Modular Forms and Applications to String Theory

With Jacobi Forms, and a brief review of Mock Modular Forms

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MODULAR FORMS & APPLICATIONS TO STRING THEORY

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These notes are consequence of my self study; which I prepared while studying the subject. I started with the motivation for Modular forms and transformations from String Theory; and then again studied Modular forms for its own mathematical sake, along with Ramanujan Conjectures and number theoretic interests. After that I again come back to physics applications of Modular forms. And at the end, a brief review on Mock Modular Forms is added for which a lecture by prof J.A. Harvey was exclusively used.

(Mock) Modular Forms & Applications to Strings

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Lec 1 Fourier Analysis, Poisson Summation Formula, Jacobi-Theta function, Modular Transformations.

Modular Forms - Tools in Physics / Mathematics

- Constraints on 2d Conformal Field Theories.
- Crucial rule in UV finiteness of String Theory.
- Anamoly cancellation - $E_8 \times E_8$, $Spin(32)/\mathbb{Z}_2$.
- Appear in BPS states counting - Special SUSY states.

$\sim$ BH Entropy.

- Appear in AdS/CFT - "Fairey tail" expansion.
- Topological Insulators.

Mathematics:

- Example of automorphic forms in Langlands program.
- Wiles proof on Fermat's Last Theory
  "Modularity Theorem"
- Number Theory $\zeta(s)$
- Moonshine - Connections between sporadic finite simple groups (Monster) and Modular forms.

Mathematical Tools:

Complex Analysis $f: \underset{z=x+iy}{\mathbb{C}} \longrightarrow \mathbb{C}$

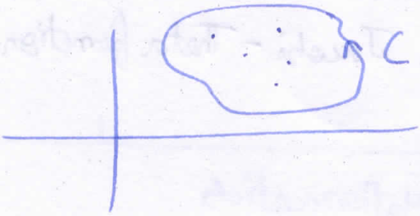
f is holomorphic

$$f(x+iy) = u(x,y) + i\, v(x,y)$$

Cauch Reimann Eqn : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$

or $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$; $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

If f is holo. except at points $a_1, \dots, a_N$



$$\oint_C f(z)\,dz = 2\pi i \sum_k \operatorname{Res}(f; a_k)$$

$\operatorname{Res}(f; a_k)$ = Coefficient of z^{-1} in Laurent expansion of f about that point.

$$f = \sum_{n \in \mathbb{Z}} c_n \cdot (z - a_k)^n$$

ie; $\operatorname{Res}(f; a_k) = c_{-1}$

Holomorphic $\Rightarrow$ Analytic

$f(z) = \sum_{t=0} a_t z^t$ converges ; and a bounded holomorphic function is constant.

Fourier Analysis $\quad f : \mathbb{R} \longrightarrow \mathbb{C}$

define $\quad \hat{f}(k) = \int_{-\infty}^{+\infty} f(x) \cdot e^{-2\pi i k x}\, dx$

also write $\quad \mathcal{F} f(k) \equiv \hat{f}(k)$

$$\mathcal{F}^2 f(x) = f(-x)$$

so: $\quad \mathcal{F}^4 = \mathbb{1}$

We will encounter functions which are their own Fourier transform.

$$\boxed{\begin{array}{l} \text{ie: if } \mathcal{F} f(x) = \lambda f(x) \\ \text{Then } \quad \lambda^4 = 1 \Rightarrow \lambda = 1, -1, i, -i \end{array}}$$

An example $f(x) = e^{-\pi x^2}$ (gaussian)

$$\Rightarrow \hat{f}(k) = \int_{-\infty}^{+\infty} dx\, e^{-2\pi i k x - \pi x^2} = e^{-\pi k^2}$$

(Fourier Transform of Gaussian is Gaussian)

~~Exercise 1~~ $\cancel{f(\tau) = e^{\pi i \tau x^2} \text{ with } \operatorname{Im}\tau > 0}$

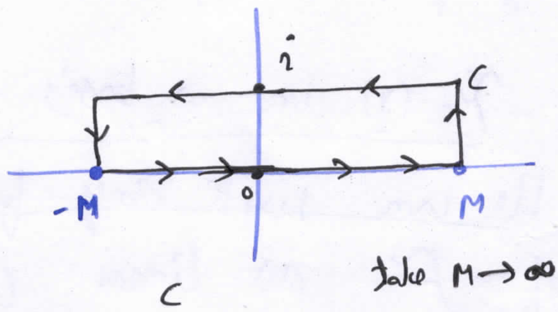
~~Then we can show~~ $\cancel{\mathcal{F} f(\tau}$

Exercise 1 $f_\tau(x) = e^{\pi i \tau x^2}$ with $\operatorname{Im}\tau > 0$

Then $\mathcal{F}(f_\tau) = \dfrac{1}{\sqrt{-i\tau}} \cdot f_{-1/\tau}$

Exercise 2 let $g(x) = \dfrac{1}{\cosh \pi x}$

By considering $\displaystyle\oint_C \frac{e^{-2\pi i k x}}{\cosh \pi x}$



We can show

$$\mathcal{F} g(x) = g(x)$$

Consider the SHO (Simple Harmonic Oscilator)

$$H = \frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2} \qquad [\hat{p}, \hat{x}] = -i \ (\hbar)$$

D : $\hat{p} \to \hat{x}$ leaves H invariant.

$\hat{x} \to -\hat{p}$ $[\hat{p}, \hat{x}]$ invariant.

(Its essentially the fourier transform; going from coordinate space to momentum space)

This suggest that; one way of understanding functions of Fourier Transform is by looking at eigenfunctions of SHO; because, since $\mathcal{F}$ commutes with H, we should be able to find eigenfunctions of both the Hamiltonian & of Fourier Transform simultaneously

Exercise 3 Show $\psi_n(x) = \langle x | n \rangle$

$$H\,\psi_n(x) = (n+\tfrac{1}{2})\,\psi_n(x) \quad ; \quad (\hbar\omega = 1)$$

Then $\mathcal{F}\,\psi_n(x) = (-i)^n\,\psi_n(x)$

So; The energy eigen functions of SHO which involves polynomials times gaussian; provide us with ∞ set of eigenfunctions of Fourier transform.

$\psi_n(x)$ are a basis for $L^2(\mathbb{R})$

We can write any function which is an eigenfunction of $\mathcal{F}$ as linear combination of $\psi_n(x)$.

Solution, Ex 1 $f_\tau(x) = e^{i\pi\tau x^2}$; $\operatorname{Im}(\tau) > 0$

$$\mathcal{F} f_\tau(k) = \int_{-\infty}^{+\infty} f_\tau(x)\cdot e^{-i2\pi k x}\,dx$$

$$= \int_{-\infty}^{+\infty} e^{i\pi\tau\cdot x^2 - i2\pi k x}\,dx$$

$$= \int_{-\infty}^{+\infty} e^{i\pi\tau\left(x^2 - \frac{2k}{\tau}\cdot x\right)}\,dx$$

$$= \int_{-\infty}^{+\infty} e^{i\pi\tau\left(x^2 - \frac{2k}{\tau}\cdot x + \frac{k^2}{\tau^2} - \frac{k^2}{\tau^2}\right)}\,dx$$

$$= e^{-i\pi\cdot\frac{1}{\tau}\cdot k^2}\int_{-\infty}^{+\infty} e^{i\pi\tau\left(x - \frac{k}{\tau}\right)^2}\,dx$$

$$\mathcal{F} f_\tau(k) = e^{i\pi\cdot\left(-\frac{1}{\tau}\right)\cdot k^2} \cdot \int_{-\infty}^{+\infty} e^{-(-i\pi\tau)\cdot(x-k/\tau)^2} dx$$

$$\Rightarrow \mathcal{F} f_\tau(k) = \frac{1}{\sqrt{-i\tau}} e^{i\pi\left(-\frac{1}{\tau}\right)k^2} \Rightarrow \mathcal{F} f_\tau(k) = \frac{1}{\sqrt{-i\tau}} \cdot f_{-1/\tau}(k)$$

Solution Ex 2 | Solution is in my complex Analysis notes.

Poisson Summation

let $g(x)$ be a function on $\mathbb{R}$ with "sufficiently fast fall off" such that $f(x) = \sum_{n=-\infty}^{+\infty} g(x+n)$ converges.

Then, we can show

$$\boxed{\sum_{n=-\infty}^{+\infty} g(n) = \sum_{m=-\infty}^{+\infty} \hat{g}(m)}$$

Proof | $f(x)$ is a periodic function with period 1

$$f(x+r) = f(x) \quad \forall\, r \in \mathbb{Z}$$

Then we can write $f(x) = \sum_{m=-\infty}^{+\infty} c_m \cdot e^{2\pi i m x}$

with $c_m = \int_0^1 f(x)\, e^{-2\pi i m x} \, dx$

$$c_m = \int_0^1 \sum_{n=-\infty}^{+\infty} g(x+n) \cdot e^{-2\pi i m x} \, dx$$

$$= \int_{-\infty}^{+\infty} g(x) \cdot e^{-2\pi i m x} \, dx = \hat{g}(m)$$

$f(0) = \sum_{n=-\infty}^{+\infty} g(n)$ from defination.

$$\Rightarrow f(0) = \sum_{n=-\infty}^{+\infty} \hat{g}(n) \Rightarrow \boxed{\sum_{n=-\infty}^{+\infty} g(n) = \sum_{n=-\infty}^{+\infty} \hat{g}(n)}$$

$$\sum_{n=-\infty}^{+\infty} \int_0^1 g(x+n)\, e^{-2\pi i m x}\, dx$$

$$= \dots + \int_0^1 g(x-1)\, e^{-2\pi i m x}\, dx$$

$$+ \int_0^1 g(x)\, e^{-2\pi i m x}\, dx$$

$$+ \int_0^1 g(x+1)\, e^{-2\pi i m x}\, dx + \dots$$

$$= \int_0^{\infty} g(x)\, e^{-2\pi i m x}\, dx$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{convergent}$$

Example of a "Modular Function"

Jacobi-theta function is

① Historically one of the Earliest.

② Template for objects to be discussed.

$$\theta(z,\tau) = \sum_{n=-\infty}^{+\infty} q^{n^2/2} \cdot y^n \quad \text{with } q = 2\pi i \tau,\ y = e^{2\pi i z}$$

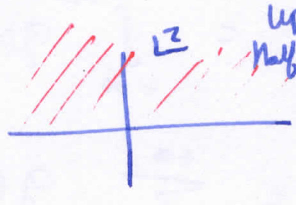
with $\tau \in \mathbb{C}$

with $\operatorname{Im} \tau > 0$

$z \in \mathbb{C}$

ie; $\tau \in \mathbb{H}$

$\mathbb{H}$



Properties of $\theta(z,\tau)$ under certain transformations $z \in \mathbb{C}$ of z and of τ.

$$\theta(z+1;\tau) = \theta(z,\tau)$$

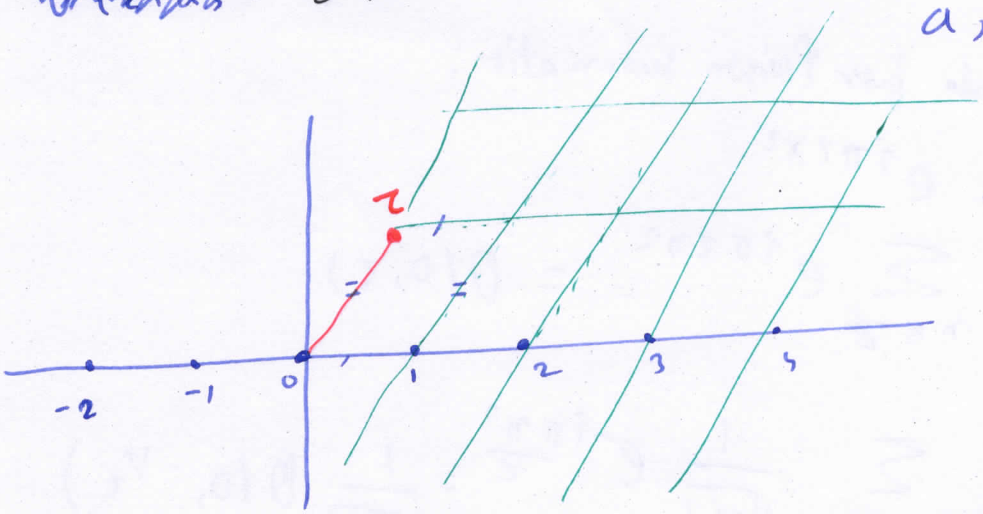
$$\theta(z+\tau,\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n(z+\tau)}\, e^{2\pi i \frac{n^2}{2}\tau}$$

$$= \sum_{n\in\mathbb{Z}} e^{2\pi i (n+1) z} \cdot e^{2\pi i \frac{(n+1)^2}{2}\tau}\, e^{-2\pi i z}\, e^{-\pi i z}$$

$$\Rightarrow \boxed{\theta(z+\tau,\tau) = e^{-2\pi i z - \pi i \tau} \cdot \theta(z,\tau)}$$

$$\theta(z+a+b\tau,\tau) = e^{-2\pi i b z - \pi i b^2 \tau} \cdot \theta(z;\tau)$$

$a, b \in \mathbb{Z}$



2d lattice

$\mathbb{Z} + \mathbb{Z}\tau$

$\rightarrow$ function comes back to itself

$\not\rightarrow$ function comes back to itself but upto a phase.

$$E_\tau = \mathbb{C} / \mathbb{Z} + \mathbb{Z}\tau = T^2$$

(identify opposite sides

T^2  .

or

Elliptic Curve

"Section of a line bundle over E_τ"

So, there is a group $\mathbb{Z}^2$ - shifts generated by $\mathbb{Z}, \mathbb{Z}\tau$.

τ transformations

First with $z=0$, (then $z \neq 0$ as exercise)

$$\theta(0,\tau) = \sum_{n} e^{\pi i \tau n^2}$$

$$\theta(0,\tau+2) = \theta(0,\tau)$$

$$\theta\left(0,-\tfrac{1}{\tau}\right) = \theta(0,\tau)\cdot\sqrt{-i\tau}$$

Also has Group Theoretical Interpretation.

Exercise 1 & Formula for Poisson summation:

Take $g(x) = e^{i\pi \tau x^2}$

$$\sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau n^2} = \theta(0,\tau)$$

$$\sum_{m\in\mathbb{Z}} \hat{g}(m) = \sum_{m\in\mathbb{Z}} \frac{1}{\sqrt{-i\tau}} e^{-i\pi\frac{n^2}{\tau}} = \frac{1}{\sqrt{-i\tau}}\,\theta(0,-1/\tau)$$

Physics: Consider the heat eqn in one dimension.

tem $T(t,x)$

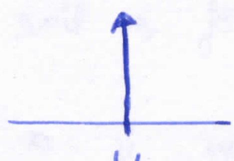
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad \alpha = \text{"Thermal diffusivity"}$$

Exercise $0 \le x \le L$, with periodic b.c. $T(t,x) = T(t,x+L)$

$\equiv$

circle with circumference L.

If $T(x,0) = T_0\,\delta(x-L/2)$



find $T(x,t)$, and express in terms of $\theta(z,\tau)$

Or Consider QM on S^1 $\qquad \varphi \sim \varphi + 2\pi$

$$\cancel{H = \frac{\hbar^2}{2Rm}} \quad H = -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\varphi^2} \qquad ; \ \psi(\varphi + 2\pi) = \psi(\varphi)$$

Compute the thermal partition function $\qquad Z(\beta) = \mathrm{Tr}(e^{-\beta H})$

(a) Using Canonical Methods - Hamiltonian. $\qquad \sim \theta(0, \tau)$

(b) Using Path Integrals. $\qquad \sim \theta(0, -1/\tau)$

This is what we get by each method

(The fact that path integral & canonical method must agree; is simply the property of Theta Function)

(modular transformation property of Theta Function)

$$\theta(z, \tau) = \sum_n e^{2\pi i z} \cdot e^{\pi i n^2 \tau}$$

$$\frac{\partial \theta}{\partial \tau} = \sum_n \pi n^2 (\quad)$$

$$\frac{\partial^2 \theta}{\partial z^2} = \sum_n -4\pi^2 n^2 (\quad)$$

$$\boxed{\frac{\partial \theta}{\partial \tau} = \frac{-i}{4\pi} \cdot \frac{\partial^2 \theta}{\partial z^2}} \quad \text{or} \quad \boxed{\frac{\partial \theta}{\partial t}(z, it) = \frac{1}{4\pi}\frac{\partial^2 \theta}{\partial z^2}(z, it)}$$

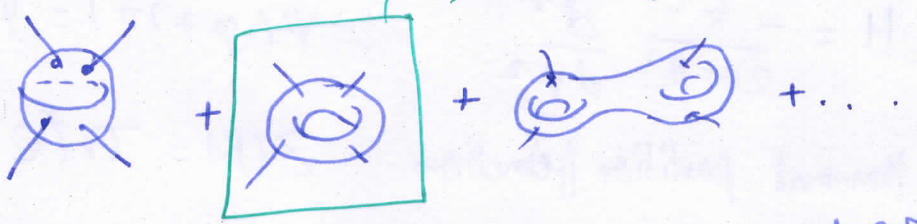
We can think of it as heat eqn or Schrodinger eqn depending on what we choose t to be; imaginary or real.

(...., Wick Rotation)

In physics: CFT string theory we encounter.

Partition function : Torus T^2

String Theory



Compute correlation functions of operators in a 2d CFT.

CFT 2d : Start with a system on a line

0 ——— L

$\rightarrow$ Impose periodic b.c. S^1

$\rightarrow$ Compute partition fun^n $Z(\beta) = Tr(e^{-\beta H})$

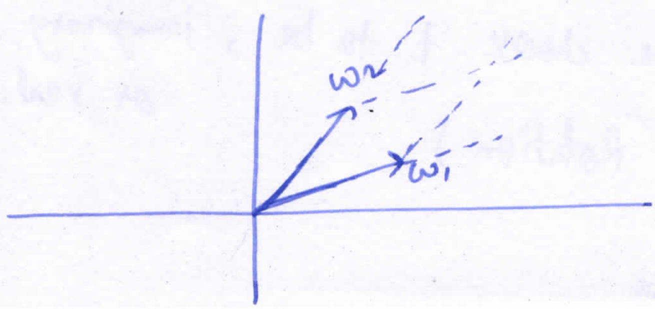
$\sim$ path integral of field $\phi(t_E)$ over Euclidean Time, with period β.

We essentially work on $S^1 \times S^1 \sim T^2$

In a conformal invariant theory on a T^2, Then we can use a mathematical result : "any metric on T^2 is conformal to the flat metric"

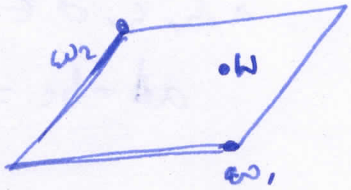
$\hookrightarrow$ Can study the system on T^2 with a flat metric.

$T^2 = \mathbb{C}/L$ ← 2 dimensional lattice.



$L_{\omega_1, \omega_2} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

Take ω_1, ω_2. let w be a point in $\mathbb{C}/L_{\omega_1,\omega_2}$



$$(\omega_1 ; \omega_2 ; w) = \omega_2 \left(\frac{\omega_1}{\omega_2} ; 1 ; w/\omega_2 \right)$$

Choose $\mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0$ (If it was not ; we can change the role of ~~orientation of the lattice~~ ω_1 & ω_2: Its basically ~~a~~ choosing an orientation for $\mathbb{C}/L_{\omega_1,\omega_2}$.

we write;

$$(\omega_1 ; \omega_2 ; w) = \omega_2 (\tau ; 1 ; z)$$

$$\tau = \frac{\omega_1}{\omega_2} \quad ; \quad z = \frac{w}{\omega_2}$$

Now we can get the same lattice by choosing a different set of Basis vectors.

