g(\bar{s})\, \partial_{\bar{s}} X)$$

So, all these currents to be conserved for f & g follow from fact that $\partial_s X$ is holomorphic.
& $\partial_{\bar{s}} X$ is anti - " .

We can think of modes as conserved charges for some symmetry.

ie; $X \to X + e^{\frac{ins}{L}}$

Then we would just look at the current $e^{\frac{ins}{L}} \partial_s X$ and then integrate it over σ from 0 to $2\pi L$.

$$\boxed{a_n = \int_{\sigma=0}^{2\pi L} e^{\frac{ns}{L}} \cdot \partial_s X \, d\sigma}$$

note)

$\hat{p}|p\rangle = p|p\rangle$

lets compute correlation functions.

sandwich in between the currents.

(Take one of the states to be vacuum for simplicity)

$$\langle p| \partial_s X(s) \partial_{s'} X(s') |0\rangle = -\frac{\delta(p)}{L^2} \sum_{n>0} n \cdot e^{\frac{n}{L}(s'+s)}$$

$$= -\frac{\delta(p)}{L^2} \cdot \frac{e^{\frac{1}{L}(s+s')}}{(e^{\frac{s}{L}} - e^{s'/L})^2}$$

This answer has divergence when ~~s approaches~~ s approaches s'.

as $s \longrightarrow s'$ $\quad -\frac{1}{(s-s')^2} + \ldots$

Define 2 dimensional Stress Tensor.

$T^{ab} = \frac{\delta S}{\delta h_{ab}}$, and it is conserved $\nabla_a T^{ab} = 0$

Once we go to flat metric, and in complex coordinates; the classical stress tensor simplifies a lot.

$T_{ss} = -\frac{1}{2}(\partial_s X)^2$ $\qquad T_{s\bar{s}} = 0$

$T_{\bar{s}\bar{s}} = -\frac{1}{2}(\partial_{\bar{s}} X)^2$

(because actually $T_{s\bar{s}}$ is equal to trace of stress energy tensor)

"In Weyl Invariant Theory; The trace of Stress Energy Tensor should vanish;

because a weyl transformation is just a variation of the metric proportional to metric itself"

(use $T^{ab} = \frac{\delta S}{\delta h_{ab}}$)

In QFT, expression like $(\partial_s X)^2$ makes no sense because we cannot multiply two operators at same position

↳ a good physical way to define it is to use regularization like point split regularization;

where we keep our operators slightly seperated from each other, & substract off divergence and then send them together

A good point splitting defination of stress tensor is

$$T = T_{ss} = -\frac{1}{2} \lim_{s' \to s} \left(\partial_s X(s)\, \partial_{s'} X(s') + \frac{1}{(s-s')^2} \right)$$

(This is not tension)

→ There is some freedom in deciding what we substract off.

But; always remember that QFT should be local.

↳ in particular it cannot depend on L!

We can think of subtracting off the whole thing in our defination $-\frac{1}{L^2} \frac{e^{1/(\ )}}{(e\ \ -e\ )} \cdot \delta(p)$

......

But its not OK; This term has L, and it depends on size of the cylinder. We cannot do that.

→ This is why we kept L untill now, to demonstrate this.

Compute expectation value of stress energy tensor.

$$\langle p|T|0\rangle = \frac{1}{2}\lim_{s'\to s}\left[-\frac{1}{L^2}\frac{e^{\frac{1}{L}(s+s')}}{(e^{s/L}-e^{s'/L})^2} + \frac{1}{(s-s')^2}\right]\delta(p)$$

$$\Rightarrow \boxed{\langle p|T|0\rangle = \frac{-1}{24L^2}\cdot\delta(p)}$$

finite ... not zero.

→ This tells us that; The vacuum on the cylinder has non-zero energy

(Stress energy tensor tells us amount of energy in state)

... called Casimier Energy.

Casimier Energy is closely related to Weyl Anamoly.

(The 24 in the RHS...)

The Classical Stress Tensor is variation of action w.r.t. metric.

The Quantum Stress Tensor includes the variation of measure too. (This is what $\frac{1}{(s-s')^2}$ was about really)

Weyl invariance is statement that Quantum Stress Tensor is traceless.

