Strong Field Gravity

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Lec 1: Introduction ; Kerr Metric , ergoregion , symmetries.

Geometric Units : $G = c = 1$

Index conventions : $a, b, c \dots$ 4 index (spacetime)

$i, j, k \dots$ 3 index (spatial)

ex $$V^a = (V^t, V^i)$$

East Coast Metric signature : $(-, +, +, +)$

Metric : $ds^2 = g_{ab}\, dx^a dx^b$

Christoffel symbol : $\Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})$

Check Indices

$$\nabla_a T^{b_1 \dots b_n}{}_{c_1 \dots c_m} = \partial_a T^{b_1 \dots b_n}{}_{c_1 \dots c_m} + \sum \Gamma^{b_j}_{ad} T^{b_1 \dots d \dots b_n}{}_{c_1 \dots c_m}$$

$$- \sum_k \Gamma^d_{a c_k} T^{b_1 \dots b_n}{}_{c_1 \dots d \dots c_m}$$

Defining this covariant derivative... $\nabla_a g_{bc} = 0$

... Then our metric is compatible with it

Reimann Tensor : $(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = R^c{}_{dab} V^d$

Ricci " : $R_{ab} = R^c{}_{acb}$, Ricci Scalar $= R = R_{ab} g^{ab}$

Einstein " : $G_{ab} = R_{ab} - \frac{1}{2} R \cdot g_{ab}$

Einstein Equation $G_{ab} = 8\pi T_{ab}$

Perfect fluid $\Rightarrow$ $T_{ab} = (\rho + P) u_a u_b + P \cdot g_{ab}$

ρ : energy density as seen by obser with four velocity u^a

P : pressure

S.E. Conservation $\nabla_a T^{ab} = 0$.
(stress Energy)

If we dont have any Energy condition, ie; conditions of matter (ie; on T_{ab}) Then we can get ~~any~~ any geometry.

So; we want to impose some energy conditions on T_{ab}.

Null Energy Condition: $T_{ab} k^a k^b \geq 0$

k^a is future pointing arbitrary null vector.

(for fluid, it is $(\rho + P)(k^a V_a)^2 \geq 0$)

Kerr Black Holes

- Only possible stationary, vacuum B.H. solutions.
- Perturbations: decay rapidly under perturbation ~~[illegible]~~
- Final state generically collapse: (M, J) + GWs.

Metric in Boyer-Lindquist Coordinates

$$ds^2 = -\left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\right) dt^2 - \frac{2a \sin^2\theta \,(r^2 + a^2 - \Delta)}{\Sigma} dt\, d\phi$$

$$+ \left(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right) \sin^2\theta \, d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma . d\theta^2$$

where $\Sigma = r^2 + a^2 \cos^2\theta$

$\Delta = r^2 + a^2 - 2Mr$

$a = \frac{J}{M}$ $J \Rightarrow$ Angular momentum of spinning black hole
$M \to$ Mass of black whole.

$\bar{a} = J/M^2$ (dimensionless number) (dimensionless spin of Black hole)

$a=0$ Kerr reduces to SCH.

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lets consider Kerr metric ~~ax~~;

~~limit~~ ~~fix a, let M→0~~ fix a, and let $M \to 0$.

Then metric takes the form

$$ds^2 = -dt^2 + \left(\frac{r^2+a^2\cos^2\theta}{r^2+a^2}\right)dr^2 + (r^2+a^2\cos^2\theta)\,d\theta^2 + (r^2+a^2)\sin^2\theta\,d\phi^2$$

$$\left.\begin{aligned} x &= (r^2+a^2)^{1/2}\cdot\sin\theta\cdot\cos\phi \\ y &= (r^2+a^2)^{1/2}\cdot\sin\theta\,\sin\phi \\ z &= r\cos\theta \end{aligned}\right\}\ \text{Ellipsoidal coordinates}$$

$\theta \Rightarrow$ polar coordinate

$\phi \Rightarrow$ azimuthal coordinate

$r=0$ gets maps to ring of radius a.

~~Symmetries of Kerr~~

Killing vector, $\mathcal{L}_k g_{ab} = 0$ Then k is killing vector.

$\Updownarrow$

ie; $k^c \partial_c g_{ab} + (\partial_a k^c) g_{cb} + (\partial_a k^c) g_{ac} = 0$

$\Updownarrow$ (can replace ∂ with ∇; Γ will cancel...)

$$\mathcal{L}_k g_{ab} = 0 + \nabla_a k_b + \nabla_b k_a$$

$$= \frac{1}{2}\nabla_{(a}k_{b)}$$

$$\Rightarrow \mathcal{L}_k g_{ab} = \frac{1}{2}\nabla_{(a}k_{b)}$$

so; $\nabla_{(a}k_{b)} = 0$

~~Killing Vectors of Kerr~~ ~~$\hat{\phi}^a$ (ϕ with hat denotes axis symmetric Killing vector)~~

Killing vectors of Kerr

$\hat{\phi}^a = (\partial_\phi)^a \iff$ Axis symmetric.

$\hat{t}^a = (\partial_t)^a \iff$ Stationary.

Killing Tensor $\nabla_{(a} K_{bc)} = 0$

Killing tensor for Kerr $K_{ab} = r^2 g_{ab} + 2\Sigma\, l_{(a} n_{b)}$

l, n are two null vectors

Horizons

There is two places where Kerr metric ~~break~~ downs in Boyler-Lind. coordinate.

$\Sigma = 0 \quad , \quad \Delta = 0 \quad \Rightarrow$ Coordinate's Breakdown.

$R_{abcd} R^{abcd}$ blows up at $\Sigma = 0$; so $\Sigma = 0$ is singularity.

so; $\Sigma = 0 \implies r^2 + a^2 \cos^2\theta = 0$

$\Delta = 0$ turns out to give Horizon of Black Hole.

A simple way to determine horizon;

look at surface of constant r, and ~~fig~~ figure out where they become null.

Surface of constant r being null is equivalent to $g^{ab}\partial_a r \partial_b r = 0$

ie; $g^{ab}\partial_a r \cdot \partial_b r = 0$

(because $\partial_a r$ is normal to constant r surface)

$\Rightarrow g^{rr} = 0 = \frac{\Delta}{\Sigma} \quad \Rightarrow \Delta = 0$

So; we have horizon at $\Delta = 0$

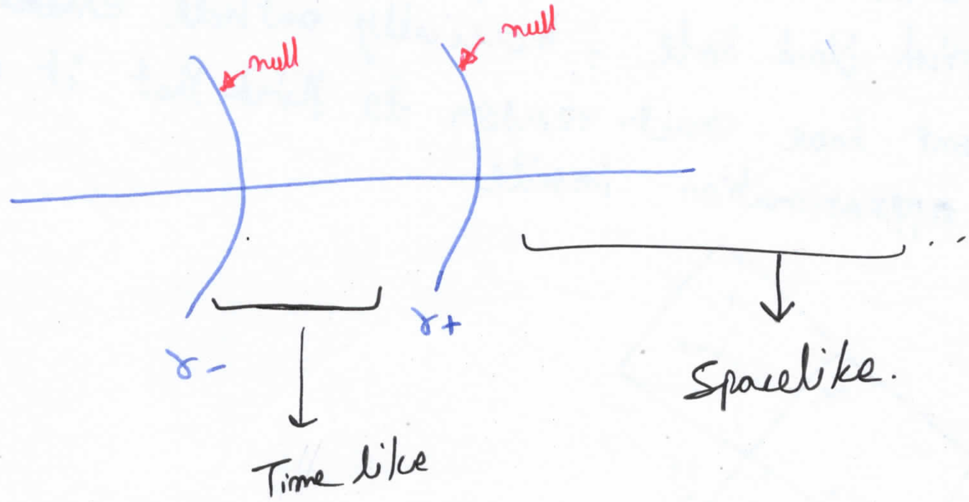
we get two solutions for r for which $\Delta = 0$.

$$\Delta = (r - r_+)(r - r_-)$$

where; $r_\pm = M \pm \sqrt{M^2 - a^2}$

$r_+ > r_-$; $r_+ \Rightarrow$ outer Horizon.

$r_- \Rightarrow$ inner "



If $a > M$, then there is no horizon to our spacetime.

↳ but still we will be then getting singularity where curvature is blowing

⟶ It will be then naked singularity. because there is no horizon to hide it from ~~are~~ observers far ~~away~~ away.

Conformal Diagram for Kerr Black Hole

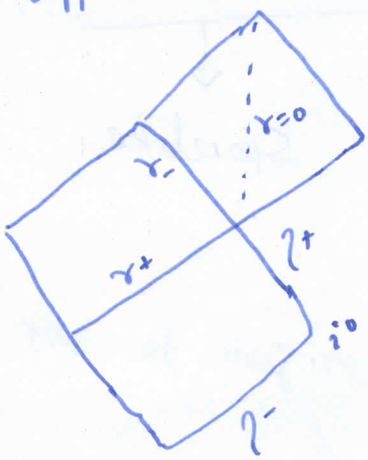
$\mathcal{I}^-$ ⇒ Past null infinity (light rays comming in from ∞ from ∞ past)

$\mathcal{I}^+$ ⇒ ~~Spat~~ Future null infinity (light rays escaping to ∞ in ∞ future)

i_0 ⇒ spatial infinity.

(like Schwarzschild we can extend the Kerr solution; and can have copy of spacetime...)

- We know kerr solution is very good approximation to astrophysical black hole; basically outside event horizon.
- We dont have much reason to think that it is good approximation inside



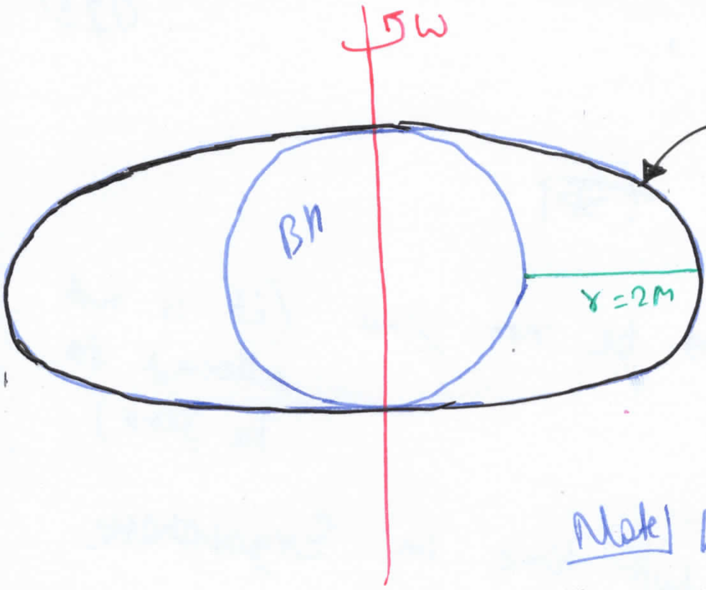
Ergosphere

$$\hat{t}^a \hat{t}_a = g_{tt} = \frac{a^2 \sin^2\theta}{\Sigma} \geq 0 \text{ at Horizon } r = r_+$$

This means time like killing vector is spacelike at Horizon except at $\theta = \pi, 0$.

Null means $\hat{t}_a \hat{t}^a = 0$ ⇒ $g_{tt} = 0$ ⇒ $\underline{\Delta = a^2 \sin^2\theta}$

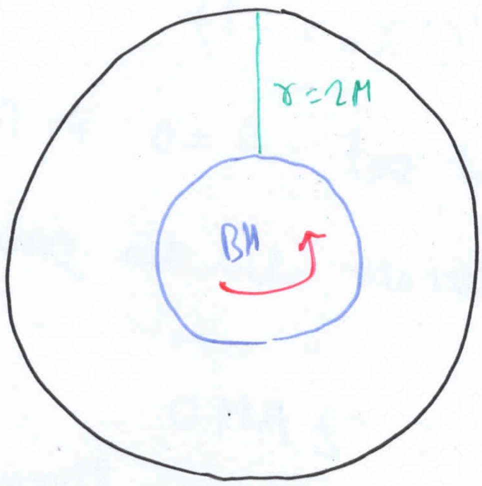
using definition of Δ, $r^2 + a^2\cos^2\theta - 2Mr = 0$.