$\omega_1 \longrightarrow a\omega_1 + b\omega_2$

$\omega_2 \longrightarrow c\omega_1 + d\omega_2$

where $a, b, c, d \in \mathbb{Z}$ s.t. $d\omega_1 \wedge d\omega_2$ area form is invariant

$$\Rightarrow ad - bc = 1$$

we see $\tau = \frac{\omega_1}{\omega_2} \longrightarrow \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2} = \frac{a\tau + b}{c\tau + d}$

$$z = \frac{w}{\omega_2} \longrightarrow \frac{w}{c\omega_1 + d\omega_2} = \frac{z\omega_2}{c\omega_1 + d\omega_2} = \frac{z}{cz+d}$$

Summary $\tau \longrightarrow \dfrac{a\tau+b}{c\tau+d}$

$a, b, c, d \in \mathbb{Z}$

$z \longrightarrow \dfrac{z}{c\tau+d}$ $ad-bc=1$

We except that these transformations leave the partition function of CFT, or one loop partition function of String Theory invariant.

We can think of it as

- Changing basis of lattice.

or

- generating diffeomorphism of T^2, which are not connected to Identity.

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a,b,c,d \in \mathbb{Z} \\ ad-bc=1 \end{array} \right\} = SL_2(\mathbb{Z})$$

$\tau \to \dfrac{a\tau+b}{c\tau+d}$ We have an action of $SL_2(\mathbb{Z})$ on the UHP $\mathbb{H}$.

τ, modulo the action of $SL_2(\mathbb{Z})$ label conformal equivalence classes of T^2 or E_τ (Elliptic curve)

τ is called Modus for E_τ.

$SL_2(\mathbb{Z})$ is called the Modular Group.

Note that, in terms of the action on τ $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ act the same.

$$\frac{a\tau+b}{c\tau+d} \qquad PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{1,-1\}$$

(ie: regard $\mathbb{1}$ & $-\mathbb{1}$ as same element)

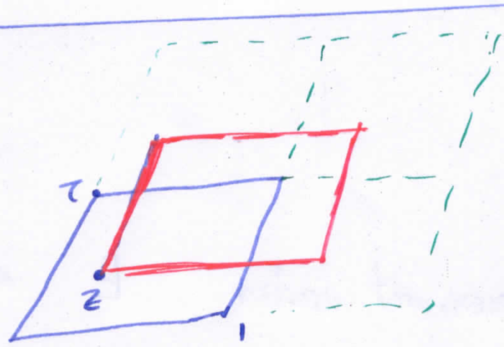
If we include the action on z, then work with $SL_2(\mathbb{Z})$ (because the transformation depends on sign)

Demand that the 1-loop partition function of string theory or CFT should be modular invariant - i.e. $SL_2(\mathbb{Z})$.

Elliptic Transformations

$$z \to z+\mu+\lambda\tau$$

Takes a point on torus & shifts by some combination of lattice vectors.



$$E_\tau = \mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$$

group : $\mathbb{Z}^2$ $\qquad \mu, \lambda \in \mathbb{Z}$

Modular Transformations

$$\tau \to \frac{a\tau+b}{c\tau+d} \qquad \text{group} : SL_2(\mathbb{Z})$$

$$z \to \frac{z}{c\tau+d}$$

Jacobi Group = Elliptic + Modular.

Jacobi group.

Thinking back for Jacobi Theta function.

- Had nice property under Elliptic Transformations.
- Modular transformation include $\tau \to \tau + 1$, and this did not left Jacobi Theta function invariant.

Can we make Functions which are either invariant or "Transform nicely" under

E Elliptic transformation,
M Modular ",
J Jacobi ".

Things invariant under E are called : Elliptic Functions
" " " M " " : Modular "
" " " J " " : Jacobi forms

There are two cases for Jacobi Forms
- Holomorphic Jacobi Forms
- Skew-Holomorphic Jacobi Forms.

In each case we have a group G; which acts on z, τ or (z,τ); and we will try to construct functions of $z, \tau, (z,\tau)$ that transforms nicely by ~~averag~~ "averaging over G".

$f + gf + g^2 f + \dots$

- get zero
- get a nice function
- Sum could diverge (Then we regularize)

Lec 2) Trig. functions, Weierstrass func, Eisenstein Series, Holo. eisenstein series of weight $2k$, Weight $2k$ action, Lipschitz Formula

Constructing functions invariant under a group action by averaging over the group.

Examples)

Function	Group	Group Action
Trigonometric	$\mathbb{Z}$	$x \mapsto x+n \ ; n \in \mathbb{N}$
Elliptic	$\mathbb{Z}^2$	$z \mapsto x + n\omega_1 + m\omega_2$, $n, m \in \mathbb{Z}$
Modular	$SL_2(\mathbb{Z})$	$\tau \to \frac{a\tau+b}{c\tau+d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$
~~Modular~~ Poincare Rademacher Series		

Consider x^{-n} , $n \in \mathbb{Z}_{>0}$

let $\varepsilon_n(x) = \sum_{r=-\infty}^{+\infty} (n+r)^{-n}$ = Average over $\mathbb{Z}$ of x^{-n}

If the sum converges, then $\varepsilon_n(x+s) = \varepsilon_n(x)$, $s \in \mathbb{Z}$

Recall: a series $\sum_{n=1}^{\infty} c(n)$ is convergent and equal to S if the partial sums $\sum_{n=1}^{N} c(n) = S_N$ converges to S as $N \to \infty$.

A series converges absolutely if $\sum_{n=1}^{\infty} |c_n|$ converges.

$\nexists$ Converges, and we can rearrange terms in the sum.

Easy to show, that $\mathcal{E}_n(x)$ converges absolutely for $n \geq 2$

lets focus on $\underline{n=1}$

$\mathcal{E}_1(x)$ does not converge absolutely $\sim \log x$ in integral.

Specify an order to the sum:

$$\mathcal{E}_1(x) = \lim_{N \to \infty} \sum_{r=-N}^{N} \frac{1}{x+r} = \lim_{N \to \infty} \left(\frac{1}{x} + \sum_{r=1}^{N} \frac{1}{x+r} + \frac{1}{x-r} \right)$$

$$= \lim_{N \to \infty} \left(\frac{1}{x} + 2x \sum_{r=1}^{N} \frac{1}{x^2 - r^2} \right)$$

This converges absolutely

$c_n \sim \frac{1}{n^2}$

(With this ordering it defines a convergent series)

We can get various properties of $\mathcal{E}_1(x)$, $\mathcal{E}_2(x)$, ... from just the defination.

We can show; $\frac{d}{dx} \mathcal{E}_1(x) = -\mathcal{E}_1^2 - \pi^2$ has a simple pole at $x = 0$

The unique solution is $\mathcal{E}_1(x) = \pi \cot(\pi x) \equiv \pi \cot(\pi x)$

$$\boxed{\mathcal{E}_1(x) = \pi \cot(\pi x)}$$

$$e(x) \overset{\text{define}}{=} \frac{\mathcal{E}_1(x) + i\pi}{\mathcal{E}_1(x) - i\pi} = e^{2\pi i x} = \cos 2\pi x + i \sin 2\pi x$$

A. Weil "Elliptic Functions according to Eisenstein & Kronecker".

Exercise For $z \in \mathbb{C}$.

We can show, $\pi \cot(\pi z) = \mathcal{E}_1(z) = \frac{1}{z} + \sum_{r=1}^{\infty} \left(\frac{1}{z+r} + \frac{1}{z-r} \right)$

by showing that the difference of LHS, RHS is a holomorphic function & bounded; hence a constant, & the constant value becomes 0.

~~Since $\mathcal{E}_1(z) = \pi \cot \pi$.~~

Since $\mathcal{E}_1(z) = \pi \cdot \cot \pi z$ is periodic, ~~which can~~ we can derive a fourier series expansion in $q = e^{2\pi i z}$

$$\pi \cot \pi z = \pi \cdot \left(\frac{2i \cos \pi z}{2i \sin \pi z} \right) = \pi i \frac{(q^{1/2} + q^{-1/2})}{(q^{1/2} - q^{-1/2})}$$

$$= \pi i \frac{(q+1)}{q-1} \Rightarrow \boxed{\pi \cot \pi z = i\pi \left(\frac{q+1}{q-1} \right)}$$

$$\pi \cot \pi z = -\pi i (q+1)(1 + q + q^2 + \ldots)$$

$$\Rightarrow \boxed{\pi \cot \pi z = \pi i - 2\pi i \sum_{m=0}^{\infty} q^m} \quad |q| < 1$$

$\hookrightarrow$ Fourier Expansion of $\pi \cot \pi z$.

Also follow from Euler's formula $\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$

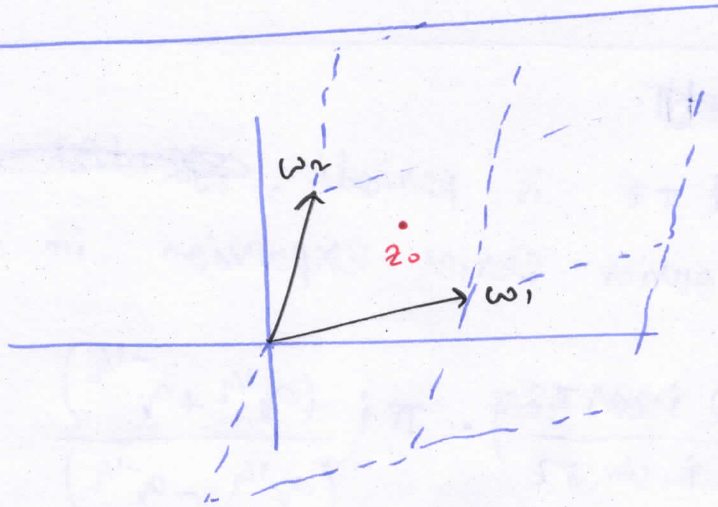
Take log of both sides & then $\frac{d}{dx}$

$$\frac{d}{dx}(\log \sin \pi x) = \pi \cot \pi x$$

$$\Rightarrow \frac{d}{dx}\left(\log \pi + \log x + \sum \log\left(1 - \frac{x^2}{n^2}\right)\right) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2}$$

And this was our original definition of $\pi \cot \pi x$, ~~using~~ in ~~terms~~ terms of averaging over ~~other~~ energies ....

Lattice



$$L_{\omega_1, \omega_2} = \{ n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z} \}$$

We ~~want~~ could construct functions that are invariant under L_{ω_1,ω_2} by just writing down double fourier series in ω_1 & ω_2 with two different variables x & y. It will be doubly periodic, but it will not be holomorphic

Can we make a holomorphic function under $\mathbb{Z}^2$?

$f(z)$ holo. s.t. $f(z + n_1\omega_1 + n_2\omega_2) = f(z)$?

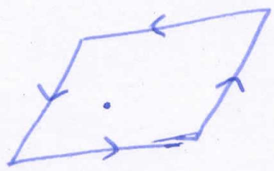
Answer No (Yes, But only if constant)

$f(z_0)$ is bounded ; $z_0 \in \square$

$\Rightarrow$ f is bounded in $\mathbb{C}$ $\Rightarrow$ f is constant.

May be we can make f "almost "holomorphic ~~with one~~

ie; ~~one simple pole in~~ ie; one simple pole in $\square$



$$\oint_{C_\square} f(z)\,dz = 2\pi i \,Res(f; z_0) \neq 0$$

but $\oint_{C_\square} f(z)\,dz = 0$ by periodicity.

So; we failed again.

So; If we have second order pole, then it can work (we know $\oint \frac{dz}{z^2} = 0$)

Try to average $\frac{1}{z^2}$ over $\mathbb{Z}^2$

ie; look at

$$\sum_{\omega \in L_{\omega_1,\omega_2}} \frac{1}{(z+\omega)^2} = \sum_{n,m \in \mathbb{Z}} \frac{1}{(z + n\omega_1 + m\omega_2)^2}$$

$$= \frac{1}{z^2} + \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \frac{1}{(z+n\omega_1 + m\omega_2)^2}$$

This will be our attempt at finding functions, which are not quite holomorphic : but has simplest pole it can have compatible with its periodicity, and we try to make sense of it.

The term $\displaystyle\sum_{\substack{n,m\in\mathbb{Z}\\(n,m)\neq(0,0)}} \frac{1}{(z+n\omega_1+m\omega_2)^2}$

diverges as $z\to 0$.

So; This term at $z=0$ is

$$\sum_{\substack{n,m\in\mathbb{Z}\\(n,m)}} \frac{1}{(n\omega_1+m\omega_2)^2}$$

first sum gives $\sim \frac{1}{x}$ $\Rightarrow$ Then second sum gives log...

So; This term is log divergent (but independent of z)

So, we can do something like Renormalization in QFT where we subtract divergences as long as it does not affect the physics.

"Here z dependence is not affected by divergences; so we subtract it & define a new function"

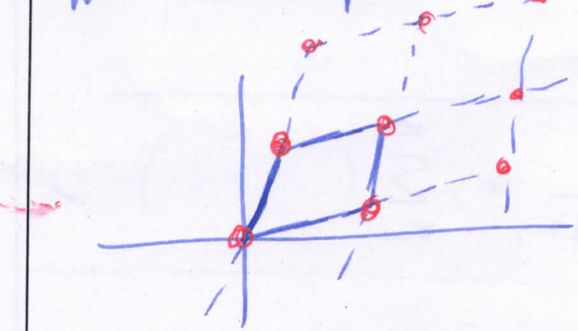
$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in L_{\omega_1,\omega_2}-\{0\}} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right)$$

Weierstrass function.

convergent at $z=0$ & periodic $\mathbb{Z}^2$

(latex code: $ \wp $ = $\wp$)

$\wp(z)$ is a meromorphic function on $\mathbb{C}$,
Invariant under $z \to z + \omega$, $\omega \in L_{\omega_1, \omega_2}$ &
has double poles at lattice points.



Exercise $\dfrac{d\wp}{dz} = -2 \displaystyle\sum_{\omega \in L_{\omega_1,\omega_2}} (z+\omega)^{-3}$ is an odd function of z

and has exactly three zeroes mod L_{ω_1, ω_2} at

$$z = \frac{\omega_1}{2}, \quad \frac{\omega_2}{2}, \quad \frac{\omega_1+\omega_2}{2}$$

Look at the Taylor series expansion of $\wp(z)$.

$$\wp(z) - \frac{1}{z^2} = \sum_{\omega \in L_{\omega_1,\omega_2} - \{0\}} \left(\frac{1}{(\omega - z)^2} - \frac{1}{\omega^2} \right)$$

$$\frac{1}{(\omega - z)^2} = \frac{1}{\omega^2 (1 - z/\omega)^2} = \cancel{\frac{1}{\omega^2}\sum_{r=0}^{\infty}} \left(\frac{z}{\omega}\right)^{2r} \quad ; \left|\frac{z}{\omega}\right| < 1$$

$$= \frac{1}{\omega^2} \sum_{r=0}^{\infty} (r+1) \cdot \left(\frac{z^r}{\omega^r} \right)$$

when r is odd, ω & $-\omega$ give opposite contribution.

$$\wp(z) - \frac{1}{z^2} = \sum_{\omega \in L - \{0\}} \sum_{k=1}^{\infty} \frac{(2k+1)\, z^{2k}}{\omega^{2k+2}}$$

$$= \sum_{k=1}^{\infty} (2k+1) \cdot G_{2k+2}(\omega)\, z^{2k}$$

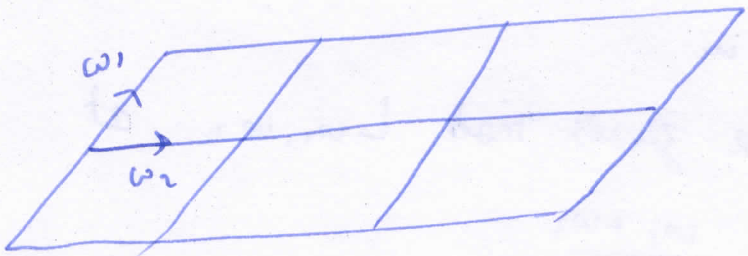
(1922)

$$\wp(z) - \frac{1}{z^2} = \sum_{k=1}^{\infty} (2k+1)\, G_{2k+2}(\omega) \cdot z^{2k}$$

where $G_{2k+2} = \sum_{\omega \in L - \{0\}} \frac{1}{\omega^{2k+2}}$

$$\frac{1}{(\omega - z)^2} = \frac{1}{\omega^2} \sum_{r=0}^{\infty} (r+1) \left(\frac{z^r}{\omega^r}\right) = \frac{1}{\omega^2} + \sum_{r=1}^{\infty} (\quad)$$

$\wp$



$\wp(z, \omega_1, \omega_2)$ ↳ Because we specify lattice under which it is invariant.

$G_{2k+2}(\omega)$ ← Coefficients functions of ω_1, ω_2.

G_{2k+2} = "Eisenstein Series" and are modular forms for $SL_2(\mathbb{Z})$.

We extract out the lattice ~~depe~~ dependance; it gives G_{2k} which are going to be modular functions essentially because they should depend simple way under modular transformation when we change basis for lattice.

$$G_{2k+2} = \sum_{\substack{n,m \in \mathbb{Z} \\ (m,n) \neq 0}} \frac{1}{(n\omega_1 + m\omega_2)^{2k+2}}$$

$$= \frac{1}{\omega_2^{2k+2}} \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \frac{1}{(n\tau + m)^{2k+2}}$$

$$\tau = \frac{\omega_1}{\omega_2}$$

define,
Holomorphic Eisenstein Series of weight $2k$ to be

$$G_{2k}(\tau) = \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \frac{1}{(n\tau + m)^{2k}}$$

$\wp(z)$ includes $G_4, G_6, G_8, \ldots$
For which the sum converges and is non zero

2 Important facts

① $G_{2k}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2k} G_{2k}(\tau)$; $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$
means that G_{2k} are modular forms of weight $2k$.

② $G_{2k}(\tau)$ can be constructed from the principle of "averaging a simple function over a group." $(SL_2(\mathbb{Z}))$

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① $SL_2(\mathbb{Z})$ is generated by

$T: \tau \longrightarrow \tau + 1$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$S: \tau \longrightarrow -1/\tau$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

any element of $SL_2(\mathbb{Z})$ can be written as a word in S & T example $ST^2ST^3STSTS^2T^3\ldots$

So, we need to show that

$$G_{2k}(\tau+1) = G_{2k}(\tau)$$

$$G_{2k}(-1/\tau) = (\tau)^{2k} \cdot G_{2k}(\tau)$$

~~Notation~~ $\cancel{\sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} = \sum{}'}$

$$\cancel{G_{2k}\left(-\frac{1}{\tau}\right) = \sum{}' \left(\frac{1}{-\frac{n}{\tau}+m}\right)^{2k}}$$

$$\cancel{\Rightarrow G_{2k}\left(\frac{-1}{\tau}\right) = \sum{}' \frac{\tau^{2k}}{(-n+m\tau)^{2k}} = \sum_{\substack{n',m' \in \mathbb{Z}(n\tau \\ (n',m') \neq (0,0)}} \frac{\tau^{2k}}{\quad}}$$

Notation $\sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} = \sum'$

$$G_{2k}\left(\frac{-1}{\tau}\right) = \sum{}' \frac{1}{\left(\frac{-n}{\tau}+m\right)^{2k}} = \sum{}' \frac{\tau^{2k}}{(-n+m\tau)^{2k}} = \sum_{\substack{n',m' \in \mathbb{Z} \\ (n',m') \neq (0,0)}} \frac{\tau^{2k}}{(n'\tau+m')^{2k}}$$

$n' = -m$
$m' = -n$

Here we are actually rearranging ; (Allowed because sum is absolutely convergent)

$$\Rightarrow \boxed{G_{2k}\left(\frac{-1}{\tau}\right) = \tau^{2k} \cdot G_{2k}(\tau)}$$

ex) $G_{2k}(\tau+1) = G_{2k}(\tau)$

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② Use $G_{2k}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2k} G_{2k}(\tau)$

Define a "Weight 2k action" of $SL_2(\mathbb{Z})$ for $f : \mathbb{H} \longrightarrow \mathbb{C}$ with $\gamma \in SL_2(\mathbb{Z})$

$$(f|_{2k}\gamma)(\tau) = (c\tau+d)^{-2k} f\left(\frac{a\tau+b}{c\tau+d}\right)$$

Then transformation law of Eisenstein series can be written as $(G_{2k}|_{2k}\gamma)(\tau) = G_{2k}(\tau)$

$$\boxed{(G_{2k}|_{2k}\gamma)(\tau) = G_{2k}(\tau)}$$

$\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$

Transformation of Eisenstein Series.

