String Theory

Bosonic Strings, Superstrings, and different types of String theories.

SHOAIB AKHTAR

STRING THEORY

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These notes are consequence of my self study; which I prepared while studying the subject. Different sources were used like books by **Schwarz & other authors** titled String Theory and M-Theory: A Modern Introduction, by **Polchinski** titled String Theory.
And some video lectures exclusively for the last section on M-Theory and 5 String theories; note that this is motivational: explicit calculations are omitted. It may be covered up in other notes properly in its full glory.

Lec 1: Why String Theory? Historical Introduction

String Theory (Quantum Gravity)

General Relativity: $S_{GR} = \frac{1}{16\pi G}\int d^4x \sqrt{-g} \cdot R$

Standard Model of Particle Physics: Quantum field theory in flat space-time.

$$S_{SM} = \int d^4x \left(-\frac{1}{4} \operatorname{tr} F^2 + \dots \right).$$

Step 1: S_{GR} as a QFT

so; $g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{\frac{1}{M_R} h_{\mu\nu}}_{\text{Fluctuation.}}$

$M_R^2 = \frac{\hbar c}{8\pi G}$; $M_R \sim 10^{18}$ GeV

Reduced Planck mass.

$M_{PL}^2 = \frac{\hbar c}{G}$; $M_{PL} \sim 10^{19}$ GeV

Plank mass.

Treat the fluctuation $h_{\mu\nu}$ as quantum field.

$$S[h] = \int d^4x \left(\underbrace{(\partial h)^2}_{\text{free Lagrangian}} + \underbrace{\frac{1}{M^2} h (\partial h)^2 + \dots}_{} \right)$$

We have to expand everything; even the $\sqrt{-g}$ term

seems that we get ∞ no. of interaction vertices.

1 loop computation

Pure gravity is 1-loop finite.

(If it was divergent; there should be some counter term which we could put into the action in order to get rid of that divergence or to compensate for that.

Then we can check that the possible counter terms that we can put to contribute to one loop contribution can be obtained by field redifinition (except for a Topological Term).

→ but there are finite numbers.

2 loop computation

We find divergence; and this will cancelled if we add $\frac{1}{\epsilon}\frac{1}{M_p^4}\int d^4x\,\sqrt{-g}\,R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\gamma\delta}R^{\gamma\delta}{}_{\mu\nu}$ to the Action.

where $\epsilon = 4 - D$.

→ This is not topological (so divergence happens ~~here~~ there) in perturbation theory

So; Pure gravity is not a finite theory; or not renormalizable because we have to add this term.

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lets add matter; and see 1-loop computation.

Gravity + SM → 1 loop divergence

Step 2: Analogy

4-Fermi Theory → Weak Interaction.

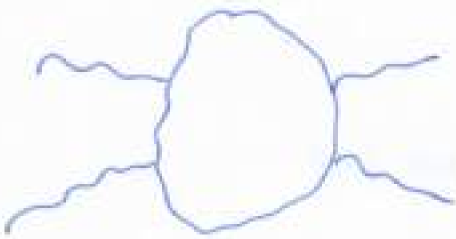
solution was Effective Theory

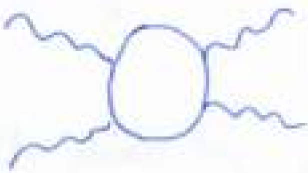
$W^{\pm}$, Z and 

$\Longleftrightarrow$ Renormalize.

Could GR be an effective theory of another QFT?

Question: Is there any QFT of gravity?





SM:  

so; we need Symmetry; that changes $\underline{\text{Bosons} - \text{Fermions}}$

Supersymmetry.

Massless Particles classified by helicity.

Graviton has helicity $h = \pm 2$.

so; $|-2\rangle$ $|+2\rangle$

The supersymmetry (the new symmetry here) acting on massless particles can be represented as operator Q. $\bar{Q}$ (hermitian conjugate of Q)

Q raises helicity by $1/2$, $\bar{Q}$ lowers helicity by $1/2$

$|-2\rangle, \; Q|-2\rangle \quad ; \quad |+2\rangle, \; \bar{Q}|+2\rangle$

$|-2\rangle, \; |-\tfrac{3}{2}\rangle \qquad |+2\rangle, \; |+\tfrac{3}{2}\rangle$

We believe CPT is symmetry of nature. So we cannot build a theory with only one kind of helicity, because it will not be CPT invariant.

$$|-2\rangle \; , \; \underbrace{|-\tfrac{3}{2}\rangle \; , \; |+\tfrac{3}{2}\rangle}_{\text{Gravitino } \psi_{\mu\alpha}} \; , \; |+2\rangle$$

$g_{\mu\nu}$

$\psi_{\mu\alpha}$ — μ: vector index, α: spinor index $\Big\} \Rightarrow$ cancels divergence at one loop.

$$\text{Superfield} \rightarrow \underbrace{g_{\mu\nu} \; , \; \psi_{\mu\alpha}}_{\text{components of Superfield}}$$

2-loop: Finite.

3-loops: Divergence.

$\Longleftrightarrow \mathcal{N}=1$ SUGRA (SUGRA = Super Gravity)

Lets say we have 2 kinds of generators Q^1, Q^2

$\mathcal{N}=2$ SUGRA & their hermitian conjugate $\bar{Q}^1, \bar{Q}^2$.

$$|-2\rangle \; , \; \underbrace{Q^I|-2\rangle}_{\psi^I_{\mu\alpha}\,|-3/2\rangle} \; , \; \underbrace{Q^1 Q^2 |-2\rangle}_{\underbrace{|-1\rangle}_{A_\mu}} \; + \text{C.P.T. friends.}$$

2-loop: Finite

3-loops: Diverges.

Maximal SUGRA

$$|-2\rangle\,,\ \underbrace{Q^I|-2\rangle}_{\text{Gravitinos}}\,,\ \underbrace{Q^I Q^J|-2\rangle}_{\text{gauge bosons}}\,,\ \ldots,\ \underbrace{Q^{I_1}Q^{I_2}\ldots Q^{I_N}|-2\rangle}_{|+2\rangle}$$

$\mathcal{N}=8$ SUGRA is the theory with maximal supersymmetry.

1-loop : finite
2-loop : finite
3-loop : Divergence (old result)

around 2006 ; Bern et. al. did 3-loop ; & find it to be finite

4-loop : finite in 2009

$\mathcal{N}=8$ SUGRA has Global Symmetry $E_{7(7)}$.

E_7 has 133 generators.

$$133 = \underbrace{N}_{N \text{ non-compact generators}} + C \longrightarrow C \text{ compact generator.}$$

$$7 = N - C$$

$N = 70$
$C = 63 = SU(8) \subset E_{7(7)}$

Quantum Gravity (Q.G.)

(A) * Low energies → GR → Macroscopic Space Time.
(B) * Meet the SM (at low energies)

Step 3| Change Paradigm.

* Check spacetime $\longrightarrow$ $h = \pm 2$ states.
* Turn on interactions $\longrightarrow$ Remains Mathematically consistent.
* (A) & (B) satisfied

String Theory (S.T.)

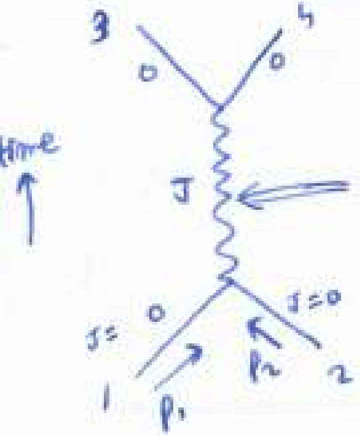
What is the big, deep new idea that led to S.T.?
Nothing, the theory was discussed by accident!

1960's| Strongly interacting Particles Mesons Massive

discovered $m^2 \propto J$
constant of proportionality $\frac{1}{\alpha'}$ ie: $m^2 = \frac{J}{\alpha'}$

ie: actually $J \sim \alpha(m^2)$
(α as a function of m^2)
α' is just derivative of α.

Scattering of $2 \rightarrow 2$.



$\left(m^2 = \frac{J}{\alpha'}, J\right) \sim \frac{t^J}{s - m_J^2}$

actually $(\text{momenta})^{2J}$

2 because 2 vertices

$s = (p_1+p_2)^2$; $t = (p_2+p_3)^2$; $u = (\quad)^2$

$\Longrightarrow$ Mandelstam Variables

mass of particle propagating.

Cubic Interaction

$S=1$ $A^2 \, \partial A$ k_μ (one power of momenta)

$S=2$ $h(\partial h)^2$ $k_\mu k_\nu$ (two " " ")

$\vdots$

$S=J$ $k_{\mu_1} \cdots k_{\mu_J}$ (J powers of momenta)

★ $A^4(s,t,u) = C_{s,t} \cdot A(s,t) + C_{t,u} A(t,u) + C_{u,s} A(u,s)$

$C_{s,t}$; $C_{t,u}$; $C_{u,s}$ → These are symmetry factors.

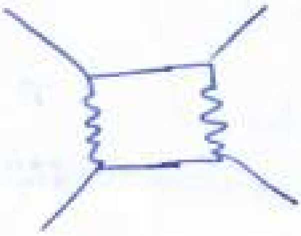
Crossing Symmetry in QFT

⟺ gives $A(s,t) = A(t,s)$

$$A(s,t)\Big|_{s\text{ channel}} = \sum_{J=0}^{\infty} \frac{t^J}{s - m_J^2}$$

$$A(s,t)\Big|_{s\text{ channel}} = \sum_{J=0}^{\infty} \frac{t^J}{s - m_J^2} \qquad A(s,t)\Big|_{t\text{ channel}} = \sum_{J=0}^{\infty} \frac{s^J}{t - m_J^2}$$

★ Loop level



large momenta behaviour

$$\sim \int^{\Lambda} d^4L \, \frac{1}{(L^2)^2} \cdot \left(\frac{(L^2)^J}{L^2} \right)^2$$

$$\sim \int^{\Lambda} d^4L \cdot (L^2)^{2J-4}$$

$J=0$ is finite

$J=1$ $\int \frac{d^4L}{L^4} \sim \int \frac{L^3\,dL}{L^4} \sim \int^{\Lambda} \frac{dL}{L} \sim \log \Lambda$

divergence ; (& we know how to deal with it)

$J \geq 2$; higher divergences.

Venerziano

~~[illegible]~~'s

Dual Hypothesis : $A(s,t)\Big|_{s\text{-channel}} = A(s,t)\Big|_{t\text{-channel}}$.

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

~~A(s,t) = ...~~

$$A(s,t) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \qquad \alpha(x) = \alpha' x + \alpha_0$$

$$\Gamma(z) = \int_0^\infty t^{z} \cdot e^{-t} \frac{dt}{t}$$

Exercise: show this satifies Dual Hypothesis. ; use $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(1) = 1$

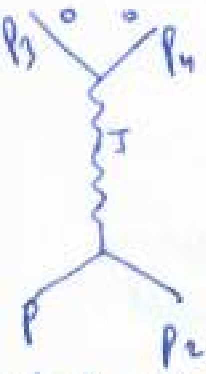
$\Gamma(z) = \frac{\Gamma(z+1)}{z}$ so; ~~[illegible]~~ $\Gamma(z)$ has pole at $z = 0$ (recall $\Gamma(1) = 1$)

→ can see that $\Gamma(z)$ has poles at $-n$. ~~[illegible]~~ where $n \in \mathbb{N} \cup \{0\}$

so; $\Gamma(-\alpha(s) - \alpha(t))$ is there in denominator to cancel poles of Neumerator.

Lec 2: Massless Fields in Curved Spacetime, Point Particle & Polyakov Actions

1960's String Theory as a theory of Strong Interactions



t-channel

$$m^2 \sim \frac{J}{\alpha'} \quad [\partial^J, \phi^2, A_{\mu_1, \dots \mu_J}]$$

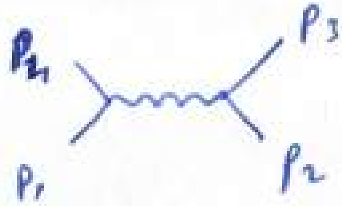
$$\rightarrow \frac{t^J}{s - m^2} \quad ; \quad \text{Veneziano} \quad A(s,t) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} \quad (*)$$

$A(s,t)\Big|_{s\text{-channel}}$ $\Rightarrow$ Contribution to amplitude from s-channel.

$A(s,t)\Big|_{t\text{-channel}}$ $\Rightarrow$ " " " " t-channel.

$$A(s,t)\Big|_{s\text{-channel}} = \sum_{J=0}^{\infty} \frac{t^J}{s - m_J^2}$$

$$A(s,t)\Big|_{t\text{-channel}} = \sum_{J=0}^{\infty} \frac{s^J}{s - m_J^2}$$



S-channel

$$A(s,t) \xrightarrow[\substack{s\to\infty \\ t\to\infty}]{} (F(\theta))^{-\alpha(s)}$$ decays exponentially!

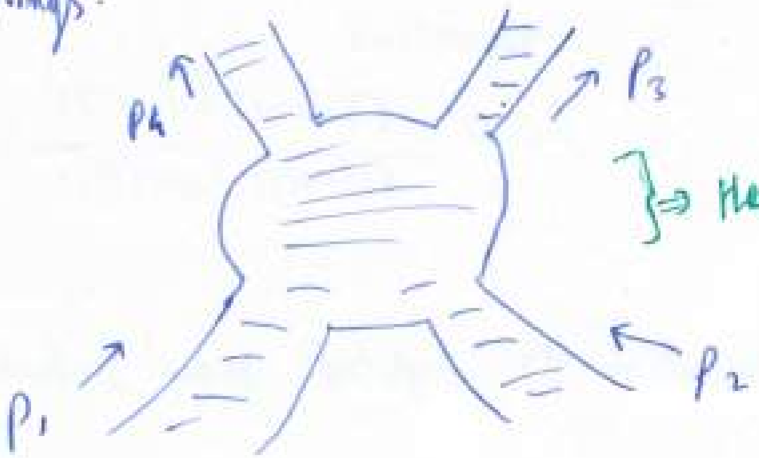
Gamma function regulates at large momentum; and gives something which depends on scattering angle

→ Exponential decay at high momenta has a very good behavior.

→ Then people tried to identify what kind of model would produce something like this as the scattering amplitude

Recall; The formula (*) was found by searching for something that would Satisfy ~~Dual~~ Dual Hypothesis. 1910

People found we can get $A_{\text{Veneziano}}$ ~~[illegible]~~ amplitude from a model where we are scattering relativistic strings.

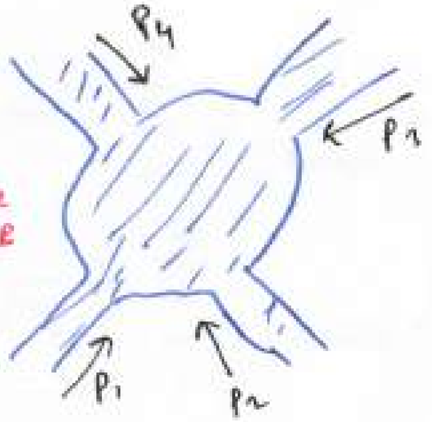


$\Rightarrow$ Here: string vibrates in infinite modes.

Its intuitively clear why this will satisfy Dual Hypothes:
make the all momentum comming inward.

$\Uparrow$ look like this ... we get t-channel

$\Rightarrow$ look like this ... we get s-channel



Yang Mills ..... SU(2) "clearly wrong"
(Y.M.) (generalizes U(1) Maxwell)

People also found $SU(N_1) \times SU(N_2) \times \ldots \times SU(N_n) \times \ldots$ gauge theories where we could have any number of Gauge groups.
And can have matter in different representations:

$\emptyset$ $\quad \square$

These formed infinite no. of families of Theories.
ie; Landscape of Theories.
(so there was an issue of ; why nature choose one out of these many possibilities)

We found from experiments.

$$A(s,t) \xrightarrow[s,t \to \infty]{} \text{Power law decay !}$$

QCD replaces this as a theory of Strong Interaction.

QCD : 1970's Asymptotic Freedom $\Rightarrow$ IR $\rightarrow$ massive particles
$\Rightarrow$ UV $\rightarrow$ SU(3) Y.M.

String Theory $\xrightarrow[\text{quantize}]{}$ Always has massless particle of $h = \pm 2$ whenever we quantize.
(people tried to remove it ; but they cannot remove it)

People faced problem in describing Strong Interactions using S.T.

In 1974 ;
Green & Schwarz : said that ; lets accept that the theory predicts $h = \pm 2$ massless. $\hookrightarrow$ Gravity.

Birth of S.T. as a theory of Q.G. (1975)

G.R. in M-dimensions + Scalar fields + Cosmological Constant (19/12)

$$S[g,\phi] = \int d^Mx \sqrt{-g}\cdot\left[R - \frac{1}{2} g^{mn} \partial_m \phi \partial_n \phi - \Lambda \right]$$

- Diffeomorphism invariant (Its not actually the symmetry; but a redundancy in the ~~meta~~ description using metric)

- Global Symmetry : $\phi \to \phi + a$

Now; replace $\partial_m \phi \partial_n \phi$ and introduce many scalar fields

$$S[g,\phi] = \int d^Mx \cdot \sqrt{-g}\cdot\left[R - \frac{1}{2} g^{mn} \partial_m \phi^i \partial_n \phi^j M_{ij} - \Lambda \right]$$

let, M_{ij} is not function of ϕ

Then still $\phi^i \to \phi^i + a^i$ is a global symmetry.

$\cancel{\phi^i \to \Lambda^i{}_j \phi^j \quad s.t.\ \Lambda}$

- $\phi^i \to \Lambda^i{}_j \phi^j$ s.t. $\Lambda^T M \Lambda = M$; Then its a global symmetry of the action

If $M = \mathbb{1}_{D\times D}$, $i = 1, .., D$, Then our global symmetry will be $SO(D)$

$$M = \left(\begin{array}{c|c} -1 & 0 \;..\; 0 \\ \hline 0 & \\ \vdots & \mathbb{1}_{(D-1)\times(D-1)} \\ 0 & \end{array}\right) \Rightarrow SO(1, D-1)$$

~~Along~~

$SO(1,D-1)$ + Translation = Poincaré group.

The internal space "i" indices one ; looks like internal Minkowski space.

If we use $SO(1, D-1)$ for internal field space, then one of the field will have negative sign for kinetic term in action.

So usually we work with $SO(D)$.

Equation of Motion for metric $\frac{\delta S}{\delta g_{mn}} = 0$

$$\Rightarrow \quad R_{mn} - \frac{1}{2} g_{mn} R + \Lambda g_{mn} = T_{mn}$$

$$T_{mn} = M_{ij} \left(\partial_m \phi^i \partial_n \phi^j - \frac{1}{2} g_{mn} g^{rs} \partial_r \phi^i \partial_s \phi^j \right)$$ Energy momentum tensor of matter theory.

$$\cancel{R_{mn} - \frac{1}{2} g_{mn} R}$$

$$R_{mn} - \frac{1}{2} g_{mn} R + \Lambda g_{mn} = T_{mn}$$

where; $T_{mn} = M_{ij} \left(\partial_m \phi^i \partial_n \phi^j - \frac{1}{2} g_{mn} \cdot g^{rs} \partial_r \phi^i \partial_s \phi^j \right)$

<u>Consider $M = 1$</u>, Then $x^0 \xrightarrow{\text{call it}} \tau$ we only have time

consider $M_{ij} \rightarrow \eta_{\mu\nu}$

our scalar fields gets the index μ: $\phi^i(\tau) \rightarrow X^\mu(\tau)$

$g = -e^2$ ← negative because of time component

metric has only one component

e is now a new field (which governs metric) & is a function of τ.

Then the action is $S[e, X]$

Note that in one dimension ; Reimann Tensor is zero. (pg 14)

$$S[e,x] = \int d\tau \cdot e \cdot \left[-\frac{1}{2} \cdot \left(\frac{-1}{e^2}\right) \dot{X}^\mu \dot{X}^\nu \cdot \eta_{\mu\nu} - \Lambda \right]$$

$$= \frac{1}{2} \int d\tau \left[\frac{1}{e} \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - 2e\Lambda \right] \qquad \partial_\tau X^\mu \equiv \dot{X}^\mu$$

$$\frac{\delta S}{\delta e} = 0 \quad \Rightarrow \quad -e^{-2} \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - 2\Lambda = 0$$

Note; this field e here has no Kinetic term in the action. (whenever we see something like this, we should integrate up)

solve for e & plug back in action.

where $\dot{X}^2 = \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}$

$$\Rightarrow e^2 = -\frac{\dot{X}^2}{2\Lambda} \Rightarrow \boxed{e = \frac{\sqrt{-\dot{X}^2}}{\sqrt{2\Lambda}}}$$

plug this in $S[e,x]$ and we get effective action.

$$S\left[e = \sqrt{\frac{-\dot{X}^2}{2\Lambda}}, x\right] = S_{eff}[X] = \frac{1}{2}\int d\tau \left(-\sqrt{2\Lambda}\sqrt{-\dot{X}^2} - \sqrt{2\Lambda}\sqrt{-\dot{X}^2}\right)$$

$$\Rightarrow \boxed{S_{eff}[X] = -m \int d\tau \sqrt{-\dot{X}^2}}$$ Effective action for just X.

where $m = \sqrt{2\Lambda}$

Non-relativistic limit

$$\eta_{\mu\nu} = \left(\begin{array}{c|ccc} -1 & 0 & \cdots & 0 \\ \hline 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{array}\right) \qquad \dot{x}^2 = -\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{d\vec{x}}{d\tau}\right)^2$$

In non-relativistic limit

$$S = -\sqrt{2\Lambda} \cdot \int dt \sqrt{1 - \left(\frac{d\vec{x}}{dt}\right)^2} \sim 1 - \frac{1}{2}\left(\frac{d\vec{x}}{dt}\right)^2 + \dots$$

$$S = \int dt \left(-\sqrt{2\Lambda} + \underbrace{\frac{\sqrt{2\Lambda}}{2} \cdot \left(\frac{d\vec{x}}{dt}\right)^2}_{} + \dots \right)$$

looks like Kinetic term of particle
if we interpret $\sqrt{2\Lambda}$ as mass of particle.

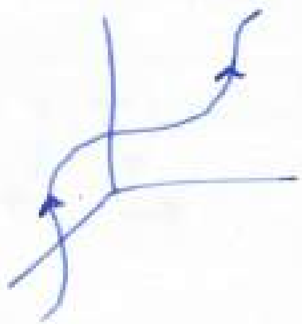
If $\eta_{\mu\nu} \rightarrow G_{\mu\nu}(x)$ some general metric

Then

$$S_{eff}[x] = -m \int d\tau \sqrt{-\dot{x}^2} \longrightarrow S[x] = -m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}^\nu G_{\mu\nu}}$$

$\hookrightarrow$ We get action for particle moving in curved spacetimes.

Equation of motion of X^μ will be geodesic equation in the curved metric $G_{\mu\nu}$.



$S[e, x] = \frac{1}{2}\int d\tau \left(e^{-1}\dot{X}^2 - m^2 e \right)$

- Diffeomorphism invariance $\tau' = \tau'(\tau)$ (redundancy)

→ can use this to gauge fix $e = 1$.

However; we can just fix $e=1$, and forget about it in Action like it never existed.

→ Because in order to preserve this constraint in time, we have to impose equations of motion. (This has to imposed as constraints)

e.o.m. : $-\dot{x}^2 - m^2 = 0$ constraint.

$$\dot{x}_\mu = \frac{\delta \mathcal{L}}{\delta \dot{x}^\mu} = P_\mu \Rightarrow \boxed{p^2 + m^2 = 0}$$

This is the constraint to be impose. (looks like dispersion relation in GR)

G.R. convention

choose $e = \frac{1}{m}$

Then ; we get $-m^2 \dot{x}^2 - m^2 = 0 \Rightarrow \boxed{\dot{x}^2 + 1 = 0}$

This is what we have seen in G.R.

$\Rightarrow \boxed{\dot{x}^2 = -1}$

Consider $n=2$ $(x^0, x^1) \longrightarrow (\tau, \sigma)$

τ ↑ time, σ ↑ space

$\phi^i \longrightarrow X^\mu(\tau, \sigma)$

$\eta_{ij} \longrightarrow \eta_{\mu\nu}$

~~Fact~~ Number of independent component of $R^{\lambda}{}_{\mu\nu\sigma}$ $= \frac{1}{12} M^2(M^2-1)$

$$R_{\mu\rho\nu\sigma} = g_{\rho\lambda} R^{\lambda}{}_{\mu\nu\sigma}$$

We can show; $R_{\rho\mu\nu\sigma} = -R_{\mu\rho\nu\sigma}$

$$R_{\rho\mu\nu\sigma} = -R_{\rho\mu\sigma\nu}$$

$R_{\rho\mu\nu\sigma} = R_{\nu\sigma\rho\mu}$; $R_{\mu a_1 a_2 a_3}$ + cyclic $(1,2,3) = 0$

$M = 2$; # d.o.f $= 1$ $\qquad g_{mn} \to h_{mn}$ (we like to call metric as h in this part of theory)

$$R_{mnpq} = \alpha\,(g_{mp}\, g_{nq} - g_{mq}\, g_{np})$$

α is our d.o.f here.

α is a scalar. $\qquad \alpha \propto \underbrace{R}_{\text{Ricci Scalar.}}$

$$R_{nq} = g^{mp} R_{mnpq} = \alpha\,(2\, g_{qn} - g_{qn}) = \alpha\, g_{qn}$$

$$R = g^{nq} R_{nq} = 2\alpha \Rightarrow \boxed{\alpha = \frac{R}{2}}$$

$$\Rightarrow \boxed{R_{nq} = \frac{R}{2} \cdot g_{qn}}$$

In $M=2$, we find $G_{mn} = R_{mn} - \frac{1}{2} g_{mn} R = 0$

so: $\Lambda g_{mn} = T_{mn}$

so: $T_{mn} = \eta_{\mu\nu}\left(\partial_m X^\mu \partial_n X^\nu - \frac{1}{2} h_{mn} h^{pq} \partial_p X^\mu \partial_q X^\nu\right)$

$T^{m}{}_{m} = h^{mn} T_{mn} = 0 \Rightarrow \mathrm{Trace}(T_{mn}) = 0$!

So; $\Lambda g_{mn} = T_{mn} \xrightarrow{\text{take trace}} \Lambda g^{m}{}_{m} = T^{m}{}_{m}$

$\Rightarrow \Lambda \cdot 2 = 0$

$\Rightarrow \underline{\underline{\Lambda = 0}}$

The only cosmological constant consistent with equation of motion is $\Lambda = 0$.

Question: Why we get Einstein Tensor $G_{mn} = 0$ for $M = 2$? What happens to gravity.

$$\underbrace{\int_{\Sigma} d\tau d\sigma \sqrt{-h} \cdot R}_{} = \chi(\Sigma)$$

make it euclidean; then do integration; Then we get Euler number of our 2d spacetime.

$\Sigma \in \{$  ,  , ... $\}$

$\tau = \sigma^0$, $\sigma = \sigma^1$

$$S[h_{mn}, X^{\mu}] = \int d^2\sigma \sqrt{-h} \left(-\frac{1}{2} h^{mn} \partial_m X^{\mu} \partial_n X^{\nu} \eta_{\mu\nu} \right)$$

This is Polyakov action.

Lec 3: Relativistic strings, Equations of Motion, Constraints, Boundary Conditions.

Polyakov Action

$$S[h_{\alpha\beta}, X^\mu] = -\frac{1}{2}\int d^2\sigma \sqrt{-h}\cdot\left(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right)$$

$\sigma^0 = \tau$, $\sigma^1 = \sigma$ X^μ refers to target space

World Sheet.

Scalar fields lives in $\mathbb{R}^{1,1}$

Target space scalar product

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} h_{\alpha\beta} h^{\rho\gamma}\partial_\rho X\cdot\partial_\gamma X$$

Equation of motion $T_{\alpha\beta} = 0$.

$$\Rightarrow \underbrace{\partial_\alpha X\cdot\partial_\beta X}_{\text{call it } G_{\alpha\beta}} = \frac{1}{2} h_{\alpha\beta} h^{\rho\gamma}\partial_\rho X\cdot\partial_\gamma X$$

$$\det(G_{\alpha\beta}) = \left(\frac{1}{2}h^{\rho\gamma}\partial_\rho X\cdot\partial_\gamma X\right)^2 \det h_{\alpha\beta}$$

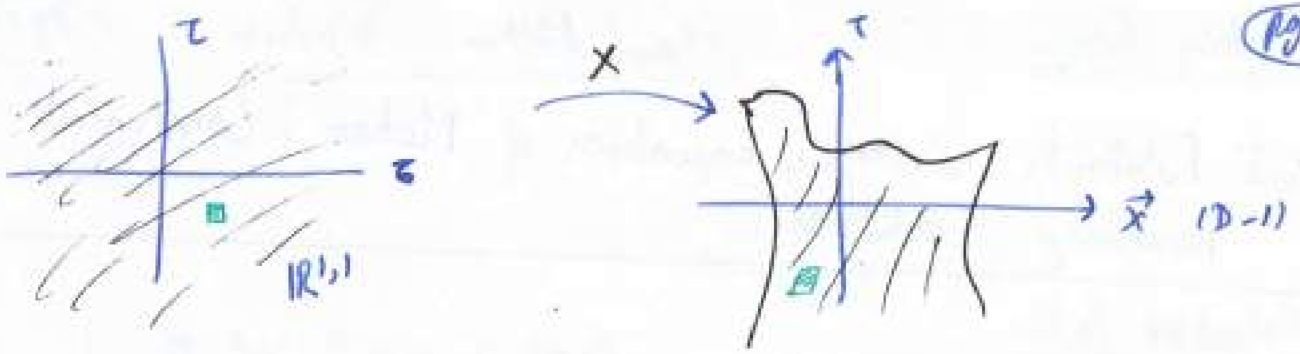
$$\Rightarrow -\det(G_{\alpha\beta}) = \left(\frac{1}{2}h^{\rho\gamma}\partial_\rho X\cdot\partial_\gamma X\right)^2 (-\det h_{\alpha\beta})$$

$$\Rightarrow \sqrt{-\det G_{\alpha\beta}} = \frac{1}{2}\cdot h^{\rho\gamma}\partial_\rho X\cdot\partial_\gamma X\cdot\sqrt{-h}$$

So; Effective action for X after integrating out h is

$$S_{eff}[X] = -\int d^2\sigma \sqrt{-\det G_{\alpha\beta}}$$

where $G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$



$\hat{G}_{\alpha\beta}$ is induced metric on ~~surface~~ world sheet from spacetime.