If we take the theory of scalar field, and compute the Quantum Stress Tensor on a ~~manifold~~ worldsheet with generic metric,

we will find $\boxed{\text{Trace (Stress Tensor)} \propto \underbrace{R\,[G_{ab}]}_{\text{Ricci curvature of metric.}}}$

- If worldsheet is flat: it is zero.
- The lack of Weyl invariance becomes visible when our worldsheet is not globally flat.

(As long as we work on cylinder; we will not see the breaking of Weyl Symmetry)

Now, Expand stress Tensor in forier modes.

set $L = 1$ from now on.

$$T = -\frac{1}{24} - \sum_n L_n e^{-ns} \;;\; \text{where } L_n = \frac{1}{2}\sum_m :a_{n-m}\, a_m:$$

~~We could cont~~

We could compute $-\frac{1}{24}$ by Zeta function regularization.

i.e. $\sum \langle 1 \, a_2 \, a_3 \, 1 \rangle = \sum_{n=1}^{\infty} n = -\frac{1}{12} \dots$

$$\bar{T} = -\frac{1}{24} + \sum_n \bar{L}_n e^{-n\bar{s}}$$

Then energy is just.

$$E = L_0 + \bar{L}_0 - \frac{1}{12}$$

$$P = L_0 - \bar{L}_0$$

This is momentum on the world sheet (not to be confused with momentum conjugate to centre of mass position of scalar field)

~~a~~ ~~says~~

$$L_0|p\rangle = \frac{1}{2}p^2|p\rangle$$

$$L_0 \, a_{-1}|p\rangle = \left(\frac{1}{2}p^2 + 1\right)|p\rangle$$

a_{-1}	raises	L_0	by	1
a_{-2}	"	L_0	"	2
$\bar{a}_{-1}$	"	$\bar{L}_0$	"	1
$\bar{a}_{-2}$	"	$\bar{L}_0$	"	2

$|p\rangle$ has no world sheet momentum.

We could compute algebra of fourier modes $L_n, \bar{L}_m$.

first calculate ~~how does L acts on~~

$$[L_n, X(s)] = e^{ns} \, \partial_s X(s)$$

Stress tensor is always the generator of translations.

Its the conserved current associated to translations of worldsheet.

Here; instead of $\nabla_a T^{ab} = 0$

the individual components are individually conserved.

$$\bar{\partial}_{\bar{s}} T_{ss} = 0$$

$$\partial_s T_{\bar{s}\bar{s}} = 0$$

This means that we can multiply T_{ss} by any function of s, and still get conserved current.

ie; $\partial_{\bar{s}} [e^{ns} T_{ss}] = 0$

$\Rightarrow$ This means; we get conserved charges by just taking fourier modes of T.

Hence L_m are conserved charges for some symmetries. The symmetries are just conformal transformations.

note;

$$L_n = \int_{\sigma=0}^{2\pi} e^{ns} T_{ss}$$

Ln is conserved charge for the conserved current T_{ss}.

What is symmetry doing?

They are just coordinate transformations $s \to s + \epsilon e^{ns}$

These are holomorphic coordinate transformations.

(Holomorphic coordinate transformations, combined with Weyl Transformation; leave the Action Unchanged)

$$s \longmapsto s + \epsilon e^{ns}$$

$$X(s) \longmapsto \underbrace{X(s+\epsilon e^{ns})}$$

This does not change the action. (its the symmetry generated by L_n's)

and can be seen in $\underbrace{[L_n, X(s)] = e^{ns}\partial_s X(s)}$

is The infinetisimal change of X

is precisely ~~this~~ the vector field ~~acting~~

$e^{ns}\partial_s$ acting on $X(s)$.

We can ~~now~~, ~~and~~ compute

$$[L_n, \partial_s X(s)\, \partial_{s'} X(s')] = \ldots$$

and then derive how L_n acts on T_{ss}

These extra terms means that stress tensor does not behave like tensor

$[\cancel{L_n,}\, T_n$

$$[L_n, T_{ss}(s)] = \underbrace{e^{ns}(2n+\partial_s)\left(T(s)\right.} + \left.\frac{1}{24}\right) + \frac{1}{12}n(n^2-1)e^{ns}$$

This is how tensor transforms under coordinate transformations.

Doing change of coordinates

$$T_{s's'}(s') = \underbrace{\left(\frac{\partial s}{\partial s'}\right)^2 T_{ss}(s(s'))}_{\text{if } T_{ss} \text{ was tensor}}$$

} Transformation under $s \to s'$ if T_{ss} was tensor.