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ex) a function 1 $\quad 1(h) = 1 \quad \forall h \in \mathbb{H}$

lets average it over "weight 2k action"

$$\mathbb{1}|_{2k}\gamma = (c\tau+d)^{2k}\,\mathbb{1}$$

We could consider $\displaystyle\sum_{\gamma \in SL_2(\mathbb{Z})} \mathbb{1}|_{2k}\gamma(\tau)$ } ⇒ but this diverges like crazy.

When $c=0, d=1$ this action does nothing

ie; $(\mathbb{1}|_{2k}\gamma)(\tau) = 1$ when $c=0, d=1$

$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \Rightarrow a=1$ ie; $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ does nothing to $\mathbb{1}$

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \tau : \tau+1 \quad \Rightarrow \quad T^b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

Notation $\Gamma = SL_2(\mathbb{Z})$

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{Z} \right\}$$

Γ_∞ is stabilizer of $\mathbb{1}$ (take $\mathbb{1}$ to itself)

$$\sum_{\Gamma/\Gamma_\infty} \mathbb{1}\Big|_{2k} \gamma$$ this makes sense.

define $$E_{2k}(\tau) = \sum_{\Gamma/\Gamma_\infty} \mathbb{1}\Big|_{2k} \gamma = \underbrace{\sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ \gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix}}}}_{} (c\tau + d)^{-2k}$$

notation! $\Gamma/\Gamma_\infty \equiv \Gamma_\infty \backslash \Gamma$

↳ Summing over all elements in $SL_2(\mathbb{Z})$ modulo terms in Γ_∞.

↳ Its averaging a function over a group; with a particular action of the group: The weight $2k$ action.

Claim $$\boxed{E_{2k}(\tau) = \frac{1}{2} \sum_{\substack{c, d \in \mathbb{Z} \\ \gcd(c,d) = 1}} \frac{1}{(c\tau + d)^{2k}}}$$

$$\underbrace{\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}}_{\Gamma_\infty} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\Gamma} = \begin{pmatrix} a + mc & b + dm \\ c & d \end{pmatrix}$$

element in Γ, & that element multiplied by an element of Γ_∞; has ~~same~~ the same lower row.

ie: Γ, $\Gamma_\infty \Gamma$ have the same lower row.

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if $\gamma' = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

with same bottom row, then

$$\gamma' = T^n \gamma \quad \text{for some } n$$

proof $(a'-a)d - (b'-b)c = (a'd - b'c) - (ad - bc)$
$= \det(\gamma') - \det(\gamma) = 0$

Since $(ad - bc) = 1$, ~~then gcd c~~

Then $\gcd(c,d) = 1$ (c & d are relatively prime.

$\therefore$ $a' - a = nc$
$b' - b = nd$ for some $n \in \mathbb{Z}$

$\Rightarrow \boxed{\gamma' = T^n \gamma}$

Relationship between G_{2k} & E_{2k} is

$$\boxed{G_{2k}(\tau) = 2\,\zeta(2k)\,E_{2k}(\tau)}$$

proof $G_{2k}(\tau) = \sum' \frac{1}{(n\tau+m)^{2k}}$

$p = \gcd(m,n)$
$m = pc$
$n = pd$ with $\gcd(c,d) = 1$

greatest common divisor.

$$\Rightarrow G_{2k}(\tau) = \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \frac{1}{(n\tau+m)^{2k}} = \sum_{p=1}^{\infty} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} \frac{1}{p^{2k}} \cdot \frac{1}{(c\tau+d)^{2k}}$$

~~But $\sum_p \frac{1}{p^{k}}$~~

But $\sum_p \frac{1}{p^k} = \zeta(2k)$ proved

Deriving Fourier Series Expansion

$E_{2k}(\tau+1) = E_{2k}(\tau)$ so can write a fourier series

in $q = e^{2\pi i \tau}$

From expansion of $\cot(\pi z)$ we have

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau+n)} = -i\pi - 2\pi i \sum_{m=0}^{\infty} q^m$$

1) Take $\frac{(-1)^{k-1}}{(k-1)!}\left(\frac{d}{d\tau}\right)^{k-1}$ on both sides

Then we find $$\boxed{\sum_{n \in \mathbb{Z}} \frac{1}{(n+\tau)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} \cdot q^m}$$

Lipschitz's Formula.

Then $$G_{2k}(\tau) = \underbrace{\sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{1}{m^{2k}}}_{\text{gives Riemann Zeta}} + \underbrace{\sum_{\substack{m,n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{(n\tau+m)^{2k}}}_{\text{Use Lipschitz formula to write fourier expansion of this.}}$$

Also use $$\zeta(2k) = (-1)^{k+1} \cdot \frac{B_{2k}(2\pi)^{2k}}{2(2k)!}$$

where B_{2k} = Bernoulli's Number.

$$G_{2k}(\tau) = \frac{(2\pi i)^{2k}}{(2k-1)!}\left(-\frac{B_{2k}}{2k} + \sum_{n=1}^{\infty}\sum_{r=1}^{\infty} r^{2k-1} \cdot q^{rn} \right)$$

→ reordering this sum we can write

$$\sum_{m=1} \sigma_{2k-1}(m)\, q^m$$

where $\sigma_{2k-1}(m) = \sum_{d|m} d^{2k-1}$

divisors of m

$$\boxed{G_{2k}(\tau) = \frac{(2\pi i)^{2k}}{(2k-1)!}\left(-\frac{B_{2k}}{2k} + \sum_{m=1} \sigma_{2k-1}(m)\, q^m \right)}$$

$$E_4(\tau) = 1 + 240q + 2160q^2 + \dots$$

$$E_6(\tau) = 1 - 504q - 16632q^2 + \dots$$

$$E_8(\tau) = 1 + \dots$$

Recap

1) Average a function over G.
   - Trigonometric Functions ($\cot \pi z$)
   - Elliptic function ($\wp(z)$)
   - Modular functions (E_{2k})

Modular Forms

Shoaib Akhtar 15/8/2020

Lec 3] $SL_2(\mathbb{Z})$, Modular form of weight k, Eisenstein Series
q-expansion of Modular form.

What is Modular form?

Special types of Analytic Functions on Upper Half Plane $\mathbb{H}$.

$$\mathbb{H} = \{x+iy : y>0\}$$

A point in $\mathbb{H}$ is denoted by τ usually.

Def^n] $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}$

(Special Linear group)

ex] $\begin{pmatrix} 5 & x \\ 12 & y \end{pmatrix} \in SL_2(\mathbb{Z})$

So solve $5y - 12x = 1$

Solutions $x = 2$, $y = 5$

Action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau \overset{\text{def}}{=} \dfrac{a\tau + b}{c\tau + d}$

Linear Fractional Transformations

$\tau \in \mathbb{H} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau \in \mathbb{H}$

Proof]
$a, b, c, d \in \mathbb{R}$
$z \in \mathbb{C}$
$z \neq -\frac{d}{c}$

Then $\operatorname{Im}\left(\dfrac{az+b}{cz+d}\right) = \dfrac{(ad-bc)\operatorname{Im} z}{|cz+d|^2}$

so if $z \in \mathbb{H}$ $\Rightarrow$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z \in \mathbb{H}$
(ie $\operatorname{Im}(z) > 0$)

Def^n) Let $k \in \mathbb{Z}$, A Modular form of weight k for $SL_2(\mathbb{Z})$ is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying.

① f is holomorphic.

② [Modularity condition] $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $\tau \in \mathbb{H}$.

③ As $\operatorname{Im}\tau \to \infty$ $(\tau \to -i\infty)$, $f(\tau)$ is bounded.

Ex) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: $f(\tau+1) = f(\tau) \ \forall \tau$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: $f(-1/\tau) = \tau^k f(\tau) \ \forall \tau$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$: $f(\tau) = (-1)^k f(\tau) \ \forall \tau$

$\Rightarrow k$ odd $\Rightarrow \underline{f \equiv 0}$ (dont have non trivial modular forms of odd weight)

$\frac{-1}{\tau}$ exchanging outside & inside (... of unit circle)

If ② [Modularity condition] holds for $f : \mathbb{H} \to \mathbb{C}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ then

② holds for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$

and holds for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$, $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^{-1}$.

Theorem) The group $SL_2(\mathbb{Z})$ is generated by

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\left(S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2\right.$ $\left(T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \ \forall n \in \mathbb{Z}\right)$

so: $\left. S^4 = I_2 \right)$ $\operatorname{ord}(S) = 4$, $\operatorname{Ord}(T) = \infty$.

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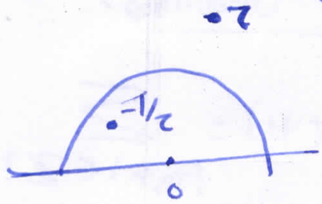
we see $S(\tau) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{-1}{\tau}$

$$\tau(\tau) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tau = \tau + 1$$



S has ord 4 as matrix.

S has ord 2 as Transformation.

To check $f : \mathbb{H} \to \mathbb{C}$ is modular form of weight k it suffices to check ①, ③

and ②' : $f(\tau+1) = f(\tau)$

$$f(-1/\tau) = \tau^k f(\tau)$$

Eisenstein Series

The only modular form of $k=2$ is zero function.

let $k \geq 4$ be even.

Define $$G_k(\tau) = \sum_{\substack{(m,n) \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k}$$

Check

- Holomorphic on $\mathbb{H}$
- satisfy $G_k(\tau+1) = G_k(\tau)$, $G_k(-1/\tau) = \tau^k G_k(\tau)$ $\forall \tau \in \mathbb{H}$
- $G_k(\tau)$ bounded as $\tau \to i\infty$

Holomorphy on $\mathbb{H}$: Converges absolutely since $k > 2$.

Converges uniformly on compact ~~sets~~ subsets of $\mathbb{H}$ $\Rightarrow$ Holomorphic.

Modularity Conditions:

$$G_k(\tau+1) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{1}{(m\tau+m+n)^k}$$

change variables $(m,n) \longleftrightarrow (m, m+n)$

$$= \sum_{\substack{(m,n) \\ \neq(0,0)}} \frac{1}{(m\tau+n)^k} = G_k(\tau)$$

using absolute convergence

$$G_k\left(\frac{-1}{\tau}\right) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{\left(m\left(-\frac{1}{\tau}\right)+n\right)^k} = \sum{}' \frac{\tau^k}{(-m+n\tau)^k}$$

$$= \tau^k \sum{}' \frac{1}{(-m+n\tau)^k} = \tau^k G_k(\tau)$$

by rearranging $(m,n) \longleftrightarrow (n,-m)$

Behavior as $\tau \to i\infty$

$$G_k(\tau) = \sum_{\substack{n\neq 0 \\ m=0}} \frac{1}{n^k} + \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}$$

since k is even $\displaystyle\sum_{n\neq 0} \frac{1}{n^k} = 2\sum_{n\geq 1} \frac{1}{n^k} = 2\zeta(k)$

$$\Rightarrow G_k(\tau) = 2\zeta(k) + 2\sum_{m\geq 1}\left(\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}\right)$$

$\hookrightarrow$ each term has τ.

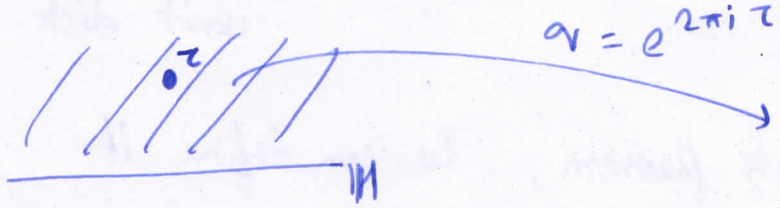
so, we expect $G_k(\tau) \xrightarrow[\tau\to i\infty]{} 2\zeta(k)$

q - expansion of a Modular form

$$f(\tau+1) = f(\tau) \quad , \quad f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)$$

ex) $e^{2\pi i(\tau+1)} = e^{2\pi i\tau}$ satisfy $f(\tau+1) = f(\tau)$ ~~is~~ is

but is not modular form



$\tau = x+iy \Rightarrow q = 2\pi i\tau$

$= e^{-2\pi y} e^{2\pi i x}$

$\Rightarrow |q| < 1$

$\Rightarrow 0 < |q| < 1$

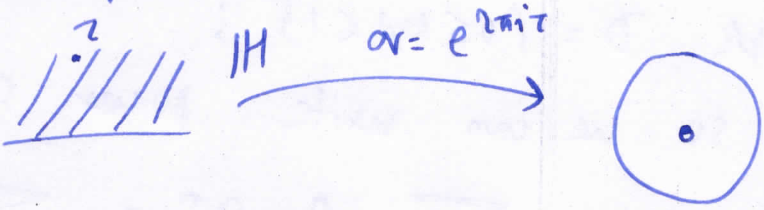
~~so~~

~~q maps $\mathbb{H}$ into unit inside unique d~~

q maps $\mathbb{H}$ inside punctured unit disk

~~q maps $\mathbb{H}$ inside unique dis~~

$q = e^{2\pi i\tau}$



$$e^{2\pi i\tau} = e^{2\pi i\tau'} \iff \tau' = \tau + n \ , \ n \in \mathbb{Z}$$

Well-defined to set $\tilde{f}(q) = f(\tau)$ where $q = e^{2\pi i\tau}$ on

$$q \in D' = \{0 < |q| < 1\}$$

So; functions on $\mathbb{H}$ can be thought of as a function of $e^{2\pi i\tau}$ (ie: function on $D' = \{0 < |q| < 1\}$

So, we can convert a modular form as a function on punctured unit disk.

as τ gets huge in $i\mathbb{R}$ direction

ie; $y \to \infty \Rightarrow |q| \to 0$

- $\tilde{f}$ is analytic function on D', D' = punctured unit disk.

By reimon Removal singularity theorem; we can define it at $q = 0$.

~~Hence $\tilde{f}(q)$ fi~~

~~Hence Modular function~~

Hence $\tilde{f}(q = 0)$ = finite

or $f(\tau \to i\infty)$ = finite.

So; Modular form can be extended to whole ~~unit~~ open unit disk $D = \{0 \leq |q| < 1\}$;

and so we can write power series around zero

$$\tilde{f}(q) = \sum_{n \geq 0} a_n q^n = \sum_{n \geq 0} a_n \cdot e^{2\pi i n \tau}$$

~~by abusive~~ by abbusive use of equation

we write $\tilde{f}(q) = f(q)$.

$$a_0 = \tilde{f}(0) = f(i\infty)$$

Lec 4) Bernoulli Numbers, Space of modular forms of $SL_2(\mathbb{Z})$ of weight k, Non holomorphic modular form, Mock Modular form, $\Gamma_0(N) \subset SL_2(\mathbb{Z})$ Modular form for $\Gamma \subset SL_2(\mathbb{Z})$ (finite index subgroup)

Modular form for $SL_2(\mathbb{Z})$ of weight k $(k \in \mathbb{Z})$

Function $f : \mathbb{H} \longrightarrow \mathbb{C}$ such that

① Holomorphic

② $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $z \in \mathbb{H}$.

③ $f(z)$ bounded as $\tau \longrightarrow i\infty$

Alternate (2') $f(z+1) = f(z)$, $f\left(-\frac{1}{z}\right) = z^k f(z)$, $\forall z \in \mathbb{H}$

(3') $f(z)$ converges as $\tau \longrightarrow i\infty$.

$f(z+1) = f(z)$ & $f(z)$ bounded as $z \to i\infty \Rightarrow f(z) = \sum_{n \geq 0} a_n e^{2\pi i n}$

$= \sum_{n \geq 0} a_n q^n$

$D = \{|q| < 1\}$

q-expansion

$a_0 = f(i\infty)$

$a_n = n^{th}$ Fourier coefficient

Ex) For even $k \geq 4$, $G_k(z) = \sum_{\substack{m,n \in \mathbb{Z} \\ \neq (0,0)}} \frac{1}{(mz+n)^k}$

$$= 2\sum_{n \geq 1} \frac{1}{n^k} + 2\sum_{m \geq 1}\left(\sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^k}\right)$$

What is its q-expansion?

$a_0 = 2\sum \frac{1}{n^k} = 2\zeta(k)$.

$$\sum_{n \in \mathbb{Z}} \frac{1}{(w+n)^k} \qquad w \in H$$

For $w \in \mathbb{H}$, and $k \geq 3$ (to get convergence)

$$\sum_{n \in \mathbb{Z}} \frac{1}{(w+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} \cdot e^{2\pi i n w}$$

How to prove this?

① Use Fourier Series

② Poisson Summation Formula

$$\sum_{\substack{n \in \mathbb{Z} \\ \uparrow \\ \text{nice}}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

fourier transform of $f : \mathbb{R} \to \mathbb{C}$ is $\hat{f} : \mathbb{R} \to \mathbb{C}$ where

$$\hat{f}(y) = \int_{-\infty}^{+\infty} f(x) \, e^{2\pi i x y} \, dx$$

here; our $f(n) = \dfrac{1}{(w+n)^k}$

Compute $\hat{f}(n)$; and find $\sum_{n \in \mathbb{Z}} f(n)$. ✓

Using this

$$G_k(\tau) = 2\zeta(k) + 2 \sum_{m \geq 1} \left(\frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} e^{2\pi i n (m\tau)} \right)$$

$$\Rightarrow G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m, n \geq 1} n^{k-1} e^{2\pi i (mn) \tau}$$

call $r = mn$

$$\Rightarrow G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{r \geq 1} \left(\sum_{\substack{d \mid r \\ d>0}} d^{k-1} \right) e^{2\pi i r \tau}$$

$$\boxed{G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) \cdot q^n}$$

where $\sigma_{k-1}(n) = \sum_{\substack{d \mid n \\ d>0}} d^{k-1}$ $\Rightarrow$ Sum of divisor of integer n to the power $k-1$; Sum of over devisors.

$\to$ q-expansion of Eisenstein Series.