Ergosphere (meet ~~the~~ black hole at $\theta = 0$ & $\theta = \pi$ and ~~from~~ the equator extends to $r = 2M$)

Note) When we let $a \to 0$ (we get SCH Black Hole); The Ergosphere moves in to touch the Horizon of Black Hole & disappears at $a = 0$.

Top down view



The word Ergo refers to work. .... it turns out that we can use BH to do work in Ergosphere.

Ergosphere is also the region where timelike observers must rotate with Black Hole

let u^a timelike (observer) so $-1 = g_{ab} V^a V^b$

in Boyer-Lindquist metric we get

$$-1 = g_{ab} V^a V^b = \underbrace{g_{tt}(V^t)^2 + g_{rr}(V^r)^2 + g_{\theta\theta}(V^\theta)^2 + g_{\phi\phi}(V^\phi)^2}_{>0 \text{ in the ergosphere}} + \underbrace{2 g_{t\phi} V^t V^\phi}_{\text{This has to become negative}}$$

(because $g_{tt} \geq 0$ in Ergosphere; and g_{rr}, $g_{\theta\theta}$, $g_{\phi\phi}$ are all positive)

$g^{t\phi} < 0$, $V^t = \frac{dt}{d\tau} > 0$,

Then $V^\phi = \frac{d\phi}{d\tau} > 0$

So; note that V^ϕ has to be non zero (it is not allowed to be zero) in the Ergosphere.

ϕ has to be changing with time in Ergosphere.

~~Now, lets consider obser~~

Now, lets consider observer with following four velocity

$$\chi^a = \alpha(\hat{t}^a + \Omega \hat{\phi}^a)$$

some normalization constant (find via $\chi^a \chi_a = -1$)

(we discussed that; we cant set $\Omega = 0$ in Ergosphere)

So, instead choose an observer which has zero angular momentum,

ie $\hat{\phi}_a \chi^a = 0$: ZAMO (Zero Angular Momentum Observer)

$$\Rightarrow g_{\phi t} + g_{\phi\phi}\Omega = 0$$

$$\Rightarrow \Omega = \frac{d\phi}{dt} = \frac{-g_{\phi t}}{g_{\phi\phi}}$$

This is coordinate frequencies of ZAMO's

now derivative w.r.t t for ZMO

as we approach the Horizon of Black ~~whole~~

then
$$\frac{-g_{\phi t}}{g_{\phi\phi}} \longrightarrow \frac{\bar{a}}{(2r_+)}$$

so; we can interpret $\frac{\bar{a}}{2r_+} \equiv \Omega_H$

Ω_H can be interpreted as Horizon frequency of B.H.

$$\chi^a \equiv \hat{t}^a + \Omega_H \hat{\phi}^a$$ } This is new Killing vector because just a linear ~~combining~~ combination of Killing vectors.

This is the killing vector that has event horizon as ~~for each~~ Killing horizon

i.e; $\chi^a \chi_a \to 0$ as $r \to r_+$

so; χ^a is basically generator of horizon for Kerr B.H.

For $\bar{a} << 1$, $r_+ \simeq (2 - \frac{1}{2}\bar{a}^2)M$

$\Omega_H \approx \bar{a}/4M$

For $\bar{a} = 1 - \varepsilon$; $\varepsilon >> 1$

$r_+ = (1 + \sqrt{2\epsilon})M$

$$\Omega_H \approx \frac{1 - \sqrt{2\varepsilon}}{2M}$$

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Lec 2: Geodesics on the Kerr Spacetime

Geodesic Equation given by $U^a \nabla_a U^b = 0$

where U^a is four ~~vel~~ velocity. $U^a = \frac{dx^a}{d\tau}$

$$U^a U_a = -1 \qquad (\text{for null geodesic RHS is } 0)$$

In generic setting $U^a \nabla_a U^b = 0$ gives 2nd order eqn for position coordinates.

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0.$$

For Kerr Spacetime we have conserved quanties because we have killing vector. These conserved quantities will be constant along geodesics.

So, Constants: $\tilde{E} = -\hat{t}^a u_a = -u_t$

remember $\hat{t}^a = (1, 0, 0, 0)$ in Boyer Lindquist

$\hat{\phi}^a = (0, 0, 0, 1)$ " " "

minus sign so that $\tilde{E}$ will be positive.

$\tilde{E}$ is actually $E/(\text{rest mass of particle})$.

because contracted with four velocity (not momentum)

$$E = m\tilde{E} = -\hat{t}^a P_a$$

$\tilde{J} = \hat{\phi}^a u_a = U_\phi$ $\Big\} \Rightarrow$ Angular momentum per unit mass

$$\tilde{E} = -(g_{tt} U^t + g_{t\phi} U^\phi)$$

$$= \left(1 - \frac{2Mr}{\Sigma}\right) \frac{dt}{d\tau} + \frac{2Mar\sin^2\theta}{\Sigma} \frac{d\phi}{d\tau}$$

$$\tilde{J} = \hat{\phi}^a U_a = U_\phi = \frac{(r^2+a^2)^2}{\Sigma} \sin^2\theta \frac{d\phi}{d\tau} - \frac{2Mar\sin^2\theta}{\Sigma} \frac{dt}{d\tau}$$

recall ; $\Delta = r^2 + a^2 - 2Mr$

$\Sigma = r^2 + a^2\cos^2\theta$

$\tilde{E} = 1$ at large r (rest mass contribution of particle $\tilde{E} = \frac{E_0}{m} = \frac{m}{m} = 1$)

$C = \underline{K_{ab}} U^a U^b$ ← This is another constant

C allows us to integrate generic non-equitorial orbit.

Killing Tensor

Equitorial case $\theta = \frac{\pi}{2}$

$U^\theta = 0 = U_\theta$

if we start with no velocity in θ direction fro $\theta = \pi/2$ plane ; Then we stay in $\theta = \pi/2$ plane.

$$\left.\begin{aligned} -1 &= g_{ab}U^aU^b = g^{ab}U_aU_b \\ &= g^{tt}(\tilde{E})^2 + g^{rr}(U_r)^2 \\ &\quad + g^{\phi\phi}(\tilde{J})^2 - 2g^{t\phi}\tilde{E}\tilde{J} \end{aligned}\right\} (*)$$

(replace lower U_t & U_ϕ with constants)

~~xxxxxxxx~~ ⟶ This gives U_r

and we know ; $U_r = g_{rr} U^r = \frac{r^2}{\Delta} \frac{dr}{d\tau}$

$$g^{tt} = -\frac{((r^2+a^2) - a^2\Delta)}{\Delta \cdot r^2}$$

$$g^{rr} = \frac{\Delta}{r^2}, \quad g^{rr} = \frac{\Delta}{r^2}, \quad g^{\phi\phi} = \frac{\Delta - a^2}{\Delta \cdot r^2}$$

$$g^{t\phi} = -\frac{2Ma}{\Delta \cdot r}$$

Equation (*) can now be written as

$$-1 = \frac{\Delta}{r^2} \cdot u_r^2 + \frac{a^2\Delta - (r^2+a^2)^2}{\Delta \cdot r^2} \tilde{E}^2 + \left(\frac{\Delta - a^2}{\Delta \cdot r^2}\right) \tilde{J}^2 + \frac{4M \cdot a}{\Delta r} \tilde{E}\tilde{J}$$

doing algebraic calculation.

$$-\Delta^2 \cdot u_r^2 = r^2\Delta + (a^2\Delta - (r^2+a^2)^2)\tilde{E}^2 + (\Delta - a^2)\tilde{J}^2 + 4M \cdot a \cdot r \tilde{E}\tilde{J} \quad (**)$$

$$= (1-\tilde{E}^2)r^4 - 2Mr^3 + [a^2(1-\tilde{E}^2) + \tilde{J}^2]r^2 - 2M(a\tilde{E} - \tilde{J})^2 r$$

$$:= \underline{V(\tilde{E}, \tilde{J}, r)}$$

This is essentially effective potential for radial motion.

$$\Rightarrow \boxed{-\Delta^2 \cdot u_r^2 = V(\tilde{E}, \tilde{J}, r)}$$

Now, find where $V=0$; This will tell turning points of orbit.

Turning points $\left(\frac{dr}{d\tau} = 0\right)$, when $V = 0$.

Bound Orbit $\tilde{E} < 1$

Then $r \longrightarrow \infty$, Then r^4 term dominates in V

i.e; $V(\tilde{E}, \tilde{J}, r) \longrightarrow (1-\tilde{E}^2)r^4 > 0$

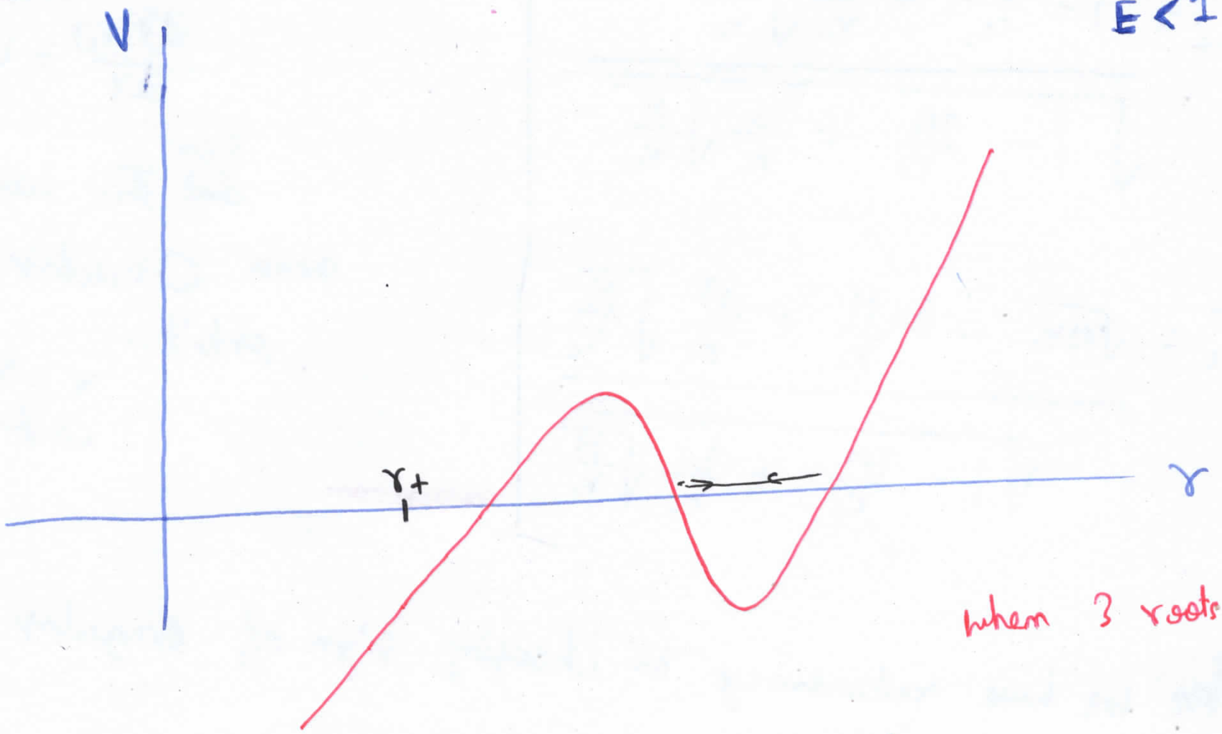
For $r = r_+$ (At Horizon)

$$(**) \Rightarrow V(r_+) = -(2Mr_+\tilde{E})^2 - (a\tilde{J})^2 + 4Ma\,r_+\tilde{E}\tilde{J}$$

$$= -(2Mr_+\tilde{E} - a\tilde{J})^2$$

$$\Rightarrow V(r_+) \leq 0$$

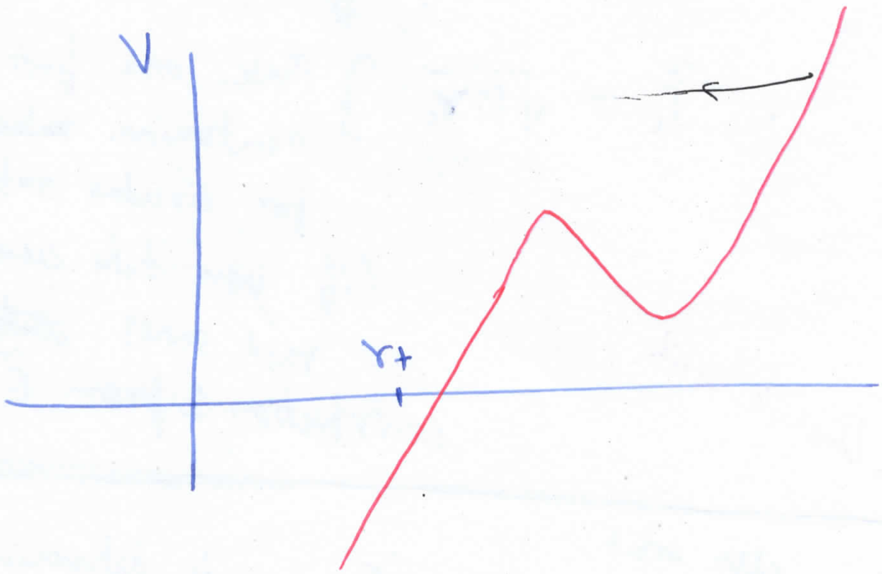
$\tilde{E} < 1$



when 3 roots.

