Cosmology

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Lec 1: Why to Study cosmology? Established facts about our Universe, Maximally symmetric Spaces

Why Cosmology?

- "RICH PHYSICS" • ZELDOVICH: "Early universe is an accelerator for poor".
- Open Arena for testing Fundamental theories like String Theory, Q.M.
- Observation Based
- New Physics likely to be discovered here. (because many puzzle emerge)
  - Λ - problem, (Dark Energy)
  - η - problem → fraction of baryons & photons
  - Lithium problem,
- All problems with Inflation.
- Jobs available in this areas (Haha..)

Established Facts

- Homogeneous & Isotropic at large scale.
- Expanding $\sim$ Hubble's Law (very well described)

} Homogeneous Universe.

- CMB (Cosmic Microwave Background) : $T \approx 2.73K$ (whole universe is filled with this radiation)
- BARYONIC Matter : 75% Hydrogen, 25% Helium,
- Dark Matter, Dark Energy (95%)

} Matter

- Small Fluctuations (like say galaxies, (Hahaha..)), ..etc.)

Galaxies

CMB fluctuation. $\delta T \sim 10^{-5}$

} Fluctuations.

Chapter I : HOMOGENEOUS UNIVERSE

- Observable Universe 3×10^4 Mpc.
- Homogeneous & Isotropic ~ scales 100 Mpc.
- Isotropy : we observe.

~~Homogeneity~~

- Copernicus Cosmological Principle ⇒ ~~Homog~~ Every observer in universe sees the same. So, have Isotropy at every point.

(~~No~~ (No ~~dat~~ experimental data ...)

↓

we are just sitting at one point.

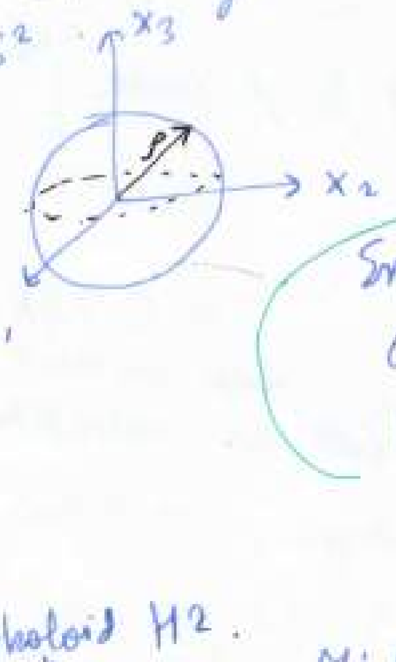
mathematically we can prove a theorem. ⟷ HOMOGENEITY

(a) Maximally Symmetric Spaces

(because we need to describe; all Homogeneous & Isotropic spaces)

Construction: Embedding in $(n+1)$ dimensional space.

Ex 1: S^2



Euclidean

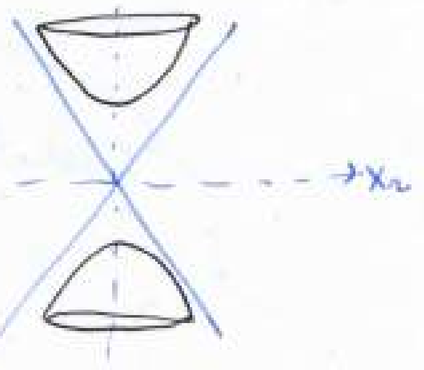
$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

Embedding : $\rho^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$

$$0 = x^1 dx^1 + x^2 dx^2 + x^3 dx^3$$

$$(dx_1^3)^2 = \frac{x_1 dx_1 + x^2 dx^2}{(\rho^2 - X_1^2 - X_2^2)}$$

Ex 2: Hyperboloid H^2.



Minkowski (Embedding in Minkowski)

$$ds^2 = (dx^1)^2 + (dx^2)^2 - (dx^3)^2$$

Embedding: $-\rho^2 = (x^1)^2 + (x^2)^2 - (x^3)^2$

General:

Signature (p, q)

$$\eta_{ab}^{(p,q)} = \text{diag}(\underbrace{+,+,\ldots,+}_{p \text{ times}}, \underbrace{-,-,\ldots,-}_{q \text{ times}})$$

X^A ; $A \in \{1, \ldots, n+1\}$: embedding space coordinates.

x^a : $a \in \{1, \ldots, n\}$: coordinates on the surface.

$x_a = \eta_{ab}^{(p,q)} x^b$; $x^2 = x_a x^a$.

---

n-dimensional, maximally symmetric geometry of signature (p,q) and curvature $K = 0, +1, -1$.

Spaces: $$ds^2 = \eta_{ab}^{(p,q)} dx^a dx^b + K \cdot (dX^{n+1})^2$$

$$ds^2 = \eta_{AB}^{(p,q)K} dX^A dX^B$$

now we are defining this new metric.

Constraint: $\eta_{AB}^{(p,q)K} X^A X^B = K\rho^2$ (Embedding equation)

$$K\rho^2 = K(X^{n+1})^2 + x^2$$

$$\Rightarrow \quad K \cdot X^{n+1} dX^{n+1} = -x_a dx^a$$

$$\Rightarrow \quad (dX^{n+1})^2 = \frac{(x_a dx^a)^2}{(X^{n+1})^2} = \frac{(x_a dx^a)^2}{\rho^2 - Kx^2}$$

$$\boxed{g_{ab}^{(p,q)K} = \eta_{ab}^{(p,q)} + \frac{K x_a x_b}{\rho^2 - Kx^2}}$$

metric on any maximally symmetric space.

$$R_{abcd} = \frac{K}{\rho^2}\left(g_{ac} g_{bd} - g_{ad} g_{bc}\right)$$

Solution of Einstein's Equations

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}}$$

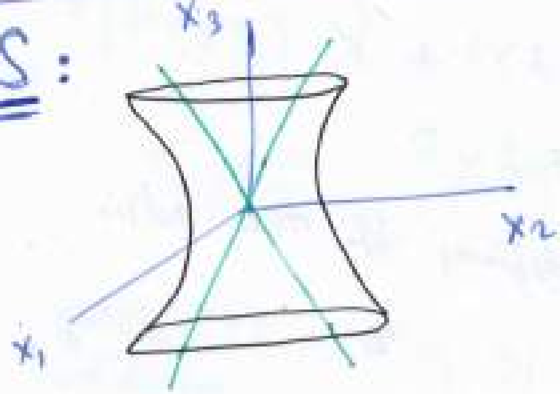
$T_{\mu\nu} = 0$; $\Lambda = \dfrac{K(n-1)(n-2)}{2\rho^2}$

TABLE

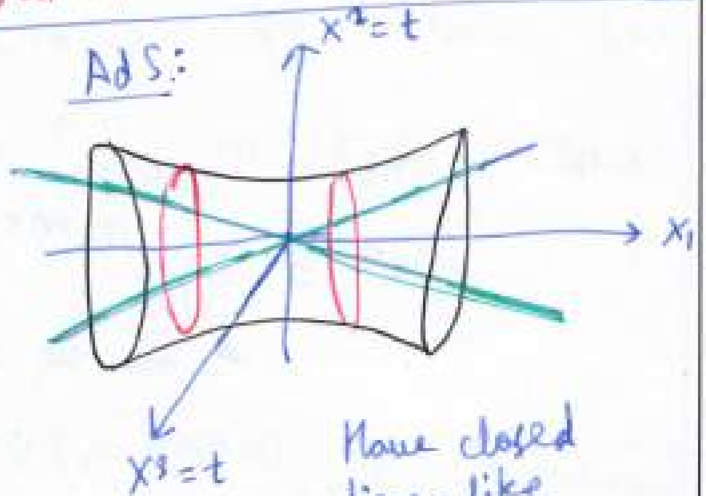
(p, q)	$k=0$	$k=+1$	$k=-1$
$(n, 0)$	Euclidean	S_n	H_n
$(n-1, 1)$	M_n	dS_n	AdS_n
$(n-2, 2)$			

END OF STORY FOR PHYSICIST.

dS:



AdS:



Have closed time-like curves in AdS.

Covering space

Why maximally Symmetric?

Symmetries correspond to killing vectors.

Killing vectors described by killing equation $\nabla_a \xi_b + \nabla_b \xi_a = 0$.

Integrability conditions $\boxed{\nabla_a \nabla_b \xi_c = -R_{bca}{}^{d} \xi_d}$

$$\nabla^2 \xi_c + R_{cd} \xi^d = 0$$

Vacuum: $\nabla^2 \xi_c = 0$ (Killing vectors satisfy wave equation in vacuum)

$\nabla \cdot \xi = 0$

$$\boxed{\underbrace{\nabla^2 \xi_a = 0}_{\text{Vacuum}} \quad , \quad \nabla \cdot \xi = 0}$$ These are similar to maxwell equations

Solution to maxwell equations is $A_a = \xi_a$.

~~(AdS):~~ $\cancel{R_{ab} \sim \Lambda g_{ab}}$

Vacuum + Λ $\qquad R_{ab} \sim \Lambda \cdot g_{ab}$

$$\boxed{(\nabla^2 + \Lambda)\, \xi_a = 0}$$ massive vector equation.

How many at most Killing Vectors ?

Initial data :
- ξ^a ... n data
- $L_{ab} = \nabla_a \xi_a$ (first derivatives)
  Anti-symmetric.
  $\binom{n}{2}$ data in n-dimensions
- $\nabla^2 \xi$ ⟸ already know ; if above data are specified. from integrability condition.

- $\nabla^3 \xi \sim (\nabla R) \cdot \xi + R \cdot \underbrace{\nabla \xi}_{L}$

Because of integrability condition; the total no. of independented datas are $\underset{(n)}{\xi^a}$ & $\underset{\binom{n}{2}}{L_{ab}}$

So, you have at most $n + \binom{n}{2}$ Killing vectors we have.

All maximally symmetric spacetime has $n + \binom{n}{2}$ independent killing vectors.

Lec 2: FRW Spacetime, Hubble's Drag, Cosmological Redshift

$$ds^2 = \eta_{AB}^{(p,q)K}\, dX^A dX^B = \eta_{ab}^{(p,q)}\, dx^a dx^b + K (dX^{n+1})^2$$

$$K\rho^2 = \eta_{AB}^{(p,q)K} X^A X^B \quad \Rightarrow g_{ab}^{(p,q)K} = \eta_{ab}^{(p,q)} + \frac{K x_a x_b}{\rho^2 - K x^2}$$

ex) S_n, H_n, E_n; M_n, AdS_n, dS_n.
Building Blocks for FRW; the metric describing our universe.

$$\boxed{n + \binom{n}{2} = \text{max no. of killing vectors} \qquad \xi^a \quad L_{ab}}$$

Constraint (surface) is invariant under

$$\tilde{X}^A = \underbrace{\Lambda^A{}_B}\, X^B \quad \text{obey} \quad \eta_{AB}^{(p,q)K} = \Lambda^C{}_A \cdot \eta_{CD}^{(p,q)K} \Lambda^D{}_B$$

$\Lambda^A{}_B$ form a representation of $O\left(p + \frac{K+1}{2},\ q + \frac{1-K}{2}\right)$ for $K \neq 0$

do infinitesimally

$$\Lambda^A{}_B = \delta^A{}_B + \lambda^A{}_B, \Rightarrow \quad \lambda_{AB} = \eta_{AC}^{(p,q)K} \lambda^C{}_B = -\lambda_{BA}$$

$\boxed{\lambda_{AB} = -\lambda_{BA}}$ the generators are anti-symmetric

index run from 1 to $n+1$

so; $\binom{n+1}{2}$ generators $\equiv \underbrace{\left(n + \binom{n}{2}\right)}_{\text{no. of Killing Vectors}}$

(b) FRW Spacetime

- Symmetry of Universe; Rotational Symmetry (Isotropy) + Translational Symmetry (Homogeneity)

KVs $\equiv$ Killing Vectors.

$\underbrace{\text{Rotational Symmetry}}_{3\,KVs} + \underbrace{\text{Translational Symmetry}}_{3\,KVs}$

Dont have "Boost" symmetry

$\Rightarrow$ There exists; preferred frame. $\approx$ called Comoving observers

This will see homogeneous & isotropic universe.

Uses coordinates t (Proper time of Comoving Observer)

$$x^i = \text{constant} \; ; \; \text{observer.}$$

In these coordinates, our universe is maximally symmetric on a spacial slice ; Spacial geometry can "Arbitrarily" stretch or shrink in time..... described by SCALE FACTOR $a(t)$

$$\boxed{ds^2 = -dt^2 + a^2(t) \cdot g_{ij}^{(3,0)K} dx^i dx^j}$$ FRW spacetime

$$\boxed{ds^2 = -dt^2 + a^2(t) \cdot \left[\delta_{ij} + \frac{K x_i x_j}{\rho^2 - K x^2} \right] dx^i dx^j}$$

$K = 0$: $\underline{E_3}$ "Flat Universe"

$K = +1$: $\underline{S_3}$ "Closed Universe"

$K = -1$; $\underline{H_3}$ "Open Universe"

Can set $\rho = 1$ (by rescaling x_i)

$$x^i = \rho \tilde{x}_i \; ; \; \underbrace{a = \tilde{a}/\rho}$$

now we see ; $\tilde{a}$ describes curvature of spacetime.

i.e. $\tilde{a}(t)$ ∴ remove tilda

∴ $\underline{a(t) : \text{Radius of Curvature}}$.

Scale Factor : determined by Matter Content (through Einstein's Equation)

$$\boxed{H = \frac{\dot{a}}{a}}$$ Hubble's Expansion Rate

$$\boxed{q = \frac{-\ddot{a}}{aH^2}}$$ Deceleration Parameter

Other Coordinates Systems

(a) Spherical. $x^1 = r\sin\theta\cos\varphi$; $x^2 = r\sin\theta\sin\varphi$; $x^3 = r\cos\theta$

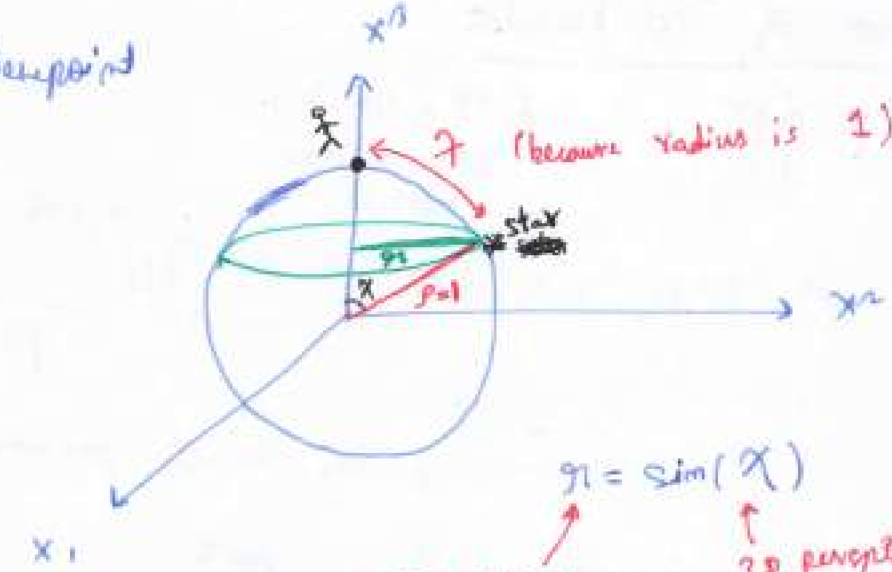
$$\boxed{ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right)}$$

(b) New Radial Coordinate : χ

χ : $r = S_k(\chi) = \begin{cases} \sin\chi & k=+1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$

$$\boxed{ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + S_k^2(\chi)\cdot d\Omega^2\right]}$$

K=+1 Embedding Viewpoint



$r = \sin(\chi)$

Embedding perspective

3D perspective

(c) Conformal Time η : $dt = a\,d\eta$

$$\eta - \eta_0 = \int_0^t \frac{dt'}{a(t')}$$

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right]$$

or $= a^2(\eta)\left[-d\eta^2 + d\chi^2 + S_k^2(\chi)\,d\Omega^2\right]$

Specifically $K=0$; FRW is conformal to Minkowski.

- Hubble Flow : $(K=0)$

Physical distance : $\boxed{x^i_p = a(t)\cdot x^i}$

$\Rightarrow$ Physical Velocity : $v^i_p = \frac{dx^i_p}{dt} = a\frac{dx^i}{dt} + \frac{\dot{a}}{a} x^i \cdot a$

$$\boxed{v^i_p = H x^i_p + a\cdot\frac{dx^i}{dt}}$$

$V^i_{pec} = a\cdot\frac{dx^i}{dt}$ ($v_{peculiar}$)

$V^i_p = H x^i_p + V^i_{pec}$

Peculiar Velocity
velocity w.r.t. comoving observer

Hubbles flow
: describes velocity due to expansion of the universe.

Motion of Test Particles

$p^\mu = \frac{dx^\mu}{d\lambda}$ $\quad (\tau = m\lambda)$ for massive particle

Then $p^2 = p^\mu p_\mu = -(p^0)^2 + \underbrace{a^2 g_{ij}^{(3,0)k}\cdot p^i p^j}_{P^2}$

$= -m^2$ $\longrightarrow$ using the λ parameter.

$$p^2 = -(p^0)^2 + P^2 = -m^2$$

Geodesic Equation $\quad p^\mu D_\mu p^\nu = 0$

Equivalently use : $L = \frac{1}{2} g_{\mu\nu} x^{\mu\prime} x^{\nu\prime}$ $\quad ' = \frac{d}{d\lambda}$ $\quad \dot{} = \frac{d}{dt}$

$$L = \frac{1}{2}\left(-t'^2 + a^2 g_{ij}^{(3,0)k} x^{i\prime} x^{j\prime}\right)$$

t : $\frac{\partial L}{\partial t} = \left(\frac{\partial L}{\partial t'}\right)' \Rightarrow$ LHS $a^2\frac{\dot{a}}{a} g_{ij} x^{i\prime} x^{j\prime} = HP^2$

RHS : $-p^{0\prime}$

$\boxed{HP^2 = -(p^0)'}$ $\boxed{-p^0 p^{0\prime} + \mathbf{P}\cdot\mathbf{P}' = 0}$

$$-\dot{P}P = -\frac{PP'}{p^0} = -p^{0\prime} \Rightarrow \boxed{p^{0\prime} = \dot{P}P}$$

$$\Rightarrow \boxed{-\dot{P}P = HP^2} \qquad \boxed{\dot{P} = -\frac{\dot{a}}{a}H}$$

$$\boxed{\frac{\dot{P}}{P} = -\frac{\dot{a}}{a}} \Rightarrow \boxed{P \sim \frac{1}{a}}$$

$\Rightarrow$ True for both massless & massive particles

Motion of Galaxies $P \simeq m v_{pec}$ $\Rightarrow \boxed{v_{pec} \sim \frac{1}{a}}$

Hubble Drag: Motion gradually comes to rest w.r.t. comoving observer.

Random Object: $v_{pec} \simeq 200$ km/s.

Radiation (Motion of Light)
$P = p^0 = E \propto \omega$; $\boxed{\omega \propto \frac{1}{a}}$ cosmological redshift.

$\lambda \sim \frac{1}{\omega} \sim a$

Redshift Parameter $$Z = \frac{\lambda_0 - \lambda_{Emitted}}{\lambda_{Emitted}}$$; $\lambda_0 \neq \lambda$ observed.

$$\Rightarrow Z = \frac{a(t_0) - a(t_E)}{a(t_E)} \qquad \boxed{Z = \frac{a(t_0)}{a(t_E)} - 1}$$

For Nearby Galaxies

~~$a(t_{EM}) = a(t_0)[1 + t_{EM} - t_0)$~~

$$a(t_{EM}) = a(t_0)\left[1 + \underbrace{(t_{EM} - t_0)}_{\text{Treating it as small}} H(t_0) + \dots \infty\right]$$

$$Z = \frac{a(t_0)}{a(t_{EM})} - 1 \simeq (t_0 - t_{EM}) H(t_0)$$

in units $c = 1$
$\therefore$ This term is distance travelled light in time $(t_0 - t_{EM})$
so; it is actually physical distance d.

$\boxed{z = dH_0}$ **Hubble's Law**

Right Interpretation is the ~~cos~~ cosmological redshift.

Originally, Hubble thought of doppler shift. $z = \frac{v}{c}$; $v = dH_0$ (1912)

using Hubble law

ie; glaxies are running faster as ~~as far the~~ farther they are.