Euler: for $k \geq 2$ even

$$\zeta(k) = \frac{(-1)^{\frac{k}{2}+1} \cdot (2\pi)^k \cdot B_k}{2 \cdot k!} \leftarrow (k!)(2)$$

where B_k is k^n Bernoulli number. , B_k is rational.

$$\frac{x}{e^x - 1} = \sum_{k \geq 0} B_k \cdot \frac{x^k}{k!} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \ldots$$

k	0	1	2	4	6	8	10	12	14
B_k	1	$-1/2$	$1/6$	$-1/30$	$\frac{1}{42}$	$\frac{-1}{30}$	$\frac{5}{66}$	$\frac{-691}{2730}$	$\frac{7}{6}$

Odd Bernoulli numbers i.e; B_k is zero except $k=1$

i.e; $B_k = 0$ for $k \in \{odd\} - \{1\}$.

$$\zeta(k) = \frac{-(2\pi i)^k \cdot B_k}{2 \cdot k!}$$

$$G_k(\tau) = 2\zeta(k) - \frac{4k\,\zeta(k)}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)\,q^n$$

Normalized weight k Eisenstein series : $E_k(\tau) = \dfrac{G_k(\tau)}{2\zeta(k)}$

$$\boxed{E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)\cdot q^n}$$

k	4	6
$-\frac{2k}{B_k}$	240	-504

$E_4 = 1 + 240q + 2160q^2 + \dots = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n$

$E_6 = 1 - 504q - 16632q^2 + \dots = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n$

$E_8 = 1 + 480q + 61920q^2 + \dots = 1 + 480\sum_{n\geq 1}\sigma_7(n)q^n$

We see that Modular forms give rise to infinite sequence of numbers with the ~~co~~ Fourier Coefficients.

Set M_k = all modular forms of weight k for $SL_2(\mathbb{Z})$

Theorem: $\dim(M_k) < \infty$ ie: finite dimensional.

ex) $\dim M_4 = 1$ $\dim M_{12} = 2$

$\dim M_6 = 1$ $\dim M_{14} = 1$

$\dim M_8 = 1$ $\dim M_k \geq 2$ for $k > 14$

$\dim M_{10} = 1$

$\dim M_0 = 1$ (constant)
$\dim M_2 = 0$
$\dim M_k = 0$ if $k < 0$

ex)

$E_8 \in M_8$

and $E_4^{\,2} \in M_8$

note) $f \in M_k$, $g \in M_l$

Then $fg \in M_{k+l}$

E_8 & E_4^2 are both in one dim M_8 and constant terms agree $\Rightarrow$ $E_4^2 = E_8$

E_4^3, E_6^2, E_{12} not equal in M_{12}

but ~~$E_{12} = a E_4^3 + b E_6^2$~~ $E_{12} = a E_4^3 + b E_6^2$

for some a & b

(can write as linear combination; because we get a basis)

$$\dim(M_k) = \begin{cases} \left[\frac{k}{12}\right] & \text{if } k \equiv 2 \mod 12 \\ \left[\frac{k}{12}\right] + 1 & \text{if } k \not\equiv 2 \mod 12 \end{cases}$$

for even $k > 0$

$[\cdot]$ is greatest integer function.

Defn) $E_2(\tau) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$

Fact) $E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) - \frac{6i}{\pi}\tau$

$\rightarrow$ This is failure for $E_2(\tau)$ to be modular form.

$E_2(\tau)$ is
- holomorphic on $\mathbb{H}$
- $E_2(\tau+1) = E_2(\tau)$
- $E_2(\tau) \to 1$ as $\tau \to i\infty$

Similar result : for $\tau \in \mathbb{H}$

$$\frac{1}{\mathrm{Im}\left(-\frac{1}{\tau}\right)} = \tau^2 \cdot \frac{1}{\mathrm{Im}(\tau)} - 2i\tau$$

just need to turn 2 to $6/\pi$

$\Rightarrow$ so; $\dfrac{3}{\pi \, \mathrm{Im}\left(-\frac{1}{\tau}\right)}$ has same rule as $E_2(\tau)$

$$\boxed{E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi \, \mathrm{Im}(\tau)}}$$

satisfy $E_2^*(\tau+1) = E_2^*(\tau)$

& $E_2^*\left(-\frac{1}{\tau}\right) = \tau^2 E(\tau)$

} satisfies Modularity condition for weight 2.

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi \cdot \mathrm{Im}(\tau)}$$

but $E_2^*(\tau)$ is not holomorphic,

Hence it is not Modular form of weight 2.

it could be called non-holomorphic modular form of weight 2.

$E_2(\tau)$ is called Mock Modular form of weight 2, or Holomorphic part of a non-holomorphic Modular form of weight 2.

Just as we make non-holomorphic correction to kill of extra term which was destroying Modularity condition.

We can make holomorphic correction also.

~~Also $F(\tau) = 2E_2(2\tau)$~~

Turns out that $2E_2(2\tau) - E_2(\tau)$ satisfies modularity condition in weight 2 for all

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ when c is even, ie; its in $\Gamma_0(2)$

(does not satisfy when c is odd)

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}$$

ie; The lower left entry is divisible by N.

$$\Gamma_0(2) = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

There are three generators for $\Gamma_0(2) \underset{\text{subgroup}}{\subseteq} SL_2(\mathbb{Z})$

Modular form for $\Gamma \underset{\text{finite index}}{\subset} SL_2(\mathbb{Z})$

is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying

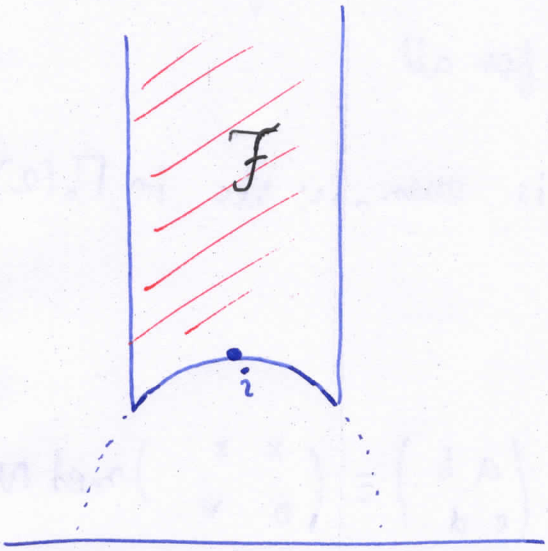
① f is holomorphic.

② $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

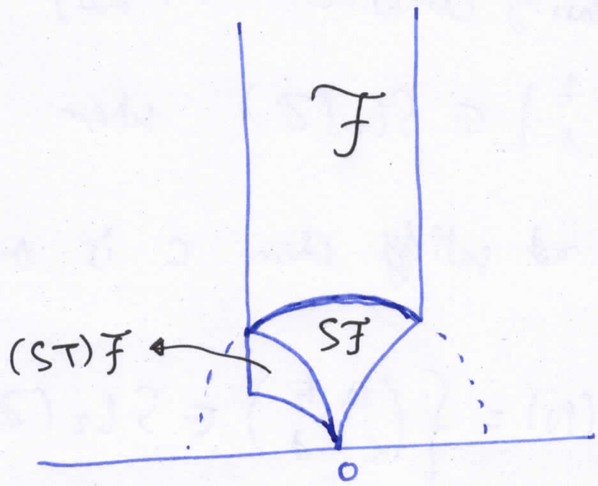
③ $\frac{1}{(c\tau+d)^k} f\left(\frac{a\tau+b}{c\tau+d}\right)$ is bounded as $\tau \to i\infty$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

This condition is called Holomorphic at cusps.



Fundamental Domain of $SL_2(\mathbb{Z})$



Fundamental domain of $\Gamma_0(2)$

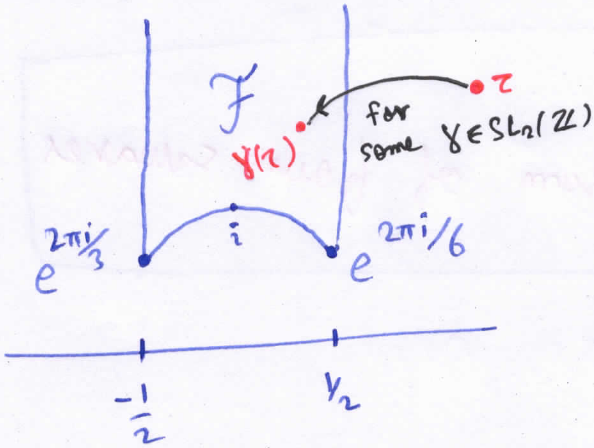
$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

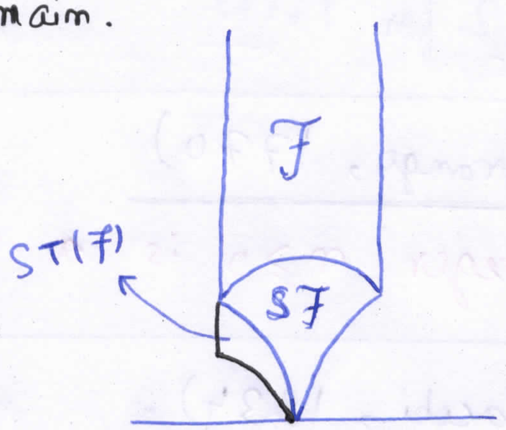
$\infty \xrightarrow[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}]{} 0$

$\Leftrightarrow$ we get a cusp.

Lec 5 | Fundamental Domain, Lagrange Theorem, Jacobi Theorem, $\dim(M_k)$, $\dim(M_k) = 0$ for $k<0$.

Meaning of Fundamental domain.





$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

for $SL_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$

$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod 2 \right\}$

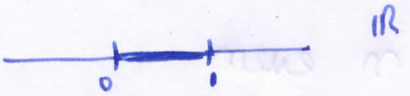
$= \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$

$ST(\mathcal{F})$

ie; Translate, then flip

$\Gamma(2)$ is subgroup of $SL_2(\mathbb{Z})$ of index 3 (have three sets there...)

ie: $\Gamma(2) \underset{3}{\subsetneq} SL_2(\mathbb{Z})$



for $\mathbb{Z}$

for $3\mathbb{Z}$

Fundamental domain of $\Gamma(2)$ is $\mathcal{F} \sqcup ST(\mathcal{F}) \sqcup S(\mathcal{F})$

(we like to have connected fundamental ~~gr~~ domain)

$$E_2(\tau) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) e^{2\pi i n \tau} \qquad (\tau \in \mathbb{H})$$

$$E_2\left(\frac{-1}{\tau}\right) = \tau^2 E_2(\tau) - \frac{6i}{\pi}\tau$$

$2E_2(2\tau) - E_2(\tau)$ satifies weight 2 modularity condition for $\Gamma_0(2) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod 2 \right\}$

$2E_2(2\tau) - E_2(\tau)$ is an example of modular form of weight 2 for $\Gamma_0(2)$

Theorem (Lagrange, 1770)

Every integer $n \geq 0$ is a sum of four squares

Theorem (Jacobi, 1834)

For $n \geq 0$,

$$r_4(n) = \#\{(a,b,c,d) \in \mathbb{Z} \mid a^2+b^2+c^2+d^2 = n\}$$

← no. of ways to write n as sum of four squares.

The formula is

$$r_4(n) = \begin{cases} 8\sigma_1(n) & \text{if } n \text{ odd} \\ 24\sigma_1(n_{odd}) & \text{if } n \text{ even} \end{cases}$$

where $n_{odd} = Od(n)$

ie; $Od(n) = \dfrac{n}{2^{b(n)}}$ where $b(n)$ is exponent of the exact power of 2 dividing n.

Jacobi Theta function $\theta(\tau) = \sum_{m \in \mathbb{Z}} e^{2\pi i m^2 \tau}$

$$= \sum_{m \in \mathbb{Z}} q^{m^2} \quad , \tau \in \mathbb{H}$$

$$\theta(\tau) = \sum_{m \in \mathbb{Z}} e^{2\pi i m^2 \tau} \qquad \tau \in \mathbb{H}$$

$$= \sum_{m \in \mathbb{Z}} q^{m^2} = 1 + 2q + 2q^4 + 2q^9 + \dots$$

$$\theta(\tau)^4 = \theta(\tau)\,\theta(\tau)\,\theta(\tau)\,\theta(\tau)$$

$$\boxed{\theta(\tau)^4 = \sum_{n \geq 0} r_4(n)\, q^n}$$

So, $\theta(\tau)^4$ is generating function for $r_4(n)$

Fact: Poisson summation will help in showing.

$$\theta(\tau)^4 \in M_2(\Gamma_0(4))$$

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod 4 \right\}$$

$$\Gamma_0(4) \subset \Gamma_0(2)$$

Fact 1 $f_1 = 2E_2(2\tau) - E_2(\tau) \in M_2(\Gamma_0(2)) \subset M_2(\Gamma_0(4))$

and $f_2 = 2E_2(4\tau) - E_2(2\tau) \in M_2(\Gamma_0(4))$

q - expansion

$$f_1 = 1 + 24 \sum_{n \geq 1} \sigma_1(n_{odd})\, q^n = 1 + 24q + \dots$$

$$f_2 = 1 + 24 \sum_{n \geq 1} \sigma_1(n_{odd})\, q^{2n} = 1 + 0 + 24q^2 + \dots$$

$$\theta(\tau)^4 = \sum r_4(m)\, q^m$$
$$= 1 + 8q + \cdots$$

using f_1 & f_2 as a basis for $M_2(\Gamma_0(4))$

Then $\theta(\tau)^4 = a\cdot f_1 + b\cdot f_2$ for some a, b

(This becomes an algebraic problem)

$a + b = 1$, $a = \frac{1}{3}$

~~b~~ $\Rightarrow$ $b = 2/3$

$$\Rightarrow \boxed{\theta(\tau)^4 = \frac{1}{3} f_1(\tau) + \frac{2}{3} f_2(\tau)}$$

using this derive Jacobi' formula.

Recall for $M_k = \{$mod. forms of weight k for $SL_2(\mathbb{Z})\}$

(its complex vector space)

We want to show for even k

$$\dim(M_k) = \begin{cases} 0 & k < 0 \\ 1 & k = 0, 4, 6, 8, 10 \\ 0 & k = 2 \\ \dim(M_{k-12}) + 1 & k \geq 12 \end{cases}$$

$$= \begin{cases} \left[\frac{k}{12}\right] & \text{if } k \equiv 2 \bmod 12 \\ \left[\frac{k}{12}\right] + 1 & \text{if } k \not\equiv 2 \bmod 12 \end{cases}$$

$\dim M_k = 0$ if $k < 0$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

let $f \in M_k$ $\Rightarrow f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$

$$\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{\mathrm{Im}\,\tau}{|c\tau+d|^2}$$

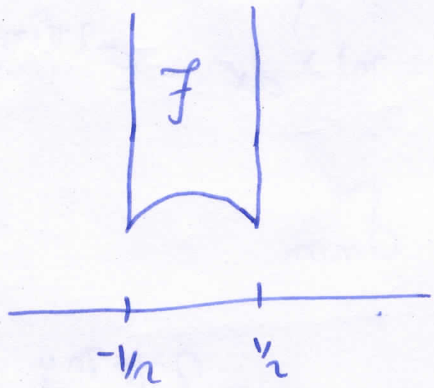
$$\Rightarrow \left|\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right)\right|^{k/2} = \frac{(\mathrm{Im}\,\tau)^{k/2}}{|c\tau+d|^k}$$

$$\left|f\left(\frac{a\tau+b}{c\tau+d}\right)\right| = |c\tau+d|^k\,|f(\tau)|$$

$$\Rightarrow \boxed{\left|f\left(\frac{a\tau+b}{c\tau+d}\right)\right| \cdot \left(\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right)\right)^{k/2} = |f(\tau)|\left(\mathrm{Im}(\tau)\right)^{k/2}}$$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Its an $SL_2(\mathbb{Z})$ invariant



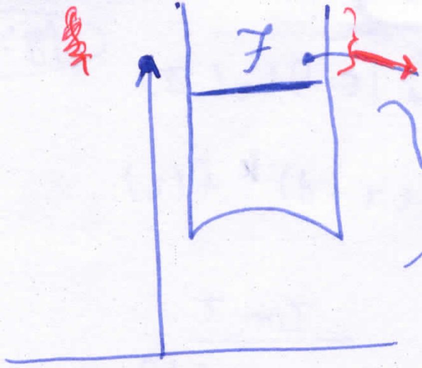
Values of $|f(\tau)|\,(\mathrm{Im}\,\tau)^{k/2}$ arise on $\mathcal{F}$

($|f(\tau)|\,(\mathrm{Im}\,\tau)^{k/2}$ is $SL_2(\mathbb{Z})$ invariant)

$|f(\tau)|\,(\mathrm{Im}\,\tau)^{k/2}$ continuous on H $\quad$ for $k < 0$

As $\mathrm{Im}\,\tau \to \infty$, $|f(\tau)|$ bounded & $(\mathrm{Im}\,\tau)^{k/2} \to 0$
(because modular form... $f(\tau)$) $\quad$ since $k < 0$

$$\Rightarrow |f(\tau)|\,(\mathrm{Im}\,\tau)^{k/2} \to 0$$



size ≤ 1

This is a compact set. (pg 50)

get $|f(z)|(\mathrm{Im}\, z)^{k/2} \le C$ on $\mathcal{F}$ for some $c > 0$

$\Rightarrow |f(x+iy)|\, y^{k/2} \le c$ for all $x+iy \in \mathbb{H}$

ie; $|f(x+iy)|\, y^{k/2} \le c$ for all $x + iy \in \mathbb{H}$

$\Rightarrow |f(x+iy)| < \dfrac{c}{y^{k+2}}$

Set $f = \sum\limits_{n \ge 0} a_n q^n = \sum\limits_{n \ge 0} a_n e^{2\pi i n z}$

Consider for $m \ge 0$, fix $y > 0$.

$$\int_0^1 f(x+iy)\, e^{-2\pi i m x}\, dx = \int_0^1 \left(\sum a_n e^{2\pi i n x}\right) e^{-2\pi i m x}\, dx$$

$$= \sum_n a_n \int_0^1 e^{2\pi i (n-m) x}\, dx \cdot e^{-2\pi i m y}$$

$$= \sum_n a_n \cdot e^{-2\pi n y} \cdot \delta_{nm}$$

$$\Rightarrow \int_0^1 f(x+iy)\, e^{-2\pi i m x} = a_m \cdot e^{-2\pi m y}$$

$$\boxed{a_m \cdot e^{-2\pi \cdot m \cdot y} = \int_0^1 f(x+iy) \cdot e^{-2\pi i m x}\, dx}$$

$$\Rightarrow \quad |a_m| e^{-2\pi m y} \leq \int_0^1 |f(x+iy)|\, dx$$
$$\leq \int_0^1 \frac{c}{y^{k/2}}\, dx$$
$$= \frac{c}{y^{k/2}}$$

$$\boxed{\forall m, \quad |a_m| \leq \frac{C \cdot e^{2\pi m y}}{y^{k/2}}}$$

we got a bound for m^{th} fourier coefficient.

let $y \to 0^+$ given $k<0$ $\Rightarrow$ $|a_m| = 0$ $\Rightarrow$ $\boxed{f \equiv 0}$

" There is modular form $\Delta(\tau) \in M_{12}$

s.t. $\Delta(\tau) \neq 0$ in $\mathbb{H}$

and $\Delta(\tau) = q + \ldots$ (no constant term)

So what?

$$f \in M_4 \Rightarrow \frac{f - \overbrace{a_0 E_4}^{\text{subtracting of the constant term}}}{\Delta} \in M_{-8}$$

This a_0 is actually constant part of f i.e.;

$f = a_0 + ?q + ??q^2 + \ldots$

neumerato satifies transformation low of M_4

denominator " " " " M_{12}

Hence $\frac{f - a_0 E_4}{\Delta}$ " " " " M_{-8}

but $M_{-8} = \{0\}$

"

$$\Rightarrow f = a_0 E_4 \Rightarrow \boxed{M_4 = \mathbb{C} E_4}$$

Lec 6 | Direct sum decomposition of M_k, $\frac{\zeta(k)}{\pi^k} \in \mathbb{Q}$, $k \geq 8$ even, Ramanujan Conjectures, Hecke Operators, L function.