The infinitesimal version of transformation of T_{ss} gives the Lie algebra we get.

In general we will find ;

$$[L_n, T_{ss}(s)] = e^{ns}(2n + \partial_s)\left(T(s) + \frac{c}{24}\right) + \frac{c}{12} n(n^2-1) e^{ns}$$

c ⇒ Central charge .

using this we finally compute

$$\underbrace{[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m,0}}_{\text{VIRASARO ALGEBRA}}$$

In the absence of this term, it will just be Lie Algebra of vector fields. (of diffeomorphism of our surface)

$$\{e^{ns}\partial_s, e^{ms}\partial_s\}_{P.B.} = (n-m)\, e^{(n+m)}\, \partial_s$$

$$\nabla_a \langle T^{ab}(v^{(1)}) O_2(v^{(2)}) O_3(v^{(3)}) \ldots \rangle = \sum_i \delta(v^{(1)} - v^{(i)})$$ } Ward identity

↳ If we define Stress Tensor in non local way, we will loose this (probably)

Lec 4: The string spectrum, BRST quantization, Cohomological field theory, a basic model.

Classical Constraints: Equation of motion: $\partial \bar{\partial} X^\mu = 0$

$\partial X^\mu \partial X_\mu = 0$ } vanishing of classical stress tensor
$\bar{\partial} X^\mu \bar{\partial} X_\mu = 0$ }

These are constraints because; if we take derivative of this; it vanishes because of ~~eq.~~ equation of motions.

⟶ Impose it as boundary conditions; and it will be maintained by evolution.

We want to impose (similar to Gupta Bleuer conditions in Quantum Electrodynamics)

$$\langle \text{PHYSICAL} | \; T(z) \; | \text{PHYSICAL} \rangle = 0$$

ie; $\boxed{L_n |\text{PHYS}\rangle = 0 \quad n > 0}$; $\boxed{\langle \text{PHYS}| L_m = 0 \quad m < 0}$

For zero modes; $\boxed{(L_0 - a)|\text{PHYS}\rangle = 0}$

~~we have~~

We included Casimir Energy here..

If $d = 26$, then $a = 1$ ie; $(L_0 - 1)|\text{PHYS}\rangle = 0$.

---

lets define null state $|\text{NULL}\rangle = L_n |\ldots\rangle$, $n > 0$

↳ state in the image of L_n

A state ~~thg~~ which is physical & null, will have zero inner product with physical state. So; we want to through it away in order to have well defined inner product on our Hilbert Space.

~~The space of Str~~

The space of string state $\mathcal{H}_{string}$.

The " " Physical " $\mathcal{H}_{phys}$

" " " Null " $\mathcal{H}_{Null}$

$$\mathcal{H}_{string} = \frac{\mathcal{H}_{PHYS}}{\mathcal{H}_{NULL}}$$

} $\mathcal{H}_{phys}$ divided or quotiented by $\mathcal{H}_{Null}$ space.

Use it, and apply to space of states of 26 free bosons on cylindrical world sheet.

Writing fock face.

$|p^\mu\rangle$ (bunch of ground states, which are eigenstates of translation of spacetime)

Then we have descendents

$a_{-1}|p^\mu\rangle$, $\bar{a}_{-1}|p^\mu\rangle$, ~~$a_{-1}$~~ etc.

$a^\nu_{-1}\, a^\mu_{-1}|\,\rangle$, $a^\mu_{-2}|\,\rangle$, $a^\mu_{-1}, \bar{a}_{-1}^\nu|\,\rangle$, ...

$(L_0 - 1)|Phys\rangle = 0$

$(\bar{L}_0 - 1)|Phys\rangle = 0$

$|Null\rangle = L_n|..\rangle$, $n > 0$ or $|Null\rangle = \bar{L}_n|..\rangle$

$$X^\mu = x^\mu + 2ip^\mu \tau + \sum \frac{a_n^\mu}{n} + \dots$$

~~This Sugg~~

Now; The commutator is

$$[a_n^\mu, a_m^\nu] = \eta^{\mu\nu} \cdot \delta_{m+n,0}$$

it has lorentz metric

Now, if we work in lorentz invariant way, we are in a risk of getting some negative normed states

<u>Ex</u>| $\langle p | a_1^\mu a_{-1}^\nu | p' \rangle = \eta^{\mu\nu} \delta^{(4)}(p - p')$

→ zero components of oscillators results in creation of states with negative norm.

