MATHS FOR QFT

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These notes are consequence of my self study; and are mostly inspired from Dr.Daniel Wohns lectures on **Maths for QFT**. This notes were made in a single blow on the midnight while waiting for Sehri in Ramazan, during the Corona Pandemic.

Tame the infinities ∞ !

— Shoaib Akhtar
1/5/2020

Maths for QFT Lec 1: Distributions, Test Functions

— Shoaib Akhtar 18/5/2020

tools + concept for QFT I, II, III

Outline : Distributions (Generalized Functions) : (2 lectures)
Asymptotic Series : (2 lectures)
"Summation" of divergent series (1 lecture)

Distributions

Motivation: ex a familiar example; point charge : has a charge distribution.

i.e.; $\rho(x) = 0$ for $\vec{x} = \vec{0}$

$\rho(\vec{0}) = \infty$.

ex $\frac{d^n}{dx^n}|x|$ at $x = 0$? ex $\int_{-\infty}^{+\infty} \frac{1}{x} dx$ ex $\frac{\delta F}{\delta f(x)}$

ex Greens function; they satisfy equation of the form $LG = \delta$.

ex existence of solutions.

Outline :
- Definations (what they are not / what they are)
- Operations
  - Derivations
  - Multiplication
  - Composition.
- Applications
  - $\frac{\delta F}{\delta f(x)}$
  - Green's functions

lecture 2

first two lectures

Most important example : Dirac delta

$$\delta(x) \overset{?}{=} f(x) = \lim_{\sigma \to 0} f_\sigma(x) \quad \text{where; e.g. } f_\sigma(x) = \begin{cases} \frac{1}{\sigma} & |x| < \frac{\sigma}{2} \\ 0 & |x| \geq \frac{\sigma}{2} \end{cases}$$

$\int_{-\infty}^{+\infty} f_\sigma(x)\, dx = 1$ for all σ

$f(x) = 0$ for $x \neq 0$.

not a function : $\delta(0)$?

Defimation: Function space is a vector space whose elements are (equivalence classes) functions with a norm that satisfies.

- positivity . $\|f\| \geq 0$
- unique identity $\|f\| = 0 \iff f = 0$
- Triangle inequality; $\|f+g\| \leq \|f\| + \|g\|$
- linearity + homogenity ; $\|\lambda f\| = |\lambda| \|f\|$

main example ; $L^2[a,b]$ set of square integrable functions

ie: $\int_a^b |f|^2 dx < \infty$. then $f \in L^2[a,b]$

note; $f_1(x) = 0$; $\|f_1\| = 0$

$f_2(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$; $\|f_2\| = 0$

almost everywhere.

we need to identify $f_1(x)$ & $f_2(x)$ if we want to read $L^2[a,b]$ as vector space ; otherwise we will not have unique identity ; say that they live in same equivalent class ..

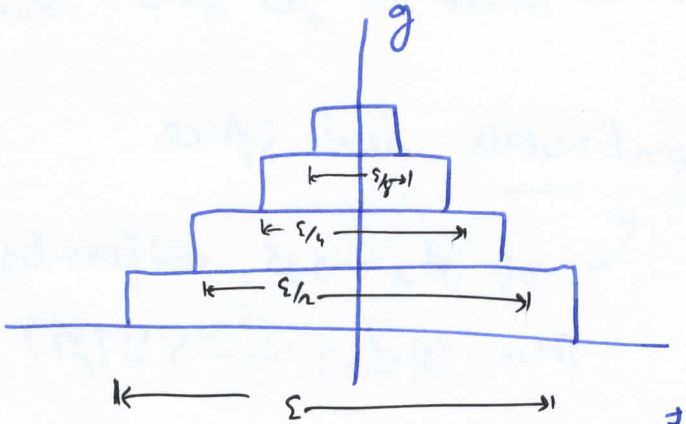
$\delta \notin L^2[a,b]$

$\therefore \|f_\sigma\|^2 = \int_{-\infty}^{+\infty} |f_\sigma(x)|^2 dx = \int_{-\sigma/2}^{+\sigma/2} \frac{1}{\sigma^2} dx = \frac{1}{\sigma}$

$\|f(x)\| \longrightarrow \infty$. so; $\delta \notin L^2[a,b]$

Choose some g_σ so that $\lim_{\sigma \to 0} g_\sigma(x) = 0$ for $x \neq 0$

$\int_{-\infty}^{+\infty} g_\sigma(x)\, dx = 1$ for all $\sigma > 0$



For each ϵ and each x we can find σ_0 so that $g_\sigma(x)$ lies inside the boxes for all $\sigma < \sigma_0$

$$\Rightarrow \int_{-\infty}^{+\infty} g(x)\,dx < A = \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \dots = 2\varepsilon$$

area of boxes

$$\Rightarrow \int_{-\infty}^{+\infty} g(x)\,dx = 0 \neq 1 = \lim_{\sigma \to 0} \int_{-\infty}^{+\infty} g_\sigma(x)\,dx$$

order of limit and integral matters.

Test functions : we will choose set $\mathcal{D}$: set of C^∞ functions with compact support

continuous with ∞ many derivatives

vanishes outside of some finite region

(The nicer your test functions are, the wilder your distributions can be)

ex e^{-x^2} not belongs to $\mathcal{D}$; because dont have compact support for it.

ex
$$f(x) = \begin{cases} e^{-\frac{1}{x^2-1}} & ; |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$\mathcal{D}^*$ = dual vector space of ~~linear~~ linear functionals of $\mathcal{D}$ to $\mathbb{R}$.

$\therefore u \in \mathcal{D}^*$ if $u(\varphi) \in \mathbb{R}$ for any test function $\varphi \in \mathcal{D}$

linear ; $u(\alpha\varphi + \beta\psi) = \alpha u(\varphi) + \beta u(\psi)$

We will put one more restriction in order to get space of distributions.

$\mathcal{D}'$ = distributions, continuous dual space.