Lec 3: FRW Spacetime, Horizons, Friedmann Equations, Cosmic Inventory

HORIZONS

Consider radial photons. $ds^2 = 0$ ∴ $d\Omega^2 = 0$

for causal structure look at null geodesics.

$$d\eta = \pm d\chi$$

(± depending on whether comming in or going out)

$$\boxed{\frac{dt}{a} = \pm d\chi}$$

Conformal time η is useful, because everything travels at 45°.

$\boxed{\text{Observational Fact} \quad \dot{a} > 0}$

- $a = a_0 \cdot \left(\frac{t}{t_0}\right)^P \sim t^P$ $\quad P =$ constant > 0 (because $\dot{a} > 0$)

a_0: a today. True in a given EPOCH of domination.

- $t \in (0, \infty)$

zero time corresponds to Big Bang

$\ddot{a} \sim P \cdot (P-1)\, t^{P-2}$

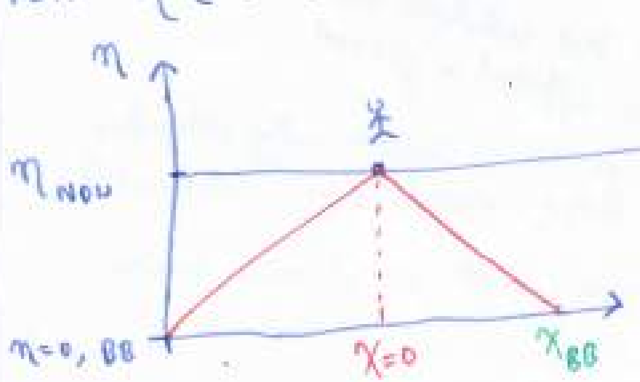
$\eta = \int \frac{dt}{a} \sim \int t^{-P}\, dt \sim \frac{1}{1-P}\, t^{1-P}$

~~ACCELERATING~~

DECELERATING $\ddot{a} < 0$

$P \in (0, 1)$

here $\eta \in (0, \infty)$



Could see everything in future.

Cannot see $\chi > \chi_{BB}$

∴ There is some horizon beyond which we cannot see

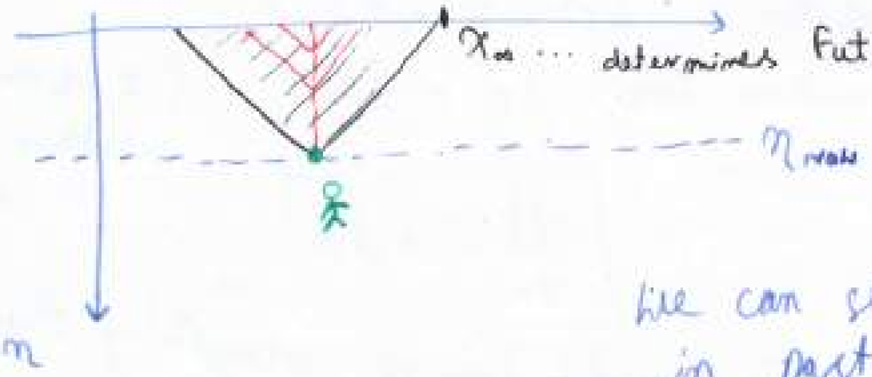
PAST PARTICLE HORIZON

(Although at later time we can see more)

Accelerating $\ddot{a} > 0$; $p > 1$

$\eta \in (-\infty, 0)$

$-\infty \equiv$ Big Bang ; $0 \equiv$ Infinite future.

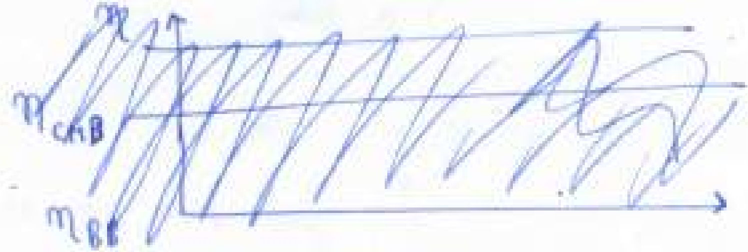


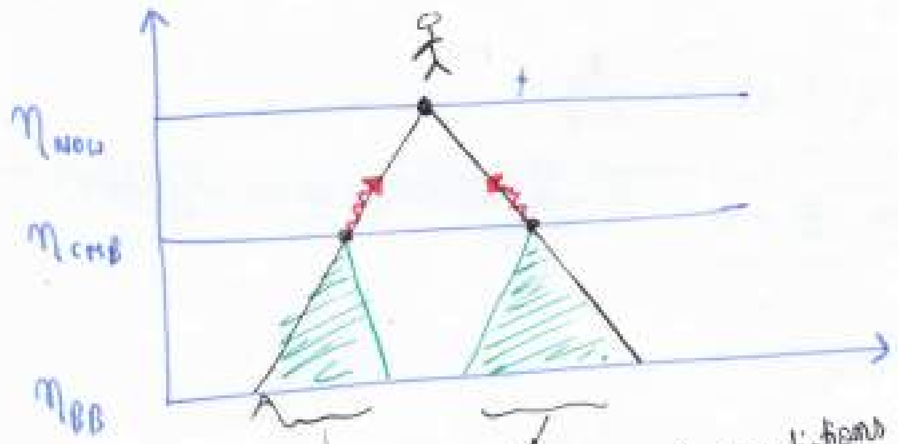
We can see everything in past.

HORIZON PUZZLE Back in 1970s.

ρ_m, ρ_r. $\Rightarrow \ddot{a} < 0$ all times.

matter radiation.





These two radiations should be different in general.

There is no reason why physics at these points are correlated.

But observation tell you that we have same temperature.

Horizon Problem Why physics is correlated?

Idea of Inflation comes in for this

Its an assumption, that just after Big Bang; universe expanded very fast ; and so $\ddot{a} < 0$ ~~toward~~ deceleration in matter & radiation was preceded by accelerating universe.

After Big Bang, for short period of time when universe expanding like crazy.

This will resolve the puzzle.

Dynamics of FRW

The most general $T_{\mu\nu}$ consistent with Homogeneity & Isotropy is that of perfect fluid.

$$\boxed{T_{\mu\nu} = P g_{\mu\nu} + (\rho + P) u_\mu u_\nu} \; .$$

where u_μ : 4-velocity of the fluid. (comoving observers)

P : Pressure.

ρ : Density.

2 independent Einstein Equations:

$$H^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2}$$

$$\dot{\rho} = -3H(\rho + P)$$

→ FRIEDMANN Equations

3 unknown a, ρ, P ; but two equations.

So cannot solve it.

... we need extra equation ... called Equation of State to solve it.

Critical Density $\rho_c(t) = \dfrac{3H^2(t)}{8\pi G_N}$

Then first Friedmann Equation reads

$$\boxed{\frac{k}{a^2} = \frac{8\pi G_N}{3}(\rho - \rho_c)}$$

Observation $\rho = \rho_c \cdot (1 \pm 0.01)$

Why ... FLATNESS PUZZLE (This is also solved by idea of Inflation)

Cosmic Inventory

To solve F.E. need to supplement "Equation of State".

$$\boxed{P = P(\rho)}$$

Typically $\boxed{P = w\rho}$ $w = \text{constant}$

for most of matter content in Universe.

2nd (FE) $\dot{\rho} = -3H(\rho + w\rho) = -3H\rho(1+w)$

$$\Rightarrow \dot{\rho} = -3\rho(1+w)\cdot\frac{\dot{a}}{a} \Rightarrow \frac{\dot{\rho}}{\rho} = -3(1+w)\cdot\frac{\dot{a}}{a}$$

$$\Rightarrow \boxed{\rho \propto a^{-3(1+w)}}$$

Typical Matter Content

Component	Example	Equation of State	Density
Matter	Galaxy, Dust, Dark Matter	$P \simeq 0,\ w = 0$	$\rho \sim a^{-3}$
Radiation	Photons, CMB, Gravitons, Neutrinos	$P = \frac{1}{3}\rho$; $w = 1/3$	$\rho \sim a^{-4}$
Cosmological Constant	Λ	$P = -\rho$; $w = -1$	ρ = constant
Curvature	K	$P = -\frac{1}{3}\rho$: $w = -\frac{1}{3}$	$\rho \sim \frac{1}{a^2}$

- K in principle can have some energy density in principle.
- Energy density of Λ does not dilute with expansion.

At given time, 1 component typically dominates; these are called EPOCHS
(ex Radiation domination, matter domination, etc.)

2 Remarks on Λ

- $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$
  where Λ is part of geometry.

but we can also write $G_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} + T_{\mu\nu}^{(\Lambda)}\right)$

where $8\pi G_N T_{\mu\nu}^{(\Lambda)} = -\Lambda g_{\mu\nu}$

→ in this form, we can view Λ as part of Energy momentum tensor.

compare it to Energy momentum tensor of perfect fluid ; so ; you have $g_{\mu\nu}$ part but no $U_\mu U_\nu$ part.

This means $(\rho + P) = 0 \Rightarrow \boxed{P = -\rho}$

$$\boxed{\rho_\Lambda = -P_\Lambda = \frac{\Lambda}{8\pi G_N}}$$

- Vacuum Fluctuations should contribute to Λ.

QFT calculation give (see Weinberg) $T_{(\text{vac})\,\mu\nu} = -\rho_{(\text{vac})}\, g_{\mu\nu}$

$$\boxed{\frac{\rho_{\text{vac}}}{\rho_{\Lambda\,\text{observed}}} \sim 10^{120}}$$ old Cosmological Constant Problem.

What is ~~...~~ cancelling larg contribution of vaccum fluctuation? No one knows the solution.

Physical derivation of $\rho = \rho(a)$ relation.

piece of FRW spacetime.

$dE = dQ - PdV$
(first law of thermodynamics)

flux of heat going through the boundary has to be zero, because every piece is same ie; $dQ = 0$.

$$d(\rho V) = 0 - P\,dV$$
$$= 0 - \rho w\,dV$$

$$\Rightarrow \quad d\rho\,V + \rho\,dV = -\rho w\,dV$$

$$\Rightarrow \boxed{\frac{d\rho}{\rho} = -(1+w)\cdot\frac{dV}{V}} \quad \rho \propto V^{-(1+w)}$$

but $V \sim a^3$ $\Rightarrow$ $\boxed{\rho \propto a^{-3(1+w)}}$

can be derived without using Friedmann Equation

Lec 4: FRW Spacetime, dynamics of scale factor, local thermal equilibrium

The second step is to solve the 2nd equation for $a = a(t) \equiv \underline{a(\eta)}$

→ "Driven Harmonic Oscillator equation"

i.e.;
$$\boxed{a(\eta) = \left[\frac{8\pi G}{3}\rho_0\, a_0^{3(1+w)}\right]^{\frac{\alpha}{2}} \cdot S_K^{\alpha}(\eta/\alpha) \qquad \alpha = \frac{2}{1+3w}}$$

; α depends on matter component

when $P = w\rho$

Simple component universe

$$\boxed{t \sim \int a(\eta)\, d\eta}$$

$\boxed{K=0}$

Component	w	Density	$a(\eta)$	$a(t)$
Matter	0	$\propto 1/a^3$	$\propto \eta^2$	$\propto t^{2/3}$
Radiation	1/3	$\propto 1/a^4$	$\propto \eta$	$\propto t^{1/2}$
Λ	-1	a^0	$\propto \frac{-1}{\eta H}$	$\propto e^{Ht}$

constant

~~exponentially~~ exponential expansion in cosmological constant dominated universe.

Current Cosmological Model

- $\underbrace{\Lambda}_{\text{Dark Energy (i.e; cosmological constant)}}\underbrace{CDM}_{\text{Cold Dark Matter}}$ model

* Relativistic Dark Matter is ruled out by observation.

- $$\Omega_I(t) = \frac{\rho_I(t)}{\rho_c(t)}$$

~~I stands for particular com~~

I stands for particular component.

$I \in \{\text{Matter}, \text{Radiation}, \Lambda, K\}$

Observation

$\Omega_m = 0.32$

$\Omega_{DM} \simeq 0.27$; $\Omega_B \simeq 0.05$

Ω_{DM} → Dark Matter (cold) i.e. non-relativistic ↪ we dont know about it

Ω_B ↪ Baryons.. normal matter ... this is what we know : and these are taken in account in Standard Model say

$\Omega_\Lambda \simeq 0.68$ (we dont know what it is)

$(\Omega_k) \leq 0.01$

$$\rho_c = \frac{3H^2(t)}{8\pi G_N} \longrightarrow (\rho_c)_0 \simeq 10^{-26}\ kg/m^3$$

(ρ_c at present)

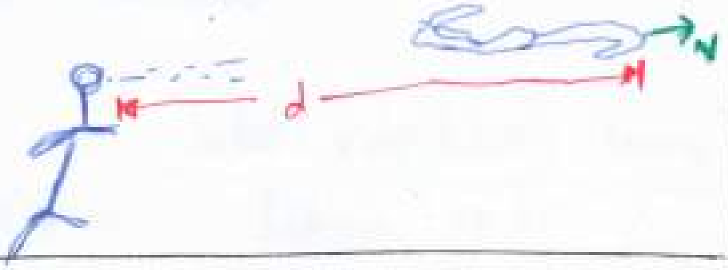
$$\left(\frac{10 \times \text{Hydrogen atom}}{m^3}\right)$$

$H_0 = 100\,h\ km/s/MpC$

(not "0" corresponds to value at todays time)

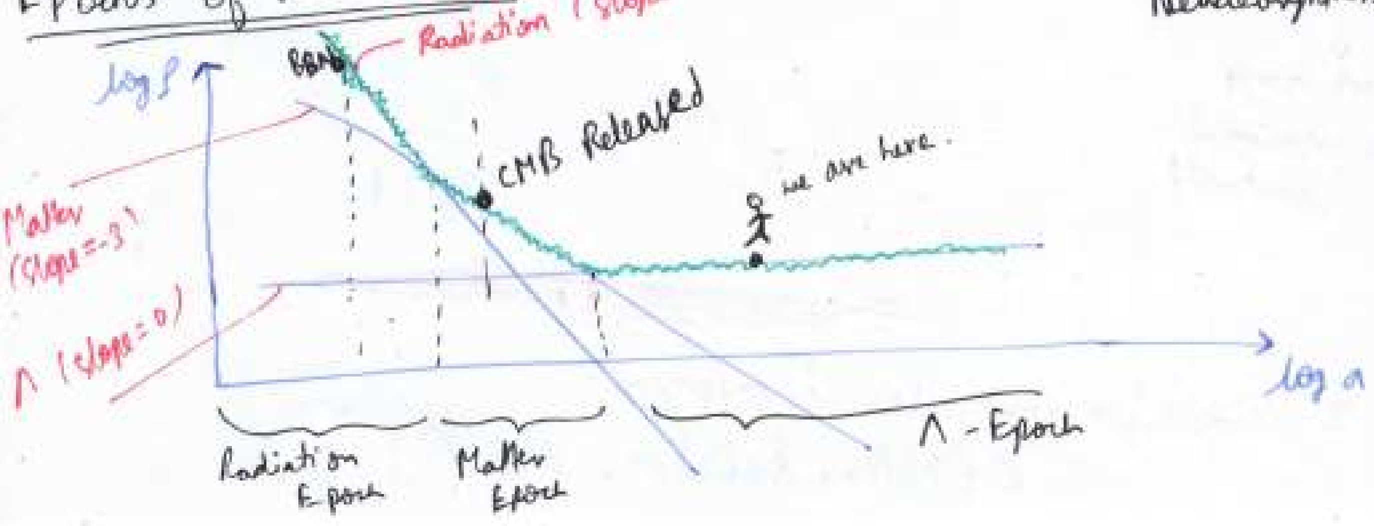
observation tells ; $h = 0.67 \pm 0.01$

$\boxed{V = H_0 d}$ for Nearby galaxies



Epochs of Domination

BBN ⇒ Big Bang Neucleosynthesis.



with inflation.

Anthropic Principle "The universe is as is, because we are here to observe it."

II) MATTER

(a) Thermodynamics in Expanding Universe

⇒ stands for "no."

• Distribution Function

$$\# \text{particles}_i = dN_i = f_i(\vec{x}, \vec{p}, t) \cdot \frac{d^3x\, d^3p}{(2\pi\hbar)^3}$$

no. of particles of a given specie in a given infinetisimal volume of phase space

Distribution function.

we will set $\hbar = 1$.

Once we find f_i, we can calculate particle density n_i

$$n_i = \int \frac{d^3p}{(2\pi)^3} f_i(t, \vec{x}, \vec{p})$$

Particle Density

: integrate out wrt 3-momenta... not doing relativistic momentum.

$$\rho_i = \int \frac{d^3p}{(2\pi)^3} E(\vec{x}, \vec{p}) f_i(t, \vec{x}, \vec{p})$$

Energy Density

Pressure $P_i = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{\vec{p}^{\,2}}{3E} \cdot f_i\,(t, \vec{x}, \vec{p})$

- How to find f_i? f_i governed by the ~~Boltzmann~~ Boltzmann equation.

$$\frac{df_i}{dt} = \frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial x^j} \cdot \dot{x}_j + \frac{\partial f_i}{\partial p_j} \cdot \dot{p}_j = COL_i(f_j)$$

→ This would be zero; if it was governed by Hamiltonian dynamics

Collision term .... This makes the system deviate from Hamiltonian system.

$$\boxed{\frac{df_i}{dt} = \frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial x^j} \dot{x}_j + \frac{\partial f_i}{\partial p_j} \dot{p}_j = COL_i(f_j)}$$

→ Most general form of Boltzman equation

Hides all dirty physics

Instead, let us consider, ~~LOCAL THERM~~ LOCAL THERMAL EQUILIBRIUM (has Maximal Entropy

$$\frac{1}{\Gamma} = \text{COLLISION TIME} << \text{EXPANSION TIME OF UNIVERSE}$$

(can be estimated by Hubbles constant $\frac{1}{H} = t_H$ .....)

$\Gamma \Rightarrow$ Reaction Rate

If this is true; then we can sort of have equilibrium.

i.e; $\frac{1}{\Gamma} << \frac{1}{H}$ or $\boxed{H << \Gamma}$

Then we can ~~use~~ use equilibrium distribution functions;

Then $$f_i = \frac{1}{e^{\frac{E-\mu_i}{T}} \mp 1}$$ $\ominus \Rightarrow$ Bosons
$\oplus \Rightarrow$ Fermions

μ_i → chemical potential

→ we use Equilibrium distribution function.

We will use ; relativistic dispersion relation: $E_i = \sqrt{\vec{p}^2 + m_i^2}$

We have : $$d^3p = d\Omega \cdot |p|^2 \, d|p|$$
$$= d\Omega \, \frac{(E^2 - m_i^2)}{\sqrt{E^2 - m_i^2}} \cdot E\,dE$$

$$\boxed{d^3p = d\Omega \, dE \cdot E \cdot \sqrt{E^2 - m_i^2}}$$ integration measure using relativistic dispersion relation.

$$n_i = \frac{1}{2\pi^2} \int_{m_i}^{\infty} \frac{\sqrt{E^2 - m_i^2} \cdot E}{e^{\frac{E-\mu_i}{T}} \mp 1} \cdot dE$$ Exact Result.

① Relativistic Limit. $T >> m_i, \mu_i$

Then $$n_i = \frac{1}{2\pi} \int_0^{\infty} \frac{E^2 \cdot dE}{e^{E/T} \mp 1}$$

$\frac{E}{T} = x$; $dE = T\,dx$ → $n_i = \frac{T^3}{2\pi} \int_0^{\infty} \frac{x^2\,dx}{e^x \mp 1}$

so; $n_i \propto T^3$

$$\int_0^{\infty} \frac{x^2\,dx}{e^x \mp 1} \quad \begin{cases} 2\,\zeta(3) & \text{for Bosons.} \\ \frac{3}{2}\,\zeta(3) & \text{for Fermions.} \end{cases}$$

$T_F = T_B$.
(fermion temperature = Boson temperature) then $n_F = \frac{3}{4} n_B$

$\boxed{n_i \propto T_i^3}$ density $n_i \propto \frac{1}{L^3}$ $L^3 \propto$ Volume

now; we are working in units $[L] = M^{-1}$

$\Rightarrow [E] \sim \frac{1}{L}$

so T_i is everything which behaves like energy

$\Rightarrow n_i \propto T^3$

$[\rho_i] \propto \frac{[E]}{[V]} \Rightarrow$ $\boxed{\rho_i \propto T_i^4}$

Same for pressure $\boxed{P_i \propto T_i^4}$

by dimensional analysis.

$\boxed{w_i = \frac{P_i}{\rho_i} = \frac{1}{3}}$ This is stat. mech prove for radiation

Lec 5: Thermodynamics in Expanding Universe : adiabatic expansion, freeze out, thermal history.

