WEAK FIELD GRAVITY & HIGHER DIMENSIONS

An advanced course in Gravity

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These notes are consequence of my self study; and are mostly inspired from Prof. Ruth Gregory lectures on **Gravitational Physics**.

Lec 1: Review of differential geometry: manifolds, tensors, differential forms.

Convention: Met signature $+ - - - \ldots$

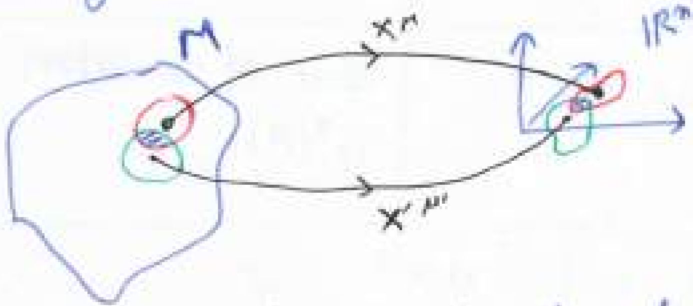
$R^a{}_{bcd} = \Gamma^a{}_{bd,c}$ etc.

$R_{ab} = R^c{}_{acb}$; $\hbar = c = 1$.

$M_p = \sqrt{\frac{\hbar c}{8\pi G}} \sim 2\times 10^{18}$ GeV $T_p = \sqrt{\frac{8\pi G\hbar}{c^5}} \sim 5\times 10^{-44}$ s.

$L_p = \sqrt{\frac{8\pi G\hbar}{c^3}} \sim 10^{-35}$ m

Recall a manifold (spacetime) is a set of events that looks locally like $\mathbb{R}^n$. (Take ∞ly differentiable)



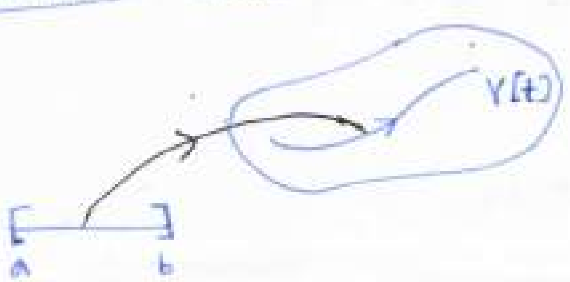
Charts map neighbourhoods of point of $\mathbb{R}^n$. On overlap of of charts, transformations are ∞ly differentiable. All charts form Atlas together. $x^\mu \longrightarrow x'^\mu$ is ∞ly differentiable.

Functions: $C^\infty(M)$ - collections of ∞ly differentiable functions on M.

$f: M \longrightarrow \mathbb{R}$.

$x^\mu \longmapsto f(x^\mu)$ (use charts to define differentiability)

Curves: These are map from $\mathbb{R}$ (or subset) to M.



$\gamma(t)$ is the curve on M.

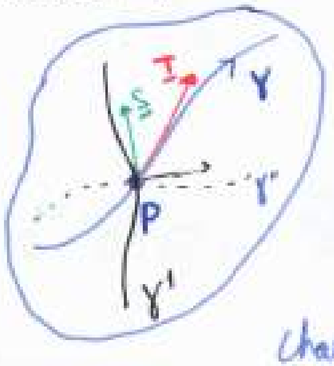
Can also write as $x^\mu(t)$ in local chart. (∞ly differentiable)

<u>Vectors</u>: Defined as tangent to curve at P.

$\underline{T}: C^\infty(M) \longrightarrow C^\infty(M)$

$f \longmapsto \frac{df}{dt}$ $\forall f \in C^\infty(M)$ at P.

associated to curve $\gamma[t]$.



These operators form a vector space associated to P.

Add vectors by composing curves in chart, scale by taking $\gamma[\lambda t]: \underline{T} \to \frac{\underline{T}}{\lambda}$

Tangent Space $T_p(M)$

Collection of tangent spaces gives tangent bundle $T(M)$

<u>Covectors</u>: Maps from tgt space to $\mathbb{R}$. | Space of co-vectors $T_p^*(M)$

$\underset{\sim}{\omega}: T_p(M) \longrightarrow \mathbb{R}$.

In more familiar language $\underset{\sim}{\omega} = \omega_\mu dx^\mu$

$\underset{\sim}{\omega}(\underline{V}) = \omega_\mu V^\mu$.

<u>Note</u> Often in physics we write T^μ for a vector.

Strictly speaking $\underline{T} = T^\mu \frac{\partial}{\partial x^\mu}$ vector is an operator.

T^μ are components of $\underline{T}$ in a particular basis

Writing T^μ is called <u>Abstract Index Notation</u>

(AIN - Penrose)

T^μ are components. $\frac{\partial}{\partial x^\mu}$ is basis; coordinate basis.

Another useful basis is orthonormal (o/n) basis.

Choose $\{\underline{e}_a\}$ at points on M.

$$g(\underset{\sim}{e}_a, \underset{\sim}{e}_b) = \eta_{ab} \qquad (g \text{ is an element of } T^*(M) \otimes T^*(M))$$

In $\mathbb{R}^3$, $\underline{e}_r = \frac{\partial}{\partial r}$; $\underline{e}_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}$; $\underline{e}_\varphi = \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$

Spherical polars (r, θ, φ) in $\mathbb{R}^3$.

These are orthonormal basis.

Coordinate basis: $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}$

FORMS// A p-form is a rank p anti-symmetric co-tensor

~~Note~~ Note that we can differentiate a scalar to get a vector.

$$f \longrightarrow \frac{\partial f}{\partial x^\mu} dx^\mu = df$$

dx^μ forms a basis for $T^*(M)$.

$$\langle \underset{\substack{\uparrow \\ \text{dual basis}}}{dx^\mu} \mid \frac{\partial}{\partial x^\nu} \rangle = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu{}_\nu .$$

P-forms are built by taking anti-symmetric products of covectors: wedge or "$\wedge$".

$\cancel{\underset{\sim}{A}, \underset{\sim}{B} \in T^*_p(M)}$

$$\underset{\sim}{\omega}, \underset{\sim}{\lambda} \in T^*_p(M) \;;\; \underset{\sim}{\omega} \wedge \underset{\sim}{\lambda} = \underset{\sim}{\omega} \otimes \underset{\sim}{\lambda} - \underset{\sim}{\lambda} \otimes \underset{\sim}{\omega}$$

$$\underset{\sim}{\omega} \wedge \underset{\sim}{\lambda} = \underset{\sim}{\omega} \otimes \underset{\sim}{\lambda} - \underset{\sim}{\lambda} \otimes \underset{\sim}{\omega}$$

In components $[\underline{A}^{(p)} \wedge \underline{B}^{(q)}]_{a_1 \dots a_{p+q}}$

$$[\underline{A}^{(p)} \wedge \underline{B}^{(q)}]_{a_1 \dots a_{p+q}} = \frac{(p+q)!}{p!\, q!} A_{[a_1 \dots a_p} B_{a_{p+1} \dots a_{p+q}]}$$

$$\underset{\sim}{A}^{(p)} \wedge \underset{\sim}{B}^{(q)} = (-1)^{pq} \underset{\sim}{B}^{(q)} \wedge \underset{\sim}{A}^{(p)}$$

Note: cannot have a form of rank greater than dimension of manifold. $n = \dim(M)$.

Rank n form is unique up to a factor $\epsilon_{abcd} = \pm 1$ ~~dependent~~ a, b, c, d odd or even permutation of 0123.

Coordinate Transformations

$x^\mu \to x'^{\mu'}$ an overlapping chart.

$\underline{T}$ is geometric, so independent of chart; but components of $\underline{T}$ change.

$$\underline{T} = T^\mu \frac{\partial}{\partial x^\mu} = T'^{\mu'} \frac{\partial}{\partial x^{\mu'}} = T^\mu \cdot \frac{\partial x^{\mu'}}{\partial x^\mu} \cdot \frac{\partial}{\partial x'^{\mu'}}$$

$$\Rightarrow \quad T'^{\mu'} = \frac{\partial x'^{\mu'}}{\partial x^\mu} \cdot T^\mu \quad \text{contravariant.}$$

Similarly $\omega'_{\mu'} = \frac{\partial x^\mu}{\partial x'^{\mu'}} \omega_\mu$ Covariant.

$$\frac{1}{n!} \epsilon_{\mu\nu\lambda\rho} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho =$$

$(n=4)$

$$= \frac{1}{n!} \det\left(\frac{\partial x}{\partial x'}\right) \epsilon_{\mu'\nu'\lambda'\rho'} dx'^{\mu'} \wedge dx'^{\nu'} \wedge dx'^{\lambda'} \wedge dx'^{\rho'}$$

Construct n-form ?? ; This is density because of the $\det\left(\frac{\partial x}{\partial x'}\right)$ factor.

(it is almost a tensor)

With a metric; we can define a 4 (or n) - form tensor.

$$\det g \xrightarrow[x \to x']{} \det(g'_{\mu'\nu'}) = \det\left[\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \cdot g_{\mu\nu}\right]$$

$$= \det\left(\frac{\partial x}{\partial x'}\right)^2 \det g$$

Define $\mathcal{E}_{\mu\nu\lambda\rho} = \sqrt{\det g}\; \epsilon_{\mu\nu\lambda\rho}$

$$= \sqrt{\det g'} \cdot \epsilon_{\mu'\nu'\lambda'\rho'} \cdot \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \cdot \frac{\partial x'}{\partial x} \cdot \frac{\partial x'}{\partial x} \cdot \frac{\partial x'}{\partial x}$$

With $\mathcal{E}$ define Hodge dual $*$

$$* : \Lambda^{(p)} \longrightarrow \Lambda^{(n-p)}$$
$$\underline{A} \longmapsto *\underline{A}$$

In components

$$(*A)_{a_1 \dots a_{n-p}} = \frac{1}{p!} \mathcal{E}_{a_1 \dots a_{n-p}}{}^{a_{n-p+1} \dots a_n} A_{a_{n-p+1} \dots a_n}$$

Exterior Derivative

Can define differentiation on forms

$$d : \Lambda^{p} \longrightarrow \Lambda^{p+1}$$

1) $\underline{d}$ reduces to $\underline{d}f$ on functions.

2) $\underline{d}$ is pseudo-Leibnitzian

$$\underline{d}(\underline{A}^{(p)} \wedge \underline{B}^{(q)}) = \underline{d}A \wedge \underline{B} + (-1)^{p} \underline{A} \wedge \underline{d}\underline{B}$$

3) $\underline{d}^2 = 0$ ⟷ These three conditions are sufficient to find d.

In components.

$$(\underline{d}\,\underline{A})_{a_1 \dots a_{p+1}} = \frac{(p+1)!}{p!} \partial_{[a_1} A_{a_2 \dots a_{p+1}]}$$

Lecture 21 Lie Derivative & Symmetries, Killing Vectors, O/N basis.

Recall, vector is an operator $f \to \frac{df}{dt}$ at a point p.

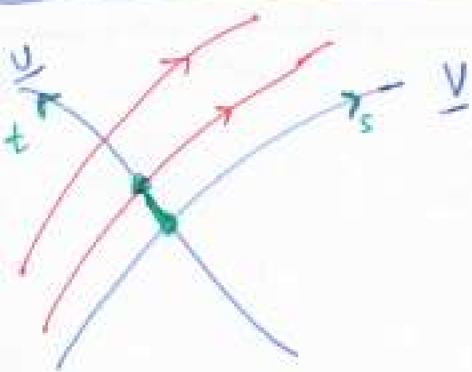
Consider $(\underline{U}\,\underline{V} - \underline{V}\,\underline{U})f$

$$= \left(U^\mu \frac{\partial}{\partial x^\mu} V^\nu \frac{\partial}{\partial x^\nu} - V^\mu \frac{\partial}{\partial x^\mu} U^\nu \frac{\partial}{\partial x^\nu}\right) f$$

$$= \underbrace{\left(U^\mu V^\nu{}_{,\mu} - V^\mu U^\nu{}_{,\mu}\right)}_{} \frac{\partial}{\partial x^\nu} f$$

↳ These are components of new vector; we call it $[\underline{U}, \underline{V}]^\nu$

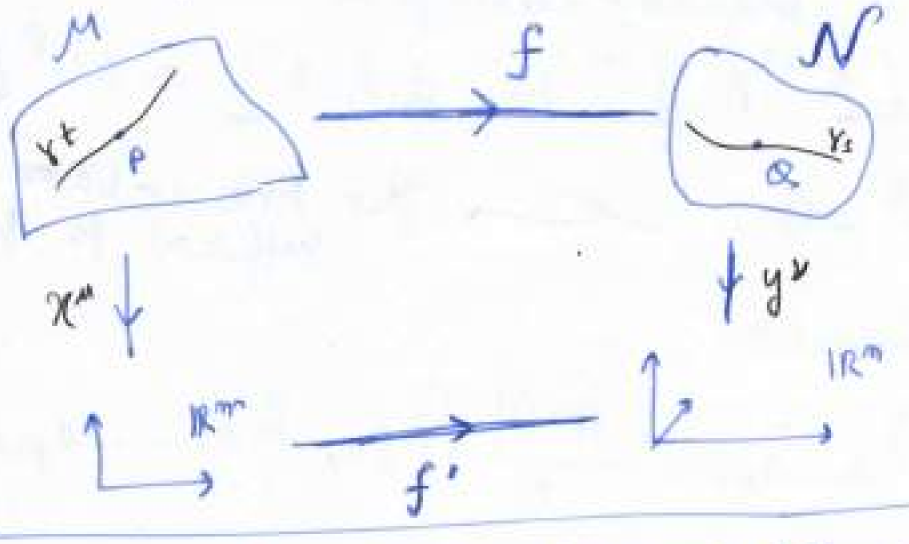
and it is Lie Bracket / commutator of $\underline{u}$ & $\underline{v}$.



Recall; $Q' = \lim_{\delta t \to 0} \frac{Q_{t+\delta t} - Q_t}{\delta t}$

? use $\underline{u}$ to transport Q's ?

In General:



If $\dim M = \dim N$; f 1-1 and onto and C^∞ etc.
Then we say f is diffeomorphism, and regard M, and N as the same

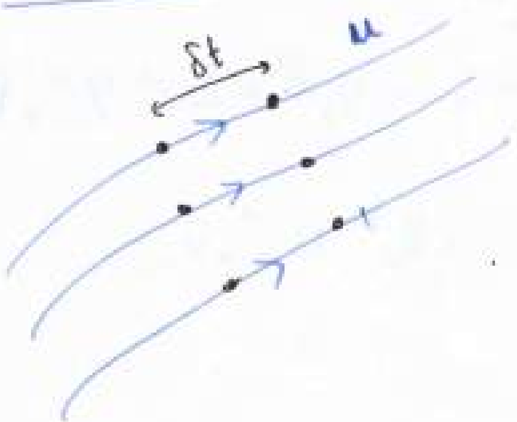
Push forward: $f_*: T_P(M) \longrightarrow T_Q(N)$

$$\underline{T} = \frac{d}{dt} \text{ on } \gamma_t \longmapsto \underline{S} = \frac{d}{ds} \text{ on } \gamma_s$$

Pull-back: $f^*: T_Q^*(N) \longrightarrow T_P^*(M)$

via preimage: via $\langle f^*(\underline{\omega}) \mid \underline{T} \rangle = \langle \underline{\omega} \mid f_*(\underline{T}) \rangle$

Return to u

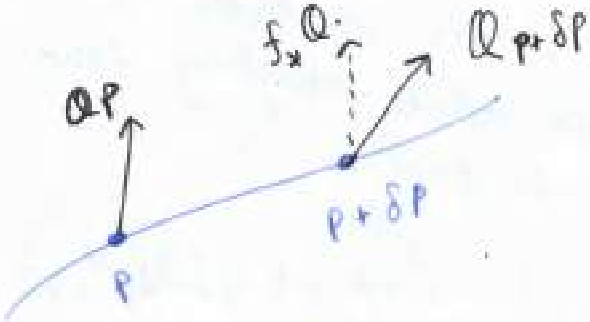


Integral curves ($\underline{u}$ tangent)

Define a coordinate transformation:

$$\boxed{x_t^{\mu} = x_0^{\mu} + \delta t \cdot u^{\mu}}$$

Can now compare at



Check for vector field $\underline{V}$

$$\underline{V}_{t+\delta t} = \underline{V}\,(x_0^{\mu} + \delta t\, u^{\mu})$$

$$= (V_0^{\nu} + V_{0\;\;,\mu}^{\nu} u^{\mu} \delta t) \cdot \frac{\partial}{\partial x_0^{\nu}}$$

Compare ~~to path~~ push forward.

$$\underline{V}_* = \frac{\partial x_t^{\mu}}{\partial x_0^{\nu}} \cdot V_0^{\nu} \cdot \frac{\partial}{\partial x^{\mu}_0}$$

$$= \left(V_0^{\mu} + \delta t \cdot \frac{\partial u^{\mu}}{\partial x^{\nu}} \cdot V_0^{\nu}\right) \frac{\partial}{\partial x_0^{\mu}}$$

$$\mathcal{L}_{\underline{u}} \underline{V} = \lim_{\delta t \to 0} \frac{\underline{V}_{t+\delta t} - \underline{V}_{t*}}{\delta t} = (u^{\nu} V^{\mu}_{\;,\nu} - V^{\nu} u^{\mu}_{\;,\nu}) \frac{\partial}{\partial x^{\mu}}$$

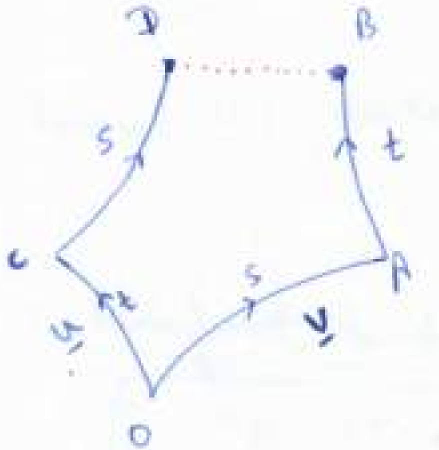
$$= [\underline{u}, \underline{V}]^{\mu} \cdot \frac{\partial}{\partial x^{\mu}}$$

This method give $\mathcal{L}_u$ in general.

$$\mathcal{L}_{\underline{u}} \underset{\sim}{\omega} = (U^\nu \omega_{\mu,\nu} + U^\nu{}_{,\mu} \omega_\nu) \, dx^\mu$$

$$(\mathcal{L}_u g)_{\mu\nu} = g_{\mu\nu,\lambda} U^\lambda + g_{\mu\lambda} U^\lambda{}_{,\nu} + g_{\lambda\nu} U^\lambda{}_{,\mu}$$

Geometrical Significance



$\vec{BD} \propto [U, V]$

we expect this

use $X_A^\mu = X_0^\mu + s V_0{}^\mu + \frac{1}{2} s^2 V_0{}^\mu{}_{,\nu} V_0^\nu$

$U_A{}^\mu = U_0{}^\mu + s V_0{}^\nu U_0{}^\mu{}_{,\nu}$

find $\vec{BD} \simeq X^\mu_A - X^\mu_B$

$$X_D{}^\mu = X_c{}^\mu + s V_c{}^\mu + \frac{1}{2} s^2 V_0{}^\nu V_0{}^\mu{}_{,\nu}$$

since it is of order s^2 we can reduce everything down to point O.

$$= (X_0{}^\mu + t U_0{}^\mu + \frac{1}{2} t^2 U_0^\nu U_0{}^\mu{}_{,\nu}) + s(V_0{}^\mu + t U_0^\nu V_0{}^\mu{}_{,\nu}) + \frac{1}{2} s^2 V_0{}^\nu V_0{}^\mu{}_{,\nu}$$

X^μ_B swaps $s \leftrightarrow t$, $u \leftrightarrow v$

$$\boxed{\begin{aligned} X_D{}^\mu - X^\mu_B &= st[U^\nu V^\mu{}_{,\nu} - V^\nu U^\mu{}_{,\nu}] \\ &= st[\underline{u}, \underline{v}]^\mu \end{aligned}}$$

If $\underline{u}, \underline{v}$ tangent to co-ordinate lines, $[\underline{u}, \underline{v}] = 0$.

Another geometric application of $\mathcal{L}$ is to symmetries of a spacetime.

Definition) A Killing Vector is a vector field along which the metric is invariant.

$$\mathcal{L}_k g = 0$$

(It represents the symmetries of spacetime)

$$(\mathcal{L}_k g)_{\mu\nu} = g_{\mu\nu,\lambda} \cdot k^\lambda + g_{\mu\lambda} k^\lambda{}_{,\nu} + g_{\lambda\nu} k^\lambda{}_{,\mu}$$

$$\Updownarrow$$

$$\nabla_{(\mu} k_{\nu)} = 0$$

e.g.1 $ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$

metric independent of φ.

define $k_3 = \frac{\partial}{\partial\varphi}$: $k^\varphi = 1$

$$\mathcal{L}_k g_{\mu\nu} = k^\lambda \cdot g_{\mu\nu,\lambda} + 0 + 0 = 0$$

Aside; also have 2 more killing vectors.

$$\cos\varphi \cot\theta \cdot \partial_\theta + \sin\varphi \cdot \partial_\varphi = k_1$$

$$\sin\varphi \cdot \cot\theta \, \partial_\theta - \cos\varphi \, \partial_\varphi = k_2$$

corresponds to rotation along different axis.

These three killing vectors obey SO(3) algebra.

∴ Vector obey SO(3) algebra:

$$[k_i, k_j] = \epsilon_{ijk} k_k$$

The Algebra of Killing Vectors is reflecting the underlying symmetries of that particular geometry.

$\mathcal{L}$ works because it has a method of connecting tangent spaces.

The connection is a structure on tangent bundle connecting tangent spaces. Tells us how a basis $\{\underline{e}_a\}$ at P links to $\{\underline{e}'_a\}$ at Q.

e.g. $\mathbb{R}^2$. $\{x, y\} \longleftrightarrow \{r, \theta\}$

Standard Cartesian — Cylindrical polars.

$\underline{e}_x = \frac{\partial}{\partial x}$, $\underline{e}_y = \frac{\partial}{\partial y}$ both coordinate & o/n.

$$\frac{\partial}{\partial r} = \frac{x}{r}\underline{e}_x + \frac{y}{r}\cdot\underline{e}_y = \underline{e}_r \quad \|\underline{e}_r\| = 1$$

$$\frac{\partial}{\partial \theta} = -y\,\underline{e}_x + x\,\underline{e}_y \quad ; \text{but } \left\|\frac{\partial}{\partial\theta}\right\| = r$$

so; ~~orth~~ orthogonal, coordinate, ~~but not o/n.~~ but not o/n.

o/n ≡ orthonormal.

---

o/n basis useful.

$$\underline{e}_{\hat{r}} = \frac{\partial}{\partial r} \; ; \; \underline{e}_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta}$$

we could write equivalently in matrix format.

$$\underline{e}_a = \begin{bmatrix} 1 & 0 \\ 0 & 1/r \end{bmatrix} \begin{bmatrix} \partial_r \\ \partial_\theta \end{bmatrix}$$

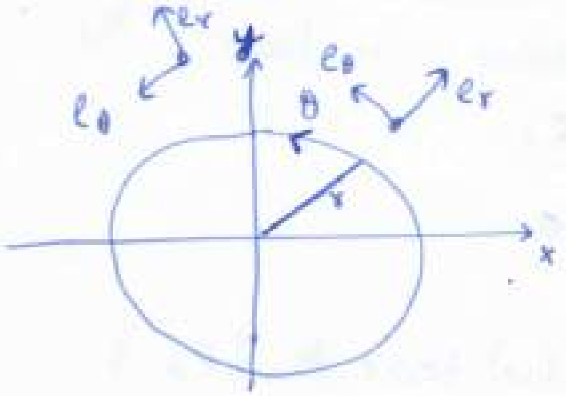
o/n basis

$e^{\mu}_{\hat{a}}$

coordinate basis

This gives us transformation between o/n & coordinate basis. In some sense it is square root of the metric.

$$\eta^{\hat{a}\hat{b}} \underline{e}^{\mu}_{\hat{a}} \underline{e}^{\nu}_{\hat{b}} \sim g^{\mu\nu}$$



$\underline{e}_x, \underline{e}_y$ same throughout $\mathbb{R}^2$.

$\underline{e}_r, \underline{e}_\theta$ not.

$$\underline{e}_r = \cos\theta \cdot \underline{e}_x + \sin\theta \cdot \underline{e}_y$$

$$\underline{e}_\theta = -\sin\theta \cdot \underline{e}_x + \cos\theta \cdot \underline{e}_y$$

$$\delta \underline{e}_r = \underline{e}_\theta \cdot \delta\theta$$

$$\delta \underline{e}_\theta = -\underline{e}_r \cdot \delta\theta$$

these are giving ~~connecting both~~ connection between tangent spaces.

$\therefore$ $\hookrightarrow$ is connection: $\left.\begin{matrix}\Gamma^r_{\theta\theta} \\ \Gamma^\theta_{\theta r}\end{matrix}\right\}$ non zero.

Lecture 3) Cartan's formalism : connection & curvature

The covariant derivatives works by adding structure (the connection) to the tangent bundle.

$$\underbrace{\nabla}_{\text{derivation}} \underbrace{\underline{e}_b}_{\text{basis vector}} = \underbrace{\Gamma^c_{\ a b}}_{\text{connection components}} \underbrace{\underline{e}_c}_{\text{basis}} \otimes \underbrace{\underline{\omega}^a}_{\text{dual basis to } \{\underline{e}_a\}}$$

connection components — scalars (set of numbers).

or, $\Gamma^a_{\ bc} = \langle \omega^a \mid \nabla_b \, \underline{e}_c \rangle$

$$\nabla_b = \underline{e}_b \cdot \underline{\nabla}$$

Dual basis : $\langle \underset{\sim}{\omega}^a \mid \underline{e}_b \rangle = \delta^a_{\ b}$

Defination) $\underline{\nabla}$ is a derivation that • commutes with contractions

• Leibnitzian • Reduces to $\underline{d}$ on functions.

We are used to $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$

For a vector,

$$\underbrace{\underline{\nabla}\,\underline{V}}_{T^*(M)\otimes T(M)} = \underline{\nabla}(V^a \underline{e}_a) \overset{②}{=} (\underline{\nabla} V^a)\,\underline{e}_a + (\underline{\nabla}\,\underline{e}_a) V^a$$

$$\overset{③}{=} \underline{d}V^a \underline{e}_a + \Gamma^b_{\ ca}\, \underline{e}_b \, \underset{\sim}{\omega}^c V^a$$

$$\boxed{\underline{\nabla}\,\underline{V} = (V^a_{\ ,c} + \Gamma^a_{\ cb} V^b)\, \underline{e}_a \, \underline{\omega}^c}$$

In G.R. we use Levi-Civita connection :
"symmetric" , $\nabla g = 0$.

Torsion ~~$T(u,v)=\nabla_u v$~~ $\underline{T}(\underline{u},\underline{v}) = \nabla_{\underline{u}} \underline{v} - \nabla_{\underline{v}} \underline{u} - [\underline{u},\underline{v}]$

almost the anti-symmetric part of connection.

$$T^a_{\ bc} = \Gamma^a_{\ bc} - \Gamma^a_{\ cb} - C^a_{\ bc}$$

where : $C^a_{\ bc} = \langle \underset{\sim}{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle$

are the structure constants of the basis $\{\underline{e}_a\}$.

∴ Torsion is a tensor.

Defination The connection 1-forms, or spin-connection are defined as $\underset{\sim}{\theta}^a_{\ c} = \Gamma^a_{\ bc} \underset{\sim}{\omega}^b$

The indices a & c are not in general tensor indices. They are just labels.

Allows us to differentiate spinors.

Cartan relates spin connection (& torsion) to derivatives of ~~{ω^}~~ $\{\underset{\sim}{\omega}^a\}$

- For metric connection $\underset{\sim}{d} g_{ab} = \underset{\sim}{\theta}_{ab} + \underset{\sim}{\theta}_{ba}$

Proof $\underset{\sim}{d} g_{ab} = \underset{\sim}{\nabla} g_{ab} = \underset{\sim}{\nabla} \langle \underline{g} | \underline{e}_a, \underline{e}_b \rangle$

because: g_{ab} are just scalars.

$$= \langle \underline{g} | \underline{e}_a, \underset{\sim}{\nabla} \underline{e}_b \rangle + \langle \underline{g} | \underset{\sim}{\nabla} \underline{e}_a, \underline{e}_b \rangle$$

(when it hits $\underline{g}$: it gives zero)

$$= \Gamma^c_{\ da} \langle \underline{g} | \underline{e}_c, \underline{e}_b \rangle \underset{\sim}{\omega}^d + \Gamma^c_{\ db} \langle \underline{g} | \underline{e}_a, \underline{e}_c \rangle \underset{\sim}{\omega}^d$$

$$= g_{cb} \underset{\sim}{\theta}^c_{\ a} + g_{ac} \underset{\sim}{\theta}^c_{\ b} = \underset{\sim}{\theta}_{ba} + \underset{\sim}{\theta}_{ab}$$

This says that metric connection has symmetric spin connection.