$\frac{V(r)}{r}$ = Cubic polynomial $\Rightarrow$ Will have 1 root or 3 root.

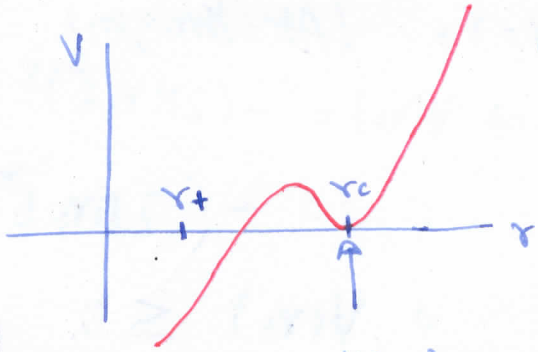
$\tilde{E} < 1$



when 1 root.

Marginal case for $\tilde{E} < 1$

when 2 roots repeated



$V(r_c) = 0$

$$\frac{dV(r_c)}{dr} = 0$$

$\Rightarrow$ This would mean Circular orbit... $r = r_c$ constant.

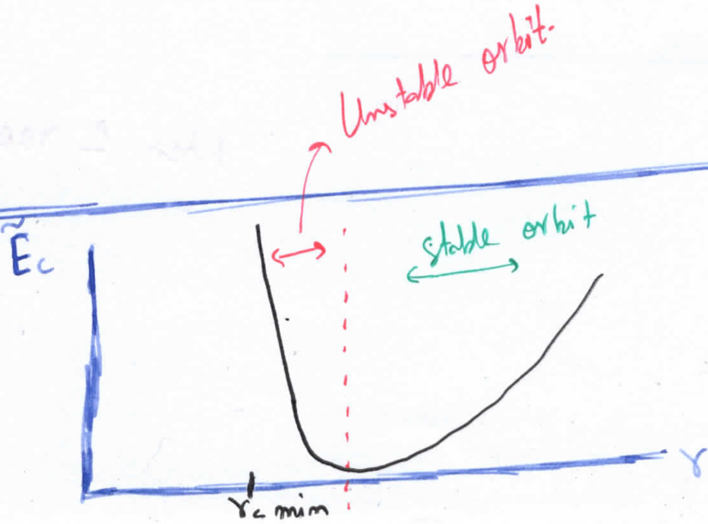
$$\tilde{E}_c = \frac{1 - \frac{2M}{r_c} + \frac{a}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$$

$$\tilde{J}_c = \frac{\sqrt{Mr_c} - 2a\frac{M}{r_c} + \frac{a^2}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$$

Here we have redundancy in choosing sign of angular momentum

For $r_c/M >> 1$ (Non relativistic limit; far away from black whole)

then $\tilde{E} = 1 - \frac{M}{2r_c}$, $\tilde{J}_c = \sqrt{Mr_c}$ } These are familiar newtonian values for circular orbit. (if you take away rest mass ~~contri~~ contribution 1 from $\tilde{E}$)



$r_{c\,min}$ is determined by making denominator of $\tilde{E}, \tilde{J}$ to be zero

$\tilde{J}_c, \tilde{E}_c \longrightarrow \infty$ when $r_c^{3/2} - 3Mr_c^{1/2} + 2a\sqrt{M} = 0$ (19.15)

Once we reach this lower value of radius,
we no longer have geodesics with finite energy

and will ~~appr~~ actually approach to limit of moving arbitrarily close to light.

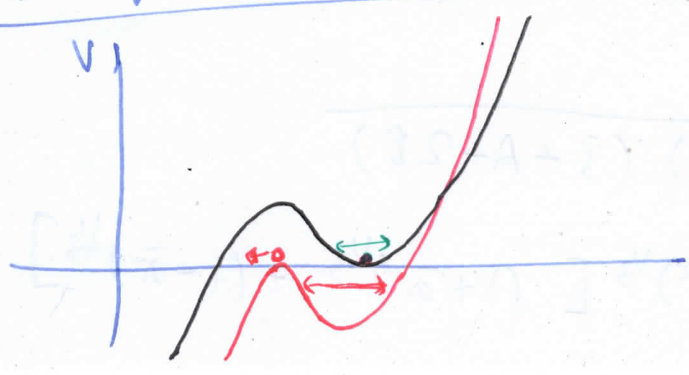
$a = 0$; $r_c > 3M$ → lower bound for circular orbit.

$\bar{a} = 1$ (orbital angular momentum & spin point in same direction) ; $r_c > M$

$\bar{a} = -1$; $r_c > 4M$
(opposite direction)

when radius have these $r_{c\,min}$ value in any case ; It will actually be photon orbit & will be followed by null geodesics.

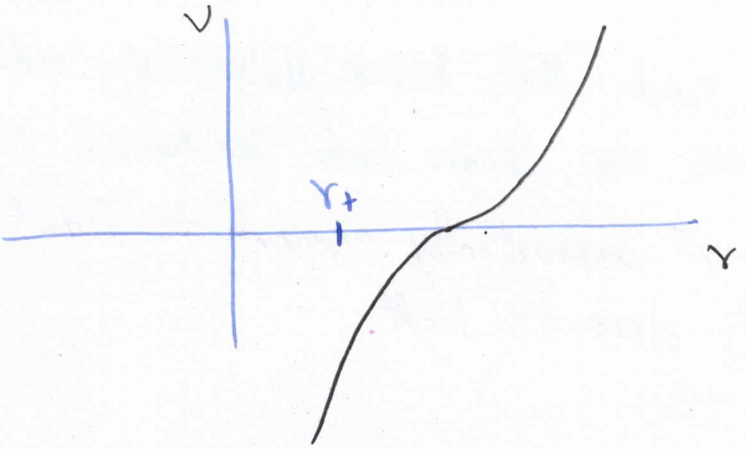
However ; not all these circular orbits are stable.
It might be true that with small perturbation it can either fall into black hole or go to ∞.



Some circular orbits are unstable.

marginal case ; when these 2 pieces merge

then we have



ISCO (Innermost Stable Circular Orbit)

we can determine value of ISCO

by $\dfrac{d\tilde{E}_c}{dr_c}(r_{ISCO}) = 0$

or

by solving $V = \dfrac{dV}{dr} = \dfrac{d^2V}{dr} = 0$

Solving we get $r_{ISCO}^2 - 6M r_{ISCO} + 8a\sqrt{M}\cdot r_{ISCO}^{1/2} - 3a^2 = 0$

$\hookrightarrow$ quartic in $r_{ISCO}^{1/2}$

for $a = 0$; $r_{ISCO} = 6M$

for $a = M$ (extremal case) ; $r_{ISCO} = M$

for $a = -M$ (retrograde case) ; $r_{ISCO} = 9M$

$$\frac{r_{ISCO}}{M} = 3 + B \mp \sqrt{(3-A)(3+A+2B)}$$

where ; $A = 1 + (1-\bar{a}^2)^{1/3}\left[(1+\bar{a})^{1/3} + (1-\bar{a})^{1/3}\right]$

$$B = \sqrt{3\bar{a}^2 + A^2}$$

we can also check ; r_{ISCO} is ~~monotonically~~ monotinically decreasing function of a.

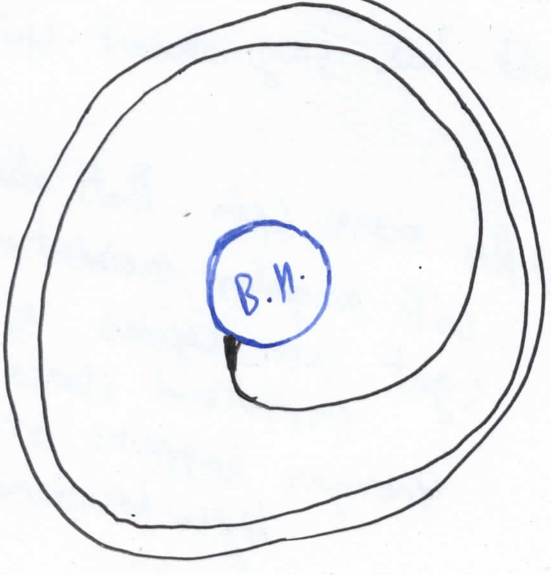
In between $r_c^{min} < r < r_{ISCO}$

we have circular orbits, but they are unstable.

Solve perturbation around unstable circular orbits.

Imagine we have small object, and its in a circular orbit around a Black hole ; And its emitting gravitational wave ; sort of loosing energy ⇒ Radius of orbit will be shrinking.

Assume, object to be small ; so that lost energy due to gravitational wave is happening very slowly ⇒ So can think of orbit being quasi circular.



At some point it will start undergoing plunch; very rapidly fall into black hole.

⟺ At what radius we expect plunch to happen for small objects ... It is r_{ISCO}.

When black hole spin is anti-allign with ~~ang~~ angular momentum of particle ⇒ Will plunch happen sooner or later.

...... occurs sooner compared to alligned case.

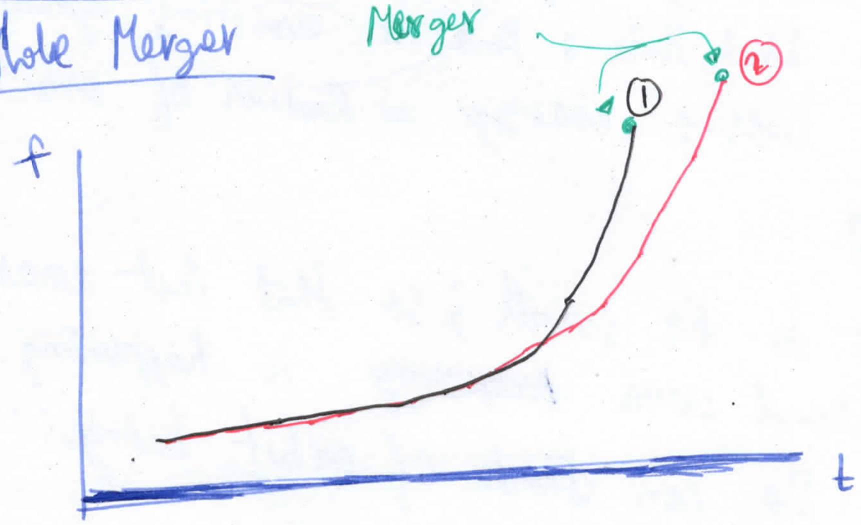
because ~~(r_c)~~ $(r_{ISCO})_{a=-1} > (r_{ISCO})_{a=1}$

⇓

Here we get more cycles.

