Statistical Physics

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STATISTICAL PHYSICS

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These notes are consequence of my self study; and are mostly inspired from David Kubiznak lectures on **Statistical Physics**. These notes also includes Statistical Field Theory, and can be effectively regarded as a book on Advanced Statistical Physics. There is also a supplementary book titled **Problems & Solutions in Statistical Physics** written by me for the present book which contains tutorials, exercises and their solution which should be read in parallel with these notes.

Lecture 1: Review of Thermodynamics: Thermodynamic Laws, Temperature, Energy, Heat & Entropy

Basic Concepts

I Preliminaries

(a) Review of Thermodynamics.

Thermodynamics: Interrelates Direct Observables. Microscopic Quantities. without any microscopic assumptions.

- New Quantities and New concepts.
  - Temperature: T (0th Law)
  - Internal Energy: U (1st Law)
  - Entropy: S (2nd Law)

- Thermodynamic (T.D.) Quantities.

INTENSIVE "Local Character"
ex Pressure (P), T, ρ

EXTENSIVE
ex Mass, (M) Volume, (V) Internal Energy (U)

Thermodynamic quantities appear in Conjugate Pairs.

ex T - S, P - V, T - A
(T: tension?; A ← surface Area)

Such that product of these quantities has unit of Energy.

★ Thermodynamic Equilibrium ≃ No further changes takes place.

A subset of this is specifically Thermal Equilibrium ~~Eat~~ (In Thermal Contact for long time)

★ Process is Reversible = Quasi-Static (slow enough) + No Hysteresis

Hysteresis example

ZEROTH LAW (helps to define temperature)

If two systems are seperately in Thermal Equilibrium with a third system, ~~they~~ then they are also in thermal equilibrium with each other.

$\Rightarrow$ Can define Emperical Temperature

Experimental Fact : State of Gas is typically characterized by two properties ~~Temperature~~ Pressure (P) and Volume (V).

Typically these can be independent.

however if it is in thermal contact with another gas then only one of them is actually independent.

"If in thermal eqn with another gas; then only one out of P, V is independent".

★ System 1 and 2 in Thermal Equilibrium with 3

Gas no. 3 will be used as Thermometer.

1-3 . $\exists$ ~~must~~ a function $F_1(P_1, V_1, P_3, V_3) = 0$
which is actually a constraint equation
constraining P_1, V_1 in terms of $P_3 V_3$.

similarly
2-3 in eqn so, $\exists$ another function $F_2(P_2, V_2, P_3, V_3) = 0$.

$\therefore$ now we can solve both equation for P_3.

$P_3 = P_3(P_1, V_1, V_3)$ (comes from $F_1 = 0$)

$P_3 = P_3(P_2, V_2, V_3)$ (" " $F_2 = 0$)

equating this we get $P_3(P_1, V_1, V_3) = P_3(P_2, V_2, V_3)$ (eq (iii))

An algebraic Relation.

lets solve this for P_1; $\Rightarrow P_1 = P_1(V_1, V_2, V_3, P_2)$ — eq (i)

1-2 by Zeroth law

this means ; $F_3(P_1, V_1, P_2, V_2) = 0$

$\Rightarrow \quad P_1 = P_1(V_1, V_2, P_2)$ — eq^n (ii)

compare eq^n (i) & eq^n (ii)

Conclusion; V_3 in (i) has to drop out

so; in eq (i) ; $P_1 = P_1(V_1, V_2, \cancel{V_3}, P_2)$

so; in $P_3(P_1, V_1, \cancel{V_3}) = P_3(P_2, V_2, \cancel{V_3})$ also

V_3 has to drop out

So; we conclude, there is some universal function f...

ie; eq (iii) implies $\boxed{f(P_1, V_1) = \hat{f}(P_2, V_2)}$

$= \theta$ (constant)

and this is the Emperical temperature.

So; we conclude $\boxed{f(P, V) = \theta}$ is our Thermal Temperature.

& we see that for each gas seperately whatever P, V be : they are always tied by θ : as $\theta = f(P, V)$ in thermal equilibrium.

Now we extend it to anything.

FIRST LAW

Step 1/ For Thermal Isolated System, there exists a function of state called the Internal Energy U, so that

$dU = dW$ — Work on the system.

(isolate the system, don't allow any heat exchange.)

Step 2 If system not thermally isolated, one can also change the state of the system without doing work. HEAT Q Measures to which extent the change is not Adiabatic. → (dont allow anything to happen with heat)

Pg 4

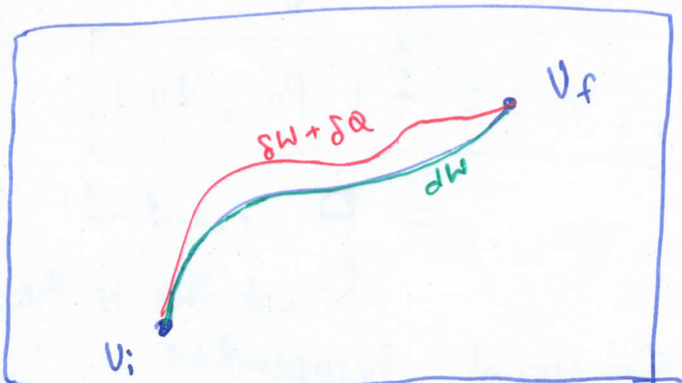
So; the first law reads $\boxed{dU = \delta Q + \delta W}$

δQ & δW are in general not total differentials.

$$U_f - U_i = dU = \delta Q + \delta W$$

δQ: Heat transferred to the system

δW: Work done on the system.



(some path in parameter space)

Only if we specify which path we are taking; we can sort of distinguish work δW from heat δQ.

Convention

* Heat flows from hotter to colder.

Examples of Work Terms (comes from fundamental Laws of physics)

(i) $\delta W = -P\,dV$ (Hydrostatic Pressure Work)

(ii) $\delta W = \tau\,dA$ (Surface tension work when expanding or contracting area)

Heat Capacity $C^{(\alpha)}_{\beta\gamma} = \dfrac{\delta Q_{\alpha\beta\ldots}}{d\alpha}$

Generalized heat Capacity

Changing variable α while keeping P & V … ~~co~~ variables constant. (to specify paths we are taking.)

$$C_V \equiv C_V^{(T)} = \left(\frac{\delta Q}{dT}\right)_V \quad \text{Heat Capacity for constant volume.}$$

$$C_P \equiv C_P^{(T)} = \left(\frac{\delta Q}{dT}\right)_P \quad \text{" " " " pressure}$$

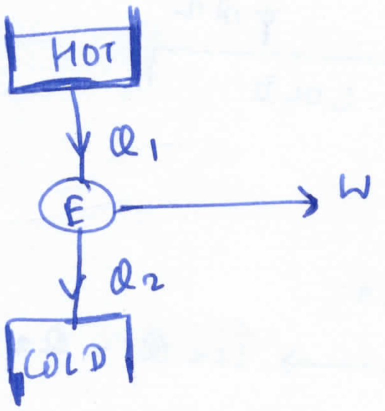
<u>Second Law</u> "Not all processes that are energetically possible actually can happen."

"Heat is a less noble cousin of work."

(you can always turn work to heat; but not always we can change heat to work)

Heat Engine (operates between two thermal boxes)

works in cycles, and returns after each cycle to its initial state.



Heat Engine (E)

~~Heat engine (E)~~

Thermal Efficiency:

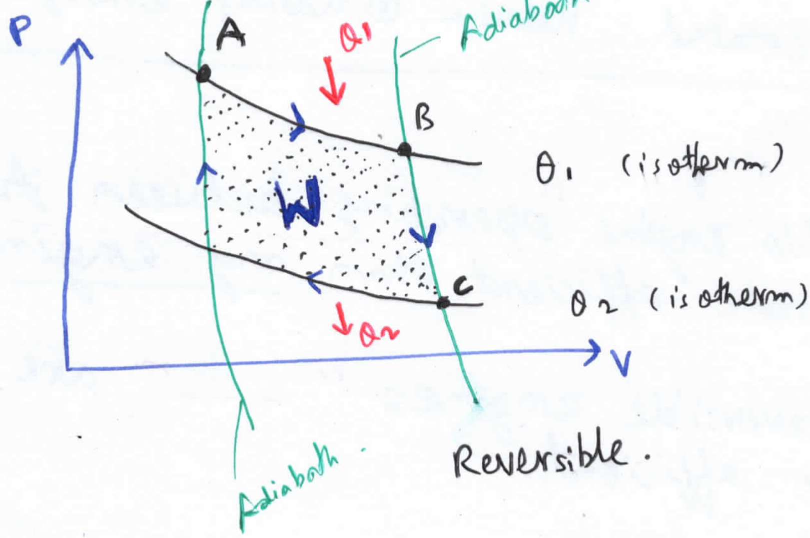
defined by

$$\eta = \frac{\text{Work Out}}{\text{Heat In}} = \frac{W}{Q_1}$$

by conservation of energy

$$\eta = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

Example of Heat Engine (CARNOT CYCLE)



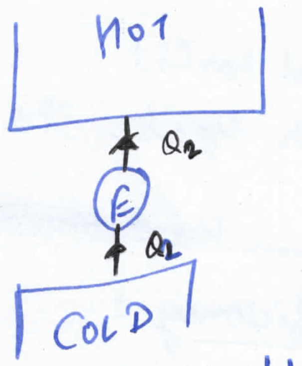
Reversible.

2nd Law (due to KELVIN) No process is possible whose sole result is the Complete Conversion of Heat into work.

2nd Law (due to CLAUSIUS) No process is possible whose sole result is the transfer of Heat from colder to hotter body.

Equivalence of statements.

If Clausius is violated,



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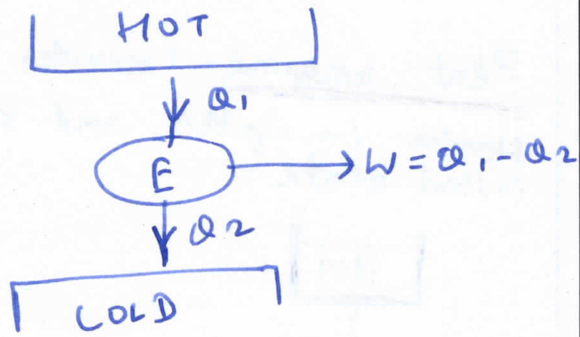
Normal Engine



then we could build engine like this.

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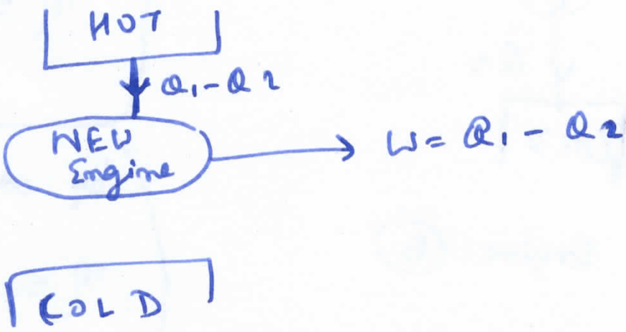


(by composing Clauses violating engine with Normal engine we created Kelvin violating energine)

Entropy

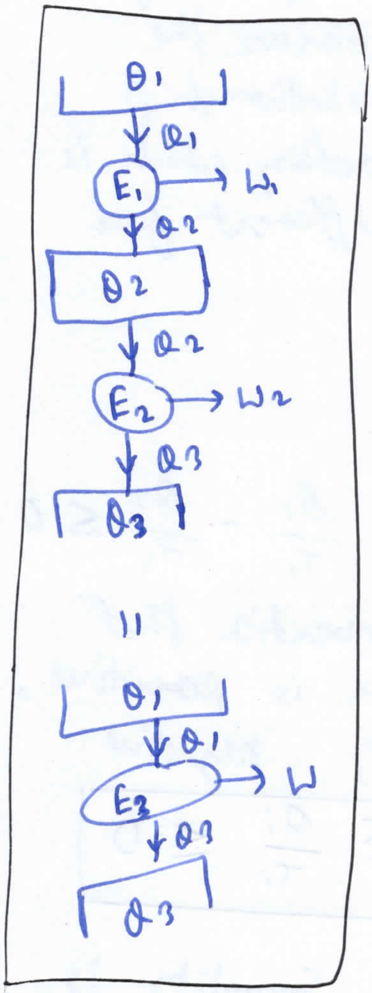
Carnots Theorem: No engine "operating between two given reserviors" can be more efficient than my engine.

COLLORARY: All reversible engines "____" are equally efficient.

Take Reversible Engine,

$$\frac{Q_1}{Q_2} = f(\theta_1, \theta_2)$$

where θ_1 and θ_2 are emperical temperature of the two reservoirs



let E_1 & E_2 be reversible engine which are composed in ~~the~~ the way as shown in diagram.

So:

$\underline{E1}$: $\frac{Q_1}{Q_2} = f(\theta_1, \theta_2)$

$\underline{E2}$: $\frac{Q_2}{Q_3} = f(\theta_2, \theta_3)$

$\underline{E3}$ $(\theta_1 \to \theta_3)$: $\frac{Q_1}{Q_3} = f(\theta_1, \theta_3)$

$$\Rightarrow \frac{Q_1}{Q_3} = f(\theta_1, \theta_3) = \frac{Q_1}{Q_2}\frac{Q_2}{Q_3} = f(\theta_1, \theta_2) \times f(\theta_2, \theta_3)$$

$$\Rightarrow \boxed{f(\theta_1, \theta_3) = f(\theta_1, \theta_2)\, f(\theta_2, \theta_3)}$$

→ we want to ~~p~~ look for a function which has this property.

Your mathematician friend will say that solution is

$$f(\theta_1, \theta_2) = \frac{T(\theta_1)}{T(\theta_2)}$$

$T(\theta_i)$ is some function of θ_i.

So, we conclude ; $\frac{Q_1}{Q_2} = \frac{T_1}{T_2}$; where $T_1 = T(\theta_1)$, $T_2 = T(\theta_2)$

$$\boxed{\eta_r = 1 - \frac{T_2}{T_1}}$$

efficiency of Reversible machine

T : Thermodynamic Temperature

~~(because it must~~ be

Thermodynamic Temperature must be a function of θ.

→ to find the function, we look at efficiency of reversible heat engine, ~~of~~ and this allows us to introduce the Thermodynamic Temperature; namely the relationship between something universal (T) and something which is an emperical thing (θ) which can be different for different substances.

Carnots Theorem: $\boxed{\eta \leq \eta_r}$

$$\Rightarrow \quad 1 - \frac{Q_2}{Q_1} \leq 1 - \frac{T_2}{T_1} \quad \Rightarrow \quad \frac{Q_1}{T_1} - \frac{Q_2}{T_2} \leq 0$$

$\Rightarrow \boxed{\frac{Q_1}{T_1} - \frac{Q_2}{T_2} \leq 0}$ if you introduce a convention that Q heat entering the engine is positive; & Q heat leaving the engine is negative then we can write more generally $\boxed{\sum_i \frac{Q_i}{T_i} \leq 0}$

OR

for closed cycle, $\boxed{\oint \frac{\delta Q}{T} \leq 0}$ Equality is only for Reversible.

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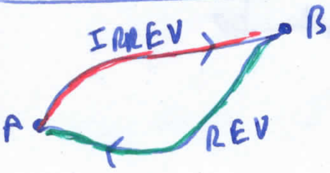
for reversible $\oint \frac{\delta Q}{T} = 0$ so; The integrand must be a total differential.

→ This motivates our defination of entropy as total differential.

DEFINATION | Entropy S as $\boxed{dS = \frac{\delta Q_r}{T}}$

$S \Rightarrow$ Function of State (that will not depend on the path on which you go)



since S is function of state; not of the path; so define it through the Reversible path

$$\oint \frac{\delta Q}{T} \le 0 \Rightarrow \underset{\text{IRR}}{\int_A^B} \frac{\delta Q}{T} \le \underset{\text{REV}}{\int_A^B} \frac{\delta Q_r}{T}$$

$$\Rightarrow \underset{\text{IRR}}{\int_A^B} \frac{\delta Q}{T} \le \int_A^B dS = S_B - S_A$$

$$\Rightarrow \boxed{\int_{A\,\gamma}^{B} \frac{\delta Q}{T} \le S_B - S_A}$$

γ is some path in parameter space.

differential version $\boxed{dS \geq \frac{\delta Q}{T}}$

In particular; for thermally isolated system $\delta Q = 0$

$\Rightarrow \boxed{dS \geq 0}$ This defines sort of ... ..."ARROW OF TIME"

"Microscopic laws typically are reversible in time; its the thermodynamics which tells you it is not".

Once we have entropy; we can write first ~~law~~ law as

$$dU = \underbrace{TdS}_{\delta Q} - \underbrace{PdV}_{\delta W}.$$

similarly; ~~$\frac{\delta Q}{dT}$~~ $C = \frac{\delta Q}{dT} = T\frac{dS}{dT}$

Third law The Entropy of an equilibrium system at $T = 0$ is exactly equal to zero.

$S = 0$ at $T = 0$. on an equilibrium system.

~~$S(T,X) = \int_0^T \frac{C(X)}{T} dT + S(0,X)$~~

$$\boxed{S(T,X) = \int_0^T \frac{C(X)}{T} dT + S(0,X)}$$

$S(0,X) = 0$

X stands for different ~~parameter~~ parametes & system.

by Third law; we say $S(0,X) = S(0)$ is independent of system (because X was something which was ~~disting~~ distinguishing the system)

Since $S(0)$ is independent of systems, we can set it to be zero. $S(0) = 0$ ~~st~~ set.

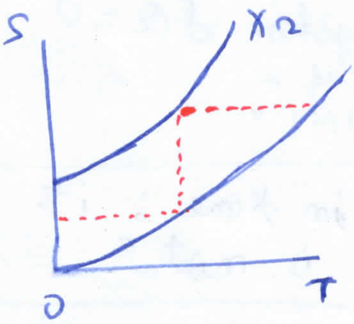
$$S(T,x) = \int_0^T \frac{C(x)}{T}\, dT$$

completely fixes S.

Fixes $S(T)$ completely ie.

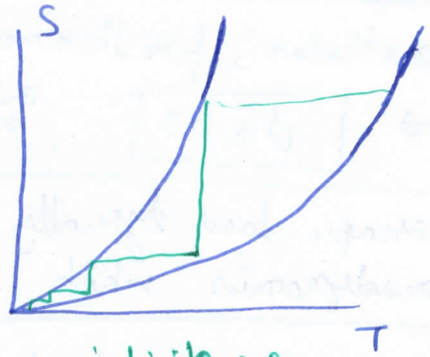
∴ "You can think of 3rd law something which fixes entropy completely".

★ As a consequence, it is impossible to cool the system to $T=0$ by finite number of ~~states~~ steps.



(if $S(0) \neq 0$)

However



infinite no. of steps needed to reach zero

Thermodynamic potentials are given by MAGIC SQUARE.

□

Most interesting for statistical physics is free energy is

$F = U - TS$ $\boxed{dF = -SdT - PdV}$

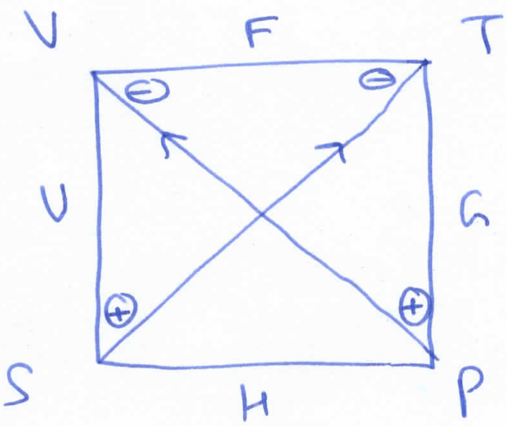
Thermodynamic Potentials

Thermodynamic potentials "determine the equilibrium state of a system under various constraints". Each potential is a function of state and has proper independent variables.

When its proper variables are held constant, stable equilibrium corresponds to the minimum of the potential.

Potential	Expression	Variables	Differential Form
Internal Energy	U	S, V	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	S, P	$dH = TdS + VdP$
Free Energy	$F = U - TS$	T, V	$dF = -SdT - PdV$
Gibbs Energy	$G = U - TS + PV$	T, P	$dG = -SdT + VdP$

Magic Square



① TD potentials are total differentials when expressed in there variables.

$$dU = TdS - PdV$$

(sign ⊕ ⊖) near potential.

[↗ ↑ carry the same info]

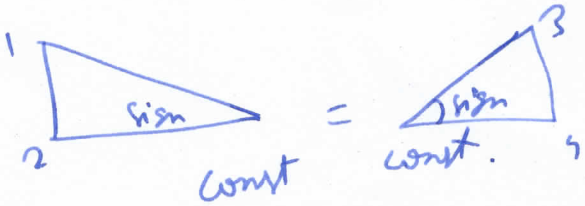
② Legendre Transform

$U(V, S) \to F(V, T)$ $F = U - TS$, $T = \frac{\partial U}{\partial S}$

(sign ⊕ ⊖) near potential.

③ Maxwells Relations

$$\left(\frac{\partial V}{\partial S}\right)_P = \left(\frac{\partial T}{\partial P}\right)_S$$



proof) $dH = TdS + VdP$; $\frac{\partial^2 H}{\partial S \partial P} = \frac{\partial T}{\partial P}\Big|_S = \frac{\partial V}{\partial S}\Big)_P$

Lecture 2: Basics of Statistical Physics; Liouville's Equation, Gibbs Distributions and Ensembles.

(b) Basics of Statistical Physics

(i) Quantum v/s Classical

In Quantum System a Microstate is described by Pure State which lives in a Hilbert Space

~~Pure Space~~
Pure State $\in$ Hilbert Space.

A Macrostate $\simeq$ Density Matrix ρ

$$\rho = \sum_{\alpha} p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$$

p_α: Probability that corresponding state $|\psi_\alpha\rangle$ happens.

It describes ensemble of pure state.

~~ρ describes~~ ρ describes a Statistical Ensemble of Quantum States.

- $Tr\,\rho = 1$ (ρ is appropriately normalized)
- Time Evolution $\dfrac{\partial \rho}{\partial t} - \dfrac{1}{i\hbar}[H, \rho] = 0$; VON-NEUMANN EQUATION

ρ is an operator.

Observable is an operator A

$\therefore$ Expectation Value $a = \langle A \rangle = Tr(\rho A)$

ESTIMATE for Validity of Classical Physics

given a gas; we ask momenta of molecule; and what ~~are~~ is the volume each molecule has.

Thermal Fluctuations can be estimated as follows.

$$\frac{(\Delta p)^2}{2m} \simeq k \cdot T \quad , \quad \Delta x = \left(\frac{V}{N}\right)^{1/3}$$

(estimate for Δp)

Δx is like 1/3 root of volume per molecule.

& we require $\Delta x \Delta p \gg \hbar$; then we are in classical regime

$\left(\frac{V}{N}\right)^{1/3} \sqrt{2mk \cdot T} \gg \hbar$

$\cancel{n = \frac{N}{V}}$ $\ni$ $\boxed{kT \gg n^{2/3} \frac{\hbar^2}{2m}}$ This is ~~the~~ condition for our Classical physics to be valid.

$n \equiv \frac{N}{V}$

* Classical physics is OK if
  - T is very high
  - n is very small.

In Classical Physics

- A Microstate describing N particles of our gas corresponds to a state in $\mathbb{R}^{6N}$ phase space. Time Evolution of this point in phase space is given by Hamilton's Equation.

- A Macrostate is described by Probability distribution (on phase space) on phase space
  w (it will be a function of phase space.

With measure $\mu = w(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n, \vec{p}_1, \ldots, \vec{p}_N)\, d\tau_N$

$$d\tau_N = \frac{1}{N!\,(2\pi\hbar)^{3N}} d^3p_1 d^3p_2 \ldots d^3p_n d^3q_1 \ldots d^3q_N$$

integration measure on phase space

This factor can be thought of as convinient chosen pre-factor so that integrating w over whole phase is 1 (for normalization)

$$\int w \, d\tau_N = 1$$

(ie; we are saying that the probability of finding the dot somewhere in phase space is 1)

$2\pi\hbar$ comes from Quantum Mechanics.

<u>Statistical Ensemble</u> $\simeq$ Having a large number of boxes with a given ~~microst~~ macrostate. If Randomly open one Box, we find a microstate with probability μ.

<u>Observable</u> is a function on the phase space $A(p, q)$

Average Value of $A = \langle A \rangle = \int A(p, q)\, w(p, q, t)\, d\tau_N$

	QUANTUM	CLASSICAL
Stage	Hilber Space $\mathcal{H}$	Phase Space Σ_N
Calculation	$Tr(\cdot)$	$\int \cdot \, d\tau_N$
Statistical Ensemble	Density Matrix : ρ Normalized $Tr\,\rho = 1$	Probability distribution w $\int w \, d\tau_N = 1$
Observable	Operator A	Phase Space Function $A(p, q)$
Average	$\langle A \rangle = Tr(\rho A)$	$\langle A \rangle = \int A(p, q)\, w(p, q, t)\, d\tau_N$

The corresponding Classical Limit of Neuman Equation is Liouville's Equation.

(iii) <u>LIOUVILLE'S EQUATION</u>

Von-Neumann : <u>QM is Unitary</u>

Classical limit: $\frac{1}{i\hbar}[\hat{q}, \hat{p}] = 1 \longleftrightarrow \{q, p\} = 1$

$$[\hat{f}, \hat{g}] \longleftrightarrow i\hbar \{f, g\}$$

$$\frac{\partial w}{\partial t} = \frac{1}{i\hbar} \, i\hbar \{H, w\} \Rightarrow \boxed{\frac{\partial w}{\partial t} = \{H, w\}}$$

Liouville's Equation,

$$\Rightarrow \frac{\partial w}{\partial t} + \{w, H\} = 0$$

$$\Rightarrow \frac{dw}{dt} = 0$$

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial t} + \{w, H\} = 0}$$ Liouville's Equation.

* Consequence of Statistical Physics

since $\frac{dw}{dt} = 0 \Rightarrow$ Thus w is an integral of motion along trajectory.

- ∴ we can solve by taking $w = \text{constant}$ (say)

If you take probability distribution as constant; in other words that you are finding ~~microstat~~ microstates with equal probability;

Then this is a statistical ensemble which you can use and is called ~~MICRO CANONICAL MICRO~~

MICRO CANONICAL ENSEMBLE.

(pain for calculation)

- $w = w$ (Integrals of motion)

$= w \, (E , \vec{P} , \vec{M})$; comes from symmetry of space...

angular momentum (not interesting because we can go to rotating frame)

~~the~~ momentum ... not very interesting because we can see the comoving frame.

$$= w(E)$$

$$\boxed{w = w(E)}$$

Two Systems

$w_{12} = w_1 w_2$ (if they are non-interacting; then it is just product)

$\Rightarrow \log w_{12} = \log w_1 + \log w_2$

$E \Rightarrow$ Total energy of the ~~system~~ system.

$E_1 \Rightarrow$ Energy of system 1

$$\log W_{12} = \log W_1 + \log W_2$$

$$\Rightarrow \log W_{12}(E)) = \log(W_1(E)) + \log(W_2(E_2))$$

looks like sum of energies.

$$\beta E = \beta E_1 + \beta E_2$$

$$\begin{array}{ccc} \| & \| & \| \\ \log W_{12} & \log W_1 & \log W_2 \end{array}$$

(looks like sum of energy)

$$\Rightarrow \log W = \beta E \Rightarrow \boxed{W = e^{\beta E}}$$

First consequence

Gibb's Distribution for Canonical Ensemble.

Second Consequence of Liouvilles Equation.

is Liouville's Theorem

$$\int w \, d\tau_N = 1 \quad \Rightarrow \quad \frac{d}{d\tau}\left(\int w \, d\tau_N = 1 \right)$$

$$\Rightarrow 0 = \int \left(\underbrace{\frac{dW}{dt} d\tau_N}_{} + w \frac{d}{dt}(d\tau_N) \right)$$

This is equal to zero

$$\Rightarrow \quad 0 = \int w \frac{d}{dt}(d\tau_N) \quad \Rightarrow \quad \boxed{\frac{d}{dt}(d\tau_N) = 0}$$

So volume element does not change as we evolve in time.

CAT PICTURE.

Behind ~~vector~~ Hamilton vector field; we can think there is ~~Unitary~~ Unitary Origin

Here we have been deriving everything ~~the~~ thing as a consequence of QM being unitary.

(iii) GIBBS DISTRIBUTION

Defination: Statistical (Von Neumann) Entropy is defined as $S = -k\,\mathrm{Tr}(\rho \log \rho)$.

set $k=1$ $\Rightarrow$ $\boxed{S = -\mathrm{Tr}(\rho \log \rho)}$

Assume system has some observables A_i have expectations Values a_i ; $a_i = \langle A \rangle = \mathrm{Tr}(\rho A_i)$; $\mathrm{Tr}(\rho) = 1$

Principle of Maximal Mess: The density matrix ρ representing the equilibrium state of a microsystem is such that it maximizes the Statistical Entropy, provided the ~~macros~~ (Macroscopical) Constraints.

This leads to Variational principle.

So; we want to Extremize functional I given by

$I = S$ subject to some constraints.

So we add lagrange multipliers.

$$I = S - \sum_i \lambda_i \left(\mathrm{Tr}(\rho A_i) - a_i \right) - \lambda_0 (\mathrm{Tr}\rho - 1)$$

$$\boxed{I = S - \sum_i \lambda_i \left(\mathrm{Tr}(\rho A_i) - a_i \right) - \lambda_0 (\mathrm{Tr}\rho - 1)}$$

We want to maximize this functional.

The thing to vary in this functional is density matrix ρ.

$$\delta I = \delta S - \sum_i \lambda_i (\mathrm{Tr}(A_i \delta\rho)) - \lambda_0 \mathrm{Tr}(\delta\rho)$$

$$\delta S = -\mathrm{Tr}(\delta\rho \log \rho) - \mathrm{Tr}(\delta\rho) = -\mathrm{Tr}(\delta\rho \log \rho + \delta\rho)$$

$\Rightarrow \delta I = Tr\left[(-\log\rho \ -1 \ \lambda_i A_i - \lambda_0)\,\delta\rho\right]$

we know $\delta I = 0$ $\quad\therefore\quad B := (-\log\rho - 1 - \lambda_i A_i - \lambda_0)$

$\Rightarrow \quad \delta I = Tr(B\,\delta\rho) = 0$

true for any $\delta\rho$.

However $\delta\rho$ is not any matrix ; if it was any matrix then we could conclude $B = 0$.

~~How~~ However ρ is Hermitial matrix

↳ there is some sutlle things to prove

but still the conclusion is same ; $B = 0$.

$$\Rightarrow \quad \log\rho = -\sum_i \lambda_i A_i - (1+\lambda_0)$$

$$\Rightarrow \quad \boxed{\rho = e^{-(1+\lambda_0)} \cdot e^{-\sum_i \lambda_i A_i}}$$

Now lets impose constraint.

$$Tr\rho = Tr\left(e^{-(1+\lambda_0)}\, e^{-\sum_i \lambda_i A_i}\right)$$

$$1 = e^{-(1+\lambda_0)}\, Tr\left(e^{-\sum_i \lambda_i A_i}\right)$$

$$\Rightarrow \quad 1 = e^{-(1+\lambda_0)} Z \ \cancel{Tr(\lambda Z)}$$

$\therefore \ \cancel{Z = Tr\, e^{-\sum \lambda_i A_i}}$

$$\Rightarrow \quad \boxed{e^{-(1+\lambda_0)} = \frac{1}{Z}} \qquad Z = Tr\left(e^{-\sum_i \lambda_i A_i}\right)$$

$$\boxed{\rho = \frac{1}{Z}\, e^{-\sum_i \lambda_i A_i}}$$

This is the density matrix consistent with all the constraints, and which maximizes the corresponding function.

Called Generalized Gibbs Distribution.

$$\boxed{Z = \mathrm{Tr}\left(e^{-\sum_i \lambda_i A_i}\right)}$$ called Partition Function.

Immediate Properties

Z does not depend on a_i but they are function of λ_i

$$\boxed{a_n = -\frac{\partial}{\partial \lambda_n} \log Z}$$

$$\boxed{S = \sum \lambda_n a_n + \log Z} \quad \boxed{\lambda_i = \frac{\partial S}{\partial a_i}}$$

fast

Homework.

Property 1 || $-\frac{\partial}{\partial \lambda_i} \log Z = -\frac{1}{Z}\frac{\partial Z}{\partial \lambda_i} = -\frac{1}{Z}\mathrm{Tr}\left(\frac{\partial}{\partial \lambda_i} e^{-\sum_i \lambda_i A_i}\right)$

$$= -\frac{1}{Z}\mathrm{Tr}\left(-A_i e^{-\sum A_j \lambda_j}\right)$$

$$\Rightarrow -\frac{\partial}{\partial \lambda_i}\log Z = \mathrm{Tr}\left(A_i \frac{e^{-\sum A_j \lambda_j}}{Z}\right) = \mathrm{Tr}(A_i \rho) = a_i$$

$$\Rightarrow \boxed{a_i = -\frac{\partial}{\partial \lambda_i}\log Z}$$

(iv) CANONICAL ENSEMBLE

The only observable is H ; $U = \langle H \rangle = \mathrm{Tr}(\rho H)$

(only energy can fluctuate)

(The only A_i here is H (Hamiltonian))

The corresponding ensemble is called Canonical Ensemble.

" " lagrange multiplier to H is $\lambda_H = \beta$ (rename to β)

Corresponding lagrange multiplier.

So; Gibbs distribution reduces to

$$\rho = \frac{1}{Z} e^{-\beta H} \qquad \text{where; } Z = \mathrm{Tr}(e^{-\beta H})$$

Connection with Thermodynamics

we define
$$\delta W \equiv Tr(\rho\, dH)$$
$$\delta Q \equiv Tr(H\, d\rho)$$

so; $dU = d(Tr\, \rho H) = Tr(\rho\, dH) + Tr(H\, d\rho)$

$$dU = Tr(d\rho\, H) + Tr(\rho\, dH)$$
$$= \delta Q + \delta W$$

$$\boxed{dU = \delta Q + \delta W}$$ This is the first law provided we call $\delta W = Tr(\rho\, dH)$ work & $\delta Q = Tr(d\rho\, H)$ Heat.

Why? (we can motivate the definations)

take $\rho = \sum p_m |m\rangle\langle m|$ $\Rightarrow \delta\rho = \sum \delta p_m |m\rangle\langle m|$

take $H = \sum l_m |m\rangle\langle m|$

(in same basis)

$$\delta Q = Tr(H\, d\rho) = Tr\left(\delta p_m |m\rangle\langle m|\; l_n |n\rangle\langle n| \right)$$

in orthogonal basis; $\langle m|n\rangle = \delta_{mn}$

$$\Rightarrow \boxed{\delta Q = \sum_m \delta p_m \cdot l_m}$$

similarly
$$\boxed{\delta W = \sum_m p_m \cdot \delta l_m}$$

Now we can see why we should call one as Work & the other as Heat.

keeping the probability ~~distribution~~ distribution fixed; we are changing the energy level.

It can easily be done by some conservative forces like Electric field.

Shift Energy levels, which can be done by conservative ~~forces~~ fields.

so we better call it work (Motivation)

δQ ↦ we keep energy level fixed.
↪ but we change probabilities
... we somehow redistribute probabilities.

→ because we change probabilities....
... because it is so dirty it is natural to call it Heat (😐 haha...)