If $\phi_n \to \phi$ uniformly.
then $U(\phi_m) \longrightarrow U(\phi)$

so; $\boxed{u \in \mathcal{D}' \quad \text{if} \quad u \text{ is } \begin{array}{l} \bullet \text{ finite} \\ \bullet \text{ linear} \\ \bullet \text{ continuous} \end{array}}$

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ex $\int \delta(x)\, \phi(x)\, dx = \phi(0) \longrightarrow$ real number, finite & continuous i.e; $\phi_n \to \phi$
→ then $\int \delta(x)\, \phi_n(x) \to \phi(0)$

Actually this is defination of delta function $\int \delta(x)\, \phi(x)\, dx = \phi(0)$; this way we dont have to define its value at any point.

We can define delta to be $\delta(\phi) = \phi(0)$:)

takes the input

output

In this sense; it is a functional.

$(\delta, \phi) = \phi(0)$:(

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ex $u(\phi) = \int (\phi(x))^2\, dx$ ← not linear ; $V(\phi) = \phi(0) - \phi'(\pi) + 37\phi''(e)$

$w(\phi) = \begin{cases} 1 & : \text{if } \phi(0) \geq 0 \\ 0 & \text{otherwise} \end{cases}$ also; $w(\phi)$ is not linear

↳ not continuous ; $\phi_m \to 0^-$;

$t(\phi) = \int \frac{\phi(x)}{x^2}\, dx$.

↳ not finite.

= : yes, it is distribution

~~ : no, it is not distribution.

functions that are locally integrable are distributions (Pg 5)

$$\int_a^b f(x)\,dx < \infty$$

for any $a, b, \in \mathbb{R}$

$$f(\phi) \equiv \int f(x)\,\phi(x)\,dx$$

~~"locally integrable functions induce distribution~~

~~locally dx integral fs~~

"locally integrable functions induce distribution"

Lec 2 : ~~Fundame~~ Operations on distributions, Applications of distributions. – Shoaib Akhtar 1/5/2020

Distribution maps test functions to $\mathbb{R}$.

Outline : Operations
- Derivation
- Multiplication.
- Composition.

} first part of lecture.

- Application to
  - $\frac{\delta F}{\delta(f)}$ + Greens functions ($LG = \delta$)

} second part of lecture

Derivation

$$\cancel{u'(x) \stackrel{?}{=} \lim_{\varepsilon \to 0} \frac{u(x+\varepsilon) - u(x)}{\varepsilon}}$$ dead end.

lets take some inspiration from locally integrable functions & the induced distribution by them.

$$u'(\phi) = \int_{-\infty}^{+\infty} u'(x)\,\phi(x)\,dx$$

compact support of $\phi(x)$

$$= u(x)\,\phi(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u(x)\,\phi'(x)\,dx$$

$$= 0 - \int_{-\infty}^{+\infty} u(x)\,\phi'(x)\,dx = -\int_{-\infty}^{+\infty} u(x)\,\phi'(x)\,dx$$

$$= -u(\phi')$$

$$\Rightarrow \underline{u'(\phi) = -u(\phi')}$$

even we derived it assuming u is a function; but we can take it as a definition even if u is a distribution and not an ordinary function.

$$\boxed{u'(\phi) \equiv -u(\phi')}$$ weak or distributional derivatives

$$\boxed{u^{(m)}(\phi) \equiv (-1)^m u(\phi^{(m)})}$$

example!! Heaviside $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

its locally integrable so it induces a distribution.

$$H'(\phi) = -H(\phi') = -\int_{-\infty}^{+\infty} H(x)\, \phi'(x)\, dx = -\int_0^{\infty} \phi'(x)\, dx$$

we can take weak derivative.

$$= -(\phi(\infty) - \phi(0))$$
$$= \phi(0) = \delta(\phi)$$

$$\Rightarrow H'(\phi) = \delta(\phi) \quad \forall\, \phi \in \mathcal{D}$$

$$\Rightarrow \boxed{H' = \delta}$$

Example: $(\ln|x|)'$

$\int_0^a \ln|x|\, dx \to \infty$ (not locally integrable)

$$(\ln|x|)(\phi) \equiv \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \phi(x) \ln|x|\, dx$$

~~with this~~ with this defination of distribution $\ln|x|$; we can now take weak derivative.

$$(\ln|x|)'(\phi) = -\lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \phi'(x) \ln|x|\, dx$$

$$= \lim_{\varepsilon \to 0} \Bigg[\underbrace{\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{1}{x}\, \phi(x)\, dx}_{(PV\frac{1}{x})(\phi)} + \underbrace{\phi(\varepsilon)\ln|\varepsilon| - \phi(-\varepsilon)\ln|\varepsilon|}_{\simeq\, 2\phi'(0)\varepsilon \ln|\varepsilon| \,\to\, 0 \text{ as } \varepsilon \to 0} \Bigg]$$

principle value distribution acting on ϕ

$$\Rightarrow (\ln|x|)' = \left(PV \frac{1}{x}\right).$$

Multiplication by a function.

lets develop the defination from ordinary functions.

~~so~~ suppose ψ ordinary function; & u distribution; $\phi \in \mathcal{D}$

$$(\psi u)(\phi) = \int_{-\infty}^{+\infty} \psi(x)\, u(x)\, \phi(x)\, dx \quad \text{if } u \text{ is a function.}$$

$$= \int_{-\infty}^{+\infty} u(x)\, (\psi(x)\, \phi(x))\, dx$$

so; we can define $\boxed{(\psi u)(\phi) \equiv u((\psi)(\phi))}$

we need to check $\psi(\phi) \in \mathcal{D}$.

so; ψ should be continuous with ∞ derivatives C^∞.

so; ψ should be C^∞ for our test function space.

but in general we need $\psi\phi$ to be a test function.

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example 1) simplify $x\delta'$

1) $-\delta$ ✓
2) $-\delta - x\delta$ ✓
3) 0
4) other.