Black Hole Merger

f ⇒ frequency



Case ① & Case ②

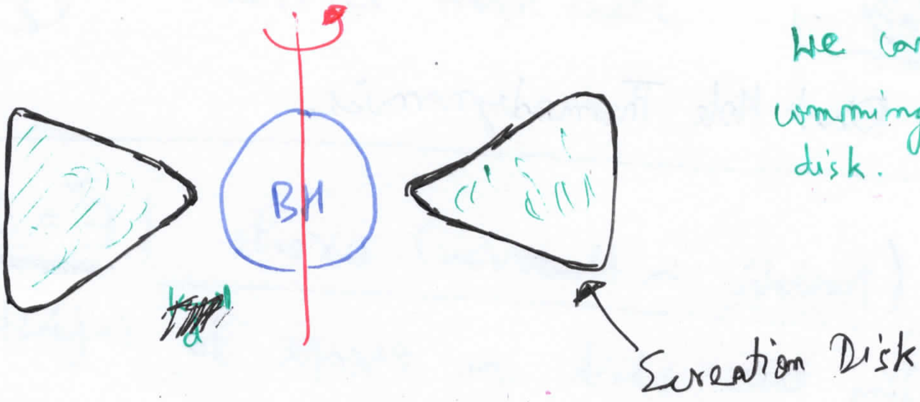
If masses of two black holes are roughly same. Then what ~~out~~ could we say about spin of BH's in these two cases

Case ② ⟹ More cycle than case ①. ⇒ Has more spin that's alligned with angular momentum (get more cycles of Gravitational Waves & merger happens at higher frequency)

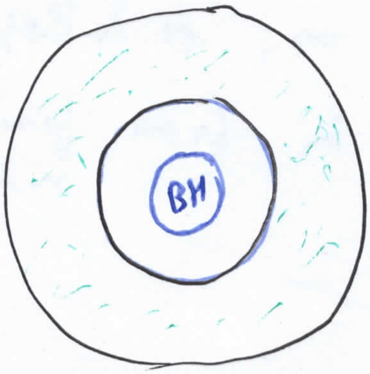
~~Case ①~~

→ This is called Orbital Hang up Effect.

We can see X-rays comming from Ecreation disk.



Side Angle View



Top View

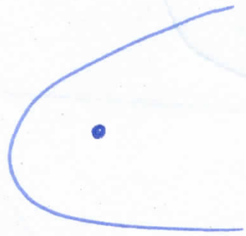
Lec 3: Review of Black Hole Thermodynamics.

Marginally bound (parabolic in Newtonian) orbits | $\tilde{E} = 1$

(These are dividing line which can escape to infinite & those which can't

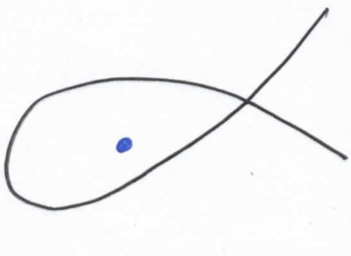
or those who come in from ∞, ~~⊗~~ & those who have to come in from finite radius)

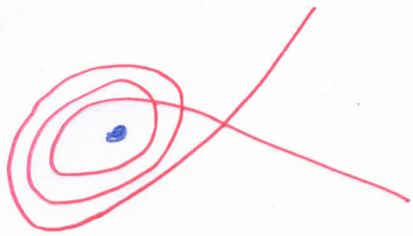
$\tilde{E} = 1$ (normalized energy)



Newtonian Theory.

In Kerr Spacetime; we can have precession.





$$V = -2Mr\left[r^2 - \frac{\tilde{J}^2}{2M} r + (a - \tilde{J})^2\right]\Bigg|_{\tilde{E}=1}$$

Again ~~we~~ we are interested in turning points; $V = 0$

$\rightarrow$ just solve the quadratic.

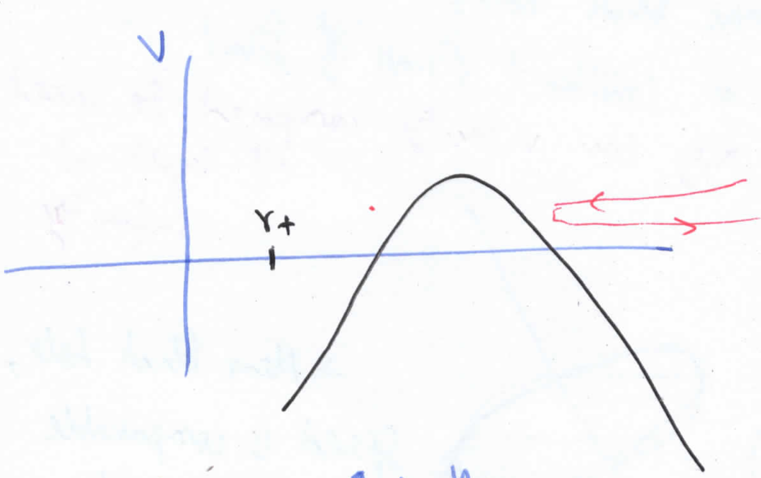
$$r_p = \frac{\frac{\tilde{J}^2}{2M} \pm \sqrt{\frac{\tilde{J}^2}{2M} - 4(a-\tilde{J})^2}}{2}$$

Periapsis Radius

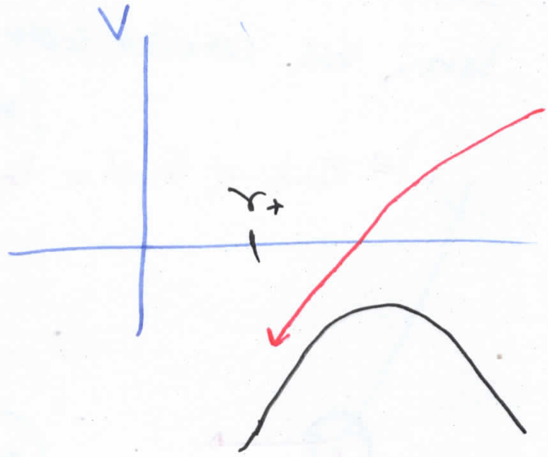
Recall; we showed $V(r_+) < 0$

also ~~too~~ have $V < 0$ for large r

Now; we can have two zeroes or no zeroes



come from ∞, And then again go to ∞.



come from ∞ & fall into black hole.

lets find the dividing line;

ie; when discriminant $= 0$ $\ni$ $\tilde{J}_m = 2M\left(1 + \sqrt{1-\bar{a}}\right)$

$(\tilde{J}_m > 0)$

Then $r_p = \frac{\tilde{J}^2}{4M} = M\left(2 - \bar{a} + 2\sqrt{1-\bar{a}}\right)$

---

for anything $r < r_p$ $\Rightarrow$ It will go to the black hole.

For $\bar{a} = -1$; Retrograde case ; $r_p = M(3 + 2\sqrt{2})$
$\simeq 5.8 M$

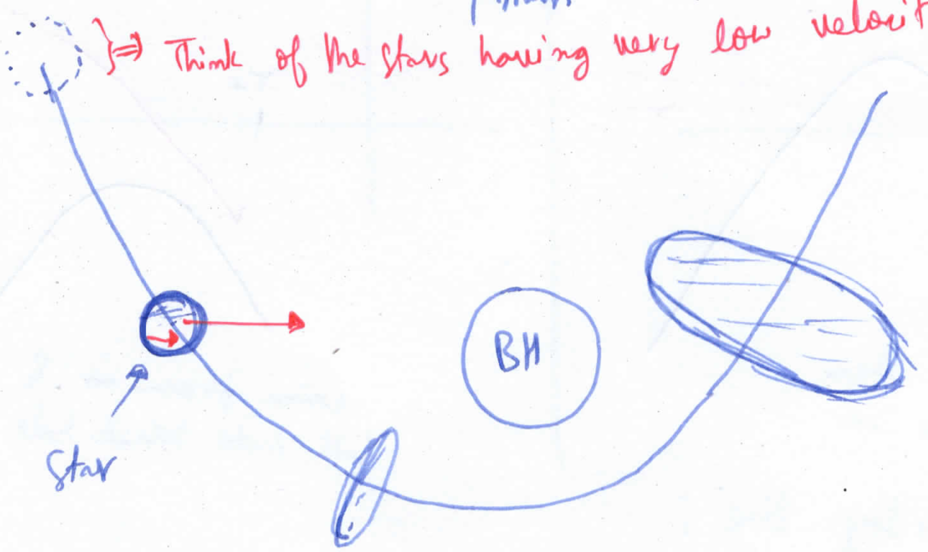
For $\bar{a} = 0$; Non spinning ; $r_p = 4M$

For $\bar{a} = 1$; Prograde case ; $r_p = M$

Tidal disruption Events

here; we consider super massive black holes
(mass = (millon) (mass of Sun)

⇒ Think of the stars having very low velocity compared to speed of light at infinity



* Near black hole, speed is comparable to speed of light.

Ignore ~~Ignore~~ the velocity at ∞; & treate it as <u>marginally bound orbit</u>.

When star gets close to BH; it experience tidal forces.

(parts of ~~star~~ star closer to BH; experience more gravity as compared to part farther away from BH)

If this tidal forces became significant to overwhelm self gravity of star ⇒ You then have what we call Tidal disruption events.

Star will get pulled apart & ~~moves~~ then moves more like cloud of dust or something like that.

↪ Then motion of stars fluid is determined by BH's gravity

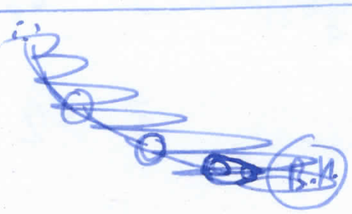
↪ & then this material can fall into BH & can form eccreation disc ; and power the greate flares that astronomer see.

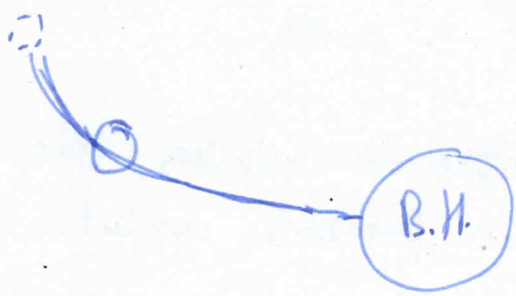
In order for this to happen

- The star should should get close enough to black Hole ; where Tidal forces are strong enough to pull apart the star.

(~~depends on~~ (depends on relative size of star compared to BH)

If star goes directly into BH, then it will not have time to form eccreation disc





What would be effect of B.H. spin on whether ~~or~~ not we get Tidal Disruption event for given star?

If star is too small compared to B.H; it wont be tidally ~~[illegible]~~ disrupted unless it gets very close to B.H.

But if we treate star spinning v/s non spinning; how would this enhance or change the ~~[illegible]~~ size of stars which can get tidaly disrupted.

(ie; given the stars; we can disrupt without having them fall into black hole)

☆ When spin alligned; Then favour disruption

parabolic orbit can basically go closer to BH & therefore can experience greater tidal force without actually falling into B.H.

Recall; Ergosphere ; $r_p < 2M$ → ~~Ergosphere at~~ Ergosphere at equator

Then $\bar{a} > 2\sqrt{1-\bar{a}}$

$$\Rightarrow \quad \bar{a} > 2 - \frac{1}{2} \approx 0.91$$

$$\Rightarrow \bar{a} > 0.91$$

for black holes with ~~spin gre~~ dimensionless spin greater than 0.91, we can have parabolic orbits that can enter the B.H. ergosphere.

(A) Penrose Process

$p^a = m\, u^a$ (rest mass)

four momentum, four velocity.

Observer with 4-velocity n^a,
Then energy that n^a will see is $E_n = -n_a p^a > 0$ as long as n^a is timelike.

E_n is not conserved

The conserved one is the one we get by contracting with the killing vector thats timelike at ∞.

$E_k = -\hat{t}_a p^a$ } Conserved.

$\hat{t}_a$ becomes spacelike in Ergosphere.

$\longleftrightarrow$ So; its possible to arrange E_k to be negative in Ergosphere

$E_k = -\hat{t}_a p^a$ can be negative in Ergosphere

Whenever we see that we can have something whose energy is negative $\Rightarrow$ This might lead us to think that somehow gain energy out of this system by still by conserving energy.