Now, to define Temperature & Entropy we need second law.

$$\delta S = -\delta \mathrm{Tr}(\rho \log \rho) = -\mathrm{Tr}(\delta\rho \log \rho + \delta\rho)$$

if $\mathrm{Tr}(\rho) = 1 \Rightarrow \underline{\mathrm{Tr}(\delta\rho) = 0}$. (because of normalization of density matrix.)

$$\Rightarrow \boxed{\delta S = -\mathrm{Tr}(\delta\rho \log \rho)}$$

∴ In canonical Ensembel.

$$\Rightarrow \delta S = -\mathrm{Tr}(\delta\rho(-\beta H - \log Z))$$

$\mathrm{Tr}(\delta\rho \log Z) = \underbrace{\log Z}_{\text{Just a number}} (\underbrace{\mathrm{Tr}(\delta\rho)}_{\to 0})$

$= 0$

$$\Rightarrow \delta S = +\mathrm{Tr}(\delta\rho \beta H)$$

$$\Rightarrow \boxed{\delta S = \beta \mathrm{Tr}(\delta\rho H)}$$

$$\Rightarrow \boxed{\delta S = \beta \delta Q}$$

δS: variational of Statistical Entropy

β: a lagrange multiplier.

Remind Clausius; $\delta S = \dfrac{\delta Q}{T}$; where T is thermodynamic Temperature.

so; we have to identify that the lagrange multiplier β is given by $\dfrac{1}{T}$.

~~so; 2nd law really provides~~ us
so; we have the total description of
Canonical Ensemble $\rho = \frac{e^{-\beta H}}{Z}$; $\beta = \frac{1}{T}$

We can make another defination,

$$\boxed{F \equiv -T \log Z} \rightarrow \text{FREE ENERGY}$$

↪ It would be nice if this free energy is the free energy we know from thermodynamics.

Is it?

looking at one of the properties $S = \sum \lambda_i a_i + \log Z$

$$S = \beta U + \log Z \quad \cancel{\Rightarrow TS = T\beta U + T \log Z}$$

$$\Rightarrow \cancel{TS = T\beta U + T\log Z \Rightarrow TS =}$$

$$\cancel{\Rightarrow T\beta} \quad TS = T\beta U + T \log Z$$

$$\Rightarrow \boxed{TS = U + T \log Z}$$

$$\boxed{-T \log Z = U - TS}$$

from the formula; we find $F \equiv U - TS$ as in TD.

so; $-T \log Z$ is good defination of FREE ENERGY

GRAND CANONICAL

Its not only energy; but no. of particles are also allowed to fluctuate.

so; corresponding partition function is given by

$$Z_G = \mathrm{Tr}\left(e^{-\beta H + \beta \mu N} \right)$$

N is the operator for particle number.

μ is the Chemical potential.

$$Z_G = \mathrm{Tr}\left(e^{-\beta H + \beta \cdot \mu \cdot N}\right)$$

Lecture 3: Thermodynamic Approach to Phase Transitions: Ideal Gas, Van der Waals Fluid, Reentrant P.T.

Recap! The equilibrium state of a system subject to some constraints $Tr(\rho A_i) = a_i$, $Tr(\rho) = 1$

Maximizes Stat. Entropy $S = -Tr(\rho \log \rho)$

$\ni$ $\rho = \frac{1}{Z} e^{-\sum_i \lambda_i A_i}$; $Z = Tr\left(e^{-\sum_i \lambda_i A_i}\right)$ Gibb's Distribution

---

(c) THERMODYNAMIC APPROACH TO PHASE TRANSITIONS

(i) Statistical Machinary at Work. ... Ideal Gas.

- Given Hamiltonian H which characterizes the system.
- First calculate Partition function Z.

  use canonical ensemble : $Z = Tr\left(e^{-\beta H}\right)$; $\beta = \frac{1}{T}$

- Calculate Free Energy to know Z

$$\therefore F = -T \log Z$$

(Thermodynamic potential free energy)

from T.D., we know ; $F = F(T, V, N)$

$$dF = -S\,dT - P\,dV + \mu\,dN.$$

$$S = -\frac{\partial F}{\partial T}\bigg|_{V,N} \; ; \; P = -\frac{\partial F}{\partial V}\bigg|_{T,N} \; ; \; \mu = \frac{\partial F}{\partial N}\bigg|_{T,V}$$

(This is how we derive properties of system starting from Statistical physics)

Example 1 Ideal Gas

N non-interacting particles.

Each of them characterized by position & momenta

$$\vec{r}_i \; , \; \vec{p}_i \qquad , \; i = 1, \dots, N$$

Say ~~a~~ all of them have same mass

So; Hamiltonian is $H = \dfrac{p_i^2}{2m}$

- $\underbrace{Z = \mathrm{Tr}\left(e^{-\beta H}\right)}_{\text{Quantum Mechanical Prescription}} = \underbrace{\int d\vec{r}_1 \dots d\vec{r}_N \, d\vec{p}_1 \dots d\vec{p}_N \, e^{-\beta H}}_{\text{Classical prescription.}}$

$$\int d\vec{r}_1 \dots d\vec{r}_N = (V)^N \; : \; V \Rightarrow \text{volume of gas.}$$

$$\Rightarrow Z = V^N \int d\vec{p}_1 \, e^{-\beta \frac{\vec{p}_1^{\,2}}{2m}} \cdot \int \dots \cdot \int \dots$$

$$\cancel{\Rightarrow Z = V^N} \Rightarrow Z = V^N \prod_{i=1}^{N} \underbrace{\int d\vec{p}_i \cdot \exp\left(-\beta \cdot \frac{\vec{p}_i^{\,2}}{2m}\right)}$$

↳ gaussian integral.

$$\cancel{Z = V^N \prod_{i=1}^{N} \left(\sqrt{\frac{2\pi m}{\beta}}\right)} \Rightarrow Z = \quad \sqrt{\frac{2\pi m}{\beta}}$$

(for each momentum on one x...)

In d dimensions:

d... spacetime dimensions $\Rightarrow \int d\vec{p}_i \cdot \exp\left(-\beta \frac{\vec{p}_i^{\,2}}{2m}\right) = \left(\sqrt{\frac{2\pi m}{\beta}}\right)^{d-1}$

$$Z = V^N \left(\frac{2\pi m}{\beta}\right)^{\frac{N(d-1)}{2}}$$

: d space-time dimension.

- Free Energy:

$$F = -T \log Z = -TN \log V - \frac{(d-1)\,TN}{2} \log\left(2\pi m T\right)$$

---

now; ~~to~~ the most ~~[illegible]~~ interesting thing is the get the equation of state for ideal gas.

$$P = -\frac{\partial F}{\partial V}\Big|_{T,N} = \frac{TN}{V} \Rightarrow \boxed{P = \frac{TN}{V}}$$

$PV = TN$

$\Rightarrow$ $PV = NRT$

$$\boxed{v = \frac{V}{N} \Rightarrow \text{specific volume}}$$

$$\boxed{Pv = T}$$

↳ Ideal Gas Law (Equation of State)

Entropy:

$$S = -\frac{\partial F}{\partial T}\Big|_{V,N} = N \log V + \frac{(d-1)}{2} \cdot N \log(2\pi m T) + \frac{(d-1) \cdot N}{2} \cdot \frac{T}{\cancel{T}}$$

$$\boxed{S = N \log V + \frac{(d-1)}{2} N \log(2\pi m T) + \frac{N \cdot (d-1)}{2}}$$

Entropy

Internal Energy

$$U = F + TS = \frac{(d-1)}{2} N T \Rightarrow \boxed{U = \frac{(d-1)}{2} N T}$$

so; U only depends on T
(it is independent of volume V)

From where does the pressure comes from?

naively we can think it comes from F; & F depends on U & S $\Rightarrow$ so; we can think there is some contribution of U to P $\Rightarrow$ so really the particles are hitting the ~~walls~~ walls. There might be some contribution from S.

∴ but it is quite puzzling that contribution from one of them is zero!!

$$P = -\frac{\partial F}{\partial V} = -\frac{\partial}{\partial V}(U - TS)$$

; U depends on T not on V

$\Rightarrow$ so; Only contribution in P is from entropy

$$\Rightarrow \boxed{P = T\frac{\partial S}{\partial V}}$$ Only contribution from Entropy

So; its really that the pressure comes from how you redistribute the particles in your room; rather than particles are actually hitting the walls. because they have some internal energy.

↳ Origin of Pressure in Ideal gas is Entropic.

- $C_V = \frac{\delta Q}{\delta T}\Big|_V = T \cdot \frac{\partial S}{\partial T}\Big|_V = \frac{(d-1)N}{2}$
- $C_p = T \frac{\partial S}{\partial T}\Big|_p = \frac{(d+1)}{2} N$

People are obsessed that (Haha...)

MYER'S LAW

$$K = \frac{C_p}{C_V} = \frac{d+1}{d-1}$$

$d=4$; $K = \frac{5}{3}$

(ii) VANDERWAALS EQUATION

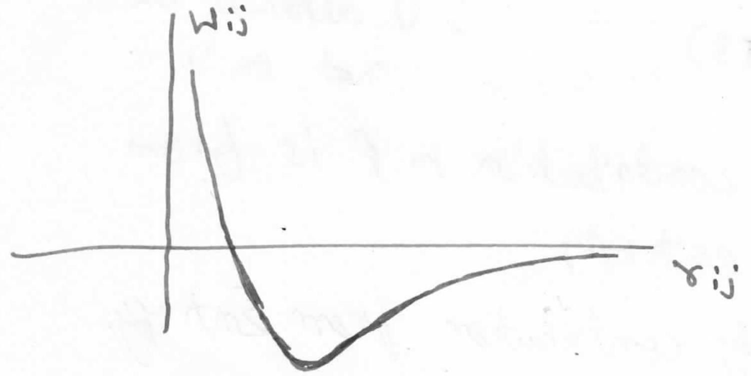
If you want an interacting gas, then it is quite complicated; Interacting Gas

$$H = \sum_i \frac{p_i^2}{2m} + \sum_{i<j} W_{ij}$$

$W_{ij}(r_{ij})$; $r_{ij} = |\vec{r}_i - \vec{r}_j|$

example: we can use the model; $W_{ij} = \frac{A}{r_{ij}^{12}} - \frac{B}{r_{ij}^{6}}$

LENNARD-JONES Potential.



Solved by Cluster Expansion

.... doing this we recover $P = nT(1 + nB_2(T) + n^2 B_3(T) + \ldots)$

$n = \frac{N}{V}$.

Van Der Walls Equation

$$\boxed{\left(p + \frac{a}{v^2}\right)(v - b) = T}$$

$a > 0$; measures attraction between particles.

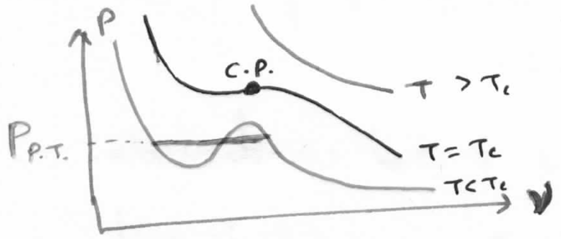
P is effectively bigger due to attraction of particle.

b : related to excluded volume of particles.

$v - b \Rightarrow$ effective volume.

Isotherms in P.V. Diagram.

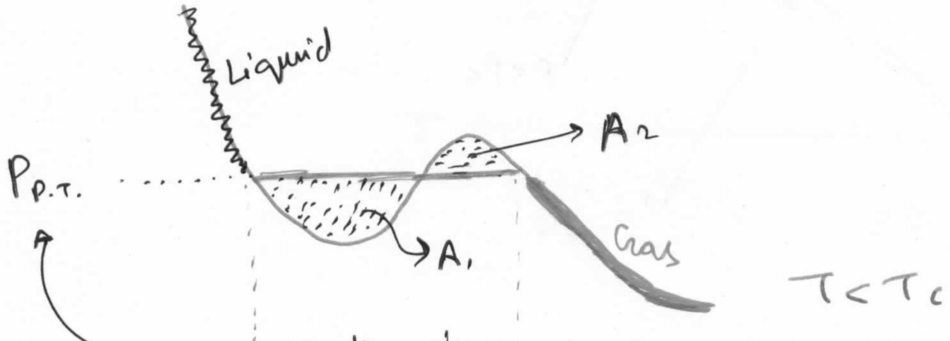
C.P. ≡ Critical Point



$T_c \Rightarrow T$ critical.

We dont see a gas doing this ; but see a phase transition ~~at T_c~~ for $T < T_c$

$P_{P.T.} \Rightarrow P$ Phase Transition



The construction is due to Maxwell ; called Maxwell's Equal Area Law.

$A_1 = A_2$.

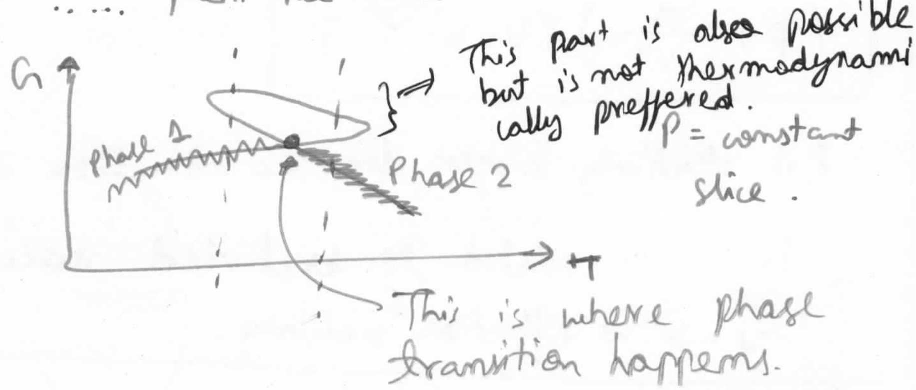
$$\boxed{\oint v\,dP = 0} \Longleftrightarrow \boxed{P_{P.T.}(v_g - v_\ell) = \int_{v_\ell}^{v_g} P\,dv}$$

Maxwell Constructions

(iii) Phase Transitions (Thermodynamic Perspective)

Characterized by Gibb's Free Energy $G = G(P,T)$

Surface $G = G(P,T)$ is continuous; but may intersect ..... then we have Phase Transition.

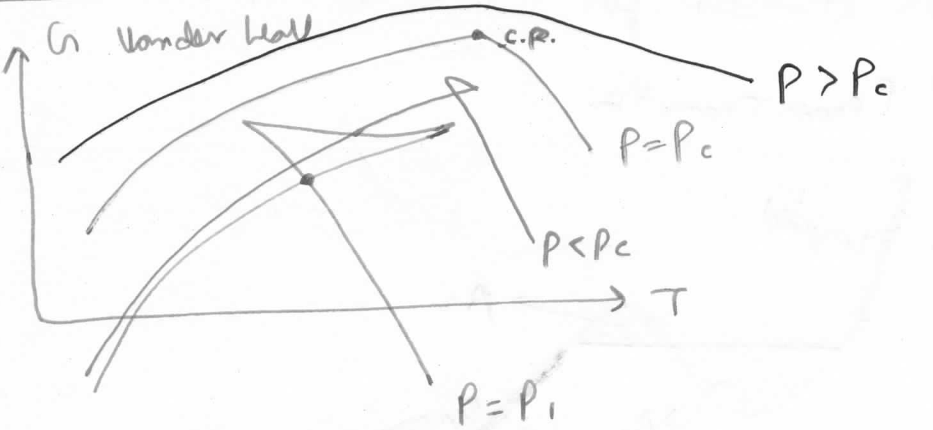


Global Minimum of G corresponds to the preferred phase.

First Order Phase Transition

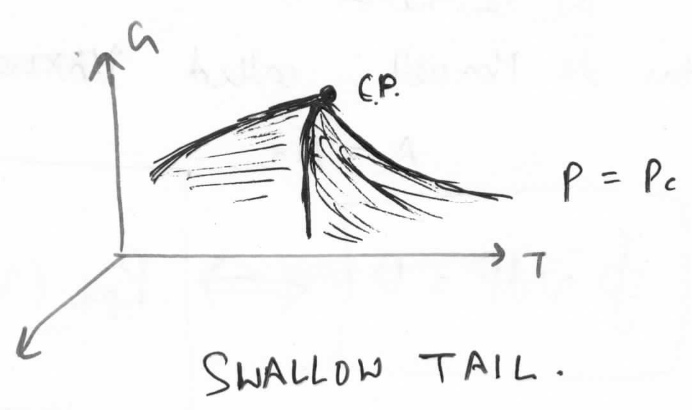
G_{Min} Continuous, G'_{Min} (derivative of G) is not continuous.

Second Order P.T.: G_{Min} continuous, G'_{Min} continuous, G''_{Min} is not.



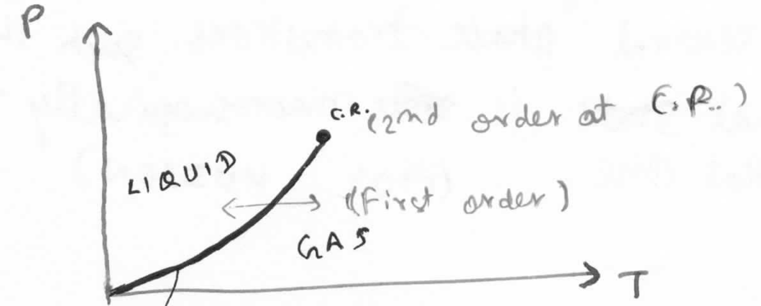
~~3-D picture~~

~~D-D~~ 3-D picture



SWALLOW TAIL.

Phase Diagram.



↳ here two phases co-exist

Co-existence line of two phases. → Shape determined from Clausius-Claperon Equation.

Applies to 1st Order Phase Transition. ←

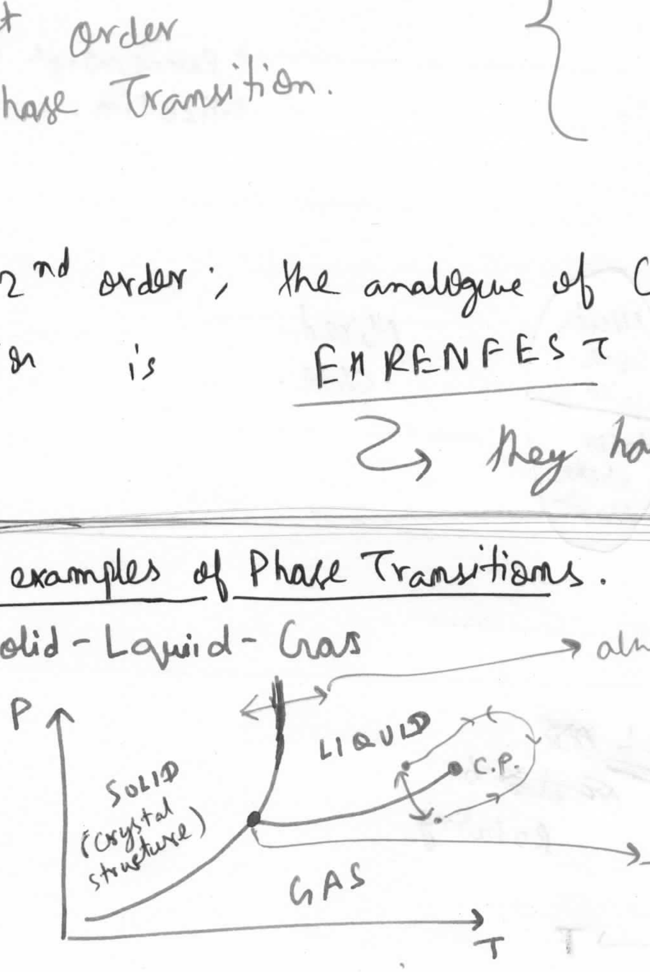
$$\left.\frac{dP}{dT}\right|_{\text{Co-existence}} = \frac{\Delta S}{\Delta V} = \frac{S_g - S_l}{V_g - V_l}$$

For 2nd order; the analogue of Claussius - Claperon equation is EHRENFEST Equation.

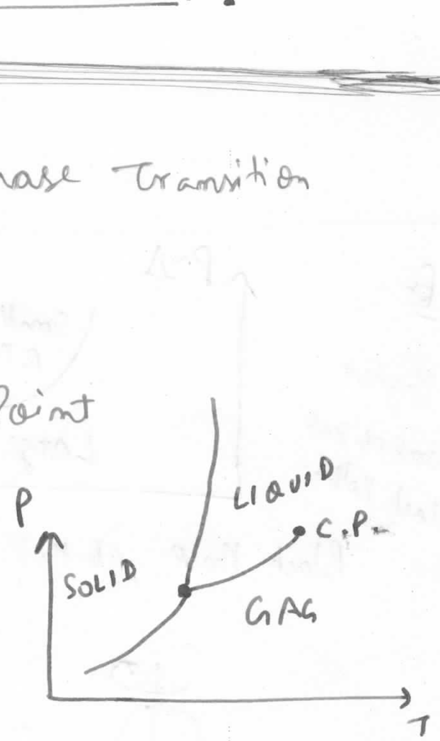
↳ they have Universality.

Other examples of Phase Transitions.

- Solid - Liquid - Gas → always Phase Transition



⇒



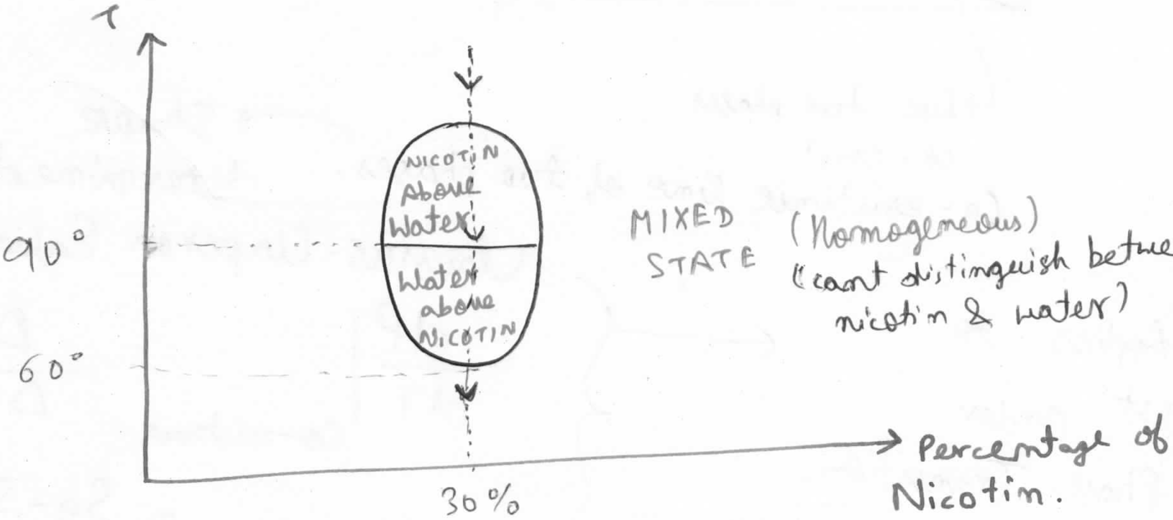
Liquid & gas are not so very different because we can go around C.P. without going through phase transition

↳ But solid is very different from them.

★ RE-ENTRANT Phase Transition.

If monotonic variation of one thermodynamic variable results in several phase transitions such that the initial and final state is macroscopically ~~same~~ similar to initial one. (1904 HUDSON)

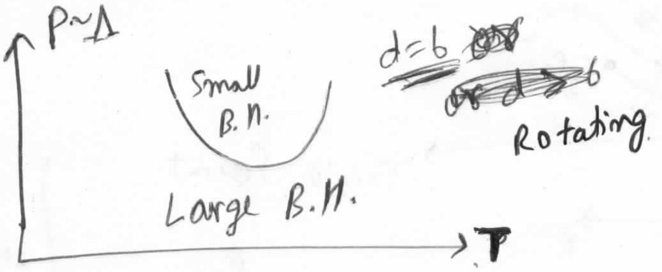
ex



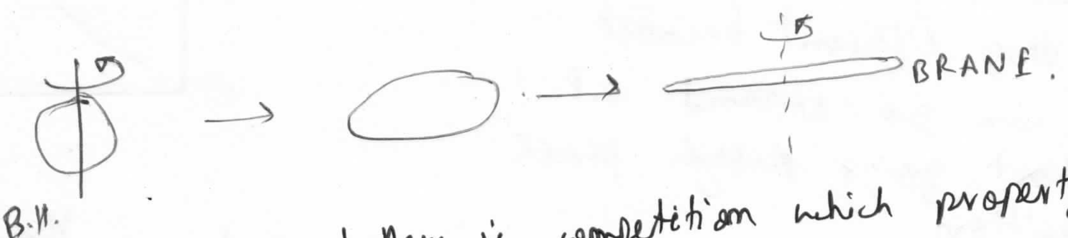
~~Mixed Water ⇒ you~~

Ex

Higher dimensions Black Holes.



Black Hole (B.H.)



and there is competition which property is wining.

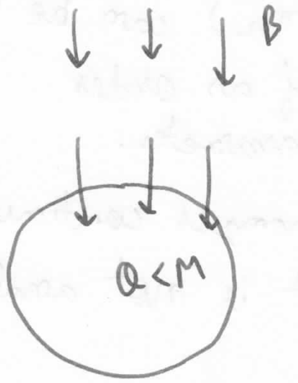
* Black Hole property of B.H. or Brane property of B.H.

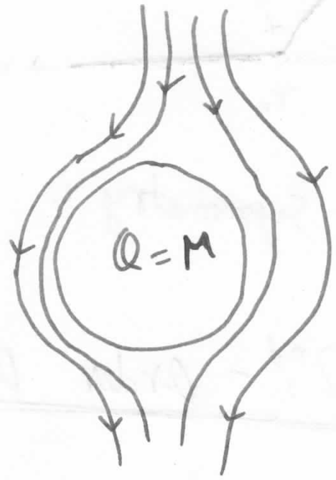
Example

Gravitational Meissner Effect. Known from 1960s.

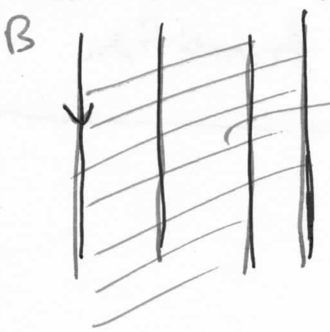
- External B.H. ($Q = M$) (Also for maximally rotating B.H.)

Magnetic field gets expelled from B.H.



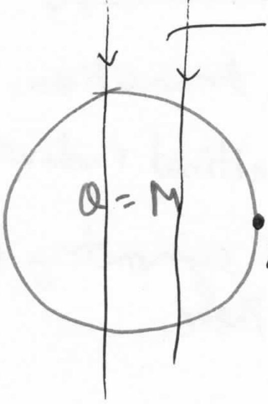


- Cosmic String



→ ϕ (a scalar fields holds together magnetic field)

↳ so that cosmic string does not stretch too far & has finite size of cosmic String.

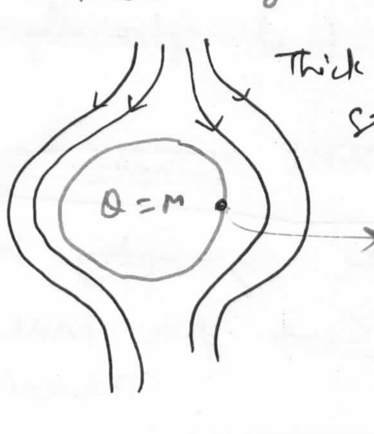


Thin Cosmic String ~~(peneter~~ (pemeterates even extremal B.H.)

$\phi(\pi/2)$ is order parameter

: If Cosmic String ~~bo~~ becomes wider than a critical radius

: Then it goes around.



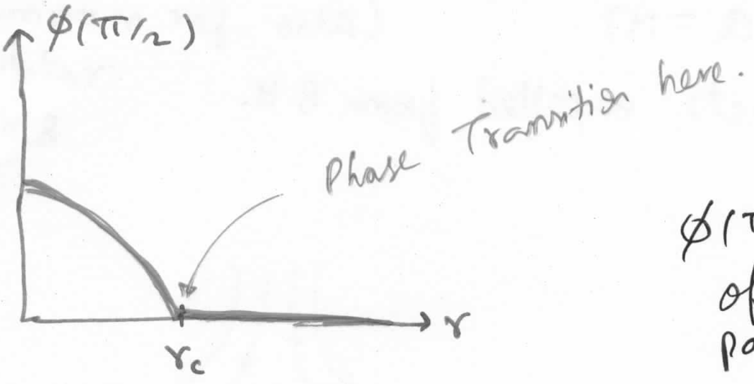
Thick cosmic string.

So; There is a phase transition.

There is a critical radius r_c

$r < r_c \Rightarrow$ penetrate

$r > r_c \Rightarrow$ expell.



$\phi(\pi/2)$ can be thought of as order parameter.

It changes continuously but it is not analytic

spherical symmetry is there

* 2nd - order phase transition here.

* Just magnetic field $\Rightarrow$ no phase transition.
* Magnetic field + scalar $\Rightarrow$ we see phase transition

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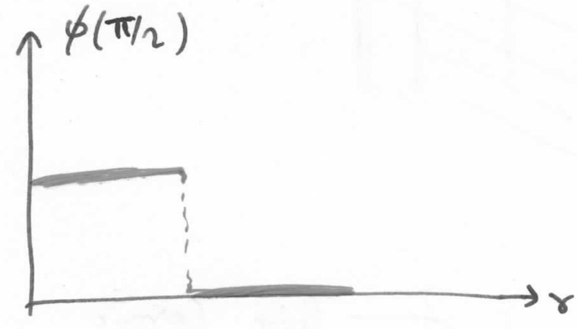
$\boxed{M = J}$



1st order ~~p~~ phase transition.

(order parameter jumps at the critical radius)

~~(not spherically symme~~

spherical symmetry is not there.

~~(more~~ ~~(more syst~~

$\boxed{\text{more symmetry in system you have} \hookrightarrow \text{you have nicer phase transition; meaning higher order phase transition)}}$

Lecture 4: Ising Model, Order Parameter, Mean Field Theory, Critical Exponent Beta.

---

2) Microscopic Theory of Phase Transition.

a) Introduction

Experimental fact: Matter organizes into phases characterized by different macroscopic properties. (still described by same Hamiltonian)

examples) Solid / Liquid / Gas:

Which macroscopic property characterize each state?

- ex Solid has "Rigidity to shear".
- ex Liquid has well defined surface, but not rigidity
- ex gas does not even has well defined surface.

example) Ferromagnetic / Paramagnetic:

- ex Ferromagnetic: has macroscopic magnetization.
- ex Paramagnet: does not.

example) fluid / super fluid ~~(conductor/superconductor)~~

example) Superconductor / Normal matter.

Superconductor has zero resistance (in the same way superfluid has zero viscosity); and expels magnetic field. known as Meissner effect.

"The phases however have to emerge from one Hamiltonian." ... same Hamiltonian works.

Different ~~phas~~ macroscopic properties are consequence of ordering.

---

ex Spins in feromagnet; atoms in crystal .. solid ..

Canonical example: ISING MODEL

(simple mathematical model to model ferromagnet & paramagnetic phase)

The model was first developed by LENZ (1920), but ISING was the one first gave the solution in 1D (1925); hence called ISING MODEL.

Pg 36

ONSAGER (1944) solution in 2d.

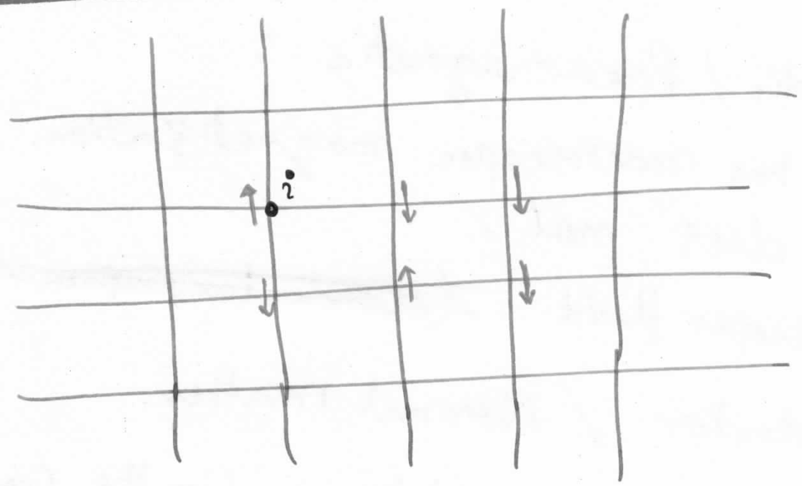
---

nobody knows how to solve in 3d. ... ongoing problem

- 4d & higher it is simple; and the solution is given by Mean Field Theory approximation is exact.

As you increase dimensions; effect of fluctuations become small ... & you get answers.

Model for FERROMAGNET



d-Dimensional lattice.

at each point of lattice you can either have spin up or down.

spins are discrete variables. σ_i

i denotes where we are in the lattice.

$$\sigma_i = \pm 1$$

$$H = -\frac{1}{2} \sum_{ij} J_{ij} \sigma_i \sigma_j$$

(you try to allign spins in same direction)

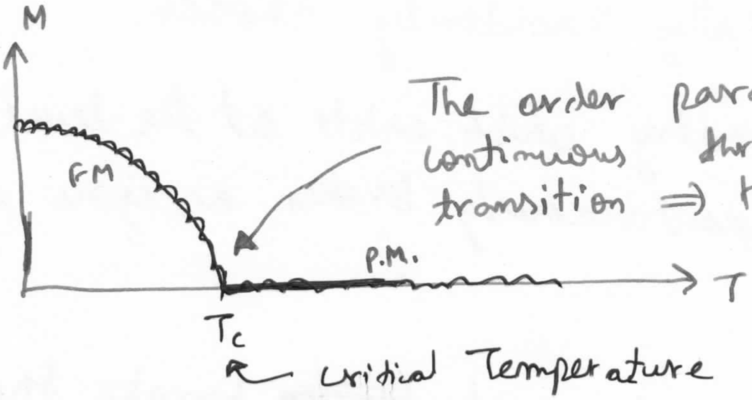
we have two spin interaction & interaction strength is described by the coupling constant J_{ij}.

(we also see, the Ising model belongs to the ~~the~~ same class as transition for water)

• 2 Phases : F.M. (Ferromagnetic) & P.M. (Paramagnetic)

distinguished by order parameter, which here is Magnetization which is given by

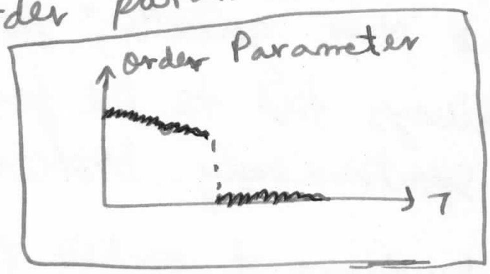
$$M = \langle \sigma_i \rangle = \begin{cases} 0 & \text{P.M.} \\ \neq 0 & \text{F.M.} \end{cases}$$



The order parameter M is continuous through the phase transition ⇒ Hence we call it Continuous Phase Transition.

Non analytic behavior at T_c.

NOTE: 1^{st} order phase Transition;
In the language of Order parameter there will be a jump.



~~Gibbs~~ when you differentiate gibbs energy you get order ~~parameter~~ parameter; so for first order P.T.; there will be a jump in order parameter during phase transition because the first derivative of Gibbs energy is discontinuous

---

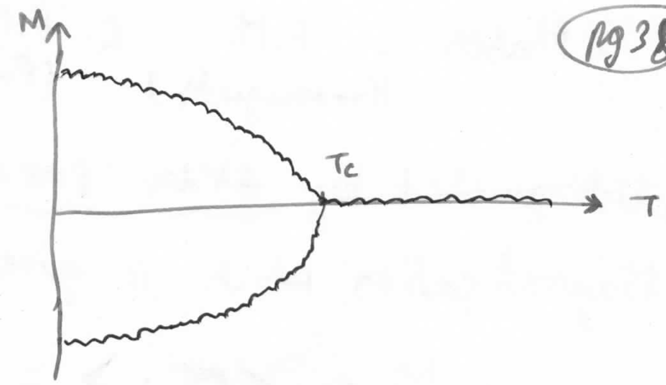
A beautiful symmetry in Hamiltonian

H is invariant under $\sigma_i \to -\sigma_i$

Z_2 ~~symm~~ symmetry.