Relativistic Limit | $T \gg m, \mu$

$n \propto T^3$, $\rho \propto T^4$, $P \propto T^4$; $w = P/\rho = \frac{1}{3}$

Non-Relativistic — $T << m$, $\mu < m$

$$n = \frac{1}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2}\, E\cdot dE}{e^{\frac{E-\mu}{T}}}$$

remove ± 1 from denominator
$m\uparrow \Rightarrow E\uparrow \Rightarrow \frac{E}{T}$ is huge ..

$$= \left|\begin{array}{l} u = \frac{E-m}{T} \\ E = uT + m \end{array}\right|$$

$$= \frac{1}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2}\, E\, dE}{e^{\frac{E-\mu+m-m}{T}}}$$

$$= \frac{1}{2\pi^2}\, e^{\frac{\mu-m}{T}}\int_0^\infty \frac{\sqrt{u^2T^2+m^2+2uTm-m^2}\cdot(uT+m)\,T\,du}{e^u}$$

$$= \frac{e^{\frac{\mu-m}{T}}}{2\pi^2}\int_0^\infty \frac{\sqrt{u^2T^2+2uTm}}{e^u}\cdot(uT+m)\cdot T\,du$$

the lowest expansion in u in numerator for estimation (because you already have e^u in denominator)

$$= \frac{1}{2\pi^2}\cdot(mT)^{3/2}\, e^{\frac{\mu-m}{T}}\int_0^\infty \frac{\sqrt{2u}}{e^u}\cdot du$$

$$\boxed{n = \frac{1}{2\pi^2}(m\cdot T)^{3/2}\cdot e^{\frac{\mu-m}{T}}\cdot\int_0^\infty \frac{\sqrt{2u}}{e^u}\cdot du}$$

$E-m$ is very small ..... haha.. because T is small
$\therefore$ although in the integration u is going to infinity.

$$\int_0^\infty \frac{\sqrt{2u}}{e^u}\,du = \sqrt{\frac{\pi}{2}}$$

$$\boxed{n \simeq \left(\frac{mT}{2\pi}\right)^{3/2}\cdot e^{\frac{\mu-m}{T}}}$$

This says some thing about childhood of Thermodynamics.
Thermodynamics hates massive particle. If a massive particle is in equilibrium with other things; it's ~~~~ abundance is exponentially suppressed.

$\boxed{\rho \simeq m n}$ $\boxed{P \simeq T \cdot n}$ $w = \frac{P}{\rho} \simeq \frac{T}{m}$

in the limit $T << m \Rightarrow \boxed{w \simeq 0}$

★ <u>Universe Expands Adiabatically</u> $S = s V = \text{constant}$

Total entropy, entropy density, Volume of universe

Can show that whatever process we have, Entropy production

$$\Delta S << S_{Total} \simeq S_{Radiation} + \underline{S_{Black\ Hole}}$$

This decouples, so not much interesting.

10^x (more photons than Baryons)

$\boxed{s \propto \frac{1}{a^3}}$ since density so $\propto \frac{1}{\text{Volume}} = \frac{1}{(\text{length})^3}$

or $\boxed{s \propto T^3}$ $\Rightarrow$ True only for relativistic particles.

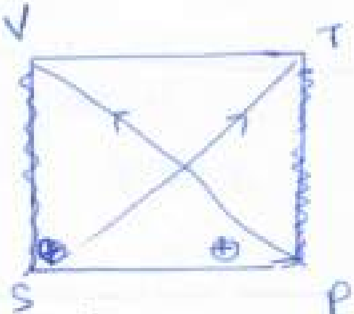
Bath of Relativistic Particles

$\rho = \rho(T)$, $P = P(T)$, $s = s(T)$

Homework 1] $\frac{\partial P}{\partial T} = \frac{\rho + P}{T} = \left(\frac{\partial S}{\partial V}\right)_P$

$= \left(\frac{\partial S}{\partial V}\right)_T$

(because P is function of T)



Magic Square

volume is independent of temperature

$S = sV \Rightarrow \left(\frac{\partial S}{\partial V}\right)_T = s$

so: Temperature dependence is there in s (ie; in Entropy Density)

$$\boxed{s = \frac{\rho + P}{T}}$$

$$s_B = \frac{2\pi^2}{45} T^3$$

for Bosons

$$s_F = \frac{7}{8} s_B$$

for Fermions

can also be derived using First Law of Thermodynamics.

Bath may include various Relativistic Species.

Let $T \equiv$ Temperature of Photons.

And $T_i \equiv$ Temperature of other species

(Everything is interacting together; relativistic species will have same temperature. That's not always the case. It may happen that some species already decoupled from the rest: There is no reaction coupling term to other species ⇒ So, they can have have different temperatures.

g_{*s} ⇒ takes into account for different species.

ie;

$$\boxed{s = g_{*s} \cdot \frac{2\pi^3}{45} T^3}$$

$$g_{*s} = \sum_{\text{Bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \sum_{\text{Fermions}} \frac{7}{8} g_i \cdot \left(\frac{T_i}{T}\right)^3$$

g_i → corresponds to how many ~~particles~~ d.o.f are there of that species. : ex for photons $g_i = 2$

g_i ⇒ corresponds to degree of freedom of i^{th} species

e.g. for graviton; $g_{graviton} = 2$

$$\boxed{\rho = g_* \cdot \frac{\pi^2}{30} T^4}$$

where;
$$g_* = \sum_{Bosons} g_i \cdot \left(\frac{T_i}{T}\right)^4 + \sum_{Fermions} \frac{7}{8} g_i \cdot \left(\frac{T_i}{T}\right)^4$$

Decoupling & Freeze Out

It is the departure from Equilibrium that makes life interesting?

if $\Gamma < H$ ... No longer we have equilibrium.

⇒ Decoupling (Freeze out)
(numbers freeze out)

If the massive particles are with the whole system; then it will eventually go to equilibrium; and we know at equilibrium massive particles are exponentially suppressed.

So; when we dont have equilibrium; then the massive particles can decouple from rest of the system & after that their no. remains constant ... we say that their number freeze out.

Example] Cold Dark Matter .... assume it corresponds to some particle X.

- For $T \gg m_X$; then the particles are relativistic & its in equilibrium.

- For $T < m_X$; X no longer relativistic.

but still χ is interacting with other species at thermal equilibrium; it is still exponentially suppressed.

- Assume not to destroy χ we have some anti-$\bar{\chi}$ ie $\bar{\chi}$

consider the reaction $\chi + \bar{\chi} \longrightarrow 2\gamma$

~~As long~~ As long as reaction is at equilibrium; the above reaction is going both ways.; we would be popping out χ & $\bar{\chi}$ from the vaccum.

But this is no longer true; The reaction actually goes one way. We are really trying to destroy as much χ as we ~~want~~ want to exponentially supress χ.

ie; we are producing photons, but not the other way round.

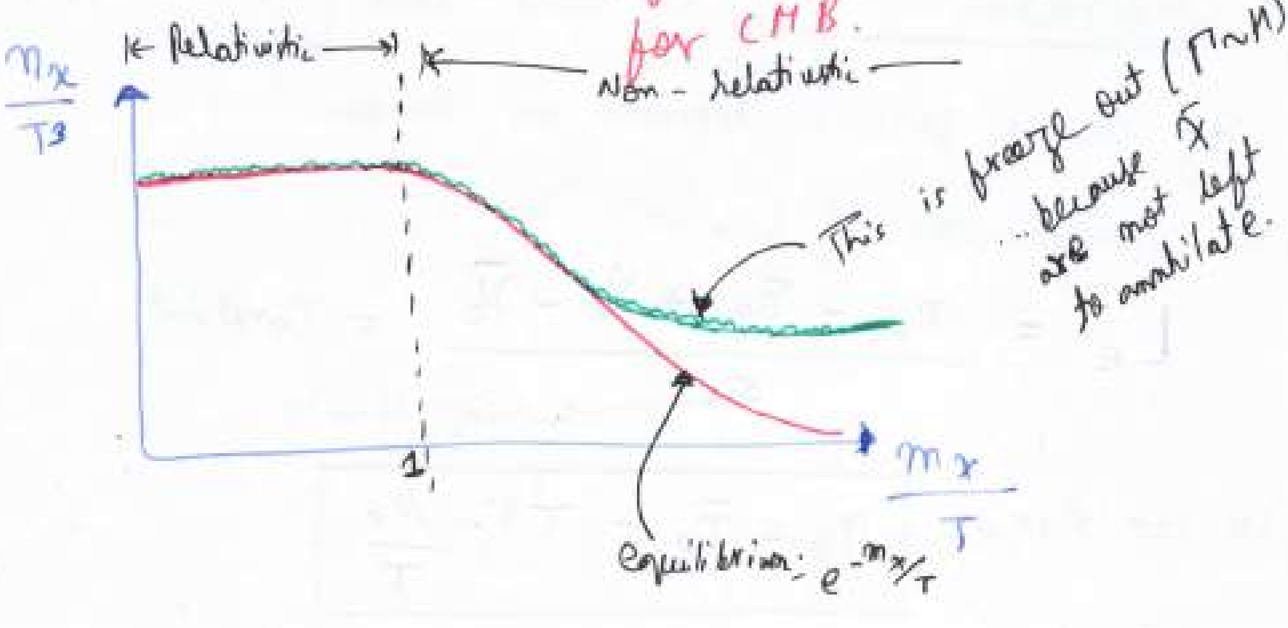
- When $\Gamma < H$, .. χ cannot find its anti-partner $\bar{\chi}$ to annhilate with .. and the reaction stops working.

ie; now; no. of χ is conserved. after this.

$$N_\chi = \text{constant}$$

or say $\quad n_\chi \propto \dfrac{1}{a^3} \propto T_{CMB}^3 \Big]$ This is Freeze out

measuring in the temperature of CMB. .. we know $T \propto \frac{1}{a}$ for CMB.



If there would not have been departure from equilibrium; then dark matter would have been exponentially suppress.

... but fortunately they departure from equilibrium; & some non-zero constant dark matter is left at last.

★ 3 Remarks on Chemical Potentials

① In QFT, particles can pop out of vacuum; unless there is a conservation law protecting particle number;

($\boxed{\mu = 0}$ .. Photons ($dE = TdS + \mu dN$)
so it does not cost any energy to create photons)

② Equilibrium process: $A + B \rightleftharpoons C + D$

$$\mu_A + \mu_B = \mu_C + \mu_D$$

ex: $e^- + e^+ \rightleftharpoons 2\gamma$ conclude $\boxed{\mu_{e^-} = -\mu_{e^+}}$ because $\mu_\gamma = 0$

$$n_A A + n_B B \rightleftharpoons n_C C + n_D D$$

$$n_A \cdot \mu_A + n_B \mu_B = n_C \cdot \mu_C + n_D \mu_D$$

③ Conservation Laws: Can find how $\boxed{\mu = \mu(T)}$

How chemical potential depends on temperature.

ex) Conservation of Lepton Number

$$L_e = \frac{n_e - \bar{n}_e + \nu_e - \bar{\nu}_e}{s} = \text{Constant}$$

s ← Entropy density.

We can show; $\boxed{n_e - \bar{n}_e = T^3 \cdot \frac{\mu_e}{T}}$

So ; we could conclude
$$\frac{\mu_e + \mu_{\nu_e}}{T} = \text{constant.}$$

(b) Brief Thermal History of Universe

$$a \longleftrightarrow T_{CMB} \longleftrightarrow t \longleftrightarrow z$$

$$\boxed{\text{Radiation Era : } T\,[\text{in MeV}] = \frac{O(1)}{\sqrt{t\,(\text{in sec})}}}$$

t (s)

- 10^{16} — Galaxies... formed by Gravitational Instability.
- 10^{12} — CMB Released ($e^- + p^+ \rightarrow H$)
- 10^{11} — Radiation-Matter Equality. (important for structure formation)
- 3 minute (0.1 MeV) — Bing Bang Nucleosynthesis (He, Li ... formed by fusing Hydrogen)
- 6 second — e^-e^+ Annhilation ($e^- + e^+ \rightarrow 2\gamma$ ... photons got energy out of this ... thats why CMB was heated up by electron & positron annhilation. thats why $T_\gamma > T_\nu$)
  - ↳ Prediction of Hot Big Bang Theory.
- 1 second — Neutrino Decoupling ($\frac{\text{Neutron}}{\text{Proton}}$ ratio freezes out)
- Dark Matter (?)

10^{-5} s (150 MeV) — QCD Phase Transition (Quarks are now confined)

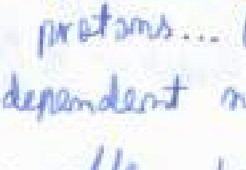
10^{-10} s (100 GeV) — Electro Weak Phase Transition.. (Higgs mechanism & stuffs like that

10^{-14} s — 10 TeV ... upto here we understand ~~physics~~ physics well.... supported by experiment ... say LHC.

?
- Baryogenesis
- $\eta = \dfrac{n_{Baryon}}{n_\gamma} \simeq 10^{-9}$

  (ie; $10^9\ \bar{p}$, we have $10^9 + 1$ p)
- Topological Defects (formation of Cosmic Strings)
- Inflation

Plank length


Big Bang

In neutrino coupled theory; ie; there can be reaction changing neutrons to protons... Once the Neutrino decouple; there are then independent neutrons & independent protons

ie; ~~then~~ after decoupling of Neutrino; Neutron proton ratio freezes out.

We know proton is stable and it can live forever.
Neutrons are very unstable (live around ~~...~~ 10 minutes)

At 1 second neutrons are seperate from protons... protons than live happily ever after. But Neutrons start decaying like crazy.

↳ So, you have only few minutes to capture his neutrons into heavier elements.

↳ because if you dont capture them in few minutes... they will simply decay & you will never see any electron again.

⟶ Thats why Big Bang Neucleosynthes have to happen very soon .... because after that you could not have any neutrons.

— Shoaib Akhtar

Brief Thermal History of Universe (A Summary)

— Shoaib Akhtar.

$a \leftrightarrow T_{CMB} \longleftrightarrow t \longleftrightarrow z$

Radiation Era : $T\,[\text{in MeV}] = \dfrac{O(1)}{\sqrt{t\ (\text{in sec})}}$

t (in second)

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- 1 second — Neutrino Decoupling ($\frac{\text{Neutron}}{\text{Proton}}$ ratio freezes out)
- Dark Matter Decoupling ?
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- 10^{-14} s — 10 TeV ... upto here we understand physics well.. supported by experiment... say LHC.

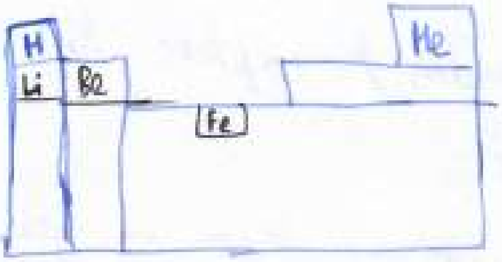
?

- Baryogenesis • $\eta = \dfrac{n_{Baryon}}{n_\gamma} \simeq 10^{-9}$
- Topological Defects (formation of Cosmic Strings)
- Inflation.

Planck scale

BIG BANG.

Lec 6: Big Bang Nucleosynthesis (BBN)



H : 75% Hydrogen
He : 25% Helium
(by mass, or number of Baryons)

~0% of everything else.

When and where produced?

He ≡ Helium.

★ 2 Possible theories?

$\boxed{\text{Theory 1}}$ He Fused in Stars.

$\boxed{\text{Theory 2}}$ He Fused in Big Bang Nucleosynthesis.

- Fe the most thermodynamically stable element.
  All things upto Fe can be fused in ~~stars~~ massive stars. ... which is something called Thermodynamic Making.

- Heavier Elements formed in Schocks.
  Example of Shock: Supernovae Explosion,
  Neutron Star Collission.

Recently we observed Gravitational Waves, γ ray burst, and saw production of gold, Platinum, Lead, etc. from Neutron Star Collission.

→ Start of the era of multi-messenger astronomy.

~~★ Cannot~~
★ Cannot explain abundance of light elements such as He, Li, Be ...

For example; consider ~~He~~ He

... ~~φ~~ it was predicted ~~to be~~ to be less than 0.5% (which is not in agreement with observed 25% abundance)

^4He

... Binding energy 28 MeV

or $7\,MeV = \frac{28}{4}$ i.e. 7 MeV per Baryon

ie $\simeq 10^{-5}$ ERG/BARYON

Assume that 25% of Baryons fused to Helium in stars during history of universe for $t \simeq 10^{10}$ years ~~(3)~~
(Universe age is 10^{10} years) $\simeq (3 \times 10^{17}\,s)$

~~Assume~~ Assume fusing these Helium in stars for whole life time of ~~the universe~~ universe; and lets assume that star are shining light by going energy from fusion.

Then luminosity of star L is

$$L = 0.25 \times N_b \times \frac{10^{-5}\,ERG}{3 \times 10^{17}\,s} \quad \left(\frac{\text{energy}}{\text{Time}}\right)$$

Total no. of baryons.

mass of proton.

M_b ~~total~~

$$M_b = N_b \cdot m_p$$

Total mass of Baryon

Luminosity to mass ratio ~~(it can)~~ (can be easily measured)

$\frac{L}{M_b}$ we predict; $\frac{L}{M_b} \simeq 0.25 \times \frac{10^{-5}\,ERG}{m_p \times 3 \times 10^{17}\,s}$

$$\frac{L}{M_b} \simeq 2.5 \frac{L_\odot}{M_\odot}$$

$L_\odot \rightarrow$ luminosity of Sun
$M_\odot \rightarrow$ Mass of Sun.

→ so; if you fuse 25% of Baryons by stars; then $\frac{L}{M_b}$ will be 2.5 ... this is very bright

→ so; we will have bright sky ... really very very bright sky & we would not see anything.

→ But we know sky is dark.
... So this is ~~to~~ really ~~top~~ high luminosity.

$$\boxed{\text{Observed} \quad \frac{L}{M_b} \leq 0.05 \frac{L_\odot}{M_\odot}}$$

→ our sky is actually quite dark
(50 times smaller than predicted)

Could fuse at most 0.5% of He in stars to match the observed data.

BBN gives the right Ball Park "Back of the envelope calculation"

Cosmology : The Expanding (Newtonian) Universe

Lec-1

— Shoaib Akhtar

Modern subject of cosmology is very new.
Thinking universe as a physical system; as a system to study mathematically with set of physical principles and equations.
We will study Universe as a system.

Observations —
(i) First observation: (may not turn out absolutely true; because it is not) is Universe is Isotropic.
(it looks like it is approximately true)

* on a whole, averaging over patches in the sky; and looking out far enough so that you get away from the immediate foreground of our own galaxy: The Universe looks pretty much the same in every direction... this is called Isotropic.

If Universe is isotropic around us; then we can bet that it is also pretty close to being Homogeneous.

↳ means it is same in every place.

*note!! here it will not matter if we call galaxy or we just call them particles... they are effectively mass points distributed throughout space. we will say particle (but we literarily mean galaxy unless explicitly stated)

*Within what we can see with telescope (Observable Universe) there are about 10^{11} galaxies.

↳ each galaxy about 10^{11} stars.

$\approx 10^{22}$ ~~stars~~ stars.

suppose you have a galaxy distribution.

so; if it looks pretty much same in every direction ⇒ then we maintain that it not only must be same in every direction but it must be same from place to place.

What it would mean for not to be same from place to place.

If it is isotropic: the only way it could not be homogeneous is if it somehow forms shell (or say spherical shells...) of sort.

so; we have geometry of some sort of shell like structure. (looks isotropic from A)

→ so; if this was the case: and if you went to some place else .. say B .. and you looked around ⇒ it would not look isotropic from there.

→ so if we dont believe that we happen to be at the centre of universe from where everything appears to be rotationally symmetric about us

→ Then we have to believe the Universe is pretty much same everywhere and that it is Homogeneous.

⇒ so; As far as we can see, the space is uniformly filled on the average with particles.

This is called Cosmological Principle

→ This is true because it has been observed to be true to some degree of approximation

Actually it is not homogeneous, but the particles tend to clusters; which on some big enough scale like a billion light years roughly (may be less); if you average over that much it looks Homogeneous.

* You can think Universe, as a uniform gas (it is ~~gas~~ ~~to~~ gas of particles (here you know that we mean actually galaxies))

↳ the gas is interacting : each particle is interacting with other particles.