$\longrightarrow$ We can do this only in Ergosphere lets suppose; we ~~have~~ are in a space ship, and running out of fuel. We see a rotating black hole nearby & we want to use it to gain some energy.

↪ Say, we point our spaceship towards the Black hole with some initial 4-momentum p^a

Initial 4 momentum : p^a

We chose this p^a ~~so that~~ such that we could enter Black holes Ergosphere.

Then take all your garbage, and toss them out of the spaceship into the B.H.

⟷ Still, we need to conserve momentum-energy.

$$p^a = P^a_S + P^a_R$$

escape B.H. piece

fall in B.H. piece.

In particular, we want to arrange $E_R = -\hat{t}_a P^a_R < 0$

So: $E_S = E - E_R > E$ (because $E_R < 0$ $\Rightarrow -E_R > 0$)

---

lets find out what is maximum efficiency we can get out of this process.

lets focus on parabolic orbits; i.e. assume $\tilde{E} = 1$, $m = 1$ ~~(for convinience~~

set $m = 1$ for convinience.

lets assume spin of B.H. is sufficiently large such that ~~these~~ these parabolic orbits can actually enter the Ergosphere

· Incomming 4-momentum given by

$$p^a = \frac{dt}{d\tau}\left(1, 0, 0, \Omega\right)$$

(consider this parabolic orbit at its turning point)

i.e; when $\dot{r}=0$.

$$\Omega = \frac{d\phi/d\tau}{dt/d\tau} = \frac{d\phi}{dt}$$

we know $-1 = p^a p_a$ (due to normalization)

hence we get ; $-1 = \left(\frac{dt}{d\tau}\right)^2 \left(g_{tt} + g_{\phi\phi}\Omega + 2\Omega g_{t\phi}\right)$ (*)

also ; $E = \tilde{E} = 1$ (because $m=1$)

· Then $\frac{d\tau}{dt} = -(g_{tt} + g_{t\phi}\cdot\Omega)$ (**) $\Rightarrow$ after dividing expression for energy by $dt/d\tau$

We can combine (*) and (**) to get an expression for Ω

ie; $-(g_{tt} + g_{\phi\phi}\Omega^2 + 2\Omega g_{tt}) = (g_{tt} + g_{t\phi}\Omega)^2$

$\Rightarrow (g_{\phi\phi} + g_{t\phi}^2)\Omega^2 + 2(g_{t\phi})(1+g_{tt})\Omega + g_{tt}\cdot(1+g_{tt}) = 0$

can solve for Ω from this quadratic equation.

$p^a_R = C_R(1, 0, 0, \Omega_R)$ split p^a like
$p^a_s = C_s(1, 0, 0, \Omega_s)$ this during ~~term~~ turning point.

p^a_R & p^a_s can have radial component such that the sum has zero radial component

$\Longleftrightarrow$ It turns out that adding radial component to either will reduce the efficiency.

So; consider here that ~~before~~ at turning point before physical splitting both components have zero radial component.

Rest mass for that component

it turns out that $C_R = \left(\frac{dt}{d\tau}\right)_R m_R$

$C_s = \left(\frac{dt}{d\tau}\right)_s m_s$.

So; we get (by conservation of p^a)

$$\frac{dt}{d\tau} = C_R + C_s \quad \text{(from time component)}$$

$$\Omega = C_R \Omega_R + C_s \Omega_s \quad \text{(from radial component)}$$

Can solve for C_s

$$\frac{dt}{d\tau}\Omega = \left(\frac{dt}{d\tau} - C_s\right)\Omega_R + C_s\Omega_s$$

$$\Rightarrow \boxed{C_s = \frac{\frac{dt}{d\tau}(\Omega - \Omega_R)}{(\Omega_s - \Omega_R)}}$$

This $\frac{dt}{d\tau}$ is for orignal incoming.

So, the energy is just $E_S = g_{ab} \hat{t}^a p_S{}^b$

$$= -C_S \left(g_{tt} + g_{t\phi} \Omega_S \right)$$

$$\Rightarrow E_S = \left(\frac{g_{tt} + g_{t\phi} \Omega_S}{g_{tt} + g_{t\phi} \Omega} \right) \left(\frac{\Omega - \Omega_R}{\Omega_S - \Omega_R} \right)$$

$$\boxed{E_S = \left(\frac{g_{tt} + g_{t\phi} \cdot \Omega_S}{g_{tt} + g_{t\phi} \cdot \Omega} \right) \left(\frac{\Omega - \Omega_R}{\Omega_S - \Omega_R} \right)}$$

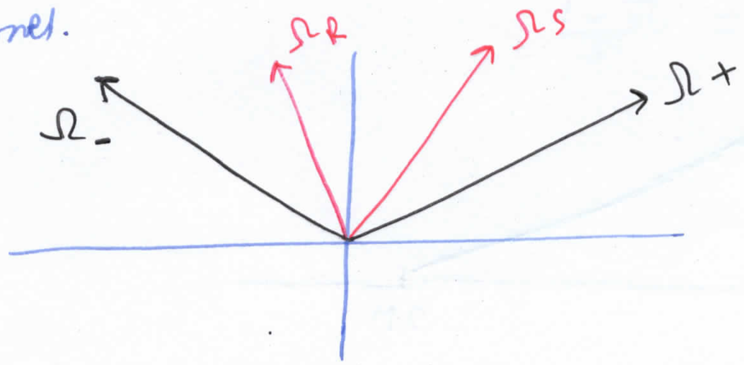
$$\Omega_\pm = \frac{-2 g_{t\phi} \pm \sqrt{4 \cdot g_{t\phi}^2 - 4 g_{tt}}}{2 g_{\phi\phi}}$$

$$\Rightarrow \Omega_\pm = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}}}{g_{\phi\phi}}$$

$$\boxed{\Omega_- < \Omega_R < \Omega_S < \Omega_+}$$

This should be be true to give time like four momentum.

These Ω_-, Ω_+ are like light cones.



If we want to maximize ~~minimize~~ efficiency of process; we take the limit, where Ω_R goes arbitrarily close Ω_-
& Ω_S " " " Ω_+.

$\Omega_s \to \Omega_+ \; , \; \Omega_R \to \Omega_s \quad ; \quad \bar{a} \to 1$

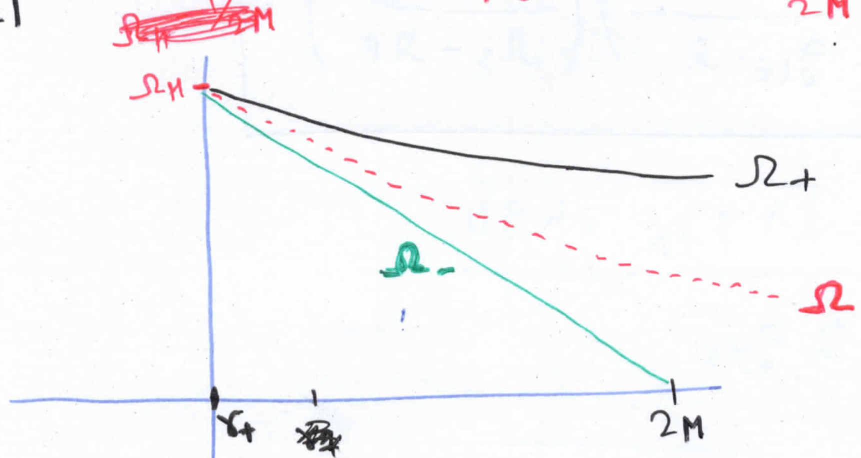
$$\eta = \frac{E_s - E}{E} = \frac{E_s - 1}{1} = E_s - 1$$

Efficiency

$$\Rightarrow \boxed{\eta = E_s - 1}$$

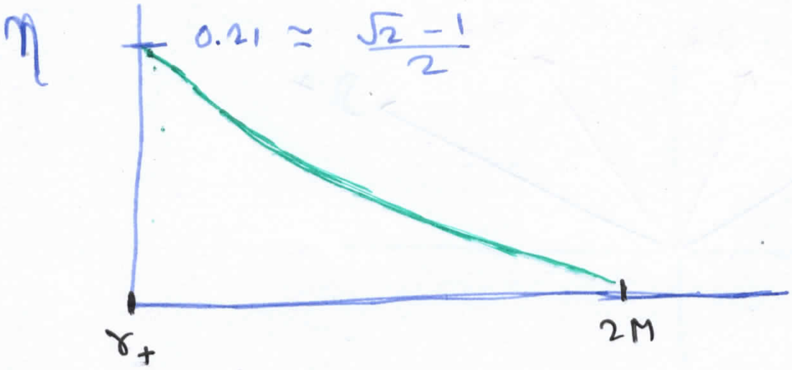
$\bar{a} = 1$

where $\Omega_H = \frac{1}{2M}$



outer edge of Ergosphere.

We have some sort of singular limit as $r \to r_+$
(at horizon all three meets)



We call $\chi^a = \hat{t}^a + \Omega_H \hat{\phi}^a$ (This killing vector is timelike outside r_+ ; & becomes null at r_+)

~~So for any p^a inside r_+~~ ;

give p^a outside horizon r_+

$p^a \chi_a < 0$ by our energy conditions.

Then $-\delta E + \Omega_H \delta J < 0$

$\delta E \Rightarrow$ energy the particle carries (conserved killing energy)

$\delta J \Rightarrow$ conserved angular momentum

So: $-\delta E + \Omega_H \delta J < 0$ for any particle ~~enters~~ that enters the Black Hole.

Penrose Process: $\delta E < 0$

This means $\delta J \cdot \Omega_H < \delta E < 0$

$\Rightarrow \Omega_H \delta J < 0 \Rightarrow \boxed{\Omega_H \delta J < 0}$

$\Longrightarrow$ So; The particle is not only carrying negative killing energy; but also negative angular momentum. So it is spinning in opposite direction of B.H.

$\Longleftrightarrow$ This will cause B.H. to spin less rapidly (because of negative δE & δJ).

$\Longleftrightarrow$ So, B.H. is losing rotational energy; and it will ~~cost~~ gained by the particle ~~p^a~~ p^a_s going out of B.H.

(New mass of B.H. is $M + \delta E$)

Lec 4: Laws of BH Mechanics
General Penrose type processes, including superradiance

B.H. Area Theorem + Thermodynamics.

Theorem due to Hawking (1971): $\delta A_{B.H} \geq 0$ with time

assumption:
(i) Null Energy Conditions.
(ii) ~~Cos~~ Cosmic Censorship. Conjecture.

Black Hole area

$$A_{BH} = \int \sqrt{g_{\theta\theta} g_{\phi\phi}}\Big|_{r=r_+} d\theta d\phi$$

$$g_{\theta\theta} g_{\phi\phi} = (r_+^2 + a^2)\sin^2\theta$$

$$\Rightarrow \boxed{A_{BH} = 4\pi(r_+^2 + a^2)}$$

recall $r_+ = M + \sqrt{M^2 - a^2}$

$$A_{BH} = 4\pi\left((M + \sqrt{M^2 - a^2})^2 + a^2\right)$$

$$\boxed{A_{BH} = 8\pi\left(M^2 + \sqrt{M^4 - J^2}\right)}$$

$J \Rightarrow$ B.H. Angular momentum

$$\frac{\delta A_{BH}}{8\pi} = 2M\delta M + \frac{1}{2}(M^4 - J^2)^{-1/2}(4M^3\delta M - 2J\delta J)$$

$$= \frac{J}{\sqrt{M^4 - J^2}}\left[\underbrace{\left(\frac{2M\sqrt{M^4 - J^2}}{J} + \frac{2M^3}{J}\right)}_{\frac{2M}{J}\left(M^2 + \sqrt{M^4 - J^2}\right) = \frac{2r_+}{a} = \Omega_H^{-1}}\delta M - \delta J\right]$$

$$\frac{\delta A_{BH}}{8\pi} = \frac{\bar{a}}{\sqrt{1-\bar{a}^2}}\left[\Omega_H^{-1}\,\delta M - \delta J\right]$$

$$\frac{\delta A_{BH}}{8\pi} = \frac{\bar{a}}{\sqrt{1-\bar{a}^2}}\left[\Omega_H^{-1}\,\delta M - \delta J\right]$$

if we want $\delta A_{BH} \geq 0$

Then $\Omega_H^{-1}\,\delta M - \delta J \geq 0 \Rightarrow \delta M - \Omega_H\,\delta J \geq 0$

~~$\Rightarrow \delta M - M\delta$~~ $\Rightarrow \boxed{\delta M \geq \Omega_H\,\delta J}$

$\Rightarrow$ This is same relation for particles crossing B.H.