If $\{\underset{\sim}{e}_a\}$ is an o/n basis.

$$g_{ab} = \eta_{ab} \Rightarrow dg_{ab} = 0$$

Cartans 1st Structural Equation

Consider $\underset{\sim}{\theta}^a{}_c \wedge \underset{\sim}{\omega}^c = \Gamma^a{}_{bc}\, \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c$

$$= \frac{1}{2}(\Gamma^a{}_{bc} - \Gamma^a{}_{cb})\, \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c$$

$\underset{\sim}{\omega}^r = \underset{\sim}{d}r$

$\underset{\sim}{\omega}^\theta = r\,\underset{\sim}{d}\theta$

$\underset{\sim}{\omega}^\varphi = r \sin\theta\, \underset{\sim}{d}\varphi$

$\underset{\sim}{e}_r = \frac{\partial}{\partial r} = (1,0,0)$ in coordinate basis

$\underset{\sim}{e}_\theta = \frac{1}{r}\frac{\partial}{\partial \theta} = (0, \frac{1}{r}, 0)$ "

$\underset{\sim}{e}_\varphi = \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} = (0, 0, \frac{1}{r\sin\theta})$ "

$\|\underset{\sim}{e}_r\| = 1$

$\|\underset{\sim}{e}_\theta\| = 1$

$\|\underset{\sim}{e}_\varphi\| = 1$

$$\underset{\sim}{g}(\underset{\sim}{e}_a, \underset{\sim}{e}_b) = \delta_{ab}$$

$$\underline{V} = V_r \underset{\sim}{e}_r + V_\theta \underset{\sim}{e}_\theta$$

$$\|V\| = \sqrt{V_r^2 + V_\theta^2}$$

$$\underset{\sim}{\theta}^a{}_c \wedge \underset{\sim}{\omega}^c = \frac{1}{2}(T^a{}_{bc} + C^a{}_{bc})\, \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c$$

$$= \underset{\text{Torsion tensor}}{\underset{\sim}{T}^a} + \frac{1}{2}\langle \underset{\sim}{\omega}^a | [\underset{\sim}{e}_b, \underset{\sim}{e}_c]\rangle\, \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c$$

$$T^a = \frac{1}{2} T^a{}_{bc}\, \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c$$

$$\underset{\sim}{\theta}^a{}_c \wedge \underset{\sim}{\omega}^c = T^a + \frac{1}{2}\langle \underset{\sim}{\omega}^a | [\underset{\sim}{e}_b, \underset{\sim}{e}_c]\rangle\, \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c$$

but, $\langle \underline{d}\,\underset{\sim}{\omega}^a |\, \underline{e}_b, \underset{\sim}{e}_c \rangle = \underline{e}_b(\langle \underset{\sim}{\omega}^a | \underset{\sim}{e}_c \rangle)$

$$- \underline{e}_c(\langle \underset{\sim}{\omega}^a | \underline{e}_b \rangle)$$

$$- \langle \underset{\sim}{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle$$

from identity.

$$\langle \underline{d}\,\underset{\sim}{\omega}^a | \underline{e}_b, \underline{e}_c \rangle = \underline{e}_b(\underbrace{\langle \underset{\sim}{\omega}^a | \underline{e}_c \rangle}_{\to 0}) - \underline{e}_c(\underbrace{\langle \underset{\sim}{\omega}^a | \underline{e}_b \rangle}_{\to 0}) - \langle \underset{\sim}{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle$$

so;

$$\langle \underline{d}\,\underset{\sim}{\omega}^a | \underline{e}_b, \underline{e}_c \rangle = -\langle \underset{\sim}{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle$$

$$\underset{\sim}{T}^a = \underset{\sim}{\theta}^a{}_c \wedge \underset{\sim}{\omega}^c - \frac{1}{2} \langle \underset{\sim}{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c$$

$$= \underset{\sim}{\theta}^a{}_c \wedge \underset{\sim}{\omega}^c + \frac{1}{2} \langle \underline{d}\,\underset{\sim}{\omega}^a | \underline{e}_b, \underline{e}_c \rangle \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c$$

$$\boxed{\underset{\sim}{T}^a = \underline{d}\,\underset{\sim}{\omega}^a + \underset{\sim}{\theta}^a{}_c \wedge \underset{\sim}{\omega}^c}$$ C1 (Cartan's 1st Structural equation)

If torsion vanishes, then $\underset{\sim}{\theta}^a{}_c \wedge \omega^c = -\underline{d}\,\underset{\sim}{\omega}^a$

and $\underset{\sim}{\theta}_{(ab)} = 0$

Curvature // defined as commutator of derivatives...

$$\underset{\sim}{R}(\underline{u}, \underline{v})\,\underline{W} = [\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[\underline{u}, \underline{v}]}]\,\underline{W}$$

$$\underset{\sim}{R} : T_p(M) \times T_p(M) \times T_p(M) \longrightarrow T_p(M)$$

physically $\underline{R}$ represents tidal forces.

In components (19.16)

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb} - C^e{}_{cd}\Gamma^a{}_{eb}$$

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb} - C^e{}_{cd}\Gamma^a{}_{eb}$$

Cartan's 2^{nd} Structural equation

Curvature 2-form is.

$$\boxed{\underline{R}^a{}_b = -\frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d}$$

(C2) $$\boxed{\underline{R}^a{}_b = \underline{d}\,\theta^a{}_b + \underline{\theta}^a{}_c \wedge \underline{\theta}^c{}_b}$$

e.g. $\mathbb{R}^2$ in polar coordinates $dr^2 + r^2 d\theta^2$

$\underline{\omega}^r = \underline{d}r$; $\underline{d}\,\underline{\omega}^r = 0$

$\underline{\omega}^\theta = r\underline{d}\theta$; $\underline{d}\,\underline{\omega}^\theta = \underline{d}r \wedge \underline{d}\theta = \frac{1}{2} e_r \wedge e_\theta$

$= -\underline{d}\theta \wedge dr$

(C1): $\underline{d}\,\underline{\omega}^a = -\underline{\theta}^a{}_b \wedge \underline{\omega}^b$ $= -\frac{1}{r} r\, d\theta \wedge dr = -\frac{1}{r}\omega^\theta \wedge \omega^r$

$\underline{\theta}^\theta{}_r = \frac{1}{r}\underline{\omega}^\theta$; no $\underline{R}^a{}_b$ ie; $\underline{R}^a{}_b = 0$.

$S^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

$\underline{\omega}^\theta = \underline{d}\theta$ $\underline{\omega}^\varphi = \sin\theta\, \underline{d}\varphi$

$\underline{d}\,\underline{\omega}^\theta = 0$ $\underline{d}\,\underline{\omega}^\varphi = \cot\theta\, \underline{\omega}^\theta \wedge \underline{\omega}^\varphi$

$\underline{\theta}^\varphi{}_\theta = \cot\theta \cdot \underline{\omega}^\varphi = \cos\theta\, \underline{d}\varphi$ | $\underline{R}^\varphi{}_\theta = \underline{d}\,\underline{\theta}^\varphi{}_\theta = -\sin\theta\, \underline{d}\theta \wedge \underline{d}\varphi$

$= \underline{\omega}^\varphi \wedge \underline{\omega}^\theta$

$R^{\hat\varphi}{}_{\hat\theta\hat\varphi\hat\theta} = 1$

Lec 4: Spherically Symmetric Spacetimes, Schwarzschild-(A)dS black holes.

Look at Static Spherically symmetric metric.

$$ds^2 = A^2(r)dt^2 - B^2(r)dr^2 - C^2(r)\underbrace{[d\theta^2 + \sin^2\theta\, d\varphi^2]}_{d\Omega^2_{II}}$$

To apply Cartan; we need to know orthonormal basis of 1-forms.

$$\underset{\sim}{\omega}^{\hat{t}} = A(r)dt$$
$$\underset{\sim}{\omega}^{\hat{r}} = B(r)dr$$
$$\underset{\sim}{\omega}^{\hat{\theta}} = C(r)d\theta$$
$$\underset{\sim}{\omega}^{\hat{\varphi}} = C(r)\sin\theta\, d\varphi$$

Spin connection

$$\underline{d}\underset{\sim}{\omega}^{\hat{t}} = A'\, dr\wedge dt$$
$$= -\frac{A'}{AB}\omega^{\hat{t}}\wedge\omega^{\hat{r}}$$
$$= \theta^{\hat{t}}{}_{\hat{r}}\wedge\omega^{\hat{r}}$$
$$\Rightarrow \boxed{\theta^{\hat{t}}{}_{\hat{r}} = \frac{A'}{AB}\omega^{\hat{t}}}$$

$$\underline{d}\underset{\sim}{\omega}^{\hat{r}} = 0.$$

$$\underline{d}\underset{\sim}{\omega}^{\hat{\theta}} = C'\, dr\wedge d\theta \Rightarrow \boxed{\theta^{\hat{\theta}}{}_{\hat{r}} = \frac{C'}{CB}\underset{\sim}{\omega}^{\hat{\theta}}}$$

$$\underline{d}\underset{\sim}{\omega}^{\hat{\varphi}} = C'\sin\theta\, dr\wedge d\varphi + C\cos\theta\, d\theta\wedge d\varphi$$

$$\Rightarrow \quad = -\theta^{\hat{\varphi}}{}_{\hat{a}}\wedge\omega^{\hat{a}} \Rightarrow \theta^{\hat{\varphi}}{}_{\hat{r}} = \frac{C'}{CB}\underset{\sim}{\omega}^{\hat{\varphi}}$$
$$\theta^{\hat{\varphi}}{}_{\hat{\theta}} = \frac{1}{C}\cot\theta\, \underset{\sim}{\omega}^{\hat{\varphi}}$$

~~Aside: $\theta^{\hat{r}}{}_{\hat{\theta}} = \eta^{\hat{r}\hat{a}}$~~

Aside: $\theta^{\hat{r}}{}_{\hat{\theta}} = \eta^{\hat{r}\hat{a}}\theta_{\hat{a}\hat{\theta}}$

$$= -\eta^{\hat{r}\hat{a}}\theta_{\hat{\theta}\hat{a}} = -\eta^{\hat{r}\hat{a}}\eta_{\hat{\theta}\hat{b}}\theta^{\hat{b}}{}_{\hat{a}}$$

$\Rightarrow \underset{\sim}{\theta}^{\hat{r}}{}_{\hat{\theta}} = -\underset{\sim}{\theta}^{\hat{\theta}}{}_{\hat{r}}$

Working in orthonormal basis : So : $g = \eta$

$\Rightarrow dg_{\hat{a}\hat{b}} = d\eta_{\hat{a}\hat{b}} = 0$

$\Rightarrow \boxed{\theta_{\hat{a}\hat{b}} + \theta_{\hat{b}\hat{a}} = 0}$

· Curvature 2-form.

$\underset{\sim}{R}^{a}{}_{b} = d\theta^{a}{}_{b} + \theta^{a}{}_{c} \wedge \theta^{c}{}_{b}$

$\underset{\sim}{R}^{\hat{t}}{}_{\hat{r}} = \underset{\sim}{d}\, \theta^{\hat{t}}{}_{\hat{r}}$

$= d\left(\frac{A'}{B} dt\right)$

$= \left(\frac{A'}{B}\right)' dr \wedge dt = \boxed{\frac{-1}{AB}\left(\frac{A'}{B}\right)' \underset{\sim}{\omega}^{\hat{t}} \wedge \underset{\sim}{\omega}^{\hat{r}} = \underset{\sim}{R}^{\hat{t}}{}_{\hat{r}}}$

$\underset{\sim}{R}^{\hat{t}}{}_{\hat{\theta}} = \theta^{\hat{t}}{}_{\hat{r}} \wedge \theta^{\hat{r}}{}_{\hat{\theta}} = \frac{-A'C'}{AB^2C} \underset{\sim}{\omega}^{\hat{t}} \wedge \omega^{\hat{\theta}}$

$\Rightarrow \boxed{\underset{\sim}{R}^{\hat{t}}{}_{\hat{\theta}} = \frac{-A'C'}{AB^2C} \omega^{\hat{t}} \wedge \omega^{\hat{\theta}}}$

& $\underset{\sim}{R}^{\hat{t}}{}_{\hat{\varphi}} = -\frac{A'C'}{AB^2C} \underset{\sim}{\omega}^{\hat{t}} \wedge \underset{\sim}{\omega}^{\hat{\varphi}}$

$\underset{\sim}{R}^{\hat{\theta}}{}_{\hat{r}} = \underset{\sim}{d}\, \theta^{\hat{\theta}}{}_{\hat{r}} + \underbrace{\theta^{\hat{\theta}}{}_{\hat{c}} \wedge \underset{\sim}{\theta}^{\hat{c}}{}_{\hat{r}}}_{\text{given zero}} = \underset{\sim}{d}\left(\frac{C'}{B} \underset{\sim}{d}\theta\right)$

$\boxed{R^{\hat{\theta}}{}_{\hat{r}} = -\left(\frac{C'}{B}\right)' \frac{1}{CB} \underset{\sim}{\omega}^{\hat{\theta}} \wedge \underset{\sim}{\omega}^{\hat{r}}}$

$\underset{\sim}{R}^{\hat{\varphi}}{}_{\hat{r}} = \underset{\sim}{d}\, \underset{\sim}{\theta}^{\hat{\varphi}}{}_{\hat{r}} + \underset{\sim}{\theta}^{\hat{\varphi}}{}_{\hat{\theta}} \wedge \underset{\sim}{\theta}^{\hat{\theta}}{}_{\hat{r}}.$

$= \underset{\sim}{d}\left(\frac{C'}{B} \sin\theta\, \underset{\sim}{d}\varphi\right) + \frac{1}{C} \cot\theta\, \underset{\sim}{\omega}^{\hat{\varphi}} \wedge \frac{C'}{CB} \underset{\sim}{\omega}^{\hat{\theta}}$

$= -\left(\frac{C'}{B}\right)' \frac{1}{CB} \underset{\sim}{\omega}^{\hat{\varphi}} \wedge \underset{\sim}{\omega}^{\hat{r}} + \frac{C'}{B} \cos\theta\, \underset{\sim}{d}\theta \wedge \underset{\sim}{d}\varphi + \frac{C'}{B} \cos\theta\, \underset{\sim}{d}\varphi \wedge \underset{\sim}{d}\theta$

$= -\left(\frac{C'}{B}\right) \cdot \frac{1}{CB} \underset{\sim}{\omega}^{\hat{\varphi}} \wedge \underset{\sim}{\omega}^{\hat{r}}$

$$\underset{\sim}{R}{}^{\hat\varphi}{}_{\hat\theta} = \underset{\sim}{d}\underset{\sim}{\Theta}{}^{\hat\varphi}{}_{\hat\theta} + \underset{\sim}{\Theta}{}^{\hat\varphi} = \underset{\sim}{d}(\cos\theta\, d\varphi) + \frac{c'}{cB}\underset{\sim}{\omega}{}^{\hat\varphi}\wedge\left(\frac{-c'}{cB}\right)\underset{\sim}{\omega}{}^{\hat\theta}$$

$$= \frac{-1}{c^2}\underset{\sim}{\omega}{}^{\hat\theta}\wedge\underset{\sim}{\omega}{}^{\hat\varphi} - \frac{c'^2}{c^2B^2}\underset{\sim}{\omega}{}^{\hat\varphi}\wedge\underset{\sim}{\omega}{}^{\hat\theta}$$

$$R^{\hat t}{}_{\hat r\hat t\hat r} = \frac{-1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right)$$

$$R^{\hat t}{}_{\hat\theta\hat t\hat\theta} = -\frac{A'c'}{AB^2c} = R^{\hat t}{}_{\hat\varphi\hat t\hat\varphi}$$

$$R^{\hat\theta}{}_{\hat r\hat\theta\hat r} = -\frac{1}{B^2}\left(\frac{c''}{c} - \frac{c'B'}{cB}\right) = R^{\hat\varphi}{}_{\hat r\hat\varphi\hat r}$$

$$R^{\hat\varphi}{}_{\hat\theta\hat\varphi\hat\theta} = \frac{1}{c^2} - \frac{c'^2}{c^2B^2}$$

→ Reiman tensor in orthonormal basis (not in coordinate basis)

For Coordinate Basis

$$\underline{R} = \underbrace{R^{a}{}_{bcd}}_{\text{components}}\ \underbrace{\underset{\sim}{e}_a\,\underset{\sim}{\omega}{}^b\,\underset{\sim}{\omega}{}^c\,\underset{\sim}{\omega}{}^d}_{\text{Tensorial part}}$$

Recall: $\underset{\sim}{e}_a = e_a{}^{\mu}\dfrac{\partial}{\partial x^\mu}$ components of orthonormal vector in coordinate basis.

$$\underset{\sim}{\omega}{}^a = \omega^a{}_\mu\, dx^\mu$$

$$\boxed{R^\mu{}_{\nu\lambda\rho} = e_a{}^\mu\,\omega^b{}_\nu\,\omega^c{}_\lambda\,\omega^d{}_\rho\,R^a{}_{bcd}}$$

$$R^t{}_{rtr} = \underbrace{e_{\hat t}{}^t}_{1/A}\,\underbrace{\omega^{\hat r}{}_r}_{B}\,\underbrace{\omega^{\hat t}{}_t}_{A}\,\underbrace{\omega^{\hat r}{}_r}_{B}\,R^{\hat t}{}_{\hat r\hat t\hat r} = B^2R^{\hat t}{}_{\hat r\hat t\hat r}$$

$$R^{tr}{}_{tr} = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right)$$

$$R^{\theta r}{}_{\theta r} = \frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB}\right) = R^{\varphi r}{}_{\varphi r}$$

$$R^{t\theta}{}_{t\theta} = R^{t\varphi}{}_{t\varphi} = \frac{A'C'}{AB^2C}$$

$$R^{\varphi\theta}{}_{\varphi\theta} = \frac{1}{C^2}\left(\frac{C'^2}{B^2} - 1\right)$$

For spherically symmetric vacuum solution...

$$R^{t}{}_{t} = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB} + 2\frac{A'C'}{AC}\right)$$

$$R^{\theta}{}_{\theta} = R^{\varphi}{}_{\varphi} = \frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB} + \frac{A'C'}{AC} + \frac{C'^2}{C^2}\right) - \frac{1}{C^2}$$

coming because of 2 sphere.

$$R^{r}{}_{r} = \frac{1}{B^2}\left(\frac{A''}{A} + 2\frac{C''}{C} - \frac{B'}{B}\left(\frac{A'}{A} + \frac{2C'}{C}\right)\right)$$

Gauge: $C = r$: area gauge.

$B = 1$: proper distance & r.

$A = \frac{1}{B}$: Schwarzschild Gauge.

$C = rB$: "ADM" gauge

$\boxed{\text{Area Gauge}}$ $C = r$, $C' = 1$, $C'' = 0$.

~~$R^{t}{}_{t}$~~ $G^{tt} = \frac{1}{r^2} - \frac{1}{r^2}\left(\frac{r}{B}\right)' = 8\pi G T^0_0$

$$\Rightarrow B^{-2} = 1 - \frac{2G}{r}\int 4\pi r^2 \underbrace{T^0_0\, dr}_{}$$

Energy between r & $r+dr$

The integral tells how much energy you get inside radius r (in some sense.

~~to~~ We can then interpret it as $M(r) = \int 4\pi r^2 T^0_0\, dr$

$B^{-2} = 1 - \frac{2GM(r)}{r}$

Vacuum Solution ; $T^0_0 = 0 \Rightarrow B^{-2} = 1 - \frac{2GM}{r}$

M an integration constant.

$R^t_t - R^r_r = \frac{2}{r}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0$ is vacuum

$\Rightarrow A' \propto 1/B \Rightarrow$ Schwarzschild ~~soln~~ solution.

$A^2 = B^{-2} = 1 - \frac{2GM}{r}$

~~What~~ What about Cosmological Constant Λ?

Cosmological Constant. $8\pi G T_{ab} = \Lambda g_{ab}$

$R^t_t = R^r_r$ even when we get cosmological constant.

$A \propto \frac{1}{B} \Rightarrow \int 4\pi r^2 T^0_0 = \frac{\Lambda}{2G}\int r^2 = \frac{\Lambda r^3}{6G}$

$$\boxed{A^2 = B^{-2} = 1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2}$$

de-Sitter has $\Lambda > 0$; $\Lambda = 3/L^2$

get 2 horizons $r \simeq 2GM$

$r \simeq L$ (at large values of r)

Cosmological Horizon at $r \sim L$

$M = 0$ gives pure d.S. ; $A^2 = 1 - \frac{r^2}{L^2}$

dS is constant curvature spacetime.

Represent ~~as 4D~~ as a 4D hyperboloid in 5D (flat) Minkowski Spacetime.

$$X^2 + Y^2 + Z^2 + U^2 - T^2 = L^2$$

The metric on dS. is inherited by projecting down the 5 dimensional flat metric onto this 4d hyperboloid.

For static slicing
$$T = L\sqrt{1 - \frac{r^2}{L^2}}\,\sinh(t/L)$$
$$U = L\sqrt{1 - \frac{r^2}{L^2}}\,\cosh(t/L)$$
$$\underline{X} = r\,\underline{n}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

FRW : global

$$\left.\begin{aligned} T &= L\sinh(\tau/L) \\ U &= L\cosh(\tau/L)\cdot\cos\chi \\ \underline{X} &= L\cosh(\tau/L)\cdot\sin\chi\cdot\underline{n} \end{aligned}\right\}\ K = 1 \text{ cosmology}$$

STATIC → FLAT
$$\tau = t + \frac{L}{2}\log\left(1\ r^2/L^2\right)$$
$$\rho = \frac{r e^{-t/L}}{\sqrt{1 - r^2/L^2}}$$

Lec 5: Black holes, horizons and Causal Structure.

Causal structure is a key feature of black holes: how do we encode this?

$$ds^2 = \left(1-\frac{2GM}{r}\right)dt^2 - \left(1-\frac{2GM}{r}\right)^{-1}dr^2 - r^2 d\Omega^2$$

$$\boxed{r_+ = 2GM}$$

Kruskals explicitly extended across horizon at r_+

① TORTOISE COORDINATES

$$r^* = \int \frac{dr}{1-\frac{2GM}{r}} = r + 2GM \log\left(\frac{r-2GM}{2GM}\right)$$

or $\boxed{r^* = r + r_+ \log\left(\frac{r-r_+}{r_+}\right)}$

Then; $ds^2 = \left(1-\frac{2GM}{r}\right)dt^2 - \left(1-\frac{2GM}{r}\right)dr^{*2} - r^2 d\Omega^2$

→ radial null geodesics $t = \pm r^* + \text{constant}$.

KRUSKAL:
$$U = -r_+ \exp\left(-\frac{(t-r^*)}{2r_+}\right)$$

$$V = r_+ \exp\left(\frac{t+r^*}{2r_+}\right)$$

$$dU\,dV = r_+^2 \exp\left[\frac{t+r^*}{2r_+} - \frac{t-r^*}{2r_+}\right]\cdot\frac{(dt^2 - dr^{*2})}{4r_+^2}$$

$$\Rightarrow dU\,dV = \frac{1}{4}e^{r^*/r_+}\left[dt^2 - \frac{dr^2}{\left(1-\frac{r_+}{r}\right)^2}\right]$$

$$\downarrow$$

$$e^{\frac{r}{r_+}}\cdot\left(1-\frac{r_+}{r}\right)\cdot\frac{r}{r_+}$$

so; $dU\,dV$ is first bit of Schwarzschild metric upto the exponential factor.

$$dU\,dV = \frac{e^{r/r_+}}{4}\left[\left(1-\frac{r_+}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{r_+}{r}\right)}\right]$$

$$\Rightarrow \boxed{ds^2 = \frac{r_+}{r}4\,dU\,dV\;e^{-r/r_+} - r^2 d\Omega^2_{II}} \Longrightarrow$$ This metric is regular at r_+

$$dU\,dV = \frac{e^{r/r_+}}{4}\cdot\left[\left(1-\frac{r_+}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{r_+}{r}\right)}\right]$$

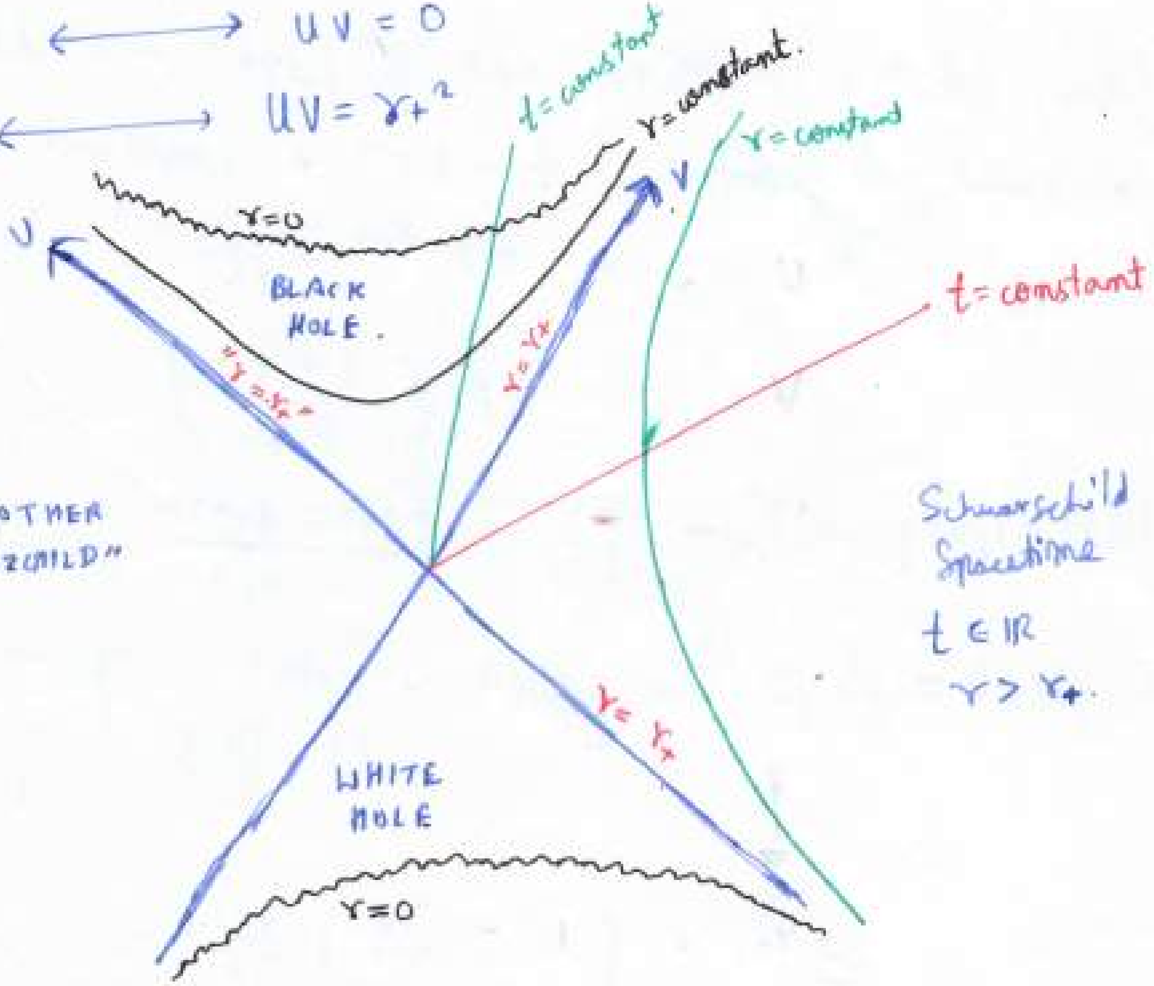
$$\boxed{ds^2 = 4\frac{r_+}{r}\cdot e^{-r/r_+}\,dU\,dV - r^2 d\Omega^2_{II}}$$

$r =$ constant. $\longleftrightarrow$ $UV =$ constant.

$t =$ constant $\longleftrightarrow$ $U/V =$ constant.

$r = r_+$ $\longleftrightarrow$ $UV = 0$

$r = 0$ $\longleftrightarrow$ $UV = r_+^2$



Null lines @ 45°.

Penrose - Carter diagrams

Compact representation of Black Hole.

Now define $p = \arctan V/r_+$, $q = \arctan U/r_+$ } maps U, V to finite range in p, q

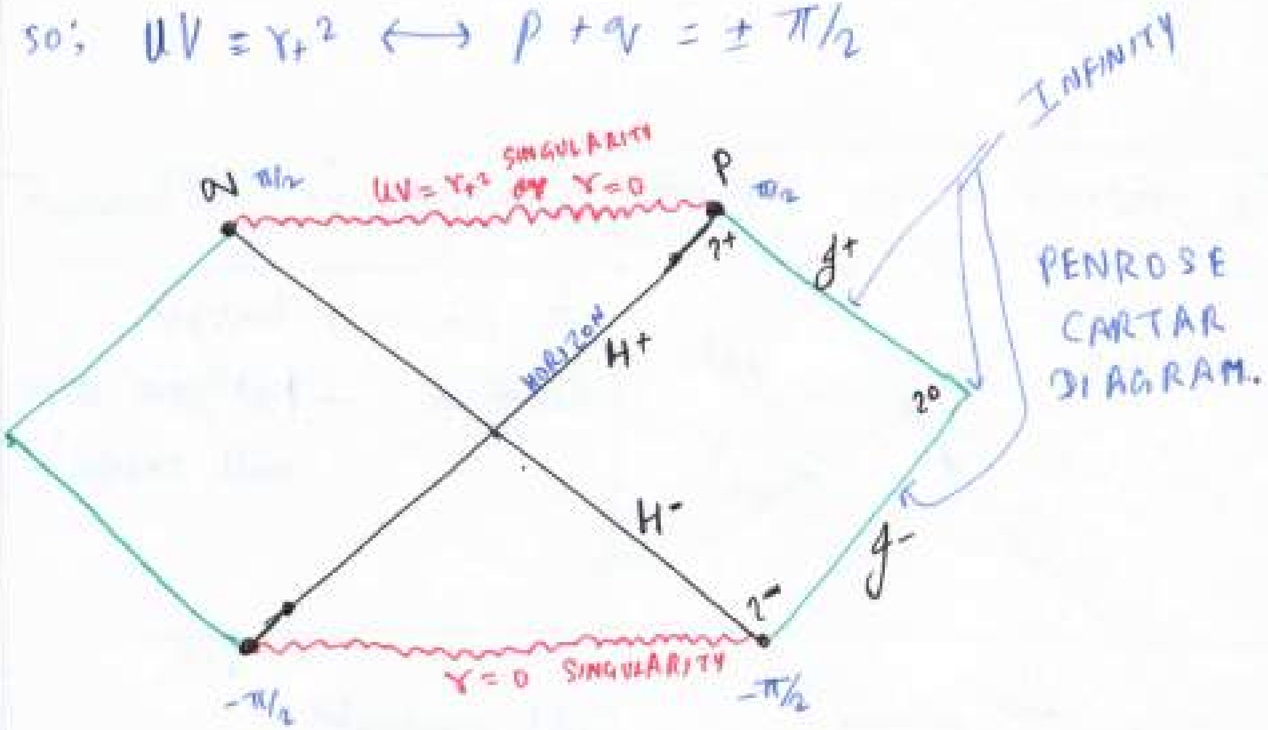
$V = 0 \longleftrightarrow p = 0$

$V = r_+ \longleftrightarrow p = \pi/4$

$V \to \infty \longleftrightarrow p = \pi/2$, etc.

$$\tan(p+q) = \frac{\tan p + \tan q}{1 - \tan p \tan q} = \frac{U+V}{r_+^2 - UV}$$

so; $UV = r_+^2 \longleftrightarrow p + q = \pm \pi/2$



$H^\pm \equiv$ event horizon $(\pm)$ / (future/past)

$i^+ \equiv$ Future timelike ∞

$i^- \equiv$ Past " ∞

$i^0 \equiv$ Spacelike ∞ (this is where you get; if you go away from black hole on special geodesic)

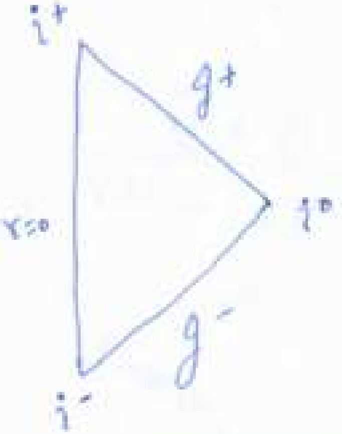
$\mathcal{J}^\pm \equiv$ Future/Past Null ∞.