∴ Galaxy on whole are not electrically charged (They are electrically neutral)

→ but Galaxies are not gravitationally neutral (They interact through Newtonian gravity ; and this is the only important force on big enough scales)

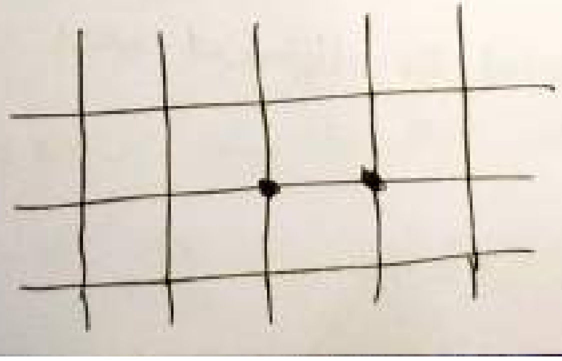
- On big enough scales where matter tends to be electrically neutral : the ~~important and~~ dominating force is gravity

- Natural thing to guess using Homogenity is that Universe is Static (Gravitational forces on a particle cancels out roughly due to uniform mass distribution) ; nothing has any net force on it.
  (But this is wrong)

- It is true that expanding universe was not understood untill Einstein created the General Theory of Relativity.
- but Newtonian gravity can also explain Expanding Universe.

We introduce some sets of coordinates.

Take space ; and introduce fictitious grid of coordinates



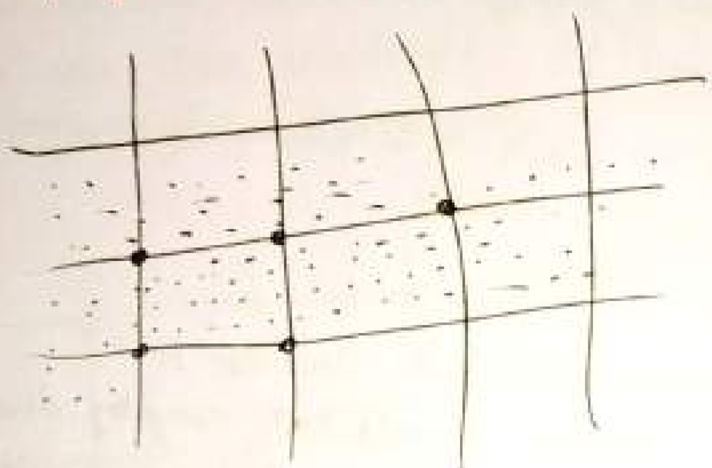
* what should we take for ~~..~~ the distance between neighbouring lattice points? ... we can take it to be like 1m, 10^3 m, 10^6 m, etc.

↳ but; smarter thing to do, them to just fix the distance between the points

We can imagine that grid points are chosen so that they always pass through some galaxies.

⇒ i.e; Galaxies in the universe provide a gride.

lets choose coordinates; so that the galaxies is sort of frozen in the grid.



Since; galaxies are nice & uniform .. no matter what happens, a given galaxy will always be at a point on the grid.

↳ if the Universe ~~universe~~ indeed expands or contracts (i.e; if galaxies are moving relative to each other) ⇒ Then the grid moves with them.

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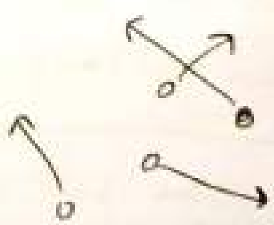
* It is not obvious that you can do that

... if the galaxies where such that they ~~..~~ were moving in different directions (random motion)



↳ Then there will be no way to fix coordinates by catching them to the galaxies.... because even at a point the different ones would be moving in different way

This is not what we observe.

What we observe is that galaxies are moving very coherently ; exactly as if they were embedded in a grid (with the grid perhaps expanding or contracting) ... the galaxies are frozen in the grid.

$\rightsquigarrow$ Any motion that takes place is because the grid is either expanding in size or contracting in size.

↳ This is an observation about the relative motion of nearby galaxies. ... nearby galaxies are moving in nice coherent way.

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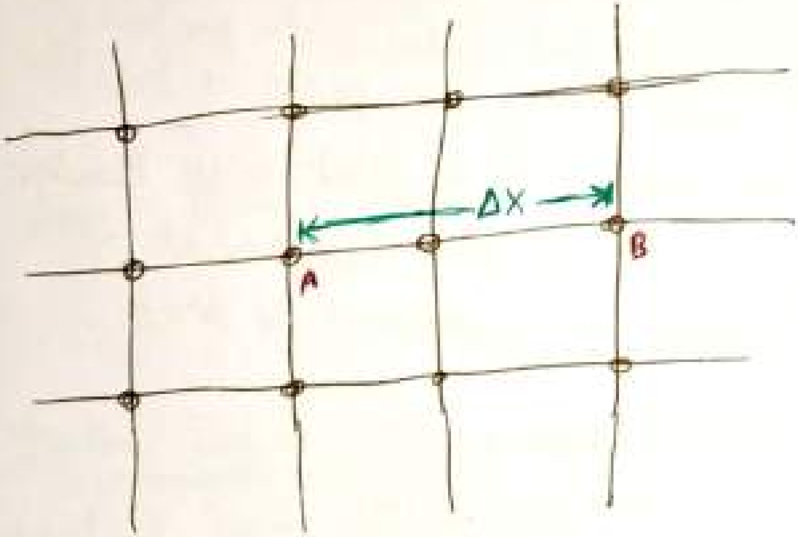
we can chose coordinates X, Y, Z ... by X, Y, Z are not measured in length (because length of grid cell may change with time).



we are going to postulate ; that the actual distance between the points (D) .. seperated by ΔX coordinate distance is proportional to ΔX .. times a scale parameter a.

$\therefore$ $D = a \cdot \Delta X$; The scale parameter "a" may or may not be just a constant.

↳ "a" may be time dependent ... lets allow that ; $a = a(t)$

$\Rightarrow$ $D_{AB} = a(t) \cdot \Delta X_{AB}$

$$D_{ab} = a(t)\sqrt{(\Delta x_{ab})^2 + (\Delta y_{ab})^2 + (\Delta z_{ab})^2}$$

∴ working for 1-D case; $\underbrace{D_{ab}}_{\text{Distance between a and b.}} = a(t)\,\Delta X_{ab}$

$$\underbrace{V_{ab}}_{\text{velocity between a \& b}} = \frac{d}{dt}\left(a(t)\,\Delta X_{ab}\right)$$

∵ since a & b are fixed in the ~~grid~~ grid ⇒ so ΔX_{ab} is not changing.

$$\Rightarrow \quad V_{ab} = \dot{a}\cdot\Delta X_{ab}$$

$$\frac{V_{ab}}{D_{ab}} = \frac{\dot{a}}{a} := H$$

→ it is called Hubble constant.

(it is constant w.r.t. relative position in space at a given point of time)

It should better be called Hubble Function:

$H(t)$ ⇒ It is independent of position & dependent on time.

$$\therefore \quad H(t) = \frac{\dot{a}(t)}{a(t)}$$

$$\Rightarrow \boxed{V = HD}$$

Hubble Law

Take a region of size $\Delta x\,\Delta y\,\Delta z$. (A region which is big enough so that we can average over the small scale structure)

How much mass is in there?

Amount of mass; ν per unit volume of grid (volume measured in grid units...)

$$\underbrace{M}_{\text{mass}} = \nu\cdot\Delta x\,\Delta y\,\Delta z$$

∴ Volume of region "V_{upl}". ; volume of same cell is:

$$\left.\begin{aligned} V &= (a\Delta x)(a\Delta y)(a\Delta z) \\ V &= a^3\,\Delta x\,\Delta y\,\Delta z \end{aligned}\right\} \text{Physical Volume.}$$

Physical density of mass. (ρ) : Density.

$$\boxed{\rho = \frac{\nu}{a^3}}$$

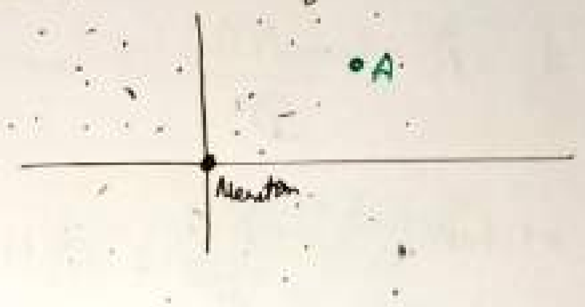
★ Amount of mass in each cell $\Delta x \Delta y \Delta z$ remains fixed; because galaxies move with the grid.

---

lets ; Newton choose to be at centre of grid.

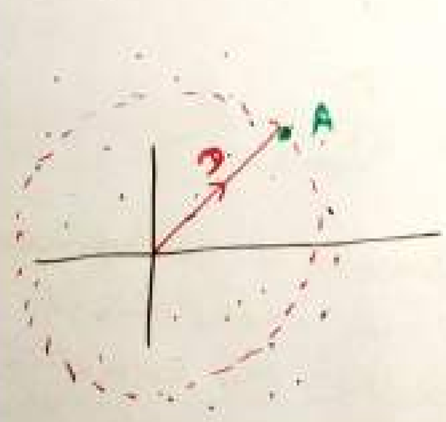
↳ and let he say that he is not moving in the grid (because he actually moves with grid)

(let him believe that he is at centre of universe for mathematical purpose... since Universe is homogeneous ⇒ any point can be taken to be centre.



~~Assum~~ ~~Assume~~ Galaxy A moves under the assumption of Newtons equation.... it says that everything gravitates with everything else.

Newtons Theorem : Given that everything is isotropic and you want to know Gravitational force in a frame of reference.
↳ then draw a sphere. ~~Cent~~ centred at origin & then take all the mass within that sphere; and pretend that it is sitting at the origin; and ignore the outside masses



$$D = \sqrt{x^2 + y^2 + z^2} \cdot a$$

$$\Rightarrow D = a(t) \cdot R$$

; w: $R = \sqrt{x^2 + y^2 + z^2}$

P38

Velocity: $V = \dot{a}(t) R$

Acceleration: $A = \ddot{a}(t) R$

R is not changing with time; because galaxy is at fixed point in this expanding lattice.

$F = \frac{-mM}{D^2} \cdot G$ — m → galaxy A mass; M → all the mass inside sphere

↳ minus sign indicates that the force is attractive... it is pulling in.

$F = \frac{-mM}{D^2} G$ ⟶ acceleration of gravity is $\frac{-MG}{D^2}$

$$\Rightarrow A = \frac{-MG}{D^2}$$

so; we have; $\frac{-MG}{D^2} = \ddot{a}(t) R$

$$\Rightarrow \frac{-MG}{a^2 R} = \ddot{a} R$$

$$\Rightarrow \ddot{a} = \frac{-MG}{a^2 R^3} \Rightarrow \boxed{\frac{\ddot{a}}{a} = \frac{-MG}{a^3 R^3}}$$ Newtons Equation.

Volume $= \frac{4}{3}\pi D^3 = \frac{4}{3}\pi a^3 R^3$

↳ Actual Physical Volume. (V_{vol}) $\Rightarrow \frac{3V}{4\pi} = a^3 R^3$

$$\Rightarrow \frac{\ddot{a}}{a} = \frac{-\frac{4}{3}\pi MG}{\frac{4}{3}\pi a^3 R^3} = \frac{-4}{3}\pi \left(\frac{M}{V_{vol}}\right) G \qquad \frac{M}{V_{vol}} = \rho$$

$$\Rightarrow \boxed{\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G \rho}$$

→ If ~~the~~ Universe is homogeneous; then ρ is independent of R ⟹ so; this equation is true for ~~all~~ every galaxy no matter how far away.

↳ The answer not depends on Newton's assumption that he was at centre.

Fundamental eqⁿ of cosmology

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G \rho}$$

↳ it also says that it is impossible to have static universe (ie; constant a) ; unless the universe is empty (empty means $\rho = 0$)

↳ so; we derived the fact that universe is not static.

ρ is not constant here; ∴ it changes with a ...

↳ lets write the equation in terms of ν; which is constant.

$$\therefore \frac{\ddot{a}}{a} = -\frac{4}{3}\pi \frac{G\nu}{a^3} \Rightarrow$$

$$\ddot{a} \Rightarrow \boxed{\ddot{a} = -\frac{4}{3}\pi \frac{G\nu}{a^2}}$$

This equation was actually discovered in the context of General Relativity by Friedman.

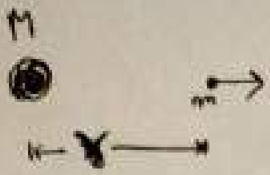
it is tradition to write the equation as "$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi \frac{G\nu}{a^3}$" } De-accelerated universe.

↳ This equation does not tells that Universe is expanding or contracting.

it tells — i) if Universe is expanding ; it tends to slow down.

ii) " " " contracting ; it speeds up.

↳ Newtons Model of Universe ... it is model which is close to being called (would have been called) Standard Model of Universe untill Accelerated Universe was discovered.



$\mathcal{E}_T = \text{Energy} = \frac{1}{2} m v^2 - \frac{mMG}{x}$

$\therefore$ if $\mathcal{E}_T$ .. particle cannot turn around

★ Energy is zero at the edge of parameter space.

$\therefore$ escape velocity: $\frac{1}{2} m v^2 = \frac{mMG}{x} \Rightarrow V = \sqrt{\frac{2MG}{x}}$ (when energy is zero)

---

similarly: ★ Universe can be above, below or at the escape velocity.

$\cdot m \quad \underset{\text{Total energy}}{E} = \underbrace{\frac{1}{2} m (\dot{a} R)^2}_{\text{Kinetic energy}} + \underbrace{\left(\frac{-mMG}{aR}\right)}_{\text{Potential Energy}}$

$\therefore$ case I $E = 0$ Universe is just on the edge.

$\Rightarrow (\dot{a})^2 R^2 = \frac{2MG}{aR} \Rightarrow (\dot{a})^2 = \frac{2MG}{aR^3}$

$\Rightarrow \frac{(\dot{a})^2}{(a)^2} = \frac{2MG}{a^3 R^3} \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{2MG \cdot \frac{4}{3}\pi}{\frac{4}{3}\pi a^3 R^3}$

$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \left(\frac{M}{V_{vol}}\right) \Rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho}$

$H(t) = \frac{\dot{a}(t)}{a(t)}$ Hubble Parameter

$\Rightarrow \boxed{(H(t))^2 = \frac{8\pi}{3} G \rho}$

Friedmann Equation

(it is not completely general; we set $E = 0$)

This universe is not going to re-collapse; but it will get asymptotically slower in expansion but never collapse back.

$\rho = \frac{\nu}{a^3}$ $\Rightarrow \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}\frac{G\cdot\nu}{a^3}$; ν is a constant

↳ The basic form of equation is

$$\boxed{\left(\frac{\dot a}{a}\right)^2 = \frac{1}{a^3}}$$

⇒ just for simplicity ; choose coordinates so that the constant $\frac{8\pi\nu G}{3}$ just becomes 1.

RHS is positive

$\frac{\dot a}{a}$ never become zero ; $\frac{\dot a}{a} = 0$ would mean universe turning around.

⇒ The expansion rate never goes to zero ; Hubble constant never changes sign. .... but it does slow down ; i.e; Hubble constant gets smaller & smaller with time.

☆ lets look for a solution of certain type.

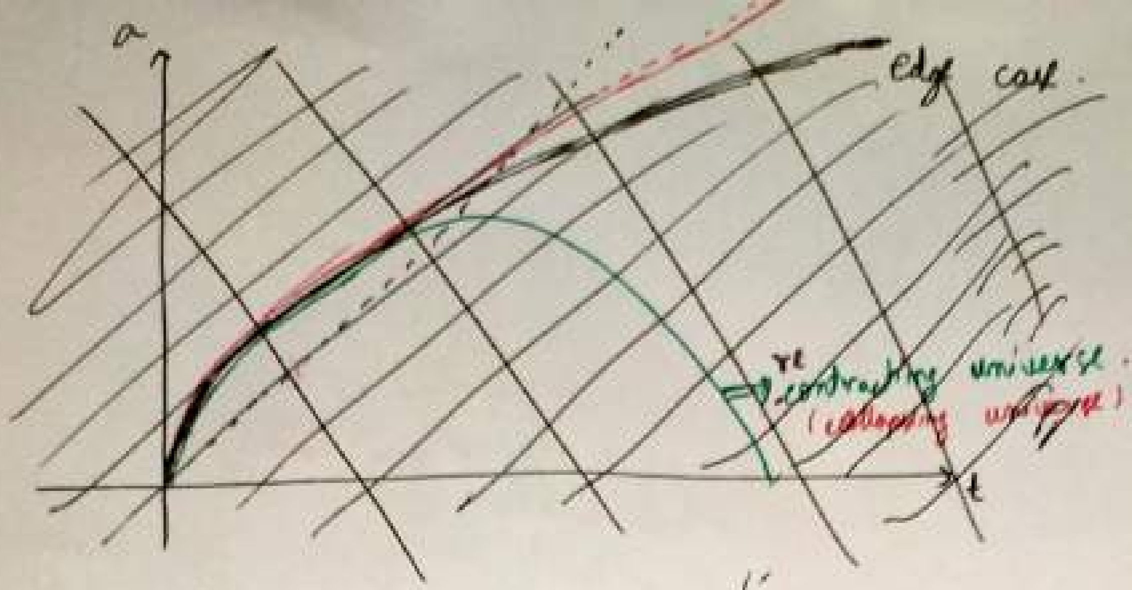
∴ $a = ct^{\alpha}$ $\Rightarrow \dot a = c\,\alpha\, t^{\alpha-1} \Rightarrow \frac{\dot a}{a} = \alpha\cdot\frac{1}{t}$

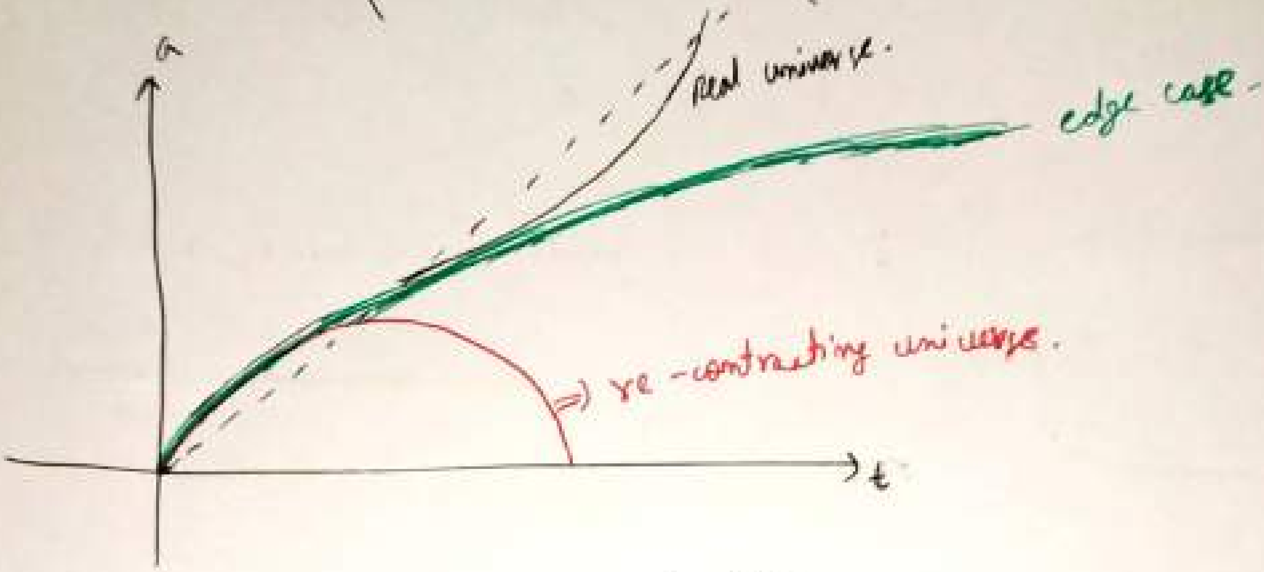
$\Rightarrow \left(\frac{\dot a}{a}\right)^2 = \frac{\alpha^2}{t^2}$ LHS is $\frac{\alpha^2}{t^2}$; $\frac{1}{a^3} = \frac{1}{c^3 t^{3\alpha}}$

$\Rightarrow \frac{\alpha^2}{t^2} = \frac{1}{c^3 t^{3\alpha}} \Rightarrow 2 = 3\alpha \Rightarrow \boxed{\alpha = \frac{2}{3}} \Rightarrow \alpha^2 = \frac{1}{c^3}$

$\Rightarrow c = \alpha^{-2/3}$

$\Rightarrow \boxed{a = \left(\frac{3}{2}\right)^{2/3}\cdot t^{2/3}}$ ⇒ This is how Newtonian universe would expand if it was right at critical escape velocity.





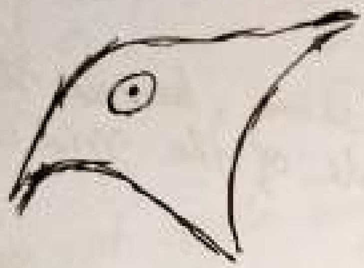
~~Real universe~~ * Real universe is accelerating.