Define $M_{ir} = \sqrt{\dfrac{A_{BH}}{16\pi}}$

called Irreducible mass

since $\delta A_{BH} \geq 0 \Rightarrow \delta M_{ir} \geq 0$

for $a=0$ (SCH); we have $M_{ir} = \sqrt{\dfrac{4\pi(2M)^2}{16\pi}} = M$

so; for SCH, its total mass can't decrease.

for extremal BH; $a=M$, $M_{ir} = \sqrt{\dfrac{4\pi(M^2+M^2)}{16\pi}} = \dfrac{1}{\sqrt{2}}M$

we it to define; the total mass of B.H. or say energy which is rotational. $\boxed{E_{rot} = M - M_{ir}}$

for $a=M$, $E_{rot} = (1 - 1/\sqrt{2})M \simeq 0.29\,M$

If we can find any astrophysical process to extract this E_{rot} ⇒ we can power very energetic events.

$\delta A_{BH} \geq 0$ reminds us of 2nd law of Thermodynamics.

2nd Law of B.H. mechanics : $\delta A_{BH} \geq 0$

Now what about 0th law.... (have to define something like temperature)

$\chi^a = \hat{t}^a + \Omega_H \hat{\phi}^a$ (recall this)

$\chi^a \chi_a = 0$ at horizon.

This means $\nabla_a (\chi^b \chi_b)$ should be normal to horizon

⟺ So this means $\underline{\underline{\chi_a \nabla_a (\chi^b \chi_b) = 0}}$

So; we can write $\nabla_a (\chi^b \chi_b)$ as

$$\boxed{\nabla_a (\chi^b \chi_b) = -2K \chi_a}$$ for some K

~~by taking~~ by taking Lie derivative w.r.t respect to χ_a

that ~~K has to be~~ $\boxed{\mathcal{L}_{\chi^a} K = 0}$

For Kerr B.H.

$$K = \frac{\sqrt{M^2 - a^2}}{2M(M + \sqrt{M^2 - a^2})} = \frac{\sqrt{M^2 - a^2}}{2Mr_+}$$

$$= \left(\frac{\Omega_H}{\bar{a}}\right)\sqrt{1 - \bar{a}^2}$$

K is constant on Horizon

K is called Surface Gravity; and plays the role of B.H. temperature.

~~0th Law of B.H. Thermodynamics~~

~~Surface gravity K is constant on B.H.~~

0th Law of B.H. Thermodynamics

Surface Gravity K is constant on Stationary B.H. horizon.

for $a = 0$ (SCH); $K = \frac{1}{4M}$ This justifies we call it Surface Gravity.

SCH $a = 0$

$V^a = ((1 - \frac{2M}{r})^{-1/2}, 0, 0, 0)$] $\Rightarrow$ four velocity for static observer in SCH

↪ has only time component.

$V^{1} = (1 - \frac{2M}{r})^{-1/2}$; where V is redshift.

$\Rightarrow V^{-1} = (1 - 2M/r)^{-1/2}$

$V \to \infty$ at horizon

Acceleration: $A_a = V^b \nabla_b V_a = \cancel{V^t \partial_t V_a}^{\,0} - \Gamma^t_{ta}(V_t V^t)^{-1}$

$$= -\Gamma^t_{ta}\,\delta^a{}_r = \frac{M}{r(r-2M)}\,\delta^r_a$$

combining redshift & acceleration gives finite quantity

$$A^a A_a \equiv a = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}}$$

$$V_a = \left(1-\frac{2M}{r}\right)^{1/2} \cdot \frac{M}{r^2\left(1-\frac{2M}{r}\right)^{1/2}} = \frac{M}{r^2}$$

$$\kappa = V_a\Big|_{r=2M} = \frac{1}{4M}$$ $\Rightarrow$ Think of taking long thin string;

and attach it to object which is hovering down very close to horizon; and see how much tension string has to exert to keep the object ~~horizon~~ (hovering over horizon) at static as measured by someone who is very far away from Black Hole holding the string.

Now; 1st Law of B.H. Thermodynamics.

$$\delta M = \frac{\kappa}{8\pi}\,\delta A_{BH} + \Omega_H\,\delta J$$

compare it to 1st Law of ordinary Thermodynamics

c.f. $dE = TdS - PdV$.

3rd Law of B.H. Thermodynamics

$$\kappa \geq 0$$

(Looking at expression for κ for Kerr; we see that as $\kappa \longrightarrow 0$; $\bar{a} \to 1$. ; reach Extremal case)

~~How to liberate rotational Energy from B.H.~~

How to liberate rotational Energy from B.H.

ex) Mechanical Penrose Processes (as discussed before)

now; we discuss more general Penrose processes.

General Penrose Process

Matter T_{ab} (ignore back reaction) (geometry will still be Kerr)

$$J_a = -T_a^{\ b}\,\hat{t}_b$$

(since, still we are on Kerr spacetime, we can use $\hat{t}_b$ to define ~~conserved~~ conserved energy current

$$\nabla_a J^a = -\underbrace{(\cancel{\nabla_a T^a_{\ b}})}_{\text{Zero because } T^a_{\ b} \text{ is stress Energy Tensor (conservation of } T_{ab})}\hat{t}^b - T^a_{\ b}\underbrace{(\cancel{\nabla_a \hat{t}^b})}_{\text{Zero because of Killing equation}} = 0$$

since $\nabla_{(a}\hat{t}_{b)} = 0$

$$\boxed{\nabla_a J^a = 0}$$

Now we can use Stokes Theorem to integrate this current on some surface

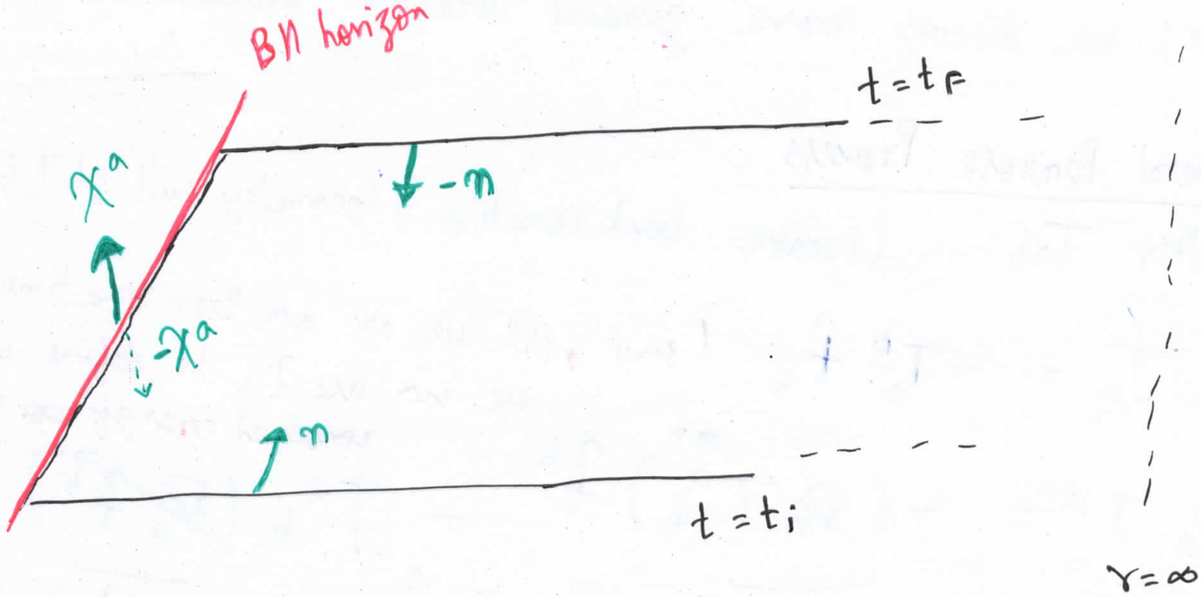
$$\boxed{\text{Stokes Theorem: } \int_M \nabla_a J^a \sqrt{-g}\, d^n x = \int_{\partial M} n_a J^a \sqrt{\gamma}\, d^{n-1}x}$$

(n_a: unit normal)

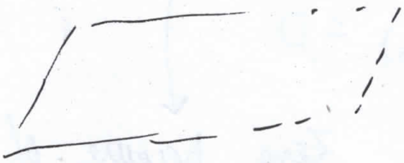
here; $\nabla_a J^a = 0 \Rightarrow \int_{\partial M} n_a J^a \sqrt{\gamma}\, d^{n-1}x = 0$

$$\int_{\partial M} n_a J^a \sqrt{\gamma}\, d^{n-1}x = 0$$

Choose M as follows :



(all the sources go to zero at $r=\infty$; basically ignore contributions from $r=\infty$)



is ∂M

use χ^a are normal to BH. horizon.

↪ we will use negative of χ^a ; ie; $-\chi^a$.

---

doing the calculation on $\int_{\partial M} n_a J^a \sqrt{\gamma}\, d^{n-1}x = 0$

$$\underbrace{\int_{t=t_f} n_a J^a \sqrt{\gamma}\, d^3x}_{T^t{}_t\sqrt{-g}} - \underbrace{\int_{t=t_i} n_a J^a \sqrt{\gamma}\, d^3x}_{T^t{}_t\sqrt{-g}} = \underbrace{-\int_{t_i}^{t_f} dt \int_{r=r_+} \chi_a J^a \sqrt{\gamma_{BH}} \cdot d\Omega}_{\Delta E_{BH}}$$

$T^t{}_t \sqrt{-g}$

↪ like energy density of Stress Energy tensor

$$E(t_f) = \int_{t=t_f} n_a J^a \sqrt{\gamma}\, d^3x$$

$$E(t_f) - E(t_i) = -\Delta E_{BH}$$

flux of energy into B.H.

If we can arrange for flux of negative energy into B.H.; Then we can have more energy than what we had before initially.

Repeat the same argument for Angular Momentum.

Also $J^\phi_a = T_{ab} \hat{\phi}^b$ conserved.

do the same argument, we get

$$J^\phi(t_f) - J^\phi(t_i) = -\Delta J_{BH}$$

Assumption NEC (Null Energy Condition)

Then $T^{ab}\chi_a \chi_b \geq 0$ here χ_a ~~are~~ is null vector.

use $\chi^a = \hat{t}^a + \Omega_H \hat{\phi}^a$

Then $(-J_b + \Omega_H J^\phi_b)\chi^b \geq 0$

integrat it over BH ~~d~~ horizon.

$$\boxed{\Delta E_{BH} - \Omega_H \Delta J_{BH} \geq 0}$$

i.e; we again get the same condition that B.H. area has to increase.

(This analysis is independent of type of matter)

Lec 5: Superradiant instability or Blandford-Znajek mechanism

Generalized Penrose Processes: we get $0 \leq \Delta E_{BH} - \Omega_H \Delta J_{BH}$.