$\cancel{(x\delta')(\phi) = \int_{-\infty}^{+\infty} \delta' (x(\phi))\, dx}$

$$x\delta'(\phi) = \delta'(x\phi) = -\delta((x\phi)') = -\delta(\phi) - \delta(x\phi') = -\delta(\phi) - \underbrace{\delta(x\phi')}_{0} = -\delta(\phi)$$

$\Rightarrow x\delta'(\phi) = -\delta(\phi) \Rightarrow \boxed{x\delta' = -\delta}$

$x\delta(\phi) = \delta(x\phi) = 0$ so; $1 \equiv 2$ (haha)

Composition

$$(u \circ f)(\phi) = \int_{-\infty}^{+\infty} u(f(x))\, \phi(x)\, dx \quad \text{if } u \text{ is a function.}$$

$y = f(x)$.

$g(y) = x$

$\cancel{= \int f(y)}$

$$\Rightarrow (u \circ f)(\phi) = \int_{-\infty}^{+\infty} u(y)\, \underbrace{\phi(g(y))\, |g'(y)|}_{\text{if Test function;}}\, dy$$

$\phi(g(y))\,|g'(y)|$ → test function } if f is C^∞
{ $y = f(x)$ has unique solution.

so: f has to be a function, which goes from $-\infty$ to $+\infty$: so: limit of $\int$: $-\infty$ to $+\infty$

$$\boxed{(U \circ f)(\phi) \equiv U((\phi \circ g)|g'|)}$$

→ sometimes possible to relax restriction.

ex $$\boxed{(\delta \circ f)(\phi) \equiv \sum_i \frac{\delta_{x_i}}{|f'(x_i)|}(\phi)}$$

$f(x_i) = 0$

$\delta_{x_i}(\phi) = \phi(x_i)$

works if f has only simple roots (otherwise denominator becomes zero)

$f(x) = x^2 - 1$; $\delta \circ f = \frac{\delta_1}{2} - \frac{\delta_{-1}}{2}$

$f(x) = x^2$; $\delta \circ f \Rightarrow$ undefined.

Applications : $\frac{\delta F}{\delta f(x)}$

$$\delta S = \int dt \, \underbrace{\frac{\delta S}{\delta q(t)}}_{} \delta q$$

functional

→ functional derivative. (its the generalization of partial derivative)

$$\delta S = \sum_i \frac{\partial S}{\partial q_i} \delta q_i$$

function

~~close~~ } ⇒ close analogy

$\Sigma \rightsquigarrow \int dt$

$\delta q(t) = \epsilon\, \phi(t)$ $\quad \phi(t) \in \mathcal{D}$

some small parameter.

$$\int \frac{\delta F}{\delta f(x)} \phi(x)\, dx = \lim_{\varepsilon \to 0} \frac{F[f+\epsilon\phi] - F[\cancel{\phi}]}{\epsilon}$$

functional derivative acting on a test function.

$$\boxed{\int \frac{\delta F}{\delta f(x)} \phi(x)\, dx = \lim_{\varepsilon \to 0} \frac{F[f+\epsilon\phi] - F[f]}{\epsilon}}$$

example 1 $F[f] = f^2(x_0)$

$$\int \boxed{\frac{\delta F}{\delta f(x)}} \phi(x)\, dx = \lim_{\varepsilon \to 0} \frac{\cancel{f^2(x_0)} + 2\varepsilon f(x_0)\phi(x_0) + \varepsilon^2\phi^2(x_0) - \cancel{f^2(x_0)}}{\epsilon}$$

$$= 2f(x_0)\phi(x_0)$$

$$= \int \boxed{2f(x_0)\delta(x-x_0)}\ \phi(x)\, dx$$

works for any ϕ

so; $\frac{\delta F}{\delta f(x)} = 2f(x_0)\delta(x-x_0)$

because of this property;

- linear
- product rule
- chain rule.

similar to partial derivatives.

Application to Greens Function

$Ly = f$ ← function

y ← solve for y.

L ← linear differential operator

$$y = L^{-1} f$$

Analogy -

Finite Dimension	Infinite dimensions
$\vec{V}$	~~fun~~ function f
components v_i	Value ~~of~~ at a point $f(x)$
matrix M	operator O
matrix element M_{ij}	Integral Kernel $(K(x,y))$

$$(M\vec{v})_i = \sum_j M_{ij} v_j$$

$$O f(x) = \int K(x,y) f(y) dy$$

Integral Kernel of operator ; & is often a distribution.

density matrix	Dirac delta δ some subtilities here

So; Greens function can be thought about integral Kernel of the inverse operator L^{-1}.

$\therefore$ $LL^{-1} = \mathbb{1}$ (identity; which is ~~direct~~ dirac delta here.

$$LL^{-1} = \delta$$

$$\Rightarrow L_x G(x,\xi) = \delta(x-\xi)$$

(Greens function equation)

~~so its called~~

Maths for QFT — Lec 3: Asymptotic Series and Stirling's Approximation. — Pg 9

- Shoaib Akhtar 1/5/2020

Asymptotics

- Gamma function
- Definations of Asymptotic series
- Stirling's Approximation.
- Saddle point approximation
  - Stoke's Phenomenon.

lec 3 and lec 4

Gamma Function

Motivation: * example for ~~Saddle~~ Saddle point approximation (SPA)

* dimensional regularization (in tutorial)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \qquad Re(z) > 0 .$$

generalization of factorial; $n! = n(n-1)!$ (functional equation)

$$\frac{d}{dt}(t^z e^{-t}) = z t^{z-1} e^{-t} - t^z e^{-t}$$

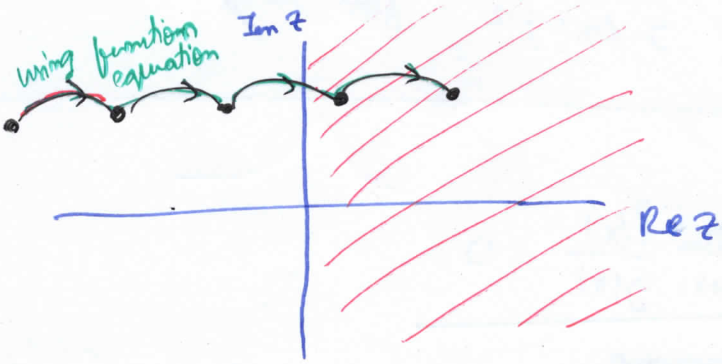
$$t^z e^{-t}\Big|_0^\infty = z \int_0^\infty t^{z-1} e^{-t} dt - \int_0^\infty e^{-t} t^z dt$$

$$0 \qquad \Rightarrow \qquad 0 = z\Gamma(z) - \Gamma(z+1)$$

$$\Rightarrow \boxed{\Gamma(z+1) = z\Gamma(z)}$$ Functional Equation.