Cosmology : Matter and Radiation Dominated Universe

Lec-2

— Shoaib Akhtar.

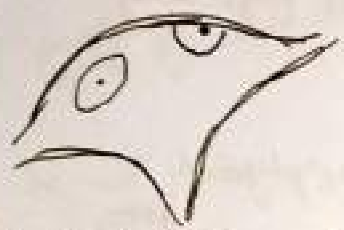
You can take small patch of it (small may mean 10 billion light years) : can study near by galaxies as long as they ~~[illegible]~~ are not moving relative to each other at high speed, using Newtons equation.



If we discover, that particles close to each other are moving close to speed ~~to~~ of light, then we have to modify the equations.

There are particles moving fast ~~compared~~ relative to us ; ~~they ex~~ photons.

~~[illegible]~~ Universe is filled with Homogeneous radiation. This Homogeneous radiation does move with the speed of light.

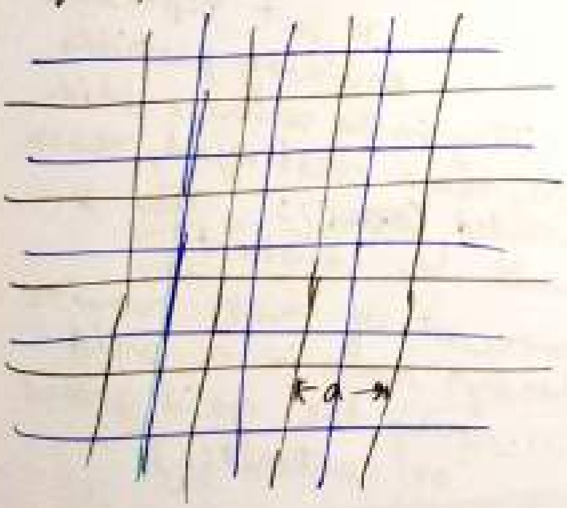


locally studying curved space.

ρ : density

a depends on the grid we laid down.

$\Leftrightarrow$ so it better be that our equations do not prefer any specific value of a.



blacks form one grid (I)

black & blue together form another grid. (dense grid) (II)

a' : distance between neighbouring point on dense grid.

clearly ; $a' = \frac{1}{2} a$

so ; at least at this stage, a does not have any physical meaning, untill you specify what the size of grid is.

Ratio of a_s may mean something.

particularly, Ratio of as at different times are recording the history of how the universe is expanding or contracting; and this is why our equations tend to only involve ratio of things with a.

eg take $\dot{a}$ $\dot{a}'$ of (II) grid will be twice as big compared to $\dot{a}$ of (I).

but $\underbrace{\frac{\dot{a}}{a} = \frac{\dot{a}'}{a'}}$ → These will be the same, because the ambiguity in the scale of the grid will cancel out.

"Things involving ratios of a; those are invariant physically meaningful thing"

eg $H = \frac{\dot{a}}{a}$

$\rho \Rightarrow$ measured in physical unit.

$$\boxed{\rho = \frac{\nu}{a^3}}$$

of the grid.

$\nu \Rightarrow$ mass contained in single cubic cell of the grid.
↳ changes as you change the grids.

equation; $\underbrace{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \cdot \rho}$

for the case of zero energy
(i.e; the situation, when every galaxy is at its escape velocity)

ν is constant with time; if ν just represents ordinary particles sitting in the universe which are never destroyed, never created ; Then ν would be constant.
(amount of mass remains always same in a grid ... only the grid grows or contracts.)

Numerical value of ν has to do with exact grid that we used.
If we change grid ; we change ν.

$$\therefore \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \cdot \frac{\nu}{a^3}$$

~~by~~ changing units of grids, changes magnitude of ν ; we can if we like, can choose $\frac{8\pi G\nu}{3}$ to be just be 1.

$$\left(\frac{\dot{a}}{a}\right)^2 = \underbrace{\frac{8\pi G\nu}{3}}\cdot\frac{1}{a^3}$$

The neumerical constant which appears here does not make any difference to the way the universe evolve.

proof// by changing definations of grid you can change the constants.

we will have a diff^n eq^n $\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{a^3}$

Solu^n// $\frac{\dot{a}}{a} = \frac{1}{a^{3/2}}$ ie: $\frac{\dot{a}}{a} = \frac{1}{a\sqrt{a}}$

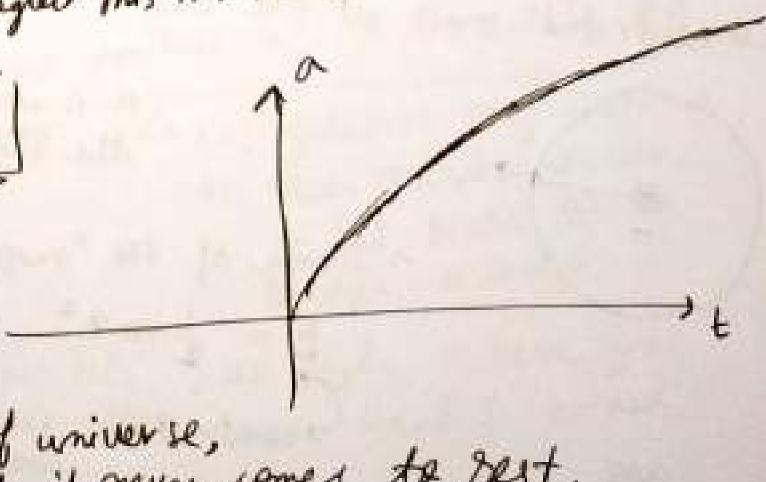
$\Rightarrow$ $\dot{a} = \frac{1}{\sqrt{a}}$ $\Rightarrow$ $\frac{da}{dt} = \frac{1}{\sqrt{a}}$

now take a to be independent variable.

ie; $\frac{dt}{da} = \sqrt{a}$; $t = \frac{2}{3}a^{3/2}$

neglect this ... ne scale

So; we have $\boxed{a = t^{2/3}}$

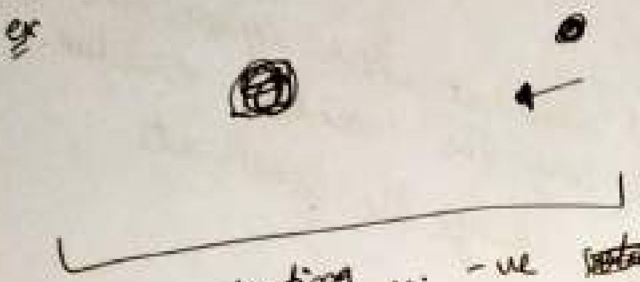


flattening of a is de-acceleration of universe, and we can clearly see that it never comes to rest.

we did not took negative square root; i.e. $\left(\frac{\dot{a}}{a}\right) = \pm \frac{1}{a\sqrt{a}}$.

$\therefore$ from the equation you cannot tell whether the universe is expanding or contracting. Both are possible from the equation.

ex



contracting ... -ve ~~solution~~ on square root

ex

expanding .... +ve ~~soluti~~ square root.

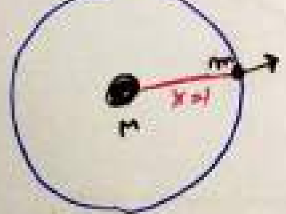
Early on, in the history of universe, the most important thing in the universe was photons.

when we say most important, we mean here the largest concentration of energy.

Earlier, the biggest or the most concentrated form of energy in the universe was Radiation.

Today, the most dominant form of energy in the universe is ~~Ate~~ just the masses of the particles. $E = mc^2$ kind of energy.

Energy of the particle moving in the ~~Fictitious~~ field of the fictitious concentrated mass at the centre.



M is combined mass of everything inside the blue sphere.

$\therefore$ Energy of the "mass m" particle is.

$$E = \frac{1}{2} m v^2 - \frac{m M G}{D}$$

constant ... it could depend on which particle we are talking about.

~~ate~~

lets take a specific particle at $X=1$ (PS 17)

- It has some specific energy E, and this will be a constant.

If the universe is homogeneous and isotropic the only real thing E could depend on is the distance away.

$$\Rightarrow \frac{V^2}{2} - \frac{MG}{D} = \frac{E}{m}$$

↳ This is also a constant

$$\Rightarrow V^2 - \frac{2MG}{D} = \frac{2E}{m}$$

; $D = ax$

$\therefore$ for $x=1$ case

$D = a$

$V = \dot{a}x$

$\Rightarrow V = \dot{a}$

$$\Rightarrow \dot{a}^2 - \frac{2MG}{a} = \frac{2E}{m}$$

↳ lets call it K (constant)

$$\text{so; } \dot{a}^2 - \frac{2MG}{a} = k \quad ; k \text{ constant.}$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 - \frac{2MG}{a^3} = \frac{k}{a^2}$$

sphere of radius a

$\therefore$ volume $= \frac{4}{3}\pi a^3$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 - \frac{2MG\,4\pi/3}{4\pi/3\, a^3} = \frac{k}{a^2} \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\cdot\left(\frac{M}{4/3\,\pi a^3}\right) = k/a^2$$

$$\Rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\cdot\rho = \frac{k}{a^2}}$$

we got Friedman Equation, but with a RHS, which knows about the total energy.

if $k>0 \Rightarrow$ T.E. was positive ie; K.E. outway P.E.

↳ i.e; galaxies may continue to ~~rec~~ receed forever, $\because$ it had beaten the escape velocity.

~~if $k<0 \Rightarrow$ T.E.~~

if $k<0 \Rightarrow$ T.E. <0 ie; more -ve P.E. ; less K.E. $\Rightarrow$ here you expect everything to go out away and then come back & crash

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \cdot \frac{\nu}{a^3} + \frac{k}{a^2}}$$

Friedmann Equation (was originally derived from General Relativity)

if $k > 0$; so: everything on RHS is positive

so; RHS > 0

& so $\left(\frac{\dot{a}}{a}\right)^2 > 0$ i.e; $\left(\frac{\dot{a}}{a}\right) > 0$

(if initially $\left(\frac{\dot{a}}{a}\right) > 0$)

$\Longleftrightarrow$ i.e.; universe continues to grow.
(but accelerating)

... when universe has grown to large size ; i.e; when a becomes sufficient large.

then ; $\frac{8\pi G}{3}\frac{\nu}{a^3} << \frac{k}{a^2}$

so; $\left(\frac{\dot{a}}{a}\right)^2 = \frac{k}{a^2}$

$\Rightarrow \frac{\dot{a}}{a} = \frac{\sqrt{k}}{a}$

(take positive square root.... initial take $\frac{\dot{a}}{a} > 0$ ... the only way it could become negative is by passing through zero.. but RHS is never zero...so it always remains positive)

$\Rightarrow \underline{\dot{a} = \sqrt{k}}$

ie; $\dot{a}$ is constant.

when a is small

$\frac{8\pi}{3} G \frac{\nu}{a^3} >> \frac{k}{a^2}$

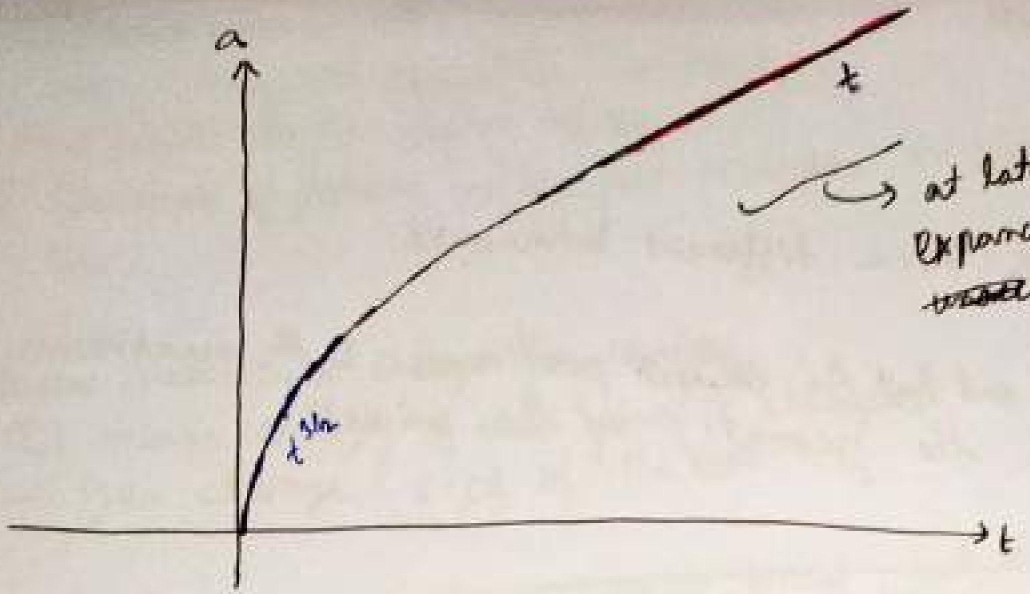
$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \cdot \frac{\nu}{a^3}$

we have studied this

...in early stage of expansion ... we found out $a \sim t^{2/3}$.

i.e; when ~~galaxies~~ galaxies get far enough away (large a) ; the effect of gravity becomes negligible & it just moves off with uniform velocity.

i.e; $\underline{a = k' t}$; k' is another constant...



→ at late time, universe expands with un-accelerated motion.

This is Matter dominated universe.

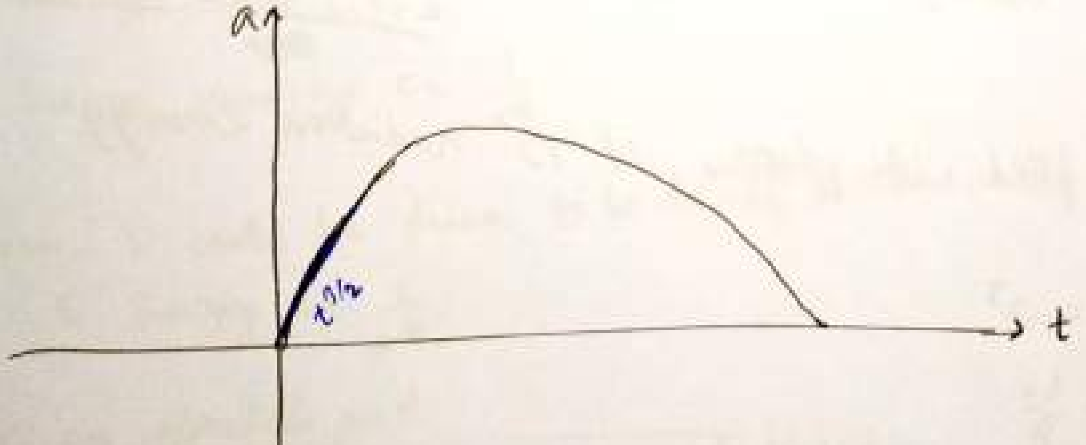
called matter dominated universe; because R.H.S. of $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{M}{a^3} + \frac{k}{a^2}$ contains a term which is just the density of ordinary non-relativistic matter.

---

when $k < 0$ (case of negative total energy)

here: RHS can become zero.

$\therefore$ somewhere between $\left(\frac{\dot{a}}{a}\right)^2$ can become zero ; this is the point where an upgoing thrown object is simply momentarily at rest.



Matter Dominated Universe

Three possibility

- $E > 0$.
- $E < 0$.
- $E = 0$.

& three different behaviours.

We will later find out that the positiveness - & the negativeness is connected with the geometry of the universe.

What happens when universe is made up of photons; i.e. radiation.

→ here we do have to think about relativity.

.. just $E = mc^2$.

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

Einstein equations are of the form, i.e: something on LHS has to do with geometry, & something on RHS has do with density of energy & momentum.

what is energy on RHS of this equation
→ it is just mass point ...

K is +ve when energy is −ve
(Kappa..)
& −ve when energy is +ve.

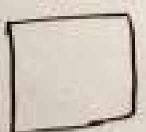
Newton ⟹ Relativity

mass density ⟹ (becomes) Energy density.

$c = 1$: speed of light.

When universe is filled with photons, it is radiation energy.

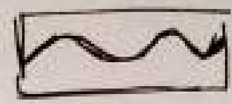
$\Delta x = 1$



$V = a^3$

$E = \frac{hc}{\lambda}$

Take a box, and expand the box slowly.
What happens to the photon inside it?
Ans Wavelength of photons inside just stretches in proportion with the box.



Because, wavelength change for photon,
This means energy of each photon changes as you change size of the box.

no. of photons in a box remains fixed, but their energy changes as you change the size of the box..

Energy of each photon $\propto \frac{1}{a}$ ⟶ i.e; size of the box.

.... you can think of this way;
space stretching; and with it the photon wavelength stretches.

The consequence of this is that energy per photon decreases like $1/a$ in contrast to the case with ordinary particle where, the mass stays the same & does not vary.

.... Total energy in the box decreases as $\frac{1}{a}$

Energy in the box $\propto \frac{1}{a}$

Energy density $\propto \frac{1}{a^4}$

here; ν could be taken to be no. of photon in a box.

& Energy $\propto \frac{1}{a^4}$

so; we effectively get

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\nu}{a^4} - \frac{k}{a^2}}$$ Radiation Dominated.

Case corresponding to zero energy. ; i.e; $K = 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\nu}{a^4}$$

by appropriate choice of size of grid; we can make $\frac{8\pi G}{3}\nu = 1$

so; we have $\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{a^4}$

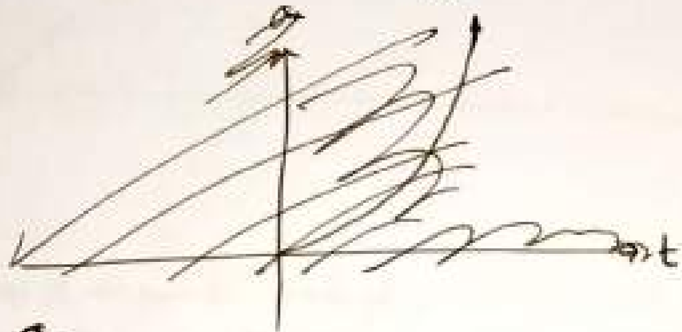
$$\Rightarrow \frac{\dot{a}}{a} = \frac{1}{a^2} \Rightarrow \dot{a} = \frac{1}{a} \Rightarrow a \cdot da = dt$$

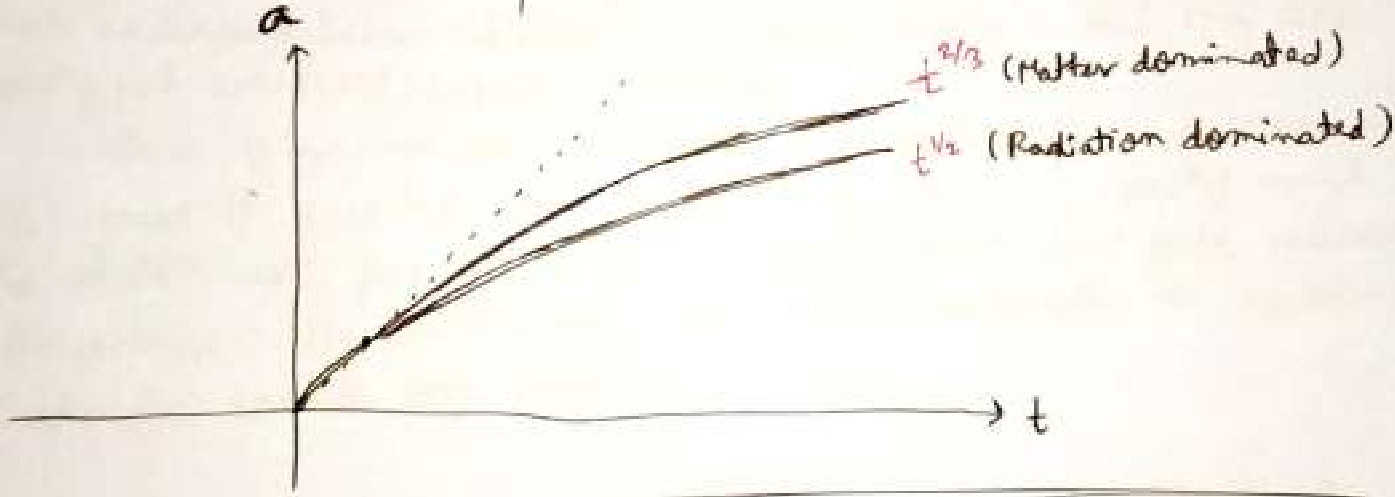
i.e; $\frac{dt}{da} = a$ so; $t = \frac{1}{2}a^2$

i.e; $t \sim a^2$

i.e; $a \sim \sqrt{t}$

$t = \frac{a^2}{2}$





Mixed case (Neither pure radiation, nor pure non-relativistic matter)

in this case, Energy density has two components.