$\int d^2\sigma \sqrt{-\det \hat{G}_{\alpha\beta}}$ is really computing area of world sheet.

$d^2\sigma \sqrt{-\det \hat{G}_{\alpha\beta}}$ local area form.

so; ~~This S_{eff} computes area~~.

This $S_{eff}[X]$ computes area in target space.

But, we dont want an action that has units of area. We ~~took~~ introduce a dimensionful constant T to compensate this.

(later we will interpret T as tension of string)

$$S_{eff}[X] = -T \int d^2\sigma \cdot \sqrt{-\det G_{\alpha\beta}} \qquad T = \frac{1}{2\pi\alpha'}$$

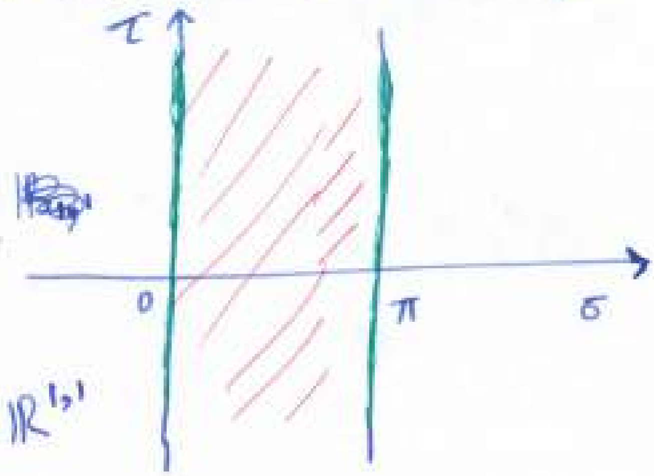
NAMBU - GOTO Action

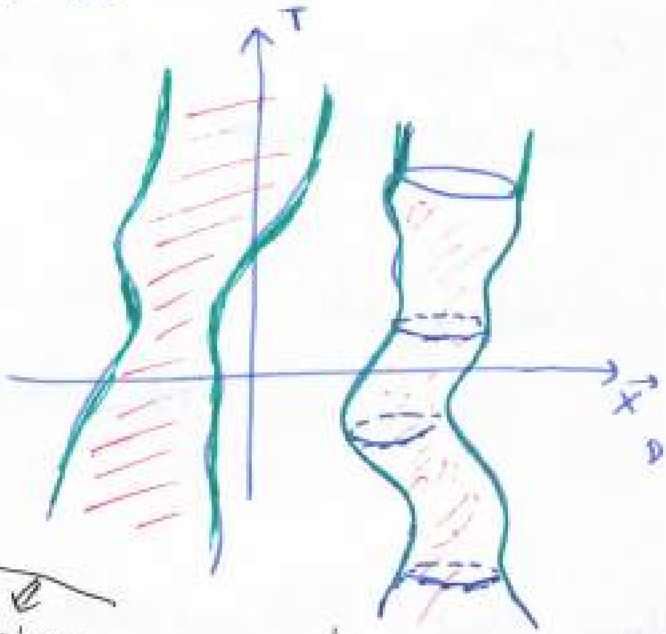
$[\alpha'] = (\text{length})^2$

$\hookrightarrow$ This will give ∞ if we integrate over whole of $\mathbb{R}^{1,1}$ (∞ area....)

So; we make some restrictions on the region of integration

Restrict σ' domain from 0 to π





open strings.

Once we do this ....
It will look like nice strings;
which describes one of our mesons

If we identify
$\sigma = 0 \sim \sigma = \pi$
Closed Strings

We could replace $\eta_{\mu\nu}$ by $\hat{G}_{\mu\nu}(X)$ general metric.

Then the formalism will describe evolution of string moving in a curved spacetime $\hat{G}_{\mu\nu}(X)$

(From the point of view of field Theory living on world sheet $\mathbb{R}^{1,1}$ (spanned by τ & σ) ; $\hat{G}_{\mu\nu}(X)$ is not a metric in any spacetime : it is just a complicated way of writing coupling between the scalar fields.

Properties of Polyakov Action:

1) Diffeomorphism invariant $\quad \tilde{\sigma}^{\alpha} \rightarrow \tilde{\sigma}^{\alpha}(\sigma, t)$

2 degrees of freedom (So we can set 2 functions in the metric)

2) Weyl invariance : so ; we can take

1 d.o.f. $\qquad h_{\alpha\beta} \to e^{\phi(\sigma,\tau)} h_{\alpha\beta} \quad , \quad \delta X^\mu = 0$

(can set 1 function)

3) Global Symmetry: $\delta X^\mu(\sigma,\tau) = a^\mu{}_\nu X^\nu(\sigma,\tau) + b^\mu$

$$\delta h_{\alpha\beta} = 0$$

so

Using Weyl + Diffeomorphism; we can Gauge fix the metric $h_{\alpha\beta}$ to be $\quad [h_{\alpha\beta}] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = [\eta_{\alpha\beta}]$

$\to$ Minkowski metric on the worldsheet.

$S_{gf}[X] \Rightarrow$ Gauge Fixed Polyakov action

is, $\quad S_{gf}[X] = +\frac{T}{2}\int d^2\sigma \, (\dot{X}^2 - X'^2)$

$\frac{\partial X^\mu}{\partial \tau} = \dot{X}^\mu \quad , \quad \dot{X}^2 = \frac{\partial X^\mu}{\partial \tau} \cdot \frac{\partial X^\nu}{\partial \tau} \cdot \eta_{\mu\nu}$

Contraction with Target space metric.

$\frac{\partial X^\mu}{\partial \sigma} = X'^\mu$

$$X'^2 = \frac{\partial X^\mu}{\partial \sigma} \cdot \frac{\partial X^\nu}{\partial \sigma} \cdot \eta_{\mu\nu}$$

Field Equations: $\delta S[X] = \pm\frac{T}{2}\int d^2\sigma \, (2\dot{X}\delta\dot{X} - 2X'\delta X')$

now integrate by parts....

$$\delta S[X] = +T\int d^2\sigma\left(-\ddot{X}\,\delta X + \frac{\partial}{\partial\tau}(\dot{X}\delta X) + X''\,\delta X - \frac{\partial}{\partial\sigma}(X'\delta X)\right)$$ pg 23

i.e. $$\delta S[X] = +T\int d^2\sigma\left(-\ddot{X}\,\delta X + \frac{\partial}{\partial\tau}(\dot{X}\,\delta X) + X''\,\delta X - \frac{\partial}{\partial\sigma}(X'\,\delta X)\right)$$

$$\Rightarrow \delta S[X] = T\int d^2\sigma\left[(\ddot{X} - X'')\cdot\delta X + \frac{\partial}{\partial\tau}(\dot{X}\delta X) - \frac{\partial}{\partial\sigma}(X'\delta X)\right]$$

$$\Downarrow$$

$$\ddot{X}^\mu - X''^\mu = 0$$

$\int d^2\sigma\,[(\ddot{X} - X'')]\delta X = 0$ for any general variation δX^μ.

i.e. $$\boxed{\left[\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right]X^\mu(\sigma,\tau) = 0}$$ Equation of motion

Recall that; gauge fixing does not comes for free; we also have to impose ~~initial conditions~~ constraints.

constraints: $T_{\alpha\beta} = 0$ come from gauge fixing.

$$T_{00} = \dot{X}^2 + \frac{1}{2}(-\dot{X}^2 + X'^2) = \frac{1}{2}(\dot{X}^2 + X'^2)$$

check $T_{00} = T_{11}$ so; impose $T_{00} = 0$

$$T_{01} = T_{10} = X'\cdot\dot{X}$$

So; Constraints are

$$X'\cdot\dot{X} = 0$$

$$\dot{X}^2 + X'^2 = 0$$

Boundary Conditions

① $\int_0^\pi d\sigma \int_{-\infty}^{\infty} d\tau \, \frac{\partial}{\partial \tau}(\dot{X} \cdot \delta X) \longrightarrow 0$

Assumption
variation vanishes
at $\tau \to \pm\infty$
i.e; $\delta X^\mu(\pm\infty) = 0$

② $\int_{-\infty}^{+\infty} d\tau \int_0^\pi d\sigma \cdot \frac{\partial}{\partial \tau}(X' \cdot \delta X)$

$$= \int_{-\infty}^{+\infty} dt \left(X' \cdot \delta X \Big|_{\sigma=\pi} - X' \cdot \delta X \Big|_{\sigma=0} \right)$$

⟹ Have to find a way to set this to be zero

(This is not easy as ① because σ varies over a compact region)

Have to set $\int_{-\infty}^{+\infty} dt \left[X'^\mu \cdot \delta X_\mu \Big|_{\sigma=\pi} - X'^\mu \cdot \delta X_\mu \Big|_{\sigma=0} \right]$

2.1 Neumann: $X'^\mu(\tau, \pi) = 0$ and/or $X'^\mu(\tau, 0) = 0$

2.2 Dirichlet $X^\mu(\tau, \pi) = X^\mu_\pi$ and/or $X^\mu(\tau, 0) = X^\mu_0$

→ setting it to be constant ; i.e; $\delta X_\mu \Big|_{\sigma=0,\pi} = 0$.

→ for open strings

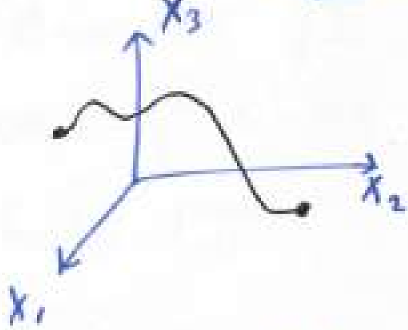
2.3 Periodic : $X^\mu(\tau, \pi) = X^\mu(\tau, 0)$

→ for closed strings.

Examples $D=4$, $\mathbb{R}^{3,1}$

(i) Open Strings

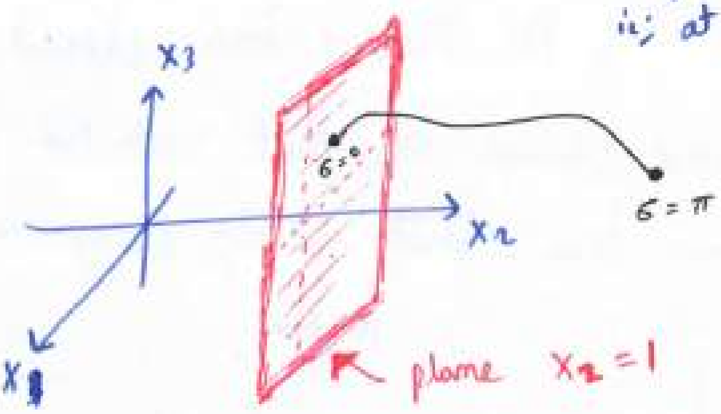
$$\begin{array}{c|ccc} & X^1 & X^2 & X^3 \\ \sigma=0 & N & N & N \\ \sigma=\pi & N & N & N \end{array}$$



Open string where end points can move freely in whole of spacetime.

(Just a snapshot at a fixed time; i.e. at fixed X_0)

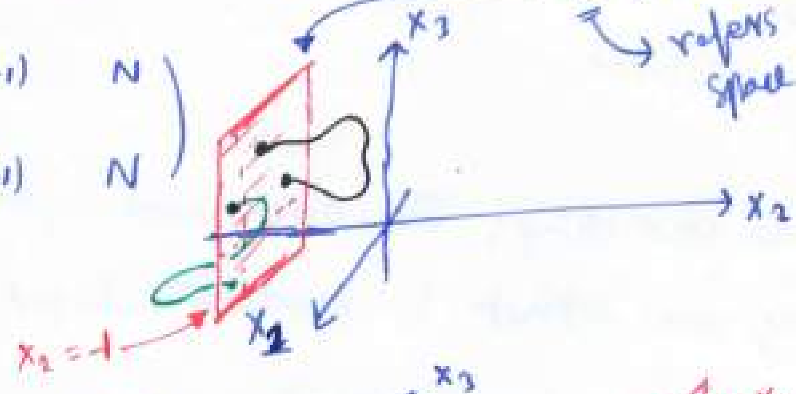
(ii) $\begin{pmatrix} N & D(+1) & N \\ N & N & N \end{pmatrix}$



Endpoint $\sigma=0$, is constraine to lie in the plane

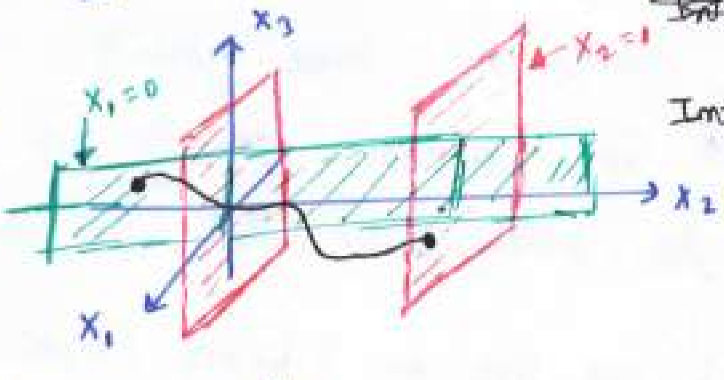
Endpoint $\sigma=\pi$, moves freely.

(iii) $\begin{pmatrix} N & D(-1) & N \\ N & D(-1) & N \end{pmatrix}$



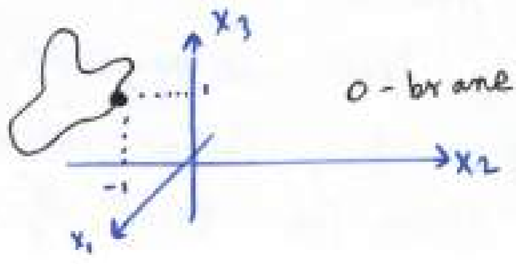
2 brane ⟶ refers to no. of space direction ~~[illegible]~~

(iv) $\begin{pmatrix} D(0) & N & N \\ N & D(+1) & N \end{pmatrix}$



~~Intersected~~ Intersecting branes.

(v) $\begin{pmatrix} D(0) & D(-1) & D(1) \\ D(0) & D(-1) & D(1) \end{pmatrix}$



0 - brane

If end points of string is restricted to live on a p-dimensional (spacial) object;

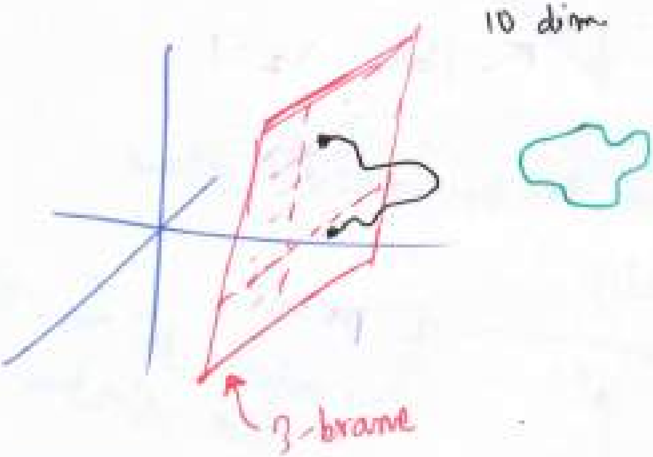
That object is called p-brane.

Discussion

If the theory has closed strings!

The closed strings are not restricted to live on brane.

It can live ~~pretty~~ freely everywhere else.



If we have open string; Then dont have gravity.

(open string can interact & produce closed strings)

Closed strings can have gravity

⟷ So; somehow gravity will behave very differently from the other fields that live on the 3-brane.

☆ If we live on 3 brane (restricted to live on 3-brane); and gravity lives anywhere else: Then the scale at which the two things interact will be very different from the point of view of the brane.

~~So; Forces that are carried by open strings~~

So; Forces on the brane that are carried by the open strings will look ~~must~~ much stronger than gravity.

(matches with observation .... solves Heirarchy problem)

We will have a way of getting Standard Model from Intersecting branes

Light Cone Coordinates

$$\sigma^{\pm} = \frac{1}{2}(\tau \pm \sigma)$$

$$\frac{\partial}{\partial \sigma^{\pm}} = \frac{1}{2}\left(\frac{\partial}{\partial \tau} \pm \frac{\partial}{\partial \sigma}\right) \qquad \partial_+ \partial_- X^\mu = 0$$

Then our field eqn looks $\boxed{\frac{\partial}{\partial \sigma^+} \frac{\partial}{\partial \sigma^-} X^\mu(\sigma^+, \sigma^-) = 0}$

Most general solution (separates in two parts) ~~$\partial_+\partial_- X^\mu(\sigma^+,\sigma^-)=0$~~

$$X^\mu(\sigma^+, \sigma^-) = \underbrace{X^\mu_R(\sigma^-)}_{\text{called Right movers.}} + \underbrace{X^\mu_L(\sigma^+)}_{\text{called Left movers.}}$$

Choose: Closed string boundary conditions

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \pi)$$

$$\left.\begin{aligned} \partial_- X^\mu &= \partial_- X^\mu_R \\ \partial_+ X^\mu &= \partial_+ X^\mu_L \end{aligned}\right\}$$ once we take derivative: These are independently periodic.

$\partial_- X^\mu_R$ & $\partial_+ X^\mu_L$ are periodic function in ~~is~~ σ_- & σ_+ respectively. Lets expand them in Fourier modes.

$$\partial_- X^\mu_R = \sum_{m=-\infty}^{+\infty} \alpha^\mu_m e^{-2im\sigma_-}$$

price we had to pay for not choosing 2π for σ' to vary

its convenient to have the fourier modes dimensionless.

So; we write. $$\partial_- X^\mu_R = l_s \sum_{m=-\infty}^{+\infty} \alpha^\mu_m \cdot e^{-2im\sigma_-}$$

string length.

doing same thing for left movers.

$$\partial_+ X^\mu_L = l_s \sum_{m=-\infty}^{+\infty} \tilde{\alpha}^\mu_m \cdot e^{-2im\sigma_+}$$

Integrating this.

$$\cancel{X^\mu_R(\sigma_-) = X^\mu_R + l_s \alpha^\mu_0 \sigma_-}$$

$$X^\mu_R(\sigma_-) = x^\mu_R + l_s \alpha_0^\mu \cdot \sigma_- + \frac{i l_s}{2} \sum_{m\neq 0} \frac{\alpha^\mu_m}{m} \cdot e^{-2im\sigma_-}$$

$$X^\mu_L(\sigma_+) = x^\mu_L + l_s \cdot \tilde{\alpha}^\mu_0 \cdot \sigma_+ + \frac{i l_s}{2} \sum_{m\neq 0} \frac{\tilde{\alpha}^\mu_m}{m} \cdot e^{-2im\sigma_+}$$

$$\Rightarrow X^\mu(\sigma_+, \sigma_-) = (x_R^\mu + x_L^\mu) + l_s(\alpha^\mu_0 \sigma_- + \tilde{\alpha}^\mu_0 \sigma_+) + \frac{i l_s}{2} \sum_{m\neq 0} \frac{1}{m}\left[\alpha^\mu_m \cdot e^{-2im\sigma_-} + \tilde{\alpha}^\mu_m \cdot e^{-2im\sigma_+}\right]$$

We want this to be periodic in σ. so: $\alpha^\mu_0 = \tilde{\alpha}^\mu_0$

~~Therefore~~ We also want X^μ to be real

~~$X^\mu(\sigma^+,\sigma^-)$~~ so: $\alpha^{\mu *}_{m} = \alpha^{\mu}_{-m}$

$\tilde{\alpha}^{\mu *}_{m} = \tilde{\alpha}^{\mu}_{-m}$

Then we get;

$$X^\mu(\sigma^+,\sigma^-) = (x_R + x_L)^\mu + 2\tau\,\alpha^\mu_0 + \frac{i l_s}{2}\sum_{m\neq 0}\frac{1}{m}\left[\alpha^\mu_m e^{-2im\sigma_-} + \tilde{\alpha}^\mu_m e^{-2im\sigma_+}\right]$$

Using Neuman (Open String)

~~So~~ we get,

$$X^\mu(\tau,\sigma) = X^\mu + l_s\cdot\alpha^\mu_0\cdot\tau + i l_s\sum_{m\neq 0}\frac{\alpha^\mu_m}{m}\cdot e^{-im\tau}\cdot\cos(m\sigma)$$

Lec 4: Closed Strings, Mode Expansion & Quatization.

Closed Strings

$$X^\mu(\sigma,\tau) = X^\mu_R(\sigma,\tau) + X^\mu_L(\sigma,\tau)$$

$$X^\mu(\sigma,\tau) = x^\mu + 2l_s\alpha^\mu_0\cdot\tau + \frac{il_s}{2}\sum_{m\neq 0}\frac{1}{m}\left[\alpha^\mu_m\cdot e^{-2im\sigma_-} + \tilde{\alpha}^\mu_m\cdot e^{-2im\sigma_+}\right]$$

(Better name for Right & left movers will be anticlockwise & clockwise.)

cc ⇒ counter-clockwise
c ⇒ clockwise

$\partial_- X^\mu_R(\sigma_-)$ are actually $\partial_- X^\mu_{R_{cc}}(\sigma_-)$

$\partial_+ X^\mu_L(\sigma_+)$ " " $\partial_+ X^\mu_{L_c}(\sigma_+)$

Noether's Theorem

(to study conserved charges in the system)

for every Global Symmetry $\longrightarrow$ Current j_α

current on the Worldsheet

$\alpha \in \{0,1\}$

conserved current $\partial^\alpha j_\alpha = 0$

So, we get Conserved charge: $\underline{Q = \int d\sigma\, j_0}$

Global Symmetries

$$\delta X^\mu = b^\mu \longrightarrow j^\mu_\alpha \underset{\text{call it}}{\Rightarrow} \overset{(\text{notation})}{P^\mu_\alpha}$$

~~We call P^μ_α instead of j^μ_α because there will be other curr~~

(We call P^μ_α instead of j^μ_α because we will be having other currents also ~~assoc~~ associated to other symmetry)

$$S = \frac{+T}{2}\int d^2\sigma\,(\dot{X}^2 - X'^2)$$

$$\boxed{P^\mu_\alpha \doteq T\,\partial_\alpha X^\mu}$$

call the charge P^μ i.e; $P^\mu = \int_0^\pi d\sigma \cdot T \cdot \dot{X}^\mu$

$$\Rightarrow \boxed{P^\mu = T\int_0^\pi d\sigma \cdot \dot{X}^\mu} \Rightarrow \text{Is it conserved?}$$

Note)
(Not everytime we have conserved current, we have conserved charge.
Recall, the proof uses the Stokes theorem to get the integral at ∞.

If the field dont vanish fast enough at ∞; or there is no ∞ (finite) .... Then we have to be careful)

Here; $$\frac{d}{dt}P^\mu = T\int_0^\pi d\sigma \cdot \underbrace{\ddot{X}^\mu}_{\text{e.o.m}} = T\int_0^\pi d\sigma \cdot \frac{\partial}{\partial\sigma} X'^\mu$$

X''^μ

$$= T\left(X'^\mu\big|_{\sigma=\pi} - X'^\mu\big|_{\sigma=0}\right)$$

$$= 0 \quad \text{for closed string.}$$

for open strings; $\frac{d}{dt}P^\mu = 0$ if both ends have Neuman condition.

In case of Dirichlet ; $\frac{d}{d\tau} P^\mu \neq 0$

Its attached to something ; & so that can provide momentum to it.

$$P^\mu = 2 l_s \, T \cdot \alpha^\mu_0 \cdot \pi \Rightarrow \alpha^\mu_0 = \frac{P^\mu}{2 l_s \cdot T \cdot \pi}$$

$$\boxed{\alpha^\mu_0 = \frac{P^\mu}{2 l_s \cdot T \cdot \pi}}$$

Under $\delta X^\mu = a^\mu{}_\nu X^\nu$ lorentz.

Then $j_\alpha^{\mu\nu} \Rightarrow$

$$\boxed{J_\alpha^{\mu\nu} = T \left(X^\mu \partial_\alpha X^\nu - X^\nu \cdot \partial_\alpha X^\mu \right)}$$

$J^{\mu\nu}$ is interpreted as Angular momentum charges in a spacetime in the target space.

Constraints: $T_{\alpha\beta} = 0 \Rightarrow \dot{X}^2 + X'^2 = 0$ & $\dot{X} \cdot X' = 0$

$$\Rightarrow \dot{X}^2 \pm 2\dot{X} \cdot X' + X'^2 = 0$$

$$\Rightarrow (X' \pm \dot{X})^2 = 0$$

$$\Rightarrow \boxed{(\partial_+ X)^2 = 0 \quad \& \quad (\partial_- X)^2 = 0}$$

$(\partial_+ X_L)^2$ & $(\partial_- X_R)^2$ are periodic in σ.

So; decompose these in fourier modes (Then it will be easier to constraints ; We just have to impose that every mode has to be zero) label the modes now as $\tilde{L}_m$ & L_m.

$$(\partial_+ X_L)^2 = 2 l_s^2 \sum_{m=-\infty}^{+\infty} \tilde{L}_m \cdot e^{-2im\sigma_+}$$

$$(\partial_- X_R)^2 = 2 l_s^2 \sum_{m=-\infty}^{+\infty} L_m \cdot e^{-2im\sigma_-}$$

Compute $\tilde{L}_m$ & L_m as a function of $\tilde{\alpha}_m$ & α_m respectively.

1) set $\tau = 0$.

2) $\displaystyle\int_0^\pi (\partial_+ X_L)^2 \Big|_{\tau=0} \cdot e^{2in\sigma} \cdot d\sigma = 2 l_s^2 \tilde{L}_n \int_0^\pi d\sigma$

$$\Rightarrow \boxed{\tilde{L}_n = \frac{1}{2 l_s^2 \cdot \pi} \int_0^\pi d\sigma \cdot (\partial_+ X_L)^2 \cdot e^{2in\sigma}}$$

$$\tilde{L}_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \tilde{\alpha}^\mu_m \cdot \tilde{\alpha}^\nu_{n-m} \cdot \eta_{\mu\nu}$$

so:

$$\boxed{\begin{aligned} \tilde{L}_n &= \frac{1}{2} \sum_{m=-\infty}^{+\infty} \tilde{\alpha}_m \cdot \tilde{\alpha}_{n-m} \\ L_n &= \frac{1}{2} \sum_{m=-\infty}^{+\infty} \alpha_m \cdot \alpha_{n-m} \end{aligned}}$$

Constraints are: $\tilde{L}_n = 0$, $L_n = 0$ $\forall n$

lets look at $n=0$: $\tilde{L}_0 = 0$

so; $\tilde{L}_0 = \frac{1}{2}\tilde{\alpha}_0^2 + \frac{1}{2}\sum_{m\neq 0} \tilde{\alpha}_m \cdot \tilde{\alpha}_{-m}$

$$\Rightarrow \tilde{L}_0 = \frac{1}{2}\tilde{\alpha}_0^{\,2} + \sum_{m>0} \tilde{\alpha}_m \cdot \tilde{\alpha}_{-m}$$

$\tilde{\alpha}_0^2 = \frac{p^2}{(2 l_s T \cdot \pi)^2} = \frac{-M^2}{(2 l_s \cdot T \cdot \pi)^2}$ use $p^2 + M^2 = 0$

$$\Rightarrow \boxed{\frac{M^2}{8\pi^2 l_s^2 \cdot T^2} = \sum_{m>0} \tilde{\alpha}_m \cdot \tilde{\alpha}_{-m}}$$

$T = \frac{1}{2\pi\alpha'}$; now if we choose $l_s^2 = 2\alpha'$

Then $\boxed{\alpha' \cdot M^2_{closed} = 4\sum_{m>0} \tilde{\alpha}_m \cdot \tilde{\alpha}_{-m}}$

... level matching.

similarly doing with L_0 we get

$$\boxed{\alpha' \cdot M^2_{closed} = 4\sum_{m>0} \alpha_m \cdot \alpha_{-m}}$$

Writing in symmetric form

$$\alpha' M^2_{closed} = 2\sum_{m>0} \left(\alpha_m \cdot \alpha_{-m} + \tilde{\alpha}_m \cdot \tilde{\alpha}_{-m}\right)$$

Quantization: actually QFT in 2 dimensions here

$$\Pi_\mu = \frac{\delta \mathcal{L}}{\delta \dot{X}_\mu} \qquad \mathcal{L} = T(\dot{X}^2 - X'^2)$$

momentum conjugate

$\Rightarrow$ call it $P^\mu(\tau, \sigma) = \dfrac{\delta \mathcal{L}}{\delta \dot{X}^\mu}$

Classical field Theory: ~~$X^\mu(\tau,\sigma)$~~,

$$\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{P.B.} = 0$$

$$\frac{\delta X^\mu(\tau,\sigma)}{\delta X^\nu(\tau,\sigma')} = \delta^\mu{}_\nu \cdot \delta_P(\sigma - \sigma')$$

periodic version of Delta functions because our variables are periodic.