Claim | There exists $\Delta(\tau) \in M_{12}$ such that $\Delta(\tau) \neq 0$ on $\mathbb{H}$ and $\Delta(q) = q + \dots$ $(a_0 = 0)$

After proving $M_k = \{0\}$ for $k < 0$, we can prove $\boxed{M_4 = \mathbb{C} E_4}$ using the claim.

More generally, for $f \in M_k$, then let $a_0 = f(i\infty)$,

has at least first order zero at ∞

so $g \equiv \dfrac{f - a_0 E_k}{\Delta} \in M_{k-12}$

has simple zero at ∞

Then $f = a_0 E_k + \Delta \cdot g$ } ~~This dec~~ This decomposition is unique.

$f \in M_k$, $a_0 \in \mathbb{C}$, $g \in M_{k-12}$

So, we have Direct Sum Decomposition.

$$\boxed{M_k = \mathbb{C} E_k \oplus \Delta \cdot M_{k-12}} \quad \text{for } k \geq 4$$

~~for $k=0$, $M_0 = \mathbb{C} \oplus \cancel{M_{-12}} \cdot \Delta$~~

for $k = 0$, $M_0 = \mathbb{C} \oplus \Delta \cdot \cancel{M_{-12}} = \mathbb{C}$

Recursive Formula.

$$\Rightarrow \boxed{\dim M_k = 1 + \dim M_{k-12} \quad \text{for } k \geq 4}$$

We can get a general formula for $\dim M_k$ (including $k=2$)

Do not need E_k for $k \geq 8$ in this proof.

Just need in each M_k some f s.t. $f(i\infty) = 1$

Ex] $k = 42 \stackrel{?}{=} 4a + 6b$ $(a, b \geq 0)$

$= 4 \cdot 9 + 6 \cdot 1$

~~take~~ $\cancel{E_4^9 \cdot E_6}$

We can also use $E_4^9 \cdot E_6 \in M_{42}$ in case of $k = 42$

in place of E_{42}.

The $\#\{(a,b) : a \geq 0, b \geq 0, 4a + 6b = k\}$

$= \dim(M_k)$

$\Rightarrow$ M_k has basis $\{E_4^a E_6^b : a, b \geq 0, 4a + 6b = k\}$

$\hookrightarrow$ They are linearly independent.

Corollary] (E_4, E_6 have fourier coefficients in $\mathbb{Q}$)

If $f \in M_k$ $(k > 0)$ and $f = \sum_{n \geq 0} a_n q^n$

has $a_n \in \mathbb{Q}$ for $n \geq 1$,

then $a_0 \in \mathbb{Q}$.

Corollary For even $k \geq 8$, $\frac{\zeta(k)}{\pi^k} \in \mathbb{Q}$

(we talk about $k \geq 8$)

Proof $G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n \in M_k$

Einstein series $k \geq 4$

$$\Rightarrow \frac{G_k(\tau)}{2(2\pi i)^k/(k-1)!} = \frac{\zeta(k)}{(2\pi i)^k/(k-1)!} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n \in M_k$$

since $\sigma_{k-1}(n) \in \mathbb{Z} \subset \mathbb{Q}$ for $n \geq 1$, we get

$$\frac{\zeta(k)}{(2\pi i)^k/(k-1)!} \in \mathbb{Q}$$

$$\Rightarrow \boxed{\frac{\zeta(k)}{\pi^k} \in \mathbb{Q} \quad \text{for even } k \geq 8}$$

We can't prove this for $k = 4, 6$ because we need E_4, E_6 has rational coefficient to get the corrolary which was based on $\zeta(6)$ & $\zeta(4)$

~~we know value of this~~

↳ can use this methods for ~~oth~~ ζ function of other fields. ....

$$E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n$$

$$\Rightarrow \frac{1}{240} E_4 = \frac{1}{240} + \sum_{n \geq 1} \sigma_3(n) q^n$$

How to build $\Delta(\tau) \in M_{12}$ with $\Delta(\tau) \neq 0$ on $\mathbb{H}$

& $\Delta(q) = q + \ldots$

Define $\theta(\tau) = \sum_{\text{odd } n \geq 1} (-1)^{\frac{n-1}{2}} \cdot n \cdot e^{\pi i n^2 \tau/4}$

Then $\theta(\tau+1) = e^{2\pi i/8}\, \theta(\tau) \Rightarrow \theta(\tau+1)^8 = \theta(\tau)^8$

Twisted Poisson Summation $\underset{\text{can use it to show}}{\Longrightarrow}$ $\theta\left(-\frac{1}{\tau}\right)^8 = \tau^{12}\, \theta(\tau)^8$

$\theta(\tau) = e^{\pi i \tau/4} - 3e^{\pi i 9\tau/4} + 5e^{\pi i 25\tau/4} + \ldots$

$\rightarrow 0$ as $\tau \longrightarrow i\infty$.

Conclusion $\theta(\tau)^8 \in M_{12}$

$\boxed{\text{Define For } \tau \in \mathbb{H}, \quad \Delta(\tau) = \theta(\tau)^8.}$

$\Delta(i\infty) = 0$

$\Delta(\tau) = \left(e^{\pi i \tau/4} - 3e^{\pi i 9\tau/4} + \ldots\right)^8$

$= e^{2\pi i \tau} + \ldots\ldots$

$= q + \ldots$

$\Delta(\tau) \neq 0$ on $\mathbb{H}$

Then show $\theta(\tau) \neq 0$ on $\mathbb{H}$

If a modular form vanishes somewhere, then due to modularity condition; it has to vanish at every point in the $SL_2(\mathbb{Z})$ orbit of the ~~first vanishing~~ point where it vanished. 1956

So, it suffices to show $\theta(\tau) \neq 0$ on fundamental domain.

~~proved~~ we can prove it using Contradiction !

There are other ways of proving non-vanishing using Algebraic Geometry.

note $\theta(\tau)$ has integer coefficients

$\Rightarrow$ $\Delta(\tau)$ also has integer coefficients

$$\cancel{\Delta(\tau) = \theta(\tau)^8 = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6045q^6 \ldots}$$

$$\Delta(\tau) = \theta(\tau)^8 = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \ldots$$

lets give names to coefficients.

$$\Rightarrow \Delta(\tau) = \sum_{n \geq 1} \tau(n)\, q^n$$

Remark $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ has discriminant $\Delta(\tau)$

(Elliptic curves has numerical invariant quantity $\Delta(\tau)$)

Ramanujan observed / Conjectured.

he observed value of $\tau(n)$ for $n \le 30$.

① first conjectured • $\tau(mn) = \tau(m)\tau(n)$ if $\gcd(m,n)=1$

~~example~~ ex) $(-24)(252) = -6048$

② second conjecture • for prime p & $r \ge 1$,

$$\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$$

③ Third Conjecture • $|\tau(p)| \le 2p^{11/2} = 2p^{5.5}$

$$\boxed{\Rightarrow |\tau(n)| \le d(n)\, n^{5.5} \text{ for all } n \ge 0}$$

$d(n) \Rightarrow$ no. of divisors of n

First two settled by Mordell & Hecke seperately
(quickly) $\hookrightarrow$ Hecke operator

Third conj. settled in 1970s by Deligne by Weil Conj.

Hecke showed $|\tau(p)| \le 2p^6$

Hecke's proof used Hecke operators $T_n : M_k \to M_k$
satisfying

- $T_m T_n = T_n T_m$
- $T_{mn} = T_m T_n$ if $\gcd(m,n)=1$
- $T_{p^{r+1}} = T_p T_{p^r} - p^{k-1} T_{p^{r-1}}$ for p prime & $r \ge 1$

Hecke operators satisfy $\qquad T_m : M_k \to M_k$

① $\quad T_m T_n = T_n T_m$

② $\quad T_{mn} = T_m T_n \quad$ if $\gcd(m,n) = 1$

③ $\quad T_{p^{r+1}} = T_p T_{p^r} - p^{k-1} T_{p^{r-1}}$ for prime p & $r \geq 1$

Note ; like These operators satisfies recursion relation much like what Ramanujan conjectured for the coefficient of Δ.

<u>Def^n</u> For prime p, let $T_p : M_k \to M_k$ by

$$(T_p f)(\tau) = p^{k-1} f(p\tau) + \frac{1}{p} \sum_{b=0}^{p-1} f\left(\frac{\tau + b}{p}\right)$$

↳ Holomorphic obviously
bounded obviously.

as $\tau \to \tau+1 \quad f(p\tau)$ is unchanged.

$f\left(\frac{\tau+b}{p}\right)$ cycling commute around as $\tau \to \tau+1$

Hence $\quad (T_p f)(\tau+1) = (T_p f)(\tau)$

$$(T_p f)\left(-\frac{1}{\tau}\right) = \tau^k (T_p f)(\tau)$$

proof <u>note</u> $\quad b=0$ ~~gh~~ gives ~~$\tau^k(f(p\tau))$~~ $\sim p^{k-1} f(p\tau)$
& rest get permutted among themselves..

$f = \sum a_n q^n$

$\Rightarrow \quad T_p f(\tau) = \sum_{n \geq 0} a_{pn} q^n + \sum_{n \geq 0} p^{k-1} a_n q^{pn}$

$$\boxed{T_p f(\tau) = \sum_{n \geq 0} a_{pn} \cdot q^n + \sum_{n \geq 0} p^{k-1} \cdot a_n \cdot q^{pn}}$$

$$M_{12} = \mathbb{C} E_{12} \oplus \underbrace{\mathbb{C}\Delta}_{f(i\infty)=0}$$

$T_p \Delta = \lambda_p \cdot \Delta$ for some $\lambda_p \in \mathbb{C}$

look at coefficient of q $\qquad \boxed{T_p \Delta = \tau(p)\Delta}$

$\tau(p) = \lambda_p$

note $\tau(1) = 1$

lets show $\tau(p^2) = \tau(p)\tau(p) - p^{11}\tau(1)$

look at coefficient of q^p in $T_p f = \tau(p) f$:

$\tau(p^2) + p^{11} = \tau(p)^2$ ✓

$$L(s, \Delta) = \sum_{n \geq 1} \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1 - \frac{\tau(p)}{p^s} + \frac{p^{11}}{p^{2s}}}$$

L function of Δ modular form

for $Re(s) > 6.5$

After ~~An~~ Analytic continuation, $s \longleftrightarrow 12 - s$

It satisfies a version of Reimann Hypothesis that all the non trivial zeroes line on the line $Re(z) = 6$

Lec 7 Dedikind η function, Eulers Pentagonal Number Theorem, Hardy-Ramanujan, Radmacher, Circle method, Farey Sequence, Ford Circle, Jacobi triple product identity, Fermions, Rational CFT, Automorphy Factor

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Lec 7 $\eta(\tau)$, growth of coefficients, more on θ-functions, Connections to simple CFTs.

Lec 8 Jacobi forms, $N=2$ Super Conformal Algebra, Elliptic genera, BH counting states.

Lec 9 Mock-Model forms, BH counting & umbral Moonshine.

Lec 10 Generalizations of Modular forms:

- Weakly holo. forms

  ie: $J(\tau) = q^{-1} + \dots$ finite # terms with negative powers of q

- Modular forms with multiplied phase
- Vector-valued mod forms

$\Big\}\ \eta(\tau), \theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}$

Dedikind η function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \qquad q = e^{2\pi i \tau}, \tau \in \mathbb{H}.$$

$\frac{1}{\eta(\tau)}$ appears in 2 important places in math / string theory.

$p(n)$ for a positive integer $=$ # ways of writing n as a distinct sum of smaller integers, ignoring order eg

n	$p(n)$
$1 = 1$	1
$2 = 2,\ 1+1$	2
$3 = 3,\ 2+1,\ 1+1+1$	3
$4 = 4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1$	5

A Generating function for $P(n)$

Claim: $$\frac{1}{\prod_{n=1}^{\infty}(1-q^n)} = \sum p(n)\, q^n$$

Write n as $\underbrace{1+1+\ldots+1}_{n}$

$$1 + q + q^2 + q^3 + q^4 + \ldots = \frac{1}{1-q}$$

no. of ways of $n = 1+1+\cdot+1$

n is even $\quad n = 2+2+2+2\ldots$

$$1 + q^2 + q^3 + q^6 + \ldots\ldots = \frac{1}{1-q^2}$$

$$(1+q+q^2+q^3+\ldots)(1+q^2+q^4+\ldots)\Big|_{q^4}$$ picking coeff. of q^4

$$= \underset{1+1+1+1}{q^4} + \underset{2+1+1}{q^2\cdot q^2} + \underset{4}{q^4} = 3q^4$$

....

It is nicer to study

$$\frac{1}{q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)} = q^{-1/24} \sum p(n) q^n$$

Because of its modular properties ;

We can use modular properties to study $p(n)$

as $n \longrightarrow \infty$.

In D spacetime dimensions, in bosonic string theory.

$$\mathrm{Tr}_{\mathrm{Fock}}\left(q^{L_0 - \frac{(D-2)}{24}}\right) = (\eta(\tau))^{-\frac{(D-2)}{24}}$$

Trace over the Fock Space

Easy : $\eta(\tau+1) = e^{2\pi i/24} \eta(\tau)$

Hard : $\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\, \eta(\tau)$

Hardest : $\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(a,b,c,d)\, \eta(\tau) \sqrt{c\tau+d}$

complicated 24^{th} root of 1.

Eulers pentagonal number Theorem

$$\prod_{n=1}^{\infty} (1-q^n) = \sum_{m=-\infty}^{+\infty} (-1)^m q^{(3m^2-m)/2}$$

Exercise Note $\frac{1}{24} + \frac{3m^2 - m}{2} = \frac{1}{24}(6m-1)^2$

and using Poisson summation; show transformation law of $\eta(-\frac{1}{\tau})$ is as claimed.

Hardy - Ramanujan.

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \quad \text{as } n \to \infty$$

and $$p(n) = \sum_{k < \alpha\sqrt{n}} P_k(n) + O(n^{-1/4})$$

Some constant which Hardy & Ramanujan determined.

but $\sum_{\text{all } k} P_k(n)$ diverges

So; They found divergent series expression for $p(n)$.

but if we compute first K terms of the series; then we could bound the ~~e~~ error.

Radmacher 1958

Exact formula

$$p(n) = \frac{2\pi}{(24n-1)^{3/4}} \sum_{k=1}^{\infty} \frac{B_k(n)}{k} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}}\sqrt{n - \frac{1}{24}}\right)$$

$I_{3/2}(x)$ = modified Bessel function. $I_\alpha(x) = i^\alpha J_\alpha(ix)$

$B_k(n)$ = Number theoretic sum.

These kinds of expansions for coefficients of ~~weakly m~~ weakly holomorphic modular forms - with exponential go under the name,

Rademacher or Poincare series

Farey tale .... AdS / CFT.

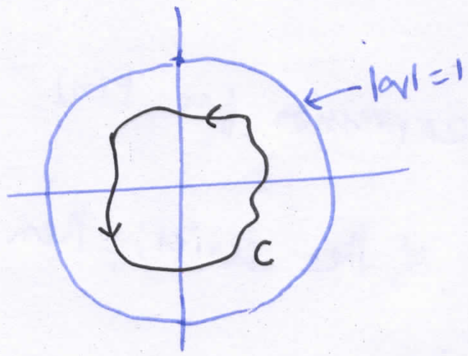
$$F(q) = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)} = \sum p(n)\, q^n$$

$$\Rightarrow \quad \frac{F(q)}{q^{n+1}} = \cdots + \frac{p(n-1)}{q^2} + \frac{p(n)}{q} + p(n+1) + \cdots$$

has first order pole as function of q

$$p(n) = \frac{1}{2\pi i} \oint_C \frac{F(q)}{q^{n+1}}$$

where c is contour inside the unit circle.



<u>Note</u> $F(q)$ has singularities at $q = 1$, $q^2 = 1$, $q^3 = 1$, ...

i.e. any N^{th} root of unity leaves $F(q)$ singular. because denominator is zero.

The general idea of the method called <u>The Circle Method</u> which is sometime associated with Hardy - Littlewood.

" " " Hardy - Ramanujan

" " " Hardy - Ramanujan - Littlewood

is to take the contour C & deform it in some way that allows us to pick out the leading singular behavior (or all of singular behavior) and allows us to estimate or evaluate the integrate.

Any $\tau = \frac{a}{c}$; $a, c \in \mathbb{Z}$

$$q = e^{2\pi i (a/c)} \quad , \quad q^c = 1$$

$\tau \to \dfrac{a\tau + b}{c\tau + d}$ Then "$i\infty$" $\longrightarrow \dfrac{a}{c}$

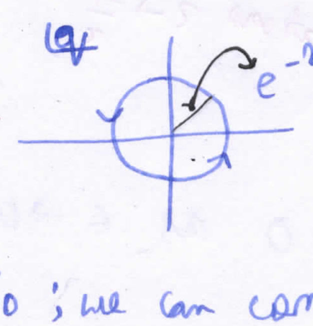
So; Modular transformations maps the point $i\infty$ to all the Rational points on the Real line ;

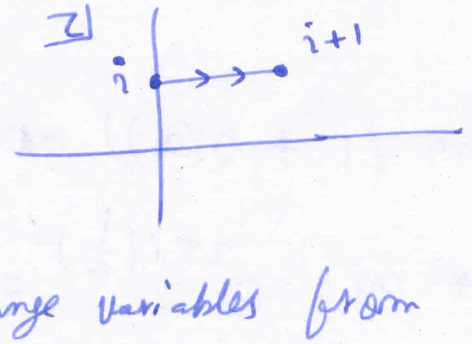
and at each one of those rational points, there is some term in the product which goes to zero in $\prod_{n=1}^{\infty}(1-q^n)$ & leads to singularity.

So; $F(q)$ has singularity at every rational point on the real axis.

Suppose C is a circle of radius $e^{-2\pi} = q_0$

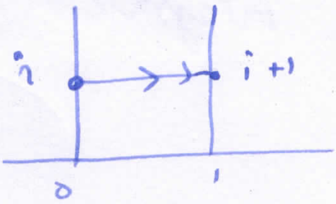
Then $\oint_C$ is τ from i to $i+1$ $\quad q = e^{2\pi i \tau}$



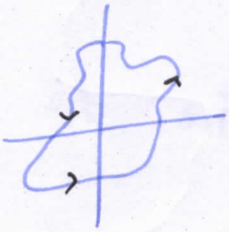


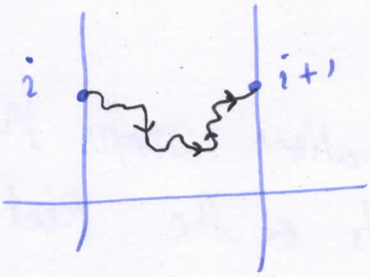
So ; we can consistently change variables from q to τ.

$$P(n) = \int_{i}^{i+1} F(e^{2\pi i \tau})\, e^{-2\pi i n \tau}\, d\tau$$



As long as we dont encounter singularities ; we are free to deform contour in any way we want.