- one for radiation ($\sim 1/a^4$)
- one for ordinary matter ($\sim \frac{1}{a^3}$)

so; the kind of equation we will have is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\xi_M}{a^3} + \frac{\xi_R}{a^4}$$

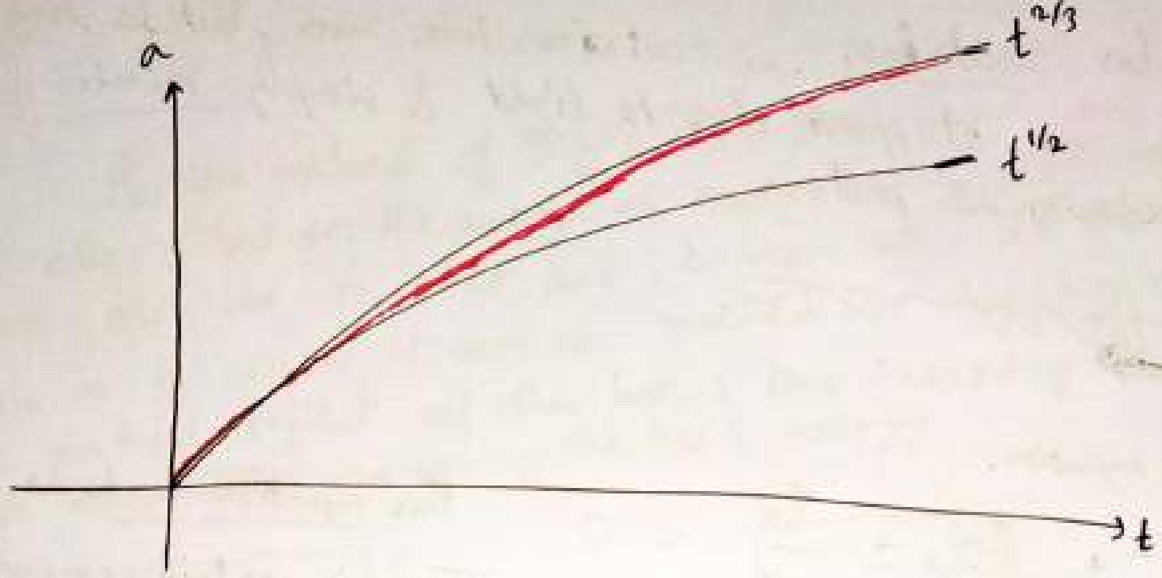
equation of motion for universe which contains ordinary non-relativistic matter + Radiation.

$\xi_M > 0$; $\xi_R > 0$ (both positive) (energy of particle at rest, and energy of photon; both are positive)

ξ_1, ξ_2 are constants.

$\frac{1}{a^3}$ is more important when a is big.

$\frac{1}{a^4}$ " " " " a is small.



When universe expands enough, things cool down and their is not much exchange between radiation & ordinary matter. And they are pretty well conserved; each one seperately.

Once things cool down to certain temperature (around 10,000 °c ...) ; but once it is cooled down to certain temperature, pretty much the photon are free streaming & pretty much dont care about the particles; the photons are too longed wavelengths to even scatter the particle very much.

There is also a third component; it is the discovered Dark Energy.

Dark Matter ~~is part~~ belongs to Matter category; it is part of ordinary matter but it simply dont react much with light...dont radiate much; so we ~~dont~~ dont see it optically.

Dark Matter is part of ξ_M.

Radiation component; ξ_R ... consists of all radiation, and all particles that are so ~~light~~ light, that they are moving close to the speed of light.

ξ_M ~~consists~~ consists of all particle which are heavy enough that they are ~~basically~~ basically at rest relative to us nearby.

ξ_R also has neutrinos ... neutrinos have mass, but so tiny that they move at speed close to light & simply mimic same behavior as photons.

Rearrange the equation.

$$-\left(\frac{\dot{a}}{a}\right)^2 + \frac{\xi_M}{a^3} + \frac{\xi_R}{a^4} = 0$$

This equation reads as,

There is certain amount of energy in mass & ration; if it changes with time, that energy gets transfered to a negative term $-\left(\frac{\dot{a}}{a}\right)^2$

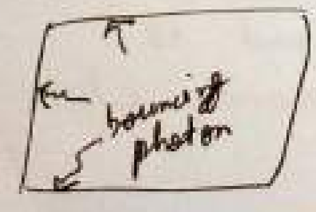
This is little bizzare; that the kinetic energy of expansion is negative $\left(-\left(\frac{\dot{a}}{a}\right)^2\right)$.

This can be read as conservation of energy.

contains three terms

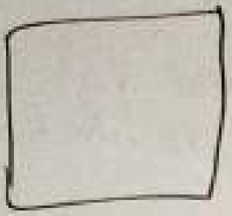
- Matter E_M
- Radiation E_R
- A term that has to do with rate of change of expansion. $E_{\star}$

Take a box; and ask what happens to energy in the box, when you expand the box?

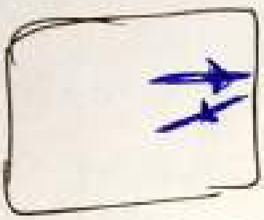


bouncing photon exerts pressure on the box & when you expand the box; it does work on the walls of the box; and will increase energy of box itself (example; can go into stretching the box)

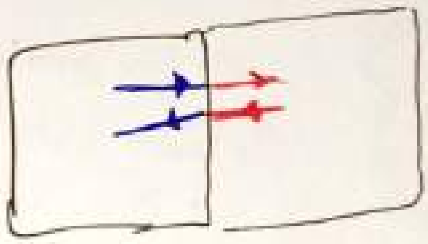
On the average; the expanding space, behaves like an expanding box. (Pg 25)



Particles instead of ~~reflecting~~ reflecting, goes from one box to the other; but on the average as many particle goes from one box to other box; & from other to the one's box. So, in a sense it can be mocked up by saying that the particles reflect off the box.; thus increasing the size of the box, increasing the box's energy.



Think as reflection.

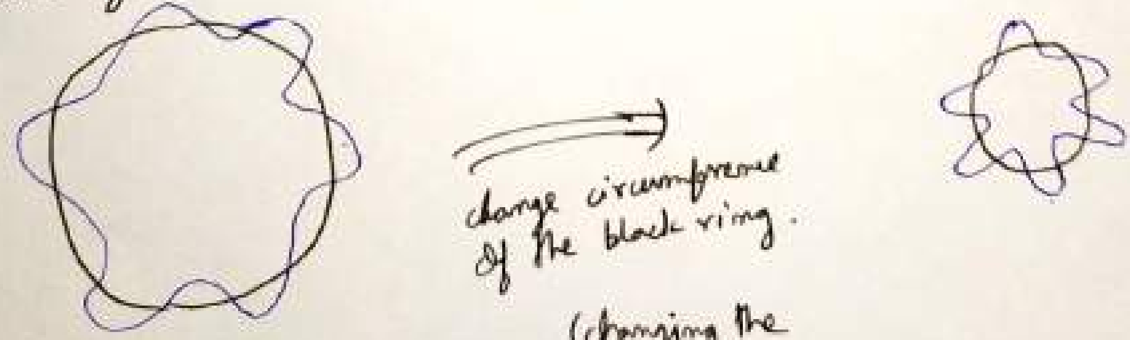


Think as particles going away from one box to the other.

Why wavelength of photon is rigidly attached with grid?

As long as you change things slowly, number of nodes remains fixed; this is one of the Adiabatic Invariant.

standing wave.



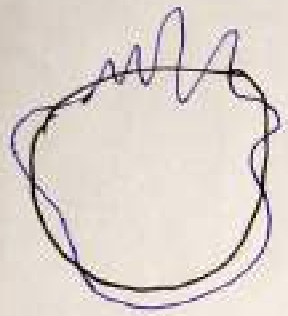
change circumference of the black ring.

(changing the circumference will not change the no. of nodes)

stretch the space on which wave is propagating on.
It can't suddenly jump. But at what point it gonna jump from 7 nodes to 8 nodes.

↳ It does not jump; what it does is the wave just get longer or shorter wavelength to match the thing on which it is on.

we are using the same phenomenon; just replace the ring with the universe itself.



works for arbitrary shaped wave also.

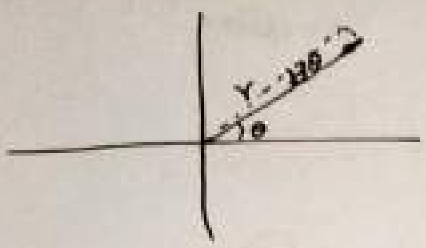
↳ work with fourier analysis.

↳ There are different wavelength components.

↳ The given wave is superposition of many other sinusoidal waves; and ~~the~~ nodes of each sinusoidal waves is adiabatic invariant; each of them's wavelength adjust accordingly (stretch ~~with~~ with space) .. and Thus the whole wave stretches along with the space.

– Shoaib Akhtar

$ds^2 = dx^2 + dy^2 + dz^2$



$\underbrace{ds^2 = dr^2 + r^2 d\theta^2}_{\text{metric in polar coordinate.}}$

a circle is 1-dimensional sphere ; and sometimes it is called 1 sphere.

polar coordinates in 2D

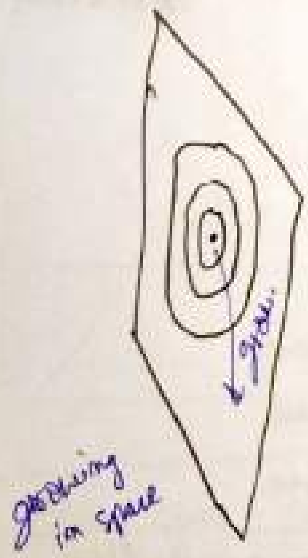
$d\Omega_1^2 = d\theta^2$

$ds^2 = dr^2 + r^2 d\Omega_1^2$: Flat

$d\theta^2$ is metric of unit circle (of unit radius)

Deformed circle could still be labelled by angular coordinates, so that equal intervals along equal angular seperations. ∴ And you will say that the metric of the deformed circle will still be the same thing, $d\theta^2$.

A circle is just space along which if you go a certain distance, you come back to same place. ~~after~~ And in unit circle case ; it is 2π.

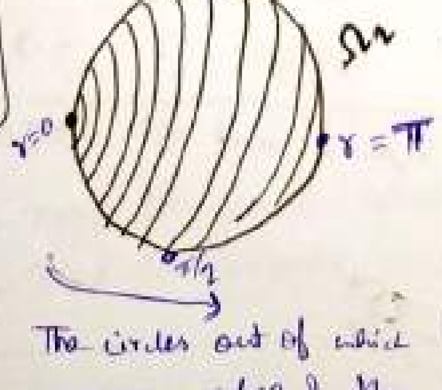


Think of the flat space as being composed of nested series of circles. ~~each one having a~~

☆ Sphere

(it is also a homogeneous surface)

2-sphere.



Metric

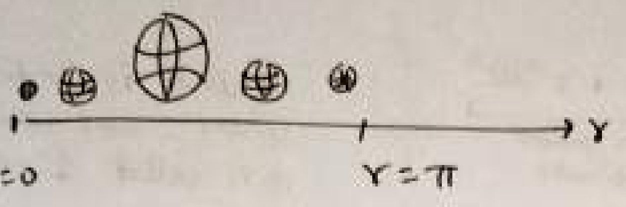
$dr^2 + \sin^2 r \, \underbrace{d\Omega_1^2}_{\text{metric of unit circle.}}$

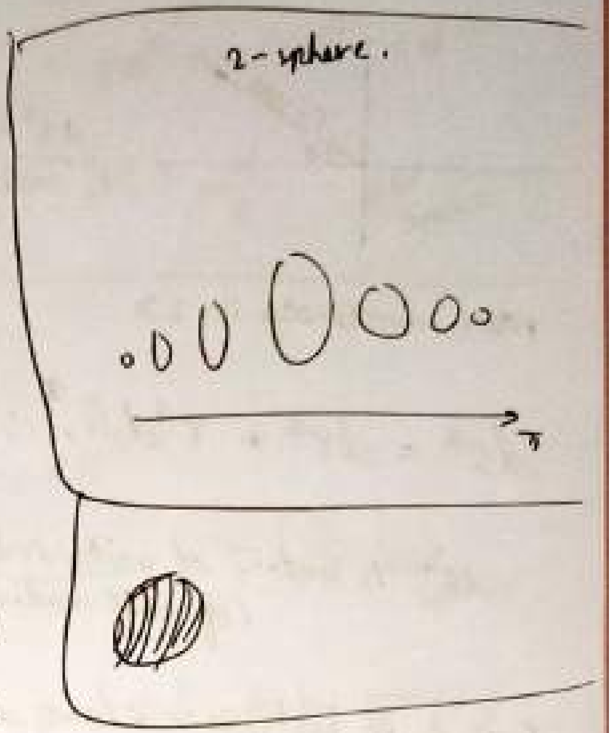
$\underbrace{d\Omega_2^2 = dr^2 + \sin^2 r \, d\Omega_1^2}_{\text{metric of two sphere.}}$

The circles out of which 2-sphere is formed, they grow, stop & the decrease in space.

Three dimensional sphere.

Series of nested 2-spheres :.... grow and collapse.





Metric of three sphere.

$$d\Omega_3^2 = dr^2 + \sin^2 r \cdot d\Omega_2^2$$

$$\boxed{\begin{aligned} d\Omega_1^2 &= d\theta^2 \\ d\Omega_2^2 &= dr^2 + \sin^2 r\, d\Omega_1^2 \\ d\Omega_3^2 &= dr^2 + \sin^2 r\, d\Omega_2^2 \end{aligned}}
\begin{aligned} &\rightarrow \text{1 sphere} \\ &\rightarrow \text{2 sphere} \\ &\rightarrow \text{3 sphere} \end{aligned}$$

$ds^2 = dr^2 + r^2 d\Omega_1^2 \Rightarrow$ flat 2-dimensional space.

$ds^2 = dr^2 + r^2 d\Omega_2^2 \Rightarrow$ flat 3-dimensional space in polar coordinates.

Standard Notation

Call metric of a a n-sphere $d\Omega_n^2$.

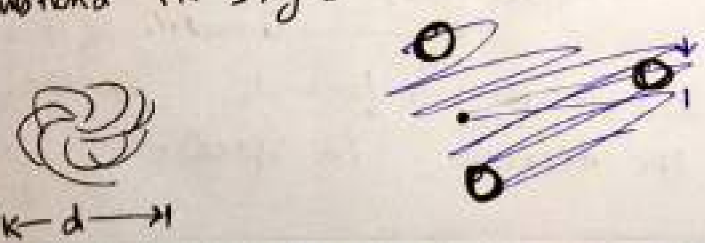
Can imagine by embedding in higher dimensions.

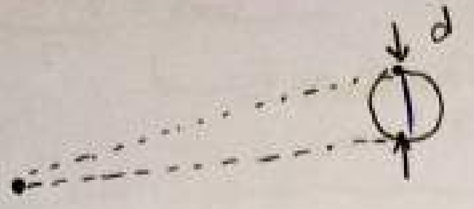
$x^2 + y^2 = 1$; 1 sphere embedded in 2-D space

$x^2 + y^2 + z^2 = 1$; 2 " " " 3-D. Space

$x^2 + y^2 + z^2 + w^2 = 1$; 3 " " " 4-D. Space

Assume all galaxies are of same size ; and ask how much angle they subtend in sky.



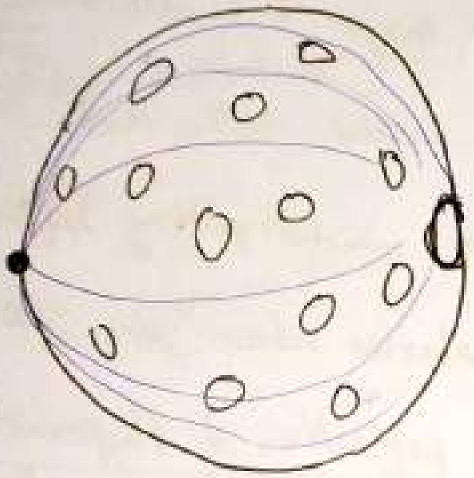


$$ds^2 = \cancel{dr^2}^{\to 0} + r^2 d\theta^2 = d^2$$

$$\Rightarrow \quad d\theta = \frac{d}{r} \quad \text{flat space}$$

---

lets do in ~~two~~ spheres. 2-sphere



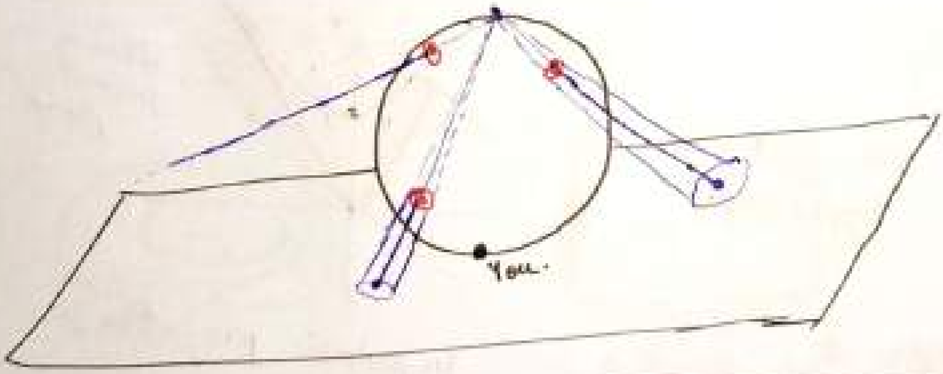
$$d^2 = \sin^2 r \, d\theta^2$$

$$\Rightarrow \quad d\theta = \frac{d}{\sin r} > \left(\frac{d}{r}\right)_{\text{flat space}}$$

Angle subtended ~~by~~ by galaxy far away would look bigger if we were on sphere.

The further toward north pole you are, the bigger the things look on plane.

Steriographic projection.



$\Rightarrow$ you can prove, circle maps to circle.

You can map a 3-sphere on infinitely flat 3-dimensional space.

sphere near you, map to small spheres.

" far away, map to bigger ones.

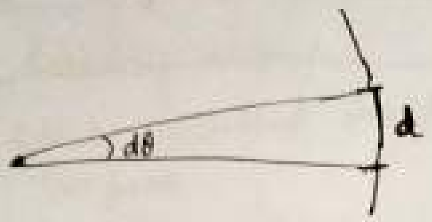
2-D hyperbolic space

$$dH_2^2 = dr^2 + \sinh^2 r \, d\Omega_1^2$$

$$\sin r = \frac{e^{ir} - e^{-ir}}{2i}$$

$$\sinh = \frac{e^r - e^{-r}}{2}$$

$$dH_3^2 = dr^2 + \sinh^2 r \, d\Omega_2^2$$



$$d^2 = \sinh^2 r \, d\theta^2$$

$$\Rightarrow d\theta = \frac{d}{\sinh r}$$

for large r : $d\theta \approx \frac{2d}{e^r}$

So, if you lived in hyperbolic world, you would notice that distant galaxies looked too small. You don't have phenomenon after which nothing lefts. Keeps growing. But you will observe more galaxies when you go far away.

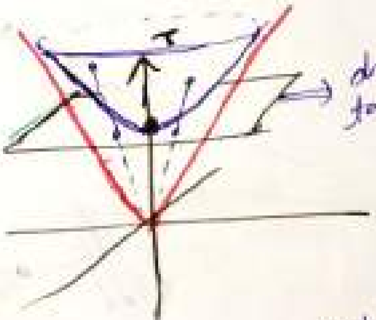
Steriographic projection of Hyperbolic plane

~~Steriographically Drawing Hyperbolic plane~~

X, Y, T space

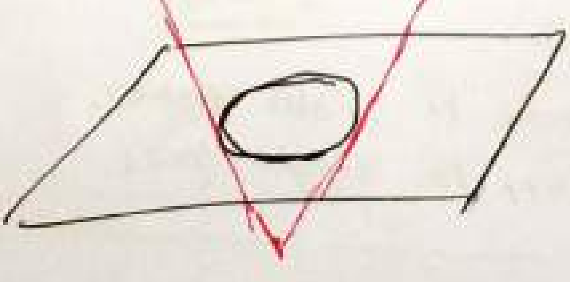
$T^2 - X^2 - Y^2 = 1$

we get hyperboloid



Hyperbolic projection.

~~T is like~~
T is the extra dimension you add so as to visualize embedding of hyperboloid.