Poisson Brackets are defined using Functional Derivatives.

$$\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{PB} = 0$$

$$\{X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\}_{P.B.} = \eta^{\mu\nu}\delta_P(\sigma-\sigma')$$

$$\{P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\}_{P.B.} = 0$$

$\delta_P(\sigma-\sigma')$ is not fourier transform of 1; but fourier decomposition of 1.

$$\delta_P(\sigma-\sigma') = \sum_{m\neq 0} e^{2i\pi m(\sigma-\sigma')}$$

Note! $P^\mu = T\dot{X}^\mu$ so; $\{X^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')\}_{P.B.} = \dfrac{\eta^{\mu\nu}}{T}\cdot\delta_P(\sigma-\sigma')$

We plug in the expansion; & get an algebra for Poisson Brackets.

$$\{\alpha^{\mu}_{m}, \alpha^{\nu}_{n}\}_{P.B.} = i\, m \cdot \delta_{m+n,0} \cdot \eta^{\mu\nu}$$

$$\{\tilde{\alpha}^{\mu}_{m}, \tilde{\alpha}^{\nu}_{n}\}_{P.B.} = i m \cdot \delta_{m+n,0} \cdot \eta^{\mu\nu}$$

$$\{\tilde{\alpha}^{\mu}_{m}, \alpha^{\nu}_{n}\}_{P.B.} = 0$$

Using these we find.

$$\{L_m, L_n\}_{P.B.} = i\,(m-n)\, L_{m+n}$$

(since this is classical: so we dont have any quantum correction: So the Central Charge term is zero here)

$$\{\cdot\,, \cdot\}_{P.B.} \longrightarrow i\,[\cdot\,, \cdot] \quad \text{Quantization}$$

define $\quad a^{\mu}_{m} = \frac{1}{\sqrt{m}} \cdot \alpha^{\mu}_{m} \quad , \quad \tilde{a}^{\mu}_{m} = \frac{1}{\sqrt{m}} \cdot \tilde{\alpha}^{\mu}_{m}$

~~From~~ ~~Eq~~

Then we find a nice algebra of oscillators.

$$[a^{\mu}_{m}, a^{\nu}_{n}] = \delta_{m+n,0} \cdot \eta^{\mu\nu}$$

we had X^{μ} will be real if $\alpha^{\mu}_{m} = (\alpha^{\mu}_{-m})^{*}$

In Quantum operators: It becomes: $\alpha^{\mu}_{-m} = (\alpha^{\mu}_{m})^{\dagger}$

so: $\boxed{a^{\mu}_{-m} = (a^{\mu}_{m})^{\dagger}}$

$$[a^{\mu}_{m}, a^{\nu\dagger}_{n}] = \delta_{m,n} \cdot \eta^{\mu\nu}$$

$$[\tilde{a}^{\mu}_{m}, \tilde{a}^{\nu\dagger}_{n}] = \delta_{m,n} \cdot \eta^{\mu\nu}$$

In quantization; we have to define Vacuum $|0\rangle$

s.t. $a^{\mu}_{m}|0\rangle = 0$; $m>0$

a^{μ}_{m} ; $m>0$: Annihilation operator

$a^{\mu\dagger}_{m}$: $m>0$: Creation operator.

Problem: $|\varphi\rangle = a^{\dagger 0}_{m}|0\rangle$ study by choosing the zeroth component.

$$\langle\varphi|\varphi\rangle = \langle 0|a^{0}_{m} \cdot a^{0\dagger}_{m}|0\rangle = \langle 0|[a^{0}_{m}, a^{0\dagger}_{m}]|0\rangle = \eta^{00} = -1$$

$\Rightarrow$ so; $\boxed{\langle\varphi|\varphi\rangle = -1}$!! Norm of the state is negative (Not Positive def

General state in the (Fock) Hilbert space is of the form

$$|\varphi\rangle = a^{\mu_1\dagger}_{m_1} a^{\mu_2\dagger}_{m_2} \cdots a^{\mu_k\dagger}_{m_k}|0\rangle$$

Then ; If an odd number of μ's are equal to zero : Then $\langle\varphi|\varphi\rangle < 0$

So; we have infinite # of negative normed state

But also have Infinite # of constraints

(we hope to get this balanced ; And we can use these constraints to project out negative normed states)

$\tilde{L}_n$, L_n are operators

So Imposing the constraint

If $|\psi\rangle$ is physical state, then $L_n|\psi\rangle = 0 \quad n>0$

we know that $L_{-n} = L_n^+$

so ; imposing $L_n|\psi\rangle = 0 \quad n>0$ is

enough ; It will translate to $\langle\psi|L_n^+ = 0$

$\Rightarrow \langle\psi| L_{-n} = 0$

$n>0$.

$\tilde{L}_n|\psi\rangle = 0 \quad n>0$; $L_0|\psi\rangle 0$

$\tilde{L}_0|\psi\rangle = 0$

There might be ordering ambiguities.

We get ordering problem only for L_0 & $\tilde{L}_0$

$$L_m = \frac{1}{2} \sum_n \alpha_n \cdot \alpha_{m-n}$$

if $m \neq 0$; Then $[\alpha_n, \alpha_{m-n}] = 0$

(dont have any ordering problem)

If $m=0$; Then $L_0 = \frac{1}{2} \sum \alpha_n \cdot \alpha_{-n}$ we have

to decide the order.

We can parametrize our ignorance of ordering upto Normal Ordering, by a number. "a".

(Since each time we use $[\alpha_n, \alpha_{-n}]$ we get a number)

$$L_0 = \frac{1}{2} \sum : \alpha_n \cdot \alpha_{-n} :$$

so; The physical state condition should be $L_0 |\varphi\rangle = a |\varphi\rangle$

(so; Physical states are actually eigen vectors of L_0 with eigen value a)

Since the system is symmetric; we should have same normal ordering constant for $\tilde{L}_0$

ie; $\tilde{L}_0 |\varphi\rangle = a |\varphi\rangle$

so; What annihilates the physical state is $L_0 - \tilde{L}_0$

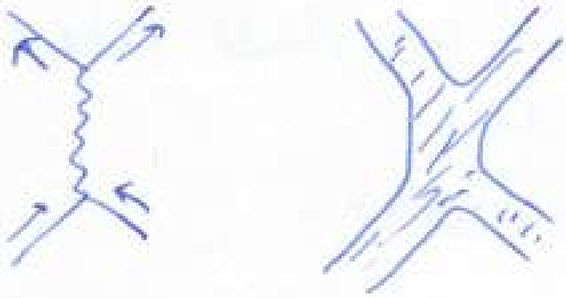
ie; $\boxed{(L_0 - \tilde{L}_0) |\varphi\rangle = 0}$ called Level Matching condition for closed string

so; We can just put level movers, without putting right movers. (There has to be some matching on the physical states)

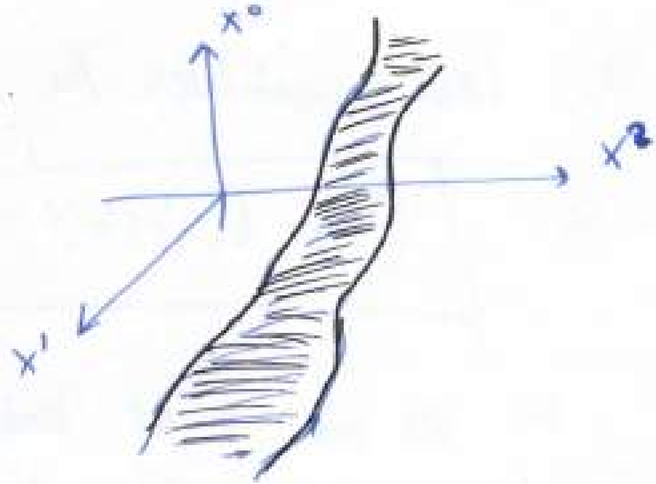
Lec 5: Quantizing Open Strings, String Spectrum, Critical Dimension.

Near 1974 : trying to describe Strongly Interacting Particles.

- Massive.
- Spin.



Usual Path: ① Nambu - Gota.



$$S = \int d^2\sigma \sqrt{-\det G}$$

$$\downarrow$$

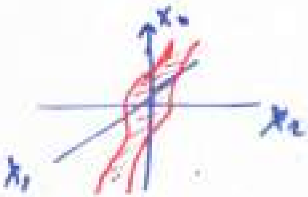
$$S_{\text{polyakov}}[h, X] = \int d^2\sigma \sqrt{-h}\,(\dots)$$

We did:

GR + Scalar fields in M dimensions + Λ

Studied the case M=1.

Studied M=2



2d has good symmetries

We dont do with $M=3$ because we dont know how to do it. (Some symmetries are lost..)

Now; we try to study quantum theory of Spolyakov

Quantization (Open Strings with Neuman boundary conditions)

$$X^\mu(\tau,\sigma) = x^\mu + l_s^2 P^\mu \cdot \tau + i l_s \sum_{m \neq 0} \frac{1}{m} \alpha^\mu_m \cdot e^{-im\tau} \cdot \cos(m\sigma)$$

define $a^\mu_m = \frac{1}{\sqrt{m}} \alpha^\mu_m \quad ; \quad m > 0$.

$$a^{\mu\,+}_m = \frac{1}{\sqrt{m}} \cdot \alpha^\mu_{-m}.$$

Then $\quad [a^\mu_m , a^{\nu\,+}_m] = \delta_{m,n} \cdot \eta^{\mu\nu}$

This looks like infinite number of Harmonic Oscillator.

But there is a problem: Unitarity
(ie; This is a theory where we could have negative probabilities) !

ex $|\psi\rangle = a^{0\,+}_m |0\rangle$ Then $\langle\psi|\psi\rangle = -1$

There are ∞ no. of negative normed state.
(we would not have much problem, if there was only one negative normed state)

Constraints:

$(\partial_+ X)^2 = 0 \;, (\partial_- X)^2 = 0$

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha^\mu_n \cdot \alpha^\nu_{m-n} \cdot \eta_{\mu\nu}$$

Imposing constraints at classical level $L_m = 0 \;\; \forall m$

In Quantum Theory, we have to impose conditions on Physical Hilbert state $L_m|\psi\rangle = 0$, $m>0$.

~~Why do~~ Why only $m>0$? $L_{-m} = L_m^\dagger$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

c → Central charge.

$$\mathcal{L} = T\int d^2\sigma\,(\dot{X}^2 - X'^2)$$

lagrangian of free bosons: ie; we have D bosons each boson has $c=1$

so; here our central charge is D.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+n,0}$$

$\langle\psi|[L_m, L_{-n}]|\psi\rangle = 0$ (Incorrect Assumption)
$L_m|\psi\rangle = 0 \;\; \forall m.$

so; $\langle\psi|[L_m, L_{-m}]|\psi\rangle = 0$

→ $\langle\psi|L_0|\psi\rangle + \frac{D}{12}m(m^2-1)\langle\psi|\psi\rangle = 0$

$\Rightarrow$ $2ma\langle\psi|\psi\rangle + \frac{D}{12}m(m^2-1)\langle\psi|\psi\rangle = 0$

⇒ where a was ordering constant.

→ $2ma\langle\psi|\psi\rangle + \frac{D}{12}m(m^2-1)\langle\psi|\psi\rangle = 0$

true for all m. so; contradiction (because a was constant.

Correct Assumption : $L_m|\psi\rangle = 0$, $m>0$

Then $\| L_{-m}|\psi\rangle\|^2 = 2ma + \frac{D\,m(m^2-1)}{12}$ ~~take $\langle\psi|\psi\rangle$~~ take $\langle\psi|\psi\rangle = 1$

$\langle\psi|[L_m, L_{-m}]|\psi\rangle = \langle\psi|L_m L_{-m}|\psi\rangle - \underbrace{\langle\psi|L_{-m}L_m|\psi\rangle}_{\text{zero using correct assumption}}$

take $m > 0$ ...

so; we get $\| L_{-m}|\psi\rangle \|^2 = 2ma + \frac{D}{12} m(m^2-1)$

---

* Infinite no. of Ghosts (negative normed states)
* " " " constraints : $L_m|\psi\rangle = 0$, $m > 0$. (quadratic constraints)

$L_m|\psi\rangle = 0$, $m > 0$ is a quadratic constraint (hard to impose)

Linearize the Constraint

Light Cone Quantization.

$X^\mu \to (X^0, X^1, \ldots\ldots, X^{D-2}, X^{D-1})$

$\eta^{\mu\nu} \to \text{diag}(-1, 1, \ldots, 1)$

~~Lcc (Light cone coordinates.~~

$\cancel{X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})}$

LCC (Light Cone Coordinates)

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$$

so; Now; $X^\mu \xrightarrow[\text{LCC}]{} (X^+, X^-, X^1, X^2, \ldots, X^{D-2})$

$$\eta = \left(\begin{array}{cc|c} 0 & -1 & 0 \cdots 0 \\ -1 & 0 & \\ \hline 0 & & \\ \vdots & & \mathbb{1}_{(D-2)\times(D-2)} \\ 0 & & \end{array}\right)$$

Whenever we have a vector U

ie; $U = (U^+, U^-, U^j)$ $\qquad j \in \{1, 2, \ldots, D-2\}$

$\hookrightarrow$ These indices are called indices in Transverse direction.

$U = (U^+, U^-, U^j)$

$V = (V^+, V^-, V^j)$

Then $U \cdot V = -U^+V^- - U^-V^+ + \vec{U} \cdot \vec{V}$

$\hookrightarrow$ usual dot product in transverse direction.

Here Negative normed states will be produced by α^+ & α^-.

Now α^+ & α^- have problem:

$$X^+(\tau, \sigma) = x^+ + l_s^2 p^+ \left(\tau + \frac{i}{l_s p^+} \sum_{m \neq 0} \frac{1}{m} \cdot \alpha^+_m \cdot e^{-im\tau} \cdot \cos(m\sigma)\right)$$

So; we have ∞ no. of bad oscillators here: α^+_m

We can get rid of it; by declaring $\tau + \frac{i}{l_s p^+} \sum_{m \neq 0} \frac{1}{m} \alpha^+_m e^{-im\tau} \cos(m\sigma)$ as some new variable.

$$S[x] = \frac{T}{2} \int d\sigma_+ d\sigma_- \, \partial_+ X^\mu \partial_- X^\nu \cdot \eta_{\mu\nu}$$

$$\begin{cases} \sigma_+ \longrightarrow \tilde{\sigma}_+ = f_+(\sigma_+) \\ \sigma_- \longrightarrow \tilde{\sigma}_- = f_-(\sigma_-) \end{cases}$$

These are conformal transformation.

$$\begin{cases} \tilde{\sigma}_+ = \tilde{\tau} + \tilde{\sigma} = f_+(\sigma_+) \\ \tilde{\sigma}_- = \tilde{\tau} - \tilde{\sigma} = f_-(\sigma_-) \end{cases}$$

Now, add them.

$$\Rightarrow \boxed{\tilde{\tau} = \frac{1}{2}\left(f_+(\sigma^+) + f_-(\sigma^-)\right)}$$

(looks the way coordinate of string looks... seperates into σ_+ & σ_- .....)

So; ~~Free~~ We are free to choose any coordinate $\tilde{\tau}$ such that it satisfies $\partial_+\partial_-\tilde{\tau} = 0$

→ This is symmetry of our theory.

$\tau + \frac{i}{l_s p^+}\sum_{m\neq 0}\frac{1}{m}\cdot\alpha^+_m\cdot e^{-im\tau}\cos(m\sigma)$ satisfies the equation (the same equation); so we can define it to be some $\tilde{\tau}$.

$$\tilde{\tau} = \tau + \frac{i}{l_s p^+}\cdot\sum_{m\neq 0}\frac{1}{m}\cdot\alpha^+_m\cdot e^{-im\tau}\cdot\cos(m\sigma)$$

Using our symmetry; we got rid of α^+.

And left with only α^- as a problem.

Constraint: $\dot{X}\cdot X' = 0$, let $U^\mu = \dot{X}^\mu$, $V^\mu = X'^\mu$

Then $-\dot{X}^+X'^- - \dot{X}^-X'^+ + \vec{\dot{X}}\cdot\vec{X'} = 0$ classical.

Recall: $\boxed{X^+(\tilde{\tau},\sigma) = x^+ + l_s^2 p^+\tilde{\tau}}$

$$\dot{X}^+ = \frac{\partial X^+}{\partial \tilde{\tau}} = l_s^2 p^+$$

$$\frac{\partial X^+}{\partial \sigma} = X'^+ = 0$$

$$-\underline{(l_s^2 p^+)} X'^- - 0 + \dot{\vec{X}} \cdot \vec{X}' = 0$$

This is the linearization happening.

$$\Rightarrow \quad X'^- = \frac{1}{l^2 p^+} \sum_{i=1}^{D-2} \dot{X}^i X'^i$$

$$\cancel{X'^- = l_s \sum_{-\infty}^{+\infty} \alpha^-_m e^{-im\tilde{\tau}}}$$

$$X'^- = -i l_s \sum_{m=-\infty}^{+\infty} \alpha^-_m \cdot e^{-im\tilde{\tau}} \cdot \sin(m\sigma)$$

$$\Rightarrow \quad \alpha^-_m = \int_0^\pi d\sigma \cdot \sin(n\sigma) \left(\qquad \right)\Big|_{\tilde{\tau}=0}$$

set $\tilde{\tau}=0$

so; we get out α^-_n.

$$\alpha^-_n = \frac{1}{l_s p^+} \cdot \left(\frac{1}{2} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} \alpha^i_{n-m} \cdot \alpha^i_m \right)$$

In quantum theory; we have to compensate normal ordering ambiguity.

$$\alpha^-_n = \frac{1}{l_s p^+} \left(\frac{1}{2} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} :\alpha^i_{n-m} \cdot \alpha^i_m: - a \cdot \delta_{n,0} \right)$$

These are our all trouble some ossillators.

Physical Hilbert Space : $|\psi\rangle = a^{i_1 +}_{m_1} a^{i_2 +}_{m_2} \cdots a^{i_l +}_{m_l} |0\rangle$

$$i_k \in \{1, \dots, D-2\}$$

We got unitarity ✓

But we spoiled Lorentz Invariance.

We made an assumption (which might not be correct in Quantum Theory) : Invariance of Action under equation ★★ on page 44.

This is true classically.

In Quantum Theory it will make an Anamoly.

lets check

Spectrum : $m=0$: $\alpha_0^- \propto p^-$ $\cancel{\ldots}$

$$p^+p^- = \frac{1}{2}\sum_{i=1}^{D-2} p^i p^i + \Big(\underbrace{\sum_{m=1}^{\infty} \sum_{i=1}^{D-2} :\alpha^i_{-m}\alpha^i_m: - a}_{m\neq 0}\Big)$$

$$\Rightarrow \quad 2p^+p^- - \frac{1}{2}\sum_{i=1}^{D-2} p^i p^i = -\alpha' p^2 = \alpha' M^2$$

$$\Rightarrow \quad \alpha' M^2 = \sum_{m=1}^{\infty} \sum_{i=1}^{D-2} :\alpha^i_{-m}\alpha^i_m: - a$$

$$\boxed{\alpha' M^2 = N - a}$$

$\alpha_m = \sqrt{m}\cdot a_m$ $\quad m>0$

$\alpha_{-m} = \sqrt{m}\cdot a_m^+$

where; $\mathbf{N} = \sum_{m=1}^{\infty} \sum_{i=1}^{D-2} m\, a^{i+}_m a^i_m$

This is our number operator.

$$\boxed{\alpha' M^2 = N - a}$$

$$N \equiv \sum_{m=1}^{\infty} \sum_{i=1}^{D-2} m \cdot a_m^{i\,+} a_m^i$$

Now we compute expectrum; level by level.

$N=0$; $\alpha' M^2 |0\rangle = -a|0\rangle \Rightarrow \boxed{\alpha' M^2 = -a}$

If $a>0$ Then issue (a could be negative)

$N=1$; $a_1^{i\,+}|0\rangle$ → These are possible states which give us $N=1$.

$D-2$ states (1 per value of i)

$$\alpha' M^2 (a_1^{i\,+}|0\rangle) = (1-a)(a_1^{i\,+}|0\rangle)$$

$$\Rightarrow \boxed{\alpha' M^2 = (1-a)}$$

So; we have $D-2$ states in this sector $N=1$

Is there any representation of the inner Lorentz group $SO(1, D-1)$;

where the index i looks like vector index.

but i has only $D-2$ d.o.f.

Is there any representation of $SO(1, D-1)$ which has $D-2$ d.o.f and looks like vector.

⟺ So; It has to be a massless representation

~~So, to~~ If we want to keep Lorentz Invariance ; then index i should behave as vector. This implies that it has to be a massless vector.

so, $\alpha' M^2 = 0 \Rightarrow \boxed{a = 1}$

So; Lorentz Invariance implies $a = 1$

But , if $a = -1$

Then $N = 1$; $\alpha' M^2 = -1$ → Then it becomes a Tachyon. !

N=2 $\quad \underbrace{a_2^{i\,+} |0\rangle}_{(D-2)} \;,\; \underbrace{a_{+1}^{i\,+} a_{+1}^{j\,+} |0\rangle}_{\frac{(D-2)(D-1)}{2}}$

Since $a = 1$ so; masses of these are

$\alpha' M^2 = 1$

$$\boxed{N = 2 \Rightarrow \alpha' M^2 = 1}$$

$a_m^{i\,+} = a_{-m}^i$! so; we usually write

$a_{-2}^i |0\rangle$; $a_{-1}^i a_{-1}^j |0\rangle$

Now: lets find how " these " can resemble themselves ~~into~~ into representation of $SO(D-1)$

→ because they are massive

Lorentz Algebra

$$J^{\mu\nu} = T\int_0^{\pi} d\sigma \, (\dot{X}^{\mu} X^{\nu} - \dot{X}^{\nu} X^{\mu})$$

$$J^{\mu\nu} = x^{\mu} p^{\nu} - x^{\nu} p^{\mu} + i \sum_{m \neq 0} \frac{1}{m} \cdot (\alpha^{\mu}_{-m} \alpha^{\nu}_{m} - \alpha^{\nu}_{-m} \alpha^{\mu}_{m})$$

Claim:

$$[J^{\mu\nu}, J^{\rho\sigma}] = \quad J + \ldots + J + \ldots J + \ldots J$$

All the commutators satisfy Lorentz Algebra;

except for the case $\mu = -$, $\nu = i$, $\rho = -$, $\sigma = j$

$$\text{Lorentz} \Rightarrow [J^{-i}, J^{-j}] = 0$$

Solving

$$[J^{-i}, J^{-j}] = \sum_{m \neq 0} \Delta_m \cdot (\alpha^{i}_{-m} \alpha^{j}_{m} - \alpha^{j}_{-m} \alpha^{i}_{m})$$

some coefficient

If $a = 1$, then $\Delta_m = \left(m - \frac{1}{m}\right) \cdot \left(\frac{26 - D}{12}\right)$

we have to set $\Delta_m = 0$, $\forall m$.

$\Rightarrow$ To recover Lorentz Invariance; we must have $a = 1$, $D = 26$

Lec 6: String Theory as a Theory of Quantum Gravity., ~~Removing~~ UV finite theory, Perturbation, Obtaining equations of motion by imposing $\beta = 0$. (Beta function)

- Negative Normed states (before imposing constraints $L_m|\psi\rangle$)
- LCQ: Linearize constraints

(light cone Quantization)

* Unitarity manifest

* Lorentz Invariance (could be broken)

$\hookrightarrow$ if conformal invariance was not broken by quantum Correction.

Spectrum $(L_0 - a)|\psi\rangle = 0$; Lorentz Invariance $\Rightarrow a = 1$

" " $\Rightarrow D = 26$

. . . .

$\alpha' M^2 = -1$ Tachyon : $N = 0$

$\alpha' M^2 = 0$ A_μ (Gauge Boson) : $N = 1$

$\alpha' M^2 = 1$ Traceless rank 2 tensor : $N = 2$

Closed Strings

LCQ ..... Lorentz Invariance $\Rightarrow a = 1$, $D = 26$

$(L_0 - a)|\psi\rangle = 0$ $\quad (\tilde{L}_0 - a)|\psi\rangle = 0$ $\Rightarrow$ so we have two number operators.

Spectrum:
$$N = \sum_{i=1}^{24} \sum_{n=1}^{\infty} (\alpha^i_{-n} \cdot \alpha^i_n)$$

$$\tilde{N} = \sum_{i=1}^{24} \sum_{n=1}^{\infty} (\tilde{\alpha}^i_{-n} \cdot \tilde{\alpha}^i_n)$$

$\tilde{N} = N$ for physical state,

(using Level Matching condition $(L_0 - \tilde{L}_0)|\psi\rangle = 0$)

$$\frac{\alpha'}{4} M^2 |\psi\rangle = (N-1)|\psi\rangle = (\tilde{N}-1)|\psi\rangle$$

Spectrum:

$N=0, \tilde{N}=0$ $\quad \frac{\alpha'}{4} M^2 = -1$ $\quad$ Tachyon $\quad |0\rangle$

(Existence of Tachyon tells that we are expanding around wrong vaccum ;

There can be standard model, where we expand around Higgs potential on the top)

$\Longleftrightarrow$ The Existence of Tachyon, is a way of telling us that we are expanding around unstable vaccum.

In open strings; vaccum decays; and all open strings are gone (decay into closed strings) and closed strings are left in background.

In closed strings; we dont know! Open Problem !!

$N = \tilde{N} = 1$ $\quad a_1^{\dagger i} \tilde{a}_1^{\dagger j} |0\rangle$ $\quad \frac{\alpha'}{4} M^2 = 0$

$\Rightarrow$ massless

24^2 states

$\hookrightarrow$ Must fall into Representation of Poincare group in 26 dimensions ... Massless representation.

$SO(1,25)$ ... little group $SO(24)$

Is there any irreducible representation of $SO(24)$ with 24^2 states? No.

So it is clear; that some pieces of the rank two tensor $a_1^{+i}\tilde{a}_1^{+j}|0\rangle$; which will not ~~transform~~ which will not mix among themselves under rotation.

What are those pieces?
If we find those pieces; we indeed find irreducible representation.

Take symmetric part $\frac{1}{2}(a_1^{+i}a_1^{+j} + a_1^{+j}a_1^{+i})|0\rangle$

(symmetric part will not mix with anti-symmetric part)

$$\frac{1}{2}(a_1^{+i}a_1^{+j} - a_1^{+j}a_1^{+i})|0\rangle \Big\} \Rightarrow \text{Antisymmetric part is}$$

Symmetric part is still reducible.

$$\star \quad \frac{1}{2}(a_1^{+i}\tilde{a}_1^{+j} + a_1^{+j}a_1^{+i}) - \frac{1}{24}\left(\sum_{k=1}^{24} a_1^{+k}\tilde{a}^{+k}\right)\delta^{ij}$$

we subtract off the trace of the tensor.

$$\star \quad \frac{1}{24}\left(\sum_{k=1}^{24} a_1^{+k}\tilde{a}_1^{+k}\right)$$

~~$\tilde{a}^{i}/\tilde{a}$~~

$$\left[\frac{1}{2}\left(a^{+i}_{1}\tilde{a}^{+j}_{1}+a^{+j}_{1}\tilde{a}^{+i}_{1}\right)-\frac{1}{24}\left(\sum_{k=1}^{24}a^{+k}\cdot\tilde{a}^{+k}\right)\delta^{ij}\right]|0\rangle$$ } Symmetric traceless rank 2 tensor of SO(24) "Graviton"

$$\left[\frac{1}{2}\left(a^{+i}_{1}\tilde{a}^{+j}_{1}-a^{+j}_{1}\tilde{a}^{+i}_{1}\right)\right]|0\rangle$$ } Antisymmetric rank 2 tensor of SO(24) "2-form B"

$$\frac{1}{24}\left(\sum_{k=1}^{24}a^{+k}\tilde{a}^{+k}\right)|0\rangle$$ } Scalar of SO(24) "Φ : Dilaton"

Massless

So; we find ;

The massless sector of our closed string has something that has a chance of being Graviton.

Antisymmetric rank 2 tensor (which can be thought as 2 form)

A massless scalar field called Dilaton

→ Why it is not graviton? Why has a chance of being Graviton.
(for now)

⟺ We have to check for interaction (then we could say something)

1974] Perhaps Quantum Gravity? (Not theory of Strong Interactions)

Aside: Conformal Invariance on the World Sheet.

* Broken : New d.o.f. Liouville field.
* Unbroken : Nice surprise.

α' has dimension.

Gravity is not CFT theory.

So; how come we have dimensionful constant α', and we still talk about conformal invariance.

☆ Its the conformal invariance on the worldsheet; not in the target space.

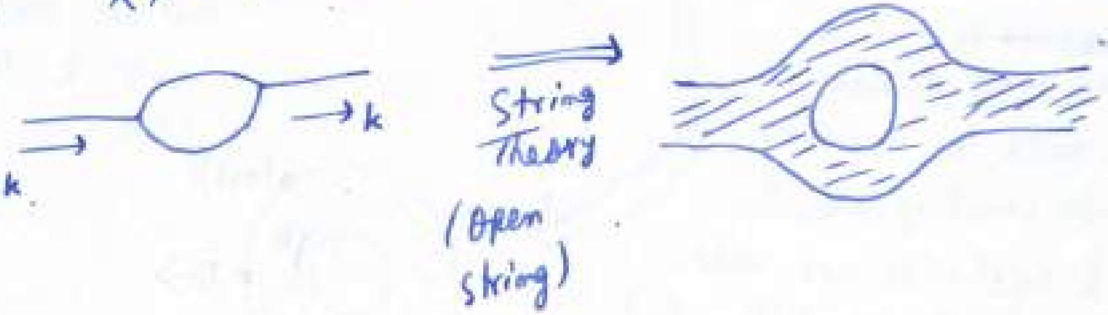
We want to get gravity in the target space. Gravity is not conformal invariant: we have dimensions of mass in our theory.

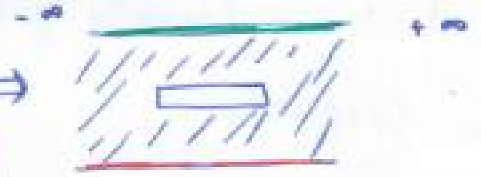
Here; we are describing a non CFT theory in a spacetime using something that can be Conforal Invariant.

So, we can use the benefits of Conformal Invariance on the worldsheet; and can describe something on spacetime which is not Conformal Invariant.

Interactions : UV behavior of the theory.

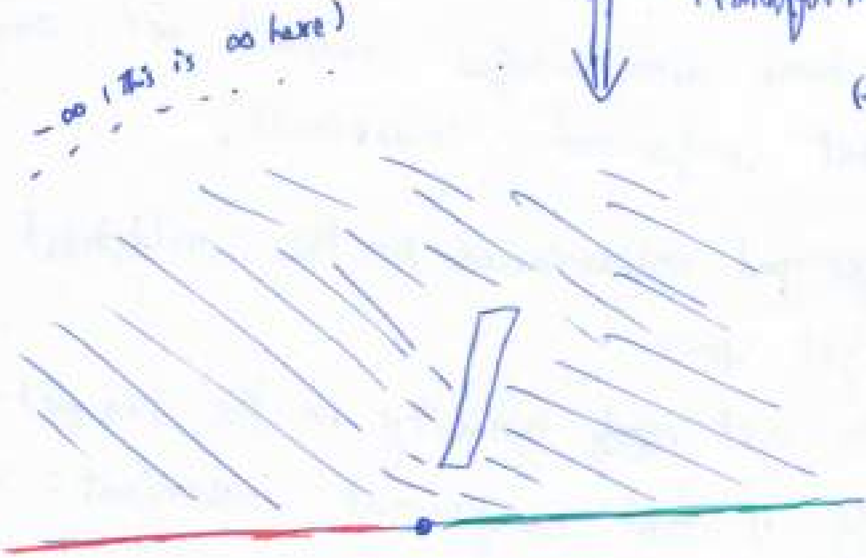
QFT : $\lambda\phi^3$ in 4D





using Conformal Transformation

using Conformal Transformation.