together with the fact $\Gamma(1) = 1$ (easy to prove)

$$\Rightarrow \quad \Gamma(n+1) = n!$$



$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\dots(z+n-1)}$$

if $Re(z) + n > 0$

$\Gamma(z)$ has no zeroes; poles at $z = -n$ for $n \geq 0$ integer.

$$z = -(n-1) + \epsilon$$

$$\Gamma(z) = \frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)\cdots(\epsilon-n+1)} \approx \frac{(-1)^{n-1}}{\epsilon(n-1)!} \quad \text{as } \epsilon \to 0$$

So, poles are simple poles and ~~red~~ residue

$$\boxed{\text{residue} = \frac{(-1)^n}{n!} \text{ at } z = -n}$$

Asymptotics

Motivation: understand behavior when parameter is large or small

Toy Model: $Z(\lambda) = \int_{-\infty}^{+\infty} e^{-x^2 - \lambda x^4} dx$ (zero dimensional path integral)

$$= \int_{-\infty}^{+\infty} e^{-x^2} \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n x^{4n}}{n!} dx \overset{?}{=} \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \frac{(-1)^n \lambda^n x^{4n}}{n!} dx$$

$$= \sum \int_0^{\infty} \frac{(-1)^n \lambda^n}{n!} e^{-u} u^{2n - 1/2} du$$

→ Asymptotic series.

$$= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \Gamma(2n + 1/2) > \underbrace{\sum_{n=0}^{\infty} (-\lambda)^n n!}_{\text{for large } n.}$$

~~does not converge~~ ↳ does not converge.

$$\therefore \Gamma(2n + \tfrac{1}{2}) \simeq \underset{(n!)^2}{\overset{}{(2n)!}}$$

$$\Rightarrow \Gamma(2n + \tfrac{1}{2}) \simeq (2n)! > (n!)^2, \text{ for large } n.$$

Definitions

1. $f(x) \underset{x \to x_0}{<<} g(x)$: if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$

example: $x \underset{x \to 0}{<<} \frac{1}{x}$: ~~true~~ ~~as~~ r

True (T) or False $\boxed{(F)}$: If $f(x) \underset{x \to x_0}{\ll} g(x)$; then $f(x) < g(x)$ for all x.

→ False.

Counter example

$x^2 \underset{x \to 0}{\ll} -1$.

$A \underset{R}{\sim} B$ asymptotic to

∴ A asymptotic to R.

$\boxed{2}$ $f(x) \underset{x \to x_0}{\sim} g(x)$

if $f(x) - g(x) \underset{x \to x_0}{\ll} g(x)$

or

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

examples: $\cos x \underset{x \to \pi}{\sim} -1$; ~~T or F~~ :

~~If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$~~,

(i) T or $\boxed{F}$: If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$, then $f(x) \underset{x \to x_0}{\sim} g(x)$

(ii) T or $\boxed{F}$: If $\lim_{x \to x_0} f(x) = y$; then $f(x) \underset{x \to x_0}{\sim} y$

~~Counter example~~ Counter examples

$\lim_{x \to 0} e^{-1/x} = 0$. : but ; nothing can be asymptotic to zero
∵ because it comes in denominator in the defination

$\boxed{3}$ Asymptotic Series $y \underset{x \to x_0}{\sim} \sum_{n=0}^{\infty} a_n (x - x_0)^n$

if $y(x) - \sum_{n=0}^{N} a_n (x - x_0)^n \underset{x \to x_0}{\ll} (x - x_0)^N \;\forall N.$

(this defination does not says that series converges,

Comments

- convergent series gets better with more terms.
- Asymptotic series better as $x \to x_0$ with a fixed no. of terms (taking more terms can be better and also bad)

If $F(\lambda) \underset{\lambda\to 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$

then $F(\lambda) + \underline{b\, e^{-c/\lambda^2}} \underset{\lambda\to 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$

$$e^{-c/\lambda^2} \underset{\lambda\to 0}{<<} \lambda^n \quad \text{for all } n$$

non-perturbative function → QFT !!!

this term is invisible to asymptotic series and is known as non perturbative function.

- $f(z) \underset{z\to z_0}{\sim} g(z)$ generally valid in wedge

$a < \arg(z - z_0) < b$ in complex plane.

(example $e^{-c/\lambda^2} \underset{\substack{\lambda\to 0 \\ \text{(imaginary)}}}{<<} \lambda^n$ will not be true)

Example: Stirling's Approximation.

Want $n! = \Gamma(n+1)$. for large real n.

we have: $\Gamma(x+1) = \int_0^{\infty} e^{-t} t^x dt$

$t = wx$

$$\Gamma(x+1) = x^{x+1} \int_0^{\infty} e^{-x(w - \ln w)} dw$$

$$= x^{x+1} \int_0^{\infty} e^{-x f(w)} \, dw$$

most of the integrals can be put in this form

expand ; $f(w)$ around minimum.

$f'(w) = 1 - \frac{1}{w}$; min at $w = 1$

$$f(w) = 1 + \frac{1}{2}(w-1)^2 + \dots \infty$$

$$\Gamma(x+1) = x^{x+1} e^{-x} \int_0^{\infty} e^{-\frac{x}{2}(w-1)^2 + \dots} \, dw$$

This is not Gaussian integral if we forget higher order terms

$$\Rightarrow \Gamma(x+1) \approx x^{x+1} \cdot e^{-x} \int_{-\infty}^{+\infty} e^{-\frac{x}{2}(w-1)^2} \, dw$$

$$= x^{x+1} e^{-x} \sqrt{\frac{2\pi}{x}}$$

$\oint$ $\int_0^{\infty} (\Xi) \, dw$ suppressed after a large deviation from some w_0

$\therefore$ $w \to 0 \Rightarrow$ it is suppressed to zero

& if we go negative ; it will be suppressed more.

so; $\int_{-\infty}^{0} (\Xi) \, dw$ will not contribute much.

$$\boxed{\frac{\Gamma(x+1)}{\sqrt{2\pi}\, e^{x+\frac{1}{2}} e^{-x}} \underset{x\to\infty}{\sim} 1 + \frac{1}{12x}}$$

if we kept $+ \dots$.