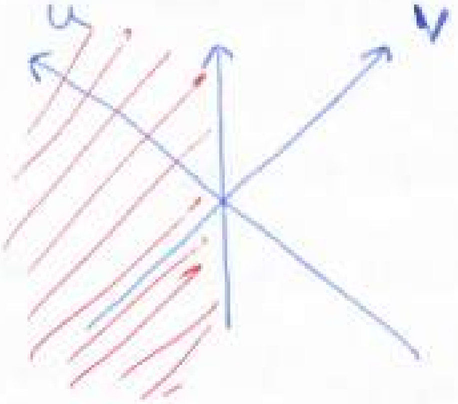
Event Horizon is boundary of the causal past of future timelike infinity.

MINKOWSKI $dt^2 - dr^2 - r^2 d\Omega^2$

$u = t - r$, $V = t + r$

$r > 0$, $V > u$



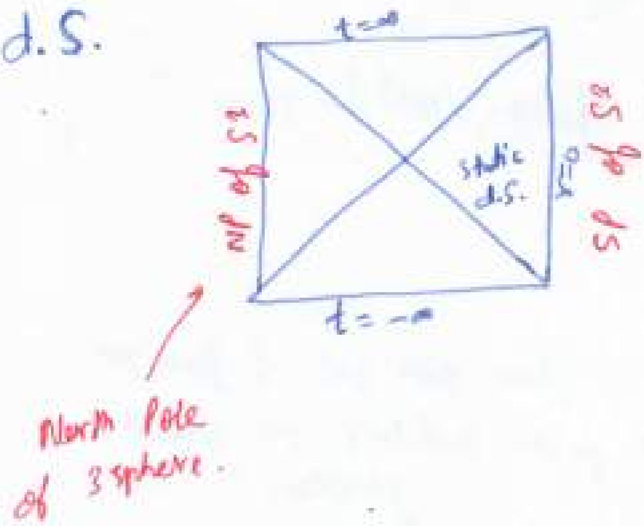


~~To construct Penrose diagram → horizon are "null crosses"~~

d.S.

To construct Penrose diagram – horizon are "null crosses"

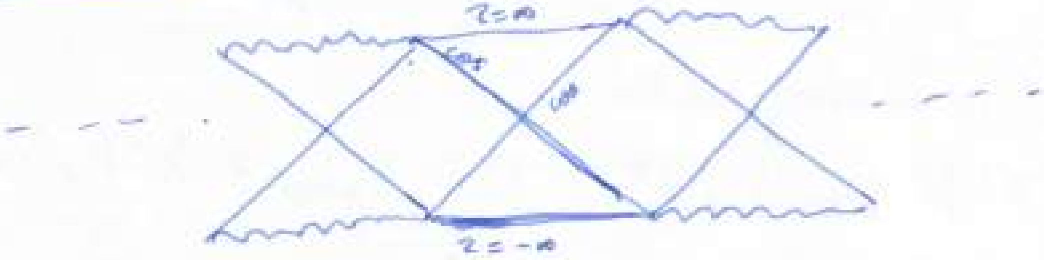
d.S.



Global coordinates

$ds^2 = dz^2 - \cosh^2(\frac{z}{2}) d\Omega^2_{\mathrm{III}}$

Schwarzschild - dS



analytically extend

Kerr Black Hole

Astro black holes rotate.

$$ds^2 = \left(1 - \frac{2GMr}{\Sigma}\right)dt^2 + 4GMar\sin\theta\, dt\, d\varphi$$
$$- \left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right]\frac{d\varphi^2}{\Sigma} - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2$$

$\Sigma = r^2 + a^2\cos^2\theta$ Boyer - Lindquist coordinates.

$\Delta = r^2 + a^2 - 2GMr$

$a = J/M$

$$ds^2 = \frac{\Delta}{\Sigma}\left[dt - a\sin^2\theta\, d\varphi\right]^2 - \frac{\sin^2\theta}{\Sigma}\left[a\,dt - (r^2+a^2)\,d\varphi\right]^2$$
$$- \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2$$

$$\Delta = 0 \longleftrightarrow r = r_{\pm} = GM \pm \sqrt{G^2M^2 - a^2}$$

notice ; $r_+ < 2GM$

$$\left\|\frac{\partial}{\partial t}\right\|^2 = r^2 + a^2\cos^2\theta - 2GMr$$

$$\left\|\frac{\partial}{\partial t}\right\|^2 = 0 \Leftrightarrow r_e = GM \pm \sqrt{G^2M^2 - a^2\cos^2\theta} \geq r_+$$

If event horizon is $\Delta = 0$ then; for $r_+ < r \leq r_e$ it is impossible to remain at rest (w.r.t. ∞) : ERGOSPHERE

Explore geodesics for $\theta = \pi/2$ (looking at equitorial geodesics)

Note, if K^μ is a killing vector.

$$\frac{d}{dt}[k_\mu \dot{x}^\mu] = \dot{x}^\nu \nabla_\nu (k_\mu \dot{x}^\mu) = \dot{x}^\nu \dot{x}^\mu \nabla_\nu k_\mu + \dot{x}^\nu k_\mu \nabla_\nu \dot{x}^\mu$$

Zero because killing vector has anti-symmetric covariant derivative.

Zero because geodesics.

$= 0$

$\rightarrow$ $k_\mu \dot{x}^\mu$ is conserved quantity.

Kerr has 2 killing vectors ∂_t, ∂_φ

$$k_t = \frac{\partial}{\partial t} \leftrightarrow \cancel{E = g_{t\mu}\dot{x}^\mu} \; E = g_{t\mu}\dot{x}^\mu$$
$$= \left(1 - \frac{2GM}{r}\right)\dot{t} + \frac{2GMa}{r}\dot{\varphi} \quad \text{at } \theta = \frac{\pi}{2}$$

$$k_\varphi = \frac{\partial}{\partial\varphi} \longleftrightarrow h = -g_{\varphi\mu}\dot{x}^\mu = \left(r^2 + a^2 + \frac{2GMa^2}{r}\right)\dot{\varphi} - \frac{2GMa}{r}\dot{t}$$
at $\theta = \pi/2$

angular momentum conserved quantity.

Now define; $P_\alpha = (E, -h) = g_{\alpha\beta}\dot{x}^\beta \quad (\alpha, \beta \in \{t, \varphi\})$

then, $P_\alpha P_\beta g^{\alpha\beta} = g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$

$$= E^2 \cdot g^{tt} - 2Eh \cdot g^{t\varphi} + h^2 g^{\varphi\varphi}$$
$$\rightarrow \frac{1}{\Delta}\left[E^2 \cdot (r^2 + a^2) + \frac{2GM}{r}(h - aE)^2 - h^2\right]$$

Geodesic equation: $P^2 + g_{rr}\dot{r}^2 = \eta$ — 1 for timelike / — 0 for null

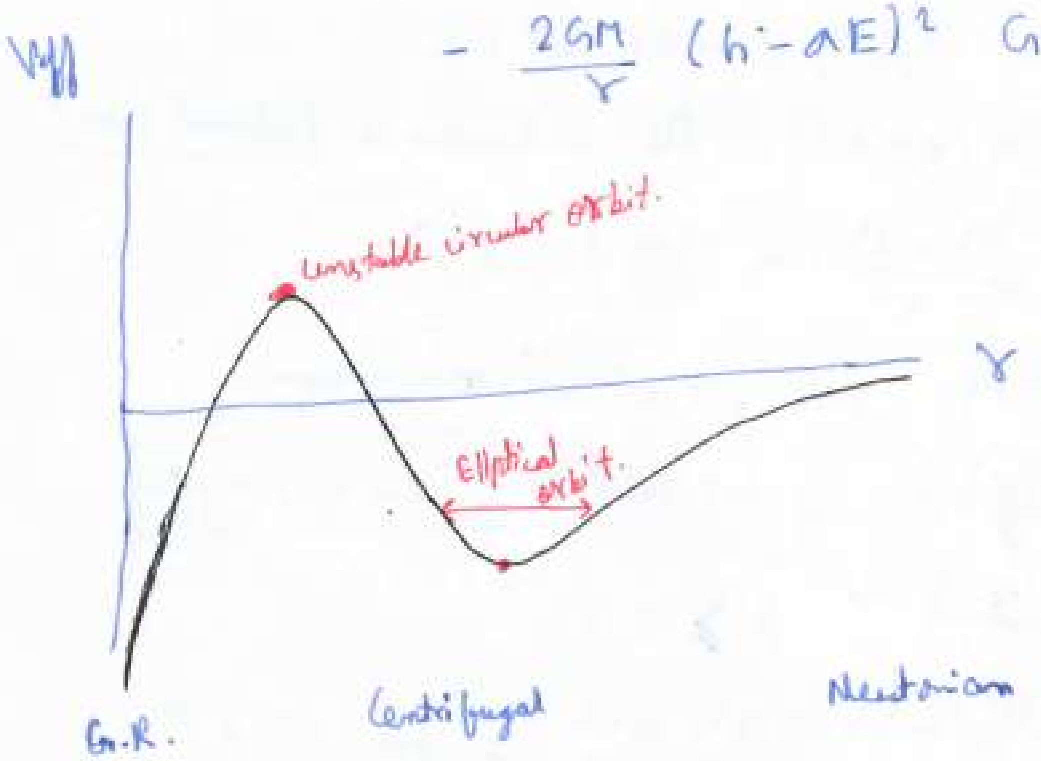
has form $\dot{r}^2 + V_{eff}(r) = 0$

$$\hookrightarrow V_{eff} = \eta \frac{\Delta}{\Sigma} - P^2 \frac{\Delta}{\Sigma}$$

$$= \eta^2 - E^2 - \frac{2GM\eta}{r} \quad \text{Newton}$$

$$+ \frac{a^2\eta + h^2 - a^2E^2}{r^2} \quad \text{Centrifugal}$$

$$- \frac{2GM}{r}(h - aE)^2 \quad \text{GR/KERR}$$



Lec 6: Rotating black holes & black hole thermodynamics

More on black holes.

Recall $\dot{r}^2 + V_{eff}(r) = 0$

$$\theta = \frac{\pi}{2} \;;\quad \eta^2 - E^2 - \frac{2GM\eta}{r} + \frac{a^2(\eta - E^2) + h^2}{r^2} - \frac{2GM}{r^3}(h - aE)^2$$

as $\Delta = r^2 + a^2 - 2GMr \to 0$

Look at Null geodesics

$\eta = 0$, $E = 1$ (by adjusting affine parameters)

for algebraic simplicity; take $GM = a$.

ie; $\Delta = (r-a)^2$ this is known as Extremal Limit.

$$1 + V_{eff} = \frac{h^2 - a^2}{r^2} - \frac{2a}{r^3}(h-a)^2$$

Initial conditions: $\dot{r} = 0$, $r = r_m$ (r minimum)

$\dot{r} = 0 \Rightarrow V_{eff} = 0$; has solution $\boxed{r_m = h - a}$

Since $r_m \geq a \Rightarrow h \geq 2a$

Now check
$$V'_{eff}(r_m) = \frac{2(h^2 - a^2)}{r_m^3} + \frac{6a(h-a)^2}{r_m^4}$$
$$= \frac{4a - 2h}{r_m^2} \leq 0$$

$\Rightarrow$ hence r increases.

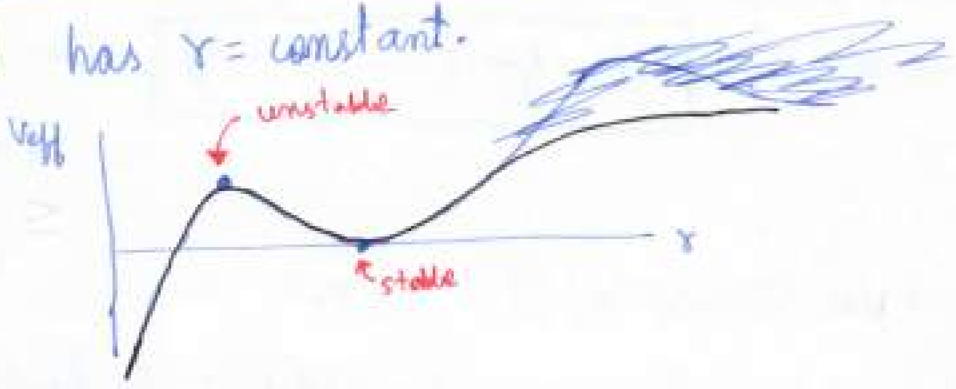
Limit at $h = 2a$, $r_m = a$.

Hence $\Delta = 0$ is event horizon.

Black Holes do not exist in vacuum. Often have an accretion disc - modeled by concentric circular orbits.

Circular Orbit has $r =$ constant.

$V_{eff} = V'_{eff} = 0$



Coordinate Picture

For simplicity take $a = 0$, $\eta = 1$, $\theta = \pi/2$

$$V = 1 - E^2 - \frac{2GM}{r} + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3}$$

$$V' = \frac{2GM}{r^2} - \frac{2h^2}{r^3} + \frac{6GMh^2}{r^4}$$

$$V'' = -\frac{4GM}{r^3} + \frac{6h^2}{r^4} - \frac{24GMh^2}{r^5}$$

$V = V' = 0$ determines h, E at given r.

$$h^2 = \frac{GMr^2}{r - 3GM} \quad ; \quad E^2 = \frac{r - 2GM}{r} + \frac{GM(r - 2GM)}{r(r - 3GM)}$$

$$V'' = \frac{2GM}{r^3} \cdot \frac{r - 6GM}{r - 3GM} \geq 0 \text{ for } r > 6GM$$

$r = 6GM$ is the limit of stable circular orbits.

Innermost Stable Circular Orbit - ISCO

For Kerr $r_{ISCO} < 6GM$

Extremal Kerr; $r_{ISCO} = a = r_H$.

Change track!

For Kerr Black Hole; event horizon at $r_+ = GM + \sqrt{G^2M^2 - a^2}$

Its area is $\boxed{A = 4\pi(r_+^2 + a^2)} = 4\pi\, 2GM r_+$

$= 8\pi GM r_+$

$\leq$ Area of Schwarzschild Black Hole.

If it was schwarzschild; $A = 4\pi r_+^2$

"Once a Black Hole starts to rotate; its horizon shrink"

$$\delta A = 8\pi(r_+ \delta r_+ + a\,\delta a)$$

$$\Rightarrow \delta A = 8\pi(r_+ \delta r_+ + a\,\delta a)$$

$$= 8\pi\left(r_+\left[G\delta M + \frac{G^2 M \delta M - a\delta a}{r_+ - GM}\right] + a\,\delta a\right)$$

$$\Rightarrow \delta A = 8\pi\left[\frac{r_+^2 G\delta M}{r_+ - GM} - \frac{GMa\,\delta a}{r_+ - GM}\right]$$

recall, $a = J/M \Rightarrow \delta a = \frac{\delta J}{M} - \frac{J}{M^2}\delta M$

$$\boxed{\delta a = \frac{\delta J}{M} - a\frac{\delta M}{M}}$$

$$\Rightarrow \frac{\delta A}{8\pi} = \frac{r_+^2 G\,\delta M + G\cdot a^2\,\delta M}{r_+ - GM} - \frac{Ga\,\delta J}{r_+ - GM}$$

$$\Rightarrow \frac{\delta A}{8\pi} = \frac{G(r_+^2 + a^2)}{r_+ - GM}\delta M - \frac{G(r_+^2 + a^2)}{r_+ - GM}\cdot\frac{a}{r_+^2 + a^2}\delta J$$

let $\Omega = \lim_{r \to r_+} - \frac{g_{\varphi t}}{g_{\varphi\varphi}}$ Angular Velocity of Event Horizon.

ANGULAR VELOCITY OF Black Hole $= \frac{a}{r_+^2 + a^2}$

Rewrite the differential equations as.

$$\Rightarrow \quad \delta M = \frac{r_+ - GM}{2\pi(r_+^2 + a^2)} \cdot \frac{\delta A}{4G} + \Omega \, \delta J$$

compare it, to $dU = T\,dS - p\,dV + \mu\,dQ$
First Law of Thermodynamics.

Suggest thermodynamic interpretation with
$M \longleftrightarrow$ Energy *

$$T = \frac{r_+ - GM}{2\pi(r_+^2 + a^2)} \quad ; \quad S = \frac{A}{4G}$$

Then $\boxed{\delta M = T\,\delta S + \Omega\,\delta J}$

For Schwarzschild; $T = \frac{1}{8\pi GM}$

Explore a bit more...
Take Schwarzschild Solution, $t \to i\tau$

$$|ds^2| = \left(1 - \frac{2GM}{r}\right) d\tau^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \, d\Omega_{II}^2$$

This is ~~[illegible]~~ Ricci flat Euclidean Solution.

What happens at $2GM$?

Try to find proper distance from $2GM$.

$\rho^2 = \lambda(r - 2GM)$

$2\rho\, d\rho = \lambda\, dr$ $\qquad 1 - \frac{2GM}{r} \simeq \left.\frac{\rho^2}{\lambda \cdot 2GM}\right|_{r \to 2GM}$

$$ds^2_{E,\,\tau,\rho} = \frac{4\rho^2}{\lambda^2} \frac{d\rho^2}{\rho^2/2GM\lambda} + \frac{\rho^2}{2GM\lambda} d\tau^2$$

$\Rightarrow \quad \lambda = 8GM \quad$; then

$$\Rightarrow \boxed{dS^2_E = d\rho^2 + \rho^2\, d\left(\frac{\tau}{4GM}\right)^2}$$

$$\text{"}\theta\text{"} = \frac{\tau}{4GM}$$

Origin of $\mathbb{R}^2$ in polars. And suggest τ is periodic with periodicity $8\pi GM$.

- Signal of field theory: at finite T, $\Delta\tau = \beta = \frac{1}{T}$

$$\boxed{T = \frac{1}{8\pi GM}}$$

Other charges? $\quad J \longleftrightarrow \Omega \quad$ Kerr

for Reisner Nordm; RN ; $\quad Q \longleftrightarrow \Phi = -Q/r_+$

What about PdV?

Pressure could be cosmological constant Λ?

$g_{tt} = 1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2 = f(r)$

Ex) Check Euclidean argument ($\rho^2 \propto (r - r_+)$) gives $T = \frac{f'_+}{4\pi}$

Horizon defined as $f(r_+) = 0$

$$(f+\delta f)(r_+ + \delta r_+) = 0$$

$$\Rightarrow \delta r_+ \cdot f_+' - \frac{2G\,\delta M}{r_+} - \frac{\delta\Lambda}{3} \cdot r_+^2 = 0$$

We will define pressure $\boxed{P = \frac{\Lambda}{8\pi G}}$ Cosmological constant tells about pressure.

$$\Rightarrow \delta M = T \cdot \frac{\delta A}{4G} + \frac{r_+^3}{6G}\delta\Lambda = T\delta S + \underbrace{\frac{4\pi}{3} r_+^3}_{V} \delta P$$

$$\boxed{V = \frac{4\pi}{3} r_+^3} \Rightarrow \boxed{\delta M = T\delta S + V\delta P}$$

may not be actual volume; but it is geometric volume inside event horizon.

This suggest M is enthalpy.

How Λ can vary; it was a constant.

$$T_{\varphi\,\mu\nu} = \partial_\mu \varphi\, \partial_\nu \varphi - g_{\mu\nu}\left(\frac{1}{2}(\partial\varphi)^2 - V\right)$$

If potential $V \gg |\partial\varphi|^2$

i.e. if fields move slowly; then $T_{\varphi\,\mu\nu}$ is potential dominated.

$T_{\mu\nu} \simeq V g_{\mu\nu}$ but if φ rolls slowly then V changes $\Rightarrow \Lambda$ changing.

$\Downarrow$

P changing.

This is Inflation.

This is how we can get varying Λ.

Lecture 7: Gravitational Action: Palatini Lemma, Gauss-Codazzi formalism, Gibbons-Hawking term.

Have seen the Einstein-Hilbert action.

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x \, \sqrt{g}\cdot R$$

review how this works

Vary using inverse metric: (use δg^{ab}).

$$\det(g) = \exp(\mathrm{tr}(\log g))$$

$$\delta \det(g) = \det(g)\cdot \underbrace{\mathrm{tr}\; g^{-1}\delta g}_{-\mathrm{tr}\, g_{ab}\delta g^{ab}}$$

$$\Rightarrow \delta \det(g) = -\det(g)\cdot \mathrm{tr}(g_{ab}\cdot\delta g^{ab})$$

$$R = R_{ab}g^{ab}$$

$$\Rightarrow \delta R = R_{ab}\,\delta g^{ab} + g^{ab}\,\delta R_{ab}$$

In coordinate basis: $\delta R_{ab} = \delta R^c{}_{acb}$

$$= \delta\left[\partial_c\Gamma^c_{ab} - \partial_b\Gamma^c_{ac} + \Gamma^c_{ce}\Gamma^e_{ba} - \Gamma^c_{be}\Gamma^e_{ca}\right]$$

$$= \partial_c\,\delta\Gamma^c_{ab} - \partial_b\delta\Gamma^c_{ac} + \delta\Gamma^c_{ce}\,\Gamma^e_{ba} + \Gamma^c_{ce}\,\delta\Gamma^e_{ba} - \delta\Gamma^c_{be}\,\Gamma^e_{ca} - \Gamma^c_{be}\,\delta\Gamma^e_{ca}$$

$$\Rightarrow \boxed{\delta R_{ab} = \nabla_c\,\delta\Gamma^c{}_{ab} - \nabla_b\,\delta\Gamma_a{}^c{}_c}$$

Palatini's Lemma

To find $\delta\Gamma$, choose Normal Coordinates (which correspond to local inertial frame)

(local inertial frame) these set Γ, but not $\partial\Gamma$, $= 0$ at a point: ie; $g_{ab,c} = 0$, not $g_{ab,cd} = 0$

$$\delta\Gamma^a_{\ bc} = \cancel{\frac{1}{2}\delta g^{ae}\left(g_{eb,c} + g_{ec,b} - g_{bc,e}\right)} \; 0 \text{ (NC)}$$
$$+\frac{1}{2}g^{ac}\left(\delta g_{eb,c} + \delta g_{ec,b} - \delta g_{bc,e}\right)$$
$$= \frac{1}{2}g^{ac}\left(\delta g_{eb,c} + \delta g_{ec,b} - \delta g_{bc,e}\right)$$

In N.C. : $\partial_a \equiv \nabla_a$

so; we can re-write.

$$\delta\Gamma^a_{\ bc} = \frac{1}{2}g^{ac}\left(\nabla_c\,\delta g_{eb} + \nabla_b\,\delta g_{ec} - \nabla_e\,\delta g_{bc}\right).$$

(This is covariant geometric expression)

↳ Since it is true in one coordinate frame ; i.e: N.C.
and it is a tensorial expression.
so: it is true in general.

$$\delta\Gamma^a_{\ bc} = \frac{1}{2}g^{ac}\left(\nabla_c\,\delta g_{eb} + \nabla_b\,\delta g_{ec} - \nabla_e\,\delta g_{bc}\right)$$

$$\delta\Gamma^a_{\ bc} = \frac{-1}{2}\left(\nabla_c\,\delta(g^{-1})^a{}_b + \nabla_b\,\delta(g^{-1})^a{}_c - \nabla^a\delta(g^{-1})_{bc}\right)$$

$\Rightarrow$ ~~$\delta R_{ab} = \frac{1}{2}(\Box$~~

$$\delta R_{ab} = \frac{1}{2}\left(\Box\,\delta(g^{-1})_{ab} + \nabla_a\nabla_b\,\delta(g^{-1}) - \nabla_c\nabla_b\,\delta(g^{-1})^c{}_a - \nabla_c\nabla_a\,\delta(g^{-1})^c{}_b\right)$$

$$\boxed{\begin{aligned}\delta R &= \Box\left(g_{ab}\,\delta g^{ab}\right) - \nabla_a\nabla_b\,\delta g^{ab} \\ &= \nabla_c\left(\nabla^c\left(g_{ab}\,\delta g^{ab}\right) - \nabla_a\,\delta g^{ac}\right)\end{aligned}}$$

Integrating up the variation of S_{EH} gives

$$\delta S_{EH} = \frac{-1}{16\pi G}\int d^4x\sqrt{g}\cdot\left(R_{ab} - \frac{1}{2}R g_{ab}\right)\delta g^{ab} - \frac{1}{16\pi G}\int_{\partial M} d^3y\sqrt{g}\; n_a\left(\nabla^a\delta g^{-1} - \nabla_b\,\delta g^{ab}\right)$$

normal

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$$\delta S_{EH} = \frac{-1}{16\pi G}\int_M d^4x\sqrt{g}\left(R_{ab} - \frac{1}{2}R\cdot g_{ab}\right)\delta g^{ab} \underbrace{- \frac{1}{16\pi G}\int_{\partial M} d^3x\sqrt{g}\cdot n_a\cdot\left(\nabla^a\delta g^{-1} - \nabla_b\delta g_{ab}\right)}_{\text{Boundary depends on } \nabla\delta g.}$$

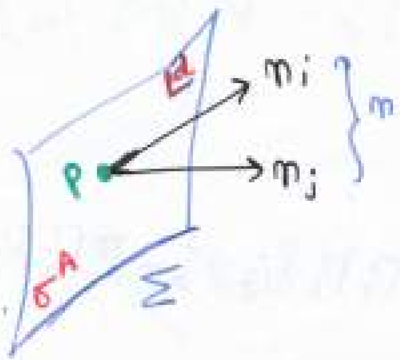
normal

Usual E.L rules would set $\delta g = 0$ on ∂M.

How to deal with a boundary?

Need to be able to describe submanifolds.

Gauss - Codazzi Formalism



Let $\Sigma \subset M$, be a manifold, dim d.

$\dim(M) = D = d + n$.

σ^A coordinates on Σ

X^μ " " M

$\mu = \{0, 1, 2, \ldots, (D-1)\}$
$A = \{0, 1, 2, \ldots, (d-1)\}$

Definition The co-dimension of Σ is $n = D - d$, & there exists n linearly independent normal vectors n_i^μ to Σ.

$$\{n_i\}_{i=1,\ldots n} \in T_p(M)$$

s.t. $n_i(\sigma^A) = 0$ $\forall$ coordinate functions on Σ

Alternatively $n_{i\mu} \frac{\partial X^\mu}{\partial \sigma^A} = 0 \quad ; i = 1, \ldots, n$

Choose $n_{i\mu} n_j{}^\mu = \pm \delta_{ij}$ (we can always choose these by choosing orthogonals ; and then re-scalling to get ortho-normal).

n^μ timelike ; $n^2 = +1$ ← (ADM if $n = 1$)

Assume $n_i{}^\mu$ spacelike ~~for~~ $\forall\, i$; then $n_i{}^2 = -1$.

~~i.e.~~ i.e. $\boxed{n_{i\mu} n_j{}^\mu = -\delta_{ij}}$

Definition The 1st Fundamental Form or induced metric on Σ is $h_{ab} = g_{ab} + n_{ia} n_{ib}$

← This is the metric which Σ inherits from M.

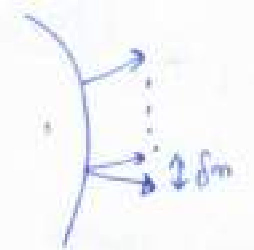
note ; $h_{ab} \in T^*(M) \otimes T^*(M)$

Can also express as element of $T^*(\Sigma) \otimes T^*(\Sigma)$ via

$\frac{\partial X^\mu}{\partial \sigma^A} : T^*(M) \to T^*(\Sigma)$ Pull Back.

$$\boxed{\gamma_{AB} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B}}$$

Definition The 2nd fundamental form or Extrinsic curvature measures how Σ curves in M. $K_{i\mu\nu} = h_{(\mu}{}^\sigma h_{\nu)}{}^\lambda \nabla_\sigma n_{i\lambda}$



(a) $$K_{i\,AB} = \frac{\partial X^\mu}{\partial \sigma^A} \cdot \frac{\partial X^\nu}{\partial \sigma^B} \nabla_{(\mu} n_{i\,\nu)}$$

$$= -n_{i\mu} D_A \cdot \frac{\partial X^\mu}{\partial \sigma^B}$$

where D is the connection inherited by Σ.

$$\boxed{D_\mu V_\nu = h_\mu{}^\lambda h_\nu{}^\sigma \nabla_\lambda V_\sigma} \quad (V_\mu n^\mu = 0)$$

Definition The <u>Normal Fundamental Form</u>

$$\beta_{\mu\, ij} = n_{i\nu} \nabla_\mu n_j{}^\nu$$

Can decompose the curvature of Σ in terms of curvature of M + extrinsic curvature.

<u>Gauss Equation</u>

$$R_{(d)}{}^a{}_{bcd} = h^a{}_{a'} h^{b'}{}_b h^{c'}{}_c h^{d'}{}_d R_{(D)}{}^{a'}{}_{b'c'd'} + K_i{}^a{}_c K_{i\,bd} - K_i{}^a{}_d K_{i\,bc}$$

<u>Gauss Equation</u>

$$\boxed{R_{(d)}{}^a{}_{bcd} = h^a{}_{a'} h^{b'}{}_b h^{c'}{}_c h^{d'}{}_d R_{(D)}{}^{a'}{}_{b'c'd'} + K_i{}^a{}_c K_{i\,bd} - K_i{}^a{}_d K_{i\,bc}}$$

Prove using Reiman identity $(D_a D_b - D_b D_a) \ldots$

Now consider $n = 1$. Σ is a "wall".