~~(bring +∞ to~~
(bring +∞ to origin)

Using Conformal Transformation

* map upper half plane to disc.

Send +∞ to -1

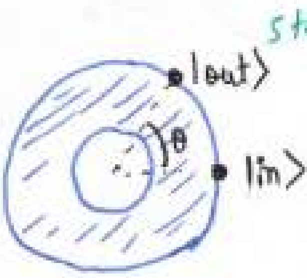
Send 0 to +1

Recall,
-1 & +1 are points where we are inserting the vertex operators; that are describing |in> & |out> state.

use Conformal Transform

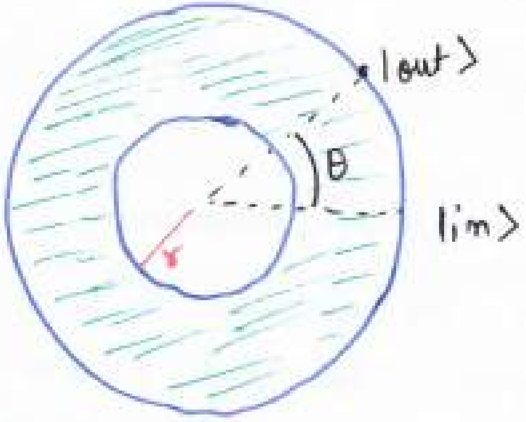
~~send it to~~

* send it to Annulus
* To make the aperture ▯ into something nice O; |in> & |out> state are now not diametrically opposite: but at some angle θ.



All the information about the aperture $\square$ is almost encoded in the angle θ, and radius of the inner circle. r

So; we have to parametrize by r & θ.



r, θ

When we do QFT, we are supposed to integrate all possible states (ie; integrate over all loop momentum; & the integration over loop momenta is what whit gives us UV divergence when we go to high momenta)

Here we have to integrate over θ & r.

* Integration over r will be dangrous $\int_0^1 dr \cdots$

Be careful with $r \to 0$ and $r \to 1$ limit.

(These are the regions where we might get UV divergences)

(If we get UV divergences; then theory might be full of them: and then we have to add ~~—~~ counter term, check it is renormalizable and if the counter term cannot be absorbed into the original Lagrangian of the theory ; Then the theory would be inconsistent)

lets see what happens here.

limit $r \longrightarrow 1$;

Very thin annulus

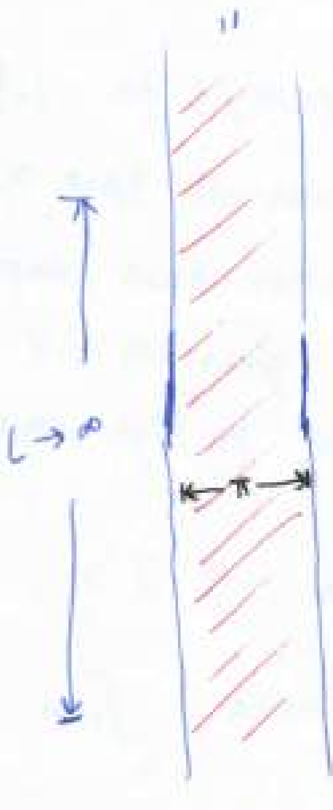
use conformal invariance

as we send $r \to 1$

we send $\varepsilon \to 0$

Do conformal transformⁿ where we keep width of "this" this to be fixed (say π) ;

and if we do this
⇒ Then Conformal Transformation will tell us that it will extend to $\pm\infty$.



This is actually "propagation of open string". from $-\infty$ to $+\infty$.

This is a propagation of physical state ; & is what we call Infrared Process.

It gives Divergence ; called Infrared Divergence

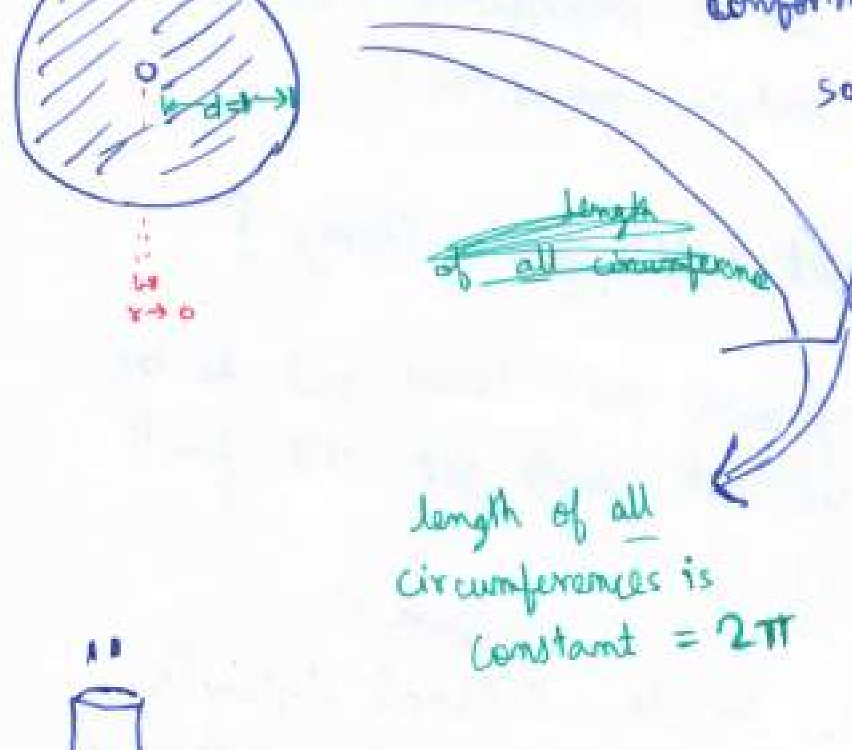
We are afraid of UV divergences.

Infrared Divergences are okay: because they have physical meaning. (when we compute any S-matrix element, and allow all the states which we cannot measure.

In a theory which has massless particles; defining a state asymptotically is something that cannot be done physically. So; we have to define wavepacket, and they ~~add~~ include infinite no. of states with very small momenta.

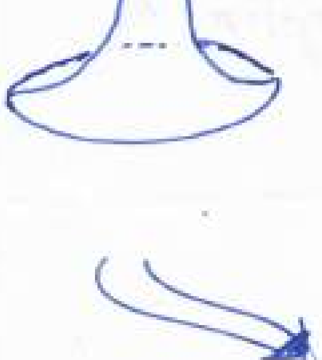
So; all these infrared Divergence cancel out)

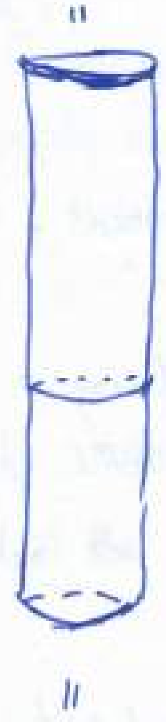
$r \to 0$ gives us UV divergence.



conformal transformation: so that length of all circumferences is fixed: constant say 2π

When we are sending $r \to 0$; The only way to keep the structure of this $(..\ 2\pi)$ will be to send it to ∞!!





looks like propagation of closed string

↪ so; we have IR divergence

(This answers a question:

If we only have open strings; Why we worry about gravity:

↪ Open strings have graviton in them; because they also have closed strings... And we realized that these states will be propagating there... And the states contain graviton)

⟹ We could get a UV finite theory!

(And if the traceless symmetric part turns out to be graviton; Then we will finally get UV finite theory of gravity)

UV divergences has to do with structure of small distances.
Here; we are able to map it which is propagation of States at long distances

Is it really a Graviton?

$$S = \frac{-1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{-h} \cdot h^{\alpha\beta} \partial_\alpha X^\mu \cdot \partial_\beta X^\nu \cdot G_{\mu\nu}(X) \quad (*)$$

Go to Polyakov action; but allow scalar fields to interact in any way they like

→ This is QFT in 2 dimensions:

and we interpret it as propagation of strings in the target space with the metric $ds^2 = G_{\mu\nu} dX^\mu dX^\nu$.

(Here we are allowing self coupling of our scalar fields through $G_{\mu\nu}(X)$)

Perturbation Theory :

(Note; its a non-linear theory; and we dont even have a canonically normalized kinetic term)

→ The way to deal with this is: Quantize the theory around some particular point (or some particular value of scalar fields)

$$X^\mu(\tau, \sigma) = \bar{x}^\mu + \sqrt{\alpha'} \cdot Y^\mu(\tau, \sigma)$$

↳ Quantum Fluctions.

So; we expand the action $*$; and treat it as action for $Y^\mu(\tau, \sigma)$

After doing this, we get:

Pg 62

$$S = \frac{-1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{-h} \cdot h^{\alpha\beta} \cdot \partial_\alpha Y^\mu \partial_\beta Y^\nu \cdot G_{\mu\nu}(\bar{x} + \sqrt{\alpha'} y)$$

$$S[Y^\mu] = \frac{-1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \cdot h^{\alpha\beta} \cdot \partial_\alpha Y^\mu \partial_\beta Y^\nu \cdot G_{\mu\nu}(\bar{x}^\sigma + \sqrt{\alpha'} \cdot Y^\sigma)$$

we want to do perturbation theory; So we can assume Y^σ is small; & can expand $G_{\mu\nu}$ in taylor series.

$$G_{\mu\nu,\rho}(\bar{x}) = \frac{\partial G_{\mu\nu}}{\partial x^\rho}\Big|_{\bar{x}}$$

~~$G_{\mu\nu}(\bar{X} + \sqrt{\alpha'} Y) = G_{\mu\nu}(\bar{X}) + \sqrt{\alpha'} \cdot$~~

$$G_{\mu\nu}(\bar{x} + \sqrt{\alpha'} \cdot Y) = G_{\mu\nu}(\bar{x}) + \sqrt{\alpha'} \cdot G_{\mu\nu,\rho}(\bar{x}) Y^\rho + \frac{\alpha'}{2} G_{\mu\nu,\rho\sigma}(\bar{x}) \cdot Y^\rho Y^\sigma + \dots$$

↳ we plug this

This gives a term which looks canonically normalized. together with Kinetic term.

$$G_{\mu\nu}(\bar{x}) \, \partial_\alpha Y^\mu \partial_\beta X^\nu$$

: This looks like almost it is canonically normalized, except that its mixing all the scalar fields. (The metric $G_{\mu\nu}(\bar{x})$ is mixing them)

→ So; we should try to diagonalize it; and then work with canonically normalized Kinetic Terms for the scalar fields Y^μ.

Riemann Normal Coordinates :

$$G_{\mu\nu}(\bar{x}) = \eta_{\mu\nu} + \frac{\alpha'}{6} R_{\mu\rho\nu\gamma}(\bar{x}) \cdot Y^\rho Y^\gamma + \dots$$

↳ This gives a nice Kinetic term $\partial_\alpha Y^\mu \partial_\beta Y^\nu \eta_{\mu\nu}$

(of course; it would be nicer if we were working in Euclidean signature)

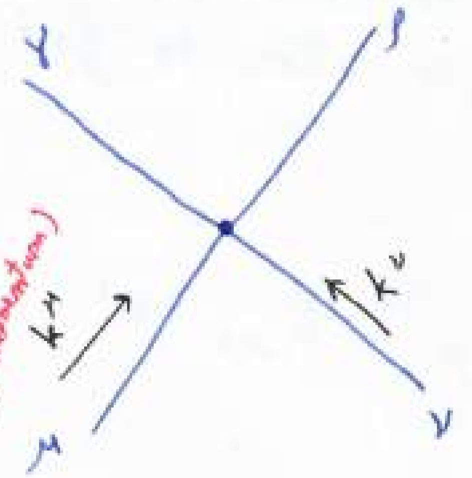
lets look at interactions.

$$\mathcal{L}_{interaction} = \frac{\alpha'}{6} R_{\mu\rho\nu\gamma}(\bar{x}) \cdot Y^\rho Y^\gamma \cdot \partial_\alpha Y^\mu \partial_\beta Y^\nu \cdot \eta^{\alpha\beta}$$

Goal : Compute Beta function.

so; first : we have to see correction to the propagator

The Feynman Rules

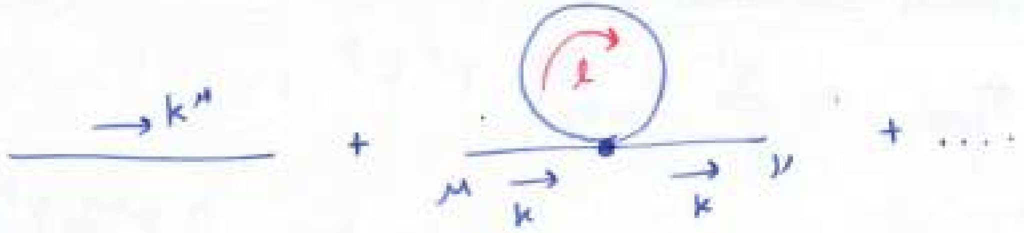


$$\frac{\alpha'}{6} \cdot R_{\mu\rho\nu\gamma}(\bar{x}) \cdot k^\mu_{\ \alpha} k^{\nu\alpha}$$

index α is 2d index.

(2d momentum)

- 2d components say labelled by α ..
- μ labels the scalar fields X^μ ....



$$= \frac{\alpha'}{6} \cdot R_{\mu\rho\nu\gamma} \cdot k^{\mu}{}_{\alpha} \cdot k^{\nu\alpha} \cdot \eta^{\rho\gamma} \cdot \int^{\Lambda} \frac{d^2 l}{l^2}$$

has logarithmic divergence.

$$\int^{\Lambda} \frac{d^2 l}{l^2} \sim \ln\left(\frac{\Lambda}{\mu_s}\right)$$

$\mu_s \ni$ scale..

We have logarithmic divergence; so we had to add ~~Quantum to~~ Counters terms in our action to get rid of this.

Note $\quad R_{\mu\rho\nu\gamma} \cdot \eta^{\rho\gamma} = R^{\rho}{}_{\mu\rho\gamma} = R_{\mu\nu}(\bar{x})$

~~So; The free part of lagrangian~~

$G_{\mu\nu}(\bar{x})$ are just coupling constants.

$$G_{\mu\nu}(\bar{x}) \longrightarrow \underbrace{\left(G_{\mu\nu}(\bar{x}) + \frac{\alpha'}{6} \cdot R_{\mu\nu}(\bar{x}) \cdot \ln \frac{\Lambda}{\mu_s}\right)}_{\text{new } G_{\mu\nu}\text{; lets call it } G_{\mu\nu}^{new}}$$

Coupling constants.

$G_{\mu\nu}^{new}$ obtained after adding counter-term.

Now; if we re-do the calculation; it will cancel the divergence precisely.

i.e. $G_{\mu\nu}(\bar{x}) \cdot \partial_\alpha Y^\mu \partial_\beta Y^\nu$

$$\xrightarrow{\text{new}} \partial_\alpha Y^\mu \cdot \partial_\beta Y^\nu \left(\eta_{\mu\nu} + \frac{\alpha'}{6} R_{\mu\nu}(\bar{x}) \ln \frac{\Lambda}{\mu_s}\right)$$

<u>β-function</u> measures ~~so~~ how coupling constant changes.

$$\beta_{\mu\nu} = \mu_s \cdot \frac{\partial G_{\mu\nu}^{new}}{\partial \mu_s}$$

At least at one loop; the new coupling constant is $G_{\mu\nu}^{new}$

$$\Rightarrow \boxed{\beta_{\mu\nu} = \frac{\alpha'}{6} \cdot R_{\mu\nu}(\bar{x})}$$

Computation of one loop β-function for ~~the~~ this theory.

For theory to be conformal; (what should happen ~~at~~ at quantum level or say loop level?) What should happen

Q. to β-function?

β-function should vanish for conformality.

Note) Target space is not conformal.

Conformality $\Rightarrow \beta_{\mu\nu} = 0$

$\beta_{\mu\nu}$ is β function in 2d theory.

∴ Its 2d QFT what we are doing here.

$\Rightarrow \underline{\underline{R_{\mu\nu}(\bar{x}) = 0}}$

↳ i.e; our spacetime (target space...) satisfies einstein's equations.

Recall that; In this spacetime; we did not have anything other than metric.

(strings are source in the metric; but we dont have any other kind of excitations:

we dont have fermions,

we dont " scalars,

we are not exciting the Dilaton,

" " " " " B-field)

We are only exciting the metric.

And this is a theory in vacuum.

& $R_{\mu\nu}(\bar{x}) = 0$ is precisely the Einsteins equation in the vacuum.

lets add more stuff to the theory

Action describing the coupling of Dilaton & B-field ~~couple~~ to string or (how they look like in 2d QFT)

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-h}\cdot h^{\alpha\beta}\cdot \partial_\alpha X^\mu \partial_\beta X^\nu \cdot G_{\mu\nu}(X)$$

$$-\frac{1}{4\pi\alpha'}\int d^2\sigma\, B_{\mu\nu}(X)\, \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta}$$

$$-\frac{1}{4\pi\alpha'}\int d^2\sigma \cdot \sqrt{-h}\cdot \Phi(\sigma^0,\sigma^1)\cdot R^{(2)}$$

(No fermions for now)

Computing the β-functions we get :

$H = dB$

$B \to 2$-form.

$H \to 3$-form.

$$\cancel{\beta^{(G)}_{\mu\nu} = \alpha'(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi}$$

$$\beta^{(G)}_{\mu\nu} = \alpha'\left(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho}\right)$$

$$\beta^{(B)}_{\mu\nu} = \alpha'\left(-\frac{1}{2}\nabla^\lambda H_{\lambda\mu\nu} + \nabla^\lambda\Phi\cdot H_{\lambda\mu\nu}\right)$$

$$\beta(\Phi) = \alpha'\left(-\frac{1}{2}\nabla^2\Phi + \nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{24}\cdot H^2\right.$$

→ Imposing conformal invariance ; will be setting all these β functions to be zero.

$$\underbrace{\beta^{(G)}_{\mu\nu} = 0 \ , \ \beta^{(B)}_{\mu\nu} = 0 \ , \ \beta(\Phi) = 0}$$

→ These looks like equations of motion for a bunch of objects; The two form & the scalar coupled to metric (gravity) in our 26 dimensional spacetime.

Given equation of motion : ~~...~~
Can we always find an action from which they arise?

Ans It's non-trivial.
Equation of motion has to satisfy some integrability conditions in order to ensure that they come from a lagrangian.

We show that : These equation of motions $\beta = 0$, satisfy integrability condition : and they come from lagrangian

$\beta = 0$ intergrate into single lagrangian.
→ This means sets of equation : All β-functions ..

$\beta = 0 \xrightarrow[\text{one loop}]{}$ Turn out to be Field Equation of a 26 dimensional Classical Gravity Theory + B + Φ

And the theory looks like :

$$S = \int d^{26}x \sqrt{-G} \cdot e^{-2\Phi} \cdot \left[R - \frac{1}{12} H^2 + 4 \partial_\mu \Phi \partial^\mu \Phi \right]$$

Now we declare ; $G_{\mu\nu}$ is really a Graviton ; And the theory really knows about Quantum Gravity.

Lec 7: String Theory as a Theory of Quantum Gravity, World sheet perturbation, Adding Fermions and getting equations of motion + Boundary conditions.

World-Sheet Theory (closed strings)

- Fields : $X^\mu(\tau, \sigma)$
- Theory lives on $\mathbb{R}^{1,1}$
- Conformal Field Theory

☆ Vanishing of β-Beta Functions

Coupling Constant here

$\alpha' \sim \frac{\alpha'}{\text{Curvature}}$ Coupling constant.

$\beta_{\mu\nu} = \alpha' R_{\mu\nu} + \alpha'^2(\ldots) + \ldots$

Space-Time Theory

- Fields : Φ (Dilaton), B_μ, $G_{\mu\nu}$ → Graviton., T → Tachyon.
- Not Conformal.
- Theory lives in $\mathbb{R}^{1,25}$

This action comes from leading order in α'.

recall; $H = dB$

$$S = \frac{1}{2\kappa_0}\int d^{26}\sqrt{-G}\cdot e^{-2\Phi}\left[R - \frac{1}{12}H^2 + 4\partial_\mu \Phi\, \partial^\mu \Phi\right]$$

This dimensionful must be related to α'.

(The only thing we have in our theory, which has dimensions)

☆ Field Equations.

→ String Frame

(when we have a factor of dilaton; $e^{-2\Phi}$)

H is a 3-form; and its the curvature of the 2-form B; B is our Gauge Potential now.

Field Redefination

using this we can go to

$$\int d^{26}x \sqrt{-G} \left[R + \ldots \right] \Big\} \text{Einstein Frame}$$

~~Shift~~

In String Frame

- Shift dilaton by a constant; $\Phi \rightarrow \langle \Phi \rangle + \Phi$

Dilaton acquires an expectation value

we call the fluctuations Φ again.

So: $$S = \frac{1}{2k_0^2 e^{2\langle \Phi \rangle}} \int d^{26}x \sqrt{-G} \cdot e^{-2\Phi} \left[R - \frac{1}{12} H^2 + 4 \partial_\mu \Phi \partial^\mu \Phi \right]$$

"So we can shift coupling constant of the theory if we shift Dilaton by a constant"

So: we ~~think~~ say, $g_c = e^{\langle \Phi \rangle}$ ~~as~~ the Coupling Constant of the theory.

Then the action looks like

$$S = \frac{1}{g_c^2}\left[\cdots \qquad \right]$$

$g_c \Rightarrow$ Coupling constant of closed string Theory.

has factor $\frac{1}{g_c^2}$ in front (looks more canonical)

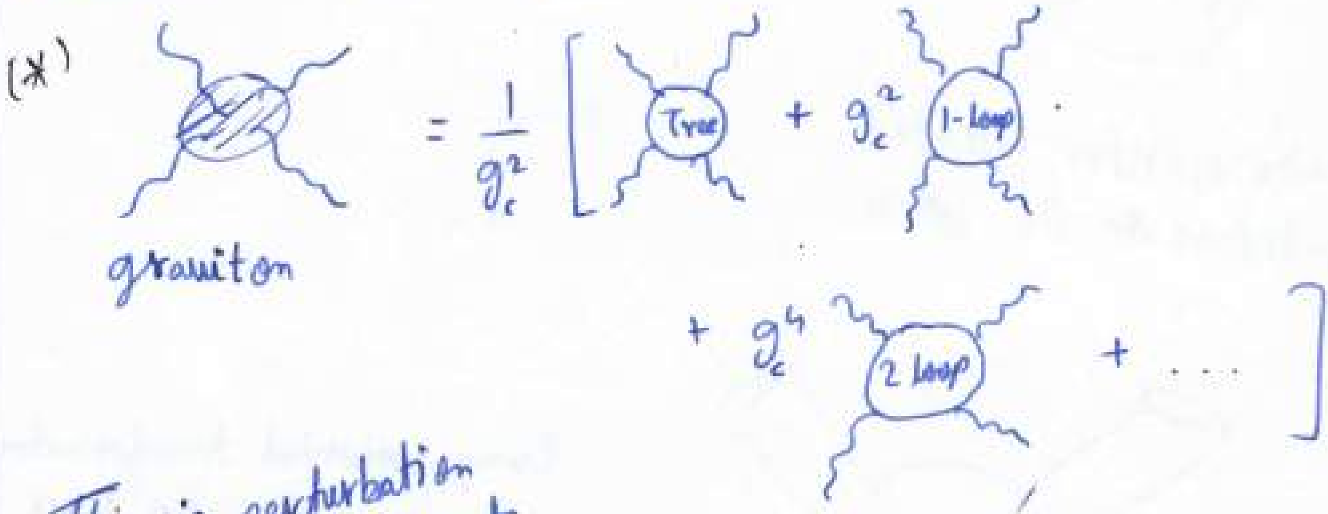
In terms of the coupling constant, we can do perturbation theory here.

~~* Lead~~

* Leading order in α'

---

* Perturbation theory in g_c



This is perturbation theory of full action to leading order in α'.

(Usually, we dont do perturbation theory with effective action: But remember, we have α' so we can control what we are doing)

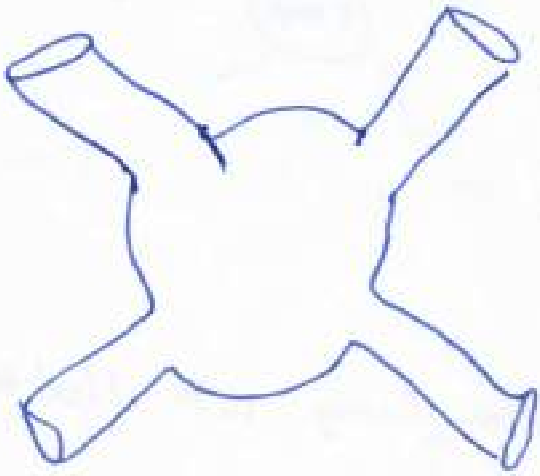
Scattering of Strings

1) Make theory Euclidean $\Rightarrow$ so that we can use CFT.

2) Vertex Operators.

The simplest worldsheet we can have if we make our theory Euclidean is Sphere.

Vertex operator inserted somewhere on the sphere.



Using conformal transformation, we can make the tubes very very thin; and so we can replace them by the vertex operators.

From the view of world sheet  is the spacetime where the theory lives.

The way we compute contribution to scattering amplitude is ~~by~~ through partition function of the string, with insertion of vertex operators.

$$Z[X] = \int [dX][Dh]\, V \dots V \cdot e^{-S[X,h]}$$

$\hookrightarrow$ This integration is done on a worldsheet that has a topology of sphere.

How we can tell that, this ~~happens there~~ happens there?

Because; Imagine that we have turned on a dilaton ~~background~~ background.

$$\int d^2\sigma\ \Phi(\sigma)\ R^{[2]} \cdot \sqrt{-h}$$

$\hookrightarrow$ Imagine this acquires an expectation value ... a number.

So; replace $\Phi(\sigma)$ by $\langle \Phi \rangle$

~~Goal: To ma~~

$\boxed{\text{Goal: To match the perturbation theory (*) that we are doing on page 71 with the perturbation theory on the worldsheet.}}$

$$\text{So; } \quad e^{\frac{-1}{2\pi} \int d^2\sigma\ \langle \Phi \rangle\ R^{[2]} \sqrt{-h}}$$

$$= e^{-\frac{\langle \Phi \rangle}{2\pi} \underbrace{\int d^2\sigma \cdot R^{[2]} \sqrt{-h}}_{\chi}}$$

$\hookrightarrow$ This term is topological invariant. Euler ~~ch~~ characteristic.

$$\propto e^{-2\langle \Phi \rangle} = \frac{1}{g_c^2}$$

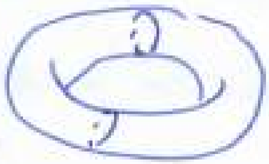
for sphere $\chi = 2$

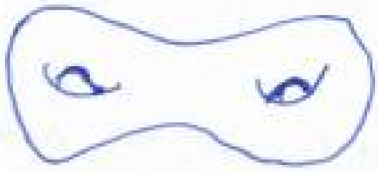
Why choosing sphere?

we could choose n-genus surface in general.

we can make more & more complicated choices for our worldsheet.

$g=0$  $\chi = 2$

$g=1$  $\chi = 0$

$g=2$  $\chi = -2$

⋮

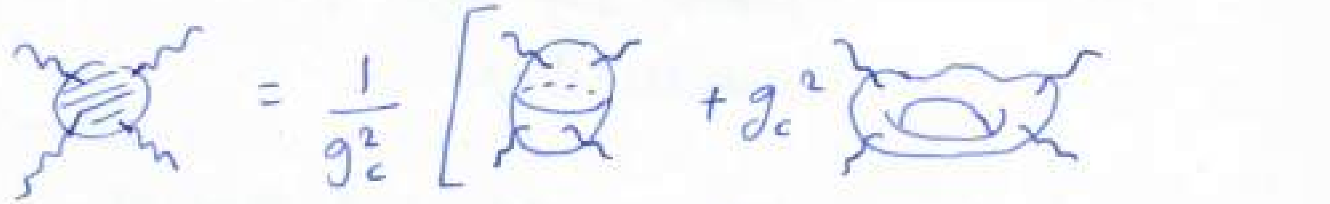
g  $\chi = 2-2g$

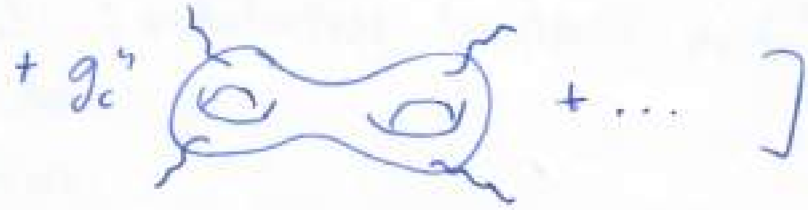
Euler characteristic of a surface of genus g is

$$\chi = 2-2g.$$

So, we see;

★ That the perturbative expansion on spacetime theory (*) as done on page 71; will correspond on the worldsheet side or the natural projection on the world sheet side will correspond to an expansion in the topology of the worldsheet. Pg 75





. This is not actually perturbative expansion: but an expansion in topologies.

In Spacetime, if we do QFT; Then how many diagrams do we have to compute if we want to ~~[illegible]~~ compute 4 particle amplitude: (at 0- loop)

(linearize G.R; and then compute)

There are many diagrams with many vertices

(It will be a very long computation)

In World sheet case; we have to compute only one diagram.  (at 0- loop)

In spacetime perturbation : The number of diagrams becomes very large if we increase loops.

But in Worldsheet perturbation: At each loop we have to compute only one diagram.

★ Power of computing scattering amplitude using String Theory methods is good.