~~actually we have~~

actually we have other solution: pointing down. (the negative part of M)



Somehow the system chooses one of the state as we reduce the temperature: either pointing up or down : and is basically randomly chosen.

So: the symmetry which exists at the level of Hamiltonian is spontaneously broken as you decrease the temperature.

∴ FM phase spontaneously ~~broke~~ breaks this symmetry. (Randomly choose up or down)

⇝ More generally, the existence of order parameter is always tied to the fact that some symmetry is spontaneously broken.

Existence of order parameter is tied to spontaneous symmetry breaking.

• Exercise | Add magnetic field to this system B: and consider finite system with N sites.

Magnetization $M = \frac{1}{N} \sum_{i=1}^{N} \sigma_i$

find probability of magnetization up & down.

∴ lets find the ratio : $\boxed{n = \frac{P_{M>0}}{P_{M<0}}}$

(Pg 39)

$$H = -\frac{1}{2}\sum_{i,j} J_{ij}\sigma_i\sigma_j - B\sum_i \sigma_i$$

$$m = \frac{P_{M>0}}{P_{M<0}} = \frac{e^{-NMB/T}}{e^{+NMB/T}} \Rightarrow \boxed{m = e^{-2NMB/T}}$$

$M = |M|$.

Limits ① N-finite and set $B \to 0$

$m = 1$; over long periods of time both up & down states equally probable.

then ~~average~~ average magnetization M has to be zero ; $M = 0$

② Thermodynamic Limit $N \to \infty$ (Then we can take $B \to 0$)

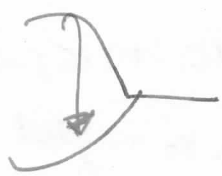
Then $\boxed{m = 0}$ or ∞ (depending on value & direction of B) $\Rightarrow$ means one of them is preffered.

$\therefore$ System remains in a state defined by previous $\underline{B}$.

; and M is now - non trivial.

If we have infinite no. of ~~states~~ sites; it will never happen that all of them swap.

Its infinitely improbable.



So; In thermodynamic limit we have phase transition & the new phase stays there forever (only in the thermodynamic limit)

~~General~~ General Conclusion Phases and Phase Transitions are defined sharply only for infinite systems.

$\therefore$ In practice ; if the system is sufficiently big: we can approx. ~~it~~ it to infinite system.

(Macroscopic system (which in practice is never infinite) ... has long time scale for which they stay in that phase)

1b) Mean Field Theory (MFT) (independent of dimensions)

- Idea: Replace all multi-body interactions with an effective one-body interaction with an average Mean Field. (Molecular)

The idea is sort of equivalent to neglecting fluctuations.

- Effectively this amounts to ~~th~~ expanding Hamiltonian H to zeroth order in fluctuations.

(4D and higher; .... it is exact)

Application to ISING

$$\sigma_i = \sigma_i - \langle\sigma_i\rangle + \langle\sigma_i\rangle = \sigma_i - M + M$$

Then

$$\sigma_i\sigma_j = ((\sigma_i - M) + M)((\sigma_j - M) + M)$$

$$= (\sigma_i - M)(\sigma_j - M) + M(\sigma_i + \sigma_j) - M^2$$

This is actually the fluctuation.

∴ if this difference is small

if ~~too~~:

$(\sigma_i - M)(\sigma_j - M)$ is of $O(\varepsilon^2)$ if fluctuation is small

so neglect this.

ie; if $(\sigma_i - M) \sim O(\varepsilon)$

$J_{ij} = J_{ji}$

Then

$$H = -\frac{1}{2}\sum_{ij} J_{ij}(\sigma_i + \sigma_j)M + \frac{1}{2}\sum_{ij} J_{ij} M^2$$

notation

$$J \equiv \sum_i J_{ij}$$

"Assuming Finite Range of Interaction".

for a given j; sum over all i.

In principle this should depend on j; but you assume lattice is uniform; so each site is exactly same as any other site.

so; J is independent of j, because lattice is uniform.

Then $\boxed{H = -MJ\sum_i \sigma_i \quad + \frac{1}{2}NJM^2}$

Mean Field Theory, Ising Hamiltonian.

↪ We replaced 2 point particle interaction with 1 spin interaction with mean field M.

<u>We replace many-brody interaction with one body interaction in some external field.</u>

<u>Applying Statistical physics machinery</u>

$$Z = Tr(e^{-\beta H}) = \sum_{\sigma_1=\pm1} \cdots \sum_{\sigma_N=\pm1} e^{-\beta H}$$

↪ sum over all possibilities (over all sites & all possible spin on those sites)

$$= \left(\prod_{i=1}^{N} \sum_{\sigma_i=\pm1}\right) e^{-\beta H}$$

$$\Rightarrow Z = \left(\prod_{i=1}^{N} \sum_{\sigma_i=\pm1}\right) e^{\beta MJ\sum_j \sigma_j - \frac{1}{2}\beta NJM^2}$$

$$= e^{-\frac{1}{2}\beta NJM^2} \prod_{i=1}^{N}\left(\sum_{\sigma_i=\pm1} e^{\frac{MJ\sigma_i}{T}}\right)$$

$$\Rightarrow Z = e^{-\frac{1}{2}\beta NJM^2} \prod_{i=1}^{N} (2\cosh(\frac{MJ}{T}))$$

$$\Rightarrow \boxed{Z = e^{-\frac{1}{2}\frac{NJM^2}{T}} \times (2\cosh(\frac{MJ}{T}))^N}$$ Partition Function.

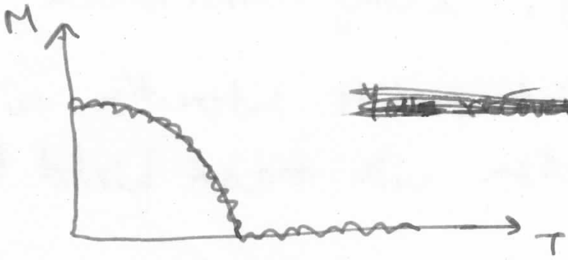
$$\boxed{F = -T\log Z = \frac{NJM^2}{2} - NT\log\left(2\cosh\left(\frac{MJ}{T}\right)\right)}$$ Free Energy

Equilibrium ⇒ Minimize F. w.r.t. M.

$$\frac{\partial F}{\partial M} = 0 \Rightarrow NJM - \frac{NT}{\cosh(\frac{MJ}{T})} \sinh(MJ/T) \times \frac{J}{T} = 0$$

$$\Rightarrow NJM - NJ\tanh(MJ/T) \Rightarrow \boxed{M = \tanh\left(\frac{MJ}{T}\right)}$$

$M = \tanh\left(\frac{MJ}{T}\right) \Rightarrow$ Tells how M looks as a function of temperature: $M = M(T)$

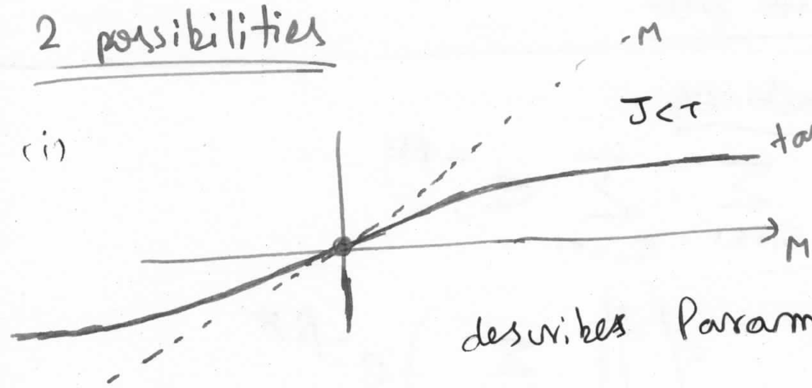


~~You recover this~~

plotting $M = M(T)$ you recover this graph.

2 possibilities

(i)



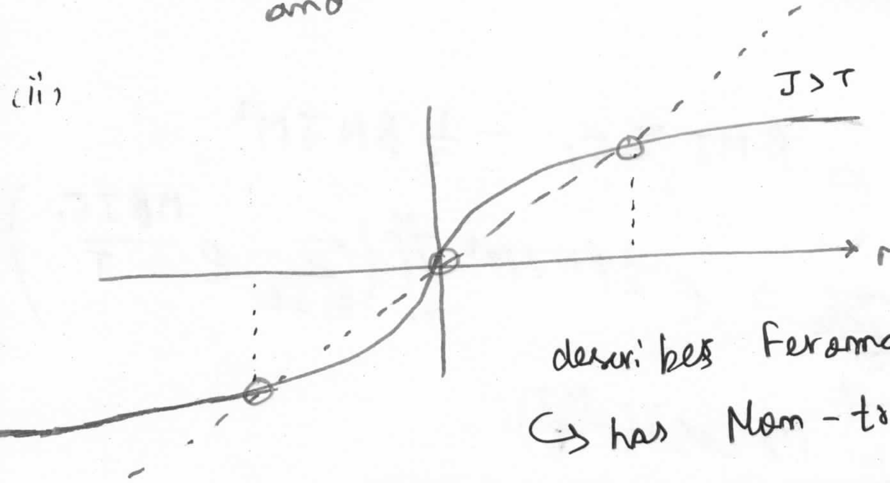
$J < T$

LHS = RHS only at origin.

describes Paramagnetic phase

and

(ii)

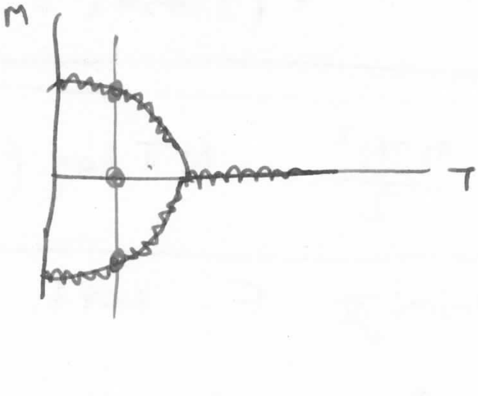


$J > T$

describes Feromagnetic phase.

↳ has Non-trivial magnetization

We always have three solutions in feromagnetic case.



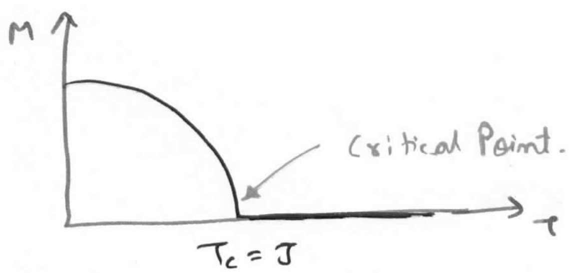
$\tanh(x) \simeq x + \ldots$

here $= \frac{MJ}{T}$ if $\frac{J}{T} > 1 \Rightarrow$ Then ferro magnetic case.

so: $\boxed{T = T_c = J}$

$\boxed{T_c = J}$ J describes how strongly the spins interact

& critical temperature will depend on strength of interaction.



... continuous phase transition ... they are simple because they are Universal.

Universality $\simeq$ " Same Critical Exponents".

Critical exponents are just numbers which describes behavior of physical quantities near Critical Point.

Exponent β : ... describes behavior of Magnetization near critical point.

↳ ie; describes behavior of Order Parameter near Critical point

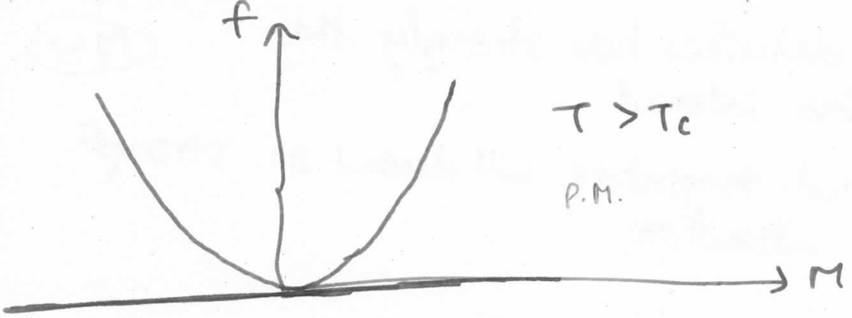
M is small near critical point, and hence we can do taylor expansion there.

~~$f = \frac{F}{N}$ (free~~ $f = \frac{F}{N}$ (free energy per lattice site)

$$\Rightarrow \boxed{f = \frac{JM^2}{2} - T \log \left(2 \cosh \frac{MJ}{T} \right)}$$

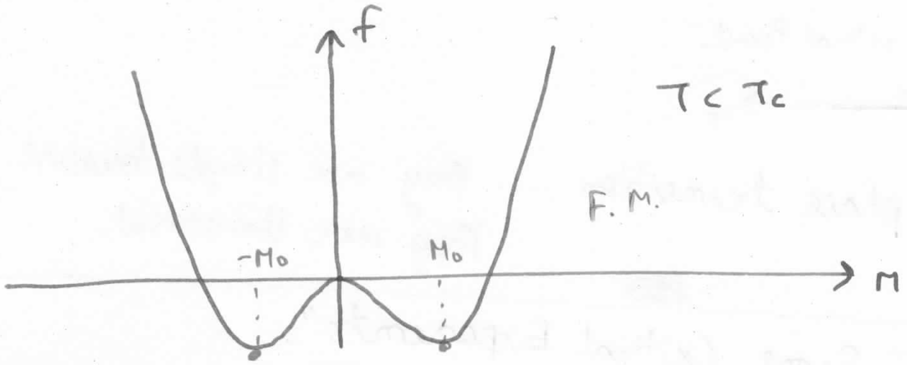
$$f \approx \frac{JM^2}{2} - \frac{1}{2} T \left(\frac{MJ}{T} \right)^2 + \frac{1}{12} T \left(\frac{MJ}{T} \right)^4 + O(M^6)$$

$$f \approx \frac{T_c}{2} \left(1 - \frac{T_c}{T} \right) M^2 + \frac{T_c^4}{12 T^3} M^4 + \dots$$



$T > T_c$

P.M.



$T < T_c$

F.M.

$$\frac{\partial f}{\partial M} = 0 = T_c\left(1 - \frac{T_c}{T}\right)M + \frac{T_c^4}{3T^3} M^3$$

set to zero to find M_0...

$\Rightarrow$ ~~$M = \pm 3(1 - T$~~

$$\boxed{M = \pm\sqrt{3\left(1 - \frac{T}{T_c}\right)}}$$

This is how M behaves in the vicinity of critical point.

$$t = \frac{T - T_c}{T_c}$$ define

↳ its a parameter which goes to zero as critical point is approached.

$$\boxed{M = \pm\sqrt{3t}}$$

$\therefore$ ~~$M \propto (-t)^\beta$~~

$\sqrt{3}$ is not universal about continuous phase transition.

$\therefore$ $\boxed{M \propto (-t)^\beta}$

$\beta = 1/2$

Critical exponent.
(prediction of MFT approximation)
(independent of d.. dimensionality of ISING Model.)

→ This will be universal.

$$\boxed{M \propto (t)^\beta}$$
near critical point

$\beta \Rightarrow$ critical exponent

Lecture 5: Critical Exponents, Landau's Theory, Hubbard-Stratonovich Transformation.

MFT Approximation $M = \langle \sigma_i \rangle$ ... order Parameter.

$\varepsilon = \sigma_i - M$ .... treated this as small.

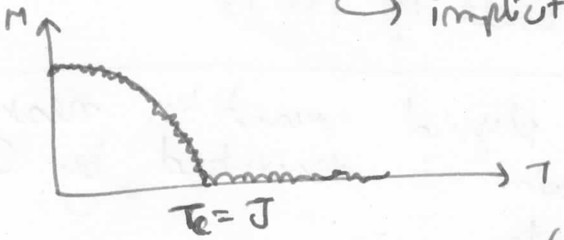
$$H = -MJ\sum_i \sigma_i + \frac{1}{2}M^2NJ$$ } Ising MFT Hamiltonian.

↳ Interactions of spins with average field.

$$f = \frac{F}{N} = \frac{JM^2}{2} - T\ln\left(2\cosh\left(\frac{MJ}{T}\right)\right)$$

Minimum w.r.t. M: $\frac{\partial f}{\partial M} = 0 \Rightarrow \boxed{M = \tanh\left(\frac{MJ}{T}\right)}$

↳ implicit relation for $M = M(T)$



Continuous phase transition (because order parameter through phase transition is continuous)

Note that f is non-analytic as a function of T $\Rightarrow$ phase transition. (irrespective of d)

Compare: 1d Ising; exact solution (tutorial)

$$f = -T\ln\left[2\cosh(J/T)\right]$$

Analytic & so no phase transition.

MFT approximation breaks down here.

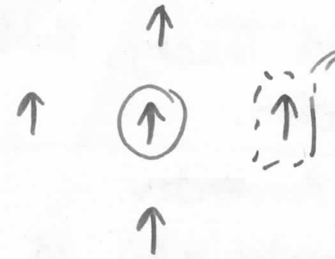
$\boxed{\text{Reason as to why MFT breaks in 1d is that the fluctuations are huge.}}$

1d: ↑ ↑ ↑ → flip $\Rightarrow$ ↑ ↑ ↓

M M (changes drastically if we flip one of them)

So: ε cannot be treated as small.

2d



→ if you flip one of them M changes; but not changes much

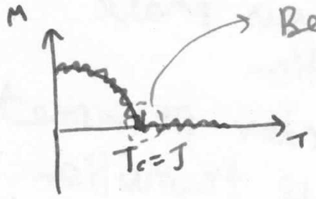
∴ Fluctuations are little bit less important.

So; actually in 2d there is phase transition. (which is correctly predicted by MFT)

... ofcourse critical exponent predicted by MFT is wrong.

$\underline{d=4}$ Upper critical dimension

($d>4$: we get "exact" result by MFT)



Behavior of physical quantities near critical point is described by Critical Exponents → tells whether something diverges or diminishes to zero around critical point.

$$t = \frac{T - T_c}{T_c}$$

if $M \propto \frac{1}{(t)^{\text{positive power}}}$ ⇒ diverge

if $M \propto (t)^{\text{positive power}}$ ⇒ converge to zero.

Expansion in small M

$$f \simeq \underbrace{\frac{T_c^2}{2T} t}_{\text{This coefficient changes sign as critical point is crossed.}} M^2 + \underbrace{\frac{T_c^4}{12T^3}}_{\text{positive definite.}} M^4 + \dots \infty$$

$$\frac{\partial f}{\partial M} = 0$$

$$\Rightarrow M \sim \sqrt{-3t} \sim (-t)^{\beta}$$

$\beta \Rightarrow$ Critical Exponent for order parameter.

here; we found $\beta = 1/2$ for 1-d ising model.

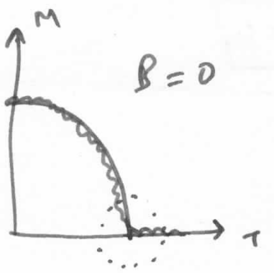
Other Critical Exponents

Susceptibility:
$$\boxed{\chi = \left(\frac{\partial M}{\partial B}\right)_T\Big|_{B=0}}$$

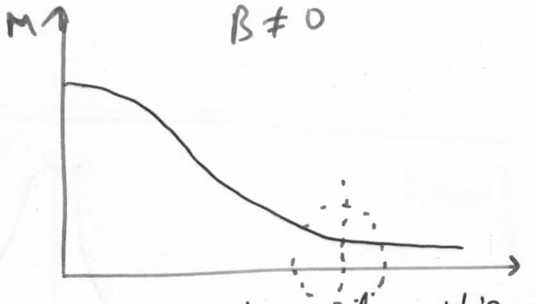
Hamiltonian MFT $\quad H = -(MJ + B)\sum_i \sigma_i + \frac{1}{2} N J M^2$ (Hamiltonian with the Magnetic field)

free energy: $\quad f = \frac{JM^2}{2} - T \log\left(2\cosh\left(\frac{MJ+B}{T}\right)\right)$

$$\frac{\partial f}{\partial M} = 0 \Rightarrow M = \tanh\left(\frac{MJ+B}{T}\right)$$



$\Rightarrow$



B small $\Rightarrow$ M small as well close to critical point.

use $\tanh(x) = x - \frac{x^3}{3} + \dots \infty$

$$\Rightarrow M \approx \frac{MJ+B}{T} - \frac{1}{3}\left(\frac{MJ+B}{T}\right)^3$$

$$\chi = \frac{\partial M}{\partial B} = \chi\frac{J}{T} + \frac{1}{T} - \left(\frac{MJ+B}{T}\right)^2\left(\chi\frac{J}{T} + \frac{1}{T}\right)$$

$B \to 0$ limit.

$$\Rightarrow \chi = \left(\chi\frac{J}{T} + \frac{1}{T}\right)\left[1 - \left(\frac{M^2J^2}{T^2}\right)\right] \qquad \because J^2 = T_c^2$$

$$\Rightarrow \boxed{\chi = \left(\chi\frac{J}{T} + \frac{1}{T}\right)\left[1 - \frac{M^2T_c^2}{T^2}\right]}$$

$T > T_c$; $M=0$; $\chi\left(1-\frac{J}{T}\right)=\frac{1}{T}$

$$\Rightarrow \boxed{\chi = \frac{1}{T-T_c}}$$

$$\chi \propto t^{-\gamma}$$

where $\gamma = 1$

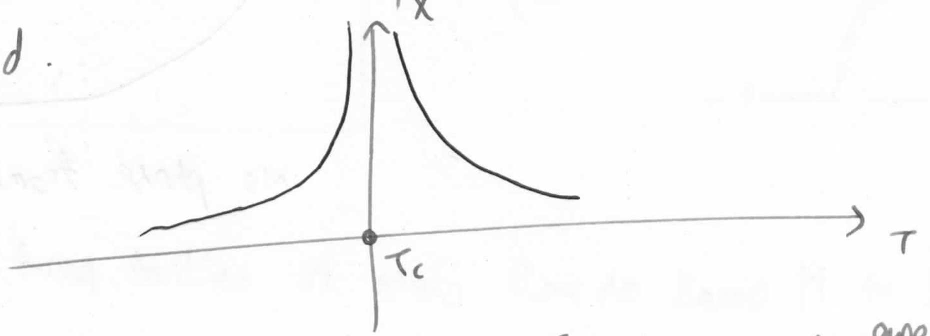
another critical exponent.

This is how susceptibility diverge as you go ~~critical~~ to critical temperature from above.

$T < T_c$; $M = \sqrt{-3t}$; $\chi \simeq \frac{1}{2}\cdot\frac{1}{T_c - T} \propto |t|^{-\gamma'}$

here ; $\boxed{\gamma' = 1}$

So; we found.



The divergence goes like $\left|\frac{1}{T-T_c}\right|$ from both side.

★ Look at $T = T_c$: How M behaves with B. (will give another critical exponent)

$$\Rightarrow M \simeq M\frac{T_c}{T_c} + \frac{B}{T_c} - \frac{1}{3}\left(\frac{MT_c}{T_c}+\frac{B}{T_c}\right)^3$$

~~higher order correction~~

$$\Rightarrow M \propto B^{1/3} = B^{1/\delta}$$

$\boxed{\delta = 3}$ δ critical exponent.

Higher order term.

Specific Heat

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V \quad ; \quad S = -\frac{\partial F}{\partial T}$$

$$\Rightarrow \quad C_V = -\frac{T}{N}\left(\frac{\partial^2 F}{\partial T^2}\right)_V$$

per lattice site

$$\Rightarrow \quad \boxed{C_V = -T \frac{\partial^2 f}{\partial T^2}\Big|_V}$$

If $T < T_c$;

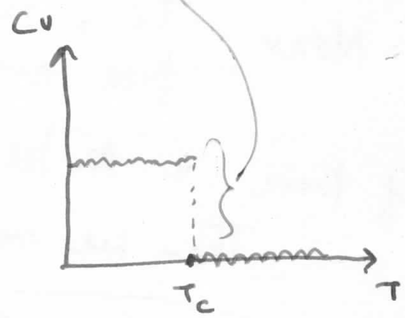
$$f \overset{M = (-3t)^{1/2}}{=\!=} -\frac{3}{4}\frac{1}{T_c}(T - T_c)^2$$

If $T > T_c$;

$$f \overset{M=0}{=\!=} 0$$

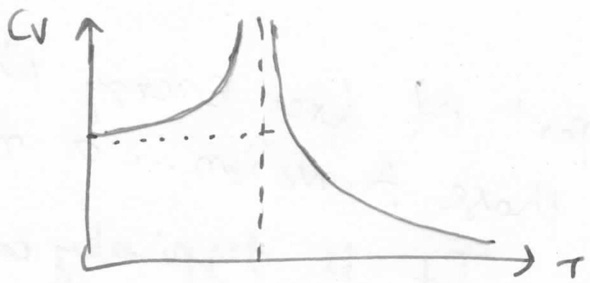
$$\therefore \quad C_V = \begin{cases} 0 & T > T_c \\ \frac{3}{2}\cdot\frac{T}{T_c} & T < T_c \end{cases}$$

ΔC_V is finite prediction of MFT



MFT predicts jump in ~~spec~~ specific Heat.

Through observation we get.



Definition

$$C_V \propto \begin{cases} |t|^{-\alpha'} & T < T_c \\ t^{-\alpha} & T > T_c \end{cases}$$

MFT prediction ; $\alpha = \alpha' = 0$; wrong prediction.

Critical exponents $\alpha, \beta, \gamma, \delta \Rightarrow$ characterize our continuous phase transition.

Landau's Theory

MFT results are very robust, and follow symmetry consideration.

Landau proposed,
All we need to know is the order parameter and the symmetries.

(in our case; $M = \langle \sigma_i \rangle$ (order parameter); & we knew H of ising model had $\mathbb{Z}_2$ symmetry $\sigma_i \to -\sigma_i$)

f is quadratic function of M $\Rightarrow$ precisely ~~but~~ because original H has $\mathbb{Z}_2$ symmetry. $\Rightarrow f(M) = f(-M)$

$\therefore$ Near T_c; we must have non-analytic behavior of free energy.

If there is ~~to~~ be phase transition;
Then we must have $f \approx r M^2 + \mu M^4$

$r \propto t$ in order to have phase transition. ~~(This captures non-analytic behavior of f)~~

we require $\mu > 0$ .. because of "Stability" property.

Prediction is Robust; many diverse systems may lead to the same "free energy".

This is just the part of free energy which is important for phase transition... a non analytic part ... It is typically called.
LANDAU'S ENERGY.

$\Rightarrow$ Should have same critical exponents if they have same free energy. It is then said that they belong to same universality class.

This is true, but critical exponents are actually predicted wrong .... They take values but not predicted by Landau's Theory.

★ MFT Predicts universality classes (Landaus Theory)

MFT is same as Landau Theory....

$\boxed{\text{For our scalar order parameter; it predicts } \alpha=0,\ \beta=1/2,\ \gamma=1,\ \delta=3}$

↳ The prediction is wrong

Experimentally; ★the value of critical exponent depend on dimensionality of space

★ Nature of order parameter (whether scalar or vector, etc)

★ Symmetry of H.

MFT or Landau theory then obviously has to be wrong; because they are irrespective of dimensions; because d never appeard in these things.

Reason; Why Landau theory or MFT breaks down is that we treated order parameter as constant. (ie; M is uniformly same throughout the lattice; that is we forbid the fluctuations & neglect the affect of fluctuations)

So, we want to tread order parameter as Thermodynamic Parameter which can fluctuate.

(c) From ISING to Field Theory

★ HUBBARD - STRATONOVICH TRANSFORMATION

We want, $Z = \sum_{\{\sigma_i\}} e^{+\frac{1}{2T}\sum_{ij}\sigma_i\sigma_j J_{ij}}$

calculate this using a trick of gaussian integration.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\phi\, e^{-\frac{1}{2}a\phi^2+b\phi} = \frac{1}{\sqrt{a}}\, e^{\frac{b^2}{2a}}$$

if $Re(a)>0$

This generalizes the higher dimensions

$$\frac{1}{(\sqrt{2\pi})^N}\int_{-\infty}^{+\infty} d\phi_1\dots d\phi_N \cdot e^{-\frac{1}{2}\phi_i A_{ij}\phi_j+\gamma_i\phi_i} = \frac{1}{\sqrt{\det A}}\, e^{\frac{1}{2}\gamma_i A^{-1}_{ij}\gamma_j}$$

$$= \frac{1}{\sqrt{\det A}}\, e^{\frac{1}{2}\gamma_i A^{-1}_{ij}\gamma_j}$$

~~Then we have~~

$\cancel{e^{\frac{1}{2}\gamma_i A_{ij}\gamma_j}}$

Then we have;

$$e^{\frac{1}{2}\gamma_i A_{ij}\gamma_j} = \frac{1}{\sqrt{\det A}}\cdot\frac{1}{(2\pi)^{N/2}}\cdot\int_{-\infty}^{+\infty} d\vec{\phi}\, e^{-\frac{1}{2}\phi_i A^{-1}_{ij}\phi_j+\gamma_i\phi_i}$$

We recognize; $A_{ij} = \frac{J_{ij}}{T}$; & $\gamma_i = \sigma_i$

now:

$$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2T}\sum_{ij} J_{ij}\sigma_i\sigma_j} = \frac{T^{N/2}}{\sqrt{\det J}}\,\frac{1}{(2\pi)^{N/2}}\sum_{\{\sigma_i\}}$$

$$\int d\vec{\phi}\, e^{-\frac{T}{2}\phi_i J^{-1}_{ij}\phi_j+\sigma_i\phi_i}$$

$$Z = \frac{T^{N/2}}{\sqrt{\det J}} \cdot \frac{1}{(2\pi)^{N/2}} \sum_{\{\sigma_i\}} \int d\vec{\phi} \, \exp\left(-\frac{T}{2} \phi_i J_{ij}^{-1} \phi_j + \sigma_i \phi_i\right)$$

do the rescalling $\phi_i \to \frac{\phi_i}{T}$

$$\Rightarrow Z = \underbrace{\frac{1}{\sqrt{\det J}} \cdot \frac{1}{(2\pi T)^{N/2}}}_{} \sum_{\{\sigma_i\}} \int d\vec{\phi} \, \exp\left[\frac{-1}{2T} \phi_i J_{ij}^{-1} \phi_i + \frac{1}{T} \phi_i \sigma_i\right]$$

This is an analytic factor

(if you want to study phase transition, this will not contribute much)

Appr.

Approach 1

Since its an analytic factor (it will give an additive analytic term in F) we set to 1.. we dont care about it.

Approach 2

lets change the integration measure.. (effectively studying analytic factor to be 1)

$$Z \overset{\text{possible non-analytic}}{\underset{\text{part}}{=\!=}} \sum_{\{\sigma_i\}} \int \mathcal{D}\phi \cdot e^{\frac{-1}{2T} \phi_i J_{ij}^{-1} \phi_j + \frac{1}{T} \phi_i \sigma_i}$$

now we could do sum easily because there is linear factor in exponential.

use $\sum_{\sigma_i = \pm 1} e^{\frac{1}{T} \phi_i \sigma_i} = 2\cosh(\phi_i / T)$

$$= e^{\log(2\cosh(\phi_i/T))}$$

$$\boxed{\begin{aligned} Z &= \int \mathcal{D}\phi \cdot e^{-S[\phi]} \\ S[\phi] &= \frac{1}{2T}\phi_i J^{-1}_{ij}\phi_j - \sum \log\left(2\cosh\left(\frac{\phi_i}{T}\right)\right) \end{aligned}}$$

$$\mathcal{D}\phi = \frac{1}{\sqrt{\det J}} \cdot \frac{1}{(2\pi T)^{N/2}}\, d\vec{\phi}\ .$$

Advantage of Hubbard - Stratonovich transformation... we transformed the problem in path integral form

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Lecture 6: Spin - Spin Correlation Function, Correlation Length, Landau - Ginzburg Functional.

At phase transition point: the theory is described by C.F.T; the theory is scale invariant ... you begin to see ~~fractals~~ fractals. "Fractals appear during phase transitions".

Spin - Spin Correlation Function

$$G_{ij} = (\sigma_i - \langle\sigma_i\rangle)(\sigma_j - \langle\sigma_j\rangle)$$

$$= \langle\sigma_i \sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle$$

Spin - Spin correlation fuction

It measures how big blocks are in our system of all of spins.

To calculate;

$$H = -\frac{1}{2}\sum_{ij} J_{ij} - \sum_i B_i \sigma_i$$

Couple the Ising model to magnetic field; which can vary from site to site in lattice

Then we can show; $G_{ij} = -T \left.\frac{\partial^2 F}{\partial B_i \partial B_j}\right|_{B=0}$

(A trick to calculate G_{ij})

$$\boxed{G_{ij} = -T \left.\frac{\partial^2 F}{\partial B_i \partial B_j}\right|_{B=0}}$$

exponential decay weighted by characteristic distance ξ

$$G_{ij} = \dots = \dots = \frac{a^3 T}{2\pi K} \frac{e^{-|\vec{r}_i - \vec{r}_j|/\xi}}{|\vec{r}_i - \vec{r}_j|^{d-2}}$$

columb like decay

$$G_{ij} \overset{M.F.T.}{=} \underbrace{\frac{a^3 T}{2\pi K}}_{\text{positive constant (not very important)}} \cdot \frac{e^{-|\vec{r}_i - \vec{r}_j|/\xi}}{|\vec{r}_i - \vec{r}_j|^{d-2}}$$

(looks like Yukawa potential)

$$\xi = \sqrt{\frac{K}{2(T-T_c)}}$$

Correlation Length. "measures how big the box are".

ξ diverges as $T \to T_c$ (means you will have blocks of the ~~sp~~ size of the system).

More Generally;

$$\boxed{G(r) \propto \frac{e^{-r/\xi}}{r^{d-2+\eta}}}$$

where $\xi \propto |t|^{-\nu}$

This introduces two more critical exponent: ν & η.

* $\nu \Rightarrow$ Critical Exponent for Correlation Length.
(prediction of M.F.T. $\nu = \frac{1}{2}$; predicts ξ diverges)

* $\eta \Rightarrow$ Anamolous Dimension.
(says that ~~usin~~ usual ~~dimen~~ dimension analysis is little bit wrong if you are near criticality; & you can pick up extra dimension from the fractal structure which is comming up ; which is described by the Anamolous Dimension.

At $T = T_c$; ξ diverges
we have observed the Fractal Structure, system is described by CFT (Conformal Field Theory)

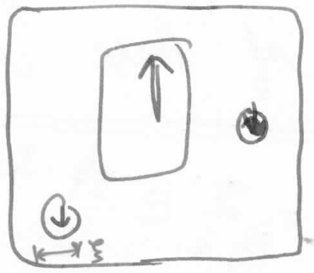
$T > T_c$

Completely random
(Random Configuration)
$\langle \sigma \rangle = 0$

$T = T_c$

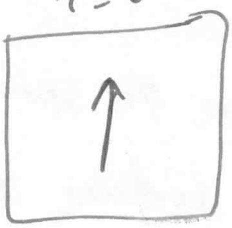
$\langle \sigma \rangle = 0$
FRACTAT structure
$(\xi \to \infty)$

$0 < T < T_c$



$\langle \sigma \rangle = 0.3$
thick blocks of ↑
∴ but also have small regions of ↓

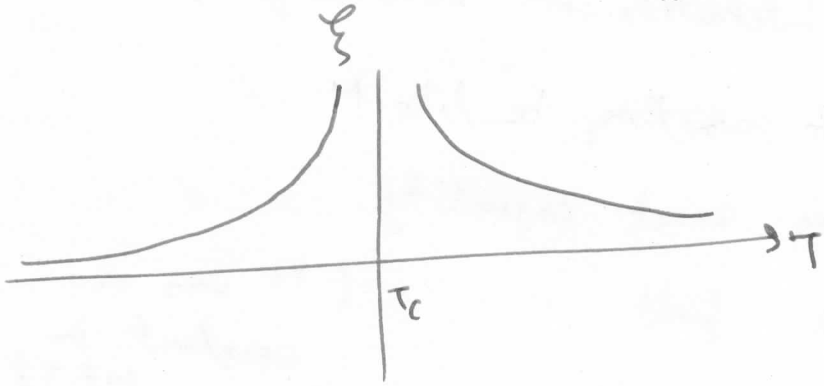
$T = 0$



$\langle \sigma \rangle = 1$

If $T > T_c$
then ξ measure the size of spin up say

if then you go to $T < T_c$
∴ ~~it~~ ξ measure the size of block which has spin down.