☆ It turns out; Once we restrict ourself to physical states & quotient by null states: all the inner products become positive definite. $\boxed{\text{Theorem}}$

(Analogous to what we do in QED)

<u>QED</u>|| ϵ^μ polarization vector.

physical condition constraint imposes $p_\mu \epsilon^\mu = 0$ (ϵ^μ perpendicular to p^μ)

p^μ ≠ momentum. ie; $\boxed{p_\mu \epsilon^\mu_{\text{physical}} = 0}$

Then quotient by polarization which are parallel to momentum.

$\epsilon^\mu_{\text{Null}} \propto p^\mu$; since $p^2 = 0$; its possible for something to be both physical & null.

ex $p^\mu = (1, 1, 0, 0)$

physical constraints makes it look like $\epsilon^\mu_{phys} = (a, a, b, c)$

The null are of form $\epsilon^\mu_{null} = (a, a, 0, 0)$

So; we are left with 2 degrees of freedom

If we are doing open strings instead of closed strings; there would be photon in the spectrum; and we would be literally doing the QED analysis here.

The first contrained we want to impose is

the vanishing of $L_0 - \bar{L}_0 = 0$ ~~for L_0 & $\bar{L}_0$ $|p^\mu\rangle$ has same eigenvalue the~~

$[L_0, a_n] = - a_n$

$\Rightarrow$ So; we can through away everything which does not have the same amount of oscillator of two sides.

$\longleftrightarrow$ Its called level matching.

Note) $|p^\mu\rangle$ has same eigenvalue for L_0 & $\bar{L}_0$

But holomorphic (like $a_{-1}|p^\mu\rangle$) and Antiholomorphic (like $\bar{a}_{-1}|p^\mu\rangle$)

oscillators has L_0 and $\bar{L}_0$ eigenvalues being non zero respectively.

So: remove things like $a_{-1}|p^\mu\rangle$, $\bar{a}_{-1}|p^\mu\rangle$

$a^\nu_{-1} a^\mu_{-1}|p^\mu\rangle$, $a_{-2}^\mu|p^\mu\rangle$ $\Big\}$ Remove

Things like $a^\mu_{-1} \bar{a}^\nu_{-1}|p^\mu\rangle$ survives.

~~Things~~ States which survives are.

$$\alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2} \bar{\alpha}_{-1}^{\mu_3} \bar{\alpha}_{-1}^{\mu_4} |p\rangle \quad ,$$

$$\alpha_{-2}^{\mu_1} \bar{\alpha}_{-1}^{\mu_2} \bar{\alpha}_{-1}^{\mu_3} |p\rangle \quad , \quad \text{etc.}$$

$|p^\mu\rangle$ lets study these states

$$(L_0 - 1)|p^\mu\rangle = \left(\frac{p^2}{2} - 1\right)|p^\mu\rangle$$

So, $p^2 = 2$ This states can be thought of as mode for scalar fields in spacetime with negative mass squared ; $m^2 = -2$

TACHYON

The other constraints are satisfied here for $|p^\mu\rangle$.

Bosonic theory is nonperturbatively unstable.

$$(L_0 - 1)\, \alpha_{-1}^{\mu} \bar{\alpha}_{-1}^{\nu} |p\rangle = 0 \quad \Rightarrow p^2 = 0$$

describes massless particle.

In order to look at other constraints; its useful to look at ~~pola~~ polarization vector.

$$[L_1, \alpha_{-1}^{\mu}] = \alpha_0^\mu$$

$$\epsilon_{\mu\nu}\, \alpha_{-1}^{\mu} \bar{\alpha}_{-1}^{\nu} |p\rangle$$

$$L_1 \left[\epsilon_{\mu\nu}\, \alpha_{-1}^{\mu} \bar{\alpha}_{-1}^{\nu} |p\rangle\right] = p^\mu \epsilon_{\mu\nu} (\bar{\alpha}_{-1}^{\nu} \cdot |p\rangle) = 0$$

So, we want $p^\mu \epsilon_{\mu\nu}$ to vanish.

Physical states satisfy $\boxed{p^\mu \epsilon_{\mu\nu} = 0}$.