Superradiance

(wave version of Penrose process)

Scalar field (massless) Ψ

satisfies massless K.G. equation: $\Box \Psi = \nabla_a \nabla^a \Psi = 0$

Stress Energy Tensor: $T_{ab} = \nabla_a \Psi \nabla_b \Psi - \frac{1}{2} g_{ab} \nabla_c \Psi \nabla^c \Psi$

Imagine solution of the form ~~$\Psi = \bar{\Psi}(r,\theta) Re[e$~~

$$\Psi = \bar{\Psi}(r,\theta)\, Re[e^{-i(\omega t - m\phi)}]$$

wave of pure frequency ω

azimuthal number (determines ... axial symmetry)

ie; $\Psi = \bar{\Psi}(r,\theta) \cos(\omega t - m\phi)$

~~integ surface~~

plug it in $$\Delta E_{BH} = \int dt \int_{r=r_+} \chi^a T_{ab} \hat{t}^b \, dS_{BH}$$

integrate over surface of B.H. horizon

~~$\Delta E_{BH} = \int dt \int_{r=r_+} [\chi^a (\nabla_a \Psi)(\nabla_b \Psi$~~

$$\Delta E_{BH} = \int dt \int_{r=r_+} \left[\chi^a (\nabla_a \Psi)(\nabla_b \Psi)\, \hat{t}^b - \frac{1}{2} \cancel{\chi_a \hat{t}^a}^{\,0} \nabla_c \Psi \nabla^c \Psi \right] dS_{BH}$$ (pg 41)

$\chi_a \hat{t}^a = 0$ because $\chi^a = \hat{t}^a + \Omega_H \hat{\phi}^a$

$\Rightarrow \chi^a \hat{t}^a = \chi^a (\chi^a - \Omega_H \hat{\phi}^a)$

$\Rightarrow \chi^a \hat{t}^a = -\Omega_H \chi^a \hat{\phi}^a$

we define Ω_H when $\chi^a \hat{\phi}_a$ goes to zero.

$\Rightarrow$

$$\cancel{\Delta E_{BH} = (\omega - m\Omega_H)\, \omega}$$

ie;

$$\Delta E_{BH} = \int dt \int_{r=r_+} dS_{BH} \cdot [\partial_t \Psi + \Omega_H \partial_\phi \Psi]\, \partial_t \Psi$$

$$= \int dt \int_{r=r_+} dS_{BH}\, (\omega - m\Omega_H)\, \omega\, \underbrace{|\bar{\Psi}|^2 \sin^2(\omega t - m\phi)}_{\geq 0}$$

~~so; ∆ sig~~ so; sign of ΔE_{BH} depends on sign of $(\omega - m\Omega_H)$

taking the convention $\omega > 0$

Then $\underline{0 < \omega < m\Omega_H} \Rightarrow \Delta E_{BH} < 0$

Superradiance Condition

hmerically; send a wave or some frequency towards the B.H., some will be absorbed & some will be scattered.

If we have spinning B.H. with the superradiance condition; it is called Superradiating Scattered.

It can actually be reflected with more energy than the incomming wave.

for wave packet $\frac{\delta E}{\delta J} = \frac{\omega}{m}$ ← frequency. ← azimuthal number.

~~Supper~~ Superradiance turns off in two limits

(i) $\omega \to m \Omega_H$ (ii) $\omega \to 0$

↳ Think of this limit as the limit where basically wavelength of the wave is very large as compared to size of B.H. horizon

↳ So wave sort of does not feel the B.H.

↳ and scatters without much flux across the horizon

We derived Superradiance for scalar waves here;

It is quite general phenomenon.

Can occur for

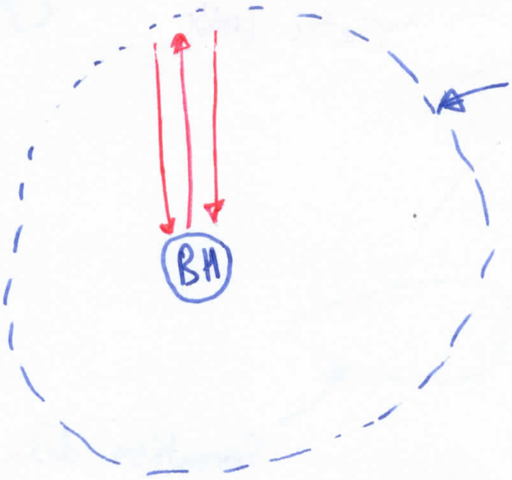
- Electromagnetic waves.
- Gravitational waves.

Superradiance as a function of spin; it maximizes for maximally spinning B.H.

$(I') = (1 + \text{Amplification})(I)$

Maximum amplification.

spin zero	$s=0$	: Scalar	: 0.3 %
	$s=1$	: Electromagnetic	: 4.4 %
	$s=2$	: Gravitational waves	: 138 %



Surround Black hole by mirror

The idea is due to Press & Teukolsky

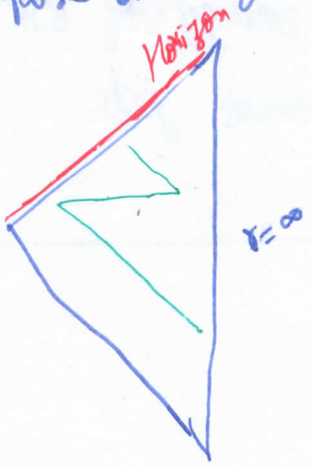
~~Black Hole Bor~~

B.H. Bomb

Consider AdS space

$r=\infty$ corresponds to time like boundary

impose boundary conditions ; like reflecting Boundary conditions



Here we can have waves, that can bounce back & forth.

And it turns out that we can get Superradiant instability in AdS this space.

so; Black Holes with ergoregion, and AdS are generically unstable

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Generalized Ergo region for Charged BH's

$$E = -\hat{t}^a p_a = \underbrace{-m\,\hat{t}^a u_a - q\,\hat{t}^a A_a}$$

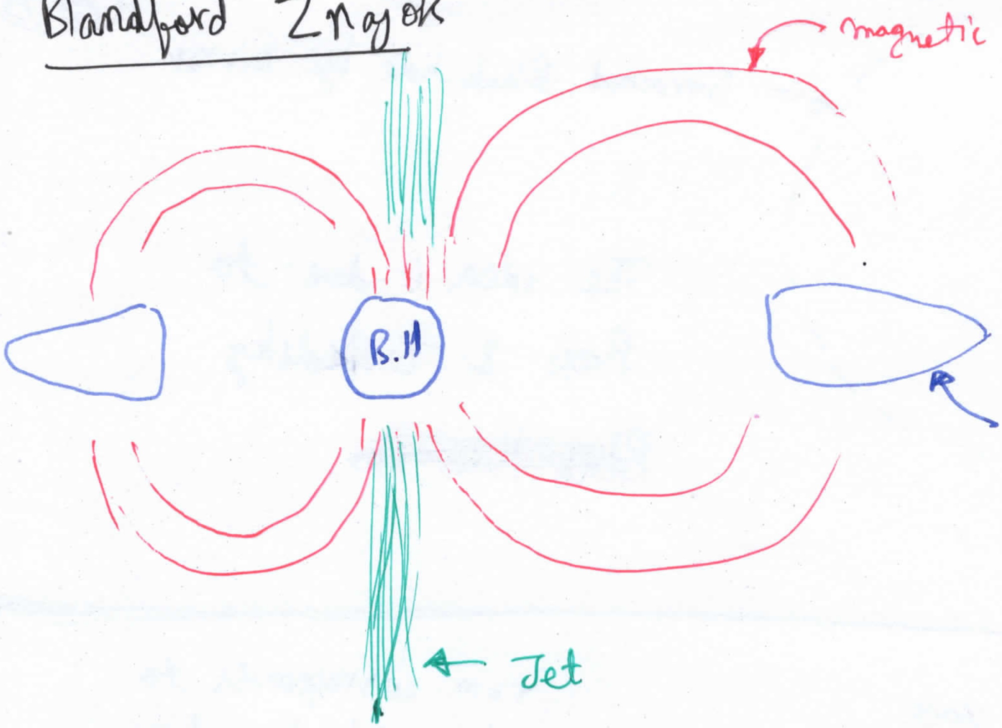
We also have this part due to charge.

where ; A_a is vector four potential associated with B.H.

If we can arrange this to be negative, then we can define effective Ergosphere for that ~~...~~ charge.

... we get similar things.....

Blandford Znajek



... baryons can source some magnetic fields

Black holes rotation will case these field lines to rotate, ... and therefore gives a pointy flux of energy that will carry away rotational energy of B.H.

↪ This is what we think powers jets.

Superradiance for massive boson

consider massic scalar field ϕ

then we have KG equation : $\Box \phi = \mu^2 \phi$.

where $\mu^2 \sim$ mass of Boson upto factor of $\hbar$)

Concentrate on non-relativistic limit of K.G.

effective compton wavelength μ^{-1} of field

consider $\mu^{-1} \gg M$; $r \gg M$ } Non-relativistic limit

(compton wavelength) $\gg$ (size of Black Hole) ; (field is far away from B.H.)

$$\Box\phi = \nabla_a(g^{ab}\partial_b\phi) = \frac{1}{\sqrt{g}}\partial_a(\sqrt{-g}\cdot g^{ab}\partial_b\phi)$$

Since ~~we~~ we are far away from B.H., we can ~~sp~~ ignore the spin of B.H.

(When you are far away from B.H. you only feel its mass; and spin falls off more rapidly)

for SCH : $\sqrt{-g} = r \cdot \sin\theta$

Then $g^{tt}\partial_t^2\phi + \partial_i\partial^i\phi - \mu^2\phi = 0$ (*)

~~where $g^{tt} \approx -(1+\frac{2M}{r})$~~ where $g^{tt} \approx -(1+\frac{2M}{r})$

(Assume: This scalar field is non-relativistic, so its momentum is much less than frequency)

$\Longleftrightarrow$ so, we only need to keep gravitational potential for time components.

~~Use ansatz : $\phi =$~~

Use ansatz : $\phi = \frac{1}{\sqrt{2\mu}}\left[\psi(x)e^{-i\omega t} + \psi(x)e^{i\omega t}\right]$

$\omega \cong \mu$ (because of non relativistic limit)

plugging it in (*) gives

$$[-(1+\frac{2M}{r})(-i\omega)^2 + \partial_i\partial^i - \mu^2]\psi = 0$$

$$\Rightarrow (\omega^2 - \mu^2)\psi = \left(-\partial_i\partial^i - \frac{2M\omega^2}{r}\right)\psi$$

$\Rightarrow$ take $\omega + \mu \approx 2\mu$

Then $$(\omega-\mu)\psi = \left[-\frac{1}{2\mu}\partial_i\partial^i - \frac{\mu M}{r}\right]\psi$$

(This equation is similar to schrodinger eqn for Hydrogen atom)

Making identification:

Fine structure constant will be : $\alpha = \mu M$

Energy : $E = \omega - \mu$

Bohr radius : Bohr radius $= \mu^{-1}\alpha \equiv r_B$

Note: we did not use any quantum mechanics.

we can label solutions by so called quantum numbers (n, l, m)

If we look at $l = m = 1$

we get $\phi \sim e^{-r/2r_B} \cdot Y^{l}_{m}(\theta, \phi) \cdot e^{-i(\omega t)}$

ω complex.

This ω ← frequency of bound state here will complex no. in general

Magnitude of Re part >> Magnitude of Imaginary part.

$$Re(\omega) = \mu\left(1 - \frac{\alpha^2}{2n^2}\right)$$

→ some correction, because the particle is bound to B.H. → so some sort of negative potential energy

also has imaginary

$|Im(\omega)|$ $\overset{\text{because}}{\Rightarrow}$ Remember we have B.H. horizon in middle ⇒ so There has to be some flux of energy across the B.H. horizon.

$$|Im(\omega)| << |Re(\omega)|$$

If frequency ω of bound state is less than $m\Omega_H$, then we can again have superradiance.

To make this instability significant; we dont want α to be very small;

ie; we want ω to be comparable to Ω_H.

$$\alpha = 0.1 \left[\frac{\mu}{10^{-12}\, eV} \right] \left(\frac{M_{BH}}{10 M_\odot} \right)$$

- We have BH of sort of 10 solar mass.
- We want the mass of boson to be somewhere in the range of 10^{-12} eV (These are very ultra light bosons; that would be relevant for having superradiance instability around Astrophysical B.H.

We dont know of any such ultralight boson in Standard Model. (They are all much more massive)

Candidates for extensions of Standard Model: things like like axion, dark photon ... Things like that, there we would have this very ultralight bosons, and they may be very weakely coupled to ~~SM~~ Standard Model.