Thats why it goes to zero on both side

M.F.T. $\eta = 0$

M.F.T. prediction.	
	$\alpha = 0$
	$\beta = 1/2$
	$\gamma = 1$
	$\delta = 3$
	$\eta = 0$
	$\nu = \frac{1}{2}$

~~7 described below~~

Only 2 are independent.
(if you can calculate 2 correctly; then you can calculate using the algebraic relations; these algebraic relation can also be wrong.)

We can show, based on very very generic arguments that for typical systems, only two out of these critical exponents are indeed independents.

$$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2T} \cdot J_{ij} \sigma_i \sigma_j}$$

using Gaussian Integral over some auxilliary fields $\phi_1, \dots \phi_N$

we get.

$$Z = \frac{1}{\sqrt{\det J}} \cdot \frac{1}{(2\pi T)^{N/2}} \sum_{\{\sigma_i\}} \int_{-\infty}^{+\infty} d\vec{\phi}\, e^{-\frac{1}{2T}\phi_i J^{-1}_{ij} \phi_j + \frac{1}{T}\sigma_i \phi_i}$$

$\sigma_i \sigma_j$ two spin interactions.

∴ but by Gaussian integral, we see the only place the spin appear is $e^{\dots \frac{1}{T}\sigma_i \phi_i \dots}$ so reduce to

Interaction of spin with fields ϕ.

(Reminds of MFT, where we removed multiparticle interaction by having one point interaction with external field, like M)

→ suggest ϕ can something be like M

∴ here; we have exact equality.

ϕ is a variable field

(M was a constant in MFT)

This is why it gives wrong critical exponents)

Partition function being written as multidimensional integral over auxilliary fields ϕ_i.

$$\boxed{\begin{aligned} Z &= \int \mathcal{D}\phi \cdot e^{-S[\phi]} \\ S[\phi] &= \frac{1}{2T}\phi_i J^{-1}_{ij}\phi_j - \sum_i \log\left(2\cosh\left(\frac{\phi_i}{T}\right)\right) \end{aligned}}$$

EXACT (NO approximation)

Saddle point (WKB) Approximation

we want to minimize S.

We want $\left.\frac{\partial S}{\partial \phi_i}\right|_{\phi_i = \bar{\phi}_i} = 0$

lets call the solution to be $\bar{\phi}_i$ (ϕ average)

$$\Rightarrow \left.\frac{1}{T} J^{-1}_{ij} \phi_j - \frac{1}{T} \tanh\left(\frac{\phi_i}{T}\right) = 0 \right|_{\phi = \bar{\phi}}$$

$$\Rightarrow \frac{1}{T} J^{-1}_{ij} \bar{\phi}_j - \frac{1}{T} \tanh\left(\frac{\bar{\phi}_i}{T}\right) = 0$$

$$\Rightarrow \boxed{\bar{\phi}_i = \sum_j J_{ij} \tanh\left(\frac{\bar{\phi}_j}{T}\right)}$$

All lattice site are same; so; $\bar{\phi}_1 = \bar{\phi}_2 = \ldots = \bar{\phi}$

$$\Rightarrow \boxed{\bar{\phi} = J \cdot \tanh\left(\frac{\bar{\phi}}{T}\right)} \qquad J = \sum_j J_{ij}$$

we have seen equation like $M = \tanh\left(\frac{MJ}{T}\right)$

if you make an identification of $\bar{\phi} = MJ$

then you return back to Mean Field Theory!

MFT is recovered by taking leading order of WKB approximation.

MFT $=$ WKB Approximation to leading order.

fields ϕ_i are confined to lattice points.

↳ so; we have something like field theory in descreate space time.

∴ we would like to take continuous limit.

close to critical temperature, we can zoom out of the system which effectively makes the system continuous. So; effectively the lattice disappear.

⟶ Continuous limit (Effectively lattice disappears)

⟹ In the end; we will recover Classical Field Theory living in continuous space.

☆ $\log\left(2\cosh(\phi/T)\right) \simeq \log(2) + \frac{1}{2}\left(\frac{\phi_i}{T}\right)^2 - \frac{1}{12}\left(\frac{\phi_i}{T}\right)^4 + \ldots$

if ϕ is like M; then close to T_c; ϕ is small

$$\log\left(2\cosh(\phi_i/T)\right) \simeq \log(2) + \frac{1}{2}\left(\frac{\phi_i}{T}\right)^2 - \frac{1}{12}\left(\frac{\phi_i}{T}\right)^4 + \ldots$$

not interested in constant term

minus is important.

<u>Lattice fourier transform</u>

$$\phi_i = \frac{1}{N}\sum_{q} \varphi(\vec{q})\, e^{i\vec{q}\cdot\vec{r}_i}$$

φ is like fourier transform of $\vec{q}$.

$\vec{q}$ momentum version of $\vec{r}$

convention; writing ϕ with argument $\vec{q}$ $\phi(\vec{q})$ means we are taking about fourier transform of $\phi(\vec{r})$

So;

$$J_{ij} = \frac{1}{N} \sum_{q} J(\vec{q})\, e^{i q \cdot (\vec{r}_i - \vec{r}_j)}$$

→ ~~hen~~ because it only depends on $\vec{r}_i - \vec{r}_j$ difference ; .. so only one summation over $\vec{q}$ works.

<u>Orthonormality relations</u>

$$\frac{1}{N} \sum_{j} e^{i(\vec{k} - \vec{k}') \cdot \vec{r}_j} = \delta_{\vec{k}, \vec{k}'}$$

$$\frac{1}{N} \sum_{q} e^{-i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \delta_{ij}$$

Lets deal with the term

$\sum_{i,j} \phi_i J^{-1}_{ij} \phi_j$ ... fourier transform everything.

$$= \sum_{ij} \underbrace{\frac{1}{N} \sum_{q} \phi(\vec{q})\, e^{i\vec{q} \cdot (\vec{r}_i)}}_{\phi_i} \underbrace{\frac{1}{N} \sum_{\vec{q}'} J^{-1}(\vec{q}')\, e^{i\vec{q}' \cdot (\vec{r}_i - \vec{r}_j)}}_{J^{-1}_{ij}} \times$$

$$\times \underbrace{\frac{1}{N} \sum_{\vec{q}''} \phi(\vec{q}'')\, e^{i\vec{q}'' \cdot (\vec{r}_j)}}_{\phi_j}$$

use then first orthonormality condition.

$$\sum_{ij} \phi_i J^{-1}_{ij} \phi_j = \sum_{ij} \frac{1}{N} \sum_{\vec{q}} \phi(\vec{q})\, e^{i\vec{q} \cdot \vec{r}_i} \cdot \frac{1}{N} \sum_{\vec{q}'} J^{-1}(\vec{q}')\, e^{i\vec{q}' \cdot (\vec{r}_i - \vec{r}_j)} \cdot \frac{1}{N} \sum_{\vec{q}''} \phi(\vec{q}'')\, e^{i\vec{q}'' \cdot \vec{r}_j}$$

doing sum over i gives $\vec{q}$ & $\vec{q}'$ become same.

sum over j " $\vec{q}'$ & $\vec{q}''$ become same.

$$= \frac{1}{N} \sum_{q, q', q''} \phi(\vec{q})\, J^{-1}(\vec{q}')\, \phi(\vec{q}'')\, \delta_{q, -q'}\, \delta_{q', q''} .$$

$$= \frac{1}{N} \sum_{\vec{q}'} \phi(-\vec{q}) \, J^{-1}(q) \, \phi(\vec{q})$$

$$\underbrace{\sum_{i,j} \phi_i \, J^{-1}_{ij} \, \phi_j}_{\Downarrow} = \underbrace{\frac{1}{N} \sum_{\vec{q}} \phi(-\vec{q}) \, J^{-1}(\vec{q}) \, \phi(\vec{q})}_{\Downarrow}$$

Sum in real space (have two sum) $\Longrightarrow$ sum in momentum space (have one sum)

This is what we achieved...

... ~~the~~ ... in momentum space it is nicer; first of all we have just one sum

$J^{-1}(\vec{q})$?

lets do some approximation.

$J^{-1}(\vec{q})$ is $\frac{1}{J(\vec{q})}$ (This is not the approximation)

$\therefore$ What is $J(\vec{q})$?

$$J(\vec{q}) = \frac{1}{N} \sum_{ij} J_{ij} \, e^{-i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)}$$

↳ Transform of J_{ij}.

Now, we want to get the continuous limit.

... if you zoom out, you ~~see~~ ~~the~~ have larger & larger distances in position space; but in momentum space you have smaller & smaller chunks.

So: $\vec{q}$ is small because we want continuum limit ~~the~~

Now we can expand the exponential in taylor series & can keep only few terms.

Then

$$J(\vec{q}) = \frac{1}{N}\sum_{ij} J_{ij}\left(\underbrace{1}_{} - i\vec{q}\cdot\underbrace{(\vec{r}_i - \vec{r}_j)}_{} - \frac{1}{2}\left(\vec{q}\cdot(\vec{r}_i-\vec{r}_j)\right)^2 + \dots\right)$$

Simple gives J

* one sum gives J
* other sum gives N

$A_{ij} = \vec{r}_i - \vec{r}_j$

of course $A_{ij} = -A_{ji}$

but J_{ij} is symmetric

so; $\sum_{ij} J_{ij}\, i\vec{q}\cdot(A_{ij}) = 0$.

(linear term does not contribute)

$$J(\vec{q}) = \underbrace{\frac{1}{N}\sum_{ij} J_{ij}}_{J = \sum_i J_{ij}} - \frac{1}{2N}\sum_{ij} J_{ij}\,|\vec{q}\cdot(\vec{r}_i - \vec{r}_j)|^2 + \dots$$

$$\Rightarrow J(\vec{q}) = J - \frac{1}{2N}\sum_{ij}\underbrace{J_{ij}\,|\vec{q}\cdot(\vec{r}_i-\vec{r}_j)|^2}_{} + \dots$$

this depends on q^2

so;

$$-\frac{1}{2}K|\vec{q}|^2$$

(Hahaha..)

$$\boxed{J(\vec{q}) = J - \frac{1}{2}K|\vec{q}|^2 + \dots}$$

K should not contain N, ... hahaha... easy to check.

$R \Rightarrow$ Range of Interaction

K is called spin stiffness.

same K which appears in correlation function.

$$K = \frac{R^2 \cdot J}{d}$$

$d \Rightarrow$ no. of dimensions.

Calculation of G_{ij} (correlation function) proceeds in similar way.

$$S = S_0 + \underbrace{S_{int}}_{\longrightarrow\ S \text{ interacting.}}$$

$$S_0[\phi] = \frac{1}{2T^2} \sum_{ij} T\, \phi_i\, J_{ij}^{-1}\, \phi_j - \frac{1}{2} \sum_i \frac{\phi_i^2}{T^2}$$

$$S_{int}[\phi] = +\frac{1}{12} \sum_i \frac{\phi_i^4}{T^4}$$

$$\cancel{S_0[\phi] = \frac{1}{2T^2N} \sum_{\vec{q}} \phi(\vec{q}) \left(\frac{T}{J(\vec{q})} - 1 \right) \phi(\vec{q}}$$

$$S_0[\phi] = \frac{1}{2T^2N} \sum_{\vec{q}} \phi(-\vec{q}) \underbrace{\left(\frac{T}{J(\vec{q})} - 1 \right)}_{\Downarrow} \phi(\vec{q})$$

$$\frac{T}{J\cdot\left(1 - \frac{1}{2}\frac{|\vec{q}|^2 K}{J}\right)} - 1$$

small if $|\vec{q}|$ is small.

$$= \frac{T}{J}\left(1 + \frac{1}{2}\frac{|\vec{q}|^2 K}{J}\right) - 1$$

in M.F.T. we know $J = T_c$

$$= \underbrace{\frac{T - T_c}{T_c}}_{t} + \frac{1}{2}|\vec{q}|^2 K \cdot \frac{T}{T_c^2}$$

$$= t + \frac{1}{2}|\vec{q}|^2 K \cdot \frac{T}{T_c^2}$$

positive quantity

use it...

$$S_0[\phi] = \frac{1}{2T^2N} \sum_{\vec{q}} \phi(-\vec{q}) \cdot \left[t + \frac{1}{2} |\vec{q}|^2 K \frac{T}{T_c^2} \right] \phi(\vec{q})$$

we want to redefine ϕ so that we get rid of this

... we want to set this to be 1.

... we absorb this in ϕ...

$$\phi \to \frac{T_c \sqrt{2d}}{R\, a^{d/2}} \phi$$

introduce new variable $r = \frac{2dt}{R^2}$

$$S_0[\phi] = \frac{1}{2} \frac{1}{Na^d} \cdot \sum_{\vec{q}} \phi(-\vec{q}) (r + q^2) \phi(\vec{q})$$

can be turned to an integral over $\vec{q}$ in the limit.

$$S_0[\phi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \phi(-\vec{q}) (r + q^2) \phi(\vec{q})$$

$$\boxed{S_0[\phi] = \frac{1}{2} \int d^d x \left(|\nabla\phi|^2 + r\phi^2 \right)}$$

$$\boxed{S_{int}[\phi] = \int d^4 x \cdot \frac{u}{4!} \phi^4}$$

Landau Ginsberg function.

$$\boxed{S[\phi] = \int d^d x \left[\frac{1}{2} |\nabla\phi|^2 + \frac{r}{2} \phi^2 + \frac{u}{4!} \phi^4 \right]}$$

This is what we have in continuum limit.

* $r \propto t$ (changes sign as you cross T_c)
* $u > 0$

Now, in Landau - Ginzberg function; what improved is that fields can spacially fluctuate because of the term $|\nabla\phi|^2$.

... we get something like scalar field theory in Euclidean Space (with imaginary time)
(no time direction in ~~field~~ this field theory; everything is just purely spacial)

The field theory lives only on same

.... if you forget the fluctuations $|\nabla\phi|^2$, you sort of return back to Landaus Theory.

$|\nabla\phi|^2$ is the main thing which Ginzberg added.

Lecture 7: Wilsonian Renormalization 1: Basic idea: Integration over fast modes, Flowing Couplings.

From Ising to Field Theory

$$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2T} J_{ij} \sigma_i \sigma_j} \simeq \int \mathcal{D}\phi \, e^{-S[\phi]}$$

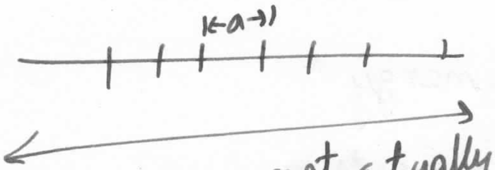
$$S[\phi] = \frac{1}{2T} \phi_i J^{-1}_{ij} \phi_j - \sum_i \log\left(2\cosh(\phi_i/T)\right)$$

$\underset{\text{Critical point}}{\text{Expansion near}}$ (effectively setting lattice spacing a goes to zero)

$$\int d^d x \left[\frac{1}{2} |\vec{\nabla}\phi|^2 + \frac{r}{2}\phi^2 + \frac{u}{4!}\phi^4 \right]$$

Landau - Ginzberg Action.

$d \Rightarrow$ no. of space dimensions.



we are not actually setting a to zero; but ~~rather~~ rather concentrating on the scale of order ξ which goes to infinity

$\Rightarrow$ This is why we can take continuum limit $\therefore$ not because $a \to 0$ but because we are looking at large scales.

$\xi \longrightarrow \infty$ large spacial scale

$\Rightarrow$ so $a \Rightarrow$ small

$$r \propto t = \frac{T - T_c}{T_c}$$ (changes sign as we cross T_c)

$u > 0$.

M.F.T. neglects fluctuations: If there are fluctuations then true T_c will be lower than predicted by T_c.

Respect symmetry of original Hamiltonian.
~~with~~ $\sigma_i \to -\sigma_i$

$\Leftrightarrow$ which translates to $\phi \to -\phi$.

So; only have even power of ϕ.

$\therefore$ add $|\nabla \phi|^2$ to add fluctuations.

$\therefore$ Is there possibility for higher derivatives ~~for~~ of ϕ; but they will not be much relevant.

- Higher powers of ϕ or ∇ are irrelevant. and so this is everything you can write down.

- **Dimensional Analysis**. " units of energy "

$[k] = \dim(k) = +1$; $k \Rightarrow$ momentum

$[x] = -1$

spatial variable.

$$[S] = 0 = \underset{-d}{[dx^d]} + \underset{+2}{[\nabla^2]} + \underset{2\times[\phi]}{[\phi^2]} \Rightarrow [\phi] = \frac{d-2}{2}$$

$$\boxed{[r] = [\nabla^2] = +2}$$

$$0 = [dx^d] + [u] + 4[\phi] \Rightarrow 0 = -d + [u] + 4 \times \frac{(d-2)}{2}$$

$$0 = [u] + (d-4)$$

$$\Rightarrow 4-d = [u]$$

define $\varepsilon = 4-d$

So; $\boxed{[u] = \varepsilon}$

Defination Coupling constant g is <u>Relevant</u> is $[g] > 0$
(it really plays role for our problem)

RELEVANT $[g] > 0$
MARGINAL $[g] = 0$
IRRELEVANT $[g] < 0$

} Classification of coupling constants } ⇒ depends on no. of dimensions

in $d=4$ dimensions ; $[r] = 2$ & $[u] = 0$

So; if you added another coupling $v\phi^6$

$\therefore$ $[v] = 6 - 2d$ — in 4 dimensions : $[v] = -2$

so; v is irrelevant by the classification

v → marginal in $d=3$
→ relevant in $d \leq 2$
→ Irrelevant $d \geq 4$

" lower dimensions you go ; you should include more terms ~~dimensions~~ in Landau - Ginsberg Functional"

So; Landau - Ginsberg as truncated here upto ϕ^4 is only good in $d=4$.

- How to calculate Z ?
  - Perturbation theory in u. (it works in Q.E.D; there the coupling constant is $\alpha = \frac{1}{137}$ .. its small..)
    (but here; it might not work ; because u may not be small)
    → This fails
  - Perturbation theory in fluctuation $\delta\phi$.
    (This also fails)

The right way to proceed is crazy way

- Do Perturbation theory in small ε. ....

.... ε - Expansion.

(similar idea is used in study of black holes; d = spacetime dimension. We see higher d black holes are simpler to study; so we study it & find results for $d=4$ by $1/d$ expansion .... (Have a look ... use it Shoaib).

here; we are solving for $d=4$ dimensions
its ~~unp~~ unphysical.. but things are easily to calculate in d... ~~B~~ to calculate for $d=2$ or $d=3$ we do expansion in d

... (expansion in ε is equivalent ~~to~~ in some sense to expansion of d; because $\varepsilon = 4-d$

(d) WILSONIAN (Momentum Space) RENORMALIZATION

PART 1| Main Idea

"It is a formal way of doing coarse graining."

Father of Renormalization is actually Cadanov; but he did it in real space which was not much helpful. It was Wilson who did it in momentum space which was more useful.

We want to calculate $Z = \int \mathcal{D}\phi \; e^{-S[\phi]}$

We want to calculate this in small steps.

When you calculate two type of divergences as in QFT; large momenta: Ultraviolet Divergences.

small momenta (because ~~the~~ ~~loops~~ in highly energy processes; we can create out of vacuum anything: The loops will cause some ~~proplo~~ problems)

Here in our subject; we are basically treating condensed matter system. In Condensed matter system only one type of divergence exist; the one for small k.

2) This is which is responsible for our ~~phase~~ ~~tr~~ renormalization of our phase transition; because we are really interested in whole system: so small k.

* For large k does not happen.
because we have lattice. so; There is a cut off on how big momentum can be. It does not make sense to probe ~~on~~ anything smaller than lattice spacing. Thats why higher momenta are cut off.

Avoiding divergences in small k.

$$\phi(x) = \int_0^{\Lambda} \frac{d^d k}{(2\pi)^d}\, \phi(k)\, e^{i\vec{k}\cdot\vec{x}}$$

This will be the problem... small k.

Λ is Natural Cut off ; $\Lambda \propto \frac{1}{a}$

$$\boxed{\int_0^{\Lambda} = \underbrace{\int_0^{\Lambda/b}}_{\text{Slow Modes}} + \underbrace{\int_{\Lambda/b}^{\Lambda}}_{\text{Fast Modes}} \quad ; \quad b > 1}$$

This we can do.

We want to avoid this!

~~$\phi = \phi_k +$~~

In momentum space ; split ϕ in $\phi_<$ (ϕ slow) & $\phi_>$ (ϕ fast)

$$\phi = \phi_< + \phi_>$$

$$\phi_<(k) = \begin{cases} \phi(k) & 0 < k < \Lambda/b \\ 0 & k > \Lambda/b \end{cases}$$

$$\phi_>(k) = \begin{cases} 0 & 0 < k < \Lambda/b \\ \phi(k) & k > \Lambda/b \end{cases}$$

Trik is to integrate fast only.

* Split S as $S = S_0 + S_{int}$

$$S_0 = \frac{1}{2}\int d^dx\,[(\nabla\phi)^2 + r\phi^2]$$

$$= \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\, \phi(-k)\,[k^2+r)\,\phi(k)$$ (by fourier transform)

$$S_0 = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\, \phi(-k)\,(k^2+r)\,\phi(k)$$

$$= \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\,(k^2+r)\,|\phi(k)|^2$$?

$\boxed{\phi(-k)\,\phi(k) = |\phi(k)|^2}$

Non interacting part.

we will see a better justification for this.

Now we will see $|\phi(k)|$; no we start plugging in the

Splitting $\phi(k) = \phi_<(k) + \phi_>(k)$

k can either be slow or fast at a time.

it cannot be both ~~simultan~~ ~~be~~ simultaneously ..

.. There will be never be mixed $\phi_<$ & $\phi_>$

because here we have only one k.

so; $S_0 = S_0[\phi_<] + S_0[\phi_>]$ because no

k_{fast} and k_{slow} mixing.

So this is something which we mean by non interactive part.

($\phi(x)$ is real $\Rightarrow$ $\phi(k) = \phi^*(k)$ ... check)

---

* $$S_{INT} = \frac{u}{4!}\int d^dx\,\phi^4$$

plug fourier transform for each ϕ.

Integration over x gives delta function over sum of ks.

$$S_{INT} = \frac{u}{4!}\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\frac{d^dk_4}{(2\pi)^d}\,\phi(k_1)\ldots\phi(k_4)\,(2\pi)^d\,\delta(k_1+\ldots+k_4)$$

$$S_{INT} = \frac{u}{4!}\int \frac{d^dk_1}{(2\pi)^d}\ldots\frac{d^dk_4}{(2\pi)^d}\cdot\phi(k_1)\ldots\phi(k_4)\cdot(2\pi)^d\cdot\delta(k_1+k_2+k_3+k_4)$$

→ have four different momentum ; out of which three are independent.

... so there can be mixing of slow & fast mode.

so;
$$S_{INT} = S_{INT}[\phi_<, \phi_>]$$

but it cant be spilled as sum.

$$S_{INT} = S_{INT}[\phi_<, \phi_>]$$

Now, we want to integrate over fast modes.

$$Z = \int \mathcal{D}\phi_>\,\mathcal{D}\phi_<\; e^{-S[\phi_<,\phi_>]}$$

→ we do this integral.

$$\Rightarrow\; Z = \int \mathcal{D}\phi_<\; e^{-\tilde{S}[\phi_<]}$$

This is how it looks after integrating out fast modes

... we get some new action $\tilde{S}$ which now depends only on slow modes.

if we are in $d=4$

∴ Then irrelevant terms will drop out.

..... more generally;

$\tilde{S}[\phi_<]$ ... has to have the same form we started with.

(can generate more terms ; but these are irrelevant)

$$\tilde{S}[\phi_<] = \frac{1}{2}\int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d} |\phi_<(k)|^2 \cdot \left(\tilde{r}_2 k^2 + \tilde{r}_0\right)$$
$$+ \frac{\tilde{r}_4}{4!}\int_0^{\Lambda/b} \frac{d^dk_1 \dots d^dk_4}{(2\pi)^{4d}} \phi_<(k_1)\dots\phi_<(k_4)\,(2\pi)^d \delta(k_1+k_2 \dots k_4)$$

orignally what stands in front of k^2 is 1 ... which is also a coupling constant. As you are integrated over fast mode; the coupling constant 1 may change to something else.

r will also chang. ; lets call $\tilde{r}_2$ the coupling constant arizing from 1

& $\tilde{r}_0$ ... arising from r.
& $\tilde{r}_4$ ... arising from u.

$$\tilde{S}[\phi_<] = \frac{1}{2}\int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d} |\phi_<(k)|^2 \cdot \left(\tilde{r}_2 k^2 + \tilde{r}_0\right) + \frac{\tilde{r}_4}{4!}\int \dots$$

$$\tilde{S}[\phi_<] = \frac{1}{2}\int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d} \cdot |\phi_<(k)|^2 \cdot (\tilde{r}_2 k^2 + \tilde{r}_0) + \frac{\tilde{r}_4}{4!}\int_0^{\Lambda/b} \frac{d^dk_1 \dots d^dk_4}{(2\pi)^{4d}} \cdot \phi_<(k_1)\dots\phi_<(k_4)\,(2\pi)^d \delta(k_1+\dots+k_4)$$

we scale the momenta & fields to get the integral from 0 to Λ back.

To bring this to exactly the same form we started with we scale.

we have to introduce new momentum $\tilde{k} = kb$

" " " " ϕ ; $\tilde{\phi}(\tilde{k}) = \frac{\phi_<(k)}{Z}$

$$\boxed{\tilde{k} = kb \;,\quad \tilde{\phi}(\tilde{k}) = \frac{\phi_<(k)}{Z}}$$

This is not partition function

The fact that you have to change fields as well, so for ~~historical~~ historical reason it is called

$$\boxed{\tilde{k} = kb \quad , \quad \tilde{\phi}(\tilde{k}) = \frac{\phi_<(k)}{Z}}$$

Wave function Renormalization.

This is not partition function. ... just a constant which will be determined later.

The fact that you have to change fields as well; so for historical reasons it is called <u>Wavefunction Renormalization</u>.

Of course we dont have any wavefunction here ... ... but any way we are normalizing some field.

$$S[\tilde{\phi}] = \frac{1}{2}\int_0^\Lambda \frac{d^d\tilde{k}}{(2\pi)^d} \cdot b^{-d} \cdot \left(\tilde{\lambda}_2 b^{-2}\tilde{k}^2 + \tilde{\lambda}_0\right) |\tilde{\phi}(\tilde{k})|^2 Z^2 + \int \ldots$$

$$= \frac{1}{2}\int_0^\Lambda \frac{d^d\tilde{k}}{(2\pi)^d} \cdot \left[\underbrace{b^{-d} Z^2 \tilde{\lambda}_2 b^{-2}}\tilde{k}^2 + \underbrace{Z^2\tilde{\lambda}_0 b^{-d}}\right] |\tilde{\phi}(\tilde{k})|^2$$

$$+ \frac{1}{4!}\underbrace{\tilde{\lambda}_4 \cdot b^{-4d} Z^4 \cdot b^d}_{} \int_0^\Lambda \frac{d^d\tilde{k}_1 \ldots d^d\tilde{k}_4}{(2\pi)^{4d}} \tilde{\phi}(\tilde{k}_1) \ldots \tilde{\phi}(\tilde{k}_4)(2\pi)^d \times$$

$$\times \delta(\tilde{k}_1 + \ldots + \tilde{k}_4)$$

Comming from δ function.

we are also changing k in δ( ) function.

use; $$\delta(f(x)) = \sum_{\{x_i\}} \frac{\delta(x - x_i)}{|f'(x_i)|}$$

so that $f(x_i) = 0$.

we have new coupling constants.

Now we have lots of freedom to set one of the coupling constant as 1 because we have also rescaled the field we set the constant which stands in front of k^2 to be 1

set
$$b^{-d} z^2 \tilde{r}_2 b^{-2} k^2 = 1$$
$$z^2 \tilde{r}_0 b^{-d} = \tilde{r}$$
$$\tilde{r}_4 b^{-4d} z^4 b^d = \tilde{u}$$

Now, drop ~ from ϕ & k....; it does not matter ~~be~~ because they are auxilliary variable which are getting integrated over $\int d\tilde{x} = \int dx$ ....

so; we require;
$$1 = b^{-d-2} z^2 \tilde{r}_2$$

whatever $\tilde{r}_2$; we ~~too~~ can write it as some function of a parameter b

$$\therefore \boxed{\tilde{r}_2 = b^{\eta} \quad ; \; b > 1}$$

choose this $b>1$ if you want it to be same as ~~the~~ our previous ~~as~~ anamalous dimension

~~η has anamalous dim~~

η is called Anamalous Dimension.
(This is same as the one we see in defination of Spin-spin correlation function)

so;
$$1 = b^{-d-2+\eta} z^2$$
$$\Rightarrow z = b^{\frac{d+2-\eta}{2}}$$

$$\boxed{z = b^{\frac{d+2-\eta}{2}}}$$

∴ we will see later.

Then $\phi(k) = b^{-\frac{d-2+\eta}{2}} \phi_<(k)$

check; prof wrote; $\phi(k) = b^{\frac{-d+2-\eta}{2}} \phi_<(k)$
check.

This is how wave function renormalizes in order to maintain that coupling constant to be equal to 1.

$\tilde{r} = Z^2 \tilde{r}_0 b^{-d} = b^{d+2-\eta-d} \tilde{r}_0$

$\Rightarrow \boxed{\tilde{r} = b^{2-\eta} \tilde{r}_0}$ $\boxed{\tilde{u} = b^{4-2\eta-d} \tilde{r}_4}$

We now have

$$S[\phi] = \int d^dx \left[\frac{1}{2}(\nabla\phi)^2 + \frac{\tilde{r}}{2}\phi^2 + \frac{\tilde{u}}{4!}\phi^4\right]$$

$S[\phi]$ after integrating ~~over~~ over fast modes (changes coupling constants)

Effectively: As we integrate over fast modes, coupling constant changes in each step.

$$(r, u) \longrightarrow (\tilde{r}, \tilde{u})$$

$\therefore$ Now, we do this repeatedly many times
... and at each step ~~r~~ r, u will change

$$(r, u) \longrightarrow (\tilde{r}, \tilde{u}) \longrightarrow (\tilde{\tilde{r}}, \tilde{\tilde{u}}) \rightarrow \ldots$$

The coupling changes
$\Rightarrow$ so we are flowing in the space of coupling constants.

A miracle can happen & ~~we may find that~~ it may happen that after ~~few step~~ some steps the coupling constants donot change any more. This is called the Fixed Point.

We may find a Fixed Point (r^*, u^*).

ie: after that; if we again do one more step

$(r^*, u^*) \to (r^*, u^*)$ .. ~~dont change~~

... dont change.

∴ ~~we are~~

In each step we are introducing new momenta $\tilde{k} = kb$; $b>1$; which means that if you think about in position space

⟶ Then in each step your new $\tilde{x} = x/b$

∴ and so this must be true for any spacial thing; in ~~po~~ particular for correlation length.

∴ In each step, the correlation length

$$\xi = \xi(r,u) \to \xi(\tilde{r}, \tilde{u}) = \frac{\xi(r,u)}{b}$$

↳ Just the consequence of how we changed the momenta

but At fixed poit we found $\xi(r^*, u^*) = \dfrac{\xi(r^*, u^*)}{b}$

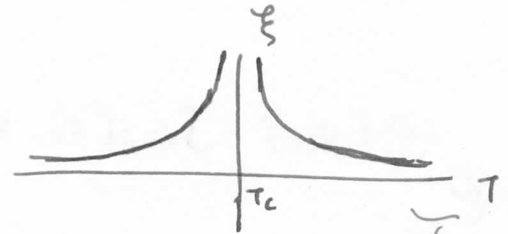
but $b > 1$

⇒ So we have two possibilities.

$$\xi(r^*, u^*) = \begin{cases} 0 & \ldots \text{ Phase } (T \neq T_c) \\ \infty & \ldots \text{ Critical Phase } (T = T_c) \end{cases}$$

∞ is interesting from the view of phase transition.

$\xi = \infty$ corresponds to critical point; so we are at $T = T_c$.



ξ goes to zero means you are in very well defined phase

$\xi = 0$ corresponds to phase; so $T \neq T_c$

ie; $\xi \to 0$ means either in the phase below T_c or above T_c (either paramagnetic or ferromagnetic phase)

How r changes as you flow in ~~space~~ space of couplings ... that will be called RG flow & the corresponding equation will be called RG equation.

... of course we are interested in when couplings dont change ... then we are at fixed point.

Part 2 Explicit Calculation.

$$Z = \int \mathcal{D}\phi_> \mathcal{D}\phi_< \, e^{-S} = \int \mathcal{D}\phi_< \, e^{-\tilde{S}[\phi_<]}$$

$e^{-\tilde{S}[\phi_<]}$; Remember $S = S_0 + S_{int}$

$= S_0[\phi_<] + S_0[\phi_>] + S_{int}[\phi_<, \phi_>]$

$$S = \underline{\underline{S_0[\phi_<]}} + S_0[\phi_>] + S_{int}[\phi_<, \phi_>]$$

$\hookrightarrow$ not touched by fast integration.

$\cancel{Z = e^{-S_0[\phi_<]}} \int \mathcal{D}\phi_>$

$$Z = \int \mathcal{D}\phi_< \, e^{-S_0[\phi_<]} \int \mathcal{D}\phi_> \, e^{-S_0[\phi_>] - S_{int}[\phi_<, \phi_>]}$$

$$Z = \int \mathcal{D}\phi_< \cdot e^{-S_0[\phi_<]} \frac{\int \mathcal{D}\phi_> \, e^{-S_0[\phi_>] - S_{int}[\phi_<, \phi_>]}}{\int \mathcal{D}\phi_> \, e^{-S_0[\phi_>]}} \times \underbrace{\int \mathcal{D}\phi_> \, e^{-S_0[\phi_>]}}_{\text{lets call this } Z_0}$$

$$\left. Z_0 \equiv \int \mathcal{D}\phi_> \, e^{-S_0[\phi_>]} \right\} \Rightarrow$$ Partition function over non interacting part over fast mode.

This is just the definition of average of $e^{-S_{int}[\phi_<, \phi_>]}$ in field theory ~~over~~ over fast mode

$$\langle e^{-S_{int}} \rangle_{FAST} = \frac{\int \mathcal{D}\phi_> \, e^{-S_0[\phi_>] - S_{int}[\phi_<, \phi_>]}}{\int \mathcal{D}\phi \cdot e^{-S_0[\phi_>]}}$$

In general: $\langle \cdot \rangle = \dfrac{\int \mathcal{D}\phi \cdot e^{-S} \times [\cdot]}{\int \mathcal{D}\phi \, e^{-S}}$

Now, we get a beautiful formula for $\tilde{S}$.

$$\boxed{\tilde{S}[\phi_<] = S_0[\phi_<] - \log\left(\langle e^{-S_{int}}\rangle_{FAST}\right) - \log Z_0}$$

Z_0 does not depends on slow modes $\phi_<$
↪ it is just a constant (so not much interesting)

~~Cumulant Expansion~~

~~given Ω be any random variable~~
~~then~~ $\cancel{\langle e^{-\Omega}\rangle = e^{\langle\Omega\rangle + \frac{1}{2}[\langle\Omega^2\rangle - \langle\Omega\rangle^2] + \dots}}$

Cumulant Expansion

given Ω be any random variable
then $\langle e^{-\Omega}\rangle = e^{\langle\Omega\rangle} + \frac{1}{2}[\langle\Omega^2\rangle - \langle\Omega\rangle^2] + \dots \infty$

Using Cumulant Expansion.

$$\boxed{\tilde{S}[\phi_<] = S_0[\phi_<] + \langle S_{int}\rangle_{FAST} - \frac{1}{2}\left[\langle S_{int}{}^2\rangle - \langle S_{int}\rangle^2\right] + \dots}$$

once we know $\tilde{S}$ after calculating,
then we can find $\tilde{r}$ & $\tilde{u}$

Then we know recursion relation between new & old couplings.