Next lecture: Saddle-point approximation.

$$I(z) = \int_C g(w)\, e^{z f(w)}\, dw \quad \text{as} \quad z \to \infty$$

C ← contour

If this complex integral, + phase changes ⇒ destructive interference.

Lec 4: Saddle-point Approximation, Stokes Phenomenon

— Shoaib Akhtar; 2/5/2020

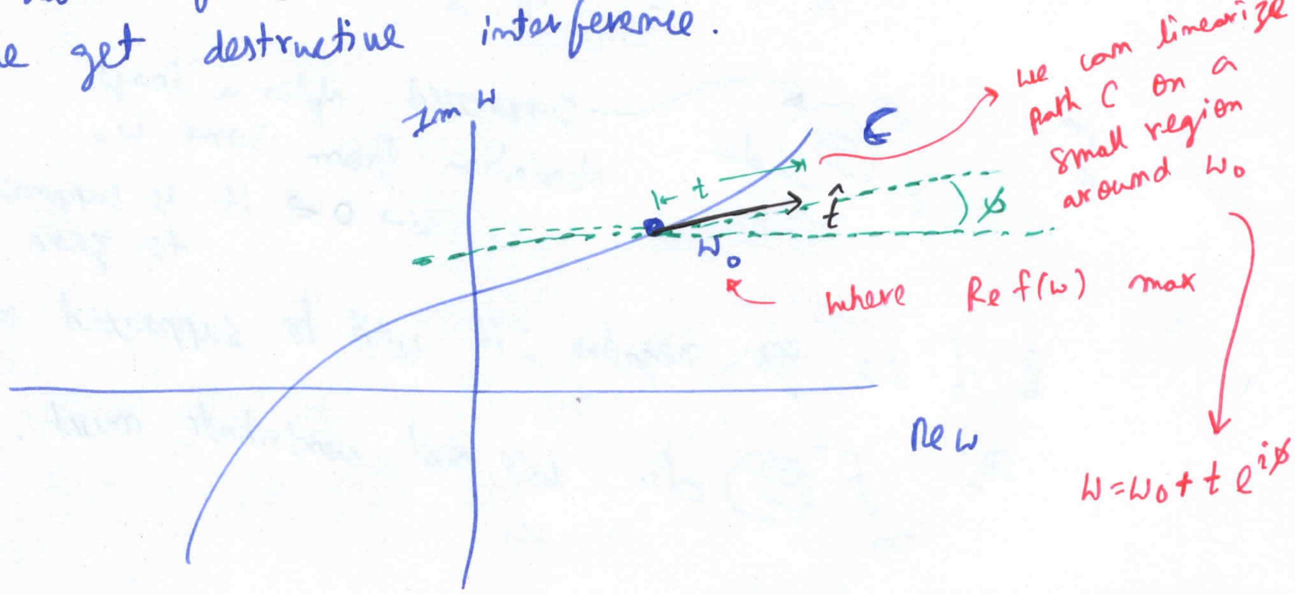
We want to approximate integral of the form

$$I(z) = \int_C g(w)\, e^{z f(w)}\, dw$$

∴ we want to understand its behavior as $z \to \infty$.

* We might expect dominant contributions to come from regions where $\mathrm{Re}(f(w))$ is maximum.

↪ works if C is a constant phase contour; otherwise we get destructive interference.



$u = \mathrm{Re}(f(w))$

$v = \mathrm{Im}(f(w))$

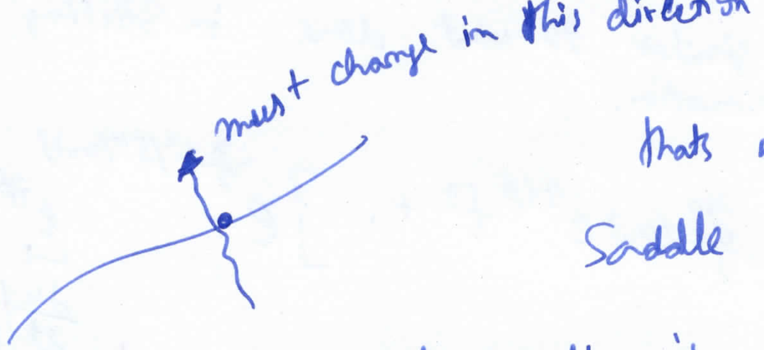
If we want constant phase curve
then $\nabla V = 0$ along the curve
(because phase is determined by V (imaginary part of $f(w)$))

also; we know; at w_0; $\underline{\underline{\nabla U = 0}}$ at w_0

$\nabla V = 0$ along curve C
$\nabla U = 0$ at w_0

$\Rightarrow f'(w_0) = 0 \Rightarrow w_0$ is critical point of $f(w)$.

∴ we know complex function dont have any maximum.
no local maximum for holomorphic function.
↪ told by Cauchy Riemann equations.



thats why: it is named
Saddle point approximation (SPA)

because w_0 is actually saddle point.

$$f(w) = f(w_0) + \frac{1}{2}(w-w_0)^2 f''(w_0) + \dots \infty$$

$f''(w_0)$ ↪ in general a complex number; $e^{i\delta_0} \| f''(w_0) \|$

locally; we should have that
$$f(w) = f(w_0) - \frac{1}{2} t^2 \| f''(w_0) \|$$

$$f(\omega) = f(\omega_0) - \frac{1}{2}t^2 |f''(\omega_0)|$$

comparing we have $(\omega-\omega_0)^2 e^{i\delta_0} = -t^2$

$$\Rightarrow e^{2i\phi} \cdot e^{i\delta_0} t^2 = -t^2$$

$$\Rightarrow \boxed{e^{2i\phi} e^{i\delta_0} = e^{\pm i\pi}}$$

↳ determines angle of constant phase contour.