Superradiance does not rely on any sort of coupling to ordinary matter. All it relies on is that they gravitate

Energy density seen by an observer with four velocity ~~is~~ n^a is ~~$n_a n_b T^{ab}$~~ $n_a n_b T^{ab}$

(we take observer to be perpendicular to slices of constant time) (This is not conserved)

~~The conserved one is $u^b T^a{}_b \hat{t}_a$~~

The conserved current is given by $T^a{}_b \hat{t}_a$

So: we can calculate conserved energy $n^b T^a{}_b \hat{t}_a$ that observer perpendicular to slices of constant time observes)

Lec 6: 3+1 formalism for decomposing spacetime

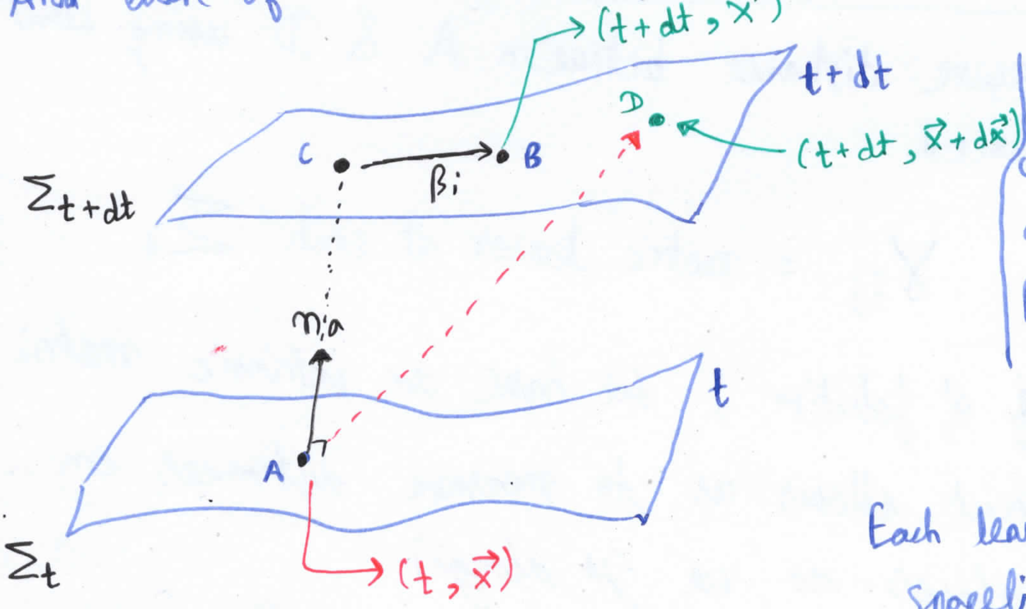
(M, g_{ab}) : Manifold ~~endowed~~ endowed with metric satisfying Einstein Equation $G_{ab} = 8\pi T_{ab}$

$\Longleftrightarrow$ We are looking forward to have some way of reinterpreting this equation in terms of something that tells us how things change in time.

So; we need to introduce some sort of time in our spacetime
(We know it is artificial; so this will imply some freedom that we can get.)

~~So, let then~~
So, re interpret (M, g_{ab}) in different way

Suppose we introduce some foliation labelled by some coordinate t.
And each of the leaves of the foliation is labelled by Σ_t.



We have complete freedom on how we label points in our spacetime.

Each leaves Σ_t is spacelike

Each leaves Σ_t is spacelike,
then its normal is timelike.

Time difference between A & C is obvious dt
because they are on leaves Σ_t & Σ_{t+dt} respectively.

~~We also~~ We can also ask what is proper time
if we go along that normal.
lets call it by $\Delta\tau$

$$\Delta\tau = \alpha \cdot dt$$

α depends on whatever point we are thinking about

α: lapse function.

α: depends on each point, and it allows us to connect the time difference as determined by foliation with the proper time.

We would also like to relate C and B
we introduce a vector β_i: called Shift Vector
(~~proper time~~ ∴ ~~The total shift is $\beta_i dt$~~

The total shift vector is $\beta_i\, dt$

We want to measure distance between A & D using what we have introduced.

We also need: γ_{ij} = metric tensor at each Σ_t

(At each leaf of foliation; we have an intrinsic metric tensor ⇒ which allows us to measure distances on the each hypersurfaces as we go along)

$$ds^2 = (\text{from } A \text{ to } C) + (\text{from } C \text{ to } B) + (\text{from } C \text{ to } D)$$

(from A to D)

$$= -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\boxed{ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)}$$

$$= g_{ab}\, dx^a dx^b$$

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \gamma_{ij}\beta^i\beta^j & \beta^i \\ \beta^j & \gamma_{ij} \end{pmatrix}$$

$$g^{ab} = \begin{pmatrix} \frac{-1}{\alpha^2} & \beta^i/\alpha^2 \\ \beta^j/\alpha^2 & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix}$$

Our Goal: To turn our understanding of spacetime (M, g_{ab}) & equations $G_{ab} = 8\pi T_{ab}$

in terms on leaves which are naturally occurring w.r.t. given initial data; also in ~~to~~ terms of ~~the~~ the free functions α, β_i which we are introducing.

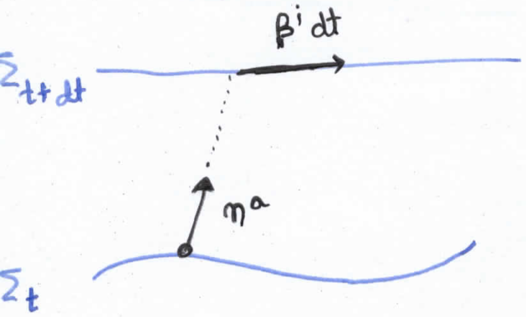
Projector: $\gamma_{ab} = g_{ab} + n_a n_b$

where n_a is unit normal orthogonal to given hypersurface.

given our signatures $(-,+,+,+)$ n^a satisfies $n^a n_a = -1$

$$n^a \gamma_{ab} = n^a g_{ab} + n_a n_a n_b = n_b - n_b = 0$$

$$\Rightarrow \boxed{n^a \gamma_{ab} = 0}$$



$n_a = (-\alpha, 0)$ normalized n_a.

choose minus sign because we want it to be future pointing

$$n^a = \left(\frac{1}{\alpha}, \frac{-\beta^i}{\alpha}\right)$$

$$\Rightarrow \boxed{\left(\frac{\partial}{\partial t}\right)^a = \alpha n^a + \beta^a}$$

Extrinsic curvature

parallel transport n^a on Σ_t

$$\boxed{K_{ab} \equiv -\gamma^c{}_a \nabla_c n_b}$$

Extrinsic Curvature

γ_{ac} is actually a projector.

we are using γ_{ac} to project result of taking gradiant of vector n_b to the hypersurface.

$$K_{ab} = -(g^c{}_a + n^c n_a) \nabla_c n_b$$

$\Rightarrow K_{ab} = -(\nabla_a n_b + n_a \underbrace{n^c \nabla_c n_b})$

↳ Acceleration of vector n
call it a_b

Then $\boxed{K_{ab} = -(\nabla_a n_b + n_a a_b)}$

if n_a happens to be geodesic
then a_b is zero.

Then $K_{ab} = -(\nabla_a n_b)$

if not, then there is component due to contribution of acceleration.

$K_{ab} = K_{ba}$ (K_{ab} is symmetric)

proof

$\left(\frac{\partial}{\partial t}\right)^a$ ⇒ vector tangent to line at the point...

$\tilde{t}$ s.t. $\nabla_a \tilde{t} = n_a \equiv \xi_a$

(define such $\tilde{t}$)

relation between $(\partial/\partial t)^a$ & n^a

$K_{ab} = \gamma^c{}_a \nabla_c n_b = \gamma^c{}_a \nabla_c \xi_b$

$\xi_b = \nabla_a \tilde{t}$ ⇒ Then $\nabla_c \xi_b = \nabla_c(\nabla_b \tilde{t})$ since $\tilde{t}$ is scalar function

then, so: $\nabla_c(\nabla_b \tilde{t}) = \nabla_b(\nabla_c \tilde{t})$

$K_{ab} = \gamma^c{}_a \nabla_b n_c = K_{ba}$

$$\mathcal{L}_n \gamma_{ab} = n^c \nabla_c \gamma_{ab} + \gamma_{ac} \nabla_b n^c + \gamma_{bc} \nabla_a n^c$$

$$= n^c \nabla_c (g_{ab} + n_a n_b) + (g_{ac} + n_a n_c) \nabla_b n^c + (g_{bc} + n_b n_c) \nabla_a n^c$$

$$\cancel{= n^c (0 + \nabla_c (n_a n_b)) + (g_{ac} + n_a n_c)} \nabla$$

$$\nabla_c (g_{ab}) = 0$$

and $n_c \nabla_b n^c = 0$ because $n_c n^c = -1$

So: $$\mathcal{L}_n \gamma_{ab} = n^c \nabla_c (n_a n_b) + g_{ac} \nabla_b n^c + g_{bc} \nabla_a n^c$$

$$= (n^c \nabla_c n_a) n_b + (n^c \nabla_c n_b) n_a + \nabla_b n_a + \nabla_a n_b$$

$$\Rightarrow \mathcal{L}_n \gamma_{ab} = (n^c \nabla_c n_a) n_b + (n^c \nabla_c n_b) n_a + \nabla_b n_a + \nabla_a n_b$$

$$\gamma_{ab} = g_{ab} + n_a n_b \Rightarrow n_a n_b = \gamma_{ab} - g_{ab}$$

So: $$\mathcal{L}_n \gamma_{ab} = (\gamma^c{}_b - g^c{}_b) \nabla_c n_a + (\gamma^c{}_a - g^c{}_a) \nabla_c n_b + \nabla_b n_a + \nabla_a n_b$$

$$= \gamma^c{}_b \nabla_c n_a + \gamma^c{}_a \nabla_c n_b - \underbrace{\nabla_b n_a - \nabla_a n_b + \nabla_b n_a + \nabla_a n_b}_{0}$$

$$\Rightarrow \boxed{\mathcal{L}_n \gamma_{ab} = \gamma^c{}_b \nabla_c n_a + \gamma^c{}_a \nabla_c n_b}$$

$$\Rightarrow \boxed{\mathcal{L}_n \gamma_{ab} = -2K_{ab}}$$

Property: $\boxed{\mathcal{L}_n \gamma_{ab} = \frac{1}{\phi} \mathcal{L}_{(\phi n)} \gamma_{ab}}$ $\forall$ function ϕ.

using it $$\mathcal{K}_{ab} = -\frac{1}{2\alpha} \mathcal{L}_{\alpha n} \gamma_{ab} = \frac{-1}{2\alpha} [\mathcal{L}_{\vec{t}} - \mathcal{L}_{\vec{\beta}}] \gamma_{ab}$$

$\mathcal{L}_{\vec{t}} \gamma_{ab} = \partial_t \gamma_{ab}$

$i, j \Rightarrow$ intrinsic to hypersurface

$a, b \Rightarrow$ run from (t, space coordinates)

Then

$$\partial_t \gamma_{ij} - \mathcal{L}_\beta \gamma_{ij} = -2\alpha K_{ij}$$

(only spacial component survive)

$$\Rightarrow \boxed{\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i}$$

D is covariant derivative associated to γ_{jk} matrix

i.e. $D \gamma_{jk} = 0$

... till now we just did kinematics.

now we will bring Einstein's equations.

$D_a T^{ab} =$ Projection of all components of derivatives

So; $^{(3)}D_a T^{ab} = \gamma^{a_1}_{a} \gamma^{b}_{b_1} \gamma^{d}_{d_1} \nabla_{a_1} T^{b_1 d_1}$

Recall Gauss - Codazzi Equation :

$$\gamma^{a_1}_{a} \gamma^{b_1}_{b} \gamma^{c_1}_{c} \gamma^{d_1}_{d} R_{a_1 b_1 c_1 d_1} \equiv \underbrace{{}^{(3)}R_{abcd} + K_{ac} K_{bd} - K_{ad} K_{bc}}_{\text{purely spacial.}}$$

~~Codazzi-Mai~~

Codazzi-Mainardi equation:

$$\cancel{\gamma^{a_1} \gamma^{b_1} \gamma^{c_1}_{c} n^d R_{a_1 b_1 c_1 d} \equiv D_b K}$$