Extend n^μ geodesically into bulk $n^\mu n^\nu g_{\mu\nu} = 1$, $\nabla_n n^\mu = 0$

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

Then $\delta(\eta^\mu \eta^\nu g_{\mu\nu}) = 0$

$$\Rightarrow \boxed{2\eta_\mu \delta\eta^\mu = \eta_\mu \eta_\nu \delta g^{\mu\nu}}$$ ⇒ Relation between $\delta\eta$ projected along η; & normal variation of $g_{\mu\nu}$

$\delta(\eta^\mu h_{\mu\nu}) = 0$ (because we need η^μ to be normal
ie; $\eta^\mu h_{\mu\nu} = 0$
$\Rightarrow \delta(\ ) = 0$)

$$\Rightarrow \boxed{h_{\mu\nu} \delta\eta^\mu = \eta_\mu \delta h^{\mu\sigma} g_{\sigma\nu}}$$ ⇒ gives Relation in terms of parallel component of $\delta\eta^\mu$ & δh

but; $h^{\mu\nu} = g^{\mu\nu} + \eta^\mu \eta^\nu$

$$\Rightarrow (g_{\mu\nu} + \cancel{\eta_\mu\eta_\nu})\delta\eta^\mu = \eta_\mu(\delta g^{\mu\sigma} + \cancel{\delta\eta^\mu}\eta^\sigma + \eta^\mu \delta\eta^\sigma) g_{\sigma\nu}$$

$$\Rightarrow \boxed{\delta\eta^\mu = \frac{1}{2}\eta_\nu \delta g^{\mu\nu}}$$ ⇒ Variation of normal is related to variation of metric.

Consider $\delta K = \delta(\nabla_\mu \eta^\mu) = \nabla_\mu \delta\eta^\mu + \delta\Gamma^\mu_{\mu\lambda}\eta^\lambda$

$$= \frac{1}{2}\nabla_\mu(\eta_\nu \delta g^{\mu\nu}) - \frac{1}{2}\eta^\lambda \nabla_\lambda \delta g^\mu{}_\mu$$

$$= \frac{1}{2}[\eta_\nu \nabla_\mu \delta g^{\mu\nu} - \nabla_\eta \delta g] + \frac{1}{2}\nabla_\mu \eta_\nu \delta g^{\mu\nu}$$

But $\nabla_\mu \eta_\nu = K_{\mu\nu}$ (because of our coordinate choice)

~~& $\delta g^{\mu\nu}\nabla_\mu \eta_\nu = \delta h^{\mu\nu}\nabla_\mu \eta_\nu$~~

$$\boxed{\delta g^{\mu\nu}\nabla_\mu \eta_\nu = \delta h^{\mu\nu} K_{\mu\nu}}$$

Boundary E-L Term

$$= \frac{-1}{16\pi G}\int d^3x \sqrt{g}\,[2\delta K - K_{ab}\delta h^{ab}]$$

Finally $\det h = \det g$; because $\eta^2 = -1$

Boundary term is then

$$= \frac{-1}{8\pi G}\,\delta \boxed{\int d^3x\,(\sqrt{h}\,K)} + \frac{1}{16\pi G}\int d^3x\,\sqrt{h}\,\delta h^{ab}\,(K_{ab} - K\cdot h_{ab})$$

→ GIBBONS - HAWKING BOUNDARY TERM.

no problem with this ∵ we have δh^{ab} ∴ this we can set to zero on boundary.

$$\boxed{S = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\,\sqrt{g}\cdot R \;-\; \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\,\sqrt{h}\,K}$$

→ gives

Einstein Equations.

Boundary (they cancel each other)

$K_{ab} - K h_{ab}$

ISRAEL EQUATION (if we keep the boundary dynamical)

Bonus Lecture 1: Gravity and field Theory Action principles, and area Domain walls and gravity.

In field theory, we use a Lagrangian formulation on the manifold; we need to be able to integrate.

$$d^4x \longleftrightarrow \text{coordinate volume}$$

but $d^4x \longrightarrow \left|\frac{\partial x}{\partial \tilde{x}}\right| d^4\tilde{x}$

However, $\sqrt{|g|}\, d^4x$ is invariant.

To vary this volume element;
use the identity $\det M = \exp(\mathrm{tr} \log M)$

$$\delta(\det M) = \det M \cdot \mathrm{tr}(M^{-1}\delta M)$$

$$\Rightarrow \delta(\sqrt{g}) = \frac{1}{2}\frac{(-1)}{\sqrt{g}} \cdot g \cdot \underbrace{g^{ab}\,\delta g_{ab}}_{\text{This is } \mathrm{tr}(g^{-1}\delta g)}$$

$$= -\frac{\sqrt{g}}{2} g_{ab}\,\delta g^{ab}$$

e.g. Massless Scalar field; $\mathcal{L}_\phi = \frac{1}{2}(\partial\phi)^2$

$$S_\phi = \int d^4x \sqrt{-g} \cdot \frac{1}{2}(\partial\phi)^2 = \int d^4x \sqrt{-g}\, \frac{1}{2}\phi_{,\mu}\phi_{,\nu} g^{\mu\nu}$$

$$\delta S_\phi = \int d^4x \Big[\sqrt{-g} \cdot \partial_\mu\phi\, \partial_\nu \delta\phi\, g^{\mu\nu} + \frac{1}{2}\sqrt{g}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right)\delta g^{\mu\nu}\Big]$$

~~Note; integrate by parts~~

$$\cancel{\delta S_\phi = \int_{\partial V} d^3x \sqrt{\ } (n^\mu \partial_\mu \phi) + \int d^4x \sqrt{g}\left[\frac{-1}{\sqrt{g}}\partial_\mu\right.}$$

Now integrate by parts

$\sqrt{g}$ corresponds to $\sqrt{-g}$..

$$\delta S_\phi = \int_{\partial V} d^3x \sqrt{g} \cdot (n^\mu \partial_\mu \phi\, \delta\phi) + \int d^4x \sqrt{g} \Big[\frac{-1}{\sqrt{g}} \partial_\mu (\sqrt{g} \partial^\mu \phi)\, \delta\phi$$
$$+ \frac{1}{2} \left[\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right] \delta g^{\mu\nu} \Big]$$

$$\frac{\delta S_\phi}{\delta \phi} = \frac{-1}{\sqrt{g}} \partial_\mu \sqrt{g} \cdot \partial^\mu \phi = -\Box \phi$$

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{2} \left[\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right] = \frac{1}{2} T_{\phi\,\mu\nu}$$

Symmetrized Energy Momentum tensor of ϕ.

$$\delta S_\phi = \int \underbrace{d^4x \sqrt{-g}}_{\text{"}dV\text{"}} \cdot \frac{\delta S}{\delta \phi} + \cdots$$

For gravitational action; we need scalar.

Normally when we write down lagrangian; we write $T - V$ (Kinetic − Potential)

~~so; we can imagine that~~

for gravity; we would like to write down kinetic term involving derivatives of g

$\Longleftrightarrow$ but there is a problem;

whatever we ~~want~~ write; must be covariant.

And we saw that; when writting covariant objects on manifold that :

we dont have any thing with first derivative of g which is covariant.

(The only think we know is Levi - Civita connection & that is not a tensor)

So, we cant construct first order kinetic term for gravity.

But ; we go forward & take second derivatives of g

i.e terms involving $\partial\partial g$ . . . like Ricci . . .

Take R (note this contains $\partial\partial g$)

$$R = R_{\mu\nu}\, g^{\mu\nu}$$

$$\delta R = \delta R_{\mu\nu}\, g^{\mu\nu} + R_{\mu\nu}\, \delta g^{\mu\nu}$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$

Compute $\delta R_{\mu\nu}$ in "normal coordinates"

(local Inertial frame) $\partial g = 0$; $\partial\Gamma \neq 0$

$$\delta R_{\mu\nu} = \delta\Gamma^{\lambda}{}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}{}_{\mu\lambda,\nu}$$

but $\underset{N.C.}{=} \delta\Gamma^{\lambda}{}_{\mu\nu;\lambda} - \delta\Gamma^{\lambda}{}_{\mu\lambda;\nu}$

Note this is tensorial equation because difference between connection is Tensor.

Recall $\delta\Gamma$ is a tensor

$$\delta R_{ab} = \nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac}$$

Convention
- using greek indices in Normal Coordinates (N.C.)
- using latin indices for general coordinates.

Rember $\partial g = 0$ in N.C.

$$\delta\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}(\delta g_{\sigma\nu,\lambda} + \delta g_{\sigma\lambda,\nu} - \delta g_{\nu\lambda,\sigma})$$

hence:

$$\delta\Gamma^a_{bc} = \frac{1}{2}[\nabla_b \delta g^a{}_c + \nabla_c \delta g^a{}_b - \nabla^a \delta g_{bc}]$$

$$\boxed{\delta\Gamma^a_{bc} = \frac{1}{2}[\nabla_b \delta g^a{}_c + \nabla_c \delta g^a{}_b - \nabla^a \delta g_{bc}]}$$

Hence $\delta R_{ab} = \frac{1}{2}[\nabla_c \nabla_a \delta g^c{}_b + \nabla_c \nabla_b \delta g^c{}_a - \nabla_c \nabla^c \delta g_{ab} - \nabla_b \nabla_a \delta g^c{}_c]$

Change to δg^{-1}.

Then $g^{\mu\nu}\delta R_{\mu\nu} = -\nabla_\mu \nabla_\nu \delta g^{\mu\nu} + g_{\mu\nu} \Box \delta g^{\mu\nu}$

This is a total derivative; but

The boundary term we get on integrating is

$$\int d^3x \; n_\mu \nabla_\nu \delta g^{\mu\nu} - n^\mu \nabla_\mu \delta g$$

we set derivatives of δg : not δg.

so; we need to set normal derivatives of δg to be zero.

We therefore write

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\cdot R$$

$$\& \ \frac{\delta S}{\delta g^{\mu\nu}} = -\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu}\right)$$

Combine with interacting scalar.

$$\mathcal{L}_\phi = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

take $V(\phi) = \frac{\lambda}{2}(\phi^2 - \eta^2)^2$

$$S_{Tot} = S_{EH} + S_\phi$$

$V(\phi)$ has two vacua $\phi = \pm\eta$

ϕ e.o.m. $\boxed{\Box\phi + 2\lambda(\phi^2 - \eta^2)\phi = 0}$

Ask that $\phi \to \pm\eta$ as $z \to \pm\infty$ in the Minkowski; in absence of G.R.

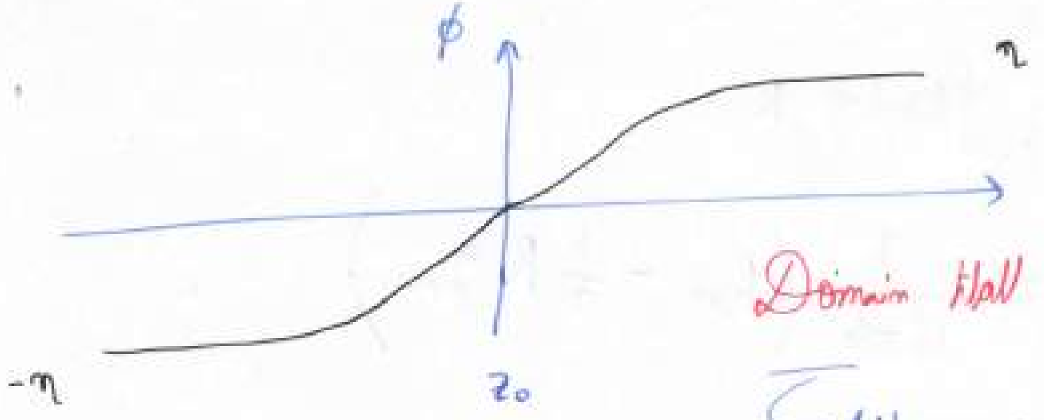
$\phi = \phi(z)$

Then $\phi = \eta \tanh[\sqrt{\lambda}\cdot\eta(z - z_0)]$ $\longrightarrow$ solution.

$\phi' = \sqrt{\lambda}\eta^2 \operatorname{sech}^2[\_\_\_\_]$

$\phi'' = -2\lambda\eta^2 \operatorname{sech}^2[\ ]\tanh[\ ]$

$= -2\lambda\eta^2(1 - \phi^2/\eta^2)\phi$ E.o.M.



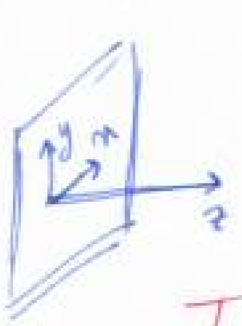
Domain Wall / Kink

(its an example of topological defect: Its a non-perturbative solution to field Theory)

Even if you try doing QFT; you cant: If you put the solution in background; you cant transition from this solution to either vacuum.

In order to go to the vacuum, the field will have to go say to $+\eta$ vacuum from $-\eta$, over an infinite amount of space: and this will cost infinite amount of energy.

So; This is Topologically stable.



$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}$$

$$T_{\mu\nu} = \phi'^2 \delta^z_\mu \delta^z_\nu + g_{\mu\nu}\left(\frac{1}{2}\phi'^2 + V\right)$$

1949

$\phi'^2 = \lambda\eta^4 \mathrm{sech}^4 [\sqrt{\lambda}\eta(z-z_0)]$

strongly localized ;

Actually $\phi'^2 = 2V$ (for that solution)

$T^0{}_0 = T^x{}_x = T^y{}_y = 2V$

$T^z{}_z = 0$.

so: It looks something three dimensional.
(nothing much going on in z direction)

Mostly a vacuum, but wall has ∞ ~~area~~ area

$$\int \rho_0 \, dz = \frac{4}{3}\sqrt{\lambda}\eta^3$$

Adding Gravity

$T^0{}_0$, etc looks like Λ in 3D space at z_0

Try $ds^2 = A^2(z)\, \gamma_{\mu\nu} dx^\mu dx^\nu - dz^2$

warning: μ indicates $t, x \dots$; not 4D!

Now, calculate Curvature.

use Cartan.

$\gamma_{\mu\nu}(x)$ arbitrary $(D-1)$ dimensional metric.

$\underline{\omega}^{\hat{z}} = dz$; $\underline{\omega}^{\hat{0}} = A(z)\, \underline{\omega}_0{}^{\hat{0}}$

$e^{\hat{0}}{}_\mu dx^\mu$

$\underline{d}\,\underline{\omega}^{\hat{z}} = 0$; $d\underline{\omega}^{\hat{a}} = \frac{A'}{A}\underline{\omega}^{\hat{z}} \wedge \omega^{\hat{a}} - \underline{\theta}_0{}^{\hat{a}}{}_{\hat{b}}\, \omega^{\hat{b}}$

(1950)

$\Rightarrow \underset{\sim}{\Theta}^{\hat{a}}{}_{\hat{b}} = \underset{\sim}{\Theta}_0{}^{\hat{a}}{}_{\hat{b}}$ & $\theta^{\hat{a}}{}_{\hat{b}} = \frac{N}{A}\underset{\sim}{\omega}^{\hat{a}}$

$\underset{\sim}{R}^{\hat{a}}{}_{\hat{b}} = \underset{\sim}{d}\,\underset{\sim}{\Theta}_0{}^{\hat{a}}{}_{\hat{b}} + \underset{\sim}{\Theta}_0{}^{\hat{a}}{}_{\hat{c}} \wedge \underset{\sim}{\Theta}_0{}^{\hat{c}}{}_{\hat{b}} + \underset{\sim}{\Theta}^{\hat{a}}{}_{\hat{c}} \wedge \theta \cdots$

$\Rightarrow \underset{\sim}{R}^{\hat{a}}{}_{\hat{b}} = \underset{\sim}{R}_0{}^{\hat{a}}{}_{\hat{b}} - \left(\frac{N}{A}\right)^2 \underset{\sim}{\omega}^{\hat{a}} \wedge \omega^{\hat{c}}\,\eta_{\hat{c}\hat{b}}$

Bonus Lecture 2: Gauss Codazzi Formalism & the Gibbons-Hawking boundary Term

Geometry of Submanifold. (Gauss Codazzi Formalism)

Recall the variational principle for S_{EH}.

$$\delta S_{EH} = \delta \int d^4x \sqrt{-g}\, R$$

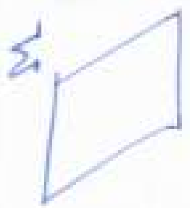
$$= \int d^4x \sqrt{-g}\, G_{ab}\, \delta g^{ab} + \int_{\partial M} d^3x \sqrt{-g} \left(\nabla_{\hat{n}} \delta g - n_a \nabla_b \delta g^{ab}\right)$$

$$\delta S_{EH} = \int_M d^4x \sqrt{-g}\, G_{ab}\, \delta g^{ab} + \int_{\partial M} d^3x \sqrt{-g} \left(\nabla_{\hat{n}} \delta g - n_a \nabla_b \delta g^{ab}\right)$$

we will add boundary term to action.

$$S_{\partial M} = \int_{\partial M} (\text{boundary term})\, \delta g^{ab}$$

Geometry of Submanifold (Gauss-Codazzi Formalism)



let $\Sigma_i \subset M$ be a submanifold of dim n ($\dim M = D$)

The codimension of Σ_i is $D-n$; & is the no. of linearly independent normals, i.e. vectors $n \in T_p(M)$ s.t. $n\, \sigma^A = 0$ $\forall$ coordinate functions σ^A on Σ.

or $n_{i\mu} \dfrac{\partial x^\mu}{\partial x^A} = 0$ (will take $n_{i\mu} n_j^{\ \mu} = \pm\delta_{ij}$)

The 1^{st} Fundamental form of Σ_i is

$$h_{ab} = g_{ab} + \sum_{i=1}^{D-n} (-1)^{\#_i} n_{ia} n_{ib}$$

\- timelike
\+ spacelike.

(The first fundamental form;
what it does is: takes the metric and kills off any piece of metric normal to Σ_i.
Subtracting off all the normal components)

h_{ab} projects out the components of g_{ab} normal to Σ, gives a local distance on Σ_i.

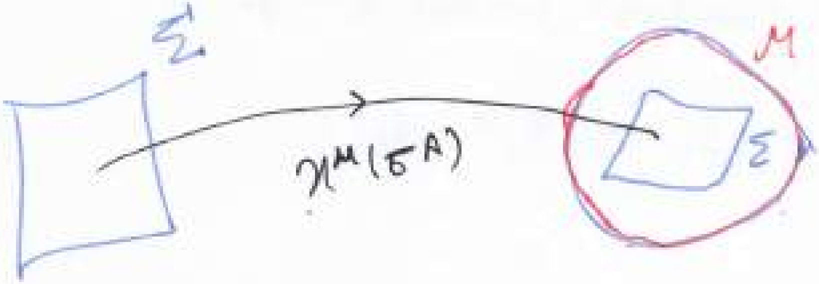
Since Σ is a manifold, we can also consider intrinsic quantities. living in $T(\Sigma_i)$

($h_{ab} \in T^*(M)$)

$$\frac{\partial x^\mu}{\partial \sigma^A} : T_p{}^*(M) \longrightarrow T_p{}^*(\Sigma_i) \quad \text{pull back map.}$$

$$\omega_\mu \longmapsto \omega_A = \omega_\mu \frac{\partial x^\mu}{\partial \sigma^A}$$

& is a projection operator on to $T_p^*(\Sigma_i)$



$\gamma_{AB} = g_{\mu\nu} \dfrac{\partial x^\mu}{\partial \sigma^A} \dfrac{\partial x^\nu}{\partial \sigma^B}$ is the intrinsic metric.

The 2^{nd} Fundamental form or Extrinsic curvature of Σ measures the "curvature" of the embedding in M.

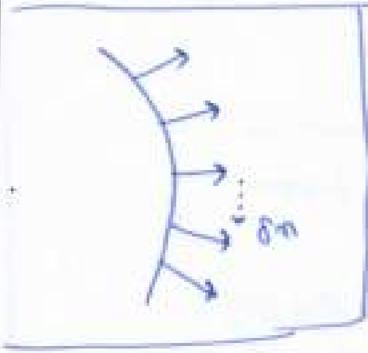
$$K_{i\,\mu\nu} = h_{(\mu}{}^{\sigma} h_{\nu)}{}^{\lambda} \nabla_\sigma n_{i\lambda}$$

or

$$K_{i\,AB} = \frac{\partial X^\mu}{\partial \sigma^{(A}} \frac{\partial X^\nu}{\partial \sigma^{B)}} \nabla_\mu n_{i\nu}$$

$$= - n_{i\nu} \; {}^{(3)}\nabla_A \frac{\partial X^\nu}{\partial \sigma^B}$$

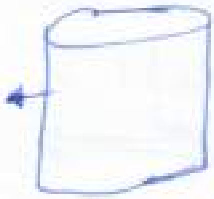
where ${}^{(3)}\nabla_A$ is the connection Σ inherits from M.



e.g. Cylinder embedded in $\mathbb{R}^3$

$\{x^2 + y^2 = a^2\} \subset \mathbb{R}^3$

$h_{ab} = \delta_{ab} - n_a n_b$



so; $h_{ab} = \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in cartesian.

$n = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$

or $\begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in polars

Extrinsic curvature : (in polar coordinates)

$K_{ab} = -\Gamma^r{}_{ab}$ in polars

$$= \begin{array}{cc} \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} r \\ \theta \\ z \end{array} \end{array}$$

Intrinsic picture; : $\sigma^A = \{\theta, z\}$

$$\gamma_{AB} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}$$

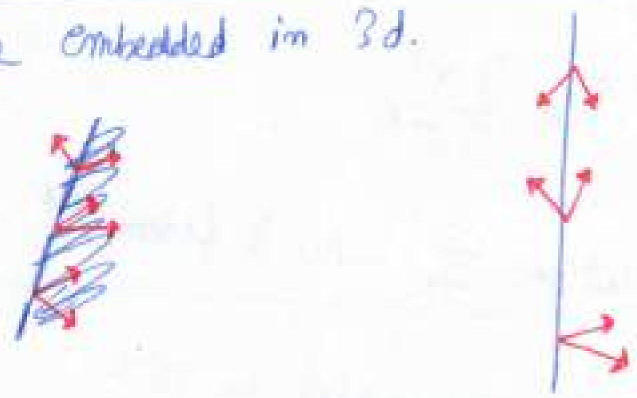
$$K_{AB} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

For Codimension > 1 ; normals could vary ...
(we can choose different set of normals)

↳ The normals can vary across Σ

---

ex) line embedded in 3d.



So; we also ~~have~~ have a fundamental form associated to it

⟷ So; it is like telling us ; gauge potential ~~associated~~ associated to that choice of normal.

The Normal Fundamental forms are the connection on the normal bundle of Σ

$$\beta_{\mu\, ij} = n_{i\nu} \nabla_\mu n_j^{\ \nu}$$

(Normal fundamental forms actually take the variation of normal & project it down transverse to Σ)

so; choose $\underline{\nabla_{n_i} n_j} = 0$ (ie; we extend normal fields geodesically off the manifold)

But; what we cant do; is that;
we cant get rid of this any twisting of my normal forces although that is potentially a gauge choice.
(what isn't a gauge choice is of course the curvature of this connection)

The curvature of Σ is now related to the curvature of M & extrinsic curvature.

Gauss Equation:

$$ {}^{(n)}R^{a}{}_{bcd} = {}^{(D)}R^{a'}{}_{b'c'd'} h^{a}{}_{a'} h^{b'}{}_{b} h^{c'}{}_{c} h^{d'}{}_{d} - \sum_{i=1}^{D-n} (-)^{\#_i} \left[K_i{}^{a}{}_{c} K_{i\,bd} - K_i{}^{a}{}_{d} K_{i\,bc} \right] $$

Proof for codimension 1.

uses Reimann identity.

$$ \left({}^{(n)}\nabla_a {}^{(n)}\nabla_b - {}^{(n)}\nabla_b {}^{(n)}\nabla_a \right) V^c = {}^{(n)}R^{c}{}_{dab} V^d $$

$$ {}^{(n)}R^{a}{}_{bcd} V^b = {}^{(n)}\nabla_c {}^{(n)}\nabla_d V^a - c \leftrightarrow d $$

$$ = h^{e}_{c} h^{f}_{d} h^{a}_{g} \nabla_e \left({}^{(n)}\nabla_f V^g \right) - c \leftrightarrow d $$

Taken higher dimensional derivative; and projected down.

$$ = h^{e}_{c} h^{f}_{d} h^{a}_{g} \nabla_e \left(h^{p}_{f} h^{g}_{q} \nabla_p V^q \right) - c \leftrightarrow d $$

do the same thing.

$$= h_c{}^e h_a{}^p h^a{}_q \nabla_e \nabla_p V^q$$

$$+ \cancel{h_c^e h_d^f h^a{}_q \nabla_e (h_p{}^p) \nabla_p V^q}$$ because $\nabla_n \cdot V = 0$

$$+ h_c{}^e h_d{}^p h^a{}_g \nabla_e (h^g{}_q) \nabla_p V^q \quad - c \leftrightarrow d$$

$h_{ab} = g_{ab} + n_a n_b$; take n ~~to~~ be spacelike for the sake of argument

$\Rightarrow \nabla_c h_{ab} = n_a K_{bc} + n_b K_{ac}$

$$^{(n)}R^a{}_{bcd} V^b = h^e{}_c h^p{}_d h^a{}_q {}^{(D)}R^q{}_{dep} V^d$$

$$+ h_c{}^e h^p{}_d h^a{}_g K_c{}^g \underbrace{n_q \nabla_p V^q}_{-V^q \nabla_p n_q = -V^q K_{pq}} \quad - c \leftrightarrow d$$

(because n & V are orthogonal)

$$= {}^{(D)}R^{a'}{}_{b'c'd'} h^{c'}{}_c h^{d'}{}_d h^a{}_{a'} V^{b'}$$

$$- V^{b'} K_{b'd} K^a{}_c + V^{b'} K_{b'c} K^a{}_d$$

(we proved for n spacelike ; and codimension 1)

Now lets look at,

how we can add a boundary term to cancel off the normal derivatives of g.

Back to Boundary Term:

Hold boundary metric fixed; only n^a varies (take spacelike for simplicity)

$$\left.\begin{array}{l}\delta n^a = -n^a n_b \delta n^b \\ \delta g^{ab} = -n^a \delta n^b - n^b \delta n^a\end{array}\right\} \Rightarrow \delta n^a = -\frac{1}{2} n^a n_b n_c \delta g^{bc}$$

$$\boxed{\delta n^a = -\frac{1}{2} n^a n_b n_c \delta g^{bc}}$$

Now consider $\delta(\nabla_a n^a) = \nabla_a \delta n^a + \delta \Gamma^a_{ab} n^b$

$$= \frac{-1}{2} \nabla_a (n^a n_b n_c \delta g^{bc}) - \frac{1}{2} \nabla_n \delta g$$

$K = \nabla_a n^a$

$$\delta(K) = -\frac{1}{2} K n_b n_c \delta g^{bc} - \frac{1}{2} n_b n^c n^a \nabla_a \delta g_{bc} - \frac{1}{2} \nabla_n \delta g$$

$$= -\frac{1}{2} K n_b n_c \delta g^{bc} - \frac{1}{2} \nabla_n \delta g$$

$$- \frac{n_b}{2} h^{ac} \nabla_a \delta g^b{}_c + \frac{1}{2} n_b \nabla_a \delta g^{ba}$$

$$\delta K = \underbrace{\frac{1}{2}[n_b \nabla_a \delta g^{ab} - \nabla_n \delta g]} - \frac{1}{2} K n_b n_c \delta g^{bc}$$

These are the terms which we want

$$- \frac{1}{2} {}^{(3)}\nabla_c (n_b \delta g^{bc})$$

$$+ \frac{1}{2} K_{ab} \delta g^{ab}$$

${}^{(3)}\nabla_c (n_b \delta g^{bc})$ gives zero contribution $\rightarrow$ because it is total derivative

because on boundary; boundary is zero.

(There is nothing to my boundary)

Hence;

$$\delta(K\sqrt{g}) = \frac{1}{2}[n_b \nabla_a \delta g^{ab} - \nabla_n \delta g] + \underbrace{\frac{1}{2}[K_{ab} - K h_{ab}]\delta g^{ab}}_{\text{This vanishes on boundary}}$$

So, the boundary term

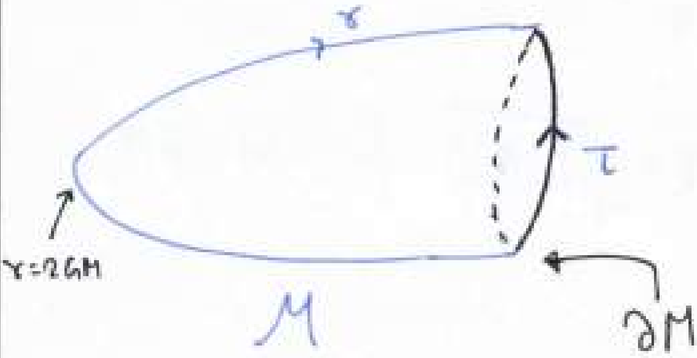
$$S_{GH} = \frac{-1}{8\pi G}\int K\sqrt{g}$$ is Gibbons-Hawking term

Lec 8: Calculation of Euclidean Action & Black-hole etropy, Israel Equations.

$$S_{G.H.} + S_{E.H.} = S$$

we finally have $S = \underbrace{\frac{-1}{16\pi G}\int_{M} d^4x \sqrt{-g}\, R}_{S_{E.H.}} \underbrace{- \frac{1}{8\pi G}\int_{\partial M} d^3x \sqrt{-h}\, K}_{S_{G.H.}}$

Recall, Euclidean Schwarzschild has periodic τ: $\Delta\tau = 8\pi GM$.



$$g_{tt} = f(r) = 1 - \frac{2GM}{r}$$

for Schwarzschild; $S_{EH} \equiv 0$ because it has zero Ricci Curvature

ie: $S_{EH}\big|_{\text{Schwarzschild}} \equiv 0$

Boundary at large $r = R$

$$ds^2_{\partial M} = \left(1 - \frac{2GM}{R}\right) d\tau^2 + R^2\, d\Omega^2_{II}$$

$d\theta^2 + \sin^2\theta\, d\varphi^2$

~~has curvature because~~

∂M does not have curvature because of $r = R$ in an intrinsic sense (because R is just a constant)

The ~~...~~ submanifold ∂M has curvature because of the $d\Omega_2^2$ piece.