$$I(z) \approx e^{zf(\omega_0)} \int_{-t_1}^{t_0} g(\omega_0 + e^{i\phi} t) e^{-\frac{z}{2}t^2|f''(\omega_0)|} \left.\frac{d\omega}{dt}\right|_{t_0} dt$$

linear for $-t_1 < t < t_0$

due to change of variable

$$\cancel{\approx e^{zf(\omega_0)} \int_{-t_1}^{t_0} \left[g(\omega_0) + \frac{g''(\omega_0)}{2} e^{2i\phi} t^2 + \dots\right] e^{-\frac{z}{2}t^2}}$$

now, ... we do similar to what done in Stirling' approximation.

$$\approx e^{zf(\omega_0)} \int_{-\infty}^{+\infty} \left[g(\omega_0) + \frac{g''}{2}(\omega_0) e^{2i\phi} t^2 + \dots\right] e^{-\frac{z}{2}t^2|f''(\omega_0)|} \underbrace{e^{i\phi}}_{\left.\frac{d\omega}{dt}\right|_{t_0}} dt$$

↳ we can extend limit of integration $-t_1$ to t_1 to $-\infty$ to $+\infty$; because only for small region the integrand is large; because of $e^{-(\equiv)}$.

* no linear term; because integral of odd function trivially vanishes.

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$$I(z) \underset{z\to\infty}{\sim} \sqrt{\frac{2\pi}{z}} \frac{e^{zf(w_0)}}{\sqrt{-f''(w_0)}} \left[g(w_0) - \frac{1}{z} \frac{g''(w_0)}{1!\,2f'(w_0)} + \dots \right]$$

phase drops out.

at some points higher derivatives of f ; $f^{(n)}$ becomes more important than $g^{(n)}$

~~$I(z) \underset{z\to\infty}{\sim} \sqrt{\frac{2\pi}{z}} e^{zf(w_0)}$~~

$$I(z) \underset{z\to\infty}{\sim} \sqrt{\frac{2\pi}{z}} \frac{e^{zf(w_0)}}{\sqrt{-f''(w_0)}} \left[g(w_0) - \frac{1}{z} \cdot \frac{g''(w_0)}{1!\,2f'(w_0)} + \dots \right]$$

Stokes Phenomenon

- abrupt change in asymptotic relations as phase of z changes

Example)

$$I(z) = \int_0^1 dt\; e^{-4zt^2} \cos(5zt - zt^3) \qquad z\to\infty \text{ (for now}$$

Naive attempt : neglect cos( )

$$I(z) \underset{z\to\infty}{\sim} \int_0^1 dt\; e^{-4zt^2} = \sqrt{\frac{\pi}{16z}}$$

critical when ; $t \underset{z\to\infty}{\sim} \frac{1}{\sqrt{z}}$ arg of cos is not small.

↳ destructive interference.

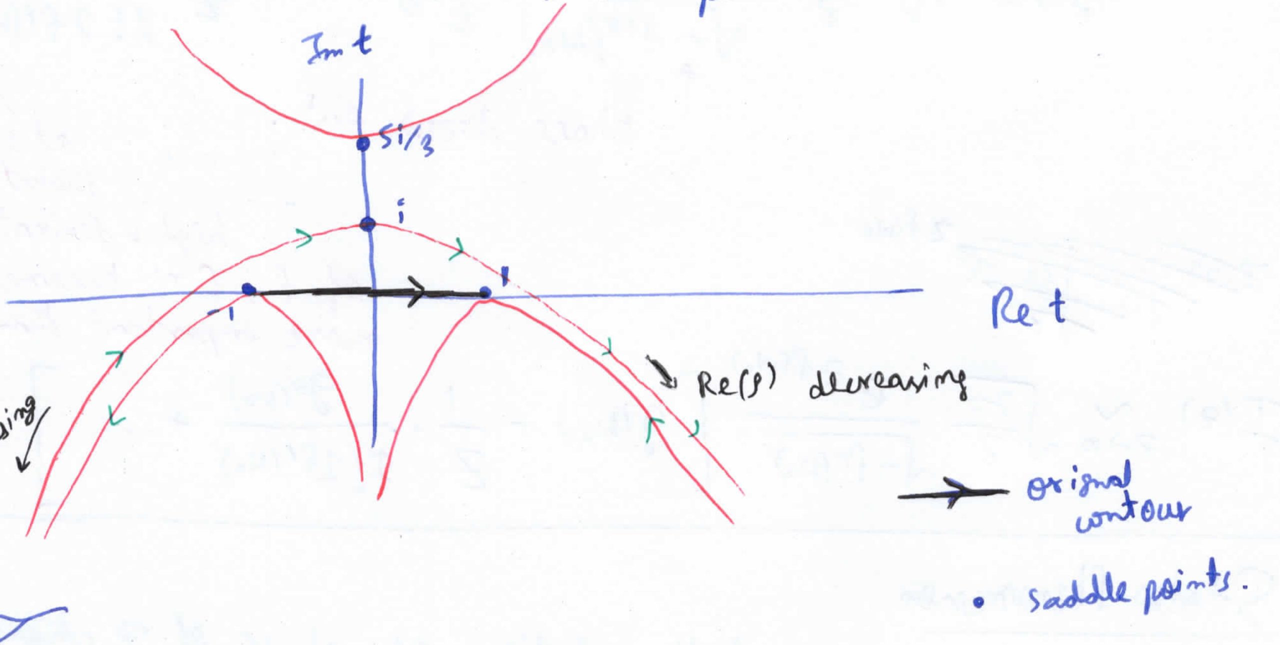
$$I(z) = \mathrm{Re} \int_0^1 dt\; e^{-4zt^2 + 5izt - izt^3}$$

$$= \frac{1}{2} \int_{-1}^1 dt\; e^{-4zt^2 + 5izt - izt^3}$$

$$= \frac{1}{2} e^{-2z} \int_{-1}^1 dt\; e^{z\rho(t)} \quad ; \rho(t) = -(t-i)^2 - i(t-i)^3$$

$f'(t) = 0$ if $t = i$, $t = 5i/3$ } saddle points.