1st ; asymptotics :

write $\tau = i\epsilon$, consider $\epsilon \to 0^+$

$$q = e^{-2\pi\epsilon} \xrightarrow[\epsilon \to 0^+]{} 1$$

$$\frac{F(q)}{q^{n+1}} = \frac{q^{\frac{1}{24} - n - 1}}{\eta(q)}$$

as $\epsilon \to 0$; $\eta(i\epsilon) = \frac{1}{\sqrt{\epsilon}}\, \eta\left(-\frac{1}{i\epsilon}\right) = \frac{1}{\sqrt{\epsilon}}\, \eta\left(\frac{i}{\epsilon}\right)$

from modular transformations $\tau \to \frac{-1}{\tau}$

$\eta(q) = q^{1/24}.(1 + O(q))$ as $q \to 0$

note ; $\frac{i}{\epsilon} \to q = e^{2\pi i \left(\frac{i}{\epsilon}\right)} = e^{-2\pi/\epsilon} \longrightarrow 0$ as $\epsilon \to 0$

So; $\eta(i\varepsilon) \sim \dfrac{e^{-\frac{2\pi}{24\epsilon}}}{\sqrt{\epsilon}}$

$$\frac{F(q)}{q^{n+1}} \sim \exp\underbrace{\left[\frac{2\pi}{24\epsilon} + 2\pi\epsilon n + \frac{1}{2}\log\epsilon\right]}_{}$$

use saddle point approximation as $n \to \infty$, $\epsilon \to 0^+$

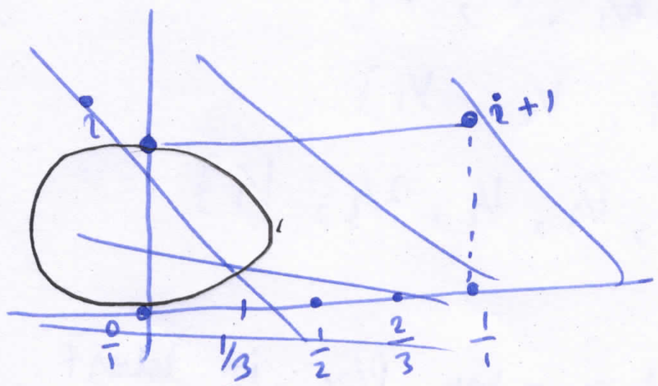
$$-\frac{2\pi}{24} + 2\pi n\epsilon^2 + \frac{\epsilon}{2} = 0$$

$$\epsilon \sim \frac{1}{\sqrt{24n}}$$

Which gives $\dfrac{F(q)}{q^{n+1}} \simeq (\quad)\exp\left[4\pi\sqrt{\dfrac{n}{24}}\right]$

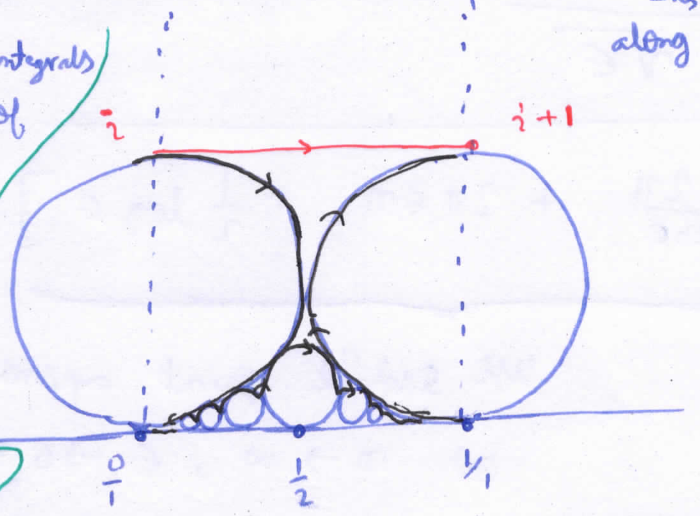
agrees with Hardy-Ramanujan

can compute this pre factor by carefully doing saddle point approximation



Rademacher (computing exact formula)

Instead of integrating along red curve; do along black curve



Ford Circles

The sum of integrals along the top of ford circles gives us the series whose sum up to infinite sum obtained by ∞ ford circle gives bounded exact formula for $p(n)$.

Sequence of fractions with denominators ~~...~~ that are bounded by some integer n. are called integers in a Farey Series

Farey Sequence The Farey sequence of order n, denoted by F_n is the sequence of completely reduced fractions between 0 and 1 which, in lowest terms, have denominators less than or equal to n, arranged in order of increasing size.

$$F_1 = \{0/1, 1, 1/1\}$$

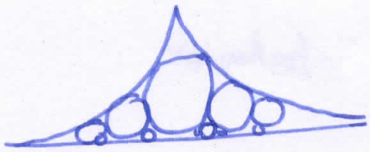
$$F_2 = \{0/1, 1/2, 1/1\}$$

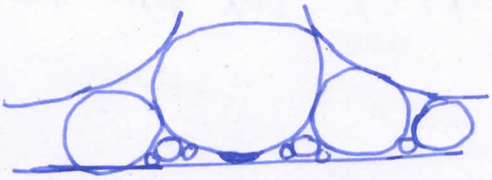
$$F_3 = \{0/1, 1/3, 1/2, 2/3, 1/1\}$$

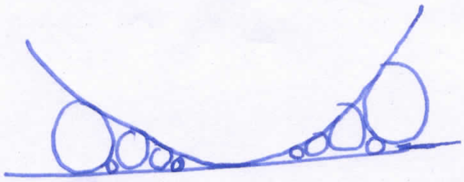
Ford Circles For every rational number p/q in lowest terms, The Ford Circle $C(p,q)$ is the circle with centre $\left(\frac{p}{q}, \frac{1}{2q^2}\right)$ and radius $\frac{1}{2q^2}$.

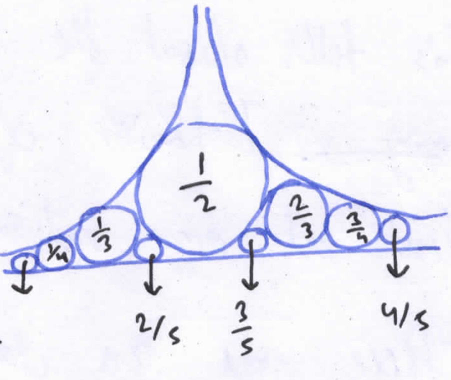
This means $C(p,q)$ is the circle tangent to the x-axis at $x = \frac{p}{q}$ with radius $\frac{1}{2q^2}$.

Every small interval of the x-axis contains points of tangency of infinitely many Ford circles.









Reference) book by "Apostal"
title "Modular functions & Dirichlet series in number theory".

What is Idea about using or integrating along Ford Circles !

The contribution is largest at rational values of τ. So, as we include more & more of the ford circles, we end up integrating along contours that are closer & closer to rational points on the axis.

for each one of the arc in the new path where we are integrating

we can use different modular transformations.

$\eta(-\frac{1}{\tau}) = (\ )\,\eta(\tau)$ use this to focus on leading behavior.

Use $\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \sqrt{c\tau+d}\;\varepsilon(a,b,c,d)\,\eta(\tau)$

$\tau \to i\infty$; $\eta\left(\frac{a}{c} + i\varepsilon\right) =$

So; This tells about the asymptotic ~~function of η~~ not ~~at any x~~ behavior of η function not only at 0 ; but at any rational number.

And these arcs go closer & closer to rational numbers.

Then we have to do some bounds & integrals....

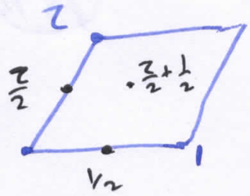
θ_1, θ_2, θ_3

Fermions NS, R ?

$\mathrm{Tr}_{\text{Fock}}\ g^{L_0} = \sqrt{\frac{\theta}{2}}$

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i \tau n^2 + 2\pi i z\cdot n}$$

$$= \sum_{n\in\mathbb{Z}} q^{n^2/2}\cdot y^n \qquad q = 2\pi i\tau, \quad y = e^{2\pi i z}$$



$$E_\tau = \mathbb{C}/L_\tau$$

$$\theta\,(z+\lambda+\mu\tau\,;\,\tau) = e^{(\;)}\;\theta(z,\tau)$$

It turns out that we can get some variance which are interesting & appear in free fermion theory by

Shifting by $1/2$, $\tau/2$ or $\frac{1}{2}+\frac{\tau}{2}$

These are called "Torsion points of order 2"

↪ ie; half lattice vectors on this elliptic curve or a torus.

The reason why we are going to shift by 2 is because these were associated to CFT, that ~~are~~ are free fermions; which has $\mathbb{Z}_2$ symmetry.

(-1) to the fermion ~~no~~ number.

Shifting by these points of order 2 ; gives the things that correspond to kind of order 2 twist of the fermion.

---

define ,
$$\theta_3(z;\tau) \equiv \theta(z;\tau) \equiv \theta_{00}(z;\tau)$$

↳ ie: shifted by $z+0\tau+0$
(shifted by $0 , 0$ in both lattice direction)

$$\theta_3(z;\tau) \equiv \theta(z;\tau) \equiv \theta_{00}(z;\tau)$$

$$\theta_{01}(z;\tau) = \theta_3(z+\tfrac{1}{2};\tau) \equiv \theta_4(z;\tau)$$

$$\theta_{10}(z;\tau) = \theta_3(z+\tau/2,\tau)\,\underbrace{e^{\pi i\tau/4 + i\pi z}}_{\text{undoes the phase transformation of } \theta_3} = \theta_2(z;\tau)$$

$$\cancel{\theta_{11}(z;\tau) = \theta_3(z+\tfrac{1}{2}+\tfrac{\tau}{2};\tau)}$$

$$\theta_{11}(z;\tau) = \theta_3(z+\tfrac{1}{2}+\tfrac{\tau}{2};\tau)\cdot i\cdot e^{\pi i\tau/4+\pi i z} = \theta_1(z;\tau)$$

These appear in partition function of free fermion theory.

Jacobi Triple product identity

"See Ginsparg's Applied CFT lectures"

$$\theta_3(z;\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}\cdot y^n$$

$$= \prod_{n=1}^{\infty}(1-q^n)(1+y\,q^{n-\frac{1}{2}})(1+y^{-1}\cdot q^{n-\frac{1}{2}})$$

or

$$\underbrace{\frac{\sum_{n\in\mathbb{Z}} q^{n^2/2}\cdot y^n}{\prod_{n=1}^{\infty}(1-q^n)}}_{1 \text{ boson } (c=1)} = \underbrace{\prod_{n=1}^{\infty}(1+y\,q^{n-\frac{1}{2}})(1+y^{-1}\cdot q^{n-\frac{1}{2}})}_{2 \text{ fermions } (c=1)}$$

Free fermion : modes b_n

$$\{b_n, b_m\} = \delta_{n+m,\,0}$$

$n \in \mathbb{Z}$ Ramond

$n \in \mathbb{Z} + \frac{1}{2}$ Neveu-Schwarz

$$L_0 = \sum_{n>0} n\, b_{-n} b_n + \begin{cases} \frac{1}{24} & R \\ -\frac{1}{48} & NS \end{cases}$$

Fock space $b_n |0\rangle = 0 \quad , n > 0$ (N.S.)

$|0\rangle$

$b_{-1/2} |0\rangle$

$b_{-3/2} |0\rangle$

Theory with 2 fermions : $b_n^i \quad i = 1, 2$

(we have two copies of b_n s)

define U(1) current

whose charge is given by :

(N.S.) $$J_0 = \sum_{\substack{n>0 \\ n \in \mathbb{Z}+\frac{1}{2}}} \bar{\chi}_{-n} \chi_n - \chi_{-n} \bar{\chi}_n$$

where χ_n is complex fermion field which is linear combination of two real fermions

$\chi_n = \frac{1}{\sqrt{2}}(b_n^1 + i b_n^2) \quad ; \quad \bar{\chi}_n = \frac{1}{\sqrt{2}}(b_n^1 - i b_n^2)$

$$\mathrm{Tr}_{NS}\, q^{L_0} y^{J_0} = q^{-\frac{1}{24}} \cdot \prod_{n=0}^{\infty} (1+y q^{n-\frac{1}{2}})(1+y^{-1} q^{n-\frac{1}{2}})$$ Pg. 74

upto factor of $q^{-1/24}$

The RHS is Triple product identity.

☆ We dont have things appearing in denominator for fermion ~~from Fermi~~ because of Fermi statistics.

So count each number of times the oscillator appears.

The power of y tell us the charge 1 or -1 every time χ & $\bar{\chi}$ acts to give state of High energy.

The L.H.S. is 1 boson on a $S^1 = \mathbb{R}/\mathbb{Z}$ such that $p \in \mathbb{Z}$

$$L_0 = \frac{p^2}{2} + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n - \frac{1}{24} \qquad J_0|p\rangle = p|p\rangle$$

$$\mathrm{Tr}_{Fock}\, q^{L_0} \cdot y^{J_0} = \frac{\theta_3(z;\tau)}{\eta(\tau)}$$

So, The Jacobbi Triple product idemity is really the statement of Relation between partition function of free boson & free fermion.

We can show using Poisson summation again.

$$\theta_i(\tau) \equiv \theta_i(z=0, \tau) \qquad i = 1, 2, 3, 4$$

$\theta_1(\tau) = 0$

$$\theta_2(-1/\tau) = \sqrt{-i\tau}\,\theta_4(\tau)$$

$$\theta_3\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\theta_3(\tau)$$

$$\theta_4\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\cdot\theta_2(\tau)$$

$$\left.\begin{aligned} \theta_2(\tau+1) &= \sqrt{i}\,\theta_3(\tau) \\ \theta_3(\tau+1) &= \theta_4(\tau) \\ \theta_4(\tau+1) &= \theta_3(\tau) \end{aligned}\right\}$$

These modular tramsformations are suggesting to think of $(\theta_2, \theta_3, \theta_4)$ as a 3-component vectors with components that mix under modular transformations.

This mixing is related to how partition functions of free fermion behave under Modular Transformation.

Notation, (Standard notation $g\,\underset{h}{\square}$: ... for computing one loop partition function in String Theory)

Computing trace in Hilbert space which is twisted by some element of group G; with an insertion of a element of group G.

$$g\,\underset{h}{\square} \overset{\text{notation}}{=} \mathrm{Tr}_{H_h}\, g\cdot q^{L_0 - \frac{c}{24}}$$

Twisted by h

$h \in G$

and $[g, h] = 0$

~~In terms of path integral,~~

In terms of Path Integral.

$$g\,\underset{h}{\square} = \int \mathcal{D}\phi \cdot e^{-S[\phi]}$$

$$\phi(\sigma_0, \sigma_1 + 2\pi) = h\,\phi(\sigma_0, \sigma_1)$$

$$\phi(\sigma_0 + 2\pi, \sigma_1) = g\,\phi(\sigma_0, \sigma_1)$$

For fermions $G = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$

with elements $\{1, (-1)^F\}$ F counts the no. of fermion oscillators we acted on states with.

or $\{P, A\}$

{periodic, Antiperiodic}

We can compute four different sectors

$$A\,\underset{A}{\square} = \mathrm{Tr}_{NS}\, q^{L_0 - \frac{c}{24}} = \sqrt{\frac{\theta_3}{\eta}}$$

$$P\,\underset{A}{\square} = \mathrm{Tr}_{NS}\, (-1)^F q^{L_0 - \frac{c}{24}} = \sqrt{\frac{\theta_4}{\eta}}$$

$$A\,\underset{P}{\square} = \mathrm{Tr}_{R}\, q^{L_0 - c/24} = \sqrt{\frac{\theta_2}{\eta}}$$

$$P\,\underset{P}{\square} = \mathrm{Tr}_{R}\, (-1)^F q^{L_0 - \frac{c}{24}} = 0$$

In CFT, we have Hilbert space of states

and we have two copies of the Virasoro algebra,

with generator L_n & $\tilde{L}_n$.

$$\cancel{\mathcal{H} = V_{ir} \otimes \tilde{V}_{ir}} \qquad \mathcal{H} = V_{ir} \otimes \tilde{V}_{ir}$$

We can always decompose our Hilbert space; into a sum of products of highest weight representation of Virasora.

V_h : $L_0|h\rangle = h|h\rangle$ Highest state representation

$L_m|h\rangle = 0 \quad m > 0$

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}_H \, q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}$$

$$= \mathrm{Tr}_H \left(e^{-2\pi \, \mathrm{Im}\tau \cdot H + 2\pi i \, \mathrm{Re}\,\tau \cdot P} \right)$$

where $H = L_0 + \tilde{L}_0 - \frac{(c+\tilde{c})}{24}$

Hamiltonian (translate in time using Hamiltonian)

$P = L_0 - \tilde{L}_0 - \frac{(c-\tilde{c})}{24}$

Translation on S^1 via momentum operator.

This partition function is supposed to have good Modular properties.

" Rational CFT"

Partition function takes the form of finite sum as follows; of functions of τ & $\bar{\tau}$

(There is a kind of split between Holomorphic & Antiholomorphic dependance)

$$\cancel{Z(\tau,\bar{\tau}) = \sum_{i=1}^{N} \chi_i(\tau)\tilde{\chi}_i(\bar{\tau})} \qquad Z(\tau,\bar{\tau}) = \sum_{i=1}^{N} \chi_i(\tau)\tilde{\chi}_i(\bar{\tau})$$

In such theories,

The modular properties are as follows.

$$\chi_i\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_j \rho_{ij}(\gamma)\, \chi_j(\tau)$$

$$\tilde{\chi}_i\left(\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = \sum_j \tilde{\rho}_{ij}(\gamma)\, \tilde{\chi}_j(\tau)$$

$\rho(\gamma)$: Matrix Representation of $SL_2(\mathbb{Z})$

$\tilde{\rho}(\gamma)$: Contragredient representation of $SL_2(\mathbb{Z})$

Such that $Z(\tau, \bar{\tau})$ is modular Invariant.

In simplest cases.

$$Z(\tau,\bar{\tau}) = \sum_{i=1}^{N} |\chi_i(\tau)|^2$$

Math - focus on holomorphic part $\chi_i(\tau)$, $\mathrm{Tr}\, q^{L_0 - \frac{c}{24}}$, etc.

Physics - Combine holo. & anti-holo. part, and often any phase under modular transformation of $\chi_i(\tau,\bar{\tau})$ is cancelled by phase of $\tilde{\chi}_i(\tau,\bar{\tau})$

Math: Modular form of weight k,

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \quad k \text{ integer}.$$

$$J(\gamma, \tau) = (c\tau + d)^k = \text{" Automorphy factor"}$$

$$f(\alpha \cdot \beta(\tau)) = f(\alpha(\beta(\tau)))$$

$$\Rightarrow \boxed{J(\alpha\beta, \tau) = J(\alpha, \beta(\tau))\, J(\beta, \tau)} \quad (*)$$

Automorphy equation

$\theta_i(\tau)$, $\eta(\tau)$ are weight $1/2$.

$$\eta(-\tfrac{1}{\tau}) = \sqrt{\tau}\ (\quad)\ \eta(\tau)$$

Are there weight $\frac{1}{2}$ modular forms;

meaning functions f such that

$$f(\gamma(\tau)) = (c\tau + d)^{1/2} f(\tau) \quad \text{for} \quad \gamma \in SL_2(\mathbb{Z})\,?$$

Ans NO !

We have to specify branch for $(c\tau + d)^{1/2}$; no matter which branch we choose, we can find modular transformation s.t. we get contradiction ~~fo~~ from Transformation law.

In other words $(c\tau + d)^{1/2}$ is not an Automorphy factor.

(it does not satisfy the relation $(*)$)

Physicist " We dont care ; we can square thing ,... "

Solution :

$$\Theta(\tau) = \theta(0, 2\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}$$

We can compute its modular transformation, (for a subgroup of modular group)

$$\Theta(\gamma(z)) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{cz+d}\, \Theta(z) \qquad \gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$

$$c = 0 \mod 4$$

$$\varepsilon_d = \begin{cases} 1 & d = 1 \mod 4 \\ i & d = 3 \mod 4 \end{cases}$$

Kronecker symbol

$$\Rightarrow \boxed{\Theta(\gamma(z)) = j_{1/2}(\gamma, z)\, \Theta(\tau)}$$

can prove, $j_{1/2}(\gamma, \tau)$ is an automorphy factor : (not for full modular group) but for a ~~sub~~ subgroup $\Gamma_0(4)$

For more general transformation,

$\Rightarrow$ go to a double cover of modular group called the Metaplectic group.