So; ∂M has curvature because ds^2 is curved (constant positive curvature space)

$\hookrightarrow$ it does not have curvature dependent on R in an intrinsic sense because R is constant

∂M does have extrinsic curvature.....

normal in $\mathcal{M}$ $\quad n \propto \frac{\partial}{\partial r}$

lets normalize the normal: $\quad n = \sqrt{1-\frac{2GM}{r}} \cdot \frac{\partial}{r}$ at ∂M

~~Extrinsic curv~~

Extrinsic Curvature:

$$K = \nabla_a n^a = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g}\, n^a) = \frac{1}{r^2}(r^2 n^r)'$$

$$= \frac{1}{r^2}(r^2 n^r)'\Big|_{r=R} = \frac{2}{r}\sqrt{1-\frac{2GM}{r}} + \frac{GM}{r^2\sqrt{1-\frac{2GM}{r}}}\Bigg|_{r=R}$$

$$K = \frac{2}{r}\sqrt{1-\frac{2GM}{r}} + \frac{GM}{r^2\sqrt{1-\frac{2GM}{r}}}\Bigg|_{r=R}$$

volume factor: $\sqrt{h}\, d^3x = \sqrt{1-\frac{2GM}{R}} \cdot R^2 \sin\theta \, d\theta \, d\phi \, dz$

Put together:

$$\int K\sqrt{h}\, d^3x = \underbrace{4\pi\beta}_{\int d\tau, d\theta, d\phi} \cdot R^2\left(\frac{2}{R}\left(1-\frac{2GM}{R}\right)+\frac{GM}{R^2}\right)$$

$$= 4\pi\beta\,[2R - 3GM]$$

Scm ~~tot~~ blows up as $R \longrightarrow \infty$.

ie; ~~B~~ This diverges as $R \to \infty$!!

Note: even if $M=0$; it is still divergent

so; This means, flat space has divergent action.

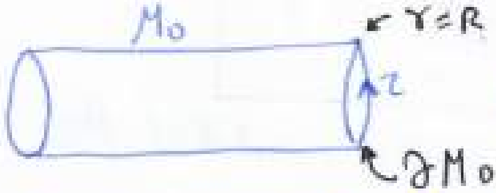
Divergent also for $M=0$; ie; flat space has divergent action.

Flat spacetime has diverging action in this particular way of writing.

Then it is just some intrinsic divergent; that we normalize.

So we take it away.

Subtract this background divergence



$$ds_0^2 = \Lambda^2 d\tau^2 + dr^2 + r^2 d\Omega_{II}^2$$

we kept a constant Λ^2 at front of $d\tau^2$.

because ~~we~~ ~~too~~, If we going to subtract off ~~to~~ background action; then the two boundary manifolds have to agree. And therefore periodicity of time has to be the same.

$$\Delta\tau = \beta$$

Choose Λ to make sure that manifolds match up at $r = R$.

$$ds^2_{\,0,\,\partial M_0} = \Lambda^2 d\tau^2 + R^2 d\Omega_{II}^2.$$

To match ∂M_0 and ∂M (the time circles)

$$\boxed{\Lambda^2 = 1 - \frac{2GM}{R}}$$

For M_0 (flat spacetime) $K_0 = \frac{2}{R}$ (because no mass term)

$$\sqrt{h_0}\, d^3x \longrightarrow 4\pi\beta \cdot \sqrt{1 - \frac{2GM}{R}}\, R^2$$

so; $$\int \sqrt{h_0}\, K_0\, d^3x = 4\pi\beta \times \frac{2}{R} \underbrace{\sqrt{1 - \frac{2GM}{R}}}_{\substack{\text{Taylor expand} \\ 1 - \frac{GM}{R} + O(R^{-1})}}$$

$$\Rightarrow \boxed{S^{(0)}_{GH} = \frac{-1}{8\pi G} \times 4\pi\beta\,[2R - 2GM]}$$

has the piece GM in it because matched the circles at $r = R$.

Hence; $$S_{\text{Schwarzschild}} = S_{GH} - S^{(0)}_{GH}$$

$$S_{\text{Schwarzschild}} = S_{GH} - S_{GH}^{(0)}$$

$$= \frac{-1}{8\pi G}\,(4\pi\beta)\times(-GM)$$

$$= \frac{\beta M}{2}$$

$$\boxed{S_{\text{Schwarzschild}} = \frac{\beta M}{2} = 4\pi G M^2}$$

This is nice & finite; dependent on periodicity of Euclidean time & Mass

Consider the partition function.

$$Z \sim \operatorname{tr} e^{-\beta H}$$

$$H \sim \int d^3x\, \mathcal{H} = \frac{1}{\beta}\int d^4x\, \mathcal{L}_E$$

Euclidean Lagrange density.

$$= \frac{I_E}{\beta}$$

I_E is Euclidean action.

So;

$$\boxed{Z \sim \operatorname{tr} e^{-I_E}}$$

At $T = \frac{1}{\beta} = 8\pi GM$ has Schwarzschild solution as a saddle point.

Entropy; $\mathbf{S} = \beta^2 \frac{\partial}{\partial\beta}\left[-\beta^{-1}\ln Z\right]$

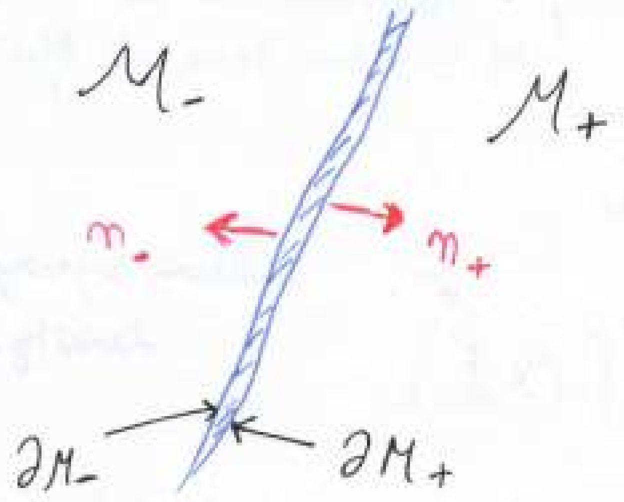
$$= \beta^2\frac{\partial}{\partial\beta}\left[I_E/\beta\right] = \beta^2\frac{\partial}{\partial\beta}\left[\frac{M}{2}\right]$$

$M = \frac{\beta}{8\pi G}$ $\ni$ $\mathbf{S} = \frac{\beta^2}{16\pi G} = 4\pi G M^2 = \frac{4\pi(2GM)^2}{4G}$

$S = \frac{4\pi(2GM)^2}{4G}$ $\boxed{S = \frac{4\pi(2GM)^2}{4G}}$

Note; In varying δS_{GH} we had a term $(K_{ab} - K h_{ab})\,\delta h^{ab}$

A "shell" in spacetime has 2 sides



~~The shell~~

The very thin shell will be a combination of the two boundries of M_- & M_+

So; If we think about this from Gauss Codazzi perspective ∴ Looking at the variation of action there

Get a term, $(K_{+ab} - K_+ h_{ab}) - (K_{-ab} - K_- h_{ab})$

flip normal n_- to get this minus sign.

(now normals point in same direction)

$\longrightarrow$ ISRAEL EQUATION

$$\Delta K_{ab} - \Delta K \cdot h_{ab} = 8\pi G\, S_{ab}$$

where $$S_{ab} = \int_{\Sigma_-}^{\Sigma^+} T_{ab}$$

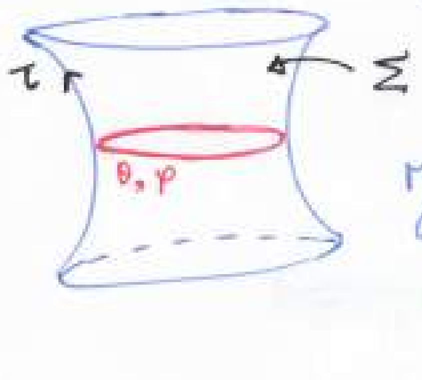
ISRAEL EQUATION.

where we take;
Energy momentum tensor for the surface (the shell) is strongly localized.
↪ Integrate Energy momentum Tensor through the Shell surface.
and it gives you S_{ab}.

ie: $T_{ab} \sim S_{ab}\,\delta(z)$; where, z is coordinate through wall

$\boxed{\text{Can take } \delta\text{-function limit for codomain 1 only.}}$

Example 1



Consider $\underline{x}^2 - \tau^2 = L^2$ in minkowski spacetime.

To get normal; we look at
$$d(\underline{x}^2 - \tau^2) = 2(\underline{x}\, d\underline{x} - \tau\, d\tau) \propto n$$

To make calculations easy; we put coordinates on this hyperboloid: τ, θ, ϕ.

Σ is $X^\mu(\sigma^A) = (L \sinh \tau/L,\ \underbrace{L \cosh \tau/L,\ \theta, \varphi}_{\text{in polars.}})$

ie; $X^\mu(\sigma^A) = \left(L \cdot \sinh \frac{\tau}{L},\ L \cdot \cosh \frac{\tau}{L},\ \theta, \varphi\right)$

$$n_\mu = \left(-\sinh\left(\frac{T}{L}\right),\ \cosh\frac{T}{L},\ 0,\ 0\right)$$

This gives $K_{\theta\theta} = -\Gamma^{\rho}_{\theta\theta}\, n_\rho$ — covariant derivatives of n ... gives Γ piece.

$$= R\, n_R$$

$$= L \cdot \cosh T/L$$

$$= -\frac{g_{\theta\theta}}{L} \Rightarrow \boxed{K_{\theta\theta} = \frac{-g_{\theta\theta}}{L}}$$

similarly; $\boxed{K_{\varphi\varphi} = -g_{\varphi\varphi}/L}$ $\boxed{K_{zz} = -g_{zz}/L}$

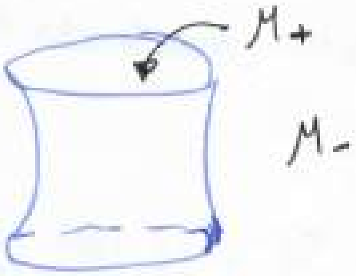
Writing curvature in terms of intrinsic components.

Then extrinsic curvature is proportional to intrinsic ~~...~~ or induced metric.

$$K_{AB} = -\frac{\gamma_{AB}}{L}$$

$$\Rightarrow \boxed{K_{AB} = K\,\gamma_{AB} = \frac{2}{L}\,\gamma_{AB}}$$

But if we take inside & outside of ~~Hyperboloid~~ Hyperbolid



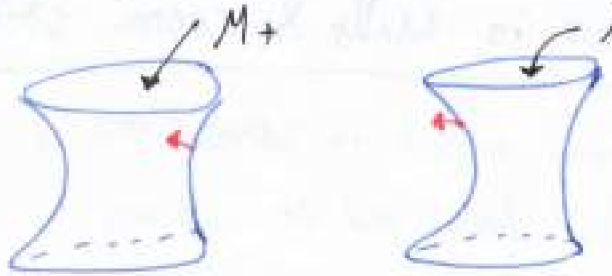
so; $K_{-AB} = K_{+AB}$

$\Downarrow$

~~How~~ Hence you get zero energy momentum tensor.

This was expected

But if we take 2 ~~[illegible]~~ interiors.



Take two copies of same setup.
(like mirror image)

~~Then K_{-AB}~~ Then $K_{-AB} = -K_{+AB}$ ~~[illegible]~~ opposite.

extrinsic curvatures are equal & opposite ← so

we get $\Delta K_{AB} - \Delta K h_{AB} = \frac{4}{L} \gamma_{AB} = 8\pi G S_{AB}$

ISRAEL Equation.

Surface with energy momentum

$$T_{ab} = \frac{1}{2\pi G L} h_{ab} \, \delta(z)$$

orthogonal coordinate

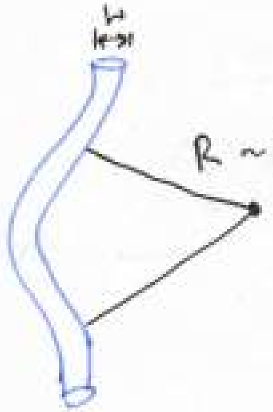
Gravitational Physics — Shoaib Akhtar 23/7/2020 (1968)

Lec 9: Physical submanifolds; domain walls & cosmic strings.

- A point particle ~~ha~~ is 1 dimensional object in spacetime, because it exists in time but localized in space.
- We could have extended objects in 3d.
  like, ~~str~~ strings, membranes or walls.

Another way submanifolds appear is as appropriate ways of describing extended objects that are localized in some direction.

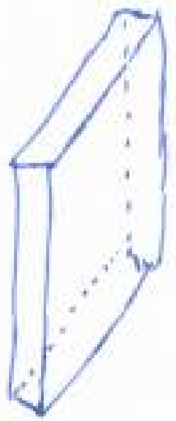
Example] String



w ⇒ width
R ⇒ radius of curvature
(... extrinsic curvature)

w is small; ie; $w << R$

Wall



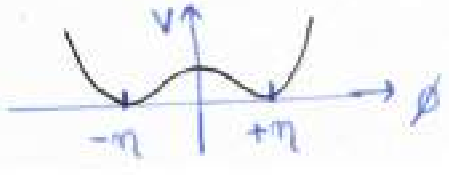
These objects, have a realization as non-perturbative solutions in QFT.

Domain Wall

consider scalar field ϕ.

$$\mathcal{L}_\phi = \frac{1}{2}(\partial\phi)^2 - V(\phi) \quad ; \quad V(\phi) = \frac{\lambda}{2}(\phi^2 - \eta^2)^2$$

$\phi \in \mathbb{R}$



potential with symmetry breaking.

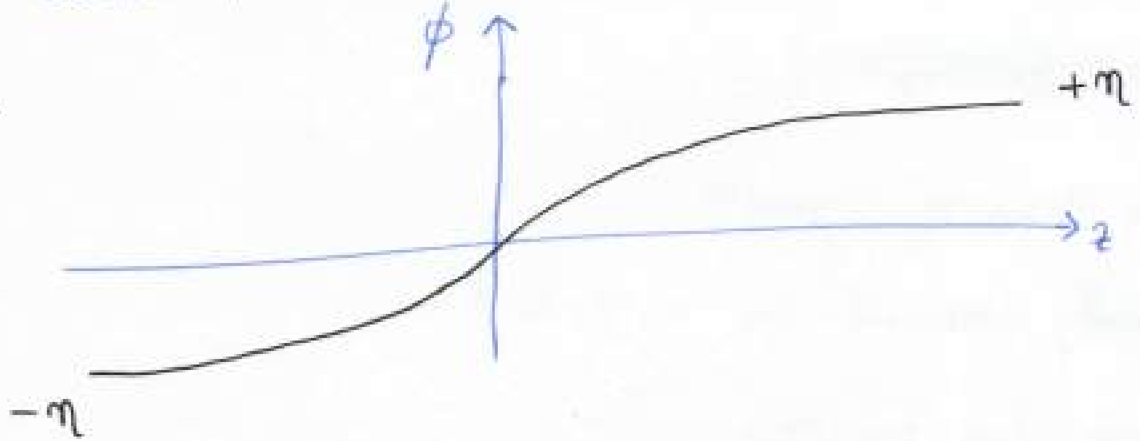
$V(\phi)$ has $\mathbb{Z}_2$ symmetry.

Once we choose a vacuum, we broke our $\mathbb{Z}_2$ symmetry ~~symmetry~~.

(Normally; we pick on vacuu & quantize around it)

But, instead

consider the solution $\phi = \eta \tanh(\sqrt{\lambda} \cdot \eta z)$



This solution interpolates from one vaccum to the other vaccum.

first of all; lets check $\phi = \eta \tanh(\sqrt{\lambda} \cdot \eta z)$ is a solution.

$$\phi' = \sqrt{\lambda}\, \eta^2 \operatorname{sech}^2(\sqrt{\lambda}\, \eta z)$$

$$\phi'' = -2\lambda \eta^3 \operatorname{sech}^2(\sqrt{\lambda}\, \eta z) \tanh(\sqrt{\lambda} \cdot \eta z)$$

for the moment; if we ignore gravity; our equation of motion is

~~Ignore~~ ~~eqn~~ ~~e.o.m~~ Ignore gravity:

e.o.m.: $\quad \Box \phi + \dfrac{\partial V}{\partial \phi} = 0$

$$-\phi'' + 2\lambda \phi \cdot (\phi^2 - \eta^2) = 0$$

$\phi'' = -2\lambda\eta^3 \cdot (1-\phi^2/\eta^2)\cdot \phi/\eta$

$\hookrightarrow$ satisfies ~~e~~ equation of motion. ✓ ... good.
in the absence of any sort
of gravitational field. "Solution in absence of gravity"

For the moment, lets proceed.

~~Solution in absence of~~

Solution in absence of gravity.

Translationally invariant in $x, y,$ & t.

Energy momentum tensor:

$$T_{\mu\nu_\phi} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(\tfrac{1}{2}(\partial\phi)^2 - V\right)$$

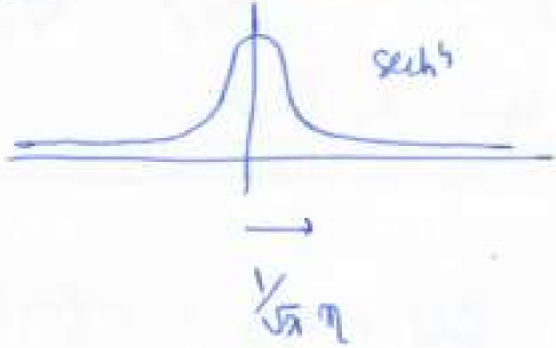
The scalar field only depends on z

so: $\phi_{,\mu} \propto \delta^z_\mu$.

$$\text{so;}\quad T_{\phi_{\mu\nu}} = \lambda\eta^4 \operatorname{sech}^4(\sqrt{\lambda}\,\eta z)\,\delta^z_\mu\delta^z_\nu - g_{\mu\nu}\left(-\lambda\eta^4 \operatorname{sech}^4(\sqrt{\lambda}\,\eta z)\right)$$

$$T_{\phi_{\mu\nu}} = 2V\cdot(g_{\mu\nu} + \underset{\eta_\mu}{\delta^z_\mu}\,\underset{\eta_\nu}{\delta^z_\nu}) = 2Vh_{\mu\nu}$$

$$\boxed{T_{\phi_{\mu\nu}} = 2Vh_{\mu\nu}}$$



T^0_0 is sharply localized with some finite width.

Now; the width is small compared to the Scale we are interested ~~in~~ in.

~~to~~ hence; localized around $z = 0$.

$$\int T^0_0 \, dz = \frac{4}{3}\sqrt{\lambda}\cdot\eta^3 = \sigma$$

energy per unit area

or Tension

It is non-perturbative; because we cannot get ~~to~~ complete vaccum everywhere without an infinite ~~area~~ amount of energy.

ie: if we wanted to make the true vacuum either $+\eta$ or $-\eta$: we will have to lift ϕ over the barrier ~~in~~ in field space (one one well to other) over infinite region of space

⟶ This will cost us infinite amount of energy ⇒ So we cant get rid of this by any sort of finite process.

~~The energy~~ So; This is topologically stable solution.

Topologically stable ... sometime it is called Domain Wall or Kink

Add gravity

energy per unit area is finite; so Total energy is infinite. because here space extends in all x, y, z direction.

? But the total energy of wall is infinite.

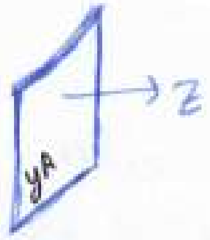
Is that going to be a problem?

lets ~~just~~ just try it any-way ......

$$ds^2 = \underbrace{A^2(z)\gamma_{AB}\,dy^A dy^B}_{\text{wall coordinate}} \quad - dz^2$$

z = distance from the wall.

$$\boxed{ds^2 = A^2(z)\gamma_{AB}\,dy^A dy^B - dz^2}$$



$z \Rightarrow$ spacelike distance from the wall

(signature $- + + + \cdots$)

Use Cartan Method to calculate curvature .....

$\underset{\sim}{\omega}^{\hat{z}} = \underset{\sim}{d}z$

$\underset{\sim}{\omega}^{\hat{A}} = A(z)\cdot e^{\hat{A}}{}_{A}\,\underset{\sim}{d}y^{A}$

orthonormal basis for γ
(o/n basis for γ)
(dreibien)

$\underline{\omega}^z = \underline{d}z \quad , \quad \underline{\omega}^{\hat{A}} = A(z)\, e^{\hat{A}}{}_A\, \underline{d}y^A$

we have $\underline{d}\,\underline{\omega}^z = 0$

$$\underline{d}\,\underline{\omega}^{\hat{A}} = \frac{A'(z)}{A(z)}\underline{\omega}^z \wedge \underline{\omega}^{\hat{A}} - A(z)\,\underbrace{\underset{\sim}{\theta}_0{}^{\hat{A}}{}_{\hat{B}}}_{\text{connection on } \gamma} \wedge \underbrace{\underline{\omega}_0{}^{\hat{B}}}_{e^{\hat{B}}{}_B\, \underline{d}y^B}$$

Now, we can read off our connection one forms.

$$\boxed{\underset{\sim}{\theta}^{\hat{A}}{}_{\hat{B}} = \underset{\sim}{\theta}_0{}^{\hat{A}}{}_{\hat{B}}} \quad ; \quad \boxed{\underset{\sim}{\theta}^{\hat{A}}{}_z = \frac{A'(z)}{A(z)}\underset{\sim}{\omega}^{\hat{A}}}$$

Now, we want to calculate curvature two forms

compute R : $\underline{R}^{\hat{A}}{}_{\hat{B}} = \underline{d}\,\underset{\sim}{\theta}_0{}^{\hat{A}}{}_{\hat{B}} + \underline{\theta}_0{}^{\hat{A}}{}_{\hat{C}} \wedge \theta_0{}^{\hat{C}}{}_{\hat{B}}$

$$+ \underset{\sim}{\theta}^{\hat{A}}{}_z \wedge \underset{\sim}{\theta}^z{}_{\hat{B}}$$

$$\Rightarrow \boxed{\underline{R}^{\hat{A}}{}_{\hat{B}} = R_0{}^{\hat{A}}{}_{\hat{B}} + \left(\frac{A'(z)}{A(z)}\right)^2 \underset{\sim}{\omega}^{\hat{A}} \wedge \underset{\sim}{\omega}_{\hat{B}}}$$

Similarly: $\underline{R}^{\hat{A}}{}_z = \underline{d}\,\underset{\sim}{\theta}^{\hat{A}}{}_z + \theta^{\hat{A}}{}_{\hat{B}} \wedge \theta^{\hat{B}}{}_z$

$= \frac{A''}{A}$

$$\Rightarrow R^{\hat{A}}{}_z = \frac{A''(z)}{A(z)}\,\underline{\omega}^z \wedge \omega^{\hat{A}} + \underbrace{A'(z)\,\underline{d}\,\omega_0{}^{\hat{A}} + A'(z)\,\underset{\sim}{\theta}_0{}^{\hat{A}}{}_{\hat{B}} \wedge \underline{\omega}_0{}^{\hat{B}}}_{A'(z)\left[\underline{d}\,\omega_0{}^{\hat{A}} + \underset{\sim}{\theta}_0{}^{\hat{A}}{}_{\hat{B}} \wedge \omega_0{}^{\hat{B}}\right]}$$

$\to$ 0 by Cartan's first equation

$$\Rightarrow \boxed{R^{\hat{A}}{}_z = \frac{A''(z)}{A(z)}\underline{\omega}^z \wedge \omega^{\hat{A}}}$$

Till now; no where we invoked dimensions being 4.
$\hookrightarrow$ so: It is true for D dimensions.

We have only allowed that $\gamma^{(y)}_{AB}$ depends on y.
(has not many assumption of the y^A part ; ....
other than it has o/n basis)

<u>Note</u>: Calculation independent of dimensionality & form of γ.

$$\Rightarrow R^{\hat{A}\hat{B}}{}_{\hat{C}\hat{D}} = \frac{1}{A^2} R_0{}^{\hat{A}\hat{B}}{}_{\hat{C}\hat{D}} + \left(\frac{A'}{A}\right)^2 \left(\delta^{\hat{A}}_{\hat{C}}\delta^{\hat{B}}_{\hat{D}} - \delta^{\hat{A}}_{\hat{D}}\delta^{\hat{B}}_{\hat{C}}\right)$$

$$R^{\hat{A}z}{}_{\hat{B}z} = \frac{A''}{A}\delta^{\hat{A}}{}_{\hat{B}}$$

So; Our **Ricci Tensor**

$$R^A{}_B = \frac{1}{A^2} R_0{}^A{}_B + \left((D-2)\left(\frac{A'}{A}\right)^2 + \frac{A''}{A}\right)\delta^A{}_B$$

$$R^z{}_z = (D-1)\frac{A''}{A}$$

Now $D=4$; and take γ to be constant curvature space

ie; $D=4$, and take γ_{AB} constant curvature K/L^2

- $K=0$ flat
- $K=1$ dS
- $K=-1$ AdS

Einstein Equation & ϕ equation.

$$G^z{}_z = \frac{3K}{L^2A^2} - \frac{3A'^2}{A^2} = 8\pi G\left(V - \frac{1}{2}\phi'^2\right)$$

$$G^0{}_0 = \frac{K}{L^2A^2} - \frac{A'^2}{A^2} - 2\frac{A''}{A} = 8\pi G\left(V + \frac{1}{2}\phi'^2\right)$$

2 einstein equation

$$\phi'' + \frac{3A'}{A}\phi' = 2\lambda\phi(\phi^2 - \eta^2)$$

1 ϕ equation.

Analytically ; consider $8\pi G\eta^2 = \epsilon << 1$

magnitude of RHS $8\pi G\,\lambda(\eta^2-\phi^2)^2 = 0\,(8\pi G\eta^2)\times\lambda\eta^2$

tells about Gravitational strength of the wall

gradiant ... how wide it .... how fast things varry

~~$\sqrt{\lambda}\,\eta$ was setting scale for tanh~~

$\sqrt{\lambda}\,\eta$ was setting scale for tanh(...) solution.

m_ϕ^{-2}

If solution is weakly gravitating ($\eta/M_p << 1$) (ie; well below plank scale)

(Then we can solve it in itterative way)

Linearization in G.R.

$\lambda\eta^2$ is actually m_ϕ^{-2}

$\therefore$ if we were thinking about it in terms of spontaneous symmetry breaking ; it would just be the mass of ϕ particle around the vacuum.

We solve for our background field in flat space $\rightsquigarrow$ calculate energy momentum $\longrightarrow$

(for $\epsilon = 0$)

$\rightsquigarrow$ plug in Einstein Equation $\rightsquigarrow$ it then gives correction to geometry to order ε

This can feedback in......

so; $A = A_0 + \epsilon A_1 + \ldots$

$\phi = \phi_0 + \epsilon \phi_1 + \ldots$

doing series expansion & solving order by order in ϵ.

$\underline{O(\epsilon^0)}$: $A_0 = 1$, $\phi_0 = \eta \tanh(\sqrt{\lambda}\cdot\eta z)$

Now we use the energy momentum we calculated to find solution for A : so the back reaction.

$\underline{O(\epsilon^1)}$: $A'^2 \sim \frac{K}{L^2} \Rightarrow K = 1$, $L = O(1/\epsilon)$

$A'' = -8\pi G V$: $O(\varepsilon)$

Solved by $A_1 = \frac{-1}{3} \log[\cosh(\sqrt{\lambda}\cdot\eta z)] - \frac{1}{3} \ln z \; \frac{-1}{12} \operatorname{sech}^2(\sqrt{\lambda}\eta z)$

so: very roughly A is. ; $A \simeq 1 - \frac{|z|}{L}$, $L = \frac{3}{\sqrt{\lambda}\eta\varepsilon}$

$\rightarrow$ smooths off around the wall..

$\rightarrow$ away from wall.

A function graph.

So the A function we get is

$$ds^2 = \left(1-\frac{|z|}{L}\right)^2 \left[dt^2 - L^2\cosh^2(t/L)\, d\Omega_{II}^2\right] - dz^2$$

(☆)

→ This is what we get for Geometry

. L is large if ε is small.

How does the geometry look like

let
$$\left.\begin{aligned} \rho &= (L-z)\cosh(t/L) \\ \tau &= (L-z)\sinh(t/L) \end{aligned}\right\} (x)$$

So, if we move away from wall; and sitting roughly at $\phi = +\eta$ or $-\eta$.

My energy momentum is vanished.

So; from the perspective of Einsteins equation, we are in vacuum; so we should have vacuum solution of Einstein Gravity.