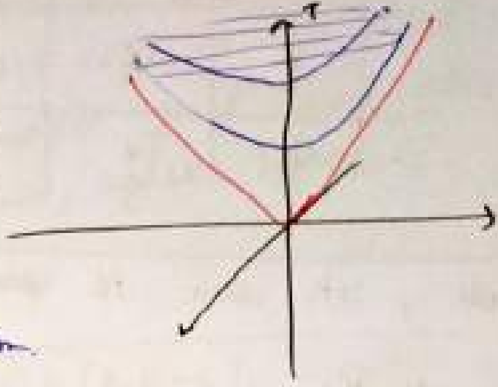


no points mapped outside the circle.

Sphere : Uniformly Positively curved space.

Hyperbolic space : " Negatively curved space.

$\hookrightarrow$ mathematically, it is space of uniform negative curvature.



Sphere : Bounding surface
Ball : solid contained inside bounding surface

Hyperbolic balls have radius associated with them.

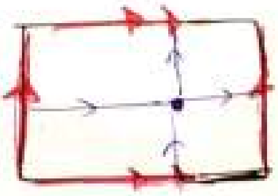
$T^2 - X^2 - Y^2 = r^2$
$\hookrightarrow$ This is the required radius.

$ds^2 = dr^2 + \sin^2 r \, d\Omega^2$
for unit sphere

$ds^2 = a^2 (dr^2 + \sin^2 r \, d\Omega^2)$
$\hookrightarrow$ for sphere of radius a.

for hyperbolic geometry
$ds^2 = a^2 (dr^2 + \sinh^2 r \, d\Omega^2)$

2-D

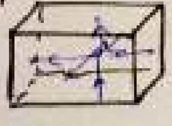


every point is same as every other point ;
$\hookrightarrow$ it is so homogeneous.

Rectangle with opposite sides identified . is topologically as torus.

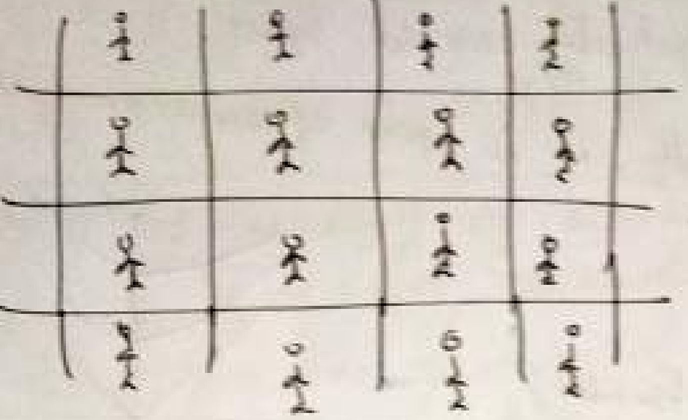
$\Rightarrow$ you can consider it flat with periodic boundary condition.

3-D torus



1-D torus is circle.

Living on torous is equivalent to



$$(x, y) \sim (x + 2\pi r_1, y)$$
$$(x, y) \sim (x, y + 2\pi r_2)$$

Now, lets come to space time.

minkowski metric ; $ds^2 = -dt^2 + \underbrace{dx^2 + dy^2 + dz^2}$
↳ metric of flat space.

for light ray: $ds^2 = 0$ along light trajectory.

$$-dt^2 + dx^2 = 0 \Rightarrow dx = \pm dt$$

(light ray moving to left or right with unit velocity)

Now, we will deform or change the kind of space-time we are talking about ;
keep time just as it is ; but substituting for the 3-D flat plane with one of the kind of geometries we discussed.

- plane
- sphere
- Hyperbolic geometry.

$\Rightarrow$ we will now also include the scale factor.

---

lets do 2-sphere.

$$ds^2 = -dt^2 + d\Omega_2^2$$

world with time in which space is just 2-sphere.

Now, we will allow the possibility, that the radius of the two sphere changes with time.

∴ ↳ to change radius of sphere, we include a^2.

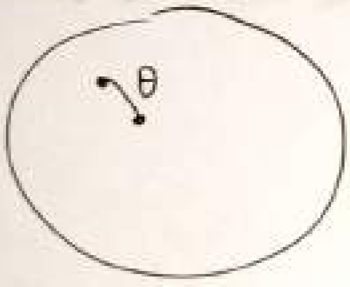
$$\underbrace{ds^2 = -dt^2 + a(t)^2 \, d\Omega_2^2}$$

↳ cosmology of world in which space is 2-d; & in which radius of universe is time dependent.

↳ This is space time geometry of ~~2nd world~~ 2-d world plus time; with time dependent radius, where space is 2-sphere.

$$ds^2 = -dt^2 + a(t)^2 \, d\Omega_3^2 \qquad (3\text{-}d) + (1 \text{ time}) \text{ world.}$$

Take any two points on sphere, seperated by angular distance θ.



The actual distance between points is $a(t)\,\theta$

$$D = a(t)\,\theta$$

$$V = \dot{a}(t)\,\theta$$ → relative velocity of two points.

$$\boxed{V = \frac{\dot{a}}{a} D}$$ → we get the same hubble law.

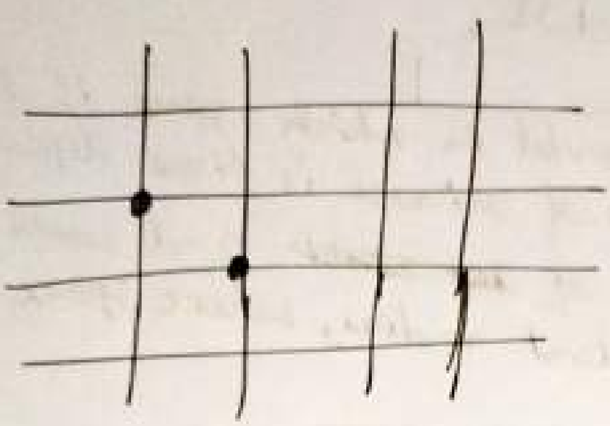
Any two point on the expanding or contracting sphere, the ~~actu~~ actual distance between them satisfy the hubble law, with $H(t) = \frac{\dot{a}}{a}$

Same thing would be true for hyperboloid space.

Space - Time metrics

$$ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2)$$

This is flat space, ordinary time but with growing or shrinking spacial geometry.



Metric of flat spacial univers, with scale factor which depends on time

---

Spherical geometry $\quad ds^2 = -dt^2 + a(t)^2 \, d\Omega_3^2$

metric of unit-3 sphere.

Hyperbolic geometry $\quad ds^2 = -dt^2 + a(t)^2 \, d\mathcal{H}_3^2$

We need equations for how $a(t)$ changes with time do have some dynamics with this cosmology

The consequence of dark energy is that it makes universe expand faster.

A homogeneous space means, that you can find a coordinate transformation which will replace any given point (as origin let say) by any other point ; such a way that the form of the metric is identical both before and after the transformation.

$ds^2 = dx_1^2 + dx_2^2$ flat space.

lets suppose, we make translation of coordinates.

$y_1 = x_1 + b_1$

$y_2 = x_2 + b_2$

what is metric in terms of y coordinate.

$dy_1 = dx_1$

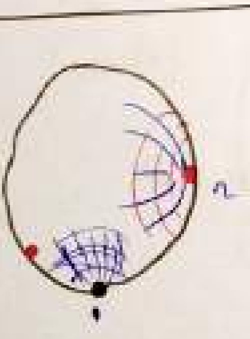
$dy_2 = dx_2$

so; $ds^2 = dy_1^2 + dy_2^2$ ↳ has exactly the same form as the metric had with x_1

$x=0$

The implication is that the neighbourhood of origin; has same property as neighbourhood of $y=0$. i.e; $x = \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix}$ i.e; we establish homogeneity.

↳ we can translate it to any point, & preserve the metric ; this is homogeneity.



$dr^2 + \sin^2 r\, d\theta^2$; metric has same form when written about 2..

⟺ so ; you can choose coordinate transformation to take point 1 to any point. It tells us that every point on the sphere is same as every other point on the sphere.

<u>1-D</u>

closed

Since any piece could be straighted out, without stretching, without deforming the metric; one dimensional spaces are all flat. They have no curvature.

How many 1-d spaces are there from intrinsic point of view.

A line segment 1-D

~~same topologically~~, topologically same.

When we talk about geometry of space, we are talking ~~about~~ about intrinsic geometry (and not the way it is embedded in some higher ~~dimen~~ dimensions for purpose of visualization)

Assume space is isotropic & homogeneous.

K stands for curvature

$K = 1$ → positively curved space (In 2-D, the analogue is sphere)

$K = 0$ → flat space.

$K = -1$

Assume space and time are not mixed with each other.

i.e; the metric has the form $ds^2 = -dt^2 + a(t)^2 \{ \_\_\_ \}$

$$\text{i.e; } ds^2 = -dt^2 + a(t)^2 \begin{cases} d\Omega_3^2 = (dr^2 + \sin^2 r \, d\Omega_2^2) \\ (dx^2 + dy^2 + dz^2) \quad ; k=0 \end{cases}$$

- $ds^2 = -dt^2 + a(t)^2 d\Omega_3^2 = -dt^2 + a(t)^2 (dr^2 + \sin^2 r \, d\Omega_2^2) \quad ; k=1$
- $ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2) \quad ; k=0$

or $= -dt^2 + a(t)^2 (dr^2 + r^2 d\Omega_2^2)$ } → 3-D polar coordinate.

$$ds^2 = -dt^2 + a(t)^2 \left(dr^2 + \sinh^2 r \, d\Omega_2^2\right) \quad ; k = -1$$

Distance between two points $\propto a(t)$ in space

$$\mathbb{D} = a(t) \, \Delta\theta$$

$\Delta\theta$ ↳ has some fixed value

$a(t)$ ↑ Scale factor

$$\therefore V = \dot{a}(t) \, \Delta\theta$$

$V = \frac{\dot{a}}{a} D$ Hubble law

$H(t) = \frac{\dot{a}}{a}$ (does not depends on position, in that ~~case~~ sense, it is constant)

↳ it does not depend on where you are, but may depend on time.

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{3} T^{\mu\nu}$$

time-time component of $T^{\mu\nu}$ is energy density.

$$\underbrace{\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}}_{\text{Geometry}} = \frac{8\pi G}{3}\rho$$

$\frac{8\pi G}{3}\rho$ → sensitive to kind of material making the universe

Two components
↳ Curvature of space has one component here.
↳ The other depends on the way thing changes with time

$$\Rightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho}{3} - \frac{K}{a^2}$$

→ Same form as we got in Newtonian example.

But here; we had an interpretation of ~~[illegible]~~ E as K ... Same equation, but here, it ~~is~~ deals with something very different.

ρ in Newtonian was ordinary mass density.

⇒ here, in General Relativity equations; ρ is both due to mass & energy, the equation apply to relativistic case also.

Although, the equations look similar.

Matter Dominated $\rho = \frac{\rho_0}{a^3}$

ρ_0 is density at the time when radius was 1

$a(t)$ is radius.

Radiation Dominated $\rho = \frac{\rho_0}{a^4}$

With these form of energy, if $K>0$ (sphere case) ⇒ corresponds to situation where universe recollapses.

If it flat ⇒ it is as if every galaxy was exactly at escape velocity; so it continues to expand ever slowing down; asymptotically coming to rest.

If $K<0$; corresponds to being above the escape velocity; & a just ~~continues to grow~~ continues to grow linearly at late times.

The important ingredient in determining, how ρ depends as function of a ; is Equation of States.

Step 1) To assume an equation of state, and derive the appropriate formula

Step 2) Derive for different kinds of material, the equation of state.

Equation of state is relationship between thermodynamical variable.

In our case Temperature will not play a major role.

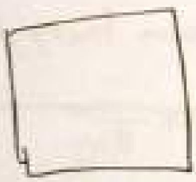
~~In particular~~ In particular, equation of state is a relationship between energy density and pressure.

$$\underbrace{\underset{\text{Pressure}}{P} = \underset{\text{constant}}{W}\ \underset{\text{energy density}}{\rho}}$$

$\Rightarrow$ Interesting cases has this form.

for matter dominated universe, things moves slowly; & thus P is negligible. so; $W = 0$, matter dominated.

☆ $W = \frac{1}{3}$, Radiation dominated



Box of gas.

$$\underset{\text{Total energy}}{E} = \underset{\text{Energy Density}}{\rho}\ \underset{\text{Volume of box.}}{V}$$

When changes take place slowly; Entropy does not change
∴ Adiabatic Change.

now; lets change volume of box.

$dE = -P\,dV$

we had $E = \rho V \Rightarrow dE = \rho\,dV + V\,d\rho$

$\therefore \rho\,dV + V\,d\rho = -P\,dV$

$\Rightarrow V\,d\rho = -(P + \rho)\,dV$

$\Rightarrow V\,d\rho = -(1+W)\rho\,dV \quad \Rightarrow \int \frac{d\rho}{\rho} = \int -(1+W)\frac{dV}{V}$

$\Rightarrow \log \rho = -(1+W)\log V + \underset{\text{constant}}{\xi'} \quad \Rightarrow \rho = \xi \cdot \frac{1}{V^{1+W}}$

$$\rho = \frac{\xi}{V^{(1+w)}}$$

Now if we think that the box is expanding with general expansion of universe, the volume of the box is simply $(a(t))^3$.

$$\Rightarrow \boxed{\rho = \frac{\xi}{a^{3(1+w)}}}$$

This what thermodynamics of a nice homogeneous material will tell us.

$w=0 \Rightarrow \rho = \frac{\xi}{a^3}$ Matter Dominated

$w=\frac{1}{3} \Rightarrow \rho = \frac{\xi}{a^4}$ Radiation Dominated.

To describe a cosmology based on some sort of energy, we need to know Equation of state in the form $p = w\rho$.

Cosmological Constant (C.C.) ; Dark Energy (D.E.) ; Vacuum Energy (V.E.)

$\hookrightarrow w = -1$ for C.C. ; D.E. ; V.E.

analogue

(it like negative pressure, pulling inward)

when $w = -1$

$\Rightarrow \rho$ is constant

this is the nature of D.E. ; It does ~~the~~ not change when you expand the size of the box.

$\hookrightarrow$ It does change energy in the box ; but not the energy density. ; because that energy density is a property of empty space, & an empty space does not dilute when you stretch it.

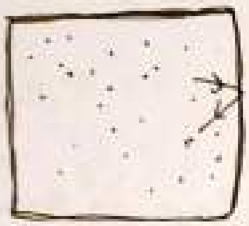
$a(t) \sim t^{2/3}$ matter dominated. (M.D) : $\rho = \rho_0/a^3$

$a(t) \sim t^{1/2}$ radiation dominated. (R.D) : $\rho = \rho_0/a^4$

We take Equation of State to be $P = w\rho$.

$$\rho = \frac{\xi}{V^{1+w}} = \frac{\xi'}{a^{3(1+w)}}$$

Equation of state : for box filled with photons.



Assume photons bounce off the wall, & loose no energy.

let ϵ = Energy per photon on average.

$\vec{\Pi}$ = momentum of characteristic particle in there.

Π = magnitude of momentum per particle.

we know: $\underbrace{\epsilon = \Pi \cdot c}$ → Relationship between energy of massles particle & its momentum.

set $c = 1$ ∋ $\epsilon = \Pi$

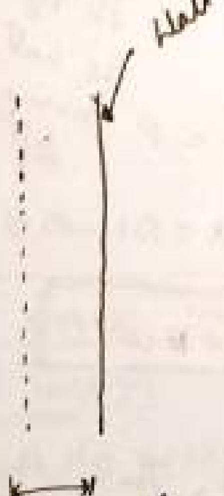
let ν = number of photons per unit volume. (density number of photon)

Δt ∋ little time interval

$$\rho = \epsilon \nu$$

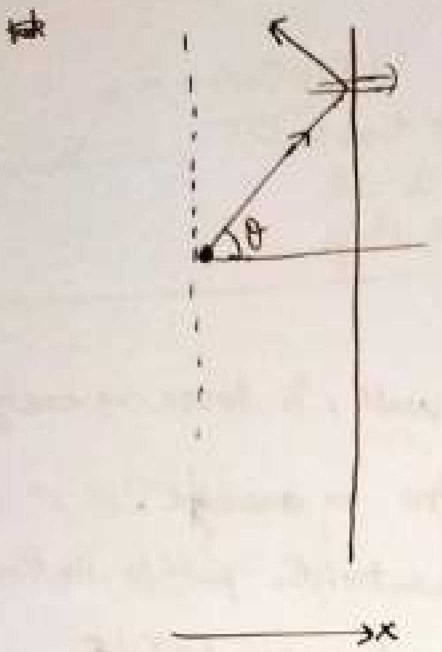
↖ energy density.

velocity of these particles (photons) is 1 (c)



$\Delta x = \Delta t$ → any particle moving to right in here with horizontal velocity will hit the wall in time Δt.

~~because, that energy density is a property of empty space,~~ Pg 42
& an empty space does not dilute when you stretch it.



hit the wall of box if $\Delta x = \Delta t \cos\theta$

Momentum transferred to the wall by a particle.

$$\Delta \Pi_x = 2\epsilon\cos\theta$$

→ due to one photon hitting the wall.

$$\frac{\Delta \Pi_x}{\Delta t} = \frac{2\epsilon \cdot \cos\theta}{\Delta t}$$

→ Force on wall : change in momentum per unit time

→ force exerted for each particle that hits the wall.

A particle moving ~~...~~ with angle θ will hit the wall if it is within $\Delta x = \Delta t \cos\theta$

∴ → no. of particle within Δx. N.

$$\underbrace{N = \Delta X A}_{\text{small volume element}} \cdot \nu$$

ν → Number of particles per unit volume

$$F = \frac{2\epsilon \cdot \cos\theta}{\Delta t}\underbrace{\left(\frac{\Delta X \cdot A\,\nu}{2}\right)}$$

$\frac{1}{2}$ because; the one moving toward the right will hit the wall; those moving toward left will not.

$\frac{F}{A} = P$

$\Delta x = \Delta t \cos\theta$

$$\frac{F}{A} = \frac{2\epsilon\nu\cos^2\theta}{2} \Rightarrow \boxed{P = \cancel{2}\,\epsilon\nu\cos^2\theta}$$

$$\boxed{P = \epsilon\nu\cos^2\theta}$$

$$P = \rho\cos^2\theta$$

→ This is pressure due to particle moving at particular angle.

now, we average over all particles moving in different directions.

$\hat{n} = (n_x, n_y, n_z)$ → unit vector along the direction of motion of the particle.

& we maintain

$$\hat{n}_x = \cos\theta \, \hat{x}$$

$$\underbrace{n_x^2 + n_y^2 + n_z^2 = 1}_{\text{because } \hat{n} \text{ is unit vector.}}$$

→ average this over all possible direction.

$$\underbrace{\langle n_x^2 \rangle = \langle n_y^2 \rangle = \langle n_z^2 \rangle}_{\text{because } x,\, y,\, z \text{ are related by rotation.}}$$

so; $3 \langle n_x^2 \rangle = 1$

$$\Rightarrow \langle n_x^2 \rangle = \frac{1}{3}$$

$$\Rightarrow \langle \cos^2\theta \rangle = \frac{1}{3}$$

here.

$$P = \rho \langle \cos^2\theta \rangle$$

so; $\boxed{P = \dfrac{\rho}{3}}$ → Equation of State for Radiation.

Under what circumstances, pressure can be negative.
Negative Pressure is called Tension.

Ex in one dimension.

spring connected which connects two ends. (it pulls the end points in) → The tension of string is effectively negative pressure.

Pressure can be negative, even when energy density is positive.

Vaccum Energy (consequence of Quantum Field Theory)

it is just the energy assigned to empty space.

We can conjecture, that empty space with just nothing in it has energy.

Vaccum energy is the energy which is simply there in empty space.

Take a box

V : volume of box

How much vaccum energy is there in this box.

Ans $\rho_0 \cdot V_x$ — volume of box.

ρ_0: vaccum energy density

Vaccum Energy Density is universal constant (does not change with size of the box)

ρ_0 : vaccum energy density ; $\underbrace{\rho_0 = \Lambda \cdot \frac{3}{8\pi G}}$

This defines Λ

$\therefore$ This Λ is called <u>Cosmological Constant</u>.

Why it is useful to define Λ ?

Friedmann eqn : $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$

Because of this it is useful to define Λ.

What kind of equation of state does Vaccum Energy correspond ?

$dE = \rho_0 dV + \cancel{d\rho_0} V$

$\Rightarrow dE = \rho_0 dV \Rightarrow \rho_0 dV = -W\rho_0 dV$

$\Rightarrow \underline{W = -1}$ for vaccum energy.

so; $\underline{P = -\rho_0}$ → Equation of state for empty universe that is governed by vaccum energy.