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$$\tilde{S}[\kappa] = S[\kappa] - \log\left(\langle e^{\ldots} \rangle\right) - \log Z_0$$

Cumulant Expansion

Given X be any random variable

Cumulant Expansion

Given X be any random variable

$\langle e^{X} \rangle = e^{\langle X \rangle + \frac{1}{2}\left[\langle X^2 \rangle - \langle X \rangle^2\right] + \ldots}$

Lecture 8: Wilson Renormalization 2: Explicit Calculation ; Cumulant expansion , RG equations

$$\tilde{S}[\phi_<] = \underbrace{S_0[\phi_<]}_{\text{0th order approximation for } \tilde{S}} + \underbrace{\langle S_{int}\rangle_{FAST}}_{1^{st}\text{ order}} \underbrace{- \frac{1}{12}\left[\langle S^2_{int}\rangle_{FAST} - \langle S_{int}\rangle^2_{FAST}\right]}_{2^{nd}\text{ order.}}$$

Zeroth Order $S_0[\phi_<] = \frac{1}{2}\int_k^{\Lambda/b} (k^2+r)|\phi_<(k)|^2$

Integral over k

$$\int_k \equiv \int \frac{d^dk_1}{(2\pi)^d}$$

First Order $S_{int} = \frac{u}{4!}\int_{k_1\ldots k_4} [\phi_<(1)+\phi_>(2)]\ldots\ldots [\phi_<(4)+\phi_>(4)]\,(2\pi)^d\delta(\Sigma k_i)$

$$= \frac{u}{4!}\int_{k_1\ldots k_4} (2\pi)^d\delta(k_1+\ldots+k_4)\cdot\Big[\underbrace{\phi_<(1)\phi_<(2)\phi_<(3)\phi_<(4)}_{\text{terms where every term is slow. A}} + \underbrace{\phi_>(1)\phi_>(2)\phi_>(3)\phi_>(4)}_{\text{term where every term is fast B}} +$$

$$+ \underbrace{4\,\phi_<(1)\phi_>(2)\phi_>(3)\phi_>(4)}_{\text{mixed with one slow C}} + \underbrace{4\times\phi_>(1)\phi_<(1)\phi_<(2)+\phi_<(3)}_{\text{mixed with one fast. D}} +$$

$$+ \underbrace{6\times\phi_<(1)\phi_<(2)\phi_>(3)\phi_>(4)}_{\text{mixed with 2 fast \& 2 slow.}}\Big] \quad E$$

now we integrate over fast mode.

B will just give a number ; no slow mode left (not interesting)

A ⇒ dont need to anything ⇒ This term survives as it is.

When we do average C & D go away : only E contributes.

Remember that doing the average means

$$\langle \cdot \rangle_{FAST} = \frac{1}{Z_0} \int \mathcal{D}\phi_> e^{-S_0[\phi_>]} \cdot (\cdot)$$

where $Z_0 = \int \mathcal{D}\phi\, e^{-S_0[\phi_>]}$

Example, $\langle \phi_>(3)\phi_>(4)\rangle = \frac{1}{Z_0}\int \mathcal{D}\phi_> \cdot \phi_>(3)\,\phi_>(4)\, e^{-\int_{\Lambda/b}^{\Lambda}(\cdots)}$

$$\Rightarrow \langle \phi_>(3)\,\phi_>(4)\rangle = \frac{1}{Z_0}\int \mathcal{D}\phi_>\, \phi_>(3)\phi_>(4)\, \exp\left(-\int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d}\cdot (k^2+r)\,|\phi_>(k)|^2\right)$$

This is just path integral formula of two point function.

Can think of it as $\phi_k \cdot A_{kk'}\phi_{k'}$

here matrix A is an operator k^2+r

QFT

we get Propagator for k_3: $G_0(k_3)$

$$G_0(k) = \frac{1}{k^2+r}$$

$$\Rightarrow \langle \phi_>(3)\,\phi_>(4)\rangle = G_0(k_3)\,(2\pi)^d\,\delta(k_3+k_4)$$

<u>Wicks Theorem</u> (multi point interactions $\Rightarrow$ product of 2 point interaction.

$\langle\phi_1 \dots \phi_k\rangle = 0$ for k odd. (because we cannot decompose it into 2 point interactions)

(C and D goes zero ~~be~~ because of this.)

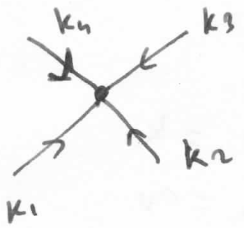
Consequently terms C & D donot contribute.
& only E is interesting.

Lets simplify our life by using Diagramatic Notation.

Feynmann Diagrams

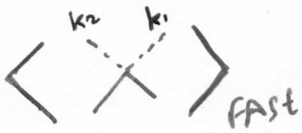

Fast Mode (or Fast particles)


Slow Mode ($\phi_<(k)$)



$$= \frac{u}{4!}(2\pi)^d\,\delta\left(\sum k_i\right)$$

Processes of Averaging means connecting two fast modes

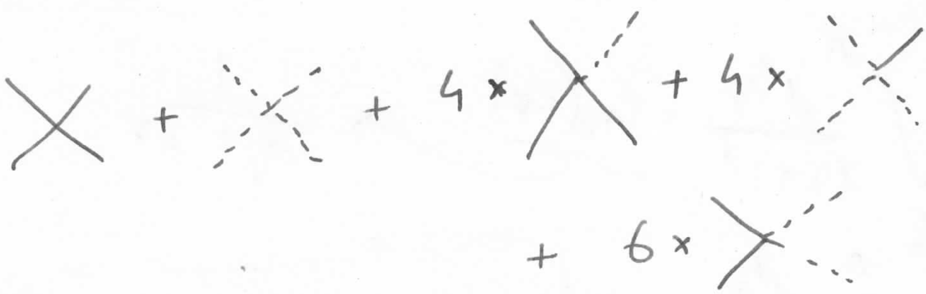
ex 

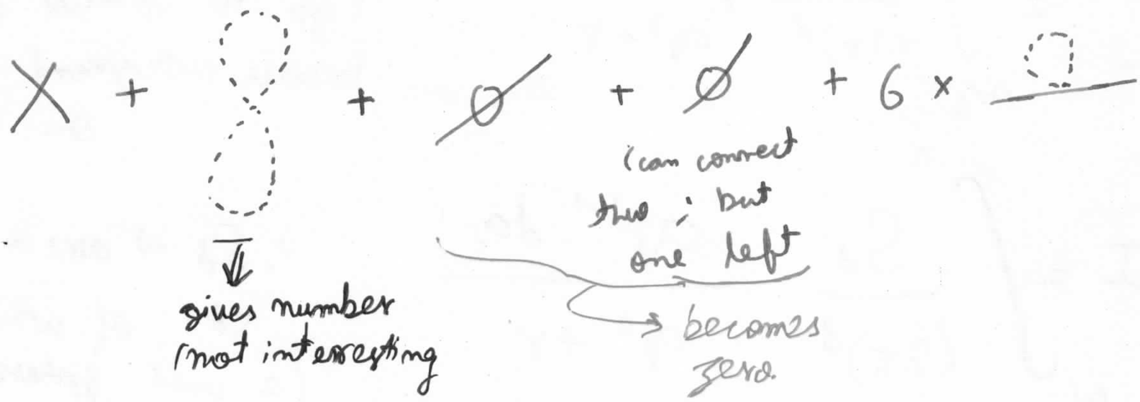
$_{\text{FAST}}$ = 

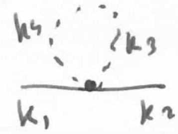
where loop over fast mode  is $= G_0(k)\,(2\pi)^d\,\delta(k_1+k_2)$

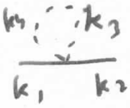
 $= G_0(k)(2\pi)^d\,\delta(k_1+k_2)$

~~Shell~~

S_{int} = 

$\langle S_{int}\rangle$ = 

$6 \times$  $= 6 \times \frac{u}{4!} \int_{k_1 \ldots k_4} (2\pi)^d \, \delta(k_1 + \ldots + k_4) \, \phi_<(k_1) \phi_<(k_2) \times G_0(k_3) (2\pi)^d \, \delta(k_3 + k_4)$

$6 \times$  $= 6 \times \underbrace{\frac{u}{4!} \int_{k_1 \ldots k_4}}_{\times} (2\pi)^d \, \delta(k_1 + k_2 + \ldots + k_4) \cdot \phi_<(k_1) \phi_<(k_2) \cdot \underbrace{G_0(k_3) (2\pi)^d \, \delta(k_3 + k_4)}$

$$= \frac{u}{4} \int_{k, q} |\phi_<(k)|^2 \, G_0(q)$$

$k_1 = -k_2 = k$
$k_3 = -k_4 = q$

The integration factorizes

$$= \frac{u}{4} \underbrace{\int_q G_0(q)}_{\text{denote this by } I} \cdot \int_k |\phi_<(k)|^2$$

$; \; I := \int_q G_0(q)$

$$I = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \times G_0(q) \longrightarrow \frac{1}{q^2 + r}$$

$$= \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \times \frac{1}{q^2 + r}$$

(go to spherical coordinates because integrand just depends on q^2

$$I = \int_{\Lambda/b}^{\Lambda} \frac{S_d}{(2\pi)^d} \cdot \frac{q^{d-1} \cdot dq}{q^2 + r}$$

$\because S_d \Rightarrow$ area of S^{d-1} of unit radius. (a unit sphere embeded in d dimensional space)

~~so~~ b has to be bigger than 1;

but we can choose b to ~~only~~ tiny bigger than 1.

Then we can use the approximation.

$$\int_a^b f(x)\,dx \simeq f(b)\cdot(b-a)$$

Now parametrize our b as

$b = e^{\Delta l}$ wher Δl is small.

then $(\Lambda - \Lambda/b) = \Lambda(1-e^{-\Delta l})$

$$= \Lambda(1-(1-\Delta l + O(\Delta l^2)))$$

$$\simeq \Lambda \cdot \Delta l$$

so; $I = \Lambda \cdot \Delta l \cdot \dfrac{S_d \cdot \Lambda^{d-1}}{(2\pi)^d(\Lambda^2+\lambda)}$

$$\Rightarrow \boxed{I = \Delta l \cdot \frac{S_d}{(2\pi)^d} \cdot \frac{\Lambda^d}{(\Lambda^2+\lambda)}}$$

$S_d \Rightarrow$ Area of unit sphere embedded in d dimensional Euclidean space.

$\langle S_{int} \rangle_{FAST} = \underset{\frac{u}{4!}\int\phi\phi\phi\phi}{\times} + 6\times \underset{\frac{u}{4} I \int \phi\phi}{-\bigcirc-}$ } First Order Term

0th + 1st order

$$\tilde{S}[\phi_<] = \frac{1}{2}\int_k \left[|\times k^2 + \left(r + \frac{u}{2} I\right)\right] |\phi_<(k)|^2 + \frac{u}{4!}\int_{k_1 \dots k_4} (2\pi)^d \delta(\Sigma k_i)\, \phi_<(1) \dots \phi_<(4)$$

Read off $\tilde{\Lambda}_2 = 1 = b^{\eta} \Rightarrow \boxed{\eta = 0}$ ~~because $b > 1$~~

$\hookrightarrow$ The ~~anatom~~ analomous dimensions comes out to be zero from ~~1~~ 0th & 1st order perturbation theory.

$\tilde{\Lambda}_0 = r + \frac{u}{2} I \Rightarrow$

$\tilde{\Lambda}_4 = u$

$\cancel{\tilde{r} = b^{2-\eta} \tilde{r}_0 b^{2-0}} =$

$\tilde{r} = b^{2-0} \cdot \tilde{r}_0 = b^2 \left(r + \frac{u}{2} I\right)$

$\tilde{u} = b^{4-d-0} \cdot \tilde{r}_4 = b^{4-d}$

Renormalization Group (R.G.) Recursion Relation (to First Order)	$\tilde{r} = b^2 \left(r + \frac{u}{2} I\right)$ $\tilde{u} = b^{4-d} \cdot u$

$\hookrightarrow$ Algebraic relation .. generally algebraic relations are difficult to deal with ... so let's turn them into differential equation.

so;

$\tilde{r} = e^{2\Delta l}\left(r + \frac{u}{2} I\right)$

Using the ~~prop~~ parametrization of b as $b = e^{\Delta l}$

$\Rightarrow \tilde{r} \simeq (1 + 2\Delta l)\left(r + \frac{u}{2} I\right)$

$\Rightarrow \tilde{r} \simeq r + 2\Delta l \cdot r + \frac{u}{2} I + (\Delta l \cdot u\, I)$

This is actually higher order ... so we will not keep this term

because $I \propto \Delta l$

$$\frac{dr}{dl} = \frac{\tilde{r} - r}{\Delta l} = 2r + \frac{u}{2} \frac{I}{\Delta l}$$

$$\Rightarrow \boxed{\frac{dr}{dl} = 2r + \frac{u}{2} \cdot \frac{S_d}{(2\pi)^d} \cdot \frac{\Lambda^d}{\Lambda^2 + r}}$$

similarly.

$$\boxed{\frac{du}{dl} = (4-d)u = \varepsilon u}$$

Differential form of R.G. relations. (to first order)

$\epsilon = 4-d$

More generally

for a coupling g :
we get relations like

$$\frac{dg}{dl} = \beta_g$$

$\longrightarrow$ we call it β_g

it is called Beta Function. for the corresponding coupling constant g.

$$\beta_r = 2r + \frac{u}{2} \cdot \frac{S_d}{(2\pi)^d} \cdot \frac{\Lambda^d}{\Lambda^2 + r}$$

$$\beta_u = (4-d)u$$

Fixed point corresponds to $\beta_g = 0$.

We like to think of l as some fictitious time or something which parametrizes the curve in the space of couplings ... How the coupling changes as you change the scale.

~~is~~ l is fictitious time which corresponds to evolution of g in the space of couplings as we change the scale.

$[r] = 2$; $[u] = 4 - d = \epsilon$ recall.

In general, we can show that in the leading order we find $\frac{dg}{dl} = \beta_g = \underbrace{[\dim g]\, g + \ldots}$ higher order terms.

if $[\dim g] > 0 \Rightarrow g$ is increasing ; Relevant

$[\dim g] < 0 \Rightarrow g$ is decreasing ; Irrelevant

Dimension of g tells whether g increases or decreases (and this is related to relevant and irrelevant of the coupling constants)

To get interesting physics we have to go to 2nd order

To get Nobel Prize we need 2nd Order

$$-\frac{1}{2}\left[\langle S_{int}^2\rangle - \langle S_{int}\rangle^2\right]$$

This can have connected & disconnected diagrams

This is very continient & actually removes all the disconnected diagrams

The result in the end is just connected diagrams .. and now we can classify them depending on how many external slow legs they have

ie; with 0, 2, 4, 6 slow legs.

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2 legs

To write this we have to chose which leg is slow leg ; so 4 possibilities

4

first S_{int} Second S_{int}

⇓ we get

~~[illegible]~~

① 4×4

② which one is this so; 3×3

$4\times4\times6\times\frac{1}{2}=48$

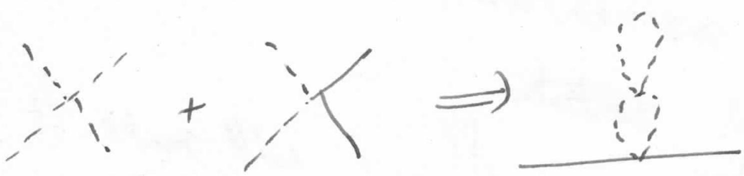
with each diagrams, comes along the combinatorical factor related to the diagram

so; $4\times4\times3\times3\times\frac{1}{2} = 72$

This is the pre-factor comming from $-\frac{1}{2}\left[\langle S_{int}^2\rangle - \langle S_{int}\rangle^2\right]$

so 72 ~~diagrams which have~~ ~~2 slow legs in~~ end ... ie;

ex



In the end we always end up with ~~s~~ two slow legs

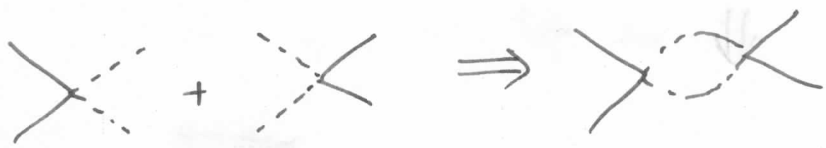
2 external leg can contribute <u>only</u> to $\tilde{r}_0$ & $\tilde{r}_2$

↳ we dont care about this, we already got these.

∴ what we care about the one with 4 external legs; which will contribute to $\tilde{r}_4$; hence contribute to u coupling.

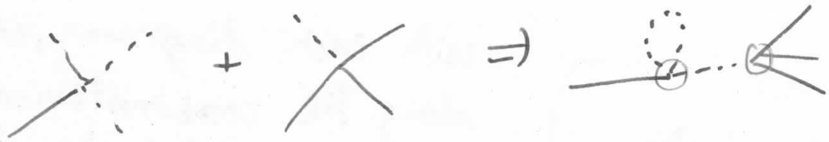
To find corrections to β_u, we need 4 legs.

ex ①



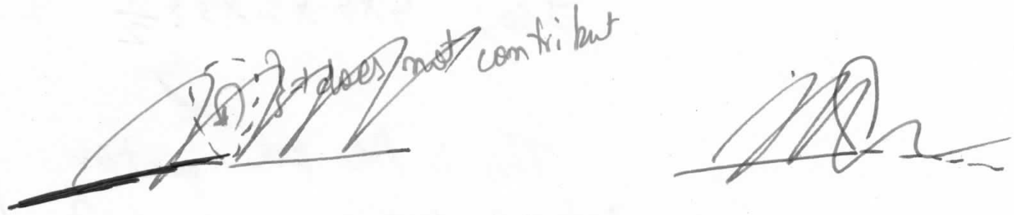
} This is interesting

②



} not much interesting)

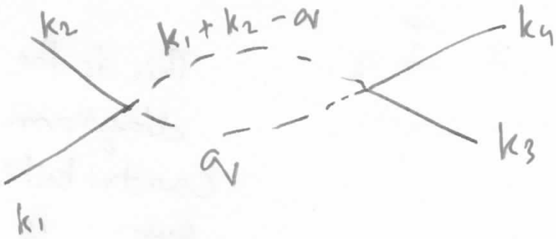
At each vertex you have to conserve total momentum



→ you cannot conserve momentum here ... violates momentum conservation because fast momentum cannot be slow momentum.

② is unphysical ↳ zero ~~contrib~~ contribution; $\delta(\ldots)$ will make it zero.

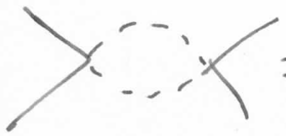
But we have to compute (1)



$^4C_2 = 6$ for ==

$^4C_2 = 6$ for == × Some factor for connecting

$$6 \times 6 \times 2 \times \frac{1}{2} = 36$$



$$= \left(\frac{u}{4!}\right)^2 \times 36 \int_{k_1,k_2..k_4,\,q} (2\pi)^d \,\delta(k_1+k_2+..+k_4)\, \phi_<(k_1)\, \phi_<(k_2)\, \phi_<(k_3)\, \phi_<(k_4) \times$$

$$\times\; G_0(q) \times G_0\left(k_3+k_4-q\right)$$

$$\left(\frac{u}{4!}\right)^2 36 \int_{k_1,\dots,k_4,\,q} (2\pi)^d \,\delta(\Sigma k_i)\, \phi_<(k_1) \dots \phi_<(k_4)\; G_0(q)\, G_0(k_3+k_4-q)$$

k_3 & k_4 are slow modes so; to get feeling q is fast mode.

$$G_0(k_3+k_4-q) \simeq G_0(-q)$$

but $G_0(-q) = G_0(q)$

$$= \left(\frac{u}{4!}\right)^2 36 \int_{k_1..k_4} (2\pi)^d \delta(\Sigma k_i)\, \phi_<(k_1)..\phi_<(k_4) \times I_2$$

This is the only diagram which contributes for our u.

where ; $I_2 = \int_q G_0^2(q)$

$$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{S_d}{(2\pi)^d} \times \frac{q^{d-1}}{(r+q^2)^2}\, dq$$

use length of interval to be small.

$$= \Delta l \; \frac{S_d}{(2\pi)^d} \times \frac{\Lambda^d}{(r+\Lambda^2)^2}$$

Correction of $\tilde{r}_4$:

$$\frac{u}{4!} - \left(\frac{u}{4!}\right)^2 \times 36 \times I_2 = \frac{\tilde{r}_4}{4!}$$

$$\tilde{u} = \tilde{r}_4\, b^{4-d} = \left(u - \frac{3}{2}u^2 I_2\right) b^{4-d}$$

Now final R.G. equation with first order correction for both r and u

$$\boxed{\begin{aligned} \frac{dr}{dl} &= \beta_r = 2r + \frac{u}{2} \cdot \frac{K_d}{r+\Lambda^2} \\ \frac{du}{dl} &= \beta_u = \varepsilon u - \frac{3}{2}u^2 \cdot \frac{K_d}{(r+\Lambda^2)^2} \\ \text{where } K_d &= \frac{S_d}{(2\pi)^d}\Lambda^d \end{aligned}}$$

corrected R.G. equation to first order.

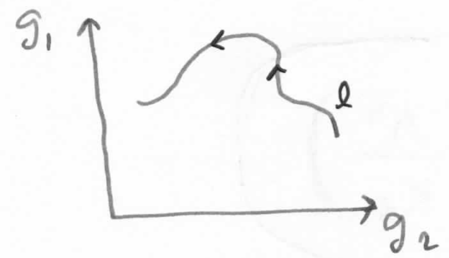
pg 95

Give the diagram which contribute to k^2 term?

 gives contribution to $V\ \phi^6$ ... but it ~~is irrelevant~~ is irrelevant.

1996

Lecture 9: Wilsonian Renormalization 3: Triumph: Wilson-Fisher Fixed Point, Critical-Exponents.



Renormalization Group flow.

Part III Fixed points in small ε - expansion.

Fixed point is when $\beta_r = 0 = \beta_u$

$(\varepsilon = 4-d)$: Expansion.

∴ We will treat ε as small & expanding everything to leading order in ε. (leading order up to $O(\varepsilon)$)

One of the fixed point is simple $(r^*, u^*) = (0, 0)$
(because each term in β_r & β_u are ~~proportional to~~ proportional to r or u)

$(r^*, u^*) = (0, 0)$ called Gaussian fixed Point
or MFT " " .

We can also have another fixed point.
if ε small then in β_u $u \propto \varepsilon$; then from $\beta_r \Rightarrow r \propto \varepsilon$

so; we know that $r \propto \varepsilon$
$u \propto \varepsilon$

so; we can neglect r in the denominator in RG equation.

Then $\beta_u = 0 \Rightarrow \varepsilon u - \frac{3}{2} u^2 \frac{K_d}{\Lambda^4} = 0$

$$\boxed{u = \frac{2}{3}\varepsilon \frac{\Lambda^4}{K_d}} \Rightarrow \boxed{r = -\frac{u}{4}\frac{K_d}{\Lambda^2}} = -\frac{\varepsilon}{6}\Lambda^2$$

So; we have another fixed point

$$(r^*, u^*) = \left(-\frac{\varepsilon}{6}\Lambda^2, \ \frac{2}{3}\epsilon \frac{\Lambda^4}{K_d}\right)$$

This is interesting one which was awarded nobel prize, called Wilson Fisher Point.

How things behave near fixed point

We want to linearize the RG equations around the critical point.

$\Rightarrow$ so, we taylor expand the β - functions

Introducing a parameter $\delta r = r - r^*$, $\delta u = u - u^*$

Now, we can write down the linearization in matrix form since we have two parameters.

$$\frac{dr}{dl} = \beta_r$$

$$\delta\left(\frac{dr}{dl}\right) = \frac{\partial \beta_r}{\partial r}\delta r + \frac{\partial \beta_r}{\partial u}\delta u$$

$$\Rightarrow \frac{d}{dl}(\delta r) = \frac{\partial \beta_r}{\partial r}\delta r + \frac{\partial \beta_r}{\partial u}\delta u.$$

$$\frac{d}{dl}\begin{pmatrix}\delta r \\ \delta u\end{pmatrix} = \begin{pmatrix} \frac{\partial \beta_r}{\partial r} & \frac{\partial \beta_r}{\partial u} \\ \frac{\partial \beta_u}{\partial r} & \frac{\partial \beta_u}{\partial u}\end{pmatrix}\Bigg|_{\substack{r = r^* \\ u = u^*}} \begin{pmatrix}\delta r \\ \delta u\end{pmatrix}$$

Taylor expansion written as matrix

This matrix is called M; which captures the behavior of RG flow close to the fixed point.

$$M = \begin{pmatrix} 2 - \frac{u}{2}\frac{K_d}{(r+\Lambda^2)^2} & \frac{1}{2}\frac{K_d}{r+\Lambda^2} \\ 3u^2\frac{K_d}{(r+\Lambda^2)^3} & \varepsilon - 3 * u\frac{K_d}{(r+\Lambda^2)^2} \end{pmatrix}$$

$$\simeq \begin{pmatrix} 2 - \frac{u}{2}\frac{K_d}{\Lambda^4} & \frac{1}{2}\frac{K_d}{\Lambda^2} \\ 0 & \varepsilon - 3u\frac{K_d}{\Lambda^4} \end{pmatrix}$$

→ This is our linearization matrix.

Evaluate M at corresponding ~~fixed~~ critical point.

(i) one for Gaussian fixed point.

(ii) another for Wilson Fisher fixed point.

Want to find eigenvalues and eigenvectors.

Gaussian Fixed Point

$(r^*, u^*) = (0, 0)$

Then our Matrix $M = \begin{pmatrix} 2 & \frac{1}{2}\frac{K_d}{\Lambda^2} \\ 0 & \varepsilon \end{pmatrix}$

eigen values 2 & ε.

$$\left.\begin{aligned} \lambda_t &= 2 \\ \lambda_{\tilde{u}} &= \varepsilon \end{aligned}\right\} \text{eigenvalues.}$$

Eigenvectors $(M - \lambda I)v = 0$

Then $V_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $V_{\tilde{u}} = \begin{pmatrix} 1 \\ \frac{2(\varepsilon - 2)\Lambda^2}{K_d} \end{pmatrix}$

In terms of new variable t & $\tilde{u}$; the matrix is diagonal ; and these are corresponding principle direction.

So; R.G. equations in terms of new variables t & $\tilde{u}$
becomes

$$\frac{dt}{dl} = \lambda_t t \quad ; \quad \frac{d\tilde{u}}{dl} = \lambda_{\tilde{u}} \tilde{u}$$

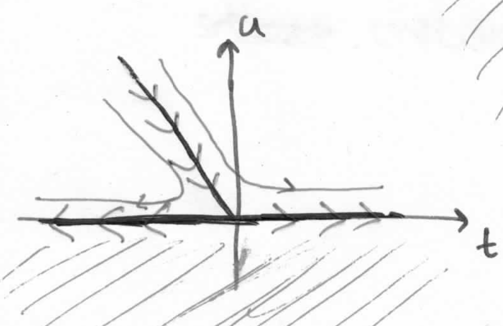
$$\Downarrow \qquad\qquad\qquad \Downarrow$$

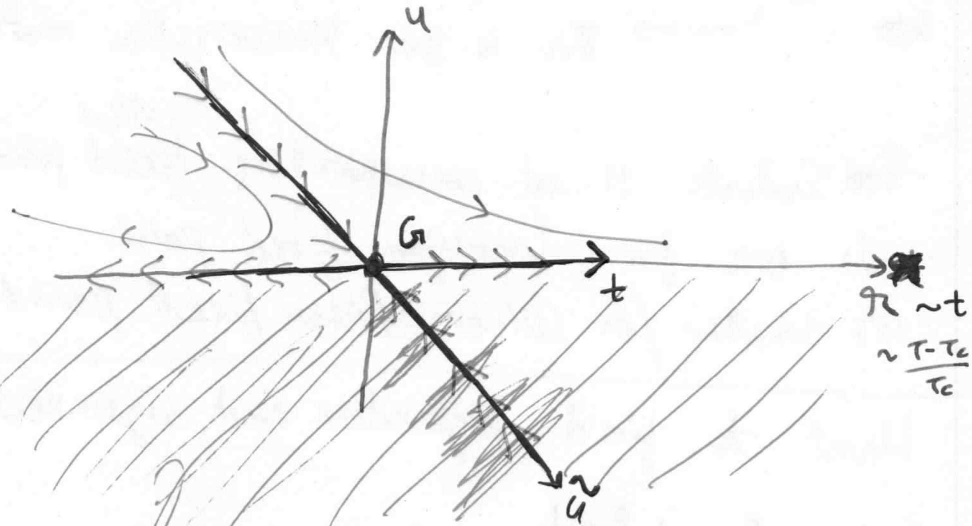
$$t = t_0 e^{\lambda_t \cdot l} \qquad\qquad \tilde{u} = \tilde{u}_0 e^{\lambda_{\tilde{u}} \cdot l}$$

$$\boxed{t = t_0 \cdot e^{2l}} \qquad\qquad \boxed{\tilde{u} = \tilde{u}_0 \cdot e^{\varepsilon l}}$$

Cases

(i) $\varepsilon < 0 \quad (d > 4)$





negative values of u is unphysical
(for landau's theory to be well defined)

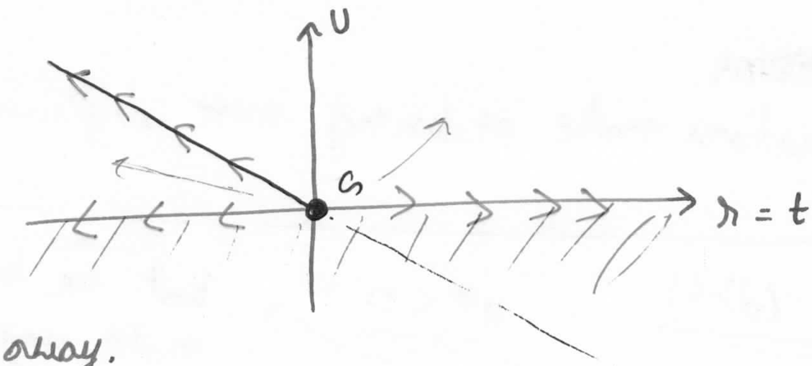
We can never go to ~~critical~~ fixed point, if we are not at $T = T_c$.

G is stable fixed point
∴ because $\tilde{u}$ direction is stable
meaning it flows in.

(r direction is always unstable ... we are not at $T = T_c$)

G describes ~~MFT~~ critical point, and ~~to~~ it is because $T=T_c$ is when $r=0$; precisely happens at $T=T_c$ as predicted by MFT.

~~Ain~~ $\underline{\varepsilon>0}$ $(d<4)$



Such G is unstable: if you are not at this G you flow away.

∴ if you are in vicinity of G; you flow away.

⟶ This is precisely the reason why MFT fails in $d<4$ dimensions.

~~because MFT ... predicts~~

Reason as to why MFT fails in $d<4$ because MFT predicts $T=T_c$ is physical ... but we found it is unstable so cant be physical.

---

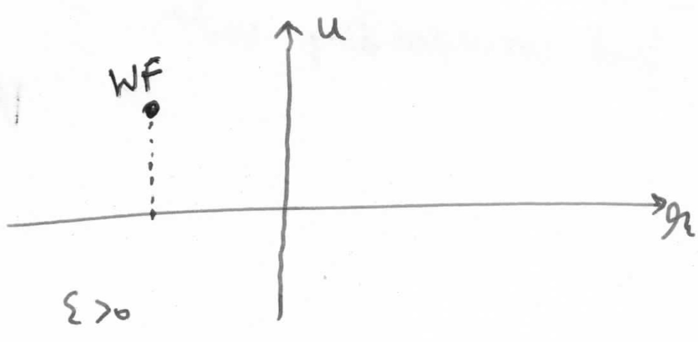
for $d=4$; we ~~cant~~ cant comment right now; we will have to go to higher orders.

Wilson Fisher Fixed Point

$$(r^*, u^*) = \left(-\frac{\varepsilon}{6}\Lambda^2, \ \frac{2}{3}\varepsilon\frac{\Lambda^4}{K_d}\right)$$

cases

(i) $\underline{\varepsilon>0}$ $(d<4)$



~~Tra~~ True transition temperature occurs for $\tilde{r} < 0$

means $T < (T_c)$ as predicted by MFT.

~~Fluctuations~~

Fluctuations make ordering more difficult ∴ $T_c < (T_c)_{MFT}$

$\varepsilon < 0$ $(d > 4)$ ... $u^* < 0$ $\longrightarrow$ but we know $u < 0$ does not make any sense.

So; Wilson Fisher fixed point does not exist in $d > 4$.

Thats why in $d > 4$, the only fixed point we have is gaussian fixed point & it is stable one; Thats why MFT is correct.

~~In~~ . In $d > 4$ Gaussian fixed point $\longrightarrow$ so MFT is correct.