(people realized that we can use the power of String Theory to resemble Feynman Diagrams)

<u>We have to check</u>

If we do the computation of  to leading order in α' we get same answer as what we get in spacetime perturbation,

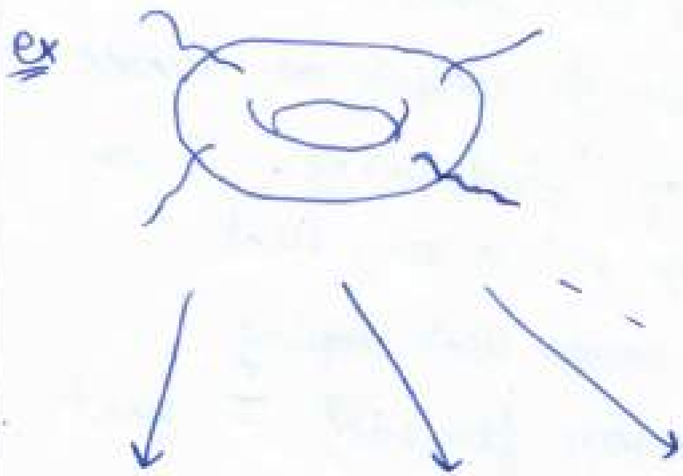
& so on.

"How does it happens that  these guys end up giving so many equivalent Feynman Digrams ?"

leading order in α' means, takes $\alpha' \to$ close to zero. So; The tension becomes very large. String with very large tension looks like a point (hard to pull apart)

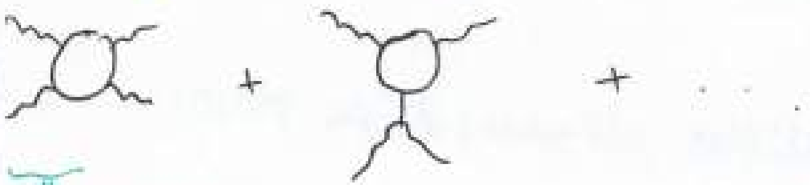
Very hard to stretch out, a string with very
large Tension.

ex



⇒ When tension is very large; This will raise contribution when Reimann Surface degenerates.

↘ degenarting Reimann surface



now it looks like particle

The reimann surface can degenerate in a many possible forms, when we take the limit $\alpha' \to 0$.

These corners in the moduli spaces of Rieman Surface are precisely in one-to-one correspond to the Feynmann diagrams we can have.

Historical Fact
Some of the first gravity amplitude, which we computed was done ~~th~~ using String Theory.

☆ In 1990's; people found that there are different string theories, And ~~magnit~~ we could show that when the coupling of one of the theories was very large; the description we got for the theory ended up ~~ex being~~

being equivalent to ~~one~~ another theory when the coupling was very very small.

" Some-how, string theories were very smart. Well, if you dont know how to compute me because I have very strong coupling; dont worry, I am equivalent to another theory with a very weak coupling. So, we can do ~~compu~~ weak coupling computations there and get answers for what I want to do "

~~These~~ These are dualities discovered in 1990s.

Towards a theory of Quantum Gravity

* Macroscopic 4 dimensional ~~S~~ Space Time.
* Standard Mode at Low Energies.
* Mathematically consistent.

Where are we?

- Good ; as far as Mathematical consistency is concerned (it allowed to remove UV divergences)

- In case of dimensions; Not good ; we got 26 dimensions.
  Kaluza Klien ...

- In terms of Standard Model; we have
  A_μ (massless gauge bosons), T (Tachyons), ...] Bosons.

Missing Fermions...

We are missing Fermions.

Where? On the worldsheet or spacetime?

We are missing Fermions in Spacetime.

How can we start putting fermions in Spacetime, or

How we can get Fermions in Spacetime?

World Sheet [Open Strings]

$$S = \frac{-1}{2\pi\alpha'} \cdot \int d^2\sigma \, \partial_\alpha X^\mu \, \partial^\alpha X_\mu$$

How can we add something to this action to produce fermions in spacetime as part of the spectrum when we quantize the theory?

We dont have any idea for now.

→ We should start with adding Fermions on the worldsheet & see what happens.

Recall $X^\mu \to$ scalar; & μ is vector index in spacetime.

Add ψ something which is ~~for~~ Fermion.

Fermion refers to statistics;

we know it has to be a spinor

(Spinor, in terms of representation;

Spinor under 2d Lorentz Group)

~~Since~~ Since, we want to make connections to space-time.

So; Its natural to consider as many Fermions, as the amount of bosons we have.

$$\psi^{\mu}.$$

So; we write the canonical action.

$$\cancel{S = \frac{-1}{2\pi\alpha'}\int d^2\sigma \left(\partial_\alpha X^{\mu}\partial^{\alpha}X_{\mu} + \psi^{\mu}\not{\partial}\psi^{\nu}\eta_{\mu\nu}\right)}$$

$$S = \frac{-1}{2\pi\alpha'}\int d^2\sigma \left(\partial_\alpha X^{\mu}\partial^{\alpha}X_{\mu} + \bar{\psi}^{\mu}\not{\partial}\psi^{\nu}\cdot\eta_{\mu\nu}\right)$$

In general, this works in any dimensions because Dirac Lagrangian is same.

where $\not{\partial} = \rho^{\alpha}\partial_{\alpha}$
$\alpha \in \{0,1\}$

In 4 dimensions; In order to write lagrangian, we have to introduce Gamma matrices.

Here we do the analogue, and call it ρ matrices & They satisfy Clifford Algebra of the spacetime we are dealing with. (which is world sheet here)

~~So~~ We have only two ρs. ρ^{α} ; $\alpha \in \{0,1\}$

$$\{\rho^{\alpha}, \rho^{\beta}\} = 2\cdot\eta^{\alpha\beta}$$

They turn out to be 2 dimensional representation of Clifford Algebra $\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $\rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\bar{\psi} = i\psi^{+}\rho^{0}$

We can impose different conditions on Spinors :

- Majorana $\longrightarrow \psi^{+} = \psi^{T}$
- Weyl . $\rho^{2} = \rho^{0}\rho^{1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (its like our γ^{5})
- Majorana & Weyl together (can't be done in 4 dimensions; but can be done in 2d)

(to not to confuse ρ^{2} with ρ squared ;
we simply write it as $\rho = \rho^{0}\rho^{1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$)

~~In order to~~
We impose Majorana & Weyl conditions together

In order to decompose something in Weyl spinors ; we ask the object to be an eigen vect of ρ with eigenvalue ± 1.

$$\rho\psi^{\mu} = \pm\psi^{\mu}$$

So ; its clear that : our 2 dimensional spinor ;
the top component will have one chirality.
& the bottom " " " other " .

So ; we can write our Dirac Spinor
as

$$\psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix}$$ ψ_-^μ & ψ_+^μ are our Weyl Spinor.

If we choose light cone coordinate (LCC)

$$\sigma^\pm = \tau \pm \sigma$$

Then ~~action~~ something nice happens (when we write in light cone coordinates)

(We will also use that these spinors are Majora)

$$S_\psi = \frac{1}{2\pi\alpha'} \int d\sigma_+ d\sigma_- \left(\psi_+^\mu \partial_- \psi_{+\mu} + \psi_-^\mu \partial_+ \psi_{+\mu} \right)$$

or writing in compact notations.

$$\cancel{S_\psi = \frac{1}{2\pi\alpha'} \int d\sigma_+ d\sigma_- \left(\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_{-\alpha} \right)}$$

$$\boxed{S_\psi = \frac{1}{2\pi\alpha'} \int d\sigma_+ d\sigma_- \left(\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_- \right)}$$

looks symmetric in + & −.

Now; we find Field Equations & the Boundary Conditions (remember; we are doing open strings)

Not writting $\frac{1}{2\pi\alpha'}$ explicity now on.

$$\delta S_\psi = \int d^2\sigma \left(\delta\psi_+ \cdot \partial_- \psi_+ + \psi_+ \cdot \delta(\partial_- \psi_+) + \delta\psi_- \cdot \partial_+ \psi_- + \psi_- \cdot \delta(\partial_+ \psi_-) \right)$$

Now we do, by parts, ...

$$\delta \partial_- \psi_+ = \partial_- \delta\psi_+$$

Integrate by parts.

$$\delta S_\psi = \int d^2\sigma \cdot \Big(\delta\psi_+ \cdot \partial_-\psi_+ + \underbrace{\psi_+ \cdot \delta\partial_-\psi}_{\partial_-(\delta\psi_+)} + \delta\psi_- \cdot \partial_+\psi_- + \underbrace{\psi_- \cdot \delta(\partial_+\psi_-)}_{-\partial_+\psi_- \cdot \delta\psi_- + \partial_+(\psi_-\delta\psi_-)}\Big)$$

$$\underbrace{-\partial_-\psi_+ \cdot \delta\psi_+}_{2(\partial_-\psi_+)\cdot\delta\psi_+} + \partial_-(\psi_+\delta\psi_+)$$

using the fact that these are fermions, & they anti-commute.

i.e. $\boxed{-\delta\psi_+ \cdot (\partial_-\psi_+) = (\partial_-\psi_+)\cdot\delta\psi_+}$

$$\Rightarrow \delta S_\psi = \int d^2\sigma \Big(2(\partial_-\psi_+)\cdot\delta\psi_+ + 2(\partial_+\psi_-)\cdot\delta\psi_- + \partial_-(\psi_+\delta\psi_+) + \partial_+(\psi_-\delta\psi_-)\Big)$$

Field Equations: $\partial_-\psi_+^\mu = 0$, $\partial_+\psi_-^\mu = 0$

(ψ_+^μ has to only be function of σ_+) (ψ_-^μ has to only be function of σ_-)

Boundary Conditions:

$$-\partial_\sigma(\psi_+\delta\psi_+) + \cancel{\partial_\tau(\ldots)} + \partial_\sigma(\psi_-\delta\psi_-) + \cancel{\partial_\tau(\ldots)}$$

Same as we did for Bosonic case.

We dont care about these: because in the integral over τ; these gives contributions at $\tau = \pm\infty$, and we assume variations vanish here.

So; only worry about derivatives w.r.t. σ.

The Boundary terms looks like

$$\underbrace{\int_{-\infty}^{+\infty} d\tau \int_0^{\pi} d\sigma \left(-\partial_\sigma(\psi_+ \delta\psi_+) + \partial_\sigma(\psi_- \delta\psi_-)\right)}$$

$$\int_{-\infty}^{+\infty} d\tau \left[\underbrace{\left(-\psi_+ \delta\psi_+\big|_{\sigma=\pi} + \psi_- \delta\psi_-\big|_{\sigma=\pi}\right)}_{0} - \underbrace{\left(-\psi_+ \delta\psi_+\big|_{\sigma=0} + \psi_- \delta\psi_-\big|_{\sigma=0}\right)}_{0} \right]$$

independently

So; The condition which we have to impose is

$$\boxed{\begin{aligned} \psi_- \delta\psi_-\big|_{\sigma=\pi} &= \psi_+ \delta\psi_+\big|_{\sigma=\pi} \\ \psi_- \delta\psi_-\big|_{\sigma=0} &= \psi_+ \delta\psi_+\big|_{\sigma=0} \end{aligned}} \quad (*)$$

We can set
$$\left.\begin{aligned} \psi_+^{\mu}(\tau,\pi) &= \pm\, \psi_-^{\mu}(\tau,\pi) \\ \psi_+^{\mu}(\tau,0) &= \pm\, \psi_-^{\mu}(\tau,0) \end{aligned}\right\}$$

to satisfy the condition $(*)$ here.

→ A relative sign on these conditions; will not have any physical effect.

Because we can always rescale all ψ_- say by -1 or all ψ_+, & nothing will change, So, one of the relative choice of sign on one of the two condition is redundant.

let $\psi_+^{\mu}(\tau, 0) = \pm \psi_-^{\mu}(\tau, 0)$

$\hookrightarrow$ let us select $\psi_+^{\mu}(\tau, 0) = +\psi_-^{\mu}(\tau, 0)$ out of these for convention (because one is redundant).

Now: $\psi_+^{\mu}(\tau, \pi) = \pm \psi_-^{\mu}(\tau, \pi)$ these two are then actually physically different.

So; we have two ways of imposing Boundary condition.

- $\psi_+^{\mu}(\tau, \pi) = +\psi_-^{\mu}(\tau, \pi)$: Ramond Boundary Conditions.
- $\psi_+^{\mu}(\tau, \pi) = -\psi_-^{\mu}(\tau, \pi)$: Neveu - Schwarz.

We can solve field equations; expand in fourier modes, & impose boundary conditions.

RNS formalism

R $\to$ Ramond

N $\to$ Neveu

S $\to$ Schwarz

Lec 8: 11 D SUGRA

M theory
11D Supergravity (11D SUGRA)

SUGRA = Supergravity
SUSY = Supersymmetric Algebra

- 5 perturbative String Theories.
- M-Theory : A duality web that relates all of the 5 perturbative String Theories to each other.

- 2 string Theories (II A, Heterotic $E_8 \times E_8$) exhibit an 11th dimension at strong coupling

- 11 d SUGRA : Low energy effective description of M theory

(Fundamental objects : membranes)
(string of String Theories are obtained by compactifying the membrane one one dimension ;
so in a sense ; String is just a membrane in which one dimension looks observable through strong coupling ....)

We only ~~study~~ study Massless particles.
(mass arises through some indirect effects)

11 is the highest dimension in which one can have consistent theory of SUGRA;

because Suppersymetric algebra is such that in dimensions higher than 11, there has to be a particle with spin greater than 2. And such theories have various problems & inconsistencies.

---

11 d SUGRA

Field Content: $M, N \in \{0, \dots, 10\}$

$g_{MN} \rightarrow$ metric (graviton)

$M \Rightarrow$ vector index
$\alpha \Rightarrow$ Spinor index

$\psi_M^\alpha \rightarrow$ Spin 3/2 fermions (gravitino) ← supersymmetric partner of graviton

$A_{MNP} \rightarrow$ Anti-symmetric tensor (3-form potential)

$\Downarrow$

A three form

Lets count the physical degrees of freedom [polarizations]

In D dimensions, the little group $SO(D-2)$

① Graviton:
# of independent degrees of freedom = # of components of a symmetric traceless $(D-2)\times(D-2)$ matrix

1) Graviton:

$$\# \text{ d.o.f.} = \underbrace{\frac{1}{2}(D-1)(D-2)}_{\substack{\#\text{ of components of} \\ (D-2)\times(D-2)\text{ symmetric} \\ \text{matrix}}} - \underbrace{1}_{\text{Trace}} = \frac{1}{2}D(D-3).$$

~~for D=11 ; we get Graviton # of d.o.f.~~ = 44

for $D=11$; we get (Graviton # of d.o.f.) = 44

Aside) Counting polarizations without mentioning the little group:

$$\boxed{(\#\text{ d.o.f.}) = (\#\text{ components}) - (\#\text{ of gauge invariances})}$$

Ex) Electromagnetic field in D dimensions.

described by vector potential ; A_M ; $M = 0, \dots, D-1$.

$$\text{Gauge Invariance: } A_M(x) \to A_M(x) + \partial_M \underbrace{\lambda(x)}_{\substack{\text{function} \\ \text{1 parameter} \\ -1}}$$

Choose gauge : $\partial_M A^M = 0$

Residual Gauge Invariance for $\Box\lambda = 0$.

(This will lead to longitudinal polarization being unphysical)

Transform to momentum space:

$$A_M(x) = \int d^D p \cdot \underbrace{\epsilon_M(p)}_{\text{Polarization vector}} \cdot e^{ip\cdot x}$$

Gauge condition: $\partial_\mu A^\mu = 0 \Rightarrow p^\mu \epsilon_\mu = 0$

Eqn of motion $\Box A_\mu = 0 \Rightarrow p^\mu p_\mu = 0$ (massless particle)

Residual gauge transformation:

$$\epsilon_\mu \to \underbrace{\epsilon_\mu + f(p) p_\mu}_{\epsilon_\mu'} \quad : \quad p^\mu \cdot \epsilon'_\mu = 0$$ (Its an on shell equation) after imposing $p^\mu p_\mu = 0$

So; we see that anything proportional to momentum is exactly the residual gauge freedom and its still a gauge artifact. Its also not physical degree of freedom. We can remove the component; by doing a shift.

$\Rightarrow$ So; The <u>longitudinal component</u> (ie; proportional to momentum) is also not physical.

In 4d: Temporal & longitudinal component of EM potential is not physical.

$\Longrightarrow$ 2 physical degrees of freedom (2 polarizations)

D dimensions : we have $D-2$ polarization.

Here; one gauge parameter allowed us to remove two degrees of freedom.

☆ For the metric; a vector gauge parameter will allow us to remove $2 \times D$ degrees of freedom.

For metric

$$g_{MN} \rightarrow g_{MN} + \partial_M \epsilon_N + \partial_N \epsilon_M$$

gauge transformation.

Choosing a gauge remove D d.o.f. (because $M \in \{0, \ldots, D-1\}$)

& the Residual gauge invariance again removes D more degrees of freedom.

So; In D dimension;

$$(\# \text{ d.o.f.}) = \frac{D(D+1)}{2} - D - D = \frac{1}{2} D(D-3)$$

---

(2) Gravitino its a vecto-spinor
(carries two indices)

$$\psi_M^{\alpha}$$

$M \ni$ vector index
$\alpha \ni$ Spinor index

Little group is $SO(D-2)$
In 11D, its $SO(9)$

Spin[9] is double cover of SO(9)

Vector of $SO(9)$: 9 components.

Spinor of $SO(9)$: $2^{\frac{8}{2}} = 16$ components.
[Spin (9)]

ψ_M^{α} : tensor product : 9×16 components

There is one more Gauge Invariance:

$$S_\psi = \text{constant} \int \bar{\Psi}_M \underbrace{\Gamma^{MNP}}_{\text{anti-symmetrized product of 3 } \Gamma' \text{ matrices.}} \cdot \partial_N \Psi_P$$

because of antisymmetry of Γ^{MNP}.

If we transform:

Gauge Invariance : $\delta \Psi^\alpha_M = \partial_M \epsilon^\alpha$ ← spinor

$\Rightarrow$ $\delta S_\psi = 0$ up to boundary terms.

(If we dont have boundary; so we have gauge invariance)

$$\#\ \underset{\text{gravitino}}{\text{d.o.f.}} = 9 \times 16 - \underbrace{16}_{\text{for } \epsilon^\alpha} = 8 \times 16 = 128$$

↳ No. of polarization for gravitino in 11 D.

---

③ <u>3-form field</u> : A_{MNP}

# of polarizations = # of components of rank 3 ant-symmetric matrices in $(D-2)$ dimensions
 $= 9$

$$= \frac{9 \times 8 \times 7}{3!} = 84$$

<u>11 d SUGRA</u> : bosonic d.o.f. = $\underbrace{84}_{A_{MNP}} + \underbrace{44}_{\text{from metric}} = 128$ same as Gravitino.

Fermionic d.o.f. = 128

for Any Supersymmetric Theory;

\# Bosonic d.o.f. = # Fermionic d.o.f

A_{MNP} ; Field Strength (4-form)

$\Downarrow$ F_{MNPQ}

Potential

$$F_{(4)} = dA_{(3)}$$

Action (only Bosonic part):

$2k_{11}^2 = 16\pi G_{11}$; $G_{11} \Rightarrow$ Newtons Constant in 11 dimensions. $F_4 = dA_3$

$$S = \frac{1}{2k_{11}^2}\int d^{11}x \cdot \sqrt{-\det g}\cdot\left(R - \frac{1}{2}F_4^2\right) - \underbrace{\frac{1}{6}\int A_3 \wedge F_4 \wedge F_4}_{\text{Charm Simon term.}}$$

$$F_4 = dA_3$$

$$F_4^2 = \frac{1}{4!}F_{M_1 \dots M_{11}} \cdot F_{N_1 \dots N_{11}} \cdot g^{M_1N_1} \dots g^{M_{11}N_{11}}$$

$$\boxed{S = \frac{1}{2k_{11}^2}\int d^{11}x \cdot \sqrt{-\det g}\left(R - \frac{1}{2}F_4^2\right) - \frac{1}{6}\int A_3 \wedge F_4 \wedge F_4}$$

Lec 9: IIA String Theory from Dimensional Reduction of 11D SUGRA

$G_{MN} \Rightarrow$ Metric in 11 Dimensions.

D = 11 SUGRA SUSY transformation rules:

$$\delta G_{MN} = \bar{\epsilon}\,(\Gamma_N \Psi_M + \Gamma_M \Psi_N)$$

$$\delta A_{MNP} = -3\,\bar{\epsilon}\,\Gamma_{[MN}\Psi_{P]}$$

$$\delta \Psi_M = \nabla_M \epsilon + \frac{1}{12}\left(\Gamma_M F^{(4)} - 3F_M^{(4)}\right)\epsilon$$

$\epsilon \Rightarrow$ SUSY parameter

$$F_{(4)} = dA_3$$

$$F^{(4)} = \frac{1}{4!} F_{MNPQ}\Gamma^{MNPQ}$$

$$F^{(4)}_M = \frac{1}{3!} F_{MNPQ}\Gamma^{NPQ}$$

$\mathcal{N} = 1$ SUSY in D = 11 : $2^5 = 32$ supercharge

$$\boxed{\begin{array}{l}\text{In } D = 4 : \mathcal{N}=1 : 2^2 = 4 \text{ supercharge} \\ 32 \text{ supercharge} \Rightarrow \mathcal{N} = 8\end{array}}$$

Book) Weinberg : "Quantum Field Theory", volume III.
Section 32.2.

Spin : Little group in 4d : SO(2)

in D dimensions :

Def^n] Spin of a representation of $SO(D-2)$ is the max. absolute value of the eigenvalues of <u>any</u> Lorentz generator in the representation.

<u>Gravitino degrees of freedom</u>:

$$\psi'_M = \psi_M - \underbrace{\frac{1}{(D-2)}\Gamma_M \cdot (\Gamma\cdot\psi)}_{\text{"Trace"}}$$: irreducible representation, spin 3/2.

$$\Gamma\cdot\psi = \Gamma^N \psi_N$$

$$9 \times 16 - \underbrace{16}_{\text{spin } 1/2} = 128$$

$$\boxed{\begin{gathered}\psi_M \to \psi_M + \partial_M \epsilon \\ \text{Gauge choice} : \Gamma\cdot\psi = 0\end{gathered}}$$

IIA String Theory (superstring..) from dimensional reduction of 11D SUGRA.

<u>Goal:</u>] Reduce $D=11$ SUGRA on an S^1.

$M = 0, \dots, 9, 11$ (no 10; so as to explicitly mention 11 here)

11 is going to be circle direction S^1 that we want to reduce on.

ν, μ : 10 d indices (IIA).

S^1 is parametrized by x^{11}.

$\mathcal{N}=2$: ϵ_1, ϵ_2 (Two different SUSY parameters) : IIA : Opposite chirality
IIB : Same chirality.

In $D=4$; γ^5 (chirality matrix) : Left & Right

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 \qquad \frac{(1\pm\gamma_5)}{2}\,\psi = \psi_{L,R}$$

In $D=10$: " γ^5 analogue is Γ^{11} "

$$\Gamma^{11} = \Gamma^0 \cdots \Gamma^9 \qquad \text{so: } \frac{(1\pm\Gamma^{11})}{2}\,\psi = \psi_{L,R}$$

Want no dependence on X^{11} :

$$ds^2_{11} = G_{MN}\, dx^M dx^N$$
$$= g_{\mu\nu}(x^\sigma)\, dx^\mu dx^\nu + f(x^\nu)\left(dx^{11} + A_\mu(x^\nu)\, dx^\mu\right)^2$$

$g_{\mu\nu} \to$ 10 dimensional metric

most general form of a metric with isometry along x^{11}.

$$\cancel{ds^2_{11} = g_{\mu\nu}(x^\sigma)\, dx^\mu dx^\nu + f(x^\sigma)\left(dx^{11} + A_\mu(x^\sigma)\, dx^\mu\right)^2}$$

$$\boxed{ds^2_{11} = g_{\mu\nu}(x^\sigma)\, dx^\mu dx^\nu + f(x^\sigma)\left(dx^{11} + A_\mu(x^\sigma)\, dx^\mu\right)^2}$$

Knowing about String Frame IIA action ; we modify the metric ansatz:

choose $f(x^\sigma) = e^{4\phi/3}$; ϕ is scalar

& rescale $g_{\mu\nu} \longrightarrow e^{-2\phi/3} \cdot g_{\mu\nu}$: $\hat{g}_{\mu\nu} = e^{-2\phi/3} \cdot g_{\mu\nu}$

$$ds_{11}^2 = e^{-2\phi/3} \cdot g_{\mu\nu} \cdot dx^\mu dx^\nu + e^{4\phi/3} \cdot (dx^{11} + A_\mu \cdot dx^\mu)^2$$

$g_{\mu\nu} \to$ 10d metric

$\phi \to$ scalar (Dilaton)

$A_\mu \to$ 10d vector (Kaluza Klein vector)

↳ This will be RR 1-form:

$$A_1 = A_\mu dx^\mu$$

occuring in type IIA.

$$R^{(11)} \to R^{(10)} + \#(\partial\phi)^2 + \# F_2^2$$

$F_2 = dA_1$

Vielbein:

By defination: $G_{MN} = E^A_M E^B_N \cdot \eta_{AB}$

↳ Flat 11d (Minkowski)

$M \ni$ Curved indices

$A \ni$ Flat (tangent space) indices

E^A_M : Vielbein.

↳ Transforms under general coordinate transformation is M.

↳ A transforms under local Lorentz Transformation.

$$E^A_M dx^M = e^{-\phi/3} \cdot e^a_\mu dx^\mu + e^{2\phi/3} (dx^{11} + A_\mu dx^\mu)$$

$a \ni$ 10d flat index

$\cancel{\eta_{AB} \cdot E^{A}_{M} dx^{M} \cdot E^{B}_{N} dx^{N} =}$ $M = \mu, 11$

$$\eta_{AB} \cdot E^{A}_{M} \, dx^{M} E^{B}_{N} \, dx^{N} = G_{MN} \cdot dx^{M} \cdot dx^{N}$$

$M = \mu, 11$

$A = a, 11$

$$E^{A}_{M} = \begin{array}{c} \begin{array}{cc} a \times \mu & a \times 11 \end{array} \\ \left(\begin{array}{c|c} e^{-\phi/3} \cdot e^{a}_{\mu} & 0 \\ \hline e^{2\phi/3} \cdot A_{\mu} & e^{2\phi/3} \end{array} \right) \\ \begin{array}{cc} \mu \times 11 & 11 \times 11 \end{array} \end{array}$$

$$\boxed{E^{A}_{M} = \begin{pmatrix} e^{-\phi/3} \cdot e^{a}_{\mu} & 0 \\ e^{2\phi/3} \cdot A_{\mu} & e^{2\phi/3} \end{pmatrix}}$$

$e^{11}_{\mu} = A_{\mu}$

Inverse matrix:

$$E^{M}_{A} = \begin{pmatrix} e^{\phi/3} \cdot e^{\mu}_{a} & 0 \\ -e^{\phi/3} \cdot A_{a} & e^{-2\phi/3} \end{pmatrix} \Bigg\} \text{Inverse Vielbein.}$$

$$E^{A}_{M} E^{M}_{B} = \delta^{A}_{B}$$

$$E^{A}_{M} \cdot E^{N}_{A} = \delta^{N}_{M}$$

$$\left. \begin{aligned} E^{\mu}_{a} &= e^{\phi/3} \cdot e^{\mu}_{a} \\ E^{11}_{a} &= -e^{\phi/3} \cdot A_{a} \\ E^{\mu}_{11} &= 0 \\ E^{11}_{11} &= e^{-2\phi/3} \end{aligned} \right\} \text{Components of Inverse Vielbein.}$$

$A_1 = A_\mu dx^\mu$

$A_\mu = E_\mu^{\ a} A_a \Rightarrow \underline{\underline{A_a}} = A_\mu E_a^{\ \mu}$

$\hookrightarrow$ 11 d flat index: "a."

$A^{(11)}_{MNP}$: $A_{\mu\nu\rho}$, $A_{\mu\nu 11}$ (Two choices)

$A_{\mu\nu\rho}$ ↓ RR 3-form potential.

$A_{\mu\nu 11} = B_{\mu\nu}$

$F_4 = dA_3$

$F^{(11)}_{ABCD} = E_A^{\ M} E_B^{\ N} E_C^{\ P} E_D^{\ Q} \cdot F_{MNPQ}$ $\qquad H_3 = dB_2$

Two options

1) $F^{(11)}_{abc11} = E_a^{\ \mu} E_b^{\ \nu} E_c^{\ \rho} E_{11}^{\ 11} \cdot F_{\mu\nu\rho 11}$

$= (e^{\phi/3})^3 e_a^{\ \mu} e_b^{\ \nu} e_c^{\ \rho} e^{-2\phi/3} \cdot H_{\mu\nu\rho}$

$F_{\mu\nu\rho 11} = H_{\mu\nu\rho}$ $\qquad \Rightarrow \boxed{F^{(11)}_{abc11} = e^{\phi/3} \cdot H_{abc}}$

2) $F^{(11)}_{abcd} = E_a^{\ \mu} E_b^{\ \nu} E_c^{\ \rho} E_d^{\ \sigma} \cdot F_{\mu\nu\rho\sigma}$

$+ 4 E^{\ 11}_{[a} E_b^{\ \nu} E_c^{\ \rho} E_{d]}^{\ \sigma} \cdot F_{11\nu\rho\sigma}$

$= e^{4\phi/3} \cdot F_{abcd} + 4 e^{\phi/3} \cdot A_{[a} \underbrace{F_{11bcd]}}_{= H_{bcd}} e^{\phi}$

$$F^{(11)}_{abcd} = e^{4\phi/3} \cdot \underbrace{\left(F_{abcd} + 4 A_{[a} H_{bcd]} \right)}_{\tilde{F}}$$

$$\Rightarrow \boxed{\tilde{F}_4 = F_4 + 4\, A_1 \wedge H_3}$$

A_1 is one form of KK vector.

<u>Integrate over x^{11}:</u>

$$S \sim \frac{1}{2k_{11}^2} \int d^{11}x \;\ldots\ldots$$

$$= \frac{1}{2K_{11}^2} \cdot 2\pi R \int d^{10}x \ldots$$

commming from S^1

So; we are actually redefining Gravitational coupling constant.

$$\Rightarrow \quad k_{10}^2 = \frac{k_{11}^2}{2\pi R}$$

$2k_D^2 = 16\pi G_D$

Newtons constant in D dimensional.