(9/8)



Constant phase contours are determined by ; $u = \mathrm{Re}(t)$, $v = \mathrm{Im}(t)$

$$\mathrm{Im}\,(f(t)) = \text{constant}$$
$$= 3uv^2 - 8uv + 5u - u^3$$

$$\mathrm{Im}\,(f(-1)) = -4$$
$$\mathrm{Im}\,(f(1)) = 4$$

This analysis shows that we should do SPA around i saddle point ; not at $5i/3$

endpoints contribute

~~$I(x) = \int f(t)\, e^{x\psi(t}$~~

$$I(x) = \int f(t)\, e^{ix\psi(t)}\, dt \underset{\substack{x\to\infty \\ x = \text{Real}}}{\sim} \frac{f(t)}{ix\psi'(t)} \cdot e^{ix\psi(t)} \Bigg|_{t=a}^{t=b}$$

works when
- RHS is non zero
- everything is C^1
- $\int_a^b |f(t)|\,dt < \infty$
- ψ is not constant anywhere

no saddle point in ~~particular~~ particular

Saddle point $\qquad \frac{1}{2} e^{-2z} \cdot \sqrt{\frac{\pi}{z}}$ $\qquad$ dominates if $|\arg z| < \arctan\left(\frac{1}{2}\right)$

$t = \pm 1$ (endpoints) $\qquad \mp \frac{i \pm 4}{68z} e^{-4z \pm 4iz}$ $\qquad$ dominates is $|\arg z| > \arctan\left(\frac{1}{2}\right)$

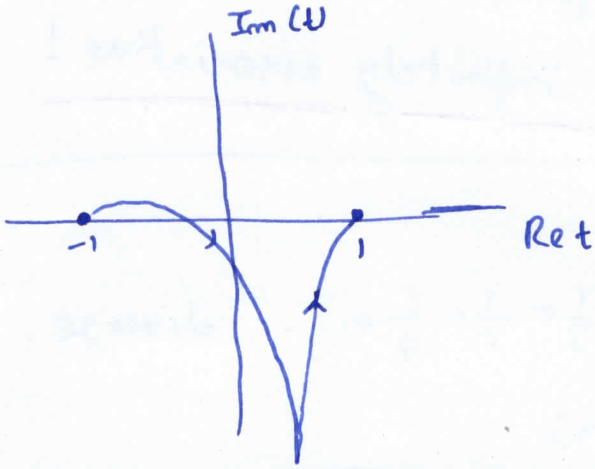
$$I(z) \underset{\substack{z \to \infty \\ \arg(z) = 0}}{\sim} e^{-2z} \sqrt{\frac{\pi}{z}}$$

$$e^{-2z} \underset{\substack{z \to \infty \\ \arg(z) \neq 0}}{\gg} e^{-4z \pm 4iz} \; ?$$

→ Not true in general.

$$\lim_{z \to \infty} \frac{e^{-4z \pm 4iz}}{e^{-2z}} = \lim_{z \to \infty} e^{-2(\pm i2 - 1)z} = 0$$

→ if $|\arg z| < \arctan\left(\frac{1}{2}\right)$



$\arg z = 1350°$

for large $|\arg z| \Rightarrow$ no saddles.

Summary: Approximate integrals using Gaussians
- results will be asymptotic.
- use constant phase contours.

Next lecture: Divergent Series

Maths for QFT   Lec 5: Divergent Series

— Shoaib Akhtar : 1/5/2020.

Motivation: - perturbative series are usually asymptotic, not convergent.

- divergent series in QFT, ie; Casimir Force.

Outline: How to "sum" a divergent series.

Perturbative series.

Examples: ① $1 + 1 + 1 + \dots$ ← someones series

② $1 - 1 + 1 - 1 + \dots$ ← Grandi series (does not converge ⇒ divergent)

different series.

③ $1 + 0 - 1 + 1 + 0 - 1 + \dots$

$$1 + (-1+1) + (-+1) + \dots = 1 + 0 + 0 + \dots = 1$$
$$(1-1) + (1-1) + (1-1) + \dots = 0$$

} assuming we are allowed to use associated property of addition, ~~we get~~ we get different answers.

↳ Conclusion: Addition is not infinitely associative!

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

has divergent sub series; $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$ diverge.

assume commutativity of addition;

∴ we can then create any number

$$\text{eg} \quad \pi = \underbrace{\underbrace{1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \dots}_{>\pi} \quad -\frac{1}{2} - \frac{1}{4}}_{<\pi} \dots + \dots$$

Conclusion: Addition is not infinitely commutative

Euler Summation:

If $y = \sum_{n=0}^{\infty} a_n$ and a_n grows like n^k.

then $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge for $|x| < 1$

$$E = \lim_{x \to 1^-} f(x)$$

factor of x^n helps the series to converge

Euler sum

ex compute $E(1-1+1-\dots)$

$$E = \lim_{x \to 1^-} \sum_{n=0}^{\infty} (-x)^n = \lim_{x \to 1^-} \frac{1}{1+x} = \frac{1}{2}$$

$$f(x) = \frac{1}{1+x} \rightarrow E = \frac{1}{2}$$

Borel Summation

$$n! = \Gamma(n+1) = \int_0^{\infty} dt\, e^{-t} t^n$$

$$1 = \int_0^{\infty} dt\, e^{-t} \frac{t^n}{n!}$$

dividing by $n!$ helps to converge (this is more powerful)

$$B(\emptyset) \equiv \int_0^{\infty} dt\, e^{-t} \sum \frac{a_n t^n}{n!}$$

Borel sum

will give finite answer even when Eulers summation fails sometimes

actually a special case : it is actually $(B(\emptyset)(1))$

... there can be one more parameter here.

ex $B(1-1+1-\dots) = \int_0^{\infty} dt\, e^{-t} \sum \frac{(-t)^n}{n!} = \int_0^{\infty} dt\, e^{-t} e^{-t} = \frac{1}{2}$

$$(B(\phi)(1)) = E(\phi)$$

whenever they both exist.