If we look for null states.

$$L_{-1}\left[\lambda_\mu \bar{\alpha}^\mu_{-1}|p\rangle\right] = p_\mu \lambda_\nu \alpha^\mu_{-1} \bar{\alpha}^\nu_{-1} |p\rangle$$

Hence Null states are the one for which polarization is proportional to p^μ.

Next level: $\epsilon_{\mu\nu}\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|p\rangle$

such that $p^2=0$ & $p^\mu \epsilon_{\mu\nu}=0$

and polarization is defined upto shift ~~of~~ to amount proportional to p.

$$\epsilon_{\mu\nu} \longrightarrow \epsilon_{\mu\nu} + \underbrace{p_\mu \lambda_\nu}_{\substack{\text{null states}\\ \text{generated by } L_{-1}}} + \underbrace{\bar{\lambda}_\mu p_\nu}_{\substack{\text{null states}\\ \text{generated by } \bar{L}_{-1}}}$$

Can decompose $\epsilon_{\mu\nu}$ into trace part

& symmetric traceless part.

& antisymmetric part.

☆ ~~$\eta_{\mu\nu}\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|p\rangle$~~ Trace Part.

☆ $\epsilon^S_{\mu\nu}\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|p\rangle$, $\epsilon^S_{\mu\nu}=\epsilon^S_{\nu\mu}$ $\epsilon^S_{\mu\nu}\eta^{\mu\nu}=0$

☆ $\epsilon^A_{\mu\nu}\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|p\rangle$, $\epsilon^A_{\mu\nu}=-\epsilon^A_{\nu\mu}$

$\epsilon^S_{\mu\nu} \to \epsilon^S_{\mu\nu} + P_\mu \lambda_\nu + P_\nu \lambda_\mu$

lets collect these polarization into a field, and see which sort of eqn does this field satisfy.

$\Box h_{\mu\nu} = 0 \qquad \partial^\mu h_{\mu\nu} = 0 \; ; \; h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu$

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

So, $\epsilon^S_{\mu\nu} \alpha^\mu_{-1} \bar{\alpha}^\nu_{-1} |p\rangle$: Spin 2 Graviton

Trace : Spin 0 Dilaton.

Within Perturbative QFT, we can prove:

Any massless spin 2 field with gauge symmetries, has to behave like Graviton.

(So at least at low energy; the modes $\epsilon^S_{\mu\nu} \alpha^\mu_{-1} \bar{\alpha}^\nu_{-1} |p\rangle$ will behave like graviton)

$\epsilon^A_{\mu\nu} \longrightarrow \epsilon^A_{\mu\nu} + P_\mu \lambda_\nu - P_\nu \lambda_\mu$

Spectrum of Open & Closed Superstrings

— Shoaib Akhtar 13/8/2020

A Construction of spectrum of Bosonic-String theory

We had our commutation relations $[a^\mu_m, a^{\nu\dagger}_n] = \delta_{m,n} \cdot \eta^{\mu\nu}$

But we find infinite no. of negative normed states, which then led to voilation of Unitarity

But, then we recall that we had classical constraint $L_m = 0 \ \forall m$

So, In Quantum Theory we impose this constraints.

note for $m \neq 0$, There is no ambiguity in the defination of L_m as an operator, because the relavant operators their commute.

Hence The constraint condition we impose on physical states are

$$L_m |\text{Physical}\rangle = 0 \qquad \forall \ m > 0$$

$$(L_0 - a)|\text{Physical}\rangle = 0$$

(here a is a parameter, which takes into account the ambiguity due to normal ordering)

(later we also find value of a)

Now, its hard to impose the above mentioned constraints because they are quadratic.

So, we go to Light Cone Coordinates, which linearizes our Constraints. (This method is called light Cone Quantization)

$x^\mu \to (x^0, x^1, \ldots, x^{D-2}, x^{D-1})$

$\eta^{\mu\nu} \to \text{diag}(-1, 1, \ldots, 1)$

$\to$ LCC (Light Cone Coordinates)

$$X^\pm = \frac{1}{\sqrt{2}} \cdot (X^0 \pm X^{D-1})$$

$$X^\mu \xrightarrow{LCC} (X^+, X^-, X^1, X^2, \ldots, X^{D-2})$$

$$\eta \overset{\text{L.C.C.}}{=} \left(\begin{array}{cc|ccc} 0 & -1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ \hline \vdots & \vdots & & & \\ 0 & 0 & & \mathbb{1}_{(D-2)\times(D-2)} & \end{array}\right)$$

In L.C.C. ; we see that negative normed states will be produced by α^+ & α^-.