<u>10d (IIA) action:</u>

$$S = S_{NS} + S_R + S_{CS}$$

S_{NS}: universal for all

S_R: depends on the type

S_{CS}: Chern Simon term.

$$S_{NS} = \frac{1}{2K_{10}^2}\int d^{10}x\,\sqrt{-\det g}\cdot \underbrace{e^{-2\phi}}_{\text{String Frame...}}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}\underbrace{H_3^2}_{\text{B field strength}}\right)$$ Pg 100

$$H_3^2 = H_{\mu_1\ldots\mu_3}\cdot H_{\nu_1\ldots\nu_3}\cdot g^{\mu_1\nu_1}\ldots g^{\mu_3\nu_3}\cdot\frac{1}{3!}$$

$$S_R = \frac{-1}{4K_{10}^2}\int d^{10}x\,\sqrt{-\det g}\cdot\left(F_2^2 + \tilde{F}_4^2\right)$$

$$F_2 = dA_1$$

$$S_{CS} = -\frac{1}{4K_{10}^2}\cdot\int B_2\wedge F_4\wedge F_4$$

$$\boxed{S = S_{NS} + S_R + S_{CS}}$$

$$S_{NS} = \frac{1}{2K_{10}^2}\int d^{10}x\,\sqrt{-\det g}\cdot e^{-2\phi}\cdot\left(R + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}H_3^2\right)$$

$$S_R = \frac{-1}{4K_{10}^2}\int d^{10}x\,\sqrt{-\det g}\cdot\left(F_2^2 + \tilde{F}_4^2\right)$$

$$S_{CS} = \frac{-1}{4K_{10}^2}\int B_2\wedge F_4\wedge F_4$$

Lec 10: RNS Formalism.

RNS String Theory

$S^{(X)} = \frac{-1}{2\pi\alpha'}\int d^2\sigma\, \partial_\alpha X^\mu \partial^\alpha X^\nu \cdot \eta_{\mu\nu}$ Has states with negative norm.

... we supplement by constraints.

Our Spacetime happened to be configuration space of this theory.

Add fermions $S^{(\psi)} = \frac{-1}{2\pi\alpha'}\int d^2\sigma\, \bar{\psi}^\mu \gamma \psi^\nu \eta_{\mu\nu}$

One way to have Fermions;

We say that, in spacetime time; at the same time as having bosonic coordinates, may be we could also have fermionic coordinates in spacetime.

Who says that the coordinates, what we call space has to be all bosonic?

Even if there were fermionic directions; that would have zero dimensions: because fermions have to satisfy Grassmann Algebra.

$\hookrightarrow$ They could not be something which we see macroscopically.

Perhaps there are also fermionic directions in spacetime. Mathematicians had studied them; They are called Super manifold.

Another Approach : (X^μ, θ)

In addition to spacetime directions that are bosonic, we also include fermionic spacetime directions)

Green - Schwarz followed this approach.

Hence its known as Green - Schwarz Formalism

Here ; we have super manifold (which is target space), and the action for the string moving in that spacetime (is induced metric on that super space)

Majorana - Weyl : $\Psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix}$

$$\int d\sigma_+ d\sigma_- \left(\psi_-^\mu \partial_+ \psi_{-\mu} + \psi_+^\mu \partial_- \psi_{+\mu} \right)$$

Field Equations : $\partial_+ \psi_-^\mu = 0$ & $\partial_- \psi_+^\mu = 0$.

Boundary Conditions (open strings)

R :

$$\psi_+^\mu(\tau, 0) = \psi_-^\mu(\tau, 0)$$

$$\psi_+^\mu(\tau, \pi) = +\psi_-^\mu(\tau, \pi)$$

Oscillator modes denoted by d.

$$\psi_+^\mu(\sigma_+) = \sum_{n \in \mathbb{Z}} d_n^\mu \cdot e^{-in\sigma_+}$$

$$\psi_-^\mu(\sigma_-) = \sum_{n \in \mathbb{Z}} d_n'^\mu \cdot e^{-in\sigma_-}$$

$\Rightarrow d_n^\mu = d_n'^\mu$ (if satisfy boundary condition)

NS :

$$\psi_+^\mu(\tau, 0) = \psi_-^\mu(\tau, 0)$$

$$\psi_+^\mu(\tau, \pi) = -\psi_-^\mu(\tau, \pi)$$

Oscillator modes denoted by b.

$$\psi_+^\mu(\sigma_+) = \sum_r b_r^\mu \cdot e^{-ir\sigma_+}$$

$$\psi_-^\mu(\sigma_-) = \sum_r b_r'^\mu \cdot e^{-ir\sigma_-}$$

Boundary condition $\psi_+^\mu(\tau, 0) = \psi_-^\mu(\tau, 0)$ implies $b_r'^\mu = b_r^\mu$

Because of Majora

$$\boxed{d^{\mu\dagger}_m = d^{\mu}_{-m}}$$

Can we impose R for some μs, and NS for some other μs? (No: because that will break Lorentz Invariance)

The NS condition

$$\psi^{\mu}_{+}(\tau,\pi) = -\psi^{\mu}_{-}(\tau,\pi)$$

is satisfied as follows.

At $\sigma = \pi$, let's add up both $\psi^{\mu}_{+} + \psi^{\mu}_{-}$

$$0 = \sum_r b^{\mu}_r \cdot e^{-ir\tau} \cdot \cos(r\pi)$$

$$\Rightarrow r \in \mathbb{Z} + \frac{1}{2}$$

so:

$$\psi^{\mu}_{+}(\sigma_+) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^{\mu}_r \cdot e^{-ir\cdot\sigma_+}$$

$$\psi^{\mu}_{-}(\sigma_-) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^{\mu}_r \cdot e^{-ir\cdot\sigma_-}$$

$\sigma_{\pm} = \tau \pm \sigma$

Reality condition: $\boxed{b^{\mu\dagger}_r = b^{\mu}_{-r}}$

Quantization

$$\{\psi^{\mu}(\tau,\sigma), \psi^{\nu}(\tau,\sigma')\} = \delta(\sigma-\sigma')\cdot\eta^{\mu\nu}$$

$$[\alpha^{\mu}_m, \alpha^{\nu}_n] = m\,\delta_{m+n,0}\cdot\eta^{\mu\nu}$$

R	NS
$\{d^{\mu}_m, d^{\nu}_n\} = \delta_{m+n,0}\cdot\eta^{\mu\nu}$	$\{b^{\mu}_r, b^{\nu}_s\} = \delta_{r+s,0}\cdot\eta^{\mu\nu}$

~~Hilbert Space~~

Hilbert Space

R

- Here also we find negative normed states.

Vacuum:

$d^{\mu}_{m}|0\rangle_R = 0 \quad \forall m>0$

note; $\{d^{\mu}_0, d^{\nu}_0\} = \eta^{\mu\nu}$

$\Rightarrow \{\sqrt{2}\, d^{\mu}_0, \sqrt{2}\, d^{\nu}_0\} = 2\eta^{\mu\nu}$

define: $\Gamma^{\mu} = \sqrt{2}\, d^{\mu}_0$

The we see that Γ^{μ} satisfies the Clifford Algebra!

$$\{\Gamma^{\mu}, \Gamma^{\nu}\} = 2\cdot\eta^{\mu\nu}$$

Note:) Nowhere we said that, we are going to get anything that has to do with spinor representation in a spacetime.

And somehow, by just imposing R boundary conditions we found that zero oscillators satisfies Clifford Algebra in spacetime.

NS

- Does $b^{0\dagger}_r|0\rangle_{NS}$ has negative norm

Take $r>0$

$|\psi\rangle = b^{0\dagger}_r|0\rangle_{NS}$

$\Rightarrow \langle\psi|\psi\rangle = {}_{NS}\langle 0|b^0_r b^{\dagger}_r|0\rangle$

$= -\langle 0|b^{\dagger}_r b_r|0\rangle + \eta^{00}\langle 0|0\rangle$

$= -1$

~~Define Vacuum~~

~~$b^{\mu}_r|0\rangle_{NS} = 0$~~

Define Vacuum

$b^{\mu}_r|0\rangle_{NS} = 0 \quad ; r>0$

$\alpha^{\mu}_{-m}|0\rangle_{NS} = 0 \ ; \ m>0$

precisely; The vacuum should be written as

$|0;k\rangle_{NS}$

$k \Rightarrow$ momentum.

(we supress k sometimes)

Number operator (RNS)

$N|0\rangle = 0$

N does not have d_0

so: $[N, d_0^\mu] = 0$

Now; if d_0^μ commutes with N; & $N|0\rangle = 0$

That; whatever our Vacuum is; It better be in a representation of Clifford Algebra.

(d_0 have to satisfy Clifford Algebra; & therefore; ~~any space~~ any space where they act on better fall in the representation of that Algebra)

$\Rightarrow$ Vacuum (or any state) must furnish a representation of the Clifford Algebra

$\dim = 2^{D/2}$

RNS Number operator continued...

Give label to the different vacuum.

$|a\rangle_R$

To furnish a representation of Algebra

$$d_0^\mu |a\rangle_R = \Gamma^\mu_{ab} |b\rangle_R$$

mixes precisely with particular representation as a matrix.

... it almost works like Fermions.

∴ But we also have to check that it propagates in spacetime as a Fermion.

$\Longleftrightarrow$ All we need is a Dirac equation is Spacetime?

Precisely & rigorously $|a\rangle_R$ should be written as

$|a;k\rangle_R$

If massless; Then Dirac Equation in spacetime, in momentum space looks $\not{k}|a;k\rangle = 0$

where $\not{k} = k_\mu \Gamma^\mu$

$\hookrightarrow$ This is dirac equation for a massless spinor in spacetime in momentum representation.

Sector with (any sector) $N=0$ $\Longrightarrow$ Must give the representation of Algebra.

$$d_0^\mu |0\rangle = \Gamma^\mu_{ab} |b\rangle$$

for R & NS both.
(for $N=0$)

<u>NS sector</u>

here $r \in \mathbb{Z} + \frac{1}{2}$; so we dont have 0-0.

So; we dont have Clifford Algebra in NS sector.

$\longrightarrow$ No Clifford Algebra $\Rightarrow$ No Fermions.

So These will be bosons.

If the vacuum is in spinner representation.
say $|a\rangle_R$.
Then after aplying ~~on~~ any number of oscillators to it with μ indices : will produces states that are in the representation ; which is a tensor product of the spinor with bunch of vectors

$\hookrightarrow$ And these will still produce Fermions.

∴ So we found a tower of Fermions.

Light Cone Quantization

$(U^\mu) = (U^+, U^-, U^i)$ where $i \in \{1, 2, \ldots, D-2\}$

U^+ & U^- where made up of U^0 and U^{D-1}.

Bosonic Action : $S = \int d\sigma_+ d\sigma_- \, \partial_- X \partial_+ X$

Dangerous oscillators : X^+ & X^- (where hidden here)

Residual symmetry : which was Conformal Symmetry

↪ This ~~was~~ was powerful enough; to allow us to eat up all the oscillators of X^+ : and we ended up with a formula ~~that~~ where X^+ did not have any oscillators

$$X^+(\tau, \sigma) = x^+ + l_s^2 p^+ \tau$$

and used the contraint $\underline{\dot{X}^\mu \cdot X'_\mu = 0}$ (This was linearized in LCC)

$$\dot{X}^\mu X'_\mu = -\dot{X}^+ X'^- - \dot{X}^- X'^+ + \sum_i \dot{X}^i X^{i\prime} = 0$$

we say $X'^+ = 0$ & $\dot{X}^+$ is a number

↪ So it linearises.

$\Rightarrow \alpha^-_n = f(\alpha^i_n)$ } so; α^- oscillators can be written only in terms of transverse oscillators.

⟹ So now ; our Hilbert space is completely made out of physical objects.

In Fermionic case

$$S = \int d\sigma_+ d\sigma_- \bar{\Psi} \gamma \Psi$$

Here; The dangerous oscillators are Ψ^+ , Ψ^- }

This we mean dirac spinors.

$\Psi^+_\pm , \Psi^-_\pm$

are μ indices

Chirality on the worldsheet.

Does this action have any symmetry which will allow us to remove some oscillators.

Yes; It has the supersymmetric version of the Conformal Group; The Super Conformal Group.

$\Longleftrightarrow$ It will allow us to set $\Psi^+_\pm = 0$

Back to the Drawing Board

we should start from

$$S = \frac{-1}{2\pi} \int d^2\sigma \sqrt{-h} \left(h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \right)$$

This is the theory which we know makes sense from the beginning.

This is actually QFT in two dimensions; where 2 D spacetime has a curved metric.

Now; we want to add fermions.

We do using <u>Diads</u> $h_{\alpha\beta} = e_\alpha^{\ a} \cdot e_\beta^{\ b} \cdot \eta_{ab}$

(called Tetrads in 4 dimension)

Here we call it Diads (because we have 2 dimensions)

$a, b \in \{0, 1\}$ are indices in the tangent space (where we have our nice Lorentz group)

And α, β are spacetime indices.

(note; $\sqrt{h} = e$)

(Once we have this; we can add fermionic term)

$\Downarrow$

$$S = \frac{-1}{2\pi} \int d^2\sigma \cdot e \left(h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \right) + \bar{\psi}^\mu \not{D} \cdot \chi^\nu \eta_{\mu\nu}$$

(*)

$\not{D} = \rho^\alpha D_\alpha$; ρ^a are analogue of γ matrices in ~~curved~~ curved space

$\{\rho^\alpha, \rho^\beta\} = 2h^{\alpha\beta}$

and ρ^α are related to ones on tangent space using diads as follows: $\rho_\alpha = e_\alpha^{\ a} \rho_a$

<u>Review</u>: $D_\alpha = \partial_\alpha + [\rho^a, \rho^b] \, \omega_\alpha^{\ ab}$

Covariant derivative

Spin connection

- Diffeomorphism + Weyl

& keep equations of motion.

$$\frac{\delta S}{\delta h^{\alpha\beta}} = T_{\alpha\beta} = \partial_\alpha X \partial_\beta X' + (\text{Fermions}) - (\text{Trace}) = 0$$

impose this as a constraint.

Constraints: $T_{\alpha\beta} = 0$ (use this to get following)

$\Rightarrow X'^- = f(X^i, \psi^i)$

$\Rightarrow \alpha_m^- = f(\alpha^i, b^i \text{ or } d^i)$

Nothing seems to kill the $\psi_\pm^-$ oscillators

→ And these will still produce negative normed states (recall $\xi^\pm$ were responsible for producing negative normed states)

→ So we have to impose supersymmetry (we dont have choice); so that we can use it to get rid of negative normed states.

So; Introduce new fields which makes the action supersymmetry.

* on page 110

So, we are going to make the Action supersymmetry ; & it is going to be our first Supergravity. (Called Supergravity because it has a gravity field $h_{\alpha\beta}$ in it; and we are going to produce a Superpartner)

Superparton of graviton ($h_{\alpha\beta}$) is Gravitino (χ_α)

Supergravity

$$\{h_{\alpha\beta}, \chi_\alpha\}$$

The action.

$$S = \frac{-1}{2\pi}\int d^2\sigma\, e\Big[h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - i\bar{\psi}^\mu \rho^\alpha D_\alpha \psi_\mu + \frac{1}{2}(\bar{\chi}_\alpha \rho^\alpha \rho^\beta \psi^\mu)\partial_\beta X_\mu + i(\bar{\psi}^\mu \psi_\mu)(\bar{\chi}_\alpha \rho^\alpha \rho^\beta \chi_\beta)\Big]$$

$$\boxed{S = \frac{-1}{2\pi}\int d^2\sigma\cdot e\Big[h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - i\bar{\psi}^\mu \rho^\alpha D_\alpha \psi_\mu + \frac{1}{2}(\bar{\chi}_\alpha \rho^\alpha \rho^\beta \psi^\mu)\partial_\beta X_\mu + i(\bar{\psi}^\mu \psi_\mu)(\bar{\chi}_\alpha \rho^\alpha \rho^\beta \chi_\beta)\Big]}$$

- Diffeomorphism + Weyl $\xrightarrow{\text{Super version}}$
- Local Supersymmetry
+
- Supersymmetric version of Weyl ; is Super Weyl

→ These will be enough to set $h_{\alpha\beta} = \eta_{\alpha\beta}$, $\chi_\alpha = 0$.

using $\chi_\alpha = 0$

The action reduces to what we had before

$$S = \frac{-1}{2\pi}\int d^2\sigma \; e\left(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - i\bar{\Psi}^\mu \rho^\alpha D_\alpha \Psi_\mu\right)$$

But, now we have to impose the constraint $T_{\alpha\beta} = 0$ (correspond to field equations of motion of $h_{\alpha\beta}$)

and $\underline{\underline{J_\alpha = 0}}$ (corresponds to field eqn of motion of $\chi_\alpha = 0$)

This is the new constraint !

(note: after imposing $h_{\alpha\beta} = \eta_{\alpha\beta}$ & $\chi_\alpha = 0$ neither the metric ~~& the graviat~~, nor the gravitino are dynamical fields)

We can use the redundancies (i) Local supersymmetry & (ii) Super Weyl to choose $h_{\alpha\beta} = \eta_{\alpha\beta}$ & $\chi_\alpha = 0$ at a particular time. But in order to make sure that these constraints stay the same as we evolve in time; we have to impose at all times the field equations that comes from the variation of $h_{\alpha\beta}$ & χ_α

ie: $T_{\alpha\beta} = 0$ and $J_\alpha = 0$.

Lec 11: Superstrings, Spacetime Fermions, Critical Dimensions.

Where are we & where we are going?

- Quantum Gravity $\Longrightarrow$ Bosonic String.

  $D = 25 + 1$

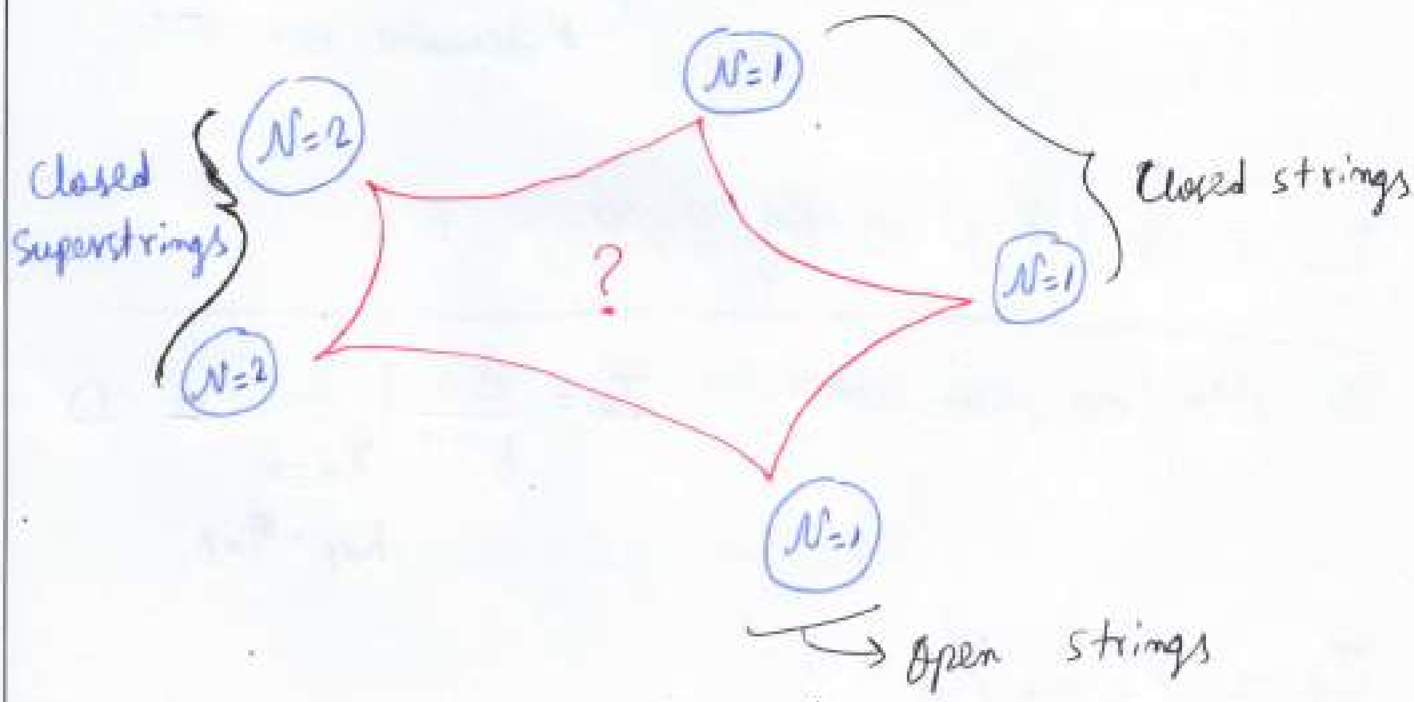
  Closed String Tachyon.

  Only Boson in Space Time.

$\Longrightarrow$ RNS String : Open String → Fermions + Bosons in spacetime

Closed String → Gravity

Goal: A classification of Physical String Theories.



Supergravity Action

$$S = \int d^2\sigma \cdot e \left[h^{\alpha\beta} \partial_\alpha X \partial_\beta X - i\bar{\psi}^\mu \not{\partial} \psi_\mu + \frac{1}{2}(\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu) \partial_\beta X_\mu + i(\bar{\psi}_\mu \psi^\mu)(\bar{\chi}_\alpha \rho^\alpha \rho^\beta \cdot \chi_\beta) \right]$$

- Diffeomorphism + Local SUSY } ⇒ allows us to set $h_{\alpha\beta} = \eta_{\alpha\beta}$, $\chi_\alpha = 0$

constraints : $T_{\alpha\beta} = 0$

- Weyl + Super Weyl

$$\left. \frac{\delta S}{\delta \chi_\alpha} \right|_{\substack{\chi_\alpha = 0 \\ h_{\alpha\beta} = \eta_{\alpha\beta}}} = 0$$

Gravitino Equation of motion

New Constraints $\sigma^\pm = \tau \pm \sigma$

$$T_{++} = (\partial_+ X)^2 + \frac{i}{2} \psi_+^\mu \partial_+ \psi_{+\mu} = 0$$

Chirality

Derivative w.r.t σ^+.

$$T_{--} = (\partial_- X)^2 + \frac{i}{2} \psi_-^\mu \partial_- \psi_{-\mu} = 0$$

We also have new constraints $J_\alpha = \left. \frac{\delta S}{\delta \chi^\alpha} \right|_{\substack{\chi_\alpha = 0 \\ h_{\alpha\beta} = \eta_{\alpha\beta}}} = 0$

$$J_+ = \psi_+^\mu \partial_+ X_\mu = 0$$

$$J_- = \psi_-^\mu \partial_- X_\mu = 0$$

LCGQ (Light cone Gauge Quantization)

$(U^\mu) = (U^+, U^-, U^i)$

$$S = \int d^2\sigma \left(\partial_+ X \cdot \partial_- X + i \psi_+ \partial_- \psi_+ + i \psi_- \partial_+ \psi_- \right)$$

Super conformal Invariance.

It is called LC Gauge; because we use redundancies & chose a gauge where we purposely fix

$$X^+(\tau, \sigma) = \bar{x}^+ + l_s p^+ \tau$$

And using the Super part of Conformal Invariance; now we can set $\psi^+_\pm(\tau, \sigma) = 0$.

But we also saw; Negative normed states could still be produced by α^-, b^- or d^-

lets see; whether our constraints can help us to fix the problem.

$$-2\partial_+ X^- \overbrace{\partial_+ X^+}^{\sim p^+} + \sum_i \partial_+ X^i \partial_+ X^i + \frac{i}{2} \cancelto{0}{\psi^+_+} \partial_+ \psi^-_+$$

$$+ \frac{i}{2} \psi^-_+ \partial_+ \cancelto{0}{\psi^+_+} + \sum_i \frac{i}{2} \psi^i_+ \partial_+ \psi^i_+ = 0$$

$$\Rightarrow \partial_+ X^- \sim \frac{1}{p^+} \left(\underbrace{\sum \partial_+ X^i \partial_+ X^i + \psi^i_+ \partial_+ \psi^i_+}_{} \right)$$

These are good oscillators: bosonic & fermionic.

This will give

$$\alpha^-_m = f(\alpha^i_n\ ,\ b_r^i \text{ or } d_r^i)$$

Now we look at $J_+ = 0$ constraints.

$$-\cancel{\psi^+}\partial_+ X^- - \psi^- \underbrace{\partial_+ X^+}_{\sim p^+} + \sum_i \psi^i \partial_+ X^i = 0$$

$\swarrow 0$

$$\Rightarrow \boxed{\psi^- \sim \frac{1}{p^+} \sum_i \psi^i \partial_+ \psi^i}$$

Hence we got Unitarity !

(At least for now; we are dealing with a theory which makes sense).

$\Longleftrightarrow$ So we can go ahead; and quantize it: and can built a Hilbert Space that makes sense.

Searching for Spacetime Fermion : Part II

Best candidate ~~are the~~ are the state in the Ramond sector: because Vaccum is a spinor in spacetime.

$|a\rangle_R \qquad a = 1, \ldots, \dim(\text{Spinor Representation})$

How can we produce dirac equation:

In Bosonic case: we took Energy momentum tensor & decomposed into Fourier modes $L_m = \int d\sigma\, T_{++}\, e^{-im\sigma}$
.... should do same thing for Fermionic currents.

Ramond lets call the fourier mode F_m

$$F_m = \int d\sigma \cdot J_+ \cdot e^{-im\sigma} = \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot d_{n+m}$$

↳ These commute; so we dont have normal ordering ambiguity.

so;

Quantum Constraints : $F_m |\phi\rangle = 0$, $m \geq 0$

Ex $|a\rangle_R$

set $m = 0$; so; $F_0 |a_R\rangle = 0$.

$$\Rightarrow \left(\sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot d_n \right) |a_R\rangle = 0$$

$$\Rightarrow \alpha_0 \cdot d_0 |a_R\rangle = 0$$ (only $n=0$ survies, ... rest annhilate)

~~$\Rightarrow F_0 |a_R\rangle = \alpha_0 \cdot d_0 |a\rangle_R$~~

$$\boxed{F_0 |a ; p\rangle_R = \alpha_0 \cdot d_0 |a ; p\rangle_R}$$

$$\alpha_0^\mu \sim p^\mu \quad ; \quad d_0^\mu \sim \Gamma^\mu$$

$$\Rightarrow \boxed{p_\mu \Gamma^\mu |a, p\rangle_R = 0}$$

☆☆☆ So we found the Dirac Equation for massless spinor in space time.

$\not{P} |\alpha; P\rangle_R = 0 \Rightarrow$ Massless Dirac Equation !

$\hookrightarrow$ We found our first honest Fermion in spacetime.

The full Ramond section will give infinite number of Fermions. And the ground state happens to ~~be~~ be. massless.

We should check our spectrum (to see whether we haven't broken our Lorentz invariance or not).

Spectrum

NS $\alpha' M^2 = N_{NS} - a_{NS}$

$$N_{NS} = \sum_{i=1}^{D-2}\left(\sum_{m=1}^{\infty}\alpha^i_{-m}\alpha_m + \sum_{r=\frac{1}{2}}^{\infty} r \cdot b^i_{-r} b^i_r\right)$$

R $\alpha' M^2 = N_R - a_R$

$$N_R = \sum_{i=1}^{D-2}\left(\sum_{m=1}^{\infty}\alpha^i_{-m}\alpha_m + \sum_{m=1}^{\infty} n\, d^i_{-n} d^i_m\right)$$

Lets study sector by sector.

NS] $N_{NS} = 0 \Rightarrow \alpha' M^2 |0\rangle_{NS} = -a_{NS}|0\rangle_{NS}$

$N_{NS} = \frac{1}{2} \Rightarrow \alpha' M^2 |pi\rangle_{NS} = \left(\frac{1}{2} - a_{NS}\right)|p^i\rangle_{NS}$

$|pi\rangle_{NS} = b^i_{\frac{1}{2}}|0\rangle_{NS}$ $\Big\} \Rightarrow$ This is a vector of $SO(D-2)$: which is little group for massless particles in D dimensions.

(since there is nothing else in $N_{NS} = \frac{1}{2}$ sector ~~apart~~ apart from $|pi\rangle_{NS}$; so $|pi\rangle_{NS}$ has to be massless representation of Poincaré group in D dimensions)

So; To restore Lorentz invariance : $a_{NS} = \frac{1}{2}$

(hence $|0;P\rangle_{NS}$ becomes Tachyon)

(at least it is an open string Tachyon)

Ramond Sector | R

Vacuum : $N_R = 0$: $\alpha' M^2 |a\rangle_R = -a_R |a_R\rangle$

$\Rightarrow a_R = 0$ ~~because of~~

Vaccum has to massless dirac spinor (if we have Lorentz invariance)

found from using constraints.

$N_R = 1$:

[redacted]

Taming the Ordering Ambiguity

NS sector | Ordering ambiguity comes from commutators & anticommutators.

$$[\alpha^{\mu}_{m}, \alpha^{\nu}_{n}] = m\, \delta_{m+n,0}\, \eta^{\mu\nu}$$

$\mu, \nu \to i, j$ (Transverse indices)

$$\{b^{\mu}_{r}, b^{\nu}_{s}\} = \delta_{r+s,0}\, \eta^{\mu\nu}$$

$\nu, \mu \to i, j$ (Transverse indices)

$$\{b^{i}_{r}, b^{i}_{-r}\} = 1$$

$$[\alpha^{i}_{m}, \alpha^{i}_{-m}] = m$$

(becomes large when m is large)

Since $[\alpha^i_m, \alpha^i_{-m}] = m$ ~~δ_{m}~~

blows up at large m ; we use a regulator!

$$[\alpha^i_m , \alpha^i_{-m}] = m \cdot e^{-m\epsilon}$$

$$\{b^i_r , b^i_{-r}\} = e^{-r\epsilon}$$

At the end, in any physical computation ; we will take limit $\epsilon \to 0$.