---

for $\varepsilon > 0$ $(d < 4)$

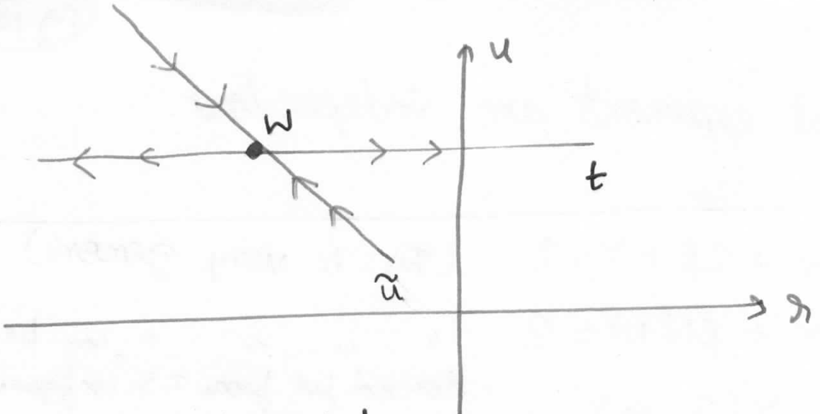
$$M = \begin{pmatrix} 2 - \varepsilon/3 & , & \frac{K_d}{2\Lambda^2} \\ 0 & , & -\varepsilon \end{pmatrix}$$

$$\boxed{\lambda_t = 2 - \frac{\varepsilon}{3}} \; ; \; \boxed{\lambda_{\tilde{u}} = -\varepsilon}$$

$$V_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then we can find corresponding vectors

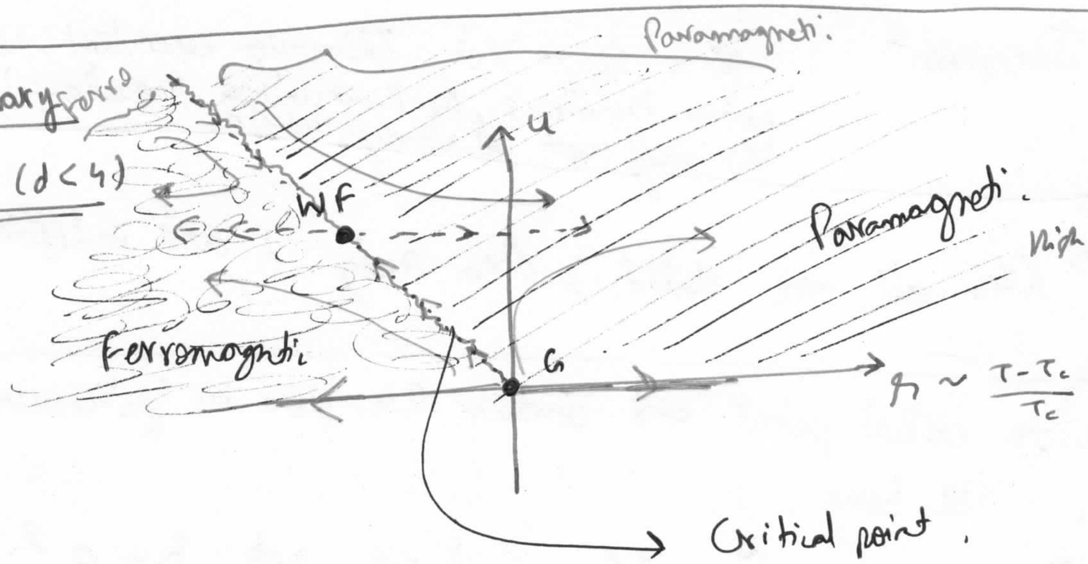
$$V_{\tilde{u}} = \begin{pmatrix} 1 \\ -\frac{4}{3} \frac{(3+\varepsilon)\Lambda^2}{K_d} \end{pmatrix}$$



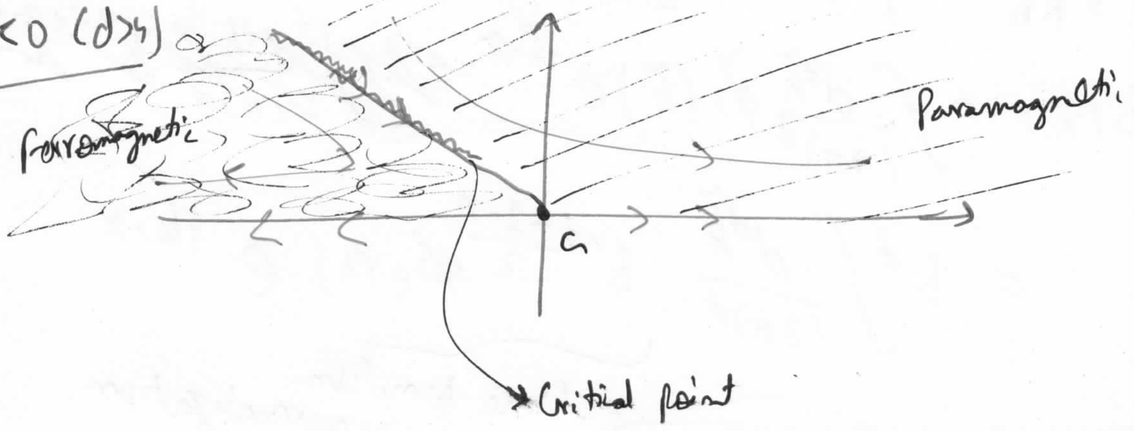
Stable fixed point.

Summary

$\varepsilon > 0$ $(d<4)$



$\varepsilon < 0$ $(d>4)$



$d>4$ MFT is correct.

PART IV Critical Exponents (Triumph)

Remind

$$C \propto |t|^{-\alpha}, \quad M \propto |t|^{\beta}, \quad \chi \propto |t|^{-\gamma}, \quad B \propto M^{\delta}$$

$$G(r) \propto \frac{\exp(-r/\xi)}{r^{d-2+\eta}}$$

$\eta \Rightarrow$ anamolous dimension.

$\xi \propto |t|^{-\nu}$

Yesterday Tutorial

Not all critical exponents are independent.

Scaling Laws:

Rush Brook : $\alpha + 2\beta + \gamma = 2$ (This is very generic)

Griffiths : $\alpha + \beta(\delta+1) = 2$ (" " " ", can be derived just from T.D. configurations)

Fisher : $\gamma = \nu(2-\eta)$ (It is little bit more subtle)

Josephson* : $\alpha = 2 - \nu d$ (dimension dependent; requires extra hypothesis, the Hyperscaling Hypothesis) (This may not be always valid)

When all are valid; then only 2 are independent.

Close to critical point and under R.G. flow in "fictitious time" l. We have

(i) $\tilde{k} = kb$, $\tilde{x} = x/b$; & we set $b = e^{l}$

(ii) $\tilde{\phi}(\tilde{x}) = \int \frac{d^d\tilde{k}}{(2\pi)^d} \tilde{\phi}(\tilde{k}) e^{i\tilde{k}\cdot\tilde{x}}$ ~~$= b^d \int \frac{d^dk}{(2\pi)^d} b^{-\frac{d+2-\eta}{2}} \phi_<(k)$~~

$$= b^d \int \frac{d^d k}{(2\pi)^d} \cdot \underbrace{b^{-\frac{d+2-\eta}{2}}}_{\text{Wave function renormalization.}} \phi_<(k)\, e^{ik\cdot x}$$

$$\boxed{\tilde{\phi}(\tilde{x}) = b^{\frac{d-2+\eta}{2}} \phi(x)}$$

This is how ϕ changes if you flow for finite l.

(iii) $\tilde{t} = t e^{\lambda_t l}$

$\tilde{u} = u e^{\lambda_u l}$

(iv) $G(x) = \langle \phi(x)\phi(0)\rangle - \langle \phi(x)\rangle\langle \phi(0)\rangle$

Now we use that G is quadratic in ϕ.

$$G(\tilde{x},\tilde{t},\tilde{u}) = \cancel{b^{d-2+\eta}}\, G\left(\frac{x}{b}, te^{\lambda_t l}, ue^{\lambda_u l}\right)$$

~~because quadratic in ϕ~~

$$G(\tilde{x},\tilde{t},\tilde{u}) = G\left(\frac{x}{b}, te^{\lambda_t l}, ue^{\lambda_u l}\right)$$
$$= \underbrace{b^{d-2+\eta}}_{\text{Quadratic in } \phi} G(x,t,u)$$

$$\Rightarrow \boxed{G(\tilde{x},\tilde{t},\tilde{u}) = b^{d-2+\eta}\, G(x,t,u)}$$

It is equivalent to saying

$$G(x,t,u) = b^{-(d-2+\eta)}\, G\left(\frac{x}{b}, te^{\lambda_t l}, ue^{\lambda_u l}\right)$$

→ This is the master formula from which you can find all critical exponents

---

let us start exactly at fixed point where $t=0=u$.

Then we get: $G(x,0,0) = b^{-(d-2+\eta)}\, G\left(\frac{x}{b},0,0\right)$

so; exactly at critical point.

$$G(x,0,0) \propto \frac{1}{x^{d-2+\eta}}$$

η is the anomolous dimension which we get from how couplings transforms

$\Rightarrow$ η is ~~anat~~ anamolous dimension as before.

$\boxed{\eta=0}$ even to second order in R.G. flow.

---

lets move away from fixed point in t direction

Now we consider: $G(x,t,0) = b^{-(d-2+\eta)}\, G\left(\frac{x}{b}, te^{\lambda_t l}, 0\right)$

true for any b.

So; the sneaky idea is to specifically choose $b = x$

Then we get

$$G(x, t, 0) = \cancel{x}^{-(d-2+\eta)} G(1, t x^{\lambda_t}, 0)$$

remember ; $b = e^{l}$; $t e^{\lambda_t l} = t (e^{l})^{\lambda_t}$

$= t x^{\lambda_t}$

$$\Rightarrow \boxed{G(x, t, 0) = x^{-(d-2+\eta)} G(1, t x^{\lambda_t}, 0)}$$

so; we require $G(1, t x^{\lambda_t}, 0)$ is some function of $F(x/\xi)$.

so; $t x^{\lambda_t}$ must be a function of $\frac{x}{\xi}$.

so; $x^{\lambda_t} \cdot t = \underline{(x t^{1/\lambda_t})^{\lambda_t}} = \left(\frac{x}{\xi}\right)^{\lambda_t}$

$$\Rightarrow \boxed{\xi \propto |t|^{-\frac{1}{\lambda_t}}}$$

The critical exponent $\boxed{\nu = \frac{1}{\lambda_t}}$

$$\nu = \frac{1}{\lambda_t} \overset{\text{W.F.}}{=} \frac{1}{2 - \frac{\varepsilon}{3}} \simeq \frac{1}{2} + \frac{\varepsilon}{12} \quad (d < 4)$$

$$\overset{G}{=} \frac{1}{2} \quad (d > 4)$$

Complete Summary (ISING in $d=3$)

Critical Exponent	MFT	RG-ε-Expansion	Numerics
α	0	$\frac{\varepsilon}{6}$ in 3D ≈ 0.17	0.11
β	$1/2$	$\frac{1}{2}-\frac{\varepsilon}{6}$ 3D $= 0.33$	0.326
γ	1	$1+\varepsilon/6$ in 3D ≈ 1.17	1.237
δ	3	$3+\varepsilon$ in 3D $= 4$	4.79
ν	$1/2$	$\frac{1}{2}+\frac{\varepsilon}{12}$ in 3D $\simeq 0.58$	0.63
η	0	0	0.036

This was good enough & so it got Nobel Prize.

Statistical Physics - Shoaib Akhtar 13/5/2020

Lecture 10: Vector Model, Goldstone Modes, CMW Theorem and Lower Critical Dimensions

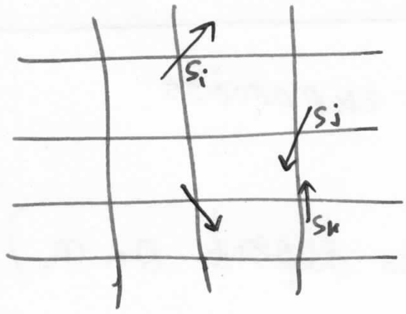
- Discrete Z_2 symmetry $\phi \to -\phi$
- Order Parameter is scalar $\phi(x)$

} Then you will have same Landau-Ginsberg functional ... hence are under same ISING UNIVERSALITY CLASS

Now, we will go beyond Ising Universality Class: where we will have more than one order parameter & different kind of symmetry.

e] Systems with Continuous Symmetry

VECTOR MODEL - n



d-dimensional cubic lattice

We can have spins at each lattice sign; and they will not only point up or down; but they can be along any directions: i.e. The spins will be arbitrary vectors

Vectors describing spins.

$$\boxed{H = -\frac{1}{2}\sum_{i,j} \vec{S}_i \cdot \vec{S}_j \, J_{ij}}$$

n-component vectors (in general n can be different from d)

Then $\vec{S}_i = (S_i^1, S_i^2, \dots, S_i^n)$

now, we require $|\vec{S}_i| = 1$

- $\underline{n=3}$: Heisenberg Model (interesting case)

Continuous $O(n)$ symmetry (n dimensional rotation)

(arbitrary global rotation where all spins are rotated by same angle & along the same axis ⇒ and the value of Hamiltonian remains unchanged)

Landau - Ginsberg

$$S = \int d^d x \left[\frac{1}{2} (\nabla \vec{\phi}) \cdot (\nabla \vec{\phi}) + \frac{r}{2} \vec{\phi} \cdot \vec{\phi} + \frac{u}{4} (\vec{\phi} \cdot \vec{\phi})^2 \right]$$

(not 4! ... for same reason ... will become clear later)

∇ in d-dimensions

$\vec{\phi}$ in n-dimensions

- In homework; we do small $\varepsilon = (4-d)$ expansion

  ⇒ Qualitatively similar to Ising; (except that; critical exponents now depend on n)
- However near $\underline{d=2}$ physics is different (very different).

Let us write $\vec{\phi} = \rho(x) \vec{n}(x)$

↳ lets parametrize the field in this way.

where $\rho(x) = \sqrt{\vec{\phi} \cdot \vec{\phi}}$ (magnitude) ; $\vec{n} \cdot \vec{n} = 1$

$\vec{n}$ ⇒ unit vector

$$\nabla \phi^a = n^a \nabla \rho + \rho \nabla n^a$$

$$(\nabla \vec{\phi}) \cdot (\nabla \vec{\phi}) = \nabla \phi^a \nabla \phi^a = \underbrace{n^a n^a}_{1} (\nabla \rho)^2 + 2\rho \nabla \rho \underbrace{n^a \nabla n^a}_{0} + \rho^2 (\nabla n^a)^2$$

$$(D\vec{\phi})\cdot(D\vec{\phi}) = (D\rho)^2 + \rho^2\, Dn_a Dn_a$$

$$\vec{\phi}\cdot\vec{\phi} = \rho^2$$

So; Now; Landau - Ginsberg Functional becomes

$$\boxed{S[\rho,\vec{n}] = \int d^dx \left[\frac{1}{2}(D\rho)^2 + \frac{1}{2}\rho^2 Dn_a Dn_a + \frac{r}{2}\rho^2 + \frac{u}{4}\rho^4\right]}$$

Landau's Theory: $f(\rho) = \frac{r}{2}\rho^2 + \frac{u}{4}\rho^4$ (lets define free energy)
(equivalent to MFT)

ρ is now order parameter here.

$f'(\rho) = 0 \Rightarrow 2\frac{r}{2}\rho + 4\frac{u}{4}\rho^3 = 0 \Rightarrow \boxed{\rho = \rho_0 = \sqrt{\frac{-r}{u}}}$ $r < 0$

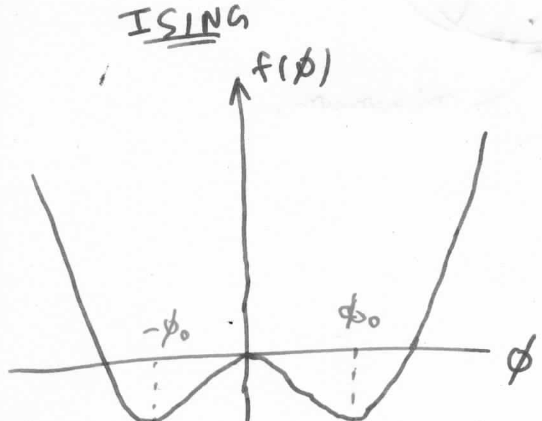
$= 0$ $r > 0$

Landau theory does not know anything about $\vec{n}$ because $\vec{n}$ only enters through the kinetic term; & we know landau theory does not have the kinetic term.

So; in this approximation in Landau's Theory; $\vec{n}$ is arbitrary.

$\vec{n}$ is not controlled by Landau's approximation.

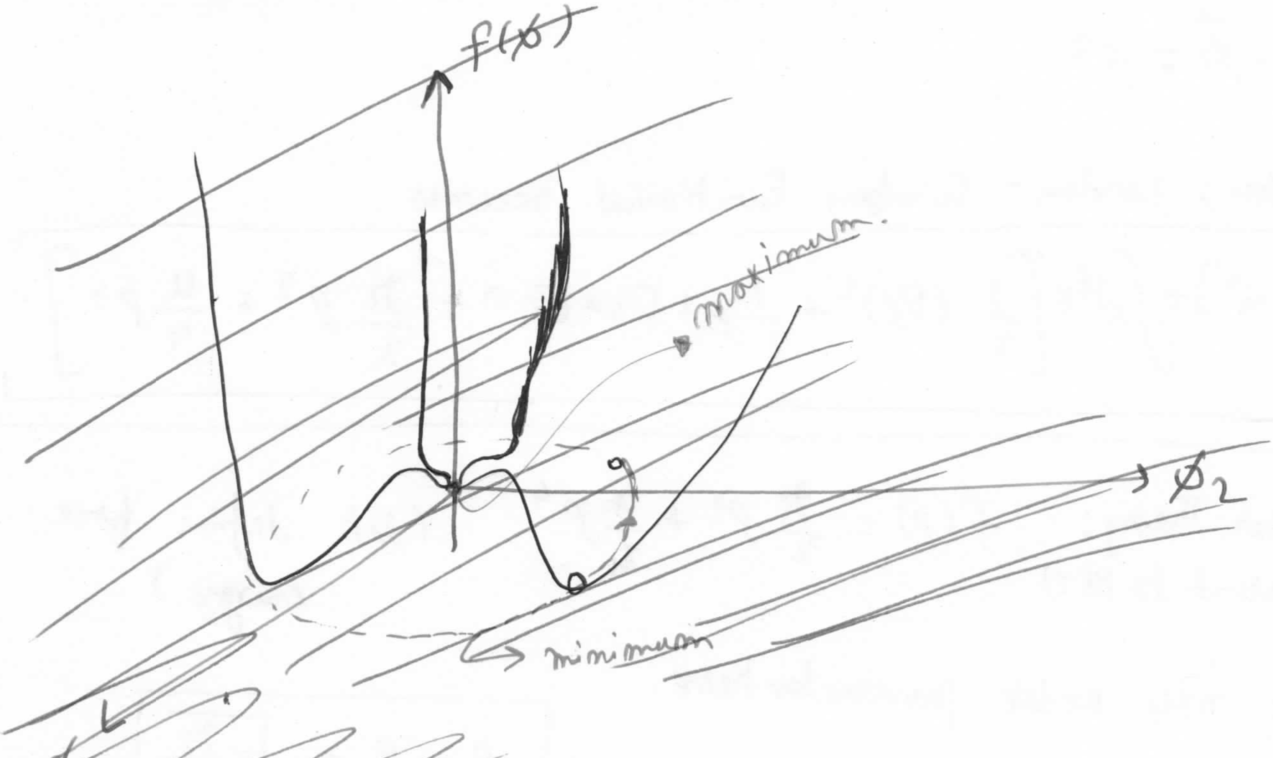
ISING



This precisely (the barier) corresponds to the fact that the symmetry we have broken is $\mathbb{Z}_2$ symmetry.

Spontaneous symmetry breaking. ; but to jump to other we have to overcome the barrier

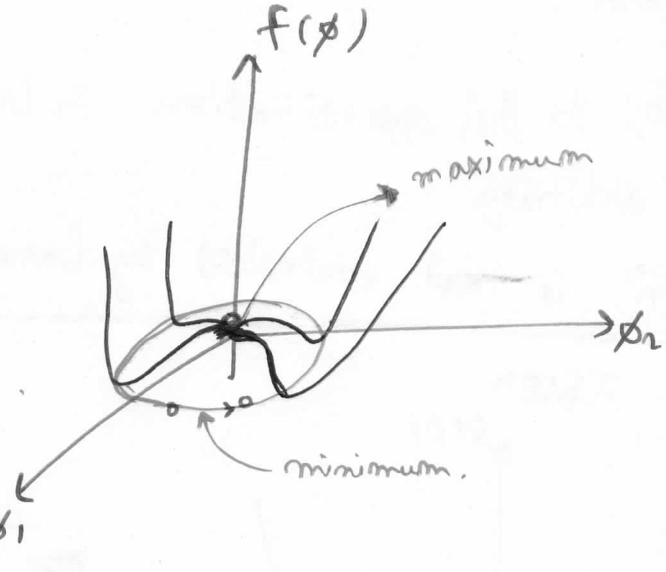
Vector ($n=2$)



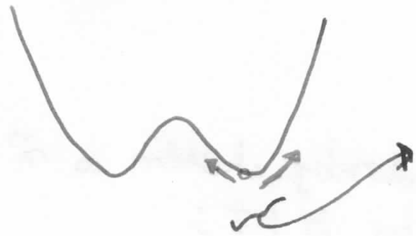
Mexican Hat

Here we can travel from one minimum to other without costing much; ie; minima connected by continuous symmetry : can go to different minimas without crossing any potential barrier.

A better diagram (try)

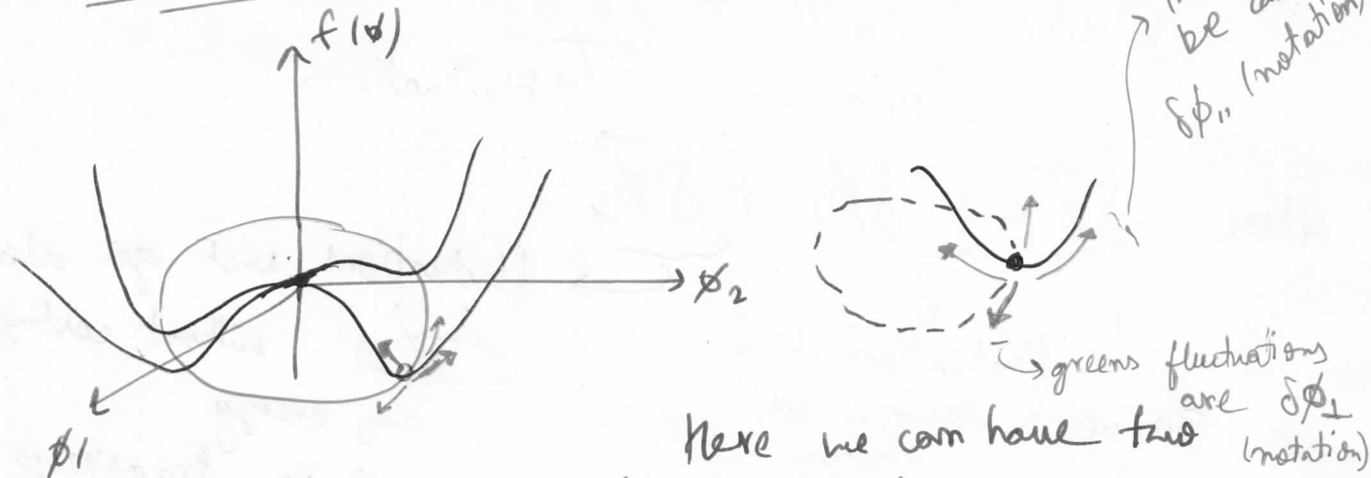


Fluctuations in Ising



once we take fluctuations into account : we can fluctuate around a minima , but the fluctuations will be such that ending up to other minima is very unlikely, because it will cause lot of energy.

Fluctuations in vector $(n=2)$



Here we can have two types of fluction.

& One of the fluctuations will not cost any energy & we can reach the other minima through thermal fluctuations without ~~...~~ costing any ~~...~~ energy.

ie; fluctuations $\delta\phi_{\perp}$ are simple

So, we see here that physics is very different as compared to Ising.

so; Through $\delta\phi_{\perp}$ we can go around minimas without costing any energy.

Small Perturbations around MFT (Landau) Approximation

Say that $\vec{n} = (1, 0, 0, \ldots, 0)$ and write

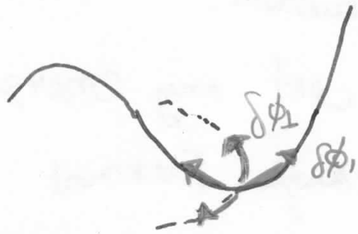
$$\vec{\phi}(x) = \underbrace{\rho_0 \vec{n}}_{\text{MFT.}}$$

(ie; at a randomly chosen $\vec{n}$ as predicted by MFT; So we chose $\vec{n}$ to be along ~~say~~ say ϕ_1 because it is completely arbitrary in MFT)

Now lets include fluctuations.

$$\vec{\phi}(x) = \underbrace{\rho_0 \vec{n}}_{\text{MFT}} + \underbrace{\rho_0 \delta\vec{\phi}}_{\text{Fluctuations.}}$$

where $\delta\vec{\phi} = \underbrace{\vec{n}\,\delta\phi_1}_{\text{fluctuations which go up \& down ... cost energy}} + \underbrace{\delta\vec{\phi}_\perp}_{\text{fluctuations wich go along without costing any energy ... its in transverse direction}}$



$$\vec{\phi}\cdot\vec{\phi} = (\rho_0\vec{n} + \rho_0\delta\vec{\phi})\cdot(\rho_0\vec{n} + \rho_0\delta\vec{\phi})$$

$$= \rho_0^2 + 2\rho_0^2\,\vec{n}\cdot\delta\vec{\phi} + \rho_0^2\,\delta\vec{\phi}\cdot\delta\vec{\phi}$$

$\boxed{\vec{n}\cdot\delta\vec{\phi}_\perp = 0}$ just by defination.

$$\Rightarrow \vec{\phi}\cdot\vec{\phi} = \rho_0^2 + 2\rho_0^2\,\delta\phi_1 + \rho_0^2(\delta\vec{\phi})^2$$
$$= \rho_0^2\left[1 + 2\delta\phi_1 + (\delta\phi_1)^2 + (\delta\vec{\phi}_\perp)^2\right]$$

$$\vec{\phi}\cdot\vec{\phi} = \rho_0^2\left(1 + 2\delta\phi_1 + (\delta\phi_1)^2 + (\delta\phi_\perp)^2\right)$$

$$(\vec{\phi}\cdot\vec{\phi})^2 = \rho_0^4\left(1 + 4\delta\phi_1 + 6(\delta\phi_1)^2 + 2(\delta\phi_\perp)^2 + \dots\right)$$

$\Downarrow$ neglecting ~~highs~~ higher order terms.

neglect terms like $(\delta\phi_\perp)^4$ because treating small purterbations.

---

now,

$$S[\phi] = \int d^dx\left[\frac{\rho_0^2}{2}(\nabla\delta\phi_1)^2 + \frac{\rho_0^2}{2}(\nabla\delta\phi_\perp)^2 + \frac{r}{2}\rho_0^2\left(1 + 2\delta\phi_1 + (\delta\phi_1)^2 + (\delta\phi_\perp)^2\right) + \frac{u}{4}\rho_0^4\left(1 + 4\delta\phi_1 + 6(\delta\phi_1)^2 + 2(\delta\phi_\perp)^2\right)\right]$$

---

look at terms with $\delta\phi_1$:

$$\frac{r}{2}\rho_0^2 2\delta\phi_1 + u\rho_0^4\cdot\delta\phi_1$$

now; we know $\rho_0^2 = -\frac{r}{u}$; plugging this here gives 0.

so; linear term in perturbation has to vanish.

(we sort of knew this ; we are expanding around minimum so the first order pertubation has to vanish)

---

looking at quadratic terms $(\delta\phi_1)^2$:

$$\frac{r}{2}\rho_0^2(\delta\phi_1)^2 + \frac{3}{2}u\rho_0^4(\delta\phi_1^2) = \rho_0^2|r|(\delta\phi_1)^2.$$

Something familiar

---

looking at $(\delta\phi_\perp)^2$:

$$\left(\frac{r}{2}\rho_0^2 + \frac{u}{2}\rho_0^4\right)(\delta\phi_\perp)^2 = 0$$

New & happy.

The fact that perturbation ~~give~~ give non-trivial contribution here upto order $(\delta\phi_1)^2$ is familar.

The fact that $\delta\phi_\perp$ vanishes upto second order is ~~new~~ new.

$$\cancel{S[\phi] = \int d^dx \left[\frac{\rho_0^2}{2} \left(\nabla\phi \right.\right.}$$

$$S[\phi] = \int d^dx \, \frac{\rho_0^2}{2} \left[(\nabla \delta\phi_1)^2 + (\nabla \delta\phi_\perp)^2 + 2|r| (\delta\phi_1)^2 \right]$$

Something like mass term

$\therefore$ The modes $\delta\phi_1$ has mass $\sqrt{2|r|}$

No mass term form $\delta\vec{\phi}_\perp$. (These modes are massles)

↳ it is result of continuous $O(n)$ symmetry breaking that we have now massless mode which can go around the minimum of the mexican hat.

These massless modes which originate in the spontaneous symmetry breaking of same continuous symmetry ~~are~~ ~~are called~~ are called Goldstone modes (Goldstone Bosons)

In general they arise when continuous symmetry is spontaneously broken.

↳ Since they are massless bosons ; it does not take any energy to create them.

<u>Write in k-space</u>

$$\delta\vec{\phi}_1(\vec{x}) = \frac{1}{V} \sum_{\vec{k}} \delta\vec{\phi}_1(\vec{k}) \, e^{i\vec{k}\cdot\vec{x}}$$

$$\delta \vec{\phi}_\perp(\vec{x}) = \frac{1}{V} \sum_{\vec{k}} \delta \vec{\phi}_\perp(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

Then ; we find that , the action in momentum space becomes

$$S = \frac{\rho_0^2}{2V} \sum_{\vec{k}} \left(k^2 |\delta\phi_1(k)|^2 + k^2 |\delta\phi_\perp(k)|^2 + 2|r|\, |\delta\phi_1(k)|^2 \right)$$

if set $k \to 0$: then, goldstone modes donot cost ~~??~~ any energy.

example 1)

We have potential term for $\delta\vec{\phi}_1$

dont have any potential term for $\delta\vec{\phi}_\perp$.

$\delta\vec{\phi}_\perp$

~~?~~ crystalization of liquid.

↳ continuous symmetry of liquid is broken & lattice is formed. ; we order our atoms in lattice.

so,

example 2) ACOUSTIC PHONONS

(breaking the continuous , translational symmetry as the ~~cryst~~ crystal crystalizes and forms a lattice)

We can show that these things have energy given by

$$\varepsilon_k = \hbar\, c_s k$$

some speed in the medium

as $k \to 0$; no energy $\Rightarrow$ So they are massless modes.

(photons in some sense are also ~~goldston~~ goldstone modes)

---

now we have name for $\delta\vec{\phi}_\perp$; they are goldstone modes.

In lower dimensions; then these massless modes can destroy any order.

↳ They are nasty things; but only in lower dimensions

Correlation Function

$\langle \phi \rangle = \rho_0 \vec{n}$

$$G_1(x-x') := \langle \delta\phi_1(x)\, \delta\phi_1(x') \rangle$$

$$= \frac{1}{V^2} \sum_k \underbrace{\langle \delta\phi_1(k)\, \delta\phi_1(-k) \rangle}_{\text{correlation function in momentum space.}} e^{i k\cdot(x-x')}$$

$$\| $$

$$\frac{V}{\rho_0^2} \cdot \frac{1}{k^2 + 2|r|} \propto \frac{1}{k^2+2|r|}$$

picking up some factor due to normalization

(~~reading~~ (reading it from action)

$\boxed{m^2 = 2r}$

$$G_1(x-x') = \frac{1}{|\vec{x}-\vec{x}'|^{d-2}} \underbrace{e^{-\frac{|\vec{x}-\vec{x}'|}{\xi}}}_{\text{because } \phi_1 \text{ modes are massive modes}}$$

} as expected, it should be like Yukawa

$$\xi = \frac{1}{\sqrt{2|k|}} \propto \frac{1}{t^\nu} \; ; \; \nu = \frac{1}{2}$$

(same as MFT prediction)

For $\delta\phi_\perp$; there won't be any exponential; because the correlation length ξ will be infinite.

On the other hand; for perpendicular modes we have

$$G_\perp(x-x') = \langle \delta\vec{\phi}_\perp(x) \cdot \delta\vec{\phi}_\perp(x') \rangle$$

~~$G_\perp(x-x') = \frac{1}{V^2}\sum_k \ldots$~~

$$G_\perp(x-x') = \frac{1}{V^2}\sum_k \underbrace{\langle \delta\phi_\perp^a(k)\,\delta\phi_\perp^a(-k)\rangle}\, e^{ik\cdot(x-x')}$$

now reading from action we get

$$\frac{V}{\rho_0^2}\cdot\frac{1}{k^2}$$ for each mode a.

(no mass term)

now; we have $(n-1)$ modes $\phi_\perp \ldots$

$$\ni \text{so;}\quad \frac{V}{\rho_0^2}\cdot\frac{(n-1)}{k^2}$$

massless $\Rightarrow$ $\xi_\perp$ diverges.

lets look at

$$G_\perp(0) = \frac{n-1}{\rho_0^2}\cdot\frac{1}{V}\sum_k \frac{1}{k^2}\,.$$

(turning to integral) $$= \frac{n-1}{\rho_0^2}\int \frac{d^dk}{(2\pi)^d}\cdot\frac{1}{k^2}$$

depends only on k^2 so we can go to spherical coordinates.

Λ $\rightarrow$ some cutoff Λ given by our lattice.

$$\ni\ G_\perp(0) = \frac{(n-1)\,S_d}{\rho_0^2\,(2\pi)^d}\int_0^\Lambda dk\cdot k^{d-3}$$

The upper limit is not a problem

The integral can be divergent..

The lower limit can be problem in lower dimensions.

... So we put an upper cutoff.

lower limit 0 ;

∴

The integral will be finite if $d > 2$

" " " " infinite when $d \leq 2$

so; The lower limit 0 is the potential problem.

---

so, $d > 2$; then contribution of goldstone modes to fluctuations is finite. ; we can have ordered phase for some temperature $0 < T_c < (T_c)_{MFT}$

but if $d \leq 2$; then the goldstone mode completely change the picture & they have infinite contribution to fluctuations.

so; we cannot order the system.
↳ The system will be disordered because of Goldstone modes.

Particle Physics ... called Coleman Theorem.

Statistical Physics ... " Mermin-Wagner Theorem.

... lets call

COLEMAN - MERMIN - WAGNER Theorem

For systems with continuous symmetry, there is no long range order at any finite temperature in $d \leq 2$. dimensions

~~we motiv~~ we motivated (but not derived) it by our previous analysis.

Two important dimensions

- Upper Critical Dimension : $d = 4$ ($d > 4$ MFT ✓ gives correct result)
- Lower Critical Dimension : $d = 2$ $\lim_{d \to 2^+} T_c(d) = 0$ for system with continuous symmetry.

Lecture 11 : Near $d=2$, Vortices, Kosterlitz - Thoules Phase Transition

C-M-W Theorem : Systems with continuous symmetry there is no long range order for $d \leq 2$.

↳ it seems like we cant have phase transition for continuous symmetry system... but we can have certain types of phase transitions.

AIM : Avade this theorem and show that we can have a phase transition in $d=2$ dimensions.

Avade means to avoid.

$\Rightarrow$ So; we will have to break some of the assumptions of CMW - Theorem to get phase transitions.

Near $d=2$ || Fluctuations of ρ around ρ_0 are not important. But fluctuations of n^a are important (because $\delta\phi_\perp$ does not cost ~~energy~~ any energy)

now, set $\rho = \rho_0$.

$$S[\vec{n}] = \frac{\rho^2}{2}\int d^dx \ \nabla n^a \nabla n^a$$

(effective Landau Ginzberg functional...)

by changing units we can achieve

NON LINEAR σ-MODEL

$$\boxed{S[\vec{n}] = \frac{1}{2T}\int d^dx \ \nabla n^a \ \nabla n^a}$$

We will take this functional to be important for study around 2- dimensions.

$\Downarrow$ Non-linear σ-Model

looks like free theory, but it is not because there is a constraint $n^2 = 1$.
(This makes things difficult)

Set up a $d = 2+\varepsilon$ expansion

We know $\lim_{d\to 2^+} T_c(d) = 0$ from CMW theorem

~~This does n~~ This does not works as good as expansion around $d=4$; ... but it works well.

We end up to following RG equation.

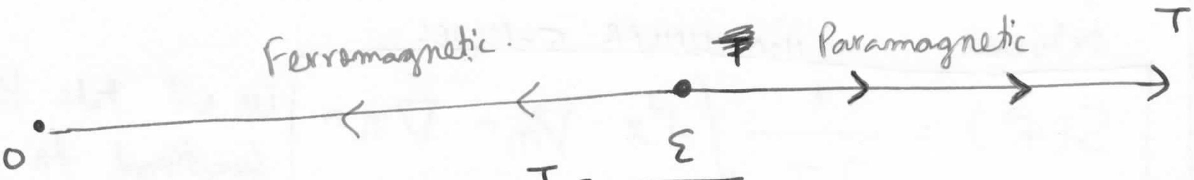
$$\boxed{\frac{d\tilde{T}}{dl} = -\varepsilon \tilde{T} + (n-2) K_d \cdot \tilde{T}^2} \qquad \varepsilon = d-2$$

$\tilde{T} = T\Lambda^{\varepsilon}$ (rescaled T)

$$K_d = \frac{S_d}{(2\pi)^d}$$

$\tilde{T} = 0$ is obvious solution of $\beta_{\tilde{T}} = 0$

The other solution is $\tilde{T}^* = \dfrac{\varepsilon}{(n-2)K_d}$



$$T_c = \frac{\varepsilon}{(n-2)K_d}$$

~~Note! d=2 exactly then~~ ~~Paramagnetic~~ ~~Ferromagnetic at T=0.~~ $\cancel{\frac{d\tilde{T}}{dl} = (n-2)K_d \tilde{T}^2}$ ~~RG equation at d=2~~

Note 1

$d=2$ exactly then

$0 \longrightarrow$ Paramagnetic $\longrightarrow T$

Feromagnetic at $T=0$

$$\boxed{\frac{d\tilde{T}}{dl} = (n-2)\, k_d \tilde{T}^2 \qquad \text{RG equation at } d=2}$$

$\hookrightarrow$ β function for Yang Mills in $d=4$

$AdS_5 \longleftrightarrow CFT_{d=4} \longleftrightarrow$ Spin Chains $d=2$

(by noting that β function is the same

Very Special Case

$d = 2 = n$; $\frac{d\tilde{T}}{dl} = 0$ ... β function vanishes upto the leading order in ε (This is what we wrote here)

$\hookrightarrow$ But it is actually true to any order in perturbation.