(Note; initially only one α^0 was problematic; but in LCC we now have to worry about two operators α^+ & α^-)

It turns out that we can get rid of one of the α^+ or α^- sets of operators, using conformal transformations on the world sheet.

We have (dealing with open strings here)

$$X^+(\tau,\sigma) = x^+ + l_s^2 p^+ \left(\tau + \frac{i}{l_s p^+}\sum_{m\neq 0} \frac{1}{m}\alpha_m^+ \cdot e^{-im\tau}\cdot\cos(m\sigma)\right)$$

Note, The action $S[x] = \frac{T}{2}\int d\sigma_+ d\sigma_-\, \partial_+ X^\mu \partial_- X^\nu \cdot \eta_{\mu\nu}$ is invariant under following.

$$\left.\begin{array}{l} \sigma_+ \longrightarrow \tilde{\sigma}_+ = f_+(\sigma_+) \\ \sigma_- \longrightarrow \tilde{\sigma}_- = f_-(\sigma_-) \end{array}\right\} \text{Conformal Transformations.}$$

where $\tilde{\sigma}_+ = \tilde{\tau} + \tilde{\sigma} = f_+(\sigma_+)$

$\tilde{\sigma}_- = \tilde{\tau} - \tilde{\sigma} = f_-(\sigma_-)$

$$\Rightarrow \tilde{\tau} = \frac{1}{2}\left(f_+(\sigma_+) + f_-(\sigma_-)\right) \Rightarrow \boxed{\partial_+\partial_-\tilde{\tau} = 0}$$

(Symmetry of our theory)

using this we get rid of α^+.

choose $\tilde{\tau} = \tau + \frac{i}{l_s p^+}\sum_{m\neq 0}\frac{1}{m}\alpha_m^+ \cdot e^{-im\tau}\cdot\cos(m\sigma)$

clearly $\tilde{\tau}$ satisfies $\partial_+\partial_-\tilde{\tau} = 0$.

Now we redefine our word sheet coordinates

$$\sigma \longrightarrow \sigma$$
$$\tilde{\tau} \longrightarrow \tau$$

Now, we are only left with α^- which are problematic.

The constraint equation $\dot{X} \cdot X' = 0$ gets linearized in L.C.C. coordinates with redefined τ.

And we finally get $\alpha^-_n = \frac{1}{l_s P^+} \left(\frac{1}{2} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} :\alpha^i_{n-m} \alpha^i_m: - a \cdot \delta_{n,0} \right)$

The Physical Hilbert space is made up of states of the form $|\varphi\rangle = a^{i_1 +}_{m_1} a^{i_2 +}_{m_2} \ldots a^{i_k +}_{m_k} |0\rangle$

where $i_k \in \{1, \ldots, D-2\}$.

And we restore unity.

But we have spoiled Lorentz Invariance.

Note] Conformal invariance of Action $S[X] = \frac{T}{2} \int d\xi^2 \, \partial_+ X^\mu \partial_- X^\nu \eta_{\mu\nu}$ is a Classical Property.

This property may not survive under quantization, and may give rise to anomaly. So, we have to keep a check on it.

Now: We start Construction of Spectrum for Open Bosonic String.

note: $\alpha' M^2 = N - a$; where $N = \sum_{m=1}^{\infty} \sum_{i=1}^{D-2} m \cdot a^{i+}_m a^i_m$

N=0 : $\alpha' M^2 |0\rangle = -a |0\rangle \Rightarrow \alpha' M^2 = -a$

(we find $a = 1$; hence N=0 level states becomes Tachyonic)

$\underline{N=1}$; $\alpha^{i\dagger}_{1}|0\rangle$ $\quad \alpha' M^2(\alpha^{i\dagger}_{1}|0\rangle) = (1-a)(\alpha^{i\dagger}_{1}|0\rangle)$

$$\Rightarrow \alpha' M^2 = (1-a)$$

The index i looks like a vector index, but has $(D-2)$ d.o.f.

So; These states have to form a representation of $SO(1, D-1)$ which has $D-2$ d.o.f.

$\Rightarrow$ so These have to be massless, hence $a=1$.