"Classical Number Operator"

$$N_{classical} = \frac{1}{2}\sum_{i=1}^{D-2}\left(\sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \alpha^i_{-m} \cdot \alpha^i_m + \sum_{r \in \mathbb{Z}+\frac{1}{2}} r \cdot b^i_{-r} b^i_r \right)$$

(It is different from Quantum Version of Number Operator :

In $N_{classical}$; we are summing over all oscillators. We dont have any reason to sum over only half of the sector.)

lets choose symmetric one for Quantum version

$$\text{"}N_{Quantum}\text{"} = \frac{1}{2}\sum_{i=1}^{D-2}\left[\sum_{m>0} \alpha^i_{-m}\alpha^i_m + \sum_{m<0} \alpha^i_{-m}\alpha^i_m + \sum_{r=1/2}^{\infty} r \cdot b^i_{-r} b^i_r + \sum_{n<0} (n+\tfrac{1}{2})\, b^i_{-(n+\frac{1}{2})} b^i_{(n+\frac{1}{2})} \right]$$

If we want to compare it to N_{NS}

Then we have to appropriately flip the ordering.

$$N_{\text{quantum}}' = \frac{1}{2}\sum_{i=1}^{D-2}\Big[2\sum_n \alpha^i_{-n}\,\alpha^i_n + \sum_{n=1}^{\infty} n\cdot e^{-n\epsilon}$$

$$+ 2\sum_{r=\frac{1}{2}}^{\infty} r\, b^i_{-r}\, b^i_r + (-1)\sum_{n=0} \left(n+\tfrac{1}{2}\right)\cdot e^{-(n+\frac{1}{2})\epsilon}\Big]$$

$$= N_{NS} + \frac{1}{2}\sum_{i=1}^{D-2}\Big[\sum_{n=1}^{\infty} n\cdot e^{-n\epsilon} + (-1)\sum_{n=0}\left(n+\tfrac{1}{2}\right)e^{-(n+\frac{1}{2})\epsilon}\Big]$$

But we said: $N_{\text{quantum}} = N_{NS} - a_{NS}$

$\Rightarrow$ we have to allow for ordering ambiguity.

$$\Rightarrow \quad -a_{NS} = \frac{1}{2}\sum_{i=1}^{D-2}\Big[\sum_{n=1}^{\infty}\left(n e^{-n\epsilon}\right) - \sum_{n=0}\left(n+\tfrac{1}{2}\right)e^{-(n+\frac{1}{2})\epsilon}\Big]$$

Compute & take the limit.

---

lets compute $\displaystyle\sum_{n=0}^{\infty}(n+\alpha)\, e^{-(n+\alpha)\epsilon} \equiv I(\alpha)$

$$\Rightarrow I(\alpha) = -\frac{\partial}{\partial\epsilon}\left(\sum_{n=0}^{\infty} e^{-(n+\alpha)\epsilon}\right) = -\frac{\partial}{\partial\epsilon}\left(\frac{e^{(1-\alpha)\epsilon}}{e^{\epsilon}-1}\right)$$

$$\boxed{\Rightarrow I(\alpha) \underset{\substack{\text{expand}\\ \text{around}\\ \epsilon=0}}{=} \frac{1}{\epsilon^2} - \frac{1}{12}\left(1-6\alpha+6\alpha^2\right) + O(\epsilon)} \quad (**)$$

Using (**) we get

$$a_{NS} = -\frac{1}{2}\cdot(D-2)\left[\left(\frac{1}{\epsilon^2}-\frac{1}{12}\right)-\left(\frac{1}{\epsilon^2}-\frac{1}{12}\left(1-3+\frac{6}{4}\right)\right)\right]$$

$$= -\frac{(D-2)}{2}\left[-\frac{1}{12}-\frac{1}{24}\right]$$

$$\Rightarrow \boxed{a_{NS} = \left(\frac{D-2}{2}\right)\cdot\frac{3}{24}}$$

But from Lorentz invariance ; we found $a_{NS} = 1$

$$\Rightarrow D-2 = 8 \Rightarrow \boxed{D=10}$$

($D=10$ is better than 26 ;
and also we know that for supersymmetry
$D=26$ too higher dimension.
For Supersymmetry we need $D \leq 11$)

$$a_{NS} = -\frac{(D-2)}{2}\left[\underbrace{-\frac{1}{12}}_{\text{like contribution from Bosons (came from } \alpha)}\quad \underbrace{-\frac{1}{24}}_{\text{like contribution from fermions. (came from } b)}\right]$$

"naively" because there will be no one to cancel $\frac{1}{\epsilon^2}$ ...!

If we "naively" drop $\frac{-1}{24}$; and keep $\frac{-1}{12}$ as Bosonic contribution ; taking $a=1$

we find $D_{Bosonic} = 26$

This means that ; (It suggests....)
There must be some way of regularizing the computation & renormalizing away ∞ ; so we get appropriate answers.

---

cancelling of $\frac{1}{\epsilon^2}$ from Bosonic & Fermionic part in NS sector shows that ; NS almost wants to be supersymmetric
(It fails to be supersymmetric because we dont get $a_{NS} = 0$.)

Ramond Sector

Bosons & Fermions precisely pair up ; and every time we commute & anti commute the other one ; we get exactly the same contribution with different signs. So they will cancel out : if we do something consistent with the fact that we found $a_R = 0$.

$\hookrightarrow$ so ; $a_R = 0$ is really manifestation of Supersymmetry on the world sheet.
SUSY !

R boundary conditions preserves supersymmetry on the worldsheet.

· NS boundary conditions dont.

Taming The Tachyon !

- Gliossi - Scherk - Olive (GSO) GSO construction.

It will be nice to have some sort of matching in R sector & NS sector's degrees of freedom.

So, we introduced an operator G_{NS}

$$G_{NS} = -(-1)^{F_{NS}}$$

F_{NS} counts number of fermionic oscillators. (counts no. of bs in a given physical state)

$$F_{NS} = \sum_{i=1}^{8} \sum_{r=1/2}^{\infty} b_{-r}^{i} b_{r}^{i}$$

$$G_R = \Gamma_{11} (-1)^{F_R}$$

$$F_R = \sum_{i=1}^{8} \sum_{m=1}^{\infty} d_{-m}^{i} d_{m}^{i}$$

(counts number of d_s)

ie; $G_R = \Gamma_{11} \, (-1)^{F_R}$

our vacuum is spinor; & so can have two chiralities.
↪ We test the chirality of something with Γ_{11} (because we are in D = 11)

GSO declared that Physical States $|\varphi\rangle$ have

$$G_{NS} |\varphi\rangle = +1 |\varphi\rangle$$
$$G_R |\varphi\rangle = +1 |\varphi\rangle$$

This clearly does the job of declaring that Tachyon is not physical.

GSO projection is consistent projection of the theorys.

(If we scatter true physical GSO states; we get GSO physical states only at the end)

↪ The two theories with GSO eigen value +1 & -1 could quite simultaneously exist.

It consistent to seperate them; & throw the other away.

GSO also solves Modular Invariance.

(Conformal Invariance at loop level is called Modular Invariance)

~~Modular Invar~~ Modular Invariance is only preserved if we impose GSO conditions. (so we could have discovered it; by computing one loop amplitude: and we would found that it has Conformal Anamoly in the form of Modular Invariance anamoly.

And then try to find a way of defing a theory that will preserve conformal invarince.

... & we would have found GSO)

↪ And then tachyon automatically gone away.

Summary | Physical RNS Open String : $D = 9+1$ (g 126)

(after GSO ; so we can produce physical theory that is consistent with interactions)

	NS	R
Massless :	A_μ gauge boson in space time	$\lambda \rightarrow$ Majorana - Weyl Spinor.

(because GSO projection choses one over the other; It makes one choice for the chirality of vacuum: and therefore state is weyl spinor. And Majorana condition is something which we imposed from the begining on the world sheet & translate into same condition on the space time fermion)

We can count no. of d.o.f for both sectors.

Lec 12 : Type IIA and IIB Superstring Theories.

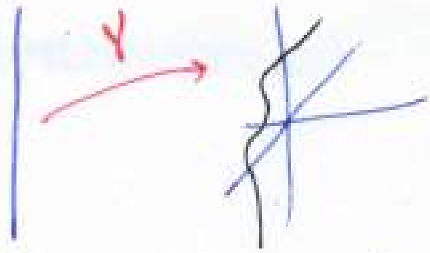
What is it that we are doing?

Analogy :

Particles

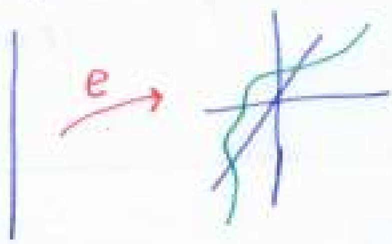
e, γ

World-Line Formalism



1+0 quantum mechanics

$a^{i}|0\rangle \to A_\mu$



$b|0\rangle \to \psi$

QFT

$\psi(x)$, $A_\mu(x)$ fields

$$A_\mu(x) = \int \frac{d^3p}{\sqrt{E}} \left[a_p^{+} \epsilon_\mu e^{ip\cdot x} + \dots \right]$$

$a_p^{+}|0\rangle$ quantum of the photon field.

$$\psi(x) = \int \dots (b^{+} \dots)$$

$b^{+}|0\rangle$ quantum of Electron.

A_μ & ψ are wavefunctions in spacetime (here these are functions in spacetime.

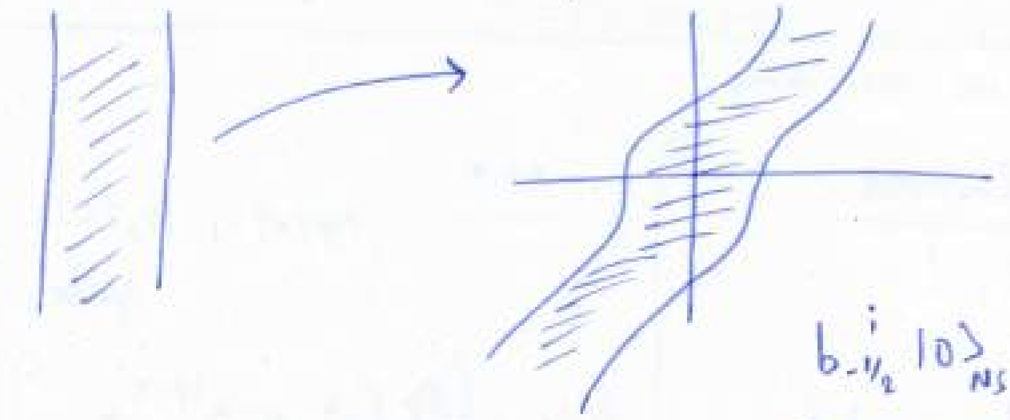
For now A_μ & ψ looks like different theories; but now we can impose rules on how these interact.

In QFT ; ~~A_μ~~

A_μ & ψ are operators in spacetime

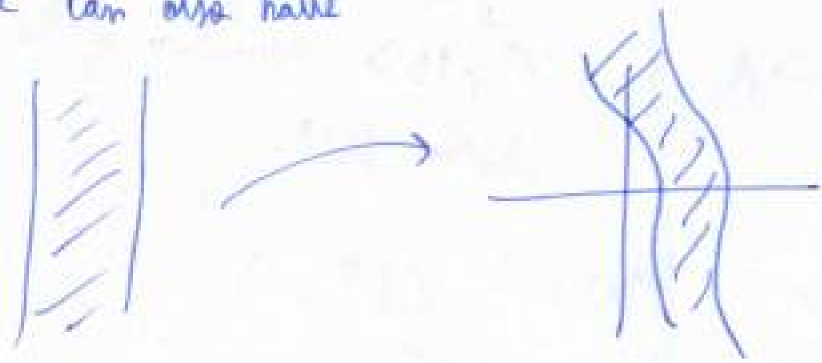
String Theory

World Sheet



$b^{i}_{-1/2}|0\rangle_{NS} \longrightarrow A^{\mu}$

We can also have



$|a\rangle_R \rightarrow \lambda$
(Fermion)

Analog of World Sheet (String Theory)

String Field Theory (SFT)

$A(X(\tau)) = \cdots$ creation operators should create a whole string (in one shot)

String Field.

The quantum of the field A should be actual strings.

Open SFT : Started by Witten

Closed SFT :

* If we had SFT, Then A_μ & ψ ~~...~~ will just be the different sector of the same theory.

For now, we dont know how to write SFT. But if we just concentrate on massless ~~sector~~ sector (just the first level of whole tower of states: We can write down field theory that would describe interaction of these particles

↳ & this is what we call Low Energy effective theory (its a field theory)

(What ever SFT is; it containes ~~...~~ infinite number of field Theories in it) ~~...~~
here; we are going to get field theory just from the first level. We are going to write full field theory.

Fields: A_μ (gauge potential)

λ (Majorana - Weyl Fermion in $D = 10$)

& effective action turns out to be

$$S = \int d^{10}x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\lambda} \not{D} \lambda \right)$$

This is just $\not{\partial}$ because we think of the field in adjoint representation.

This action is invariant under following transformations.

$$\delta A_\mu = \bar{\epsilon} \Gamma_\mu \lambda \qquad \delta \lambda = F_{\mu\nu} [\Gamma^\mu, \Gamma^\nu] \epsilon \qquad \text{SUSY}$$

where ϵ is a Majorana - Weyl spinor hence ϵ has 16 real components.

S is invariant under SUSY upto total derivative.

↳ only one spinor; $\mathcal{N} = 1$ SUSY in 10 dimensions

↳ 16 supercharges (we call each compoment a supercharge)

In 4D ($\mathcal{N}=4$ super Yang Mill)

Q

$|-1\rangle$, $\underbrace{Q|-1\rangle}_{|-\frac{1}{2}\rangle}$

If Q^I : $I \in \{1,2,3,4\}$

Then $|-1\rangle$, $Q^I|-1\rangle$, $Q^{IJ}|-1\rangle$, $\underbrace{Q^{IJK}|-1\rangle}_{|+\frac{1}{2}\rangle}$, $\underbrace{Q^1Q^2Q^3Q^4|-1\rangle}_{|+1\rangle}$

$\underbrace{|-1\rangle}_{\text{1 states}}$, $\underbrace{|-\frac{1}{2}\rangle^I}_{\text{4 states}}$, $\underbrace{|0\rangle^{IJ}}_{\text{6 scalars}}$, $|+\frac{1}{2}\rangle$ 4 states , $|+1\rangle$ 1 states

$\downarrow$ A_μ , $\downarrow$ ψ^I , $\downarrow$ $(\phi^1, \phi^2, \dots, \phi^6)$

(As a field theory $|-1\rangle$ & $|+1\rangle$ will produce single Gauge Boson A_μ)

We have 10 dimensional action $S=\int d^{10}x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{i}{2}\bar{\lambda}\not{D}\lambda\right)$

(invariant under $A_\mu \to A_\mu + \partial_\mu \phi$.

If we want to produce theory in 4 dimensions; we must compactify all directions on circle which we dont want ..... Kaluza Klien Reduction on torus of 6 dimensions, ie: on $T^6 = S^1 \times S^1 \times \dots \times S^1$

$(A_\mu^{10D}) \longrightarrow (A_0, A_1, A_2, A_3, \overbrace{A_4, A_5, A_6, A_7, A_8, A_9})$

from the point of view of 4 dimension lorentz group

(We only care about 4 dimensional Lorentz invariance; because we are breaking Lorentz invariance in 10 dimensions by compactifying down on circles.

We dont have L.I. in full space; we only have L.I. on the four directions which we are not compactifying)

$$M^{1+9} = \mathbb{R}^{1,3} \times T^6$$

from 4D Lorentz group, or say from 4D point of view

- (A_μ^{4D}) ie; A_0, A_1, A_2, A_3 is a vector.
- $A_4, A_5, \dots, A_9$ are scalars. ie; $\phi^1, \phi^2, \dots, \phi^6$.

So our first QFT that we get in 4d, which comes from String Theory can be set to be $N=4$ super Yang mill.

<u>Although its not quite Super Yang mill:</u>

because we have 1 gauge group; ie; U(1) gauge ~~gauge~~ theory.

⟶ We should better call $N=4$ super maxwell.

"$N=1$ super maxwell goes down to $N=4$ super maxwell in 4 dimensions."

Closed Super String

Bosonic Sector (same as before)

Fermionic Sector
$$\delta S\Big|_{\substack{\text{Boundary}\\ \text{Conditions}}} = \int_{-\infty}^{+\infty} d\tau \Big[(\psi_+ \delta\psi_+ - \psi_- \delta\psi_-)\Big|_{\sigma=\pi} - (\psi_+ \delta\psi_+ - \psi_- \delta\psi_-)\Big|_{\sigma=0} \Big] = 0$$

- - - - = cancelation for ~~open~~ closed strings.

〰〰 = " " open strings.

We can choose $\psi_+^\mu(\tau,\sigma) = \pm\, \psi_+^\mu(\tau, \sigma+\pi)$

$$\psi_-^\mu(\tau,\sigma) = \pm\, \psi_-^\mu(\tau, \sigma+\pi)$$

Periodic : Ramond = R

Anti-periodic : NS

$\psi_+ \Rightarrow$ left movers

$\psi_- \Rightarrow$ Right movers.

We have 4 sectors: we can make choices for ψ_+^μ & ψ_-^μ independently.

With the choice of sectors also comes how many different theories we can have. How many of these sectors can be put together in such a way that they interact in consistent way.

It turns out there are two possibilities of combining sectors.

There are two possible theories; which are characterized by basically what we do, to the R vacuum state in each sector.

Left Movers — Right Movers

R - R

R - NS

NS - R

NS - NS

Theories: (choosing chirality of Ramond vacua)

Left movers		Right movers

$$\left(|+\rangle_R \,,\ \tilde{b}^i_{-\frac{1}{2}} |0\rangle_{NS} \right) \otimes \left(|+\rangle_R \,,\ b^i_{-\frac{1}{2}} |0\rangle_{NS} \right)$$

$$\left(|-\rangle_R \,,\ \tilde{b}_{-\frac{1}{2}} |0\rangle_{NS} \right) \otimes \left(|+\rangle_R \,,\ b^i_{-\frac{1}{2}} |0\rangle_{NS} \right)$$

These are two inequivalent theories (each of these theories will have 4 sectors)

~~Note) $|+\rangle_R$ $|+\rangle_R$ is equivalent~~

Note) The theory $|-\rangle_R \cdot\cdot \otimes |-\rangle_R$ is equivalent to $|+\rangle_R \cdot\cdot \otimes |+\rangle_R$

" " $|+\rangle_R \cdot\cdot \otimes |-\rangle_R$ " " " $|-\rangle_R \cdot\cdot \otimes |+\rangle_{NS}$.

Theory 1 II B Chiral Supergravity Theory

Massless States → Irreducible representations of SO(8)

R-R $\quad |+\rangle_R \otimes |+\rangle_R \rightarrow 1 \oplus 28 \oplus 35 \Rightarrow \varphi, B^{R-R}_{\mu\nu}, D_{\mu\nu\rho\sigma}$

axion, 2-form, 4-form

NS-R $\quad \tilde{b}^i_{-\frac{1}{2}} |0\rangle_{NS} \otimes |+\rangle_R \rightarrow 8_+ \oplus (56)_+$

R-NS $\quad |+\rangle_R \otimes b^i_{-\frac{1}{2}} |0\rangle_{NS} \rightarrow 8_+ \oplus (56)_+$

ψ^I, χ^I_μ

dilatons, gravitinos

$I \in \{1, 2\}$

NS-NS $\quad \tilde{b}^i_{-\frac{1}{2}} |0\rangle_{NS} \otimes b^j_{-\frac{1}{2}} |0\rangle_{NS} \quad 8_v \otimes 8_v = 1 \oplus 28 \oplus 35$

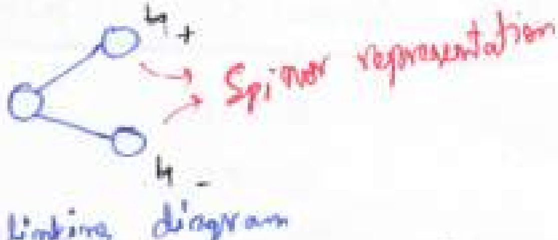
v reminds of vector; Φ, $B_{\mu\nu}$, $G_{\mu\nu}$

We need all four sectors for the theory to be consistent.

Toy Model

SO(6)

$\hookrightarrow 4_+ \otimes 4_-$

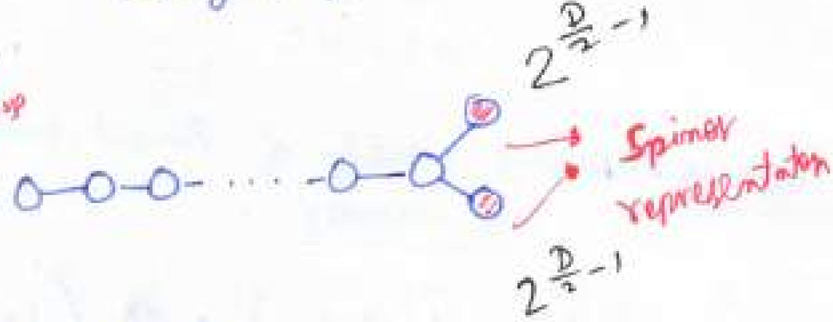


$\square \otimes \square = \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square\square$

$4\times4 = 16 \quad 6 \quad 10$

$A_\mu \oplus A_{\mu\nu\rho}$

SO( )



SU(4)

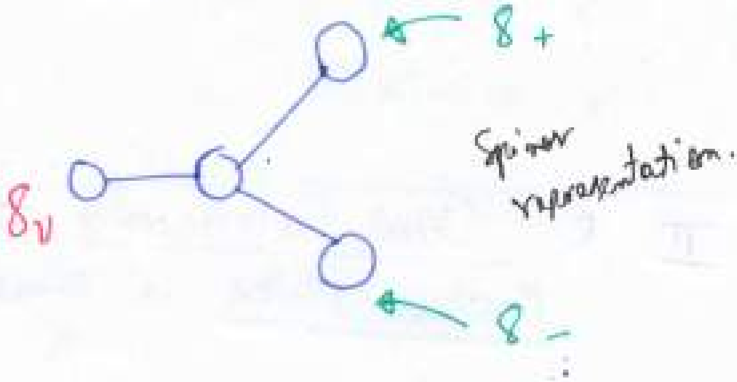
fundamental representation : writes as $\square$

4

$\bar{\square}$

4

SO(8)



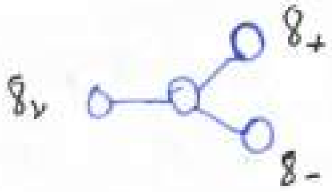
$* A = A$

$* A = -A$

$A_{\mu\nu\rho}$: $6\,\frac{\times 5\times 4}{3\times 2\times 1} = 20 = 10 + 10$ $\quad A_{3\text{-form}} = A_3^+ + A_3^-$

~~SO(8)~~

SO(8)



$8_- \otimes 8_- = 1 \oplus 28 \oplus 35_-$

$8_+ \otimes 8_+ = 1 \oplus \underbrace{28}_{B_{ij}} \oplus \underbrace{35_+}_{\text{Self dual 4-form}}$

$8_+ = |+\rangle_R$

$8_- = |-\rangle_R$

$8_v = b^i_{-\frac{1}{2}}|0\rangle_{NS}$

$A_{ijkl} \to 70 = 35_+ \oplus 35_-$

$8_v \otimes 8_v = 1 \oplus 28 \oplus \underbrace{35}_{g_{ij}}$

$8_- \otimes 8_+ = 8_v \oplus \underbrace{56_v}_{A_{ijk}}$

$8_v \otimes 8_+ = 8_+ \oplus 56_+$

$8_v \otimes 8_- = 8_- \oplus 56_-$

~~Theory 2~~

~~R-R : $|-\rangle_R \otimes |+\rangle_R \to 8_v \oplus 56_v$~~ ($A_\mu$, $C_{\mu\rho\gamma}$)

~~NS-R : $b_{-\frac{1}{2}}|0\rangle_{NS} \otimes |+\rangle_R \to 8_+$~~

II A

Non chiral ~~supersymmetry~~ supergravity theory.

Theory 2

R-R $\quad |-\rangle_R \otimes |+\rangle_R \longrightarrow 8_v + 56_v$ ($\to A_\mu$, $\to C_{\mu\rho\gamma}$)

NS-R $\quad \tilde{b}_{-\frac{1}{2}}|0\rangle_{NS} \otimes |+\rangle_R \longrightarrow 8_+ \oplus 56_+$

R-NS $\quad |-\rangle_R \otimes b_{-\frac{1}{2}}|0\rangle_{NS} \longrightarrow 8_- \oplus 56_-$

$\left.\begin{matrix} \psi_{(+)} \; , \; \chi^{(+)}_\mu \\ \psi_{(-)} \; , \; \chi^{(-)}_\mu \end{matrix}\right\}$ ψ → Dilatini, χ → Gravitini

NS-NS $\quad \tilde{b}_{-\frac{1}{2}}|0\rangle_{NS} \otimes b_{-\frac{1}{2}}|0\rangle_{NS} \longrightarrow 1 \oplus 28 \oplus 35 \to \Phi, \; B^{NS-NS}_{\mu\nu}, \; G_{\mu\nu}$

By no. of gravitino in the theory, we can tell about amount of Supersymmetry.

2 Supersymmetry ⟷ 2 super partner for graviton.

So; 2 will not tell us how to differentiate the Theory.

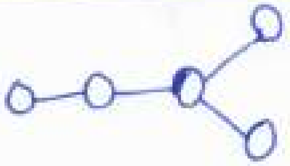
II

The one which has both chirality is A theory ; II A
The one " " Same " " B " ; II B

II B ⇒ Chiral because if we apply parity transformation that reverses chirality, the theory does not stay the same.

II A ⇒ here chirality will flip them; it will be non-chiral.

Lec 13: D branes, T duality, U(N) Gauge Group from Superstrings.



$$\frac{2\alpha^i \cdot \mu^j}{(\alpha^i)^2} = \delta_{ij}$$

$\alpha^i \rightarrow$ Simple roots

$\mu^j \rightarrow$ Fundamental weights

$j = 1, \dots, r$ ← rank.

$A_n \Rightarrow n$-form

Massless

II A

R-R $\quad |-\rangle \otimes |+\rangle \rightarrow A_1, A_3$

NS-R $\quad \tilde{b}^i_{-\frac{1}{2}}|0\rangle \otimes |+\rangle \rightarrow \psi_+, \chi_{+\mu}$

R-NS $\quad |-\rangle \otimes b^j_{-\frac{1}{2}}|0\rangle \rightarrow \psi_-, \chi_{-\mu}$

$\psi_\pm \rightarrow$ Dilatini, $\chi_{\pm\mu} \rightarrow$ Gravitini

NS-NS $\quad \tilde{b}^i_{-\frac{1}{2}}|0\rangle \otimes b^j_{-\frac{1}{2}}|0\rangle \rightarrow \Phi, B_2^{NS-NS}, G_{\mu\nu}$

II B

R-R $\quad |+\rangle \otimes |+\rangle \rightarrow A_0, \underset{B_2^{R-R}}{A_2}, A_4^+$

NS-R $\quad \tilde{b}_{-\frac{1}{2}}|0\rangle \otimes |+\rangle$

R-NS $\quad |+\rangle \otimes b_{-\frac{1}{2}}|0\rangle$

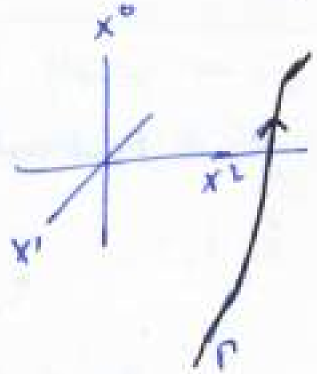
$\Big\} \; \psi^I_+, \chi^I_+ \quad I = 1, 2$

NS-NS $\quad \tilde{b}^i_{-\frac{1}{2}}|0\rangle \otimes b^j_{-\frac{1}{2}}|0\rangle \rightarrow \Phi, B_2^{NS-NS}, G_{\mu\nu}$

What is the meaning of all these differential forms?

Hint 1: In 4D $A_1 = A_\mu dx^\mu$ gauge potential.

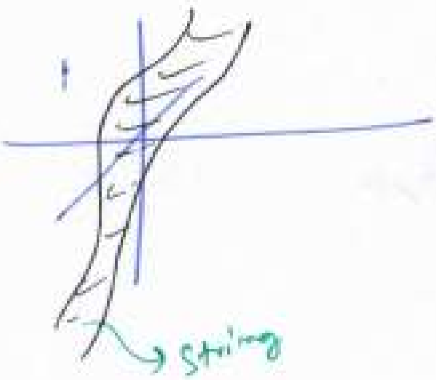
ψ wavefunction of an electrically charged particle.



$$\psi \to e^{e\int_\Gamma A_\mu dx^\mu} \cdot \psi$$

World line $\int A_\mu \dot{x}^\mu d\tau$

In 10 D $B = B^{NS-NS}_{\mu\nu} dx^\mu \otimes dx^\nu$



$$\int_\Sigma B_{\mu\nu} dx^\mu dx^\nu = \int B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta} d^2\sigma$$

Hint 2: 4D Maxwell ; Vacuum $d*F = 0$
$dF = 0$

Part I

In the presence of charges $d*F = j_e$ or $(d*F = *j)$
$dF = 0$

$$\Rightarrow \quad F = dA$$

If we had magnetic charge: $d*F = 0$ Then $F = dA$ not globally,
$dF = J_m$ true locally $F = dA$

Hint 2: Part II

Electric charge $\longrightarrow A_1 \longrightarrow F_2 = dA_1$

Magnetic charge $\longrightarrow \tilde{A}_1 \longrightarrow \tilde{F}_2 = * F_2 = d\tilde{A}_1$

D Electro-magnetic duality in D dimensions.