In this case; if we have vacuum solution of Einstein gravity that looked like z^2 times something ⟹ Its kind of not clear there should be any curvature sitting outside away from wall.

lets do the coordinate transformation (x)

$\rho^2 - \tau^2 = (L-z)^2$

$ds^2 = d\tau^2 - d\rho^2 - \rho^2 d\Omega_{II}^{\,2}$ (Minkowski in polar coordinates)

but; $z=0 \longleftrightarrow \rho^2 - \tau^2 = L^2$ (hyperboloid)

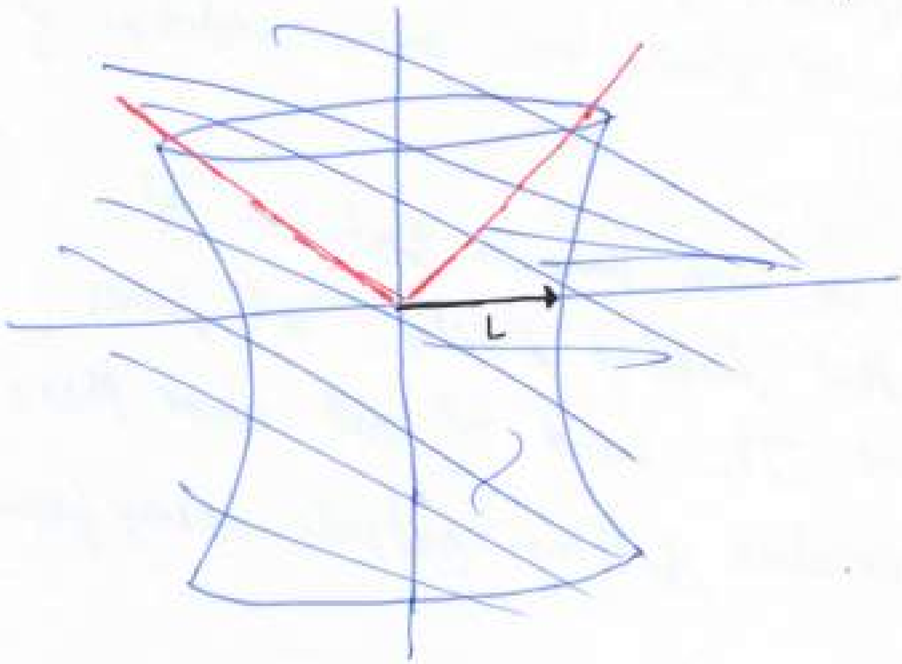
So; we got a curious ~~so~~ situation ~~so~~ where we might think there is problem at $z=\pm L$ (from equation ☆)

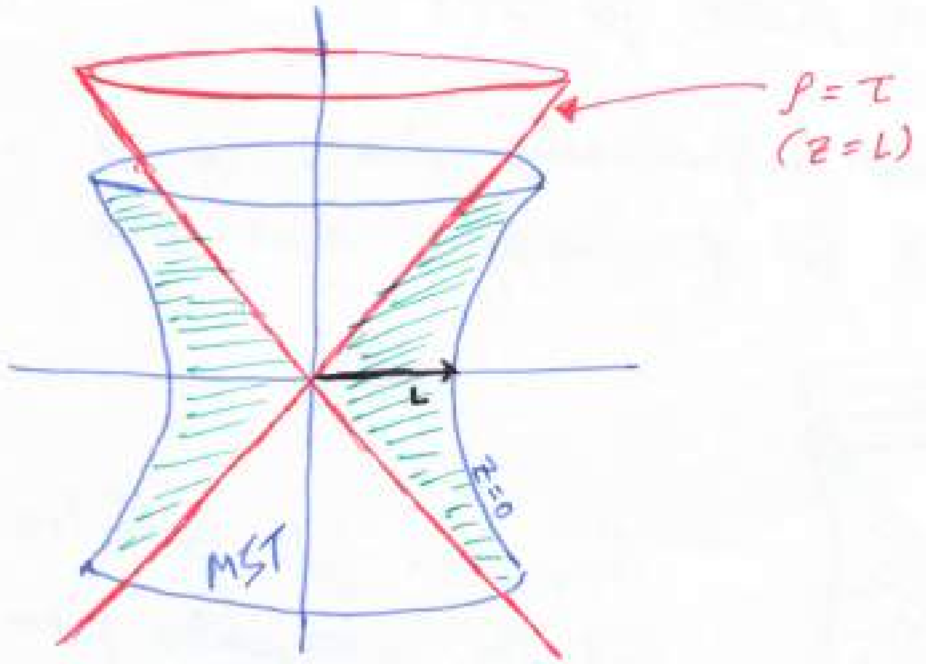
but doing coordinate transformation (*); we see that

- $z=0$ looks like hyperboloid in new coordinates.
- $z=L$ looks like is just $\rho^2 = \tau^2$

~~what if this thing has ∞ area~~ ?

~~We argue; that it does not have ∞ area.~~





The coordinates cover the shaded green region.

This shows that $z = L$ is horizon (not a singularity)

$z = L$ is a horizon.

by writing down the coordinate transformation (*) we have explicitly analytically continued across this horizon.

Inside here we just have Minkowski Space Time (MST)

This is only half the picture for $z > 0$

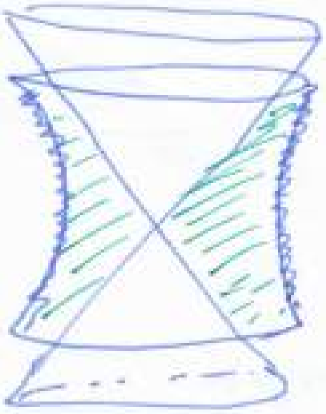
Infact if we look at $z < 0$; we would flip the sign in the defination of ρ and τ.

and it would now be $(L+z)\cosh(-)$

$(L+z)\sinh(-)$

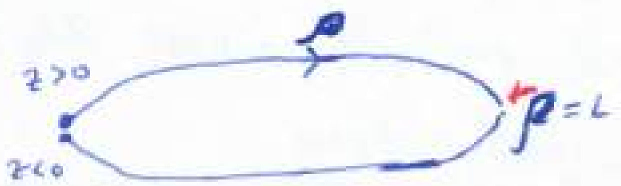
So; its the same picture for $z<0$

The wall is not infinitesimally thin; so we get little bit of fuzzyness near $z=0$.



So; our full geometry actually interpolates from interior of one hyperboloid to the interior of other.

If we look at $\tau = t = 0$

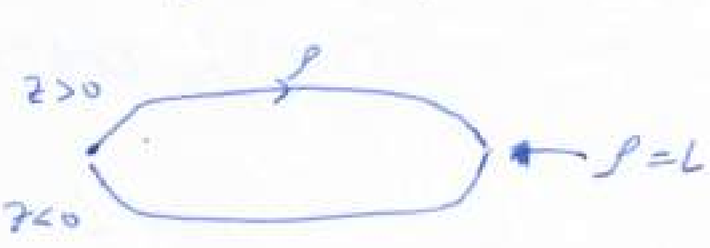


This is what happens along t = constant surface if we glue these things together & smudge out due to wall thickness

Has topology S^3

ie: it is compact.

So, the total energy of my wall is finite because the wall no longer itself has ∞ area.



"Gravity has solved infinite energy by spontaneously compactifying space."

Check, Israel Equation $8\pi G\sigma = 4/L$

$$\Rightarrow 8\pi G \cdot \frac{4}{3}\sqrt{\lambda}\,\eta^3 = \frac{4\sqrt{\lambda}\cdot\eta\,\epsilon}{3}$$

$\epsilon = 8\pi G\eta^2$

✓ fits together

This says that: The linearized calculation we did here is totally ~~co~~ consistent with Israel Approach.

In many ways, the wall behaves like a universe itself: The ~~wall~~ wall behaves like dS space and its energy momentum is Cosmological Constant.

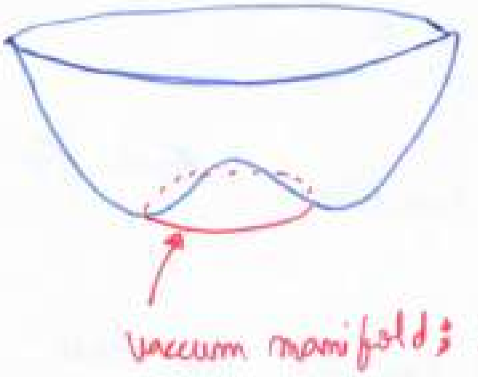
If we go beyond; and look at perturbations around the wall spacetime: we find is that these perturbations behaves like standard lowered dimensional Einstein gravity perturbation within that wall space.

We begin to see that, although we start off with higher dimensional gravity solution; we constructed something which is lower dimensional, and its behaving almost as if it is own little spacetime with its own little energy momentum & its own gravity.

↪ This leads naturally to extra dimensions in gravity.

In 4D

we can have **strings**, these also corresponds to solutions of QFT.



$\phi \in \mathbb{C}$

and ϕ seeing a sort of ~~pot~~ potential.

String: $\phi \sim \eta f(r) e^{i\theta}$

$A_\mu = \frac{1}{e} (P(r) - 1) \partial_\mu \theta$

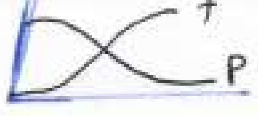
we kind of wind ~~around~~ around the vaccum ~~~~ manifold. In order to remove this winding number; we have to lift over the potential barrier along an infinite length

So; we have String being topologially stable.

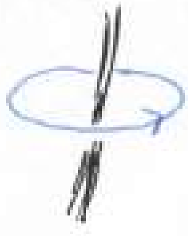
This can happen in U(1) Abelian Higgs model

Here; The functions giving us the profile are no longer analytic likee ~~tanh~~ tanh( ) function.

we have sort of



The gauge field kind of components for the winding; but since at $r=0$ we have polar coordinate singularity : we can't unwind this gauge field. It is like, localized magnetic flux of Z particle



(Magnetic flux is kind of confined in sort of Higgs core)

Lec 10 : Kaluza - Klein Theory, K K Black holes

~~4~~ 4 is an interesting no. of dimension ~~to~~ for gravity.

~~It is the~~ 4 is the lowest dimension in which gravity freely propogates. We can think of it as for gravitational waves. There are no gravitational waves in 3 dimensions.

4 is the lowest dimension in which spacetime can be Ricci flat, but not Reimann flat:
which are tidal forces essentially.

In 3d dimensions; we still have gravity but its trivial. We only get spacetime curvature where you have energy momentum: So its all localized where you have got matter.

But in 4 dimensions; we have ~~a~~ tidal forces which is like the ~~the~~ free squeezing or deforming of space.

If you go up from 4 dimensions: We find that from geometry side things are very similar; but not the physics.

In 4d; we get centrifugal repulsion looking like $1/r^2$ & gravitational attraction looking like $1/r^2$.
So; we can have these two balancing out and can have orbits.
In 5 or higher dimensions; gravitational

attraction goes like $\frac{1}{r^3}$, $\frac{1}{r^4}$, ...
& so you no longer get this balance; and you dont have solar system for example.
(Also ~~yo~~ our shoe laces will become undone in higher dimensions ☺)

42.

If extra dimensions; then must explain why we "see" 4.

KK Theory does this by having small extra dimensions and geometry independent of these

We have something like $ds_5^2 = ds_4^2 \underbrace{- L^2 d\chi^2}_{\text{size } L}$

set χ has periodicity 2π.

(The extra dimension is ~~cut~~ curled up on a scale of L)

(for many years it was assumed that this scale was plank scale; ~~so~~ but it does not have to be)

Wavefunctions that depend on χ have large masses.

$$\Psi(x, \chi) \sim \Psi(x) e^{i\chi}$$

Now if we look at ~~the~~ 5 dimensional wave operator acting on Ψ; and peel off the bit which depends on 4 dimensions $\Rightarrow$ It then looks ~~like~~ like the equation of motion for massive particle.

$$\nabla_5^2 \Psi \sim (\Box \psi + \underbrace{L^{-2}}_{m^2} \psi) e^{i \chi_m}$$

If we think L to be small ; The mass is very big.

So ; We have the following understanding:

Yes things could depend on extra dimensions but they will have very high masses. So ; if the extra dimension were something like planck scale , we would be talking about planck mass excitations which is way beyond any energy scale we are interested in.

What we are ~~interest~~ interested in something which happens at low energy , and ~~therefore we d~~ ~~therefore~~ therefore we dont have anything dependent on χ.

we could also have $\Psi(x, \chi) \sim \psi(x) e^{in\chi}$

Then $\nabla_5^2 \Psi \sim (\Box \psi + n^2 L^{-2} \psi) e^{in\chi_m}$

So ; we get tower of massive modes.

If L is small (very small) ; These modes will have high mass.

For us ; it is relevant because it can describe low energy physics : below the energy scale set by size of internal manifold.

What about gravity?

$$ds_5^2 = \hat{g}^{(4)}_{\mu\nu}\, dx^\mu dx^\nu - e^{2\sigma(x)}[\,dz + A_\mu dx^\mu]^2$$

size of internal direction.

$A_\mu = A_\mu(x)$

We split up our general metric in this way.

(It is general for now...)

we have

$$\Rightarrow \sqrt{g_5} = e^{\sigma}\sqrt{\hat{g}_4}$$

$\Rightarrow$ This is why we splitted like that to get this beautiful relation (sets ~~scen~~ scene for computing curvature)

$\hat{g}^{(4)}_{\mu\nu}$: Tensor

A_μ : vector

σ : Scalar

So, with ~~respect~~ respect to 4 dimensional spacetime; 5 dimensional metric decomposes into ~~3~~ 3 natural degrees of freedom: a tensor, a vector & a scalar.

Use Cartan (hand-out) ... (handout printed by Prof. David) example with off-diagonal terms

$$\underset{\sim}{\omega}^a = e^a{}_\mu dx^\mu \quad ; \underset{\sim}{\omega}^5 = e^\sigma [\underline{d} z + \underline{A}_\mu \underline{d} x^\mu]$$

$$\left.\begin{aligned} \underline{d}\underset{\sim}{\omega}^a &= -\underset{\sim}{\theta}_0{}^a{}_b \wedge \underset{\sim}{\omega}^b \\ \underline{d}\underset{\sim}{\omega}^5 &= \sigma_{,a}\, \underset{\sim}{\omega}^a \wedge \underset{\sim}{\omega}^5 + e^\sigma \underline{F} \end{aligned}\right\} \text{Algebraic Relations}$$

$$\rightsquigarrow \underline{\underset{\sim}{\theta}^5{}_a = \sigma_{,a}\, \underset{\sim}{\omega}^5 + \frac{1}{2} e^\sigma F_{ab}\, \underset{\sim}{\omega}^b}$$

F lies in four dimensional subspace of 2 forms.

also ; we get

$$\rightsquigarrow \underline{\underset{\sim}{\theta}^a{}_b = \underset{\sim}{\theta}_0{}^a{}_b + \frac{1}{2} e^\sigma F^a{}_b\, \underset{\sim}{\omega}^5}$$

Computation of Riemann is straightforward.

$$R^a{}_{bcd} = \hat{R}_0{}^a{}_{bcd} + \frac{1}{4} e^{2\sigma} (F^a{}_c F_{bd} - F^a{}_d F_{bc} + 2F^a{}_b F_{cd})$$

$$R^5{}_{a5b} = -\nabla_a \nabla_b \sigma - \nabla_b \sigma \nabla_a \sigma - \frac{1}{4} e^{2\sigma} F_{ac} F_b{}^c$$

$\hookrightarrow$ These are the ones in which we are interested.

$$\Rightarrow R_5 = \hat{R}_4 + \underbrace{\frac{1}{4} e^{2\sigma} F^2}_{\text{like maxwell piece}} - 2e^{-\sigma} \Box e^\sigma$$

giving the einstein action S_5

$$S_{5,EH} = \frac{-1}{16\pi G_5}\int d^4x\, dz \sqrt{g^4}\left[e^{\sigma}\hat{R}_4 + \frac{1}{4}e^{3\sigma}F^2 - 2\Box e^{\sigma}\right]$$ Pg 88

$$S_{5,EH} = \frac{-1}{16\pi G_5}\int d^4x\, dz \sqrt{g^4}\left[e^{\sigma}\hat{R}_4 + \frac{1}{4}e^{3\sigma}F^2 - 2\Box e^{\sigma}\right]$$

$\int dz$ gives length scale L

This is total derivative piece ; so no Kinetic Term for scalar

?

It looks like Einstein gravity but there are few key differences

① Scalar fields : we cant just through them away ; It is multiplying $\hat{R}_4$; also hiding with the maxwell piece.

When you do variational principle to find new Einstein equation ; we will find that derivative of σ will come up naturally as part of the integration by parts

So; it is misleading to say : " Just because ~~be there~~ there is no explicit kinetic term for scalar ; scalar does not have equations of motion "

This is an example of Scalar - Tensor gravity

(The nature of gravitational interaction is not just spin 2 ; it has got scalar field in it as well)

The scalar multiplies the Ricci tensor, & derivatives appear in the "Einstein equations" through integration by parts. 89

$\hat{g}_{ab}$ is said to be the "<u>Jordan frame</u>".
(It is a choice of metric in which gravitational ~~action~~ action has scalar multiplied)

We can conformally transform our metric

$$\hat{g}_{ab} = \Omega^2(x)\, g_{ab}$$

Then $\hat{R} = \Omega^{-2}(R - 6\Omega^{-1}\Box\Omega)$

choose $\Omega = e^{-\sigma/2}$ (a particular value of ~~conformal~~ conformal factor related to scalar field)

$$e^{\sigma}\cdot\sqrt{\hat{g}}\,\hat{R} = e^{\sigma}\,\underbrace{\Omega^4\sqrt{g}}\;\underbrace{\Omega^{-2}(R - 6\Omega^{-1}\Box\Omega)}$$

$$= \underbrace{\sqrt{g}\;(R}_{\text{Standard Einstein Action}} + 3\Box\sigma \underbrace{- \tfrac{3}{2}(\nabla\sigma)^2}_{\text{Kinetic term}})$$

"Conformal transformations should not changed degrees of freedom"

~~to~~ We got gradient term ~~to~~ ~~[illegible]~~ in our E.O.M. by integration by parts on the Jordan frame; we will still be getting gradients on the new frame, which we call Einstein Frame.

Final step of getting K.K. action : redefine your scalars.

Redefine $\varphi = \frac{\sigma}{\sqrt{3}}$ to canonically normalize

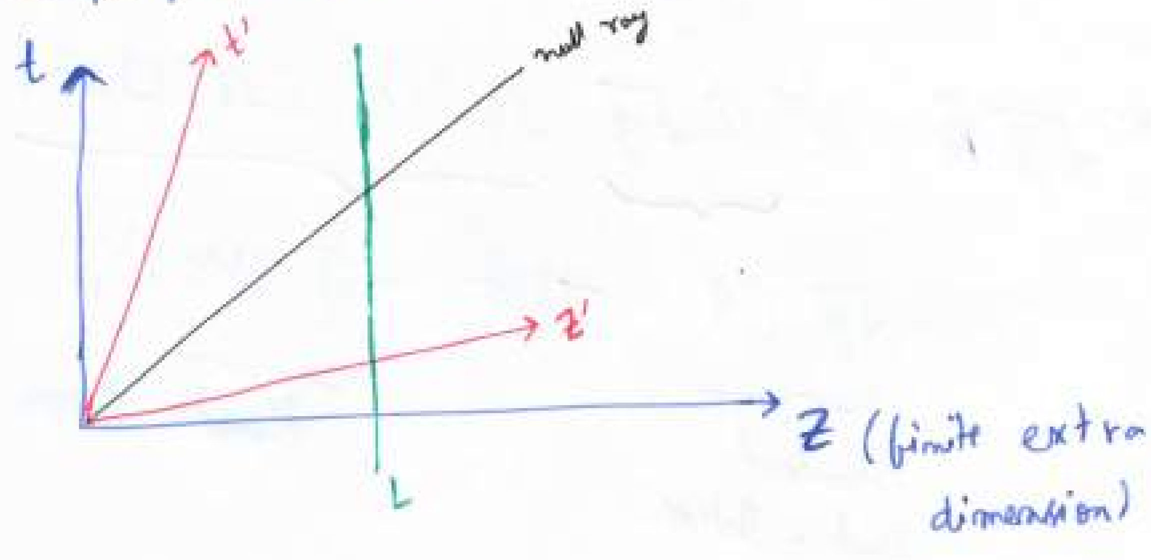
Kinetic terms

$$S = \frac{1}{16\pi G_5/L} \int d^4x \sqrt{g}\left(-R + \frac{1}{2}(\partial\phi)^2 - \frac{e^{\sqrt{3}\varphi}}{4} F^2\right)$$

$$ds^2 = e^{-\sqrt{3}\phi}\, \hat{g}^{(4)}_{\mu\nu} dx^\mu dx^\nu - e^{2\sqrt{3}\phi}\left[dz + A_\mu dx^\mu\right]^2$$

KK Black Holes

Note: Compactification picks a rest frame.



$z \sim z + L$ (at constant T)

If we do Lorentz transformations,

$$\left.\begin{aligned} t' &= \gamma(t - vz) \\ z' &= \gamma(z - vt) \end{aligned}\right\} \text{ Then } z \sim z + L \Rightarrow \begin{aligned} z' &\sim z' + \gamma L \\ t' &\sim t - v\gamma L \end{aligned}$$

So; we are identifying points at different times.

What does this do to our black hole solutions?

lets start by getting Black hole solution; rather Black Brane solution,

because we started off by saying that geometry could not depend on extra dimension.

So; we cant have a Black Hole in 3d; it got to be translationally invariant in z-direction.

& fortunately it turns out that,

SCH $\times \mathbb{R}$ is solution of 5D gravity

(Black String...)

(SCH ≡ Schwarzschild abbreviation.)

$$ds^2 = \left(1-\frac{2GM}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2GM}{r}\right)} - r^2 d\Omega_{II}^2 - dz^2$$

Black String.

This is simplest KK Black hole solution; uncharged; nothing happening in z direction.

---

now lets do boost on that;

and then identify (and see what we have)

Boost before identifying.

$$ds^2 = \left(1-\frac{2GM}{r}\right)\frac{(dt+v\,dz)^2}{1-v^2} - \frac{dr^2}{1-\frac{2GM}{r}} - \frac{(dz+v\,dt)^2}{1-v^2} - r^2 d\Omega_{II}^2$$

Now identify; $z \sim z + L$, and do the compactification.

and rearrange to get. (boost in z direction, and then do identification).

$$ds^2 = \left(1-\frac{2GM}{(1-v^2)r+2GMv^2}\right)dt^2 - r^2 d\Omega_{II}^2 - \frac{dr^2}{1-\frac{2GM}{r}}$$

$$- \left(1+\frac{2GMv^2}{(1-v^2)r}\right)\left[dz + \frac{2GMv\,dt}{(1-v^2)r+2GMv^2}\right]^2$$

When we rearrange; and complete the square for the dz part of the metric: we know that the gauge field $A_\mu dx^\mu$ is just going to have a t component.

This says that: boost in z direction has given us effective electric charge.

This shows; we can identify velocities in extra direction with electric charge in the sense that it is Kaluza Klein electric charge.

Velocity in extra dimension $\longleftrightarrow$ electric KK charge

Now; its natural to redefine coordinates the new metric we got after boost to make it look like old one say.

Note! The black string with no velocity in extra dimension; and the black string which is ~~going &~~ moving in extra ~~dimension~~ dimension are two distinct solutions.

So; The velocity in the extra direction is showing up as an electric charge from 4d perspective (because we got A_t comming from lorentz transformation)

Write $\hat{r} = r + \dfrac{2GMv^2}{1-v^2}$; $q = \dfrac{2GMv}{1-v^2}$

$\hat{M} = \dfrac{2GM}{1-v^2}$

→ Transformation from Jordon frame to Einstein frame.

Then; get the following for einstein frame metric

$$g_{\mu\nu}\, dx^\mu dx^\nu = \left(1-\frac{2G\hat{M}}{\hat{r}}\right)\left(1-\frac{vq}{\hat{r}}\right)^{-1/2} dt^2 - \left(1-\frac{vq}{\hat{r}}\right)^{3/2} \hat{r}^2\, d\Omega_{II}^2$$

$$- \left(1-\frac{2G\hat{M}}{\hat{r}}\right)^{-1}\left(1-\frac{vq}{\hat{r}}\right)^{1/2} d\hat{r}^2$$

→ K.K. electric Black hole.

This look similar to Reiner-Nordstom black hole;
but has some differences.

Just like R-N. ; we can have extremal limit for K.K. electric black hole $\Rightarrow$ But the extremal limit here where q goes to M ; Then this is not singular

... There are differences in terms of properties of Black Hole

---

For a magnetic black hole.

$$\underline{F} = Q \sin\theta \, \underline{d}\theta \wedge d\varphi$$

Take the field strength; and make it proportional to the area form on the sphere surrounding your object. (in this case it is 2 sphere)

$$\underline{d}\,\underline{F} = 0$$

but $\underline{F} \neq \underline{d}\,\underline{\omega}$ expect locally.

This is an example of form which is closed, ie $\underline{d}\underline{F} = 0$

but not exact ie; not expressable as
$\underline{F} = \underline{d}\,\underline{\omega}$

When you have closed forms, which are not exact

$\Longleftrightarrow$ It is ~~~~ usually a sign of something happening with the topology of the ~~space~~ space.

In this case, our topology is non-trivial.

We got non-trivial 2-sphere: we got event horizon; if we ~~~~ try & shrink 2-spheres we hit the event horizon.

~~So~~ We can of course write $\underset{\sim}{F} = d\underset{\sim}{A}$ locally.

ie; locally, ~~A~~

N, S ~~a~~ stands for North & South.

$$\underset{\sim}{A}_N = Q\,(1-\cos\theta)\,\underset{\sim}{d}\varphi$$
$$\underset{\sim}{A}_S = -Q\,(1+\cos\theta)\,\underset{\sim}{d}\varphi$$
$$\underset{\sim}{A}_S = \underset{\sim}{A}_N - 2Q\,\underset{\sim}{d}\varphi$$

we can write; magnetic solution as this ... gauge part

A_N works everywhere ~~at~~ around the north pole; but the south pole axis

& A_S works everywhere, but the north pole axis.

And the two are related by gauge transformations

$$\underset{\sim}{A}_S = \underset{\sim}{A}_N - 2Q\cdot\underset{\sim}{d}\varphi$$

With this Ansatz;
find the solution

$$ds^2 = \left(\frac{r-r_+}{r-r_-}\right)dt^2 - \frac{dr^2}{\left(1-\frac{r_+}{r}\right)} - r(r-r_-)\, d\Omega_{II}^2$$

$$- \left(1 - \frac{r_-}{r}\right)\left[dz_N + Q(1-\cos\theta)\, d\varphi\right]^2$$

$$Q \downarrow \sqrt{r_+ r_-}$$

$$\underline{A}_N$$

We have two horizon; inner & outer

We could also have the south pole patch

$Z_S = Z_N + 2Q\cdot\varphi$ for S pole patch.

$= Z_N + 2Q(\varphi + 2\pi) \sim Z_S + L\cdot n \;; n\in\mathbb{Z}$

If we go round in the φ direction (round the circle) we go to $\varphi + 2\pi$.

(The two periodicity has to be in allignement)

This tells us that ;

$$4\pi Q \in L\cdot\mathbb{Z} \quad (\text{because } z \sim z+L)$$

This means that charge is quantized.

(similar to dirac quantization of electric charge ...)

Extremal Limit

$r_+ = r_-$; $Q = r_+ = \frac{L}{4\pi}$

let $\chi = \frac{4\pi z}{L}$ so; The peridicity of χ is now 4π

$(\Delta\chi = 4\pi)$

$$ds^2 = dt^2 - \frac{dr^2}{1 - Q/r} - \left(1 - \frac{Q}{r}\right)\left[r^2 d\Omega_{II}^2 + Q^2 (d\chi_N + A_N)^2\right]$$

$\Leftrightarrow$ so; if $r_+ = r_-$; our time coordinate now becomes an extra coordinate tacked on to the geometry; and we have some non-trivial spacial manifold which a kind of just exists in time.

so; when $r \longrightarrow Q$ (what happens)

$$\rho^2 = 4Q(r - Q)$$

$$ds^2 \sim dt^2 - d\rho^2 - \underbrace{\frac{\rho^2}{4}\left[d\theta^2 + \sin^2\theta\, d\varphi^2 + (d\chi + (1-\cos\theta)\, d\varphi)^2\right]}_{\text{Origin of } \mathbb{R}^4 \text{ in Euler Angles}}$$

(used Euler angles to describe three spheres)

$$\left.\begin{aligned} x + iy &= \rho \cos\frac{\theta}{2} \cdot e^{i\chi/2} \\ z + iw &= \rho \sin\theta/2 \cdot e^{i(\varphi + \chi/2)} \end{aligned}\right\} (*)$$

While doing the transformation (*) :

fibring S^2 by S^1 with a "twist"

(when we take KK compactification ; ρ adds a little manifold ; here its a little circle)

and this "lifts" S^2 to an S^3 (instead of $S^2 \times S^1$) (because of twist)

This is an example of HOPF FIBRATION.

Symmetries of S^3, $SO(4) \sim SO(3) \times SO(3)$

or $SU(2) \times SU(2)$ locally algebra is the same.

We see this ;

Transformations acting Left & Right.

This gives two $SO(3)$ Killing Algebra.

Lec 11: Black Branes : higher-dimensional Schwarzchild , p-form charges, D3 branes, black string instability

Have seen black string $SCH \times \mathbb{R}$

$$ds^2 = \left(1-\frac{2GM}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2GM}{r}\right)} - r^2 d\Omega_{II}^2 - dz^2$$

$$f(r) \longrightarrow K - \frac{2GM}{r} - \frac{\Lambda}{3}r^2$$

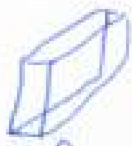
K is curvature of space $d\Omega_{II}^2$ surrounding the black at constant distance from Black Hole.

In 4d we have black hole.
In higher dimension; say 5d we have black string.
In " " we have more possibility of "black holy" kind of thing.

We can add more extra dimension (sort of flat space)

$$ds^2 = \left(1-\frac{2GM}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2GM}{r}\right)} - r^2 d\Omega_{II}^2 - dz_1^2 - \dots - dz_m^2$$

$\longrightarrow$ also a solution of Einstein equation.



Can add extra dimensions to sort of end ~~with~~ up with Black String, Black membrane, etc.

Recall the domain wall metric

$$ds^2 = A^2(z)\, \gamma_{\mu\nu}\, dx^\mu dx^\nu - dz^2$$

Using Cartan we found the connection.

$$\left.\begin{array}{l} \underset{\sim}{\theta}{}^a{}_b = \underset{\sim}{\theta}{}_0{}^a{}_b \\ \underset{\sim}{\theta}{}^a{}_z = \frac{A'}{A}\underset{\sim}{\omega}{}^a \end{array}\right\} \rightarrow \begin{array}{l} R^\mu{}_\nu = \frac{1}{A^2}R_0{}^\mu{}_\nu + \left((D-2)\frac{A'^2}{A^2} + \frac{A''}{A}\right)\delta^\mu_\nu \\ R^z{}_z = (D-1)\frac{A''}{A} \end{array}$$

This suggest that we can Generalize to higher dimensional thing (by simply bolding together ~~which have their own~~ different subspaces that have their own warp factors)

$$ds^2 = A^2(z)\underbrace{g_{\mu\nu}\,dx^\mu dx^\nu}_{\substack{\text{first space} \\ \text{containing time} \\ (1+P)\text{ dimensional}; \; 0,1,\dots P \\ \text{(Lorentzian)}}} - dz^2 - C^2(z)\underbrace{\gamma_{\alpha\beta}\,dy^\alpha dy^\beta}_{\substack{n \text{ dimensional} \\ + + \dots + \\ \text{(Euclidean.)}}}$$

$$ds^2 = A^2(z)g_{\mu\nu}\,dx^\mu dx^\nu - dz^2 - C^2(z)\gamma_{\alpha\beta}dy^\alpha dy^\alpha$$

Its quite clear; from the way it is set up:

At the connection level; the μ & α part of the metric dont mix.

g and γ don't mix at connection level.