Hence, Lorentz Invariance $\Longrightarrow a=1$

Now; we finally find the allowed value of D.

Note] $J^{\mu\nu} = T\int_0^\pi d\sigma\,(\dot{X}^\mu X^\nu - \dot{X}^\nu X^\mu)$

$$= x^\mu p^\nu - x^\nu p^\mu + i\sum_{m\neq 0}\frac{1}{m}\left(\alpha^\mu_{-m}\alpha^\nu_m - \alpha^\nu_{-m}\alpha^\mu_m\right)$$

These $J^{\mu\nu}$ classically satisfy Lorentz Algebra.

During Quantization; Anomalies can develop.

We find except for $\mu=-$, $\nu=i$, $\rho=-$, $\sigma=j$ case:

The Algebra is satisfied.

We get, $[J^{-i}, J^{-j}] = \sum_{m\neq 0}\Delta_m\cdot\left(\alpha^i_{-m}\cdot\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right)$

If $a=1$, then $\Delta_m = \left(m - \frac{1}{m}\right)\left(\frac{26-D}{12}\right)$

Lorentz Invariance $\Longrightarrow \Delta_m = 0 \rightarrow \boxed{D=26}$

Conclusion] To recover Lorentz Invariance (so that we dont get Anomalies) ; we are forced to choose $a=1$, and $D=26$ for Bosonic String Theory

Construction of Spectrum for Closed Bosonic Strings

We do LCQ (Light Cone Quantization)

Lorentz Invariance $\Rightarrow$ $a=1$, $D=26$.

Now, we have two number operators in this case

$$N=\sum_{i=1}^{24}\sum_{n=1}^{\infty}(\alpha^i_{-n}\cdot\alpha^i_n)$$

$$\tilde{N}=\sum_{i=1}^{24}\sum_{n=1}^{\infty}(\tilde{\alpha}^i_{-n}\cdot\tilde{\alpha}^i_n)$$

$$\Leftarrow \begin{cases}(L_0-a)|\psi\rangle=0\\(\tilde{L}_0-a)|\psi\rangle=0\end{cases}$$

So we have two number operators.

$a=\tilde{a}$ because of symmetry between left & right movers in closed strings.

Using level matching condition $(L_0-\tilde{L}_0)|\psi\rangle=0$

we get $\tilde{N}=N$

$$\frac{\alpha'}{4}M^2|\psi\rangle=(N-1)|\psi\rangle=(\tilde{N}-1)|\psi\rangle$$

<u>$N=\tilde{N}=0$</u> ; $\frac{\alpha'}{4}M^2=-1$ Tachyonic.

<u>$N=\tilde{N}=1$</u> ; $a^{+i}_1\tilde{a}^{+j}_1|0\rangle$; $\frac{\alpha'}{4}M^2=0$ massless.

So, these must fall into Representation of Poincaré group in 26 dimension (massless representation)

The little group of $SO(1,25)$ is $SO(24)$

We don't have any irreducible representation of $SO(24)$ with 24^2 states!

So, we take symmetric traceless, antisymmetric and the traceful irreducible pieces of $SO(24)$

$$\left[\frac{1}{2}(a^{+i}_1\tilde{a}^{+j}_1+a^{+j}_1\tilde{a}^{+i}_1)-\frac{1}{24}\left(\sum_{k=1}^{24}a^{+k}_1\cdot\tilde{a}^{+k}_1\right)\delta^{ij}\right]|0\rangle$$

These are <u>Graviton</u> (They are symmetric, traceless rank 2 tensor of $SO(24)$)

$\left[\frac{1}{2}\left(a_1^{\dagger i}\tilde{a}_1^{\dagger j} - a_1^{\dagger j}\tilde{a}_1^{\dagger i}\right)\right]|0\rangle$ "2-form B" : Antisymmetric rank 2 tensor of SO(24)

$\frac{1}{24}\left(\sum_{k=1}^{24} a_1^{\dagger k}\tilde{a}_1^{\dagger k}\right)|0\rangle$ "Φ : Dilaton" : Scalar of SO(24)

The massless sector of closed Bosonic string has

(i) Graviton
(ii) 2-form B field
(iii) Φ : Dilaton

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[b] Spinors in 2d

(Note that our Spinors live on the worldsheet $\psi(\tau, \sigma)$; so we have to discuss about spinors in 2 dimensions) (i.e. we are discussing World sheet Theory)