β function vanishes for any T ; so for any T we have like critical point (because fixed point)

? Line of critical points for every T ?

Writing : $\vec{n} = \cos\theta\, \hat{x} + \sin\theta\, \hat{y}$

$$\nabla \vec{n} = -\sin\theta \cdot \nabla\theta\, \hat{x} + \cos\theta \cdot \nabla\theta\, \hat{y}$$

We want $\nabla n^a \nabla n^a$; so square $(\nabla \vec{n})^2$

$$\nabla\vec{n} \cdot \nabla\vec{n} = \sin^2\theta\, (\nabla\theta)^2 + \cos^2\theta\, (\nabla\theta)^2$$

$$= (\nabla\theta)^2$$

$$\boxed{S[\theta] = \frac{1}{2T}\int d^2x\,(\nabla\theta)^2}$$

Landau - Ginzberg for $n=2$ in 2 dimensions.

looks like free theory; but not exactly because θ is angle & it is compactified from 0 to 2π

ie; $\theta \in (0, 2\pi)$ ... compact ... so not a free theory.

lets pretend θ is not compactified

so; we can calculate correlation functions.

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

so; ~~the~~ we calculate $G(x) = \langle e^{i\theta(x)}\, e^{-i\theta(0)} \rangle$

$$= \dots =$$

since the integral diverges. ... we introduce a cutoff for lattice spacing.

$$= \left(\frac{a}{x}\right)^{\frac{T}{2\pi}}$$

$$G(x) = \langle e^{i\theta(x)}\, e^{-i\theta(0)} \rangle = \left(\frac{a}{x}\right)^{\frac{T}{2\pi}}$$

Compare with $G(x) \propto \dfrac{e^{-x/\xi}}{x^{d-2+\eta}}$

we notice that we miss exponential part.

$\longrightarrow$ Comparing these we would conclude

$\boxed{\xi \longrightarrow \infty}$; ie; There is no mass of these things.

$\boxed{\eta = \dfrac{T}{2\pi}}$ Critical exponents depend on T.

- line of critical points with T-dependent critical exponent.

- $G(x \to \infty) \Longrightarrow \underbrace{0}_{} \neq M^2$

  ↳ so there is no long range order (because if there were ; then $G(x\to\infty)$ should go to M^2)

  ↳ ~~Agrees~~ Agrees with CMW - Theorem.

↳ All these results are provided θ not compact.

This is <u>physically</u> wrong.

because we are finding that for any temperature $\xi \to \infty$. (and we cannot have ~~this~~ this)

We must have T_c such that the correlation length becomes finite above $T > T_c$

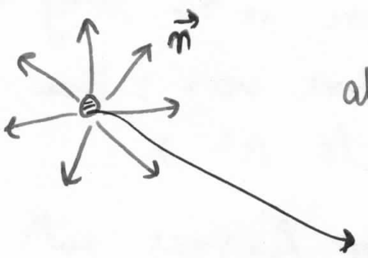
Cannot have $\xi \to \infty$ for any T.

We must have T_c such that

$$G(x) \propto \frac{1}{x^{\eta(T)}} \longrightarrow G(x) \propto \frac{e^{-x/\xi}}{x^{\eta(T)}}$$

$$\xi = \begin{cases} \infty & T < T_c \\ \text{Finite} & T > T_c \end{cases}$$

By assuming θ non-compact we neglected a possibility of having vortices. , ie; VORTEX



all spins going out of a vortex

~~to~~

Vortex Core.

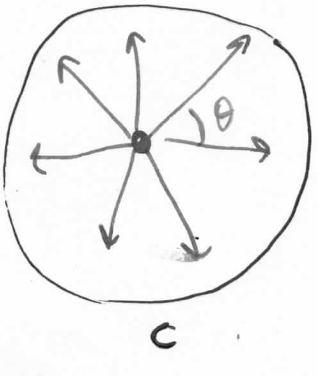
Vortex is something which is called Topological Defect ; & we can detect it from very very far away...

... The presence of vortex is through stokes theorem or gauss theorem & with them we can detect vortices.

ex charge in a volume ; we can calculate flux ~~fro~~ very very far away & calculate charge.

↳ same thing we do ; but in less dimensions...

... use stokes theorem instead of gauss theorem



C

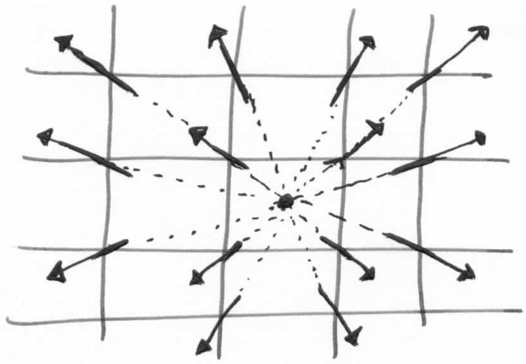
~~to~~ (go around circle C)

$$\boxed{\oint_C \nabla\theta \cdot d\vec{l} = 2\pi m}$$

m is winding number & characterizes strength of the vortex

$m = \ldots -2, -1, 0, 1, 2, 3, \ldots$

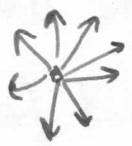
In Lattice



all the spins in the whole lattice has to twisted due to presence of vertex in the middle.

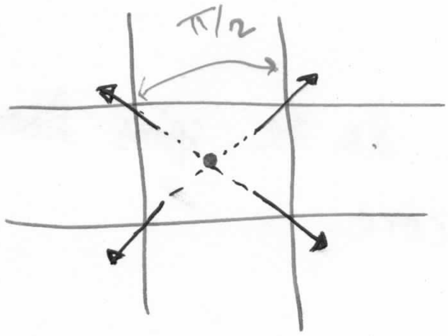
(so; it is energetically very costly to create vortex b because we to twist all spins)

In continuous limit



$\nabla\theta \propto \frac{1}{r}$; close to origin $(r=0)$ this is problematic.

~~Renormalizing~~ when integrating energy we introduce cutoff a.



The biggest gradient you can have for spins if you are in lattice is something like $\frac{\pi/2}{a} = \frac{\pi}{2a}$

Continuous limit breaks down at core.
But lattice ensures everything is well defined. } → so you can't have infinite gradient max is $\sim \frac{\pi}{2a}$.

Lets estimate the energy of the vortex.

here Action is actually the hamiltonian.

$$H = \frac{1}{2}\int d^2x \, (\nabla\theta)^2$$

$$\delta H = \frac{1}{\cancel{2}}\int d^2x \, (\nabla\delta\theta)\,\nabla\theta$$

$$= -\frac{1}{\cancel{2}}\int d^2x \, (\nabla^2\theta)\,\delta\theta \quad \text{(doing by parts)}$$

~~As~~ now $\delta H = 0 \Rightarrow -\int d^2x \, (\nabla^2\theta)\,\delta\theta = 0$

$\Rightarrow$ so we get that vortex is described by $\boxed{\nabla^2\theta = 0.}$

now; still we have the constraint which defines the vortex

$$\oint_C \nabla\theta \cdot d\vec{l} = 2\pi m \qquad \text{(constraint for non-trivial vortex)}$$

so; we have to solve

$$\nabla^2\theta = 0 \, , \quad \oint_C \nabla\theta \cdot d\vec{l} = 2\pi m \quad \text{simultaneously.}$$

remember θ is the angle how do the spins ~~[illegible]~~ actually twist around the core.

In polar coordinates; (r,ϕ)

we have $\boxed{\theta = m\phi}$ will solve both the equations.

indeed $\nabla\theta = \frac{m}{r}\hat{\phi}$ ... & satisfies.

Energy $\quad E = \frac{1}{2}\int d^2x\ (\nabla\theta)^2$

$$= \frac{1}{2}\int_0^{2\pi} d\phi \int_a^L dr\ \frac{m^2}{r^2}\, r$$

in polar coordinates

* L size of the system
* a lattice spacing.

$$\boxed{E = m^2\pi \ln\left(\frac{L}{a}\right)}$$

This is how much energy it cost to create a vortex

costs lots of energy to ~~create~~ create a vortex;
(need to turn all spins in lattice)
however it can still be thermodynamically possible.

At finite T; to find if vortices are possible, consider the free energy $\quad F = E - TS$

Entropy; $\quad S = \log(\text{no. of states})$ $\quad \sim$ how many places we can place the vortices.

$$\simeq \log\left(\text{no. of places we can place ~~the~~ a vortex}\right)$$

$$= \log\left(\frac{L}{a}\cdot\frac{L}{a}\right)$$

$\left(\frac{L}{a}\right)^2$ = no. of cells we have.

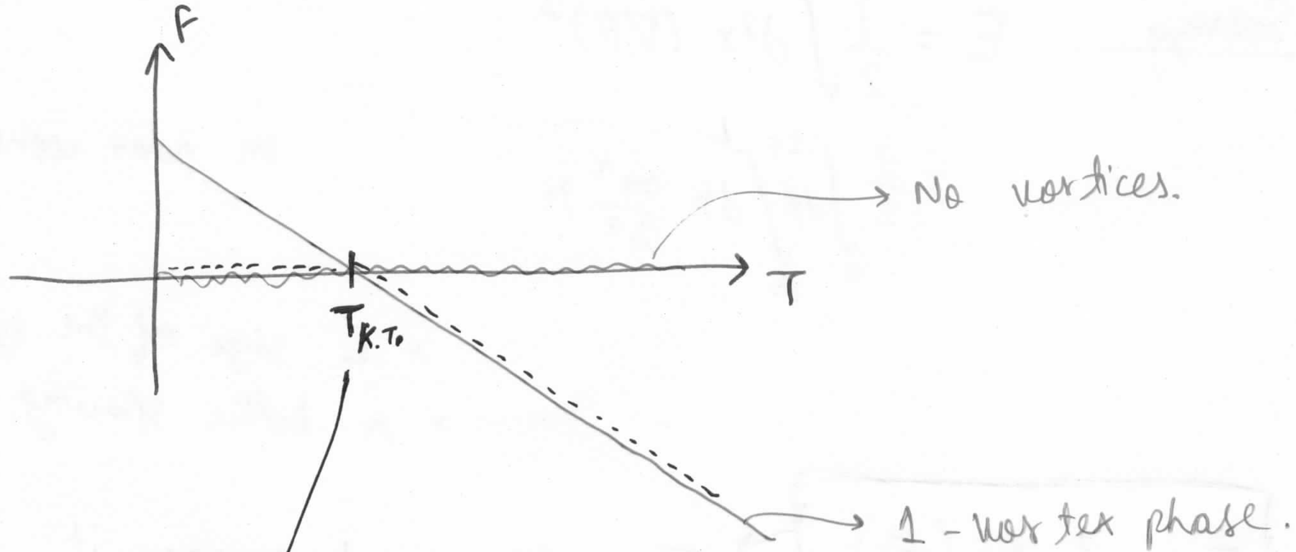
$$\Rightarrow \boxed{S = 2\ln\left(\frac{L}{a}\right)}$$

so; entropy is also huge.

$$\boxed{F = (\pi m^2 - 2T)\ln\left(\frac{L}{a}\right)}$$

simplest possible vortex

$m = 1$; $\Rightarrow$ $F = (\pi - 2T)\log(L/a)$



so ; There exists a critical temperature $T_{K.T.}$

$\therefore$ $F = 0 \Rightarrow \boxed{T_{K.T.} = \frac{\pi}{2}}$

$T > T_{K.T.} = \frac{\pi}{2}$; we can have vortices.

$T_{K.T.}$; K.T. for Kosterlitz & Thoules

$T_{K.T.}$: Kosterlitz & Thoules temperature.

awarded Nobel Prize in 2016.

... also DUNCAN HALDANE.

It is not a standard phase transition; but it is a Topological Phase Transition.

Phase Transitions → 1st order : ... latent heat

Phase Transitions → 2nd order : Typical Ising Model

Phase Transitions → Topological (no order, but we can have phase transitions)

Phase Transitions → Glassy Phase Transition.

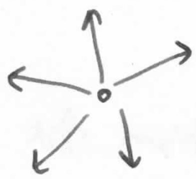
Vortex and Anti-vortex

There are like charge.

vortex is characterized by positive m.

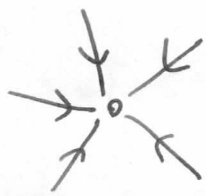
Anti-vortex " " " negative m.

Remark Can have anti-vortices ($m<0$)



$m=+1$

Vortex



$m=-1$

Anti-vortex.

We can sort of pop out of the vaccum, i.e. the lattice, both of vortex & anti-vortex simultaneously.

Its like, if you have vaccum fluctuation; i.e. you have vaccum, then you ~~are~~ ~~create~~ create a charge and anti-charge ... it does not cost much energy.

... We cant measure these effects ~~if~~ if we are too far.

To create vertex & anti-vortex pair costs lot less energy than creating just one vortex or anti-vortex.

$\boxed{\text{Seperating vortex \& anti-vortex pair costs } \cancel{\text{more}} \text{ energy}}$

So; even below T_{KT}, can have vortices present in ~~the~~ spin chain : but these are bound to be in the form of pair.

Can have vortices in pairs for $T < T_{K.T}$ but not isolated ones.

$\therefore$ Isolated vortices can only happen for $T > T_{K.T.}$ where you have enough energy.

$$\xi = \begin{cases} \infty & T < T_{K.T.} \\ \text{FINITE} & T > T_{K.T} \end{cases}$$

People sometime, give a physical interpretation the ξ as distance between two vortices.

- $T > T_{K.T}$ we can have free vortices; ~~so~~ so you can have finite distance.
- for $T < T_{K.T.}$ you dont have any free vortices, so distance between them is ~~that~~ technically infinite.

~~Az~~ ☆ Azimuth (search on internet)

lecture 12: Monte Carlo Methods, Sampling, Markov Chain, Single-Spin-Flip Metropolis Algorithm.

Main Reference:
Newman & Barkema, "Monte Carlo Methods in Statistical Physics", chapters 1-3.
(Added reference in lec 13; 1-4, 8.3.

Our goal is to develop numerical methods to study phenomena such as phase transition for many-body systems of N interacting particles.

Recall,
$$\langle Q \rangle = Tr(Q \rho)$$

In canonical ensemble
$$\rho = \frac{1}{Z} e^{-\beta H} \quad \text{, means } \langle Q \rangle = \frac{1}{Z} Tr(Q e^{-\beta H})$$

remind; $\beta = \frac{1}{k_B T}$ (we will set $k_B = 1$)

$Z = Tr(e^{-\beta H})$ is the partition function.

For classical systems;
$$\langle Q \rangle = \sum_\mu Q_\mu p_\mu \quad ; \quad Z = \sum_\mu e^{-\beta E_\mu}$$

where $p_\mu = \frac{1}{Z} e^{-\beta E_\mu}$

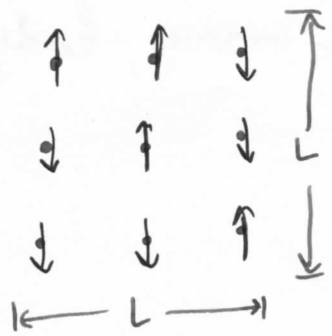
μ represents a state of the system.
E_μ is the energy of state μ.

We would like to study expectation values and, ultimately, phase transitions in the thermodynamic limit. $(N \to \infty)$

For example,

$$\langle E \rangle = \frac{\sum_{\mu} E_{\mu} e^{-\beta E_{\mu}}}{\sum_{\mu} e^{-\beta E_{\mu}}}$$

Lets consider; how to evaluate $\sum_{\mu}$ directly for a nearest-neighbour (nn) Ising model. on a d-dimensional Cubic lattice with linear length L.



$N = L^d$
no. of spins

On each site i, there is a binary variable $\sigma_i = \pm 1$.

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

sum over nearest neighbours.

- $\sum_{\langle ij \rangle}$ denotes a sum over nn pairs.
- J is a coupling strength.
- h is an external field.

Exact solutions are known for
- $d=1$ (see Tutorial 2)
- $d=2$ (Onsagers, 1944)

The total # of ~~states~~ acceptable states is 2^N.

ie: $2^{(L^d)}$

For $N=25$, evaluating $\sum_{\mu}$ requires at least 1 minute.

N	time
25	~ 1 min
50	~ 60 years
80	~ 7×10^{10} years (longer than age of universe!)

We are limited to lattices ~~to~~ to $L = 7$ or 8 in $d = 2$.

Classical Monte Carlo (MC) Methods

Idea: Handle this exponential complexity of calculating

$$\langle Q\rangle = \frac{\sum_\mu Q_\mu e^{-\beta E_\mu}}{\sum_\mu e^{-\beta E_\mu}}$$

by considering only M states $\mu_1, \mu_2, \dots, \mu_M$ selected at random. We would like to get an accurate estimate for $M << 2^N$ for large N.

For example; if we choose each state μ_m $(1 \le m \le M)$ ~~from~~ uniformly from the 2^N possible states, then we can estimate $\langle Q\rangle$ from

$$Q_M = \frac{\sum_{m=1}^{M} Q_{\mu_m} e^{-\beta E_{\mu_m}}}{\sum_{m=1}^{M} e^{-\beta E_{\mu_m}}}.$$

"Estimator"

Lets consider E_μ for the Ising model in $d=1$.

$\uparrow \downarrow \downarrow \dots \uparrow$ $\quad H = -J\sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}$ $\quad$ ($h=0$, open boundary conditions)

From Tutorial 2, we know

$$Z = 2\,[2\cosh(\beta J)]^{N-1} \Rightarrow \langle E\rangle = -\frac{\partial \log Z}{\partial \beta}$$

$$= -(N-1)\cdot J\cdot \tanh(\beta J)$$

• More generally, if the states are taken from a probability distribution W_μ, then the estimator for $\langle Q\rangle$ is

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$$Q_M = \frac{\sum_{m=1}^{M} Q_{\mu_m} W_{\mu_m}^{-1} \cdot e^{-\beta E_{\mu_m}}}{\sum_{m=1}^{M} W_{\mu_m}^{-1} \cdot e^{-\beta E_{\mu_m}}}$$

Important Sampling

We would like to sample the more "important" ~~states~~ (high probable) states more frequently.

~~In practice~~ In particular, if $W_\mu = P_\mu = e^{-\beta E_\mu}/Z$, then

$$Q_M = \frac{1}{M} \sum_{m=1}^{M} Q_{\mu_m}.$$

But how do we choose states from this distribution when we dont know Z?

Markov - Chain Monte Carlo

Idea: Given state μ_m, move to the next state μ_{m+1} according to transition probability $\underbrace{T(\mu_m \rightarrow \mu_{m+1})}$

(does not depend on history kind of thing)

$\rightarrow$ does not depen on $\mu_1, \mu_2, \ldots, \mu_{m-1}$

$$\sum_\nu T(\mu \rightarrow \nu) = 1$$

(transition probabilities also have to normalized)

The resulting set $\{\mu_1, \mu_2, \ldots, \mu_n\}$ is called a Markov Chain of states.

To reach states from a target probability distribution W_μ, an algorithm must satisfy two conditions

① <u>Ergodicity</u> : given μ and ν, it must be possible (non-zero probability) to reach ν from μ.

$$\mu \rightarrow \underbrace{\ldots\ldots}_{\text{intermediate states.}} \rightarrow \nu$$

② <u>Detailed Balance (DB)</u>:

The rates of transitions into and out of any state μ must be equal. (if they are not equal, then our probability distribution would be changing with time)

(ie; otherwise W_μ would change with time)

so;

$$\underbrace{\sum_\nu W_\nu T(\nu\to\mu)}_{\substack{\text{Rate of transition}\\ \text{into } \mu \text{ from some}\\ \text{state}}} - \underbrace{\sum_\nu W_\mu T(\mu\to\nu)}_{\substack{\text{Rate of transition out of } \mu\\ \text{to some other state } \nu.}} = 0$$

One possible way to satisfy this condition:

$$\boxed{W_\mu T(\mu\to\nu) = W_\nu T(\nu\to\mu)}$$ Detailed Balance (DB) condition.

all our algorithm will satisfy this.

So, when $W_\mu = e^{-\beta E_\mu}/Z$

$$\frac{T(\mu\to\nu)}{T(\nu\to\mu)} = e^{-\beta(E_\nu - E_\mu)}$$

Notation, $T(\mu\to\nu) = \underbrace{g(\mu\to\nu)}_{\text{Selection Probability}} \underbrace{A(\mu\to\nu)}_{\text{Acceptance probability.}}$

- $g(\mu \to \nu)$: Probability of proposing the move $\mu \to \nu$ (probability that given state μ, our algorithm suggest moving it to state ν)

- $A(\mu \to \nu)$: Probability of ~~acceptance~~ accepting the move

  (A is probabilistic; but in general we dont have to always ~~accep~~ accept the move)

We want $A(\mu \to \nu)$ to be as close to 1 as possible so that we sample as many different states as possible.

Single - Spin - Flip Algorithms for the Ising Model

Given a state μ_m of N spins at step m of the Markov chain; propose a new state μ_{m+1} that differs from μ_m by a single spin flip at some lattice site i.

ex

$$\begin{matrix} \uparrow & \uparrow \\ \downarrow & \uparrow_i \end{matrix} \quad \xrightarrow[\sigma_i \to -\sigma_i]{\text{proposed move}} \quad \begin{matrix} \uparrow & \uparrow \\ \downarrow & \downarrow_i \end{matrix}$$

Lets say that the lattice site i is chosen with equal probability such that

$$g(\mu \to \nu) = \begin{cases} \frac{1}{N} & \text{if } \mu \text{ and } \nu \text{ differ by a single-spin flip (ssf)} \\ 0 & \text{if } \mu \text{ and } \nu \text{ differ by more than on ssf.} \end{cases}$$

By DB, for such a ssf

$$e^{-\beta(E_\nu - E_\mu)} = \frac{A(\mu \to \nu)\frac{1}{N}}{A(\nu \to \mu)\frac{1}{N}} = \frac{A(\mu \to \nu)}{A(\nu \to \mu)}$$

Let's consider two options for choosing the acceptance probabilities.

<u>Option # 1</u> $A(\mu \to \nu) = A_0 \cdot e^{-\frac{1}{2}\beta(E_\nu - E_\mu)}$

We are free to choose A_0 provided $A(\mu \to \nu) \le 1$ for any ssf. Since the minimum energy difference for a ssf on a d-dimensional cubic lattice is $-4Jd$.

eg

in <u>2d</u>.

so: $A_0 = e^{-2\beta J d}$

$$A(\mu \to \nu) = e^{-\frac{1}{2}\beta(E_\nu - E_\mu + 4Jd)}$$

<u>Option #2</u> Metropolis Algorithm (1953)

For a general proposed move $\mu \to \nu$ (not necessarily a ssf) and target distribution W_μ.

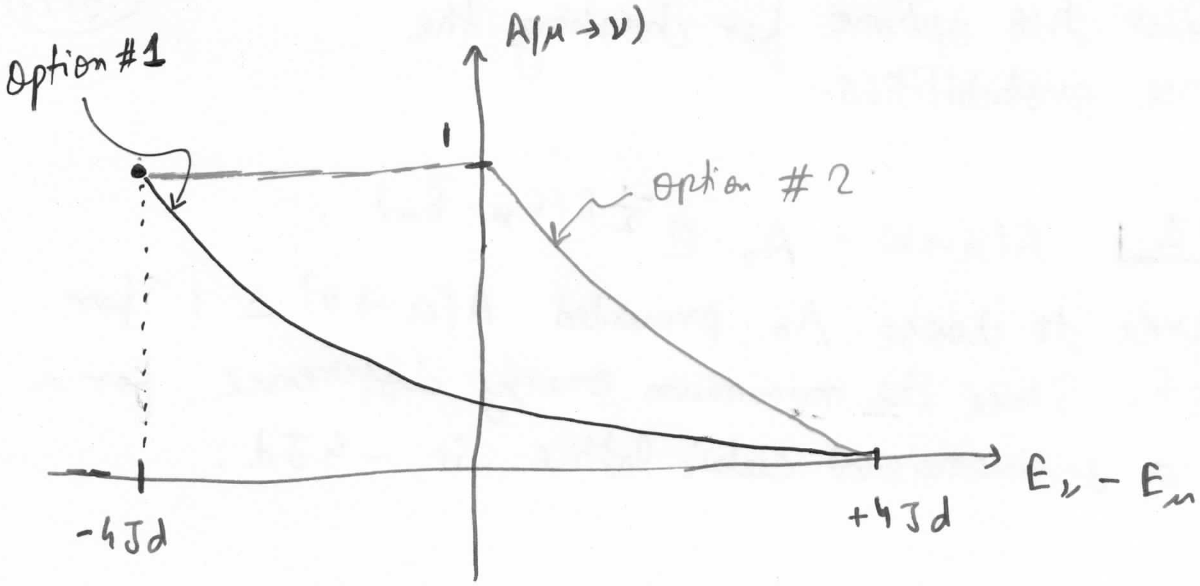
$$A(\mu \to \nu) = \min\left(1, \frac{g(\nu \to \mu)}{g(\mu \to \nu)} \cdot \frac{W_\nu}{W_\mu}\right)$$

For the ssf $\left(g(\mu \to \nu) = \frac{1}{N}\right)$ and $W_\mu = e^{-\beta E_\mu}/Z$:

This tells

$$A(\mu \to \nu) = \begin{cases} e^{-\beta(E_\nu - E_\mu)} & \text{if } E_\nu - E_\mu > 0 \\ 1 & \text{otherwise.} \end{cases}$$

Exercise: show that this choice for $A(\mu \to \nu)$ satisfies DB.



We would like to use option #2 as compared to option #1 because acceptance ratio is higher.

Single-Spin-Flip Metropolis Algorithm for the d-dimensional Ising Model.

① Generate a random initial state μ_1

② Choose a site i of the lattice with equal probability

③ Calculate the energy difference ΔE associated with flipping $\sigma_i \to -\sigma_i$

④ Generate a random ~~number~~ number $r \in [0, 1)$ from a uniform distribution.

⑤ If $\Delta E \leq 0$ or $r < e^{-\beta \Delta E}$, accept the flip and get $\mu_{m+1} \neq \mu_m$.
Otherwise, reject the flip ($\mu_{m+1} = \mu_m$)

Repeat many times from ② with μ_{m+1}.

Statistical Physics - Shoaib Akhtar 13/5/2020.

lecture 13| Equilibration and Measurement, Monte Carlo for the 2nd Ising Model, Finite-Size Scaling.

Specifically, we studied the single-spin flip (ssf). Metropolis algorithm, where we get μ_{m+1} from μ_m by ~~changing~~ choosing site i at random and accepting $\sigma_i \to -\sigma_i$ with probability $A(\mu_m \to \mu_{m+1})$ given as

$$A(\mu_m \to \mu_{m+1}) = \min\left(1, e^{-\beta(E_{\mu_{m+1}} - E_{\mu_m})}\right)$$

This is optimal.

From DB: $\dfrac{A(\mu \to \nu)}{A(\nu \to \mu)} = e^{-\beta(E_\nu - E_\mu)}$ we have this constraint.

For any μ, ν one of $A(\mu \to \nu)$ or $A(\nu \to \mu)$ is equal to 1. and we basically cant do better than that ...

Outline:

- Data Analysis practices
  - ↳ Equilibration
  - ↳ Measurement correlations
- Code: MC for the 2D Ising Model.
- Cluster Algorithms.
- Estimating T_c from specific heat and susceptibility.
- Finite-size scaling.
- May be: XY model.

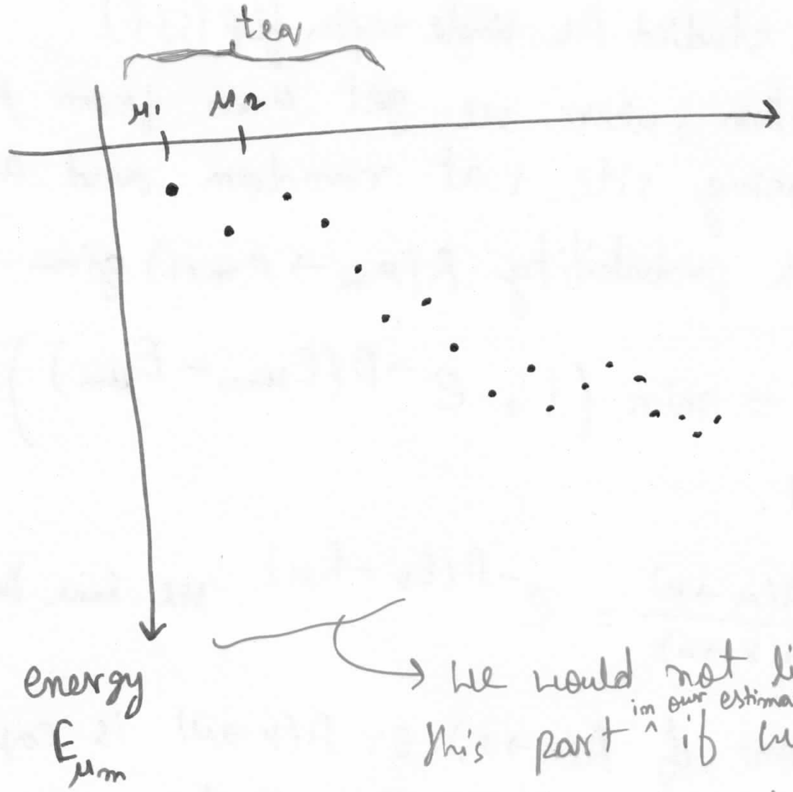
Equilibration

Recall we start our sampling from any μ_1

In many cases, μ_1 and the subsequent $\mu_2, \mu_3, \dots$

might have very low P_μ.

For example, we might see the following for the energy of our samples v/s "Markov time".



We would not like to include this part in our estimation if we are trying to calculate an estimator for any expectation value

lets call it equilibration time t_{eq}

$$\boxed{Q_M = \frac{1}{M - t_{eq}} \sum_{m = t_{eq}+1}^{M} Q_{\mu_m}}$$

Measurement

When calculating Q_M, we should ideally use samples μ_m that are statistically independent.

(When we use ssf algorithm; our neighbouring states will be very correlated.. will be very similar to each other)

13/4/11

For ssf algorithms, we usually do a "sweep" of $O(N)$ (order N) proposed ~~updates~~ updates before performing a measurement.

Then μ_m and μ_{m+1} are seperated by many ssf's.

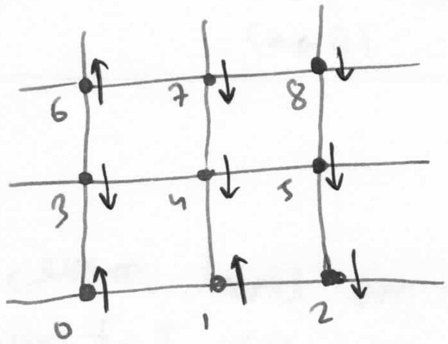
The number of updates required to get independent samples depends on T.

See section 3.3.1 of Newman & Barkema.

~~Code for today's tut tutorial Ising 2d~~

Code for todays tutorial ising2d_mc.py

Ising model on an L×L lattice with periodic boundary conditions and $h = 0$.



$L = 3$

$N = 3 \times 3$

$\mathcal{H} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$

we have a given state in our system;

Here we put periodic boundary condition

ie; to say, 5 is also a nearest neighbour of 3.

We use an array called "spins" to store the current state:

$$\text{spins} = [+1, +1, -1, -1, -1, -1, +1, -1, -1]$$

+1 for up spin & −1 for down spin.

In order to calculate energies, we need to know the nn's of each site i. Store this in a two-dimensional "neighbours" array.

For example, when $L=3$;

neighbours $[2] = [\underset{\text{right}}{4}, \underset{\text{up}}{6}, \underset{\text{left}}{5}, \underset{\text{down}}{0}]$

The code implements a ssf Metropolis algorithm to sample states μ and measure their energy and magnetization.

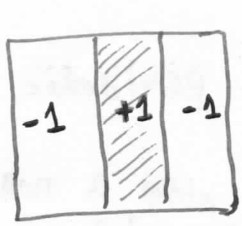
We do this for different temperatures.

The ssf algorithm has problems:

① It takes long time to generate independent samples. (especially near T_c)

② It can get stuck in local energy minima at low T, where it is very ~~so~~ unlikely to accept moves that raise the energy. $\left(e^{-\beta \Delta E} \to 0 \text{ when } \underset{(\beta \to \infty)}{T \to 0} \text{ and } \Delta E > 0\right)$

For example, consider this state:



This is obviously not the ground state

here; there is no local move, no single spin flip that you can come up with that ~~would~~ would ever lower the energy to get us to the ground state.

↳ To get from this state to the ground state; we first have to lower the energy which is very unlikely from this algorithm.

There is no possible ssf that lowers the energy of ~~the~~ this configuration (but it is not the ground state)

In the limit of low temperature; we will never accept the move that raises energy because $e^{-\beta \Delta E} \to 0$ as $\Delta E > 0$ & $T \to 0$ ie; $\beta \to \infty$

so; means our ~~...~~ this algorithm can get stuck in this situation.

Cluster Algorithms

Idea: Probabilistically build clusters of either all up or all down spins and propose flipping the entire cluster

↳ Swendsen and Wang (1987)
↳ Wolff (1989)

Estimating T_c from specific heat and susceptibility

Recall that near a continuous phase transition.

$C_V/N \sim |t|^{-\alpha}$ (in the vicinity of phase transition) → Specific heat per spin

$\chi/N \sim |t|^{-\gamma}$ → Susceptibility per spin

$\xi \sim |t|^{-\nu}$ → Correlation length.

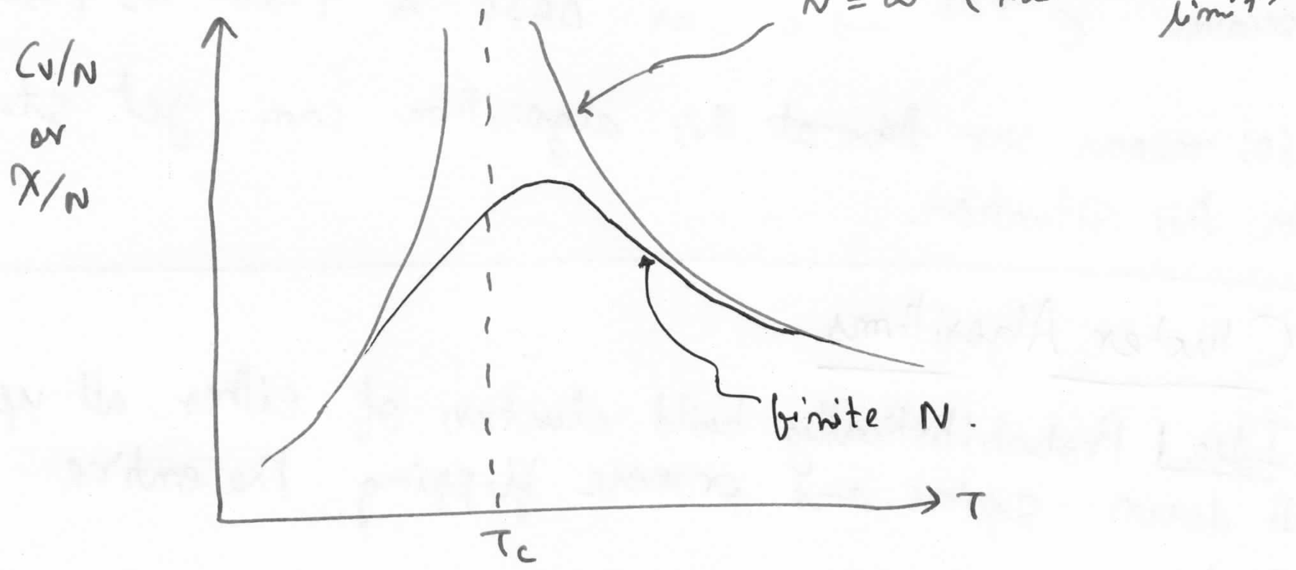
where, α, γ, ν are critical exponents.