We will need to have pretty exact theory of all of quantum fields in nature to be able to compute what ρ_0 is.

~~lets study~~ lets study; pure Empty space.

six cases ; $\Lambda = \pm, 0$
$K = $ (negative, 0, positive)

so we have six cases $\underline{\Lambda \times K}$ to study.

$\Lambda = \pm$
$K = -1, 0, +1$

& we have the equation $\underbrace{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \cdot \rho_0 - \frac{K}{a^2}}_{\text{for Empty space}}$

$\therefore$ note; $\frac{8\pi G}{3} \rho_0 = \Lambda$

so; $\left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \frac{K}{a^2}$

Λ is also the thing, which is famously called Dark Energy.

$\Lambda < 0, K > 0$
Friedman eq^n makes no sense

$\Lambda < 0 ; K = 0$
no solution.

~~$\Lambda < 0, K > 0$~~

take $\Lambda > 0, K = 0$

$\frac{\dot{a}}{a} = \sqrt{\Lambda} \Rightarrow \dot{a} = \sqrt{\Lambda}\, a \Rightarrow \frac{da}{dt} = \sqrt{\Lambda}\, a$

$\Rightarrow \underline{a(t) = a(0)\, e^{\sqrt{\Lambda} \cdot t}} \rightarrow$ a grows exponentially with time.
The universe exponential. expands.
This is the case of positive vaccum energy & no curvature.

here; $H(t) = \frac{\dot{a}}{a} = \sqrt{\Lambda} \Rightarrow \underline{H = \sqrt{\Lambda}}$

Hubble

$\Rightarrow \underline{a(t) = a(0)\, e^{H \cdot t}}$

This is spacetime, which exponentially expands; and is called ~~De sitter space~~ de Sitter space

This geometry is not geodesically complete. There are trajectories back into the past which go infinity past in finite proper time. ... some of geometries are missing.

put other kinds of matter, say Radiation $\cancel{\frac{P_R}{a^4}}$ (ρ_R)

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda + \frac{\rho_R}{a^4}$$

for large a (at very late time)

$\therefore \Lambda >> \frac{\rho_R}{a^4}$

↳ and hence universe will exponentially expand.

$\therefore$ at very early times; (small a)
Λ is not much important

$\therefore$ at very late time Λ dominates everything; both matter and radiation.

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda + \frac{\rho_R}{a^4} + \frac{\rho_M}{a^3}$$

Universe is in process now of making transition from being Matter dominated to vacuum dominated.

We have not yet seen; genuine exponential expansion..

It is called accelerating (the exponential expansion) because of the simple reason, that if a increases exponentially; and we ~~do~~ calculate acceleration $\ddot{a}$; this is also increasing exponentially.

So, the universe is expanding in an exponential accelerated way.

Observation at present time confirms acceleration;
↳ the more precession we get, we begin to see more exponential it gets.

Calculating vacuum energy of quantum field,
it is positive for bosons.
it " negative for fermions.

There is no known theory which is consistent with the world as we know it today with zero vacuum energy.

What is mysterious about dark energy : is it small value.
; although if it was large ; we would have discovered it.

Big Rip would have happened ; if w was more negative than -1.
↳ & there is no know theory (which agrees with everything we know about nature) which predicts $w << -1$.

Really fundamental constants of nature are $c, \hbar, G$.
↳ we say that they are fundamental, because there is sense in which they are universal things.

Plank System ; they are units of length, mass & time which corresponds to setting $G, \hbar, c$ equal to 1 numerically.

$l_p \sim 10^{-33}$ cm ; plank volume : 10^{-99} cm^3.

$m_p \sim$. ; $\frac{m_p}{(l_p)^3} \Rightarrow$ ~~the~~ unit of energy density.

→ This is huge energy density.
↳ this is natural unit of energy density that occurs in fundamental physics.

Vacuum energy density not as big as $\frac{m_p}{(l_p)^3}$; and it is about 10^{-123} smaller than $\frac{m_p}{(l_p)^3}$

→ This is what we might care ... what it would have been in Natural units.

What is mysterious about Vacuum energy or Dark Energy is ~~the~~ not that it is there; but it is almost not there (very small).

~~[illegible]~~ W is now measured to about 10%.

ie; -1 to within 10%.

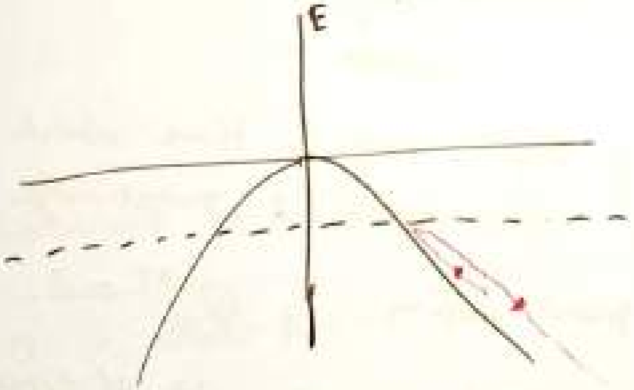
$\Lambda > 0$; $K = +1$ (spherical universe, positively curved universe, with positive cosmological constant)

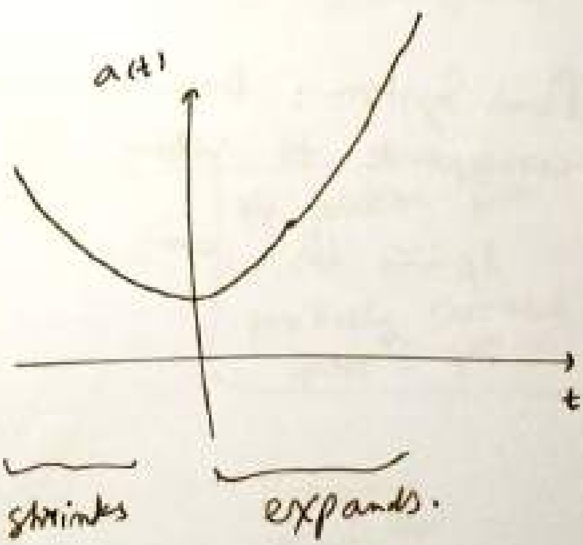
$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \frac{1}{a^2}$ $\Rightarrow$ $\dot{a}^2 - \Lambda a^2 = -1$

Now; think of a as ~~coordinate~~ coordinate of a particle

$\therefore$ $\dot{a}^2 \propto$ Kinetic Energy

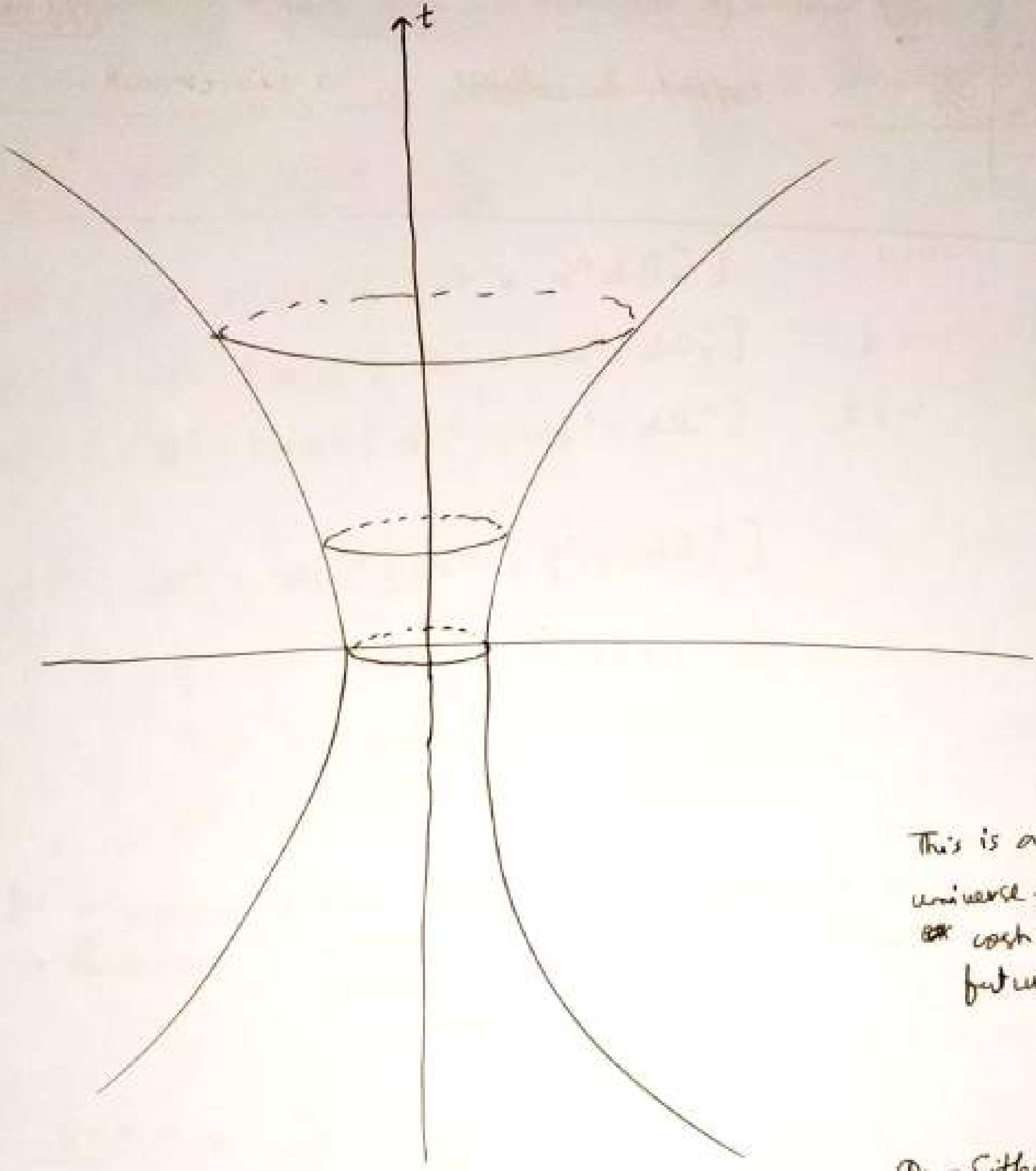
$-\Lambda a^2 \propto$ like potential energy





solution

$$a = \frac{1}{\sqrt{\Lambda}} \cosh(\sqrt{\Lambda}\, t)$$



This is a strange universe... increases ~~ex~~ cosh(...) in future...

De-Sitter space.

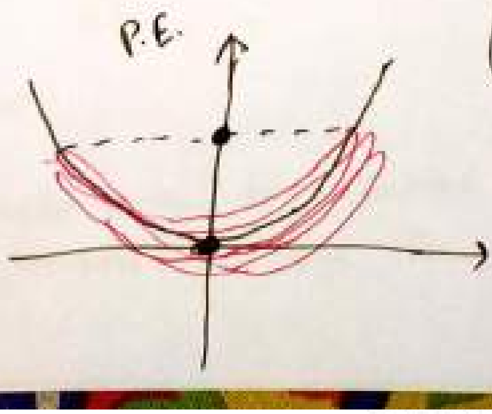
Universe will be a bounce off

$\Lambda < 0$; ~~~~ $K = -1$

set $\Lambda = -1$

No solution for $\Lambda < 0$ and positively curved space

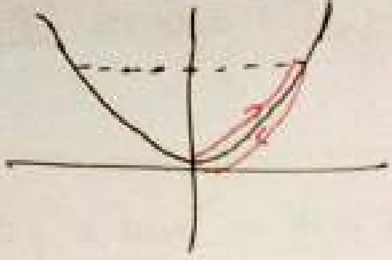
$$\left(\frac{\dot{a}}{a}\right)^2 = -1 + \frac{1}{a^2}$$



Harmonic oscillator

$$\Rightarrow (\dot{a})^2 + (a)^2 = 1$$

Energy conservation for harmonic oscillator.



expand & collapse ... a big crunch.

-Shoaib Akhtar

$$\left(\frac{\dot{a}}{a}\right)^2 = H(t)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$$

$$ds^2 = -dt^2 + a(t)^2[dr^2 + r^2 d\Omega_2^2] \qquad K=0$$

$$ds^2 = -dt^2 + a(t)^2[dr^2 + \sin^2 r\, d\Omega_2^2] \qquad K=+1$$

$$ds^2 = -dt^2 + a(t)^2[dr^2 + \sinh^2 r\, d\Omega_2^2] \qquad K=-1$$

$$ds^2 = -dt^2 + a(t)^2[dr^2 + \xi^2(r) d\Omega_2^2]$$

$$\xi(r) = \sin r \qquad ; k=+1$$

$$\xi(r) = \sinh r \qquad ; k=-1$$

$$\xi(r) = r \qquad ; k=0$$

The only sources we know today ;.. contributions to ρ

- Radiation
- Matter (ordinary non-relativistic particle)
- Vacuum

$$\therefore \frac{8\pi G}{3}\rho = \frac{C_R}{a^4} + \frac{C_M}{a^3} + \Lambda$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} = \frac{C_R}{a^4} + \frac{C_M}{a^3} + \Lambda - \frac{K}{a^3}$$

$$\Rightarrow H^2 = \frac{C_R}{a^4} + \frac{C_M}{a^3} + \Lambda - \frac{K}{a^3}$$

↳ H is constant over space.

Think that H(t) is contant over some time period ; today the value of H is effectively is constant in some time period.

Think of the equation, $H^2 = \frac{C_R}{a^4} + \frac{C_M}{a^3} + \Lambda - \frac{K}{a^2}$ as a constraint equation in todays time ; ie; H is constant locally in time.

* Cosmic Distance ladder.

{The density of energy in universe today, on the average is about 1 proton per cubic meter. 26 - September - 2019.

→ C_M

C_R is negligible. ; $C_R << C_M$

$$\Omega_R = \left(\frac{C_R}{a^4}\right)\cdot\frac{1}{H^2}$$

$$\Omega_M = \left(\frac{C_M}{a^3}\right)\cdot\frac{1}{H^2}$$

$$\Omega_\Lambda = (\Lambda)\frac{1}{H^2}$$

$$\Omega_K = \left(\frac{-K}{a^2}\right)\frac{1}{H^2}$$

} Ratio value at todays time

and the rule is $\boxed{\Omega_R + \Omega_M + \Omega_\Lambda + \Omega_K = 1}$

earlier (around when Leonard Susskind was young)

people thought $\Omega_R \approx 0$

& Ω_M is what you measure when you see ~~[illegible]~~ matter (which interacts with light)

↳ roughly 1 proton per cubic meter. Ω ~~roughly~~ 1

∴ and people believed $\Omega_\Lambda = 0$ / → Ω_Λ was not discovered.

∴ and so people thought to compute Ω_K

ie; $\Omega_K = 1 - \Omega_M$.

Ω_M was like $\frac{1}{30}$ (small) ; & the only conclusion was that $\Omega_K \approx 1$

conclusion; $K < 0$ $\therefore$ ~~$\frac{1}{a^2} = \frac{1}{H^2}$~~

and $\frac{1}{a^2} = H^2$

$a \sim t^{2/3}$ (Matter dominated)

$\frac{\dot{a}}{a} = \frac{2}{3} t^{-1/3} \cdot t^{-2/3} = \frac{2}{3} t^{-1}$

so, in Model of Matter dominated universe is $H(t) = \frac{2}{3t}$

so, $H_{TODAY} = \frac{2}{3 \cdot (\text{Age of Univers})}$

$$\boxed{H_{Today} = \frac{2}{3 \cdot (\text{Age of Universe})}}$$

for Matter Dominated Universe.

T := Age of Universe.

$H_{Today} = \frac{2}{3T}$

for matter dominated universe.

$\therefore \frac{1}{a^2} = \frac{1}{T^2}$

$\Rightarrow$ telling

(Size of Universe) $\sim$ (Age of Universe) c

↳ Distance light will travel in age of universe.

↳ This was something which was expected to be true on the basis of ~~GR~~ cosmology 40 years ago.

- Radiation negligible.
- Matter being substantial, but still small.
- $\Lambda = 0$.

Ordinary matter made out of electrically charged particles that radiate's when its accelerated, or when you collide with it. Its called Luminuous (because it radiates)

↳ Its kind of matter which we know about direct telescopic observation.

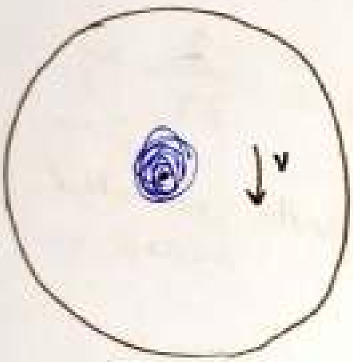
There is another kind of defination of matter, which is, How much it gravitates.

First it was thought that galaxies were made ~~of~~ of luminous matter & mass of galaxy was just mass of luminous matter.

People then observed that ~~ground~~ some cluster of galaxies were misbehaving, i.e; they were not properly following Newtons laws, or Einsteins laws.

→ and people then ~~sujje~~ sujjested that there was some more matter beside what is observed.

Rotation Curves of Galaxy.



lets begin by assuming something which would follow if all of the matter was luminous.

The fact is that most of the luminous matter in the galaxy is in the centre of the galaxy.

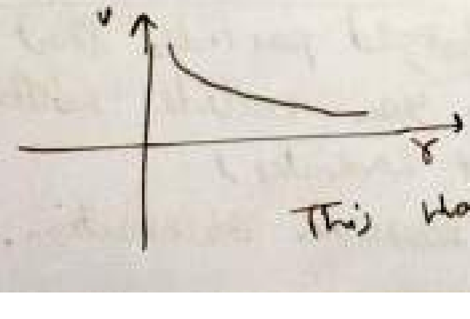
look at stars at different raddi & their tangential velocity.

For gravitational purposes, the outer parts of galaxy are moving under the influence of central force provided by mass near center. (Although this is wrong, in todays time)

let: M be mass at centre.

$$\frac{MG}{r^2} = \frac{v^2}{r} \Rightarrow v^2 = \frac{MG}{r} \Rightarrow v = \sqrt{\frac{MG}{r}} \quad \text{i.e: } v \sim \frac{1}{\sqrt{r}}$$

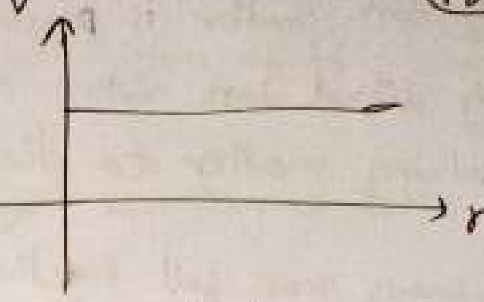
This is keplers planetary law.



This was expected.

What was observed, is that velocity is pretty much constant

~~not oon~~ (not angular velocity, but tangential velocity)



Linear velocity.

Now, lets think, how Newton would have analysed this.

let say mass is distributed. i.e.; look there is more mass out there.

let $M(r)$: amount of mass contained within a sphere out to radius r.

$$\frac{MG}{r^2} = \frac{v^2}{r} \Rightarrow \frac{M(r)G}{r^2} = \frac{v^2}{r}$$

$$\Rightarrow v^2 = \frac{M(r)G}{r} \qquad \text{if } v \text{ is constant} \quad \Rightarrow \underline{M(r) \sim r}$$

Also, we know from the motion of stars perpendicular to the plane of galaxies that it looks as if this distribution of mass is spherical in character; rather confirming the flat spiral shape of galaxy.

Now, by knowing amount of luminous matter & $M(r) \sim r$ we can find how Dark Matter (non-luminous matter) is distributed.

(Dark Matter) = (6 or 5 or 10) (Luminous matter)

~~3 Check roughly from~~

it is order of magnitude 10 times larger than luminous matter.

Why dark matter is bigger in size.
Why didnot dark matter also fall together with the same way as luminous matter to the centre of galaxy.

Luminous mass fell together by loosing energy; it lost energy to radiation and among other things; and in the process of colliding, radiating & loosing energy; it fell into centre.

So, what we need to expect then about dark matter is that whatever it is, it is much more weakly interacting.
In particular, it is not electrically charged.

the dark matter particles which make up ~~hallo~~ halla over there; the hallow matter had not collapssed like luminous matter over these years.
⟶ means that

(There is some degree, to which they do follow each other; i.e. they interact; but very weakly)
↳ (in terms of so electrical force)

It is very likely that, the galaxies formed by first the dark matter collecting in these great big halos & then the baryons, proton, electron... falling into the dark matter hallos; & then stars & galaxy are formed.

∴ They interact through gravitational forces.

Dark matter also tend to cluster because of gravitation.

Dark matter also did not make, $\Omega_M = 1$.
(so no hope for K becoming positive)