Lec 8) Jacobi forms, Applications in String Theory & BPS counting state, Elliptic genus,

Jacobi Forms

Motivation) String theory is a correct theory of quantum gravity is the detailed microscopic understanding of Black Hole Entropy.

$$S_{BH} = \frac{A}{4} \sim \text{microscopic degeneracy.}$$

~~Stroninger - Vafa~~ Stroninger - Vafa

II B string Theory on $\mathbb{R}_{time} \times \mathbb{R}^4_{space} \times S^1 \times K3$

4d Calabi Yau Space

There are Black ~~Solution~~ Hole solution carrying 3 charges with $S_{BH} \neq 0$ and are BPS

(i.e: preserve some of spacetime supersymmetries)

Microscopic: n-units of momentum on S^1

where Q_1 - D1 - branes wrap $\mathbb{R}_t \times S^1$

Q_5 - D5 - branes " $\mathbb{R}_t \times S^1 \times K3$

$$S_{BH} = \log(d(Q_1, Q_2, n)) = 2\pi\sqrt{Q_1 Q_5 - n}$$

for $n \gg Q_1 Q_5$

A) Argue that low-energy dynamics on $\mathbb{R} \times S^1$, a CFT on

$$(K3)^Q / S_Q$$

Permutation group on Q objects

Orbifold CFT (where Orbifold group is Permutation group)

B) Show BPS states are counted by the elliptic genus of this CFT.

c) Compute $d(Q_1, Q_5, n)$ using analysis similar to that for $p(n)$ done.

Hardy – Ramanujan – Cardy

Extensions, two analyzed in much detail.

(i) $N=8$ SUSY IIB on $\mathbb{R}_t \times \mathbb{R}^3 \times T^6$

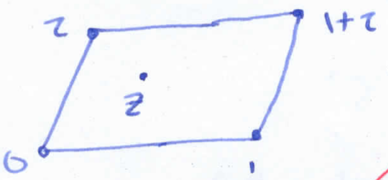
count $\frac{1}{8}$ BPS states. } Jacobi form

(2) $N=4$ SUSY IIB on $\mathbb{R}_t \times \mathbb{R}^3 \times K3 \times T^2$

count $1/4$ BPS states. } Jacobi form

Refinement involves mock modular forms.

Jacobi Forms



$E_\tau = \mathbb{C} / (\mathbb{Z} + \tau\mathbb{Z})$

$E: \quad z \longrightarrow z + \mu + \lambda\tau \quad , \mu, \lambda \in \mathbb{Z}$

$\tau \longrightarrow \tau$

Elliptic

$\longrightarrow$ Elliptic functions.

Modular $\quad \tau \longrightarrow \dfrac{a\tau + b}{c\tau + d}$ $\quad z \longrightarrow \dfrac{z}{c\tau + d}$ $\Bigg\}$ $f\left(\dfrac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k} f(\tau)$

$\longrightarrow$ Modular forms.

Elliptic transformations (E) & Modular transformations (M) are closely connected.

- E moves us in $\mathbb{C}$ by lattice points
- M corresponded to changing the basis of lattice vector but preserving the lattice between choosing two different bases.

$\Longrightarrow$ So, its natural to ask, is there a nice theory of function z & τ, that transform under both E & M

$\phi(z,\tau)$ transform under both E & M? <u>Yes</u>.

$$\theta(z;\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2 + 2\pi i n z}$$

$$\theta(z+\mu+\lambda\tau;\tau) = e^{-2\pi i \lambda z - \pi i \lambda^2 \tau}\,\theta(z;\tau)$$

$$\theta(0;\tau+2) = \theta(0;\tau)$$

$$\theta(0,-1/\tau) = \sqrt{\frac{\tau}{i}}\,\theta(0;\tau)$$

can generalize this to $\theta\left(\frac{z}{\tau}, \frac{-1}{\tau}\right)$ using poisson summation

$$\frac{\partial\theta}{\partial\tau} = \frac{-i}{4\pi}\frac{\partial^2\theta}{\partial z^2}$$ heat equation

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☆ <u>Holomorphic Jacobi forms</u> — Eichler + Zagier.

(appears often in BH problems, CFT)

☆ <u>Skew-Holomorphic Jacobi forms</u> — Skoruppa

<u>E-A</u>; defined a Jacobi form of weight k and index m for $SL_2(\mathbb{Z})$ is a holomorphic function

$$f : \underset{\tau}{\mathbb{H}} \times \underset{z}{\mathbb{C}} \longrightarrow \mathbb{C}$$

$$M: \phi(\gamma(z,\tau)) = (c\tau+d)^k e^{2\pi i m \cdot \left(\frac{c z^2}{c\tau+d}\right)} \phi(z,\tau)$$

$$E: \phi(z+\lambda\tau+\mu;\tau) = e^{-2\pi i m \cdot(\lambda^2\tau + 2\lambda z)} \cdot \phi(z,\tau)$$

Jacobi Group.

$$\gamma(z,\tau) = \left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$$

$$\gamma \in SL_2(\mathbb{Z}) \qquad k, m \in \mathbb{Z}$$

Note: $\phi(z,\tau+1) = \phi(z,\tau)$
$\phi(z+1,\tau) = \phi(z,\tau)$ $(*)$

$\Rightarrow \quad q = e^{2\pi i\tau}, \quad y = e^{2\pi i z}$

If $\phi(z,\tau)$ is a weight k, index m Jacobi form

Then we can write $\phi(z,\tau)$ as

$$\phi(z;\tau) = \sum_{n,r} c(n,r) \cdot q^n \cdot y^r$$

follows from $(*)$

Use E to find relations between $c(n',r')$ & $c(n,r)$

$$\phi(z+\lambda\tau+\mu;\tau) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \cdot \phi(z,\tau)$$

$$\sum_{n,r} c(n,r)\, q^n y^r = e^{2\pi i m(\lambda^2\tau + 2\lambda z)} \sum_{n,r} c(n,r) q^n (yq^\lambda)^r$$

$$\Rightarrow \sum_{n,r} c(n,r)\, q^n y^r = q^{m\lambda^2} y^{2m\lambda} \sum_{n,r} c(n,r)\, q^n \cdot (yq^\lambda)^r$$

$$= \sum_{n,r} c(n,r)\, q^{\underbrace{n+r\lambda+m\lambda^2}_{n'}} y^{\underbrace{r+2m\lambda}_{r'}}$$

$$\Rightarrow c(n,r) = c(n+r\lambda+m\lambda^2, r+2m\lambda) = c(n',r')$$

$$r' = r + 2m\lambda \equiv r \mod 2m.$$

$n' = n + r\lambda + m\lambda^2$

check: ~~$4n'm - 4nm - r^2$~~

$$\boxed{4n'm - r'^2 = 4nm - r^2}$$

$\Rightarrow$ $C(n,r)$ with same $r \bmod 2m$

and same $D = 4mn - r^2 =$ "The discriminant"

are equal.

$$\sum_{n,r} C(n,r)\, q^{n + r\lambda + m\lambda^2}\, y^{r + 2m\lambda}$$

$$= \sum_{n', r'} C(n' - r\lambda - m\lambda^2, r' - 2m\lambda)\, q^{n'}\, y^{r'}$$

$C(n,r) \equiv C(D, \tilde{r})$ $\quad \tilde{r} = r \bmod 2m$

D: $4mn - r^2$; $\tilde{r}$: defined mod $2m$

or $C_{\tilde{r}}(D)$

So,

$$\sum_{r} \longrightarrow \sum_{\tilde{r}=0}^{2m-1} \sum_{\substack{r \in \mathbb{Z} \\ = \tilde{r} \bmod 2m}} \qquad n = \frac{D + r^2}{4m}$$

sum on r, replaced by

Doing this we find,

$$\phi(z,\tau) = \sum_{\tilde{r} \bmod 2m} \underbrace{\sum_{D} C(D,\tilde{r})\, q^{D/4m}}_{h_{\tilde{r}}(\tau)} \underbrace{\sum_{\substack{r \in \mathbb{Z} \\ r = \tilde{r} \bmod 2m}} q^{r^2/4m}\, y^r}_{\theta_{m,\tilde{r}}(z;\tau)}$$

$$\Rightarrow \phi(z,\tau) = \sum_{\tilde{\gamma} \bmod 2m} h_{\tilde{\gamma}}(\tau) \cdot \theta_{m,\gamma}(z;\tau)$$

~~Exercise~~

$\theta_{m,\gamma}(z;\tau)$ are very natural generalization of Theta function

↳ we can analyze their modular transformation properties using Poisson Summation.

⇔ The only difference is that, now they are vector valued. ie; when we do modular transformation, the components mix into each other according to some matrixes.

$$\theta_{m,\gamma}\left(-\frac{z}{\tau}, \frac{-1}{\tau}\right) = \sqrt{-i\tau} \cdot e^{2\pi i m z^2/\tau} \cdot \sum_s S_{rs}\, \theta_{m,s}(z,\tau)$$

$$\theta_{m,\gamma}(z, \tau+1) = \sum_s T_{rs} \cdot \theta_{m,s}(z,\tau)$$

Then, we can show,

$$T_{rs} = e^{2\pi i \cdot \left(\frac{r^2}{4m}\right)} \cdot \delta_{rs}$$

$$S_{rs} = \frac{1}{\sqrt{2m}}\, e^{2\pi i \cdot (rs)/2m}$$

Modular transformation of $\phi(z;\tau)$ implies that $h_{\tilde{\gamma}}(\tau)$ are also vector valued modular transformation of weight $k - 1/2$

: ~~& $\theta_{m,\gamma}$~~ & $\theta_{m,\gamma}(z;\tau)$ of weight $\frac{1}{2}$.

This defines us a map.

$$\text{map} : \begin{pmatrix} \text{Jacobi forms} \\ \text{weight } k \end{pmatrix} \longrightarrow \begin{pmatrix} \text{vector valued modular forms} \\ \text{of weight } k - \frac{1}{2} \end{pmatrix}$$

Weak Jacobi form $c(n, r) = 0$ whenever $n < 0$
(ie; no negative powers of q)

Strong Jacobi form $C_f(D, r) = 0$ whenever $D < 0$

Jacobi Cusp forms $C_f(D, r) = 0$ whenever $D \leq 0$

Connection between:

$N = 2$ Superconformal algebra properties $\longleftrightarrow$ Elliptic property of Jacobi forms.

via Elliptic genus = Jacobi form

$\hookrightarrow$ and this object is counting function for State ~~space~~ preserving supersymmetry, ie; BPS.

In String Theory there are algebra which are supersymmetric version of non-supersymmetric ~~conformal~~ conformal algebra

ie, $N = 0, 1, 2, 4$ super conformal algebras which occurs in the study of String Theory on various compact manifolds that are used for String Compactification.

$N = 2$ - Calabi - Yau spaces

$N = 4$: Hyper Kahler - K3 - 4-real dimension Calabi Yau space.

$\mathcal{N}=0$ or Virasoro Algebra L_n ; $n \in \mathbb{Z}$

$$[L_n, L_m] = (m-n) L_{n+m} + \frac{c}{12} m (m^2-1) \delta_{m+n,0}$$

In CFT / String Theory L_n, $\tilde{L}_n$.

$\underline{\mathcal{N}=1}$ add some fermionic operators

G_r $\quad r \in \mathbb{Z}$ Ramond

$\quad r \in \mathbb{Z} + \frac{1}{2}$ Neveu - Schwarz.

$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r}$$

$$\{G_r, G_s\} = 2 L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right) \delta_{r+s,0}$$

$\underline{\mathcal{N}=2}$ $G_r^{\pm}$ and $U(1)$ current J_n

$$[L_m, J_n] = -n J_{m+n}$$

$$[J_m, J_n] = \frac{c}{3} m \delta_{m+n,0}$$

$$\{G^+_r, G^+_s\} = \{G^-_r, G^-_s\} = 0$$

$$\{G^+_r, G^-_s\} = L_{r+s} + \frac{1}{2}(r-s) J_{r+s} + \frac{c}{6}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}$$

$$[J_m, G^{\pm}_r] = \pm G^{\pm}_{m+r}$$

Schwimmer + Seiberg.

"Spectral Flow" - is isomorphism between R, $N=2$ algebra $(r \in \mathbb{Z})$ and NS $(r \in \mathbb{Z} + \frac{1}{2})$

and by doing this isomorphism twice gives us map $R \to R$, $NS \to NS$

$$L_n \to L_n + \mu J_n + \frac{c}{6}\mu^2 \delta_{n,0}$$

$$\boxed{\mu = \frac{1}{2}}$$ for $R \to NS$

$$J_n \to J_n + \frac{c}{3}\mu \cdot \delta_{n,0}$$

$$G_r^{\pm} \longrightarrow G^{\pm}_{r\pm\mu}$$

We discover that the algebra is left invariant under above transformation, which is called "Spectral Flow"

So; There is a kind of symmetry which preserves the Algebra

Can consider $\mu \in \mathbb{Z}$ for $R \to R$ or $NS \to NS$.
(it ~~stg~~ reshuffels L_n & J_n & $G_r^{\pm}$; and still preserves the algebra)

---

① Define Elliptic Genus
② Show spectral flow of $N=2$ SCA is E transformation of a Jacobi form

SCA ≡ Super Conformal Algebra.

Witten index in SUSY QM

$\{Q, Q^+\} = H$, $(-1)^F$

States $(-1)^F |b\rangle = +|b\rangle$ ≡ bosonic state
$(-1)^F |f\rangle = -|f\rangle$ ≡ fermionic state.

$Q|b\rangle \to |f\rangle$
$Q|f\rangle \to |b\rangle$ if Energy $E \neq 0$, $H|b\rangle = E|b\rangle$

and in this theory

$$\mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\beta H} = n_{\text{bosonic}}(E=0) - n_{\text{fermionic}}(E=0)$$

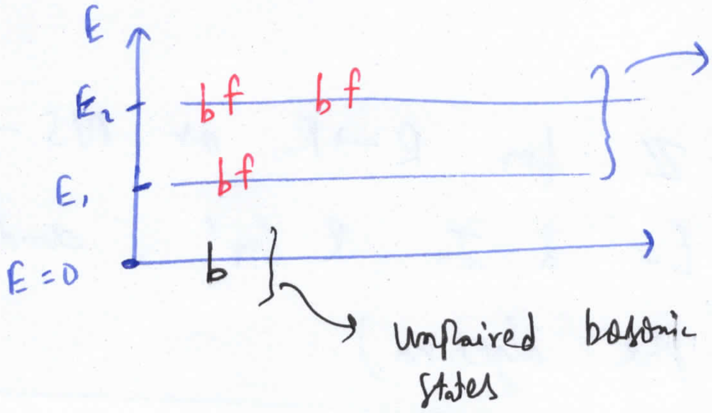
for all $E \neq 0$, $\quad Q|b\rangle \longleftrightarrow |f\rangle$.

and they ~~go~~ get cancelled up in trace because of $(-1)^F$

So,

$$\mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\beta H} = n_{\text{bosonic}}(E=0) - n_{\text{fermionic}}(E=0)$$

If $H|b\rangle = 0$; Then we can show $Q|b\rangle = 0$.



here we have states which are paired, with equal numbers of bosons & fermions

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Given a Calabi-Yau manifold X of real dimension $2m$, there is a $N=2$ SCA, with $c = 6m$.

Then, $Z_{ell}(X; z; \tau) = \mathrm{Tr}_{\mathcal{H}_{RR}}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} (-1)^{J_0 - \tilde{J}_0} \cdot e^{2\pi i z J_0}$

$$\boxed{Z_{\text{elliptic}}(X; z; \tau) = \mathrm{Tr}_{\mathcal{H}_{RR}}\, q^{L_0 - \frac{c}{24}} \cdot \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \cdot (-1)^{J_0 - \tilde{J}_0} \cdot e^{2\pi i z J_0}}$$

$(-1)^{J_0} \sim (-1)^F$ on left movers (holo...)

$(-1)^{\tilde{J}_0} \sim (-1)^{\tilde{F}}$ on right movers (anti-holo.)

SUSY $(N=2)$ pairs all states which are not ground states with $L_0 = \frac{c}{24}$ or $\tilde{L}_0 = \frac{\tilde{c}}{24}$ into pairs with

Opposite $(-1)^F$.

R-movers $\Rightarrow$ only $\tilde{L}_0 = \frac{\tilde{c}}{24}$ contribute

L-movers; we inserted a factor of $e^{2\pi i z J_0}$, and non ground states are paired into states with ± 1 eigenvalues under $(-1)^{J_0}$, but they can have different J_0 eigenvalues $\Rightarrow$ so they don't cancel in the trace.

This arguments implies

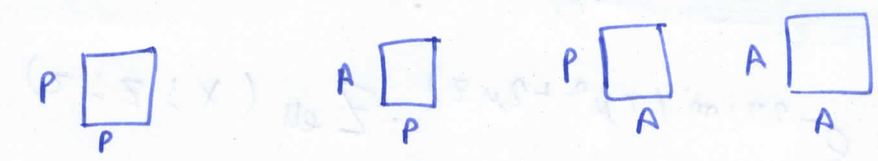
① $Z_{ell}(X; z, \tau)$ Elliptic genus is independent of $\bar{\tau}$ (because only contributing thing is $\tilde{L}_0 - \frac{\tilde{c}}{24}$ & this is 0)

$\Rightarrow$ holomorphic function of z & τ.

② Recieves contributions from R-moving ground states, but arbitrary L-moving states

$\Longrightarrow$ $\frac{1}{4}$ BPS states.

preserving $\frac{1}{4}$ of spacetime SUSY in Type II String theory.



invariant under $SL_2(\mathbb{Z})$

mix under $SL_2(\mathbb{Z})$

So, Elliptic genus $Z_{ell}(X_i; z, \tau)$ has both (Pg 92)

- Modular properties — (from Path Integral formalism)
- Elliptic " — (spectral flow of $N=2$ SCA)

These two combined facts tells us that, <u>$Z_{ell}(X; z, \tau)$ is actually Jacobi form</u>.

Spectral flow by integer μ with $c = 6m$.

$$L_0 \longrightarrow L_0' = L_0 + \mu J_0 + m\mu^2$$

$$J_0 \longrightarrow J_0' = J_0 + 2m\mu$$

Under these transformation; we have

$$e^{2\pi i \tau L_0} \cdot e^{2\pi i z L_0} \rightarrow e^{2\pi i \tau (L_0 + \mu J_0 + m\mu^2)} \times e^{2\pi i z (J_0 + 2m\mu)}$$

$$= e^{2\pi i \tau L_0} e^{2\pi i z J_0 (z + \mu\tau)} \times e^{2\pi i \tau m\mu^2} e^{2\pi i m (2\mu + z)}$$

This tells that

$$\cancel{Z_{ell}(X; \tau; z + \mu\tau) = e^{-2\pi i m (\tau\mu}}$$

$$Z_{ell}(X; z + \mu\tau; \tau) = e^{-2\pi i m (\tau\mu^2 + 2\mu z)} \cdot Z_{ell}(X; z; \tau)$$

$$Z_{ell}(X; z + \lambda, \tau) = Z_{ell}(X; z, \tau) \quad ; \lambda \in \mathbb{Z}$$ (because J_0 has integer eigenvalues)

And then slight computation gives.

$$\cancel{Z_{ell}(X; \frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d})}$$

$$Z_{ell}\left(X; \frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = e^{2\pi i m \frac{c z^2}{c\tau+d}} \cdot Z_{ell}(X; z, \tau)$$

$\Rightarrow$ $Z_{ell}(X)$ is a Jacobi form of weight 0 and index $m = \frac{\text{Complex dim}(X)}{2}$

example) $Z_{ell}(K3, z, \tau)$ is weight 0, index 1.