- $t = \frac{T - T_c}{T_c}$ (a quantity which parametrizes how close we are to phase transition)

For the 2D Ising Model, C_V/N and χ/N diverge at T_c

However; on a finite ~~system~~ -size lattice, C_V/N &

χ/N diverge will peak near T_c but will not diverge.



One can show, $C_V = \frac{\partial \langle E \rangle}{\partial T} = \frac{\langle E^2 \rangle - \langle E \rangle^2}{T^2}$

$$\chi = \left.\frac{\partial \langle M \rangle}{\partial h}\right|_{h=0} = \underbrace{\frac{\langle M^2 \rangle - \langle M \rangle^2}{T}}_{\text{exercise ... prove it.}}$$

Finite - Size Scaling

↳ Technique for calculating T_c and Critical exponents from calculations on finite lattices.

Note Since $\chi/N \sim |t|^{-\gamma}$ and $\xi \sim |t|^{-\nu}$;

$$\frac{\chi}{N} \sim \xi^{\gamma/\nu}$$

↳ this is for finite system.

ξ kind of measures distance on lattice over which spins are going to be correlated. But if we are on lattice (finite); then ξ is limited by size of order L (size of system)

On a finite system, correlations cannot exceed $O(L)$ and so

$$\frac{\chi}{N} \sim \begin{cases} \xi^{\gamma/\nu} & \text{when } \xi << L \\ L^{\gamma/\nu} & \text{when } \xi >> L \end{cases}$$

Can write as: $\frac{\chi}{N} = \xi^{\gamma/\nu} f\left(\frac{L}{\xi}\right)$

where $f\left(\frac{L}{\xi}\right) = \begin{cases} \text{constant} & \text{when } \xi << L \\ \left(\frac{L}{\xi}\right)^{\gamma/\nu} & \text{when } \xi >> L \end{cases}$

Lets eliminate ξ:

$$\frac{\chi}{N} = L^{\gamma/\nu} \cdot \left(\frac{L}{\xi}\right)^{-\gamma/\nu} f\left(\frac{L}{\xi}\right)$$

$$= L^{\gamma/\nu} \left(L|t|^{\nu}\right)^{-\gamma/\nu} f\left(L|t|^{\nu}\right)$$

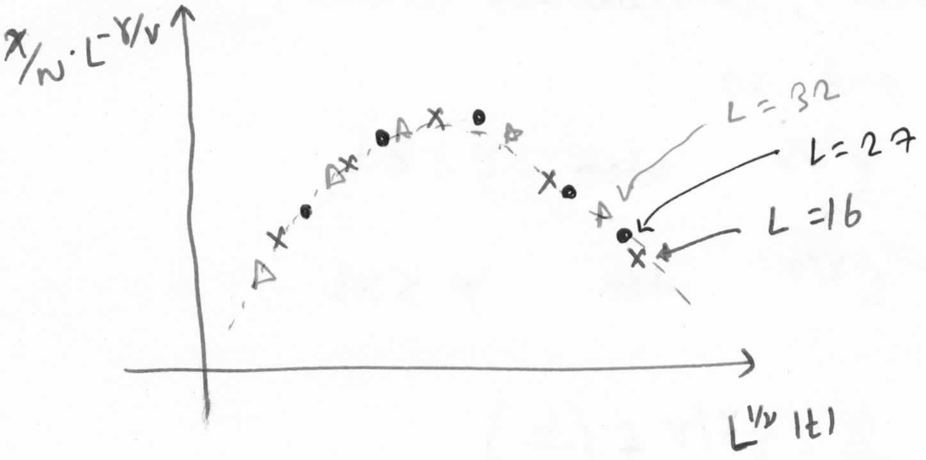
$$= L^{\gamma/\nu} \tilde{f}\left(L^{1/\nu}|t|\right)$$

where $\tilde{f}(x) \equiv x^{-\gamma} f(x^{\nu})$

$$\Rightarrow \frac{\chi}{N} L^{-\gamma/\nu} = \tilde{f}\left(L^{1/\nu}|t|\right)$$

Plots of $\frac{\chi}{N} L^{-\gamma/\nu}$ v/s $L^{1/\nu}|t|$ for various system sizes L should all collapse onto the same curve in the vicinity of T_c.

From this collapse, we can estimate γ, ν and T_c.

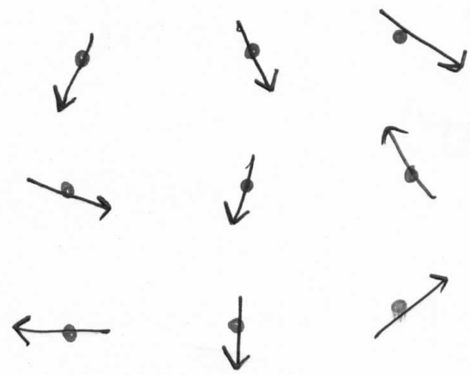


XY Model

$\vec{S}_i$ are allowed to rotate on circle. $(n=2)$

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

$$\theta_i \in [0, 2\pi)$$

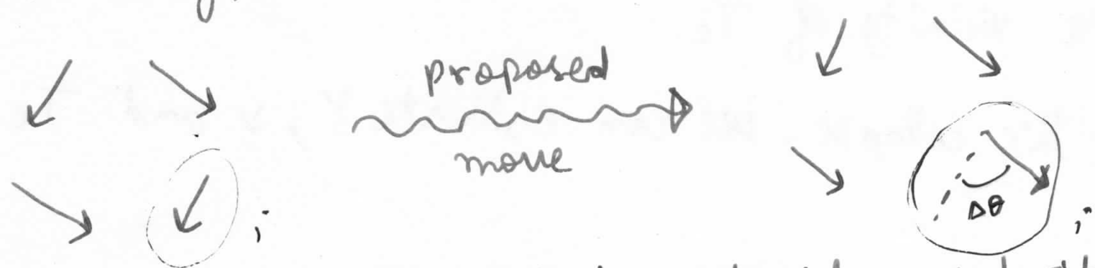


In 2D, there is no phase transition to a long-range ordered phase (M-W theorem).

Instead, there is a topological K.T. transition.

Single-spin update

Similar to Ising, but involve a random rotation.



~~choose site i random; & rotate~~ it randomly by some angle.

choose site i at random; and rotate it by some random angle.

Propose the move $\theta_i \rightarrow \theta_i + \Delta\theta$

where $\Delta\theta \in [0, 2\pi)$ is selected from a uniform distribution.

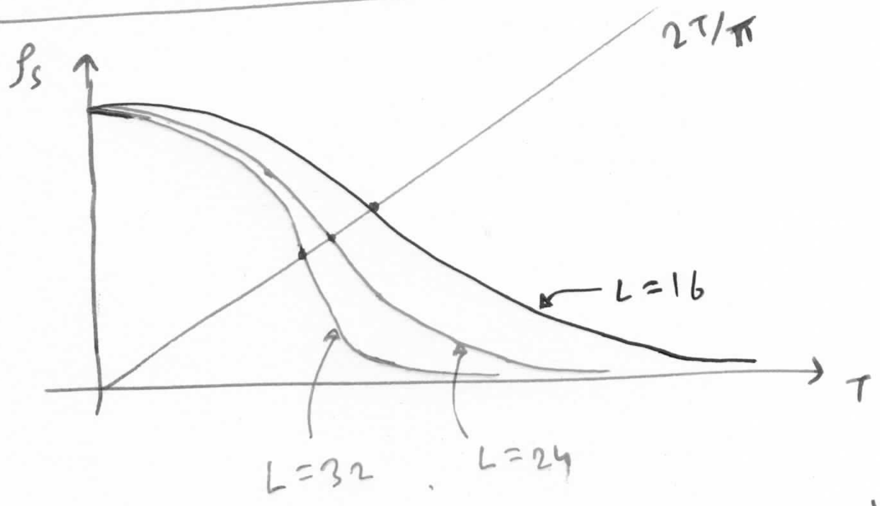
At the KT transition, it is known that

$$\rho_s(T_{KT}) = \frac{2T_{KT}}{\pi}$$ (Nelson and Kosterlitz 1977)

spin stiffness.

(this is something we can measure)

Plots of $\rho_s(T)$ v/s T should intersect the line $2T/\pi$ at $T = T_{KT}$ when $L \rightarrow \infty$.



Study the location of the crossing points as a function of size and extrapolate to $L \rightarrow \infty$.

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Bonus Lecture 14: Black Hole Thermodynamics.

1(a) Motivation

Schwarschild Metric: $ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2$; $f = \left(1 - \frac{2M}{r}\right)$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

Asymptotic mass (Total Energy)

In G.R. it is very problematic to define energy. Gravity actually behaves holographically. It means that you cannot define energy locally. But you can say only what is energy inside some volume of space.

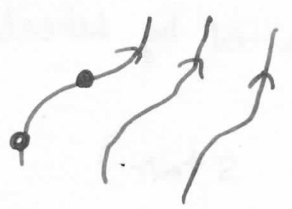
So we dont talk about local mass. Talk about Asymptotic mass.

We inclose a big sphere ; & talk about how much mass is inside that spacetime.

by Noether's theorem, if there is symmetry $\leftrightarrow$ conserved quantity.

If this symmetry is something like time translation, then conserved quantity is something like mass of the spacetime.

Symmetries in G.R. $\longleftrightarrow$ Killing Vectors



$\mathcal{L}_\xi\, g_{\alpha\beta} = 0$ (symmetry of metric)

$D_\mu \xi_\nu + D_\mu \xi_\nu = 0$ (Killing vector equation)

Schwarzschild is static, no time dependance.

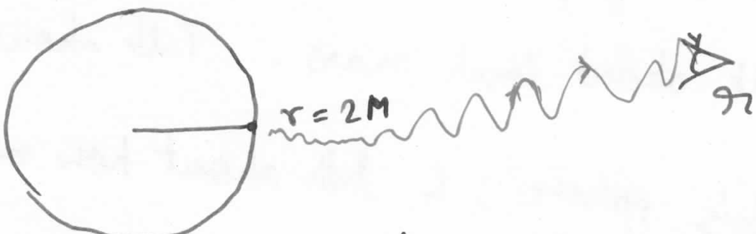
So; there is a killing vector $k = \frac{\partial}{\partial t}$

Komar Mass

$$M = \frac{-1}{8\pi} \int_{S^2_\infty} * dk$$

$* dk$ is 2 form ... integrate over 2-manifold S^2 at ∞.

- Black Hole Horizon $\boxed{r_+ = 2M}$ " Boundary of Black Hole"



(send radial photons)

$$\frac{\nu_r}{\nu_{r_+}} = \frac{\Delta \tau_{r_+}}{\Delta \tau_r} = \frac{\sqrt{f(r_+)}}{\sqrt{f(r)}} \cdot \frac{\Delta t}{\Delta t}$$

$$\Rightarrow \frac{\nu_r}{\nu_{r_+}} = \frac{\sqrt{f(r_+)}}{\sqrt{f(r)}}$$

but $f(r_+) = 0$

$$\Rightarrow \boxed{\frac{\nu_r}{\nu_{r_+}} = 0}$$

so; you can think of ~~of~~ r_+ as surface of infinite redshift.

This is why it is called the BLACK HOLE.

(termed coined by Wheeler)

(people ~~ea~~ earlier called it Frozen star)

Surface Area $\quad A = 4\pi r_+^2$

because of the chosen Area Gauge for r.

i.e.; surface of sphere is ~~such as~~ such as it is in flat space.

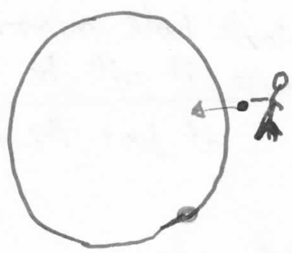
$$dt = 0 = dr$$

So; The induced metric on the horizon is

$$d\gamma^2 = r_+^2 \, d\Omega^2$$

$$A = \int \sqrt{\gamma} \cdot d\theta \, d\phi = \int_{\theta,\phi} r_+^2 \sin\theta \, d\theta \, d\phi = 4\pi r_+^2$$

Surface Gravity



what is ~~force~~ force to hold a particle at a fixed r on the Horizon (just outside the Horizon)

On the Horizon, the force will be infinity (otherwise you could escape from it)

An observer at infinity can hold a particle on the horizon using a massless string; with finite force (due to red shift)

~~Recon~~ F

Finite because, you are balancing whatever ∞ hold at horizon with ~~to~~ ∞ redshift at observer at infinity ... & so force can be finite.

The better will ~~be~~ to calculate acceleration.

K : surface gravity. acceleration
(It is like gravitational acceleration of the particle on the horizon ; however as percieved at infinity)

($m = 1$)
(mass of test particle to be hold)

We can show,

$$K = \frac{f'(r_+)}{2}$$

(for any spherically symmetric spacetime)

$$\Rightarrow K = \frac{1}{2}\left(\frac{d}{dr}\left(1 - \frac{2M}{r}\right)\right)\Bigg|_{r=r_+} = \frac{1}{4M} = \frac{1}{2r_+}$$

$$\boxed{K = \frac{1}{4M} = \frac{1}{2r_+}}$$

Think of K as ; gravitational acceleration on the horizon of black hole normalized to infinity (otherwise it will be ∞ at just the horizon)

Observation// use newton's gravity.

~~so; $g = \frac{M}{r^2}$~~ so; $g_{Newton} = \frac{M}{r^2}\Big|_{r=r_+}$

$$= \frac{M}{r_+^2}$$

remember $r_+ = 2M$

$$\Rightarrow g_{Newton} = \frac{1}{4M}$$

(The results coincides with the Newtonian theory ; ~~but~~ but of course the interpretation is slightly different)

Not generally true.

... but ~~of~~ spherically symmetric spacetime are almost Newtonian.

<u>Observation</u>

$$A = 4\pi r_+^2$$
$$M = r_+/2$$
$$K = \frac{1}{2r_+}$$

now lets play with differentials

$$\boxed{dM = \frac{K}{2\pi}\frac{dA}{4}}$$

... interesting observation

... begining of Black Hole Thermodynamics.

(b) <u>Black Hole Mechanics</u> (1973)

Bardeen, Carter, Hawking considered most general Black Hole (it does not only have mass M, but it can also rotate with some angular momentum J, can also have charge Q)

... No hair theorem

(cant have scalar field...

... will not have any ~~sc~~ scalar hair:

scalar field ~~would~~ would either actually go to infinity and radiat away or it would collapse into the B.H; swallowed by the B.H; and there would be no any trace of it.)

so; we cannot have scalar field, i.e; scalar hair; which would be another charge which will be characterising the B.H.

0th Law : K = constant on the horizon.

(all these laws are proved by purely using geometrical arguments by these gentlemens)

1st Law : $$dM = \frac{K}{2\pi} \cdot \frac{dA}{4} + \underbrace{\Omega \, dJ + \phi \, dQ}_{\text{Work terms.}}$$

Our black hole can have J; & the thermodynamic conjugate quantity to that is like angular velocity Ω

$\Omega \Rightarrow$ angular velocity of Horizon.

$Q \xleftrightarrow[\text{Quantity}]{\text{Conjugate}}$ Electrostatic potential ϕ.

$J \Rightarrow$ Angular Momentum

2nd Law : $\boxed{dA \geq 0}$ Hawking himself showed this.

3rd Law : It is impossible to reduce K to zero in finite no. of steps.

These laws were proved on Geometrical Arguments.

These looks like ~~Ther~~ laws of Thermodynamics.

If we Identify $\boxed{K \sim T}$?

& $\boxed{A \sim S}$?

~~Then these laws looks like laws of thermodynamics.~~

Then these laws will just be laws of Thermodynamics.

Can this be true?

if we say this; then Black Hole would have finite temperature. This does not make any sense because in order to have finite temperature you have to radiate something.

But classically B.H. dont radiate.

Classically, this is impossible ~~becau~~ because Black Hole is an ultimate sponge.

so; T has to be equal to 0 ; $\underline{T=0}$ no other way.

(c) Black Hole Thermodynamics.

The key idea comes from Wheeler,

he asked his graduates students what happens if we through a cup of coffee into black hole.

it initially had entropy (the cup of coffee)

but B.H. has zero temperature. So where does the entropy go.

<u>Wheeler's Cup of Tea</u>: where does the entropy go?

He asked to Bekenstein.

Bekenstein concluded $S \propto A$ so that B.H. can have entropy.

↳ People did not believed it at that point untill Hawking ~~showed~~ one year later showed that indeed if you take quantum affects into consideration, then ~~you have~~ B.H.s have non-trivial temperature.

Hawking (1974) Hawking considered QFT in curved background. (The calculation is complicated and actually wrong)

and showed

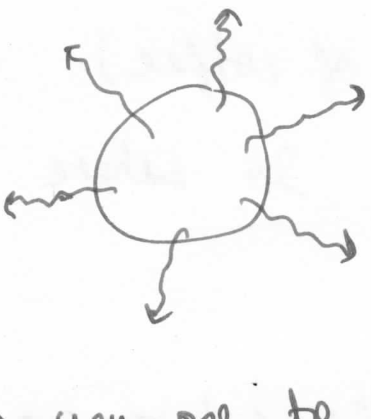
$$T = \frac{\hbar K}{2\pi k_B}$$

B.H. actually radiats if you take Quantum affects

to fix with Bekenstein; it also came out $S = \frac{A}{4\hbar G}$

$$\boxed{T = \frac{\hbar K}{2\pi k_B}} \quad \boxed{S = \frac{A}{4\hbar G}}$$

⇒ Tells that really; Quantum Mechanically B.H. radiates particles.



its like Black Body.

as you go to infinity, the Black Body becomes Grey Body: ~~[illegible]~~ b

Grey body at infinity : because some of the modes are scatter back.

... The particles which are created, they are to pass over a potential; so there some are scattered back.

At infinity you observe $T = \frac{\hbar K}{2\pi k_B}$; ~~...~~ and if you neglect scattering back at infinity; then you will see black body with temperature $T = \frac{\hbar K}{2\pi k_B}$

- Derivation criticized by many people (you cannot neglect back reaction near horizon)
- But, Euclidean Path Integral gives same result upto leading order.

LQG, String... all these ~~results give same approaches~~ approaches gives same result upto the leading order.

It can be shown that Hawking Radiation is a kinematical effect. (Does not require Einstein Equations.)

→ Analogue Systems.

- (Surface Waves on water)

→ measured ✓

(d) Euclidean Trick

★ Thermal Green functions have periodicity in imaginary time. (comes from huge similarity between path integral & partition function as we known in Statistical physics)

$\tau = it$

$$\boxed{G(\tau) = G(\tau + \beta)}$$

if this is ~~tr~~ true then we can show the periodicity is $\beta = \frac{1}{T}$ (will be shown in QFT 2)

~~We can study this we can study f~~

We can study fields in the vicinity of Euclideanized Schwarschild black hole. We find they have periodic greens function $\Rightarrow$ So, ~~these~~ this means that they were in thermal bath.

In particular; consider the Gravitational Field.

$\hookrightarrow$ Euclideanized Schwarschild Metric.

$$ds_E^2 = f\,d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2$$

(Euclidean version of Schwarschild Metric)

Near Horizon; we can write $f(r) = f(r_+) + (r - r_+) f'(r_+) + \dots \infty$

$f(r_+) = 0 \quad \Rightarrow \quad f(r) = \Delta r\, f'(r_+) + \dots \infty$

$\Delta r = r - r_+ \quad \Rightarrow \quad \boxed{f(r) = 2 \Delta r K}$

so; close to horizon; metric looks like.

$$ds_E^2 = 2 \Delta r K d\tau^2 + \frac{dr^2}{2\Delta r K} + r_+^2 d\Omega^2$$

$\frac{dr^2}{2\Delta r K}$ depends only on r; so its a total differential.

so: let $d\rho = \frac{dr}{\sqrt{2K\Delta r}}$ (we can integrate it & get ρ)

by integrating we find $\boxed{\Delta r = \frac{K}{2} \rho^2}$

$$\boxed{ds_E^2 = K^2 \rho^2 d\tau^2 + d\rho^2 + \underbrace{r_+^2 d\Omega^2}}$$

not much interesting, ... just a sphere of radius r_+.

Almost

$$ds_E^2 = \underbrace{\rho^2 d\psi + d\rho^2} + r_+^2 d\Omega^2$$

This is just 2D space written in polar coordinates (ρ, ψ)

$\rho \Rightarrow$ radial ; $\psi \Rightarrow$ angle.

so; $\boxed{\psi = K\tau}$

The sneak argument is ; $\rho^2 d\psi + d\rho^2$ will be non-singular only if ψ has periodicity 2π.

Otherwise, we have conical singularity at origin $\rho = 0$.

so; non-singular at $\rho = 0$ only if $\psi \sim \psi + 2\pi$.

$\rho = 0$ means $r = r_+$

(ie; we are ~~dep~~ demanding that there will be no singularity at $r = r_+$)

$\Longleftrightarrow$ And its ok ; there is no true singularity at horizon ; there is just coordinate singularity.

but ; $\psi = K \cdot \tau \Rightarrow \tau \sim \tau + \frac{2\pi}{K}$

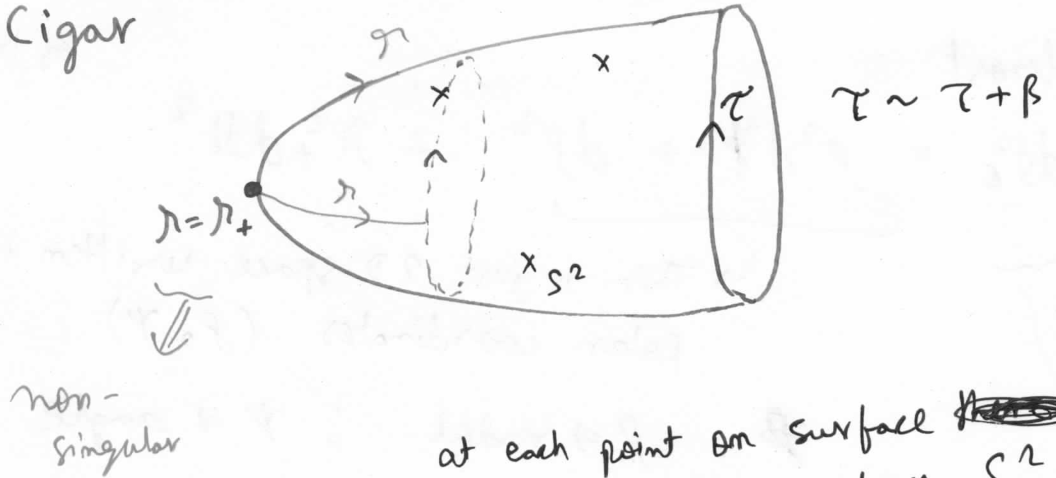
if there is periodicity of β imaginary time
↪ so there must be temperature ; $\beta = \dfrac{2\pi}{k}$

$$\boxed{T = \frac{K}{2\pi}}$$

Black hole has temperature.

How does Euclidean Geometry looks like (in τ and r)

$\simeq$ Cigar



at each point on surface ~~...~~ is of Cigar is a sphere S^2 sitting there.

Geometry in Euclidean space.

Partition Function

here field is metric.

$$Z = \int \mathcal{D}g \cdot e^{-S_E[g]}$$

path integral over metric

• Euclidean action for gravitational field.

WKB approximation. $\simeq e^{-S_E[g_{classical}]}$

$$Z = \int \mathcal{D}g \, e^{-S_E[g]} \underset{\sim}{\overset{WKB}{}} e^{-S_E[g_{classical}]}$$

This is the first approximation for Quantum Gravity that we have here.

$$S_E = \int_{\Omega} \frac{d^4x\sqrt{-g}}{16\pi G} \cdot R \quad \Bigg\}$$ Einstein - Hilbert Action

~~Einstein Hilbert Action~~

just writing this's variational principle is not well posed

(say when Ω has boundary)

so; $$S_E = \int_{\Omega} \frac{d^4x\sqrt{-g}}{16\pi G} R \qquad \left(+ \ \delta g = 0 \quad + \ \delta(\partial g) = 0\big|_{\partial\Omega} \right)$$

But it kills many solutions. as we know from Classical Mechanics.

... so we put in boundary terms in action.

$$S_E = \int_{\Omega} \frac{d^4x\sqrt{-g}}{16\pi G_N} R + \underbrace{\int_{\partial\Omega} \frac{d^3x\ \sqrt{h}\ K}{8\pi G_N}}$$

~~K ⇒ extensi~~
$K \Rightarrow$ extensive curvature.

Gibbons - Hawking term
(makes variational principle well defined)

Now ; to calculate actual value of partition function ; we have to find actual value of action

... & they come out to be infinite even for flat space.

so; we add another boundary term to tune the value of action ; which is hiding in K_0.

define K_0 such that S_E vanishes for flat space.

$$S_E = \int_{\Omega} \frac{d^4x \sqrt{-g}}{16\pi G_N} \cdot R + \int_{\partial\Omega} \frac{d^3x \sqrt{h}\, K}{8\pi G_N} \underbrace{- \int_{\partial\Omega} \frac{d^3x \sqrt{h}\, K_0}{8\pi G_N}}_{\text{To tune value of } S_E}$$

(WKB or Geometric Optics Approximation ; are same thing)

After doing calculation for Euclineatized Schwarschild. we get

$$\boxed{S_E^{\text{Schwarschild}} = \frac{\beta M}{2}}$$

boundary integrals $\int_{\partial\Omega}$ has d^3x

& one of them is τ

so; $\int d\tau$ must give β.

---

so; now we calculate free energy.

$$\boxed{Z = e^{-\beta M/2}}$$

$$\boxed{F = -T \log Z = \frac{M}{2}}$$

$$S = -\frac{\partial F}{\partial T} = \frac{1}{16\pi T^2} = \pi r_+^2 = \underbrace{\frac{A}{4}}_{\text{Bekenstein.}}$$

remember $T = \frac{1}{8\pi M}$

Note) Euclidean geometry has nothing inside horizon.
→ It looks like cigar...
... we make the geometry nice and smooth at horizon & so whatever was inside horizon sort of disappeared.

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so; in the integral; we are integrating what ~~happe~~ is happening outside horizon.

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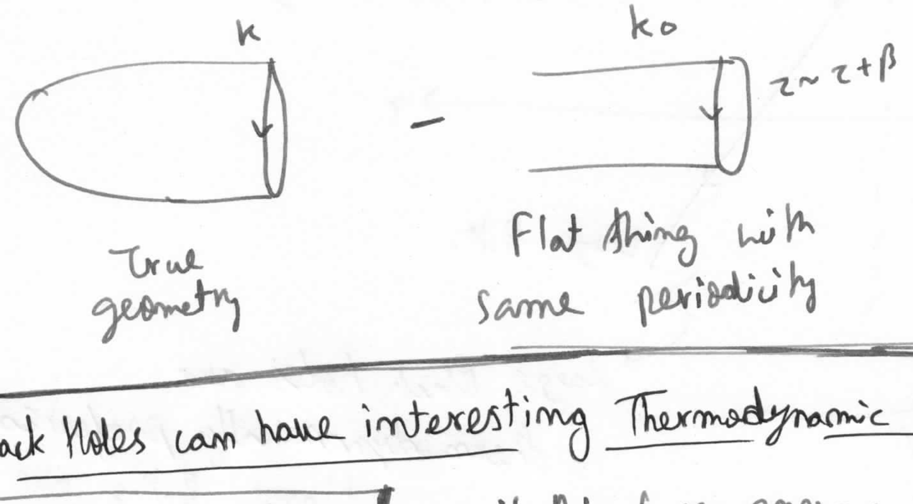
what effectively we do



$$= \frac{M}{2} \cdot \beta$$

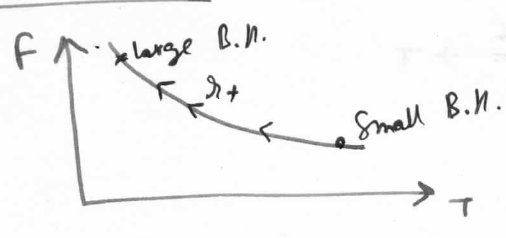
Black Holes can have interesting Thermodynamic Phase Transition.

$$\boxed{F = M - TS}$$

mass

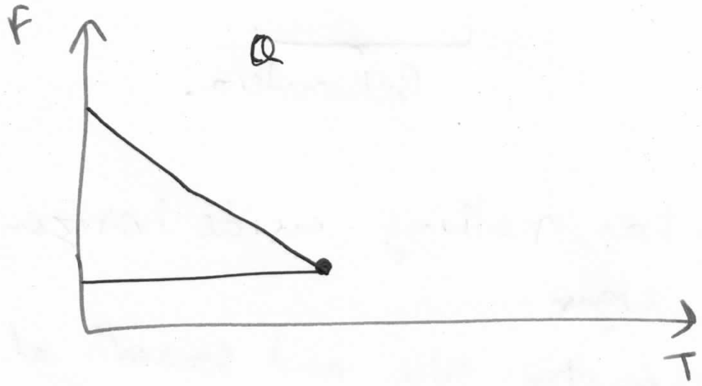
if this free energy is doing something bad; then we know that there is phase transition.

Schwarzschild B.H. $F = F(T, \ldots)$ → other parameters.



This is boring.

Schwarzschild + Charge (Charged B.H.)



Not very interesting.

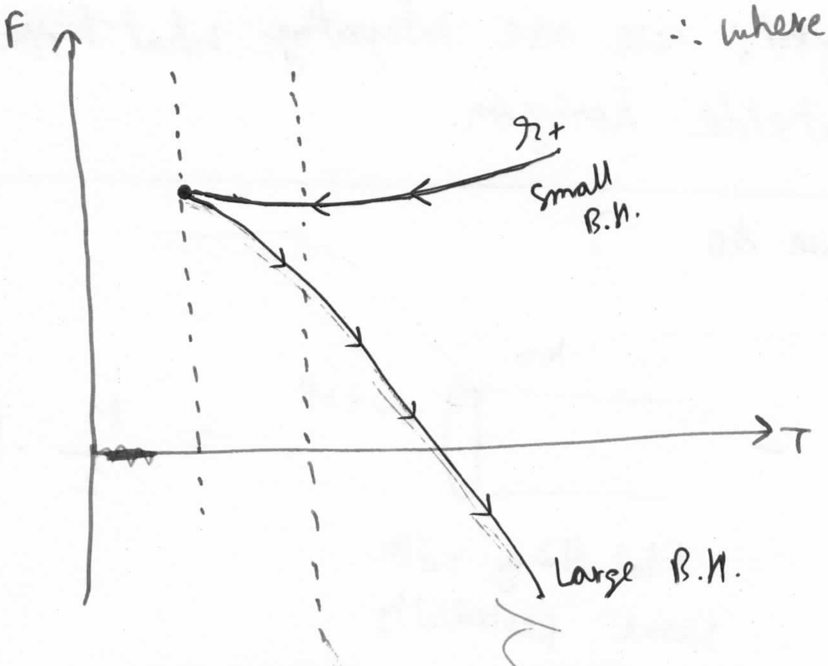
Becomes very interesting ~~go~~ as you go to AdS.

Schwarzschild + $\Lambda < 0$

$$f = 1 - \frac{2M}{r} + \frac{r^2}{l^2}$$

$\therefore$ where $\Lambda \sim -\frac{1}{l^2}$

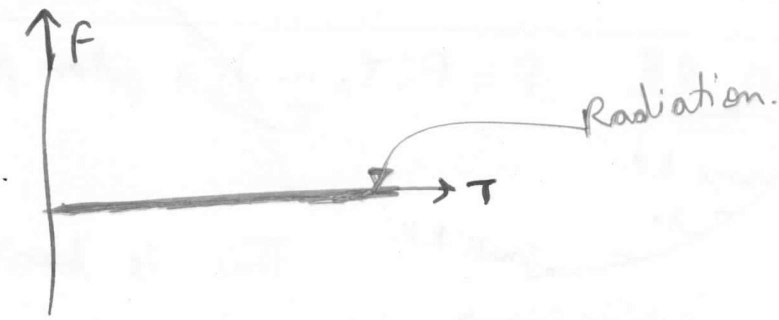
(cosmological setting ... cosmological constant characterized by length scale l)



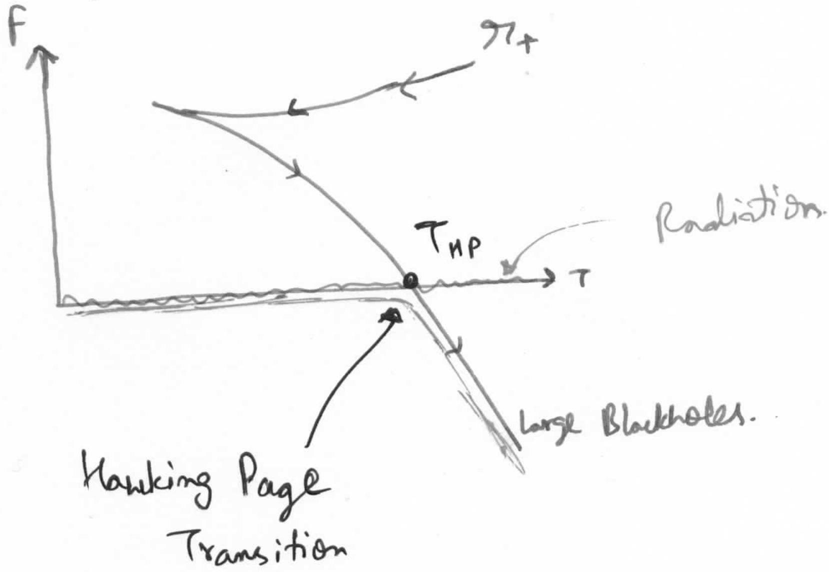
large Black holes are thermodynamically preferred.
(large B.H.s are preffered)

System = BH + Radiation

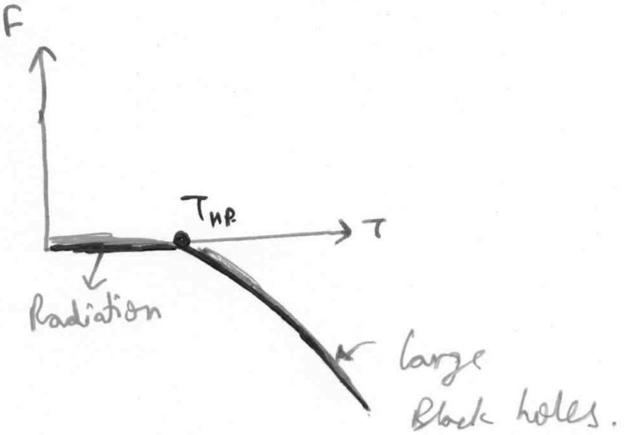
Radiation



SO; F of system = BH + Radiation.



Radiation to large Black Hole phase transition.



ADS/CFT ... DECONFINEMENT of Quarks.

THANK

YOU

Looking at the world from a different viewpoint changes the physics of it.

--Shoaib Akhtar