If we keep electric charge point; ie: A_1 (one form)

$$(*F)^{\mu_3 \dots \mu_D} = \frac{1}{\sqrt{-g}} \epsilon^{\mu_1 \mu_2 \dots \mu_D} F_{\mu_1 \mu_2}$$

Then: Electric charge $\rightarrow A_1 \longrightarrow F_2 = dA_1$

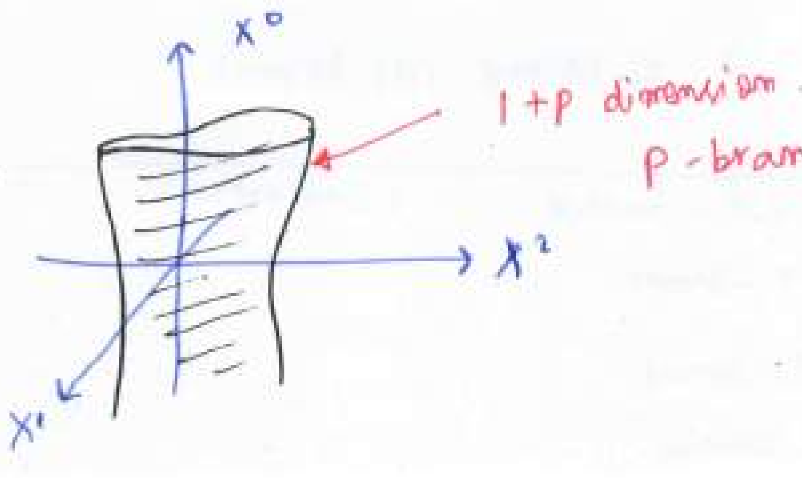
Magnetic charge $\rightarrow \tilde{A}_{D-3} \longrightarrow \tilde{F}_{D-2} = *F_{D-2} = d\tilde{A}_{D-3}$

In general $\quad A_{1+p} \quad (1+p)$ - form

(gauge potential) $\quad$ time

The gauge potential A_{1+p} couples to something as

$$\int_{V_{p+1}} A_{\mu_1 \dots \mu_{1+p}} \cdot \partial_{\alpha_1} X^{\mu_1} \dots \partial_{\alpha_{p+1}} X^{\mu_{p+1}} \cdot d^{p+1}\sigma$$



P-brane $\xrightarrow[\text{couples to}]{}$ A_{1+p} $\longrightarrow$ Field Strength F_{p+2}

$$\downarrow$$

$$*F_{p+2} = \tilde{F}_{D-(p+2)}$$
$$= dA_{D-(p+3)}$$

$\underline{(D-p-4) \text{ brane}}$

↳ It is Magnetic dual.

$$D-(p+3) = \underbrace{D-(p+4)}_{\Downarrow} + 1$$

This is our p dual i.e. $\tilde{p}$

$$\Rightarrow \boxed{\tilde{p} = D-p-4}$$

IIA R-R

$A_1 \xrightarrow[\text{couples to}]{}$ 0-brane $\quad D = 10$

$\tilde{p} = 10-0-4 = 6$

$\rightsquigarrow$ 6-brane

Dual (magnetic dual)

$A_3 \longrightarrow$ 2-brane

$\rightsquigarrow$ 4-brane.

NS-NS $\quad B_2^{NS-NS} \longrightarrow$ 1-brane (Fundamental String)

$\rightsquigarrow$ 5-brane (NS brane)

IIB R-R

$A_0 \longrightarrow$ (-1)-brane (Instanton)

$\rightsquigarrow$ 7-brane

$A_2 \longrightarrow$ 1-brane

$\rightsquigarrow$ 5-brane

$A_4 \longrightarrow$ 3-brane

$\rightsquigarrow$

NS-NS

$B_2 \longrightarrow$ 1-brane } NS

$\rightsquigarrow$ 5-brane

Summary | IIA : 0, 2, 4, 6, "8"

IIB : -1, 1, 3, 5, 7, "9"

These are missing.

Puzzle

R-p-branes.

Big Puzzle : T-duality

Take IIA on $\mathbb{R}^{1,8} \times S^1_R$ ← radius R

$\longrightarrow X^9$

$X_R^9 \longrightarrow -X_R^9$ $\qquad X_L^9 \longrightarrow X_L^9$

$\Downarrow$ Right movers $\qquad \Downarrow$ left movers.

$\psi^9 \longrightarrow -\psi^9$

Ramond Sector

$d_0^9 \longrightarrow -d_0^9$ $\qquad \Gamma^{11} = \Gamma^0 \dots \Gamma^{99}$

? $\qquad \downarrow$

$\Gamma^9 \longrightarrow -\Gamma^9$ $\qquad -\Gamma^{11}$

So: chirality flips

T-duality : flips the chirality of the Ramond ground state in the Right sector.

~~New theory : $\mathbb{R}^{1,8} \times S^1_{1/R}$~~

New theory : $\mathbb{R}^{1,8} \times S^1_{1/R}$ $\qquad$ IIB compactified on $\mathbb{R}^{1,8} \times S^1_{1/R}$

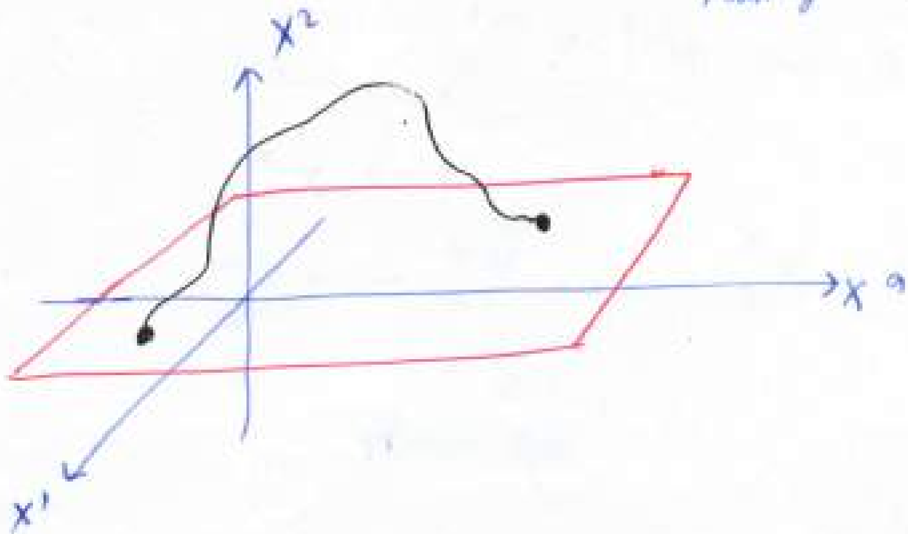
Puzzle : $R \to 0 \Rightarrow$ IIB on $\mathbb{R}^{1,9}$

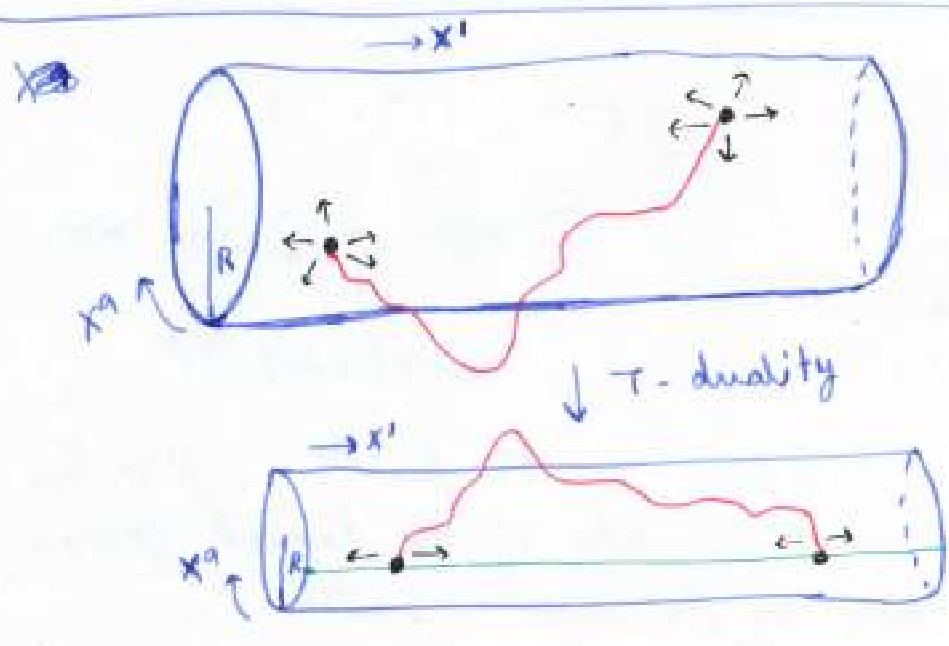
Hint : Boundary conditions for Open Strings..

	0	1	2	3	4	5	6	7	8	9
$\sigma = 0$	X	X								X
$\sigma = \pi$	X	X								X

X : Neumann

Nothing : Dirichlet.





So:

(both ends same conditions)

	0	1	2	3	4	5	6	7	8	9
2 Brane	x	x								x
1 brane	x	x								

T-duality

T duality.

Wraps S^1_R

Started with II A ; with D-2-brane

II A $\xrightarrow{\text{through duality as below}}$ II B

D2 brane
- Wraps S^1_R → D 1-brane
- Does not wrap S^1_R → D3-brane

Does not wrap S^1_R

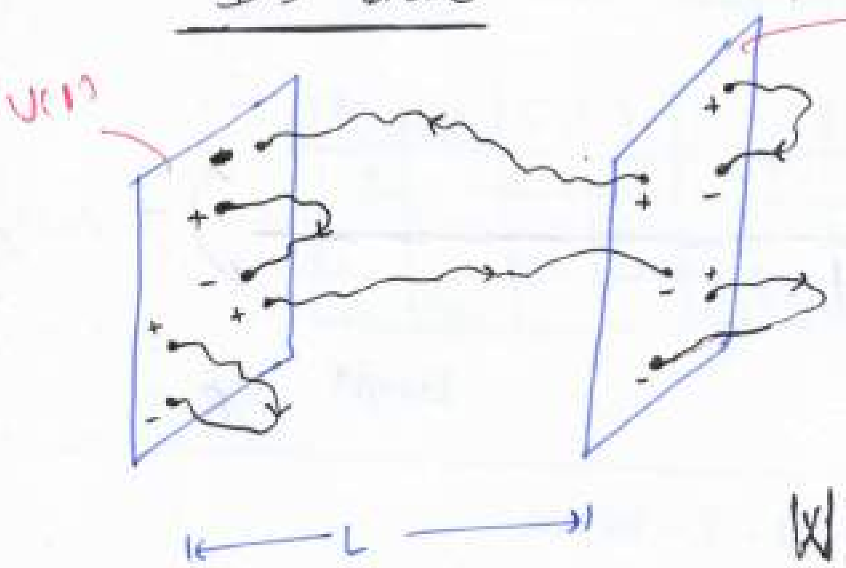
(only action happening in 9th direction)

	0	1	2	3	4	5	6	7	8	9
D2 brane	x	x		x						
D3-brane	x	x		x						x

T-duality

D-brane
- → Gauge Groups
- → $\boxed{\text{AdS/CFT}}$

D3 - brane



U(1) × U(1)

Massive particles

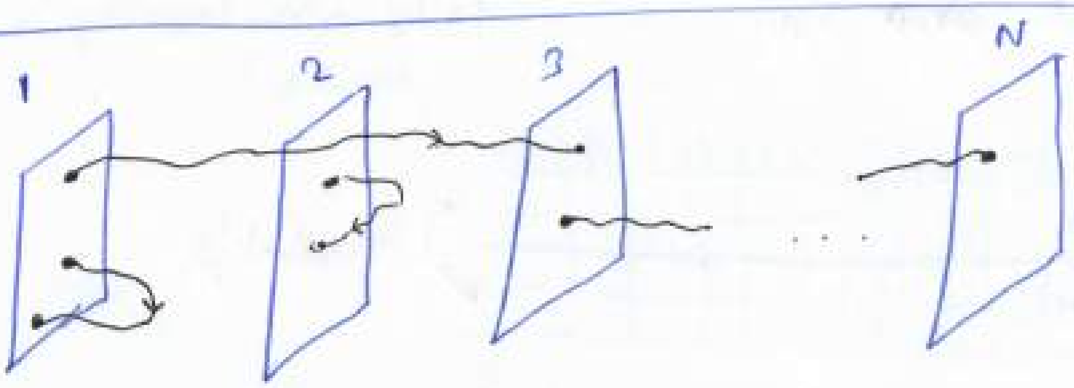
$W^{\pm}$: Massive Vector bosons.

$W^{\pm}$: Massive Vector Bosons.

Taking $L \to 0$; $U(1) \times U(1)$ becomes $U(2)$

(When we go from $U(2)$ to $U(1) \times U(1)$; we are doing some Columb branch....)

$U(2) \supset SU(2)$



~~Coming back ∃ negative roots~~
~~Going out ∃ positive roots.~~

This starts to look like

U(N) gauge group.

* Chan - Paton Factors.

Lec 14: M Theory & 5 String Theories.
Rough sketch of K3 & CY_3-fold.

11 Dimension SUGRA $\mathcal{N}=1$
or 32 Supercharges

11 D SUGRA on $\mathbb{R}^{1,9} \times S^1_R$

$$ds^2 = e^{-2\Phi/3}(g_{\mu\nu}dx^\mu dx^\nu) + \underbrace{e^{4\Phi/3}}_{= R^2}(dx_{10} + A_\mu dx^\mu)^2$$

Low energy IIA $\xrightarrow[\text{massless modes}]{}$ IIA SUGRA $\mathcal{N}=2$.

$$\chi_M \longrightarrow \chi_\mu \quad \mu \in \{0,1,\dots,9\} \to \chi^+_\mu, \chi^-_\mu$$
$$\chi_{10} \Longrightarrow \chi^+_{10}, \chi^-_{10}$$

$\boxed{1995}$

$$g_s = e^{\langle\Phi\rangle}$$
← string coupling in spacetime.

$$g_s^{2/3} = R_{10}$$

Crazy Idea: ∃ a theory in 11 dimensions which has a weakly coupled description around flat spacetime with lowest modes, those of 11D SUGRA.
M-Theory.

11 dimensions

G_{MN} , χ_M , A_{MNP}

(3 form) gauge potential

$\hookrightarrow$ So it couples to $(1+2)$ object;

i.e; 2 brane.

So; The theory contains a Membrane;

we call it M-2-brane.

Magnetic dual of this is 5 brane.

$$\text{M Theory} \begin{cases} \rightarrow \text{M-2 brane} \\ \rightarrow \text{M-5-brane} \end{cases}$$

$A_3 = A_{1+2}$ "p

$\rightarrow$ 2-brane

$\rightsquigarrow$ 5-brane

$$\text{M-2 brane} \begin{cases} \xrightarrow{\text{Wrap}} \text{" String} \\ \xrightarrow[\text{Not to wrap}]{\text{"}} \text{2-brane} \end{cases}$$

(1-brane)

(In type II A we had only one String: Fundamental string)

Fundamental String.

2-brane (The only kind of 2 brane we have in II A is D-2-brane.

$$\text{M-2 brane} \begin{cases} \xrightarrow{\text{Wrap}} \text{Fundamental String (NS-1-brane)} \\ \xrightarrow[\text{Not to wrap}]{} \text{D-2-brane} \end{cases}$$

Magnetic Dual.

$$\text{M-5-brane} \begin{cases} \xrightarrow{\text{Wrap}} \text{D-4-brane} \\ \xrightarrow[\text{Not to wrap}]{} \text{NS-5-Brane} \end{cases}$$

Motivation : D0? "D8"?
D6 ?

---

11 dim M-Theory

$\downarrow S^1_R$

10 dim IIA $g^{2/3} = R$ IIB

$\text{IIA} \xrightarrow{S^1_R} \text{II}$, $\text{IIB} \xrightarrow{S^1_{1/R}} \text{II}$

9 dim II

What else ?

M-Theory

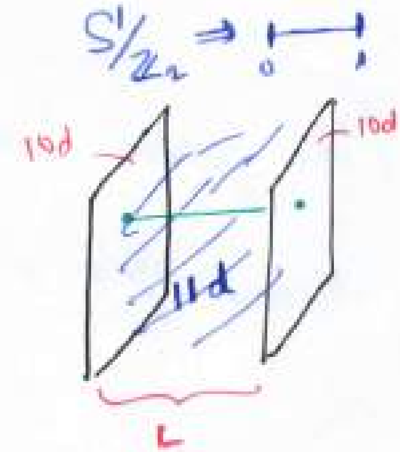
$\downarrow S^1/\mathbb{Z}_2$

10 dim

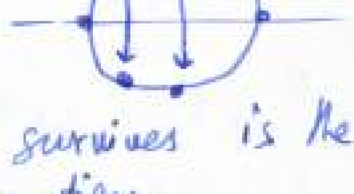
$\mathcal{N}=1$
SUGRA

$\chi^{(+)}_\mu , \ \lambda^{(-)} = \chi^{(-)}_{11}$

$S^1/\mathbb{Z}_2 \Rightarrow$



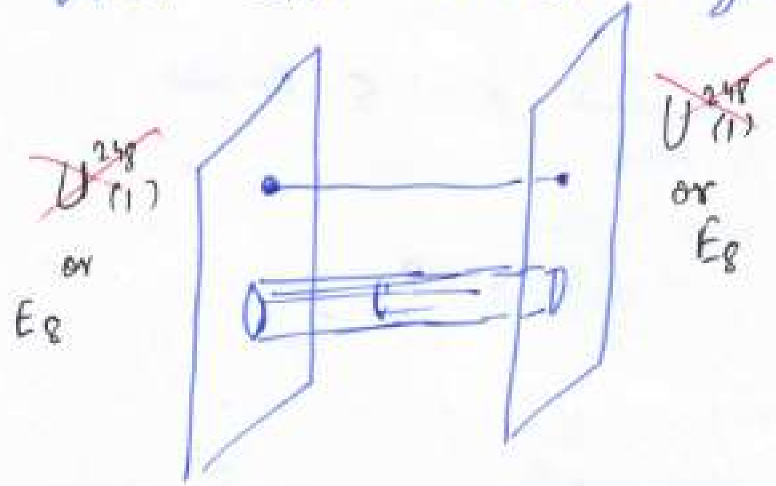
$S^1/\mathbb{Z}_2$



$$\chi_M \rightarrow \begin{cases} \Gamma^{11} \chi_\mu \\ -\Gamma^{11} \chi_{10} \end{cases}$$

When we go down to 10 d : The only things which survives is the one which is invariant under this transformations.

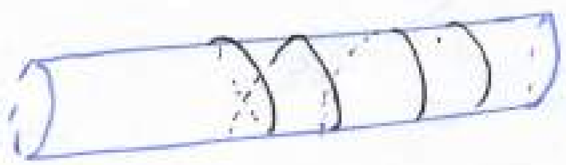
due to Anomalies at the end of the world, $\mathcal{N}=1$ SYM. & $E_8 \times E_8$ gauge group.



Torus Compactifications

Bosonic (closed) string on $\mathbb{R}^{1,24} \times S^1_R$

$$X^{25}(\tau + \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi R m$$



Condition on momentum: $\frac{1}{2}(P_L^2 - P_R^2) = n\, m$

momentum.

More general Compactification:
Compactifying on T^d torus

Compact direction denoted by X^I
μ indices is for non compact directions.

$$I = 1, \dots, d$$

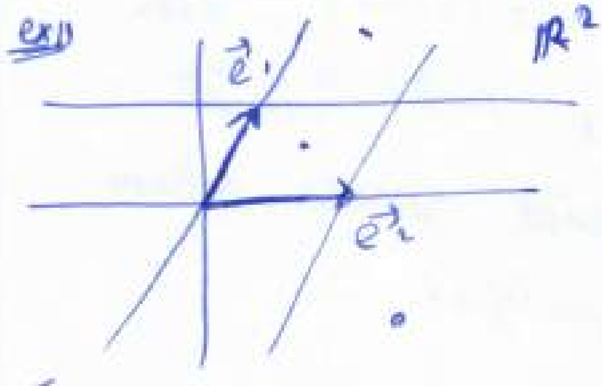
$$T^d = (S^1)^d$$

So; " $X^I(\tau, \sigma + \pi) = X^I(\tau, \sigma) + 2\pi \cdot R \cdot m\, e^I$ "

$\longrightarrow$ We write it in vector notation...

$$\vec{X}(\tau, \sigma+\pi) = \vec{X}(\tau, \sigma) + 2\pi R \cdot m \cdot e^{I}$$

exp



Torus.

Writing it clearly...

$$\vec{X}(\tau, \sigma+\pi) = \vec{X}(\tau, \sigma) + 2\pi R \sum_{i=1}^{d} m_i \vec{e}_i$$

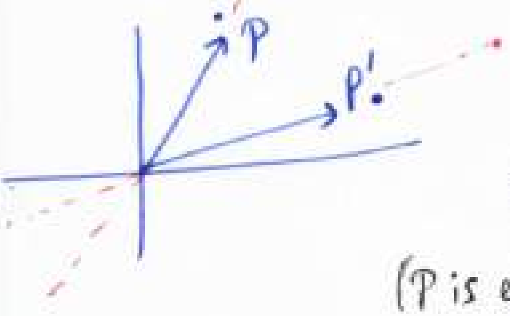
& we have. $$\boxed{\frac{1}{2}\left(\vec{P}_L^{\,2} - \vec{P}_R^{\,2}\right) = \sum_i^d n_i m_i}$$

$\Rightarrow \vec{P}_L^{\,2} - \vec{P}_R^{\,2} = 2 \sum_i n_i m_i \quad (*)$

Define a vector in $\mathbb{R}^{d,d}$. $P = (P_L, P_R)$

where metric in that space is metric $= (\underbrace{1, 1, \ldots, 1}_{d}, \underbrace{-1, -1, \ldots, -1}_{d})$

Lattices in $\mathbb{R}^{d,d}$ that are "even".



$P \cdot P' \in 2\mathbb{Z}$

$P \in \Gamma^{d,d}$

(P is element of lattice)

This is the brilliant use of factor of 2 on RHS of $(*)$

Quantum Consistency requires. $\Gamma^{d,d} = (\Gamma^{d,d})^*$

The lattice is self dual.

Now we have a math problem

Find all self dual $(\Gamma^{d,d}) = (\Gamma^{d,d})^*$ even lattices i.e. $\forall P' \,\&\, P \in \Gamma^{d,d}$ s.t. $P \cdot P' \in 2\mathbb{Z}$:

and this will give us possible spaces where our String Theory can be compactified.

Γ = even & self dual (we want this)

General |

Γ^{d_1, d_2} even & self dual ; $d_1 - d_2 \equiv 0 \mod 8$

we have have solution of these lattices.

~~II 10 D → too many → Right movers → $b^i_{-1/2}|0\rangle_{NS}$, $|0\rangle_R$~~

~~Left movers~~

~~Bosonic 26 D → zero SUSY → compactify on $\Gamma^{16,8}$~~

~~→ on T^{16} → 10 dim~~

II 10 D

Next page

II 10D $\xrightarrow{\text{Too many SUSY}}$ Right movers $\longrightarrow$ $\begin{Bmatrix} b^i_{-\frac{1}{2}}|0\rangle_{NS}, |a\rangle_R \\ \alpha^I_{-1}|0\rangle, \alpha^{\mu}|0\rangle \end{Bmatrix}$ $\longrightarrow$ $\mathcal{N}=1$ SUGRA

Bosonic 26D $\xrightarrow{\text{zero SUSY}}$ left movers $\Gamma^{16,0}$ $\nearrow$

$\Gamma^{16,0}$ $\xrightarrow{\text{maths}}$ $\Gamma_{SO(32)}, \Gamma_{E_8}\times\Gamma_{E_8}$

Bosonic 26D $\longrightarrow$ on T^{16} $\longrightarrow$ 10 dimensions.

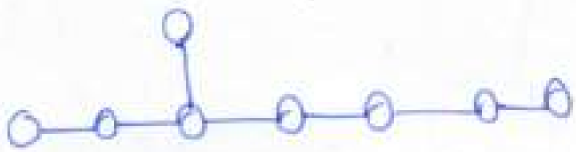
$E_8 \times E_8$ SYM

or

SO(32) SYM

Heterotic $E_8\times E_8$ on a circle.

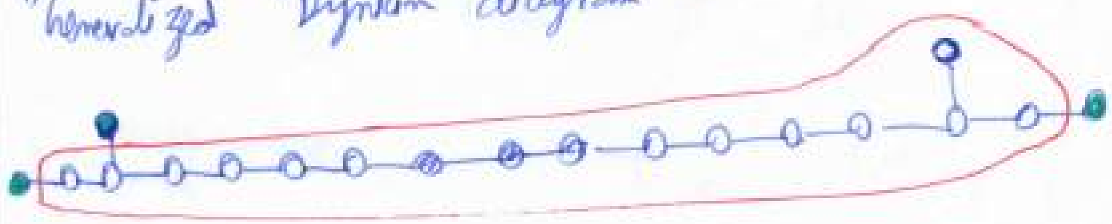
$$\underbrace{\underbrace{\Gamma_{E_8}\times\Gamma_{E_8}}_{\Gamma^{8,0}\times\Gamma^{8,0}}}_{\Gamma^{16,0}} \times \Gamma^{1,1} \longrightarrow \Gamma^{17,1}$$



E_8

E_8

"Generalized Dynkin diagram"



complete by adding three nodes

$SO(32) \longrightarrow 16$ $\qquad$ $\Gamma_{SO(32)} \times \Gamma^{1,1}$

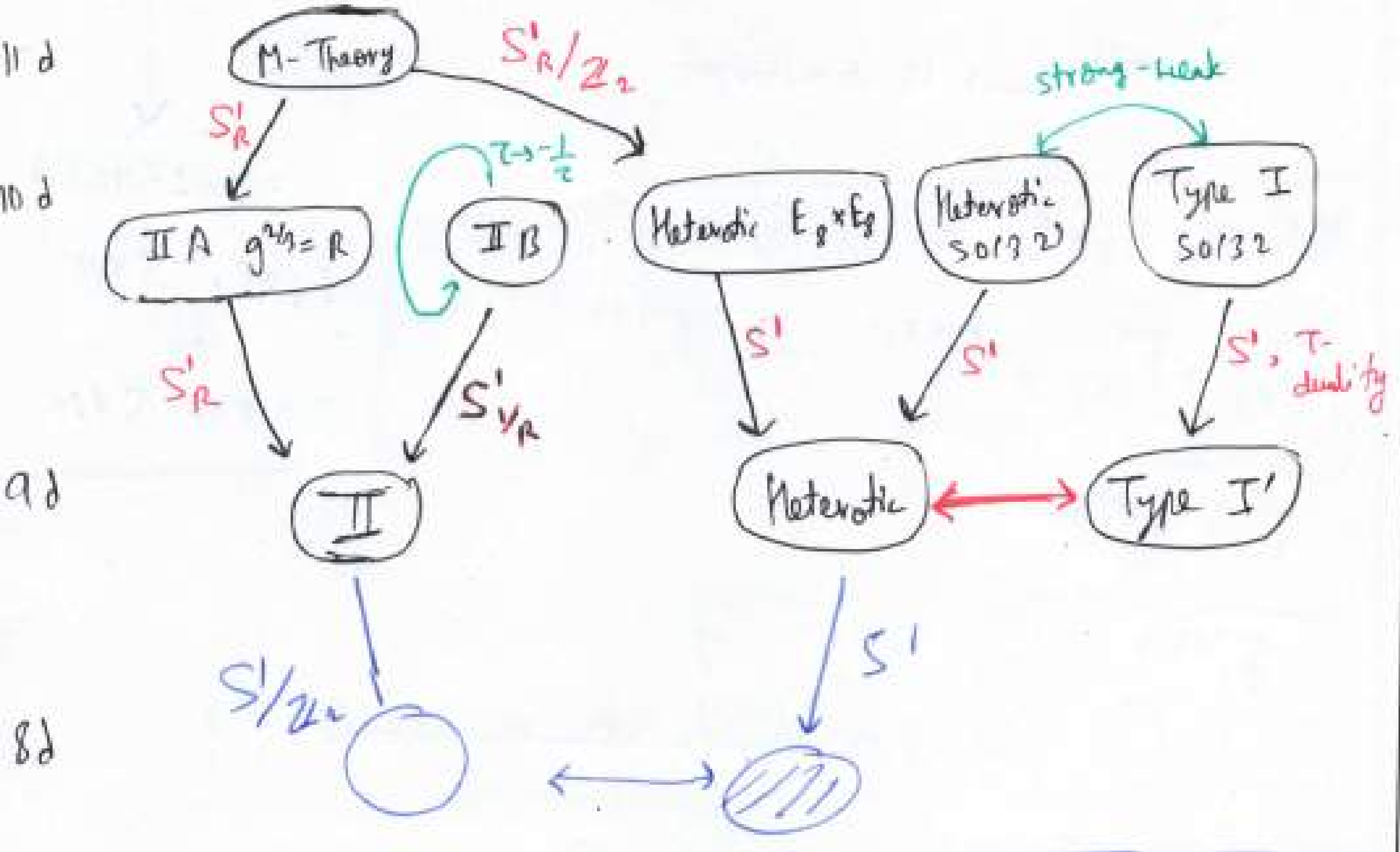
3 left ...

· When we compactify on circle, The two heterotic string theory become same (or belong to same family of objects)

Pg 152

$\tau \to -\frac{1}{\tau}$ dual, as a strong - weak coupling pair.

IIB is self dual ; $\text{IIB} \circlearrowleft \tau \to -\frac{1}{\tau}$



K3 breaks half of the supercharges ; & reduces ~~32 super~~ 32 supercharges to 16 supercharges.

$$2^r \text{ form } \underset{B}{} + \Gamma^{16,3} \longrightarrow \Gamma^{20,4}$$

: K3 $\Rightarrow$ Complex dim $= 2$

Real dim $= 4$

- K3 breaks half of the supercharges.

$$\begin{array}{c} \Gamma^{16,0} \\ \Gamma^{4,4} \\ \hline \Gamma^{20,4} \end{array}$$

32 supercharge Type IIA $\xrightarrow{K3}$ 16 supercharges

Heterotic $E_8 \times E_8$ $\xrightarrow{T^2}$ 16 supercharges

16 supercharges $\longleftrightarrow$ 16 supercharges (Strong - Weak)

16 supercharges $\xrightarrow{K3}$ 8 supercharge : 4D.

---

32 supercharge Type IIA $\xrightarrow{\text{Calabi - Yau manifolds 3 folds}}$ 8 supercharges

$\dim_{\mathbb{R}} 6$

$\dim_{\mathbb{C}} 3$

Heterotic $E_8 \times E_8$ $\xrightarrow{T^2}$ 16 supercharge $\xrightarrow{K3}$ 8 supercharge : 4D

8 supercharges $\longleftrightarrow$ 8 supercharge : 4D

Calabi Yau breaks one quarter of super charges.

$CY_{3\text{-fold}} \equiv$ Calabi - Yau 3 - fold.

Heterotic $E_8 \times E_8$ $\xrightarrow{CY_{3\text{-Fold}}}$ 4 dimensions, 4 super charges $\boxed{\text{S.M}}$

1995

NATURE PLAYS MUSICAL TONES

THANK YOU

Extended objects helps in removing divergences and show us different realm of reality.
Shoaib Akhtar