Generic Summation

(desirable properties which we would like to our summation procedure have)

1. $\mathcal{S}(a_0 + a_1 + \dots) = S = a_0 + \mathcal{S}(a_1 + a_2 + \dots)$
2. $\mathcal{S}(\Sigma(\alpha a_n + \beta b_n)) = \alpha \mathcal{S}(\Sigma a_n) + \beta \mathcal{S}(\Sigma b_n)$: linearity

---

example

$$S = \mathcal{S}(1 - 1 + 1 - 1 + \dots)$$

$$S = 1 + \mathcal{S}(-1 + 1 - 1 + 1 - \dots)$$

$$\Rightarrow S = 1 - \mathcal{S}(\cancel{1} - 1 + 1 - 1 + \dots) \Rightarrow S = 1 - S \Rightarrow S = \frac{1}{2}$$

$$\boxed{\mathcal{S}(1 - 1 + 1 - 1 + \dots) = \frac{1}{2}}$$

"Broad class of summation method gives same value"

Perturbation Theory

$$f(g) \quad \underset{g \to 0}{\sim} \quad \sum_{n \geq 0} a_n g^n \equiv \phi(g)$$

f: exact quantity of interest; g: perturbation parameter

$$\begin{tikzcd} f(g) \arrow[r, bend left, "\text{perturbation theory}"] & \phi(g) \arrow[l, bend left, "?"] \end{tikzcd}$$

Dyson's Argument

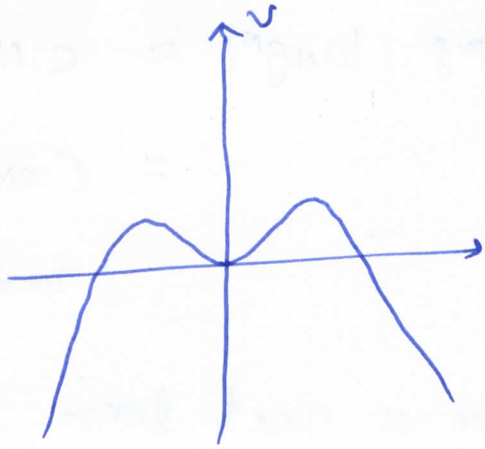
(why asymptotic series are usually divergent from physical viewpoint?)

Imagine Quantum Mechanical (QM) particle in $V(q) = \frac{q^2}{2} + \frac{g}{4} q^4$

$g > 0$

stable

$\mathrm{Im}(E_0) = 0.$



$g < 0$

unstable

quantity of interest $f(g) = E_0(g)$ ← ground state energy

non-perturbative effect. → tunnelling.

$\mathrm{Im}(E_0) \neq 0$

So: if $\phi(g)$ has finite radius of convergence

$\Longrightarrow$ describes $g > 0$ and $g < 0$.

contradiction.

So: $\phi(g)$ must have zero radius of convergence.

Non Perturbative physics $\longrightarrow$ divergence of ϕ.

Optimal Truncation

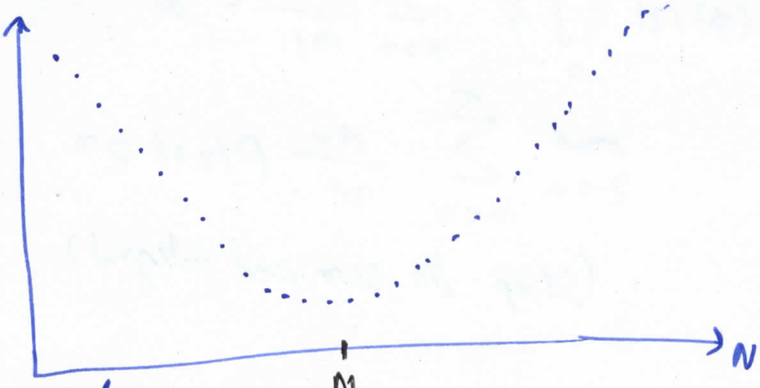
Typically in QM or QFT

$$a_n \underset{n \to \infty}{\sim} A^{-n} n!$$

N_* ← optimal truncation is usually to keep terms up to the smallest term

$$\left| f - \sum_{n=0}^{N} a_n g^n \right|$$

Nature of graph is so, because usually the series is diverging ... so has minima at some point.



minimize $|a_N g^N| \approx C N! \left|\frac{g}{A}\right|^N$

$\simeq C \exp\left(N(\log N - 1 - \log\left|\frac{A}{g}\right|)\right)$ Stirling

minimized at $N_* \simeq \left|\frac{A}{g}\right|$ for $N >> 1$

error $\simeq$ next term

$\approx e^{-|A/g|}$ non-perturbative : ambiguity

For many purposes; "divergent series converge faster than convergent series"

(because they dont have to converge : haha)

Borel Summation

$\phi(g) = \sum a_n g^n$

Borel Transform : $\hat{\phi}(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$

Borel Sum : $B(\phi)(z) = \int_0^\infty e^{-t} \hat{\phi}(zt)\, dt$

The reason Borel summation procedure is interesting ; is because Borel Sum of a series has same asymptotic expansion as the original series when both exist.

ie; Matches with origi~~nal~~ series when both exist.

proof:

$$B(\phi)(z) = \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{a_n t^n z^n}{n!}\, dt$$

$$\underset{z\to 0}{\approx} \sum_{n=0}^{\infty} \frac{a_n}{n!} \Gamma(n+1) z^n = \phi(z)$$

(swap the sum and integral)

$$f(g) \underset{\text{Borel Sum}}{\overset{\text{perturbation theory}}{\rightleftarrows}} \phi(g)$$

$\hat{\phi}(t)$ has singularities on real line, whenever there are non-perturbative contributions to f.

"If orignal series has finite radius of convergence, then Borel sum matches the function in that region"

↳ We can often use $B(\phi)/z)$ to analytically extend it beyond that.

↳ The case of interest for us is when our orignal series is divergent and has zero radius of convergence.

Summary:

- Distributions
- Asymptotics
- Divergent series.

Tame the beast;
give meaning to
infinities!!

— Shoaib Akhtar