The non-zero pieces are as follows:

$$\underset{\sim}{\theta}{}^{\hat{\mu}}{}_{\hat{\nu}} = \underset{\sim}{\theta}{}_0{}^{\hat{\mu}}{}_{\hat{\nu}} \qquad \underset{\sim}{\theta}{}^{\hat{\alpha}}{}_{\hat{\beta}} = \underset{\sim}{\theta}{}_0{}^{\hat{\alpha}}{}_{\hat{\beta}}$$

$$\underset{\sim}{\theta}{}^{\hat{\mu}}{}_{\hat{z}} = \frac{A'}{A}\underset{\sim}{\omega}{}^{\hat{\mu}} \qquad \underset{\sim}{\theta}{}^{\hat{\alpha}}{}_{\hat{z}} = \frac{C'}{C}\underset{\sim}{\omega}{}^{\hat{\alpha}}$$

2 forms are largely the same,

but have

$$\underset{\sim}{R}^{\hat{\mu}}{}_{\hat{\alpha}} = \underset{\sim}{\theta}^{\hat{\mu}}{}_{\hat{z}} \wedge \underset{\sim}{\theta}^{\hat{z}}{}_{\hat{\alpha}}$$

$$\boxed{\underset{\sim}{R}^{\hat{\mu}}{}_{\hat{\alpha}} = \frac{A'C'}{AC}\, \underset{\sim}{\omega}^{\hat{\mu}} \wedge \underset{\sim}{\omega}_{\hat{\alpha}}}$$

After calculations we finally get

$$R^{\mu}{}_{\nu} = A^{-2} R_0{}^{\mu}{}_{\nu}(g) + \left(\frac{A''}{A} + p\frac{A'^2}{A^2}\right)\delta^{\mu}{}_{\nu} + n\frac{A'C'}{AC}$$

$$R^{\alpha}{}_{\beta} = C^{-2} R_0{}^{\alpha}{}_{\beta}(\gamma) + \left(\frac{C''}{C} + (n-1)\frac{C'^2}{C^2}\right)\delta^{\alpha}{}_{\beta} + (p+1)\frac{A'C'}{AC}$$

$$R^{z}{}_{z} = (p+1)\frac{A''}{A} + n\cdot\frac{C''}{C}$$

building blocks for finding (simple) brane solutions

Reimann tensor tend to split in two parts;

The curvature is splitted; (the curvature is inherited if g or γ metric does; and pieces due to the warp factor A or C }

This is what you get; (say the g space of γ space changing as we move in z-direction)

& ; and also the last piece ~~A~~ mixing $\frac{A'C'}{AC}$

(cross talk between two subspaces)

Note: for the moment we are not doing rotations...

e.g.) $\cancel{\text{SH}}$ $\underline{SCH_D}$ (schwarzschild solution in D dimensions)

$$\underline{A^2 dt^2 - dz^2 - C^2 d\Omega^2_{D-2}}$$

here $n = D-2$, $P = 0$

~~So~~ now; we change gauge~~s~~ and get it in following form $C = r$, $dz = \frac{dr}{B}$ (*)

This gauge (*) splits the solution in more familiar form.

$$C' = \frac{dc}{dz} \longrightarrow B \; ; \; C'' \longrightarrow B\frac{dB}{dr}$$

From Ricci's

$$G^0_0 = -(D-2)\frac{B}{r}\frac{dB}{dr} - \frac{(D-2)(D-3)}{2r^2}(B^2-1)$$

but $G^0_0 = 8\pi G_D T^0_0 + \Lambda$ (by Einstein's equation)

(we get possibly the same thing what we got for normal SCH solution)

$$\Rightarrow B^2 = 1 - \frac{16\pi G}{(D-2)r^{D-3}}\underbrace{\int r^{D-2}\, T^0_0\, dr}_{} + \left(\frac{-2}{(D-2)r^{D-3}}\int r^{D-2}\Lambda\right)$$

This is total energy divided by area A_{D-2} of $D-2$ sphere (at least in linearized sense) M/A_{D-2}

for total mass $\int T^0_0$ (volume)

$$\text{volume} = dr\, r^{D-2} \cdot (\text{Angles})$$

The integral over $\int(\text{Angles})$ gives A_{D-2}.

So; we end up with.

$$B^2 = 1 - \frac{16\pi G}{(D-2) A_{D-2}} \cdot \frac{M}{r^{D-3}} - \frac{2\Lambda r^2}{(D-2)(D-3)}$$

$A_{D-2} \Rightarrow$ area of S_{D-2}. ($D-2$ sphere)

If $\Lambda = 0$, this gives us the SCH radius in terms of mass

$$r_s = \left(\frac{16\pi G M}{(D-2) A_{D-2}}\right)^{1/D-3}$$

Note; Λ piece remains proportional to r^2

Λ is kind of not changing so much.
The only thing that is changing is specific relation between Λ and AdS or dS length scale.

Λ remains an r^2 dependent, dimension shows up in relation to radius of curvature.

$$\frac{1}{l^2} = \frac{2|\Lambda|}{(D-2)(D-3)} \quad (\text{AdS or dS})$$

$$B^2 = 1 - \frac{16\pi G}{(D-2) r^{D-3}} \int r^{D-2}\, T^0_0\, dr + \left(\frac{-2}{(D-2)\, r^{D-3}} \int r^{D-2} \Lambda\right)$$

Charged Black Hole

Charge ?

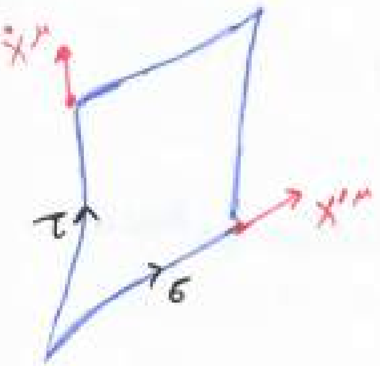
In 4D, a point particle comes electric (and magnetic) charge

$$\mathcal{L}_0 \sim q\int d\tau\, \dot{X}^\mu A_\mu \qquad \int^{x^\mu(\tau)}$$

Thinking classically;

if we want point particle with charge; we add an extra piece to Lagrangian which would be coupling the tangent vector to gauge field A_μ (and this will tell how to modify our geodesic equations)

How to construct Charged String:



string worldsheet.

We get two vectors $\dot{X}^\mu$ & X'^μ that tell us about the string: They are parallel to string worldsheet.

This suggests that we have two index object (one index for each tangent vector)

How could that be gauge field & how could it make sense:

Suggests : $\int d\tau\, d\sigma\, \dot{X}^\mu X'^\nu B_{\mu\nu}$

If we make $B_{\mu\nu}$ a 2-form

Then our gauge transformation is $\underline{B} \longrightarrow \underline{B} + \underline{d}\,\underline{A}$

(The 2 form has a gauge transformation just like the gauge transformation for electromagnetism

$$A \to A + d\phi.$$

where ϕ is scalar)

& our field strength is going to be $\underline{H} = \underline{d}\,\underline{B}$

$$\left.\begin{aligned} &\underline{B} \longrightarrow \underline{B} + \underline{d}\,\underline{A} \\ &\underline{H} = \underline{d}\,\underline{B} \end{aligned}\right\}$$

(Analogue of $\underline{A} \to \underline{A} + d\varphi$
$\underline{F} = \underline{d}\,\underline{A}$

Now, we see how to construct general electrically charged objects by simply instead of having 2 forms, we have $(p+1)$ forms.

In general $\underline{A}^{(p+1)}$ gives an electrically charged p-brane.

(This is one way of extending charges ; .. or say $U(1)$ gauge group to higher dimensions by simply extending the notion of what we mean by gauge potential)

What do we mean by Magnetic Solution?

Magnetic Solution

In 4D electromagnetism.

$F_{\theta\phi} \sim Q \sin\theta$

so: $\underline{F} \sim Q\,\underline{\epsilon}_2$ $\qquad \underline{\epsilon}_2 \Rightarrow$ Unit S^2 are form.

In general, $\underline{H} \propto \underline{\epsilon}_{p+2}$; ~~area form~~

where $\underline{\epsilon}_{p+2}$ is area form on $(p+2)$ - share.

(Now think of $(p+2)$ ~~sphere~~ sphere surrounding our black-brane ; and therefore this means that we got $(p+2)$ dimension surrounding the brane ; one dimension which is z or r coordinate

so; $(p+2)+(1) = (p+3)$

So; The dimensionality of the object is $D-(p+3)$

which is $D-(p+4)$ brane.

Low Energy string supergravity has severfal form fields

$$S = -\int d^{10}x \sqrt{g}\cdot e^{-2\phi}\left[R + (\nabla\phi)^2 - \frac{1}{12} H^2_{abc} \right)$$

When we take an effective low energy gravity theory from String Theory we ~~at~~ always get this initial piece & which is comming from basic **excitation** of basic fields that live on the String **World** sheet)

H is of course the field strength of the B

So, the string always has $B_{\mu\nu}$ living on its world sheet.

But depend on other supersymmetries we ask for we get the following schematic for.

$$(-)^{p+1} \frac{2(\partial A_p)^2}{(p+1)!}$$

~~p odd IIA.~~
~~p even IIB.~~

p odd IIA
p even IIB

If we think again going back to electric & magnetic solutions in 4D,
point objects (0-branes) have both e, Q charges

e → electric charge
Q → magnetic charge.

In general D dimensions, to have same dimensionality for electric & magnetic charges

$$p-1 = D-p-3$$

$$\Rightarrow \boxed{p = \frac{D-4}{2}} \Rightarrow \underline{p = 3 \text{ in } 10\,D}$$

⟶ This gives ~~self~~ self-dual ~~3-~~ 3-brane in IIB

$$ds^2 = \left(1 - \frac{r_+^4}{r^4}\right)^{1/2}(dt^2 - d\underline{x}^2) - \frac{dr^2}{\left(1-\frac{r_-}{r}\right)^2} - r^2 d\Omega^2_{\text{V}}$$

$$\underline{F} = Q\,(\underline{\epsilon}_5 + *\,\underline{\epsilon}_5)$$ (it is self dual)

$$\underline{F} = *\,\underline{F}$$

$$r_+^4 = 4\pi g_s N \alpha'^2$$

g_s: string coupling

N: no. of unit charged D3's.

In 4 dimensions; we have point particle, and it can be electrically or magnetically charged.

If we were in 5 dimensions; our electric charged object in Einstein - Maxwell theory will still have to be a point particle because we still get that A_μ coupled to $\dot{x}^\mu$.

But, the magnetic object would now be a string.

$F_{\mu\nu} \propto$ 2 sphere.

a 2-sphere in 4 spatial dimension surrounds an extended object.

So; If we change gauge

$$\rho^4 = r^4 - r_+^4$$

we get a kind of usual expression.

$$ds^2 = \left(\frac{\rho^4 + 4\pi g_s N\alpha'^2}{\rho^4}\right)^{+\frac{1}{2}}(dt^2 - d\underline{x}^2) - \left(\frac{\rho^4 + 4\pi g_s N\alpha'^2}{\rho^4}\right)^{+1/2} d\rho^2$$

$$- \left(\rho^4 + 4\pi g_s N\alpha'^2\right)^{1/2} d\Omega^2_{\text{V}}$$

$$\cancel{ds^2 = \left(\frac{\rho^4+4\pi g_s N\alpha'^2}{\rho^4}\right)^{-1/2}(dt^2-d\underline{x}^2) - \left(\frac{\rho^4+4\pi g_s N\alpha'^2}{\rho^4}\right)^{-1/2} d\rho^2 - (\rho^4+4\pi g_s N\alpha'^2)^{1/2} d\Omega^2_{\mathrm{V}}}$$

$$ds^2 = \left(\frac{\rho^4+4\pi g_s N\alpha'^2}{\rho^4}\right)^{+1/2}(dt^2-d\underline{x}^2) - \left(\frac{\rho^4+4\pi g_s N\alpha'^2}{\rho^4}\right)^{-1/2} d\rho^2 - (\rho^4+4\pi g_s N\alpha'^2)^{1/2} d\Omega^2_{\mathrm{V}}$$

Here we got an offset from $\rho = 0$

(what happens as we go to $\rho = 0$? which is event horizon or would be event horizon of this object))

lets suppose N is large.

$$4\pi g_s N\alpha'^2 \equiv R^4$$

and now look at $\rho << R$.

so; we have

$$ds^2 \approx \underbrace{R^2\left[\frac{(dt^2-d\underline{x}^2)}{\rho^2} - \frac{\rho^2 d\rho^2}{R^4}\right]}_{} - R^2 d\Omega^2_{\mathrm{V}}$$

This is $AdS_5 \times S_5$

$R^2\left[\frac{dt^2-d\underline{x}^2}{\rho^2} - \frac{\rho^2 d\rho^2}{R^4}\right]$ looks like AdS_5

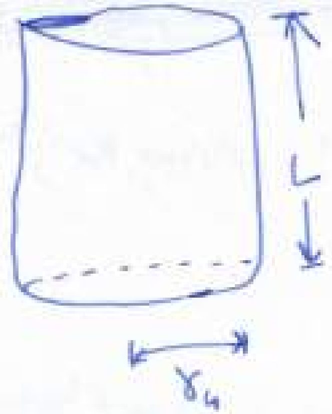
$R^2 d\Omega^2_{\mathrm{V}}$ is like S_5..

Stability?

look at KK black string.

Think about Black string in KK sense; and look at length L of Black String.

$SCH \times \mathbb{R}$



mass $M = \dfrac{r_4 L}{2G_5}$

Entropy $S_{BS} = \dfrac{\pi r_4^2 L}{G_5}$

entropy of Black String.

Now; lets look at 5d Black Hole solution SCH_5

Black Hole with radius r_5

$$\text{mass}: M = \frac{3 \times 2\pi^2 \times r_5^2}{16\pi G_5}$$

$$= \frac{3\pi r_5^2}{8 G_5}$$

$$\Rightarrow M = \frac{3\pi r_5^2}{8G_5}$$

$$\text{Entropy}: S_{BH} = \frac{\pi^2 r_5^3}{2G_5} = \frac{\pi^2}{2G_5}\left(\frac{8G_5 M}{3r}\right)^{3/2}$$

Now lets compare the entropies.

$$\frac{(S_{BH})_{SCH_5}}{(S_{BH})_{SCH \times \mathbb{R}}} = \frac{\frac{\pi^2}{2G_5}\left(\frac{8G_5 M}{3\pi}\right)^{3/2}}{\frac{4\pi M^2 G_5}{L}} = \frac{\sqrt{\frac{8}{27\pi}}}{\sqrt{G_5 M}} \cdot L$$

$$\frac{(S_{BH})_{SCH_5}}{(S_{BH})_{SCH \times \mathbb{R}}} = \sqrt{\frac{8L}{27\pi G_5 M}}$$

~~If $L > \frac{27}{8}\pi G_5 M$~~

~~The black hole will have bigger entropy~~ as compar.

If $L > \frac{27}{8}\pi G_5 M$: Then Black Hole SCH_5 will have bigger entropy as compared to Black String $SCH \times \mathbb{R}$

Black Hole is preferred.

From entropy thermodynamic perspective; If we try to make the string too long and thin ⇒ it will ~~...~~ prefer to clumb into a black hole.

This entropy argument suggests that Black String is unstable.

Suggests an instability of black string.

Confirmed by ~~B~~ Linearized analysis. (RG + Laflamme)

and full non-linear numerics ~~...~~ (Pretoni, Lehner)

There is an instability:

Black String

makes event horizon kind of wobble.

So, Black String is definitely unstable in 5 dimensions,
but it seems like at the end points of unstability,
that the Black String fragments into Black holes ;
and if this is the case, then at the moment of
fragmentations we should get sort of instantaneous
naked singularities
(looks like cosmic censorship is violated in higher
dimentions)

Lec 12: Warped compactifications: Rubakov - Shaposhnikov & Randall - Sundrum toy models.

Warped Compactification

(an alternative way to hide extra dimensions)

Alternative way of hiding extra dimensions is to confine particle physics to a submanifold.

by doing confinement; rather than the process of things not depending on extra dimensions which we did in KK theory.

Toy model (Rubakov + Shaposhnikov)

Domain wall $\phi \sim \eta \tanh(\sqrt{\lambda}\cdot\eta z)$

Couple the scalar to fermion

$$\mathcal{L}_\Psi = i\bar{\Psi}\Gamma^a \nabla_a \Psi - g\phi\bar{\Psi}\Psi$$

Ψ is 5 dimensional fermion.
Γ^a is " " Γ matrices.

~~Normally we expect~~ because ϕ profile is non-trivial the lowest energy profile for Ψ is non-trivial - a condensate.

$$i\gamma^5\Psi' = g\eta\tanh(\sqrt{\lambda}\cdot\eta z)\Psi$$

$$\rightarrow \Psi = \Psi_0\left[\operatorname{sech}(\sqrt{\lambda}\eta z)\right]^{g/\sigma}$$

if $i\gamma^5\Psi_0 = -\Psi_0$ chiral.

how strongly localized will depend on coupling g.

ϕ shifting in background $\to$ provides shift in mass for spinor.

Also if $\not{\partial}\Psi_0 = 0$, this solves 5D equations.
(The Background of domain wall provided the shift in, what we might think is lowest energy is for Ψ profile)
$\Longleftrightarrow$ We get non trivial profile around the wall, and call it Condensate.

ie; Also, if $\not{\partial}\Psi_0 = 0$, this solves 5D equations. A massless chiral fermion "confined" to brane.

$\Longleftrightarrow$ Provides notion for localizing particle physics ✓

What about Gravity?

Toy model (Randall - Sundrum)
domain wall in AdS space.

Brane is flat

$$ds^2 = A^2(z)\,\eta_{\mu\nu}\,dX^\mu dX^\nu - dz^2$$

∞ extra dimension

$e^{-2k_c|z|}$ or $e^{-2|z|/l}$

$z=0$

sharply peaked warp factor around $z=0$.

∞ extra dimension; geometry depends only on Z.
(completely opposite to KK compactification)

Recall Gauss Equations:

$$ {}^{(4)}R^{a}{}_{bcd} = h^{a}_{a'} h^{b'}_{b} h^{c'}_{c} h^{d'}_{d} \, {}^{(5)}R^{a'}{}_{b'c'd'} $$
$$ - K^{a}{}_{c} K_{bd} + K^{a}{}_{d} K_{bc} $$

Israel Equation: $\Delta K_{ab} - \Delta K \cdot h_{ab} = 8\pi G S_{ab}$

Now; we will show in a general way; how we might get 4D gravity on this sharply localized solution inspite of the fact there is ∞ extra dimension.

$$ S_{ab} \xrightarrow[\text{will be}]{} (\underbrace{\sigma h_{ab}}_{\text{wall part}} + \underbrace{T_{ab}}_{\text{matter on Brane}}) $$

Assume that, the bulk remains 5D pure AdS.

$D=5$, ${}^{(5)}R_{abcd} = \frac{1}{\ell^2}[g_{ac}g_{bd} - g_{ad}g_{bc}]$

where $\Lambda = -6/\ell^2$

RS has spacetime $\mathbb{Z}_2$ - symmetry around brane.

If $T_{ab} = 0$,

Then $K^{+}_{ab} = -\Gamma^{z}_{ab} n^{+}_{z} = \frac{1}{2} g_{ab,z} = -\frac{1}{\ell} h_{ab}$

$K^{-}_{ab} = -K^{+}_{ab}$ ($\mathbb{Z}_2$ symmetry)

Israel Equation for $\mathbb{Z}_2$ symmetry is

$K^{+}_{ab} - K^{+} h_{ab} = \left(\frac{3}{\ell} h_{ab}\right) = 4\pi G \sigma h_{ab}$

Now add T_{ab}:

$$K^+_{ab} = 4\pi G\left(S_{ab} - \frac{1}{3} S h_{ab}\right)$$ (By taking trace of Israel equation...)

$$\Rightarrow K^+_{ab} = 4\pi G\left(-\frac{6}{3} h_{ab} + T_{ab} - \frac{1}{3} T h_{ab}\right)$$

$$= -\frac{1}{\ell} h_{ab} + 4\pi G\left(T_{ab} - \frac{1}{3} T h_{ab}\right)$$

$$\boxed{\begin{aligned} &\text{Einstein 4D gravity; } G_{ab} = 8\pi G_{(4)} T_{ab} \\ &\text{or } R_{ab} = 8\pi G_{(4)}\left(T_{ab} - \frac{1}{2} T h_{ab}\right)\end{aligned}}$$

<u>Contract Gauss Equation</u>: (& average over brane)

(If we contract the Gauss eqn; we still kind of get hidden δ function in there because 5d geometry has discontinuity in derivative of warp factor)

We are kind of taking average; rather than difference across the brane.

$${}^{(4)}R_{ab} = \frac{3}{\ell^2} h_{ab} + K^+_{ac} K^{+c}{}_b - K^+ K^+_{ab}$$

~~[illegible]~~ use Israel equation to replace K^+..

$${}^{(4)}R_{ab} = \frac{3}{\ell^2} h_{ab} + \left(\frac{1}{\ell} h_{ac} - 4\pi G\left[T_{ac} - \frac{1}{3} T h_{ac}\right]\right)\left(\frac{1}{\ell} h^c_b - 4\pi G\left[T^c_b - \frac{1}{3} T h^c_b\right]\right)$$
$$- \left(\frac{4}{\ell} + \frac{4\pi G}{3} T\right)\left(\frac{1}{\ell} h_{ab} - 4\pi G\left[T_{ab} - \frac{1}{3} T h_{ab}\right]\right)$$

$$^{(4)}R_{ab} = \frac{8\pi G_5}{\ell} T_{ab} - \frac{4\pi G_5}{\ell} T \cdot h_{ab}$$
$$+ (4\pi G_5)^2 \left[T_{ac} T^c{}_b - \frac{1}{3} T T_{ab} \right]$$

$$= 8\pi G_4 \left(T_{ab} - \frac{1}{2} T h_{ab} \right) + \underbrace{(4\pi G_5)^2 \tau_{ab}}_{O(T^2_{ab})}$$

so: $G_{(4)} = \frac{G_{(5)}}{\ell}$

if doing perturbation expansion.

To leading order, we get 4D einstein gravity. living on the brane.

Note: $G_{(4)}$ is a derived quantity

Newtons constant.

ℓ gives us a shift in Planck scale.

Heuristically: $M_p^2 = M_D^{D-2} L^{D-4}$

If we integrate over some extra dimension give us effective cutoff because of the warping (this is what AdS is doing; Negative curvature is kind of putting us in a box)

OR

because in KK theory & we have finite size manifold

Then we could say $\underset{\text{planck mass}}{M_p^2} = \underset{\text{higher dimensional planck scale}}{M_D^{D-2}} L^{D-4}$

$L \Rightarrow$ length scale which represents volume of internal manifold.

Turn around:

$$L \sim \left(\frac{M_P}{M_D}\right)^{\frac{D-2}{D-4}} \cdot \underbrace{M_P^{-1}}_{\sim L_P}$$

true quantum gravity scale.

putting in numbers:

$$L \sim \left(\frac{10^{16}\,TeV}{M_D}\right)^{1+\frac{2}{n}} \cdot L_P \qquad (n = D-4)$$

(no. of extra dimensions.)

$$L \sim \left(\frac{TeV}{M_D}\right)^{1+\frac{2}{n}} \cdot 10^{\frac{32}{n}-17}\ cm$$

as $n \uparrow \Rightarrow L$ decreases

$$\boxed{n = 6 \;:\; L \sim 10^{-13}\ cm\,, \quad M_D \sim 10\,TeV}$$

Instead of having KK theory; we have some kind of confinement mechanism. We then have much larger ~~mechanism~~ than we originally thought. That larger internal manifold gives us the volume factor which takes us between fundamental planck scale & our 4d planck scale.

This alters ~~the~~ ~~pr~~ magnitude of hierarchy problem in field theory.

(This is why people originally did it)

Randel - Sundram (RS)

Shiromizu - Maida - Sasaki

4D effective Einsteins equation in the form

$$G_{ab} = 8\pi G_{(4)} T_{ab} + O(T_{ab}^2) + \mathcal{E}_{ab}$$

projection of 5D Weyl tensor.

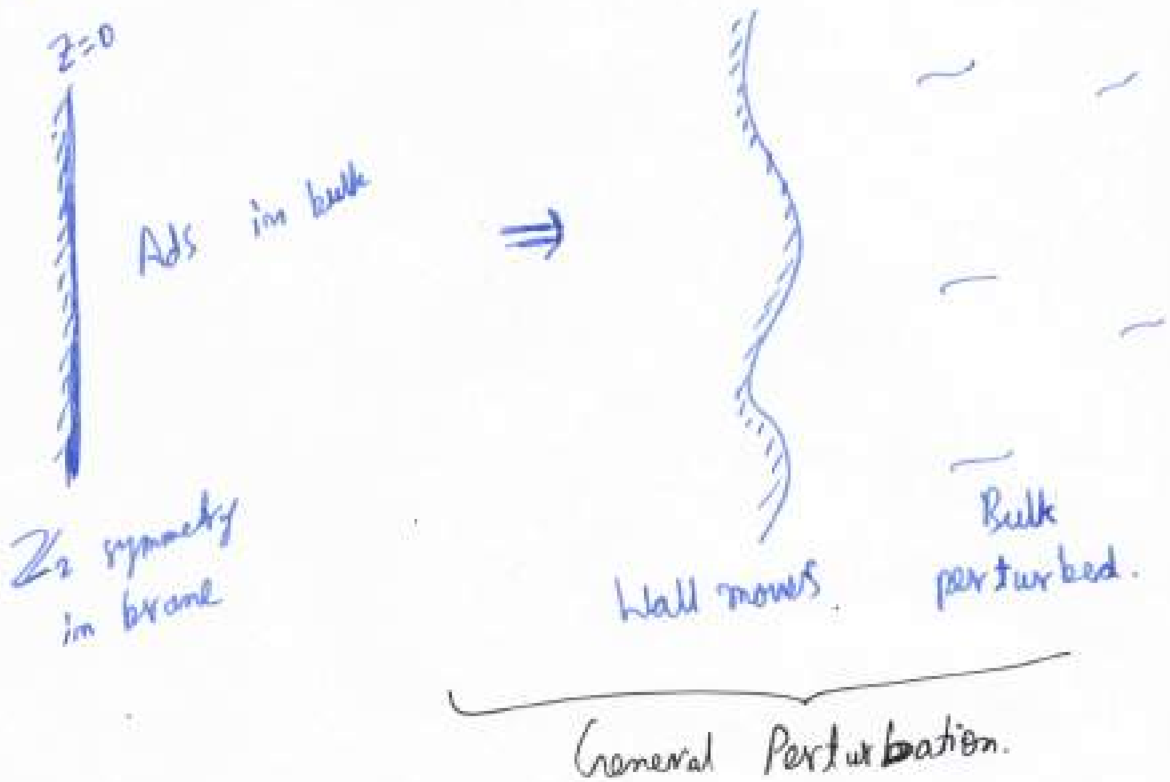
$$\mathbf{E}_{ab} = C_{acbd}\, n^c n^d$$

Bulk perturbations appear here.

$\mathcal{E}_{ab}$ is unknown here.

So; It is a non-perturbative relation

In general, can perturb both brane and bulk



But ~~that that~~ we can always write

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu - dz^2 \quad \Big\} \text{ Gaussian Normal}$$

within some range moving away, near the wall $z = 0$

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Lec 13: Gravitational Perturbation theory, perturbing Randall - Sundrum

Because exact solutions hard to find, perturbation theory is useful in GR

- Solar system tests.
- Cosmological P.T.
- Gravitational waves.

Idea: perturb around background

$$g_{ab} = g_{0\,ab} + h_{ab}$$

now δg_{ab}!

Reminder; we have seen

$$\delta\Gamma^a_{bc} = \frac{1}{2}\left(\nabla_c h^a{}_b + \nabla_b h^a{}_c - \nabla^a h_{bc}\right)$$

$$\delta R_{ab} = \frac{1}{2}\nabla_c \nabla_a h^c{}_b + \frac{1}{2}\nabla_c \nabla_b h^c{}_a - \frac{1}{2}\Box h_{ab} - \frac{1}{2}\nabla_a\nabla_b h$$

we want to swap up the order because we would like to have $\nabla_a(\nabla \cdot h_b)$

use Riemann Identity to swap.

$$\delta R_{ab} = \frac{1}{2}\nabla_a\nabla_c h^a{}_b + \frac{1}{2}R^c{}_{dca}h^d{}_b + \frac{1}{2}R_{bdca}h^{cd}$$
$$+ \frac{1}{2}\nabla_b\nabla_c h^c{}_a + \frac{1}{2}R^c{}_{dcb}h^d{}_a + \frac{1}{2}R_{adcb}h^{cd}$$
$$- \frac{1}{2}\Box h_{ab} - \frac{1}{2}\nabla_a\nabla_b h.$$

$$\delta R_{ab} = -\frac{1}{2}\Box h_{ab} - R_{acbd}h^{cd}$$
$$+ R_{d(a}h^{d}{}_{b)} + \nabla_{(a}\nabla^{c}\bar{h}_{b)c}$$

where $\bar{h}_{ab} = h_{ab} - \frac{1}{2}h g_{ab}$

we write this in the following way;

$$\delta R_{ab} = -\frac{1}{2}\Delta_L h_{ab}$$

where Δ_L is the Lichnerowicz operator

$$\Delta_L h_{ab} = \Box h_{ab} + 2R_{acbd}h^{cd} - 2R_{(a}{}^{c}h_{b)c} - 2\nabla_{(a}\nabla^{c}\bar{h}_{b)c}$$

$$\delta R_{ab} = -\frac{1}{2}\Delta_L h_{ab}$$

$$\Delta_L h_{ab} = \Box h_{ab} + 2R_{acbd}h^{cd} - 2R_{(a}{}^{c}h_{b)c} - 2\nabla_{(a}\nabla^{c}\bar{h}_{b)c}$$

Linchervonicz operator Δ_L is spin 2 wave operator in curved background

---

This is wave equation for gravity.

Now lets decide; how much it represents physical degree of freedom & how much of it represents gauge degree of freedom

Reimann, and Ricci piece are physical

We have to think about $\bar{h}$ piece.

Coordinate transformations are generated by vector fields.