Eidler - Zagier (A theorem proved by these people)

A Jacobi form of weight k, index m can be constructed as a product of $E_4(\tau)$, $E_6(\tau)$,

$$\varphi_{-2,1}(z;\tau) = \theta_1^2(z;\tau)\,(\eta^6(\tau)),$$

a jacobbi form of index +1 & weight −2

$$\varphi_{0,1}(z;\tau) = 4\left(\frac{\theta_2(z;\tau)^2}{\theta_2(0;\tau)^2} + 2\to 3 + 2\to 4\right),$$

Jacobi form ~~funct~~ of weight 0, and index 1

and $\varphi_{-1,2}(z;\tau) = \frac{\theta_1(2z;\tau)}{\eta^3(\tau)}$.

+ sums of ~~sa~~ terms that have same weight & index

In particular,

$$Z_{ell}(K3 ; z, \tau) = \text{constant} \cdot \varphi_{0,1}(z ; \tau)$$

because there is no other weight 0, index 1 jacobi form we can make by taking product of functions as mentioned in Eichler – Zagier theorem.

To find the constant ;

use $\quad Z_{ell}(K3 ; z = 0 ; \tau) = \chi(K3) = 24$

(after proving it ... then use!)

Euler number of $K3$

This gives our constant to be 2

$$\Rightarrow \boxed{Z_{ell}(K3 ; z, \tau) = 2\, \varphi_{0,1}(z ; \tau)}$$

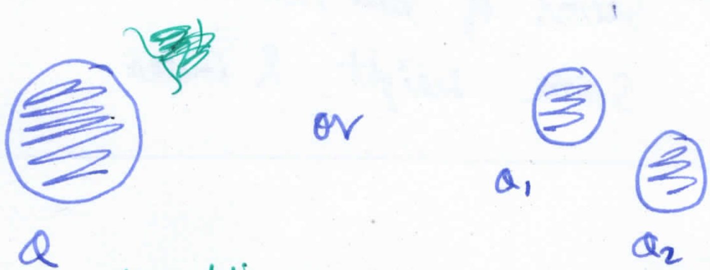
In counting BPS / BH states;

we are actually counting microscopic BPC at weak coupling,

and

computing the entropy of the black hole with the same charges at strong ~~coupl~~ coupling.

B.H. description



$Q_1 + Q_2 = Q$

Single center solution

Two - centre solution.

$\underline{N=8}$ IIB on $\mathbb{R}^{3,1} \times T^6$ This subtlety does not occur ; because there is a counting function, not quite the elliptic genus ; but is a Jacobi form $\varphi_{-2,1}$.

$\underline{N=4}$ This subtely does occur ; and picking out the single centered B.H. in terms of microscopic counting function - Jacobi form is subtle problem.

Dabholkar, Murthy, Zagier. showed.

Single centered B.H. $\longleftrightarrow$ Mock Modular forms (Ramanujan)

Lec 9 / Mock Modular Forms, Weight k laplacian.

1920 last letter of Ramanujan to Hardy

Ramanujan wrote some formulas (did not proved anything)

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)(1+q^2)^2} + \dots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2 \cdots \cdots (1+q^n)^2}$$

mock in literature means something like fake or not real.

$$w(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2 \dots (1-q^{2n+1})^2}$$

(Ramanujan wrote 17 of such formulas (here I write only 2))

Ramanujan said that (in the letter) these are "order 3 mock theta functions".

Then he wrote down some other ones called "order 7", "order 5".

But he did not define what order was!

Then number of mathematicians worked on it

Watson, Andrews (Number Theorist), Dyson, other.

Main mystery: What kind of modular properties do these have?

people found $f(q)$, $q = e^{2\pi i z}$

$$f(\tfrac{-1}{z}) = (\quad) f(z) + (\text{weird integral})$$

History of Lost Notebook of Ramanujan

When Ramanujan died, he had 100 losely pages of all the result he had found which were not published.
~~th~~ When Ramanujan died, those pages went to his widow; and she gave it to some Indian Academic Institution.
→ They sent to Hardy.
Hardy somehow lost interest in them; it then went to Watson, ... then Watson died, then some one else got them, ... & again ...

After long series, these pages ended up in the library. But the Mathematical world did not knew about it. They lost track of it.
Then George Andrew in Cambridge asking somebody about some problem, he was told ~~told~~ to go & see papers of Witekers.

And he went and realized those pages were of Ramanujans.
So in 1998 (around) there was big flory in math world that Lost Notebook of Ramanujan had been found.
And Ramanujan in his notebook had introduced some mock theta functions.

2002 Ph.D Thesis of S. Zwegers.

2005 paper Brunier – Funke "Harmonic Maass forms"

Mock Modular forms (it generalizes examples found by Ramanujan)

On the UHP (Upper Half Plane) $\mathbb{H}$, with constant negative curvature metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad \tau = x + iy$$

$SL_2(\mathbb{R})$: $\tau \to \frac{a\tau + b}{c\tau + d}$ acts as isometries

Laplacian $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ $SL_2(\mathbb{R})$ invariant.

"weight k" Laplacian $\Delta_k = y^{2-k} \cdot \frac{\partial}{\partial \tau} y^k \frac{\partial}{\partial \bar{\tau}}$

$\frac{\partial}{\partial \tau} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$; $\frac{\partial}{\partial \bar{\tau}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

We can show that Δ_k takes objects (not nec. holo.) that transforms like weight k modular form to weight k modular object

(weak) Maass form is a function (not nec. holo.) on $\mathbb{H}$. (with some growth conditions at cusps)

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \rho\,(c\tau + d)^k f(\tau)$$ under modular transformation

$\uparrow$ multiplier system

Such that $\Delta_k f = \lambda f$

~~If $\lambda = 0$, f is a "weight k harmonic Maass f~~

If $\lambda = 0$, f is a "weight k harmonic Maass form".

$\hat{M}_k$ = Space of harmonic (weak) Maass form of weight k.

$M_k^!$ = Space of weakly holomorphic modular forms of weight k (holo. except at $z \to i\infty$. $q^{-1} \dots$)

Clearly any element of $M_k^!$ is in $\hat{M}_k$,

because Δ_k has $\frac{\partial}{\partial \bar{z}}$ in it on far right.

$\Delta_k = (\cdot\cdot) \frac{\partial}{\partial \bar{z}}$

$\hookrightarrow$ These derivatives will kill elements of $M_k^!$ because they are holomorphic, and pull it in $\hat{M}_k$ space.

So; we have a map between $M_k^!$ & $\hat{M}_k$

just by inclusion; $M_k^! \hookrightarrow \hat{M}_k$.

Given a $\hat{f} \in \hat{M}_k$, define a map

$$S(f) = y^k \frac{\partial f}{\partial \bar{z}}$$

We claim that $S(F)$ is a weakly anti-holomorphic modular form of weight $2-k$.

- It is clear that $S(f)$ is anti-holomorphic.

$$\Delta_k f = 0 = y^{2-k} \frac{\partial}{\partial \tau} y^k \frac{\partial}{\partial \bar{\tau}} f = y^{2-k} \frac{\partial}{\partial \tau} S(f)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial \tau} S(f) = 0}$$

- Need to check that $S(f)$ has weight $2-k$

This means that we have a map between $\hat{M}_k$ &

$\bar{M}^!_{2k}$ (This map is called Shadow map)

This bar means anti-holomorphic.

Shadow map.

$$M^!_k \hookrightarrow \hat{M}_k \xrightarrow{S} \bar{M}^!_{2-k}$$

Theorem (Brunier - Funke)

$$0 \longrightarrow M^!_k \hookrightarrow \hat{M}^!_k \xrightarrow{S} M^!_{2-k} \longrightarrow 0$$

This is an exact sequence.

~~Need to show~~ Prove that for every $h \in \bar{M}^!_{2-k}$ there is a $\hat{f} \in \hat{M}_k$ with $S(\hat{f}) = h$ (Hard part of theorem)

Pg 101

Suppose we have a function $g(\bar{\tau}) \in \bar{M}^!_{2-k}$ and is a cusp form, vanishing as $\bar{\tau} \to i\infty$ (which is often the case in many examples)

We can ~~p~~ invert the shadow map.

$g^*(\tau, \bar{\tau}) \in \hat{\mathbb{M}}_k$ such that

not complex coordinate

$$y^k \frac{\partial}{\partial \bar{\tau}} g^*(\tau, \bar{\tau}) = g(\bar{\tau})$$

$$g^* = -(2i)^k \int_{-\infty}^{+\infty} \frac{g(-\bar{z})}{(z+\bar{\tau})^k} \, dz$$

Given a $\hat{f} \in \hat{\mathbb{M}}_k$ such that $S(\hat{f}) = g(\bar{\tau})$

We can define $\hat{f} = f + g^*$ or $f = \hat{f} - g^*$

function f ~~to~~ to be $f = \hat{f} - g^*$.

This f is a Mock Modular form of weight k.

---

f is holomorphic (even if ~~not~~ f not holomorphic)

$$\frac{\partial}{\partial \bar{\tau}} f = \frac{\partial \hat{f}}{\partial \bar{\tau}} - \frac{\partial}{\partial \bar{\tau}} g^* = \frac{1}{y^k} S(\hat{f}) - \frac{1}{y^k} g(\bar{\tau}) = 0$$

because g is shadow of f.

$\hat{f}$ is modular (ie; transforms like a weight k modular form, but not holomorphic)

f is holomorphic, but not modular.

Mock modular form has to do with tension between to having a function which is holomorphic & having a function which is modular.

There are situations in both mathematics & physics where we can't reconcile this tension between holomorphy & Mock Modularity.

Caution: Much of the literature defines

$$S(f) = y^k \overline{\frac{\partial}{\partial \bar{z}} f}$$

so that $\hat{M}_k \xrightarrow{S} M^!_{2-k}$

"Abstract defination" but it is not constructive.

Examples

① Any weakly holomorphic modular form of weight k, $f \in M^!_k$ is a mock-modular form.

We write a pair (f, g)

mock modular form (weight k)

Shadow (weight $2-k$ anti hole. modular form)

i.e.: $j(\tau) = q^{-1} + 196884\, q + \dots$ is a weight 0 mmf (mock modular form) with vanishing shadow.

Since j has weight 0, its shadow would have to have weight 2; but there are no weight 2 modular forms

② Eisenstein series E_{2k} for $2k = 4, 6, 8, \dots$

$$E_{2k} = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} \frac{1}{(c\tau+d)^{2k}} = \frac{1}{2\zeta(2k)} G_{2k}(\tau)$$

$$G_{2k} = \sum_{\substack{n,m \in \mathbb{Z} \\ \neq (0,0)}} \frac{1}{(n\tau+m)^{2k}} = \sum{}' \frac{1}{(n\tau+m)^{2k}}$$

Modular properties involved reordering of terms in the sum (allowed by absolute convergence).

~~G_2~~) In $G_2(\tau)$ we are not allowed to re order terms in the sum. So there is we find slight deviation from modularity.
(because its not absolutely convergent)

Consider

$$G_{2,\epsilon}(\tau) = \sum{}' \frac{1}{(m\tau+n)^2 |m\tau+n|^{2\epsilon}} \qquad \tau \in \mathbb{H},\ \epsilon > 0,\ \epsilon \text{ small..}$$

converges absolutely.

so; $G_{2,\epsilon}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 |c\tau+d|^{2\epsilon} \cdot G_{2,\epsilon}(\tau)$

define

$$I_\epsilon(z) = \int_{-\infty}^{+\infty} \frac{dt}{(z+t)^2 |z+t|^{2\epsilon}}$$

consider:

$$G_{2,\epsilon} - 2\sum_{m=1}^{\infty} I_\epsilon(m\tau) = \underbrace{2\sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}}}_{\text{comes from } m=0 \text{ term in } G_{2,\epsilon}} + \sum \ldots$$

ie:

$$G_{2,\epsilon} - 2\sum_{m=1}^{\infty} I_\epsilon(m\tau) = \underbrace{2\sum_{n=1} \frac{1}{n^{2+2\epsilon}}}_{m=0 \text{ term in } G_{2,\epsilon}} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{+\infty} \left\{ \frac{2}{(m\tau+n)^2 |m\tau+n|^{2\epsilon}} - 2\int_n^{n+1} \frac{dt}{(m\tau+t)^2 |m\tau+t|^{2\epsilon}} \right\}$$

$m \neq 0$ terms in $G_{2,\epsilon}$

use $\displaystyle\sum_{n\in\mathbb{Z}} \int_n^{n+1} = \int_{-\infty}^{+\infty}$

This is well defined for $\epsilon \to 0$. (we get something absolutely convergent)

So: set $\epsilon = 0$ to evaluate it

$$2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{+\infty} \left(\frac{1}{(m\tau+n+1)} - \frac{1}{(m\tau+n)} \right)$$

This telescopes to zero.

$$\boxed{\sum_{n=-N}^{N} (a_{n+1} - a_n) = -a_N + a_{N+1} \to 0 \text{ if } a_N \xrightarrow[N\to\infty]{} 0}$$

define $\hat{G}_2(\tau) = \lim_{\epsilon \to 0} G_{2,\epsilon}(\tau)$

$$= G_2(\tau) + \lim_{\epsilon \to 0} 2 \sum_{m=1}^{\infty} I_\epsilon(m\tau)$$

~~where:~~ $\cancel{G_2(\tau) = \sum_{n \neq 0}}$

where $G_2(\tau)$ is defined in the following order of sum.

$$G_2(\tau) = \sum_{n \neq 0} \frac{1}{n^2} + \sum_{\substack{m \neq 0 \\ n \in \mathbb{Z}}} \frac{1}{(m\tau + n)^2}$$

$G_2(\tau)$ is holomorphic

$\hat{G}_2(\tau)$ transforms like a weight 2 modular form.

Ex show $\lim_{\epsilon \to 0} \sum_{m=1}^{\infty} I_\epsilon(m\tau) = \frac{-\pi^2}{2 \operatorname{Im} \tau}$

$$\boxed{\hat{G}_2(\tau) = G_2(\tau) - \frac{\pi}{\operatorname{Im}(\tau)}}$$

holomorphic

not holomorphic, but transforms like a weight 2 modular forms.

$$\Delta_2 \hat{G}_2 = \frac{\partial}{\partial \tau} y^2 \frac{\partial}{\partial \bar{\tau}} \hat{G}_2 = \frac{\partial}{\partial \tau} y^2 \left(\frac{i}{2} \frac{\partial}{\partial y} \left(-\frac{\pi}{y} \right) \right)$$

$$= \frac{\partial}{\partial \tau} y^2 \cdot \frac{1}{y^2} = 0$$

$\Rightarrow$ $\hat{G}_2$ is a weight 2 Harmonic Maass form.

And shadow.

$$S(\hat{G}_2) = y^2 \frac{\partial}{\partial \bar{\tau}} \hat{G}_2 = \frac{i\pi}{2}$$ a constant (so a weight 0 modular form)

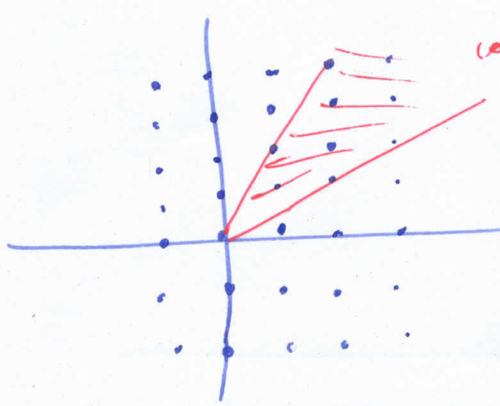
so; G_2 or E_2 are called quasi modular forms.

Zwegers: gave 3 constructions of mmf.

1. Appell-Lerch sums
   - related to characters of the $N=4$ superconformal algebra — String Theory on K3 manifold.

2. Meromorphic Jacobi forms
   $\phi(z,\tau)$ with M and E transformation properties but with poles in z.

3. Θ functions associated to lattices of signature $(1,r)$



$$\Theta = \sum_{\nu \in L} q^{\nu^2/2}$$ diverges

we can modify it to something which is not holomorphic anymore; by putting in convergence factor depending on $\tau, \bar{\tau}$ and on two vectors c_1 & c_2

2) Mero Jacobi forms shows up

A) Counting of BH states for IIB on $K3 \times T^2$

- Dabholkar
- Murthy
- Zagier.

~~Dabholkar~~
~~Murthy~~
Zagier

B) Umbral Moonshine.

- Eguchi, Ooguri, Tachikawa.

extended by Cheng, Duncan, J.H.

In counting BH, there is a counting function.

$$\frac{1}{\Phi_{10}} = \sum_{m=1}^{\infty} p^m \psi_m(\tau, z)$$

weight -10, index m, weak Jacobi forms with a double pole $\sim \frac{1}{z^2}$ as $z \to 0$.

DMZ showed.

$$\psi_m(\tau, z) = \underbrace{\psi_m^P(\tau, z)}_{\text{double pole, elliptic property}} + \psi_m^F(\tau, z)$$

double pole, elliptic property $\sim$ Appell-Lerch Sum

ψ_m^F ↳ Mock Jacobi Form

$$= \sum_{r \bmod 2m} h_r(\tau)\, \theta_{m,r}(\tau, z)$$

$h_r(\tau)$ → Vector valued mock modular form.

BH single centre

Q

double centered.

Single centered calculated by
coeff of $\psi_m^F(z;\tau)$

CDH :
Considered Jacobi forms with a first order pole at $z=0$, with slowest growth of coefficients

$$\psi_m^F = \sum_r \underbrace{H_r^{(m)}}_{\text{v-v mmf}} \theta_{m,r}(\tau, z)$$

v-v mmf (vector valued mmf)

growth condition $q^{1/4m} H_r^{(m)}(\tau) = O(1)$ as $\tau \to i\infty$ for all r.

m=2) $H_1^{(2)} = 2q^{-\frac{1}{8}}(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \dots)$

dimensions of irrep of M_{24} sporadic group (EOT)

m=3) $H_1^{(3)} = 2q^{-1/12}(-1 + 16q + 55q^2 + \dots)$

$H_2^{(3)} = 2q^{2/3}(10 + 44q + 110q^2 + \dots)$

moonshine for ~~2~~ $2 \cdot M_{12}$ ← Mathew group

~~He showed here are~~
23 examples of $(\underset{\text{mmf}}{H^k_r}, \underset{\text{groups}}{G^*})$

labelled by X = Niemeier lattices rank 24
even self-dual lattices.

Math tools: Modular forms, Jacobi forms, Mock Modular forms, Siegel forms, Rademacher sums, ..

• BH counting.

• Moonshine - links special modular objects with ~~spor~~ sporadic groups.

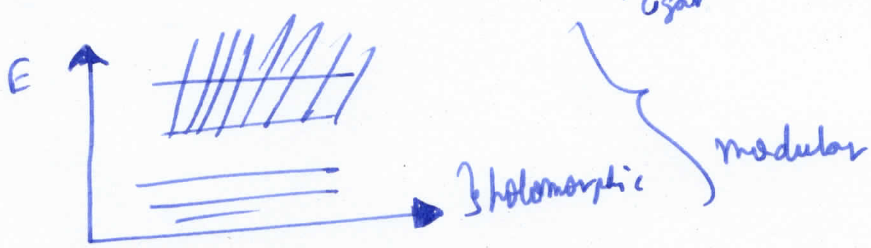
connection

$$\mathbb{R}^{24}/\Lambda_{\text{Niemer}}$$

$$G^* = \mathrm{Aut}(\Lambda_{\text{Neimer}})/W_X$$

Elliptic genus of non-compact

"cigar" $SL_2(\mathbb{R})/U(1)$



Pg 100

PATTERN IN NUMBERS ARE POWERFUL

THANK YOU

The results of Number Theory seems to govern Physics of various system beautifully.

Shoaib Akhtar