Now consider gauge transformation.

$$X^a \longrightarrow X^a + \xi^a \qquad \xi^a \text{ is small.}$$

$$g_{ab} \longmapsto g_{ab} + \underbrace{\mathcal{L}_\xi g_{ab}}_{h_{\xi ab}}$$

$h_{\xi ab}$ ↦ This is gauge perturbation (not physical)

but it is perturbation in components of g_{ab}.

changing coordinates produces a "perturbation".

lets check : $\bar{h}_{\xi ab} = 2\nabla_{(a}\xi_{b)} - (\nabla\cdot\xi)\cdot g_{0\,ab}$

(we will use g_{ab} instead of g_{0ab} when meaning is clear)

$$\Rightarrow \nabla^a \bar{h}_{\xi ab} = \Box\xi_b + \underbrace{\nabla^a\nabla_b\xi_a - \nabla_b\nabla^a\xi_a}$$

use Reimann Identity to reduce it to Ricci term.

$$\nabla^a \bar{h}_{\xi ab} = \Box\xi_b + R^a{}_b\xi_a$$

Well posed PDE ; so can solve given initial conditions.

we can solve for ξ satisfying

$$\nabla^a \bar{h}_{\xi ab} = \Box\xi_b + R^a{}_b\xi_a = -(\nabla^a \bar{h}_{ab})$$

Can choose a gauge in which $\nabla_a \bar{h}^a{}_b = 0$

This is De Donder gauge.

Is there any Remaining Gauge Freedom.

$$x^a \longrightarrow x^a + \chi^a$$

s.t. $\Box \chi^a + R^a{}_b \chi^b = 0$

(satisfying massless wave equation)

D linearly independent solutions.

---

lets count, ~~left~~ degrees of freedom.

$h_{ab} \longleftrightarrow \frac{D(D+1)}{2}$ components (because $h_{ab} = h_{ba}$ symmetric)

However, we chose gauge $\nabla_a \bar{h}^{ab} = 0 \longleftrightarrow$ D constraints.

(use these constraints to write some of these components in terms of others)

finally we have got χ^a remaining d.o.f $\longleftrightarrow$ D

So; what is left over now is

$$\frac{D(D+1)}{2} - 2D \quad \text{physical d.o.f.}$$

$$\frac{D^2 + D - 4D}{2} = \frac{D(D-3)}{2} \quad \text{physical d.o.f.}$$

~~The phys.~~ The no. of physical solutions of Lichnerowicz operator Δ_L is given by $\frac{D(D-3)}{2}$ in D dimensions.

$D = 4 \longrightarrow$ 2 degrees of freedom $\leftarrow$ 2 polarization $+ , \times$

$D = 5 \longrightarrow$ 5 " " "

$D = 10 \longrightarrow$ 35 " " "

$D = 4$: 2 d.o.f

$+$:

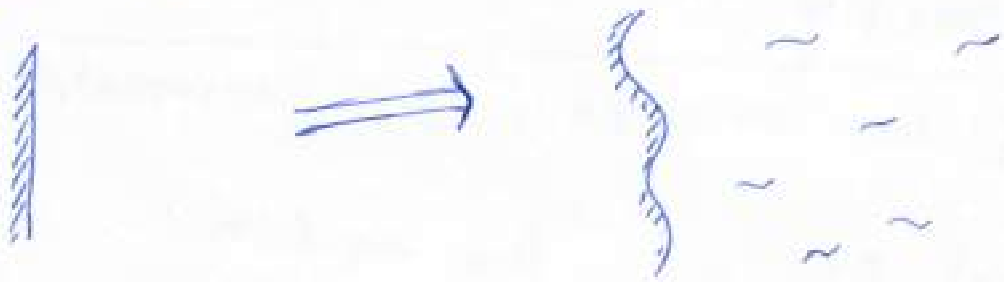
$\times$:

Return to R.S.

we could always choose the gauge where extra dimension is orthogonal & measuring proper distance.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu - dz^2 \quad \text{(G.N.)}$$

(gaussian normal)



To set wall at $z = 0$, must gauge transform RHS picture.

R.S. background (g_0)

$$ds^2 = e^{-2|z|/\ell} \cdot \eta_{\mu\nu} dx^\mu dx^\nu - dz^2$$

to keep in G.N. gauge.

make the wall flat again by doing translation in z-directions

$\xi^z = f(z^\mu)$ and add ξ^μ.

$$\delta g_{z\mu} = \xi_{z,\mu} + \xi_{\mu,z} - 2\Gamma_\mu{}^\nu{}_z \xi_\nu = 0$$

Solved by $\xi_\mu = 2\ell f_{,\mu} + \chi_\mu(x)$

we could also add another field depending on x.

$$\delta g_{\mu\nu} = 2\xi_{(\mu,\nu)} + \Gamma^z_{\mu\nu} f$$

Then

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \ell f_{,\mu\nu} - \frac{2}{\ell} f \cdot g_{\mu\nu}$$

Transverse trace free $(h^\lambda{}_\lambda = 0 , \nabla_\mu h^\mu{}_\lambda = 0)$

Lich op. ∇_L has zz, $z\mu$, $\mu\nu$ components.

<u>Scalar component zz</u> $\quad$ A is warp factor

$$A^{-2}\left[A^2\left(A^{-2} h^\lambda_\lambda\right)'\right]' = 4\,\delta(z)\,\partial^2 f = -\frac{2}{3}\delta(z)\cdot 8\pi G\, T^\lambda{}_\lambda$$

Matter on brane with non-zero T induces an f - "Brane Bending"

Now equating these we get

$$\Rightarrow \boxed{\partial^2 f = -\frac{8\pi G}{6} T^\lambda{}_\lambda}$$

This gives what f is.

$$\mu\nu : A^{-2}\partial^2 h^{TT}_{\mu\nu} - A^{-2}\left[A^4\left(A^{-2}h^{TT}_{\mu\nu}\right)'\right]' = -16\pi G\delta(z)\underbrace{\left[T_{\mu\nu} - \frac{T}{3}\eta_{\mu\nu}\right]}_{\text{This comes from } \delta R_{\mu\nu}}$$

$-2\,\delta R_{\mu\nu}$ (after doing algebra)

3 comes because we are in 5D.

Construct solutions. (we fourier transform)

$$h^{TT}_{\mu\nu} = u_m(z)\, e^{ipx}\, \epsilon_{\mu\nu}$$

polarization tensor

eigen functions because of warping in z direction

This is just standard momentum decomposition of the graviton.

$$\partial^2 h^{TT}_{\mu\nu} = -m^2 h^{TT}_{\mu\nu} \qquad (p^2 = m^2)$$

~~tells~~ tells something about mass in 4 D dimensions.

We solved the homogeneous problem, i.e; ~~set~~ (set $T_{\mu\nu} = 0$ and Israel Boundary Conditions)

$$e^{2|z|/l}\left[m^2 \cdot u_m + \left\{e^{-4|z|/l} \cdot \left(e^{2|z|/l}\, u_m\right)'\right\}'\right] = 0$$

This is now a ~~straightforward~~ Sturm-Liouville problem.

~~change variables~~ ~~$\zeta = l\, e^{|z|/l}$~~ now; change variable $\zeta = l \cdot e^{|z|/l}$

$$\frac{\partial}{\partial z} \longrightarrow (\pm)\,\frac{\varsigma}{l}\,\frac{\partial}{\partial \varsigma}$$ transforms this

to $$u_m'' + \frac{u_m'}{\varsigma} + \left(m^2 - \frac{4}{\varsigma^2}\right)u_m = 0$$

a Bessel Equation.

Eigenfunctions are

$$u_m = \sqrt{\frac{ml}{2}} \cdot \frac{\left[\, J_1(ml)\, J_2(m\varsigma) - N_1(ml)\, J_2(m\varsigma)\right]}{\sqrt{J_1^2(ml) + N_1^2(ml)}}$$

at $z=0$ $\varsigma = l$: Boundary condition.

Now we can construct propagator from eigenfunctions

$$\cancel{G_R(x,x') = -\int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot(x-x')}\left[\frac{A^2(z)A^2(z')}{l(p^2-(\omega+i\varepsilon))} + \int_0^\infty dm\, u_m(z)\right]}$$

$$G_R(x,x') = -\int \frac{d^4p}{(2\pi)^4}\, e^{i\,p\cdot(x-x')}\left[\frac{A^2(z)A^2(z')}{l(\underline{p}^2-(\omega+i\varepsilon)^2)} + \int_0^\infty dm\,\frac{u_m(z)\,u_m(z')}{m^2+p^2-(\omega+i\varepsilon)^2}\right]$$

apart from $A^2(z)A^2(z')$
looks like something
independent of independent
dimensions ... ~~do~~ depends
on 4 D..

19/29

Perturbation in the brane due to matter on the brane

$$G_R(x^\mu, 0, x^{\mu'}, 0) = \underbrace{\frac{1}{e} D_0(x-x')}_{\text{massless propagator.}} + \underbrace{\int_0^\infty dm\, U_m(0)^2\, D_m(x-x')}_{\text{integral over continuum of massive states.}}$$

Hence we can find out $h^{TT}_{\mu\nu}$

$$h^{TT}_{\mu\nu} = \underbrace{\frac{-16\pi G}{l}\int D_0(x-x')\left[T_{\mu\nu} - \frac{1}{3}T\eta_{\mu\nu}\right]}_{\text{zero mode piece}} - \underbrace{16\pi G\int dm \ldots}_{\text{dependent on massive mode}}$$

This does not looks correct because it is not full perturbation

---

⇒ full perturbation have to take into account f

$$h_{\mu\nu} = h^{TT}_{\mu\nu} - \frac{2f}{l}\eta_{\mu\nu} + l f_{,\mu\nu}$$ ← gauge.

and this f has to satisfy another massless wave equation

we end up with

$$h_{\mu\nu} = \frac{-16\pi G}{l}\int D_0(x-x')\left[T_{\mu\nu} - \frac{1}{2}T\eta_{\mu\nu}\right] + \int dm \ldots$$

↳ has right structure in leading order piece.

↳ This is another way of accessing the fact that gravity is 4 dimensional on the brane.

⇒ it also shows how leakage in extra dimension is occuring.

The eigenfunctions tells how things can leak off the ~~brane~~ brane.

~~For a source on~~

For a source $\sim M\,\delta^{(3)}(\underline{x})$ on brane

(for a localized source

$$g_{tt} \sim 1 - \frac{2G_N M}{r}\left(1 + \frac{2l^2}{3r^2} + \cdots\right)$$

The leading piece we get; the leading order at large r ...... & find correction to my Newtonian potential is now power law.

↳ often when you see extra dimension or extra interaction, we will ~~see the~~ find that correction is more like Yukawa like correction to the gravitational interaction. with exponential.

But here we find its parallel ~~The reason is;~~ power law.

This illustrates example of using perturbation theory.

Lec 14: Gravitational instantons : false vacuum decay

OR : Decay of the false vacuum

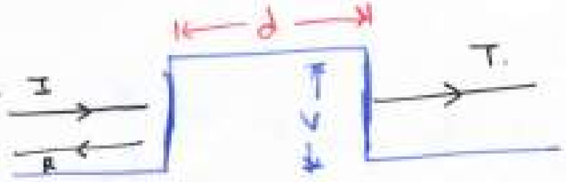
Instanton = solution to Euclidean field equations

Perturbative $\sim O(\hbar)$

Non-Perturbative $\sim O(1/\hbar)$

Paper : Coleman PRD 15 2929 '77
+de Lucca PRD 21 3305 '80

Tunnelling



we solve Schrodinger's Equation

$$\Psi \sim e^{-iEt/\hbar}\, e^{i\Psi/\hbar}$$

solve at fixed energy
and use boundary condition at barrier.

This gives probability of ~~transition~~ transmission

$$\text{as} \quad |T|^2 = \frac{1}{1 + \dfrac{V_0^2 \sinh^2 \Omega d}{4E(V_0 - E)}} \sim e^{-\Omega \cdot d}$$

$$\Omega^2 = 2m(V_0 - E)/\hbar = \frac{1}{\hbar}\int_0^d \sqrt{2m(V_0 - E)}$$

Transmission probability dominated by exponential.

Now if we input wavefunction into Schrodinger Operator.

$$\frac{-\hbar^2}{2m}\Psi'' + V\Psi = \left[-\frac{i\hbar}{2m}\psi'' + \frac{\psi'^2}{2m} + V\right]\Psi$$
$$= E\Psi$$

The equation of motion looks like

$$V + \frac{\psi'^2}{2m} = E \quad , \quad \psi' = \pm\sqrt{2m(E-V)} \cdot x$$

Consider a particle moving in a potential well (classical)

$$\frac{1}{2}\dot{x}^2 = \Delta V$$

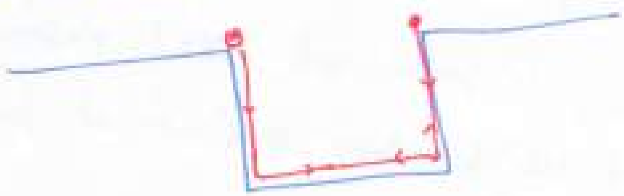
↓ use it

$$\int \sqrt{2\Delta V}\, dx = \int 2\Delta V\, d\tau$$

$$= \int \left(\Delta V + \frac{1}{2}\dot{x}^2\right) d\tau$$

looks like standard Lagrangian (has Kinetic & potential term)

Our exponent integral can be interpreted as motion in an inverted potential. (since ~~is~~ ΔV comes with plus sign) in Euclidean Time.



⇒ The action of this process happens to agree with the ~~process~~ exponent we want

In this case it is fictitious process; but its kind of linking the notion of analytic continuation looking at some Classical Trajectory; computing the action of the ~~tra~~ trajectory & identifying the exponent we want.

For a general potential, amplitude for decay given by solving classical motion in inverted potential (to leading order)

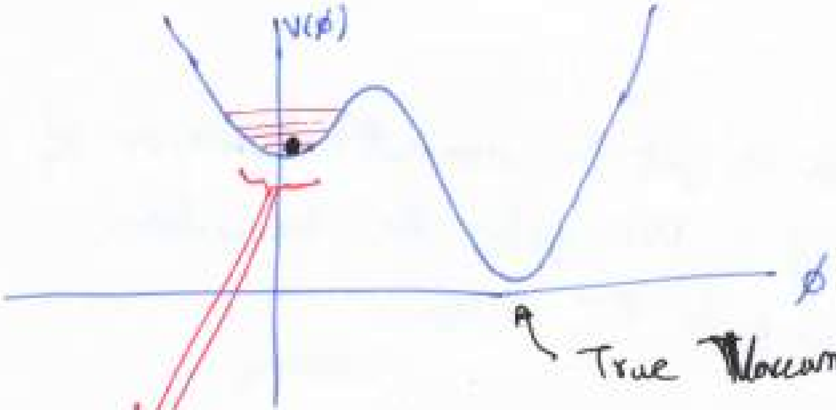
⇒ look at Euclidean solution; compute Euclidean action & take its ~~exponentation~~ exponential

⇘ And this gives us the probability of decaying

False Vacuum Decay

Take a scalar field

two minimum,
- one global
- one local.



looks like local vacuum (but there is small probability of exiting & going to the true vacuum (by quantum fluctuations))

⟷ We call it False Vacuum (The local minima one)

The global minima one is called True Vacuum

⟶ false because it looks like vacuum but is not

Colmen … considered $V(\phi) = \frac{\lambda}{2}(\phi^2 - \eta^2)^2 - \frac{\epsilon}{2\eta}(\phi - \eta)$

double well potential with tilt (tilting with the linear term)

$\updownarrow \epsilon$ has notion of difference in vacuum energy.

We kind of cooked this so that

$\phi \simeq -\eta$ is False vacuum F.V.

& $\phi \simeq \eta$ is True " T.V.

F.V. is unstable.

The idea is: In order to get the dominant behavior of this probability for decay; we solve the Euclidean field equations & compute the Action.

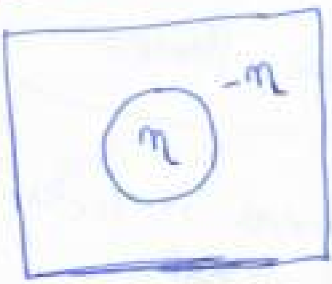
To compute exponent for decay, solve Euclidean eqns.

$$\frac{d^2\phi}{d\tau^2} + \underline{\nabla}^2\phi = \frac{\partial V}{\partial \phi} = 2\lambda(\phi^2 - \eta^2) - \frac{\epsilon}{2\eta}$$

[illegible] (2x)

[illegible] page

$\phi'' + \frac{3}{\rho}\phi'$



Solving for ϕ with just this on RHS gives $\tanh(\sqrt{\lambda}\,\eta z)$ profile & this was Domain wall.

By symmetry; look for a hyperspherical solution

$$\phi\left(\sqrt{\tau^2 + \underline{x}^2}\right)$$

And we guess its going to be close with the wall form $\tanh(\sqrt{\lambda}\,\eta z)$

$\rho = \sqrt{\tau^2 + \underline{x}^2}$

Guess $\phi \sim -\eta \tanh\left(\sqrt{\lambda}\,\eta\,(\rho - \rho_0)\right)$

$\rho \to \infty \quad : \quad \phi = -\eta \qquad$ F.V.

$\rho = 0 \quad : \quad \phi \simeq +\eta \qquad$ T.V.

~~Coleman's argument~~;

Coleman's Argument: Wall is thin, so approximate by a wall of tension at R.

~~bubble~~

To form the bubble, there is cost which is making the wall.

Cost: $\sigma \cdot (\text{Area}) \qquad$ Area of 3 sphere is $2\pi^2 R^3$

$= \sigma \cdot 2\pi^2 R^3$

Gain in energy : $\epsilon \cdot (\text{volume})$

$= \epsilon \cdot \frac{\pi^2}{2} R^4$

Solution is stationary w.r.t. R

$6\pi^2 \sigma R^2 = 2\pi^2 \epsilon R^3 \quad \Rightarrow \quad \boxed{R = \frac{3\sigma}{\epsilon}}$ (***)

* if ϵ small, then R is big.

Action, $B = \frac{\pi^2}{2}\left(4\sigma R^3 - \epsilon R^4\right)$

$= 27 \frac{\sigma^4}{\epsilon^3} \pi^2 \underbrace{(4-3)}_{1} = 27 \frac{\sigma^4}{\epsilon^3} \pi^2$

Bubble Action.

Why we expect bubble to be spherical?

Because we what we are doing is actually have competetion between Surface Area & Volume.

And we kind of know spheres are most efficient in generating best $\frac{\text{Surface Area}}{\text{Volume}}$ ratio

The process will be dominated by spherical bubble

... Coleman first paper.

Coleman Second Paper .....

Coleman + De Lucca : False vacuum energy gravitates.

Add gravity - use Israel equations for thin wall.
To calculate Euclidean action, subtract the FV manifold action from Bubble Action.

Euclidean dS
constant +ve curvature

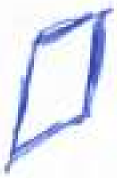
S^4

F.V.

radius of this is proportional to $\frac{1}{\sqrt{8\pi G\varepsilon}}$

→ so size of sphere tells about vacuum energy

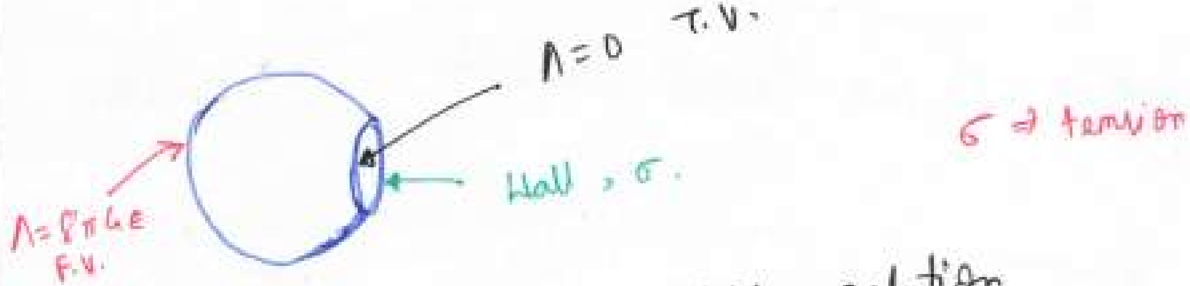
If tunnelling to $\Lambda = 0$ (flat space)



The analogue of 

we have ; inside the bubble flat
outside " " sphere.

Take sphere, and cut it.



This is geometry ... the bubble solution.

Calculate action,

$$S_B = \frac{\pi L^2}{G} \frac{(4\pi G \sigma L)^4}{[1 + (4\pi G \sigma L)^2]^2}$$

Check : $G \to 0$ (should match with previous result)

note ; $8\pi G \epsilon = 3/L^2$

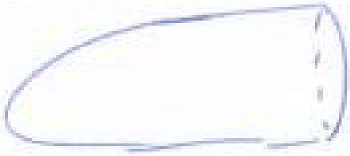
so as $G \to 0 \Rightarrow S_B \longrightarrow (4\pi)^4 \cdot \left(\frac{3}{8\pi G \epsilon}\right)^3 \cdot \pi \sigma^4$

$$= \frac{27}{2} \pi^2 \sigma^4 / \epsilon^3$$

Adding an impurity makes phase transition easier to complete.

Obvious impurity in Gravity is Black Hole.

Euclidean b.h. (for schwarzschild)



$$\Delta \tau \sim \frac{1}{8\pi G M}$$

If we have Cosmological constant

$\Lambda > 0$, SdS
(Schwarzschild dS spacetime)

Then $\Delta \tau_{BH} \neq \Delta \tau_{CEH}$

CEH ⇒ Cosmological Event Horizon.

⇒ This means it has Conical Deficit.

BH ⇒ Black Hole

$\Delta \tau_{BH}$
$\downarrow$
periodicity of τ comming from BH.

So, we have mismatch
between natural periodicity
that makes event Horizon ~~sing~~ non-singular
& natural periodicity that makes Cosmological event Horizon non-singular

$\Longleftrightarrow$ Euclidean Space has Conical Deficit

Cost : $\sim \sigma \cdot R^2 \cdot \beta \rightarrow \propto$ area ...

Gain : $\sim \epsilon \cdot R^3 \cdot \beta \rightarrow \propto$ volume ..

we still get $R_c \propto 2 \frac{\sigma}{\epsilon}$

critical radius.

so; Action now goes as ~~[illegible]~~

$$B \sim \frac{\sigma^3}{\epsilon^2} \times \beta .$$

Probability for decay $\sim e^{-B}$

→ For black holes with $M < 10^7 M_p$
decay dominates over evaporation & half life is $\sim 10^{-22}$ s

Seed Black Hole M_+
Remnant M_-

$$B \propto M_+ \Delta M$$

Lec 15 | Beyond Einstein gravity, Modifying gravity with scalars Chameleons. Gravitational wave.

Why try other theories?

- Cosmology — Λ C D M

Cosmological constant ~ 70%

Dark Matter ~ 28%

SM (standard model model) ~ 5%

- String Theory
  - Extra dimensions
  - Module

$>$ Beyond Einstein.

Dark Energy

Cosmology: FRW metric

$$ds^2 = dt^2 - a^2(t)\, d\underline{x}_k^2$$

cosmological time

Scale factor

Constant curvature space

$k=1$ closed S^3

0 Flat $\mathbb{R}^3$

-1 Open $\mathbb{H}^3$

Friedman equations
$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G\rho}{3}$$

$$\dot\rho + 3\frac{\dot a}{a}(p+\rho) = 0$$

Perfect fluid $P = w\rho$

$\rho_m \propto a^{-3}$ Dust $P=0$

$\rho_r \propto a^{-4}$ $w=1/3$

$\rho_\Lambda \propto a^0$ $w=-1$

$H = \frac{\dot{a}}{a}$

~~$H^2 = \ldots$~~

$$H^2 = H_0^2 \sum \Omega_i (1+z)^{3(1+w_i)}$$

$\downarrow$
fractional density

$$\Omega_i = \frac{8\pi G \rho_i}{3H_0^2} \quad \text{at } t_0 \text{ (now)}$$

and $\Omega_k = \frac{-k}{a_0^2 H_0^2}$ (curvature contribution)

and z is redshift ; $(1+z) = \frac{a_0}{a}$

~~SN surveys~~

SN Surveys (Super Nova) ; $d_L = (1+z) d_{phys}$

$$= (1+z)\int_0^z \frac{dz}{H}$$

To leading order $d_L = \frac{z}{H_0}$

but to next orders, dependence of H on z comes in.

SN fainter than expected. So H dominated by an Ω with $w \simeq -1$

Can have $w < -1$

? Modify gravity or ~~...~~ particle sector.

Modify gravity with scalars.

$$S_{JBD} = \frac{1}{16\pi}\int\sqrt{g}\left[-\Phi R + \omega\frac{(\nabla\Phi)^2}{\Phi}\right]$$

Φ ... gives background Newtons Constant.

Brans Dicky

$$\rightarrow \quad G_{ab} = \frac{8\pi}{\Phi}T_{ab} + T_{\Phi\, ab}$$

$$T_{\Phi\, ab} = \Phi^{-1}\nabla_a\nabla_b\Phi + \omega\frac{\nabla_a\Phi\nabla_b\Phi}{\Phi^2} - g_{ab}\left(\Phi^{-1}\Box\Phi + \frac{\omega}{2}\cdot\frac{(\nabla\Phi)^2}{\Phi^2}\right)$$

$$R + 2\omega\frac{\Box\Phi}{\Phi} - \omega\frac{(\nabla\Phi)^2}{\Phi^2} = 0$$

so $$\boxed{\frac{\Box\Phi}{\Phi} = \frac{8\pi T}{3+2\omega}}$$

$\Rightarrow$ scalar field couples to Trace of Energy momentum tensor; ~~to~~ more matter there is, stronger the scalar field react.

$\Longleftrightarrow$ So, Newton's constant depends on what else is around.

If scalar field is reacting to matter, so it means in our Solar system its going to react to local density of environment.

Putting in particle physics way; we got scalar forces.. We have fifth forces.

Screening BD like theory, and replace Φ with..

$$\Phi \rightarrow A^{-2}(\phi)$$

and Einstein frame

In Einstein frame,

$$S = \int \underbrace{-\frac{M_P^2}{2}R + \frac{1}{2}(\partial\phi)^2 - V}_{\text{Gravity sector}} + \mathcal{L}_m[\Psi, A^2 g_{\mu\nu}]$$

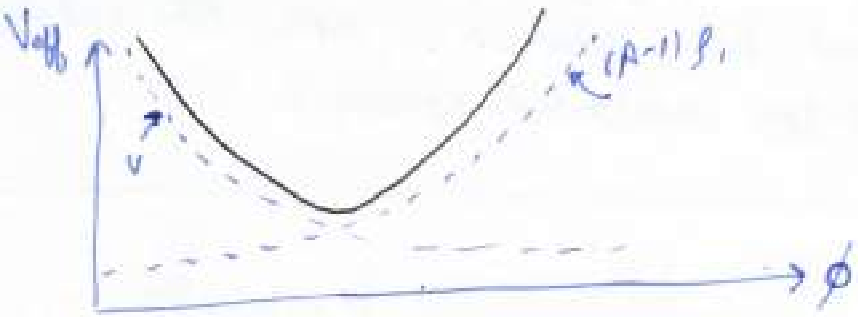
In Einstein frame matter couples ϕ.

Scalar equation, $\Box\phi + \underbrace{\frac{\partial V}{\partial\phi} + \frac{\partial A}{\partial\phi}}\rho = 0$

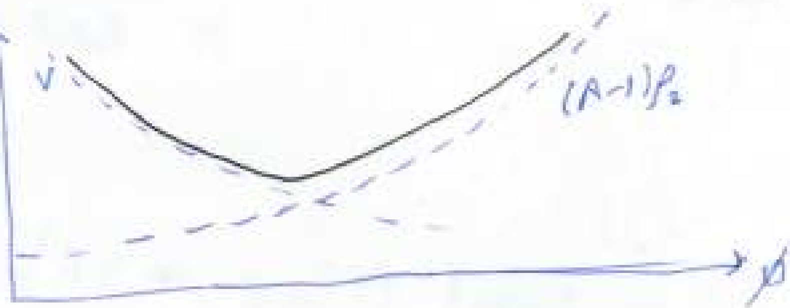
cosmological density ρ_m/A of fluid (or matter)

$$\Downarrow$$

$$\frac{\partial}{\partial\phi}\left[V + (A-1)\rho\right] = \frac{\partial V_{eff}}{\partial\phi}$$



m^2_{eff}



V

m^2_{eff}

So, denser the region, the stronger the lift is in effective potential.

"Chameleon"

This is simple example where we come across notion of ~~sch~~ screening mechanism;
because you have some sort of scalar which comes in the gravity sector as a scalar tensor theory : so we have

conformal factor dependent on our scalar.

If we look at physics in Einstein frame; our scalar field, while it still couples to the trace of energy momentum.

But Trace term gives something which depends on density of environment.

So: it is possible, that we have an effective potential such that if we are in the galaxy or the solar system, the ambient density is sufficiently ~~h~~ to not to disturb solar system test.

However on cosmological density scale; The scalar mass becomes lower, such that solution is more like rolling scalar field which has accelerated expansion.

Other modifications:

- Modify Newtonian ~~Dynamics~~ Dynamic — Dark Matter
  - × Bullet Cluster.

- Modify $\mathcal{L}$agrangian — $f(R)$
  - → Gauss Bonnet $(R^2_{abcd} - 4R^2_{ab} + R)\Phi$
  - → $R^2_{\mu\nu\lambda}$ —

careful to get 2^{nd} order equations

- Massive gravity, Galileons, Lorentz violation ...

Test against:

Cosmology: Structure formation

~~matter~~ - matter clumps under gravity

+ Numerical Simulation

Large Scale Structure Survey.

- Evolution of perturbation spectrum ~ CMB

+ Numerical Simulation CMB FAST.

Solar system tests

Screened scalars +

- main sequence stars
- Neutron stars.
- Pulsars
- Black Holes. - Atom interferometry.

Black Holes - Imaging a black hole?

Etreme environment form accretion.

Light can have an interesting path from black hole.

Recall for ~~EM~~ Schwarszchild, γ

$$V_{eff} = -1 + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3} \qquad \dot{r}^2 + V_{eff} = 0$$

Photon initially tangent

$$\frac{h^2}{r^2}\left(1 - \frac{2GM}{r}\right) = 1$$

$$h^2 \sim (2GM)^3/\epsilon \qquad \text{if } r_0 \sim 2GM + \epsilon$$

$$\text{and } \dot{\phi} = h/r^2 \sim \frac{1}{\sqrt{2GM\epsilon}}$$

photon "hugs" black hole
before escaping
... Strong Lensing

~~leads to interest~~ leads to interesting images

GRAVITY IS BEAUTIFUL

THANK YOU

Shoaib Akhtar