

Lec - 1

Classical Physics, consists of equations which tell how to update the state at any given time, & is completely deterministic.

Once ~~physics~~ the physics moves past the range of parameters that has to do with ordinary experience,
 ex, objects which are so small, that they are far beyond what ordinary physics is about.
 ex, or velocities which are so fast that they are far beyond ordinary velocities.

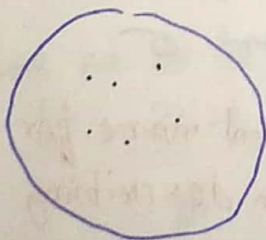
Whenever we move out of the range of parameters that we are familiar with ordinary experience, we inevitably run into things that we cannot visualize.

There is no substitute for the process of abstracting & using mathematics to describe things which are beyond our ability to directly visualize.

Closed system is one which at least temporarily is not in interaction with anything else; It is isolated,
 & so it can be described by itself as an alone entity with nothing else ~~involved~~ involved.

Space of States

ex



Space of states of a die

(• consists of six points)

ex



Space of states of a coin

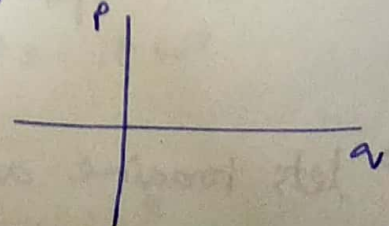
(* consists of two points)

ex

Abstract state of a particle moving along a line.

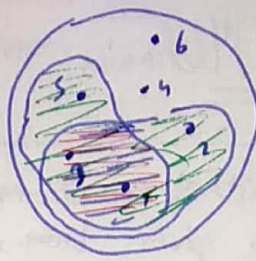
→ phase space

(the space consists of the set of possible positions of the particles & the possible momenta)



continuously infinite set of two dimensions.

- Concept of and
(Intersection)



- concept of or
(Union)

*Inclusive or means it can be both

generally or means inclusive or

Logic of Quantum Mechanics is different
→ it is not based on the idea that the state of the system is a set

c-bit $\Rightarrow$ simply the idea of a coin
(Heads or Tails) ... classical coin.
(c stands for classical)

q-bit $\Rightarrow$ A q-bit is a system which can have two states
(Analogue of classical coin) and q stands for quantum.

ex qubit

~~assuming the motion of qubit is not moving~~

q-bit when its measured is either Heads (H) or Tails (T)

but, let's describe it in other way

lets ~~give~~

H : $\sigma = 1$: $\uparrow$

T : $\sigma = -1$: $\downarrow$

→ let's give it a degree of freedom (a mathematical degree of freedom) and we will call that mathematical degree of freedom σ .

(assuming, loss of motion of qubit is that nothing happens to it)

description in terms of picture.
(just a notation)
(no. implication here of anything geometric for the moment)

σ is a traditional name for the degree of freedom describing a q-bit.

→ σ can either take value 1 or -1.

now, let's imagine an experiment.

* An experiment involves more than just a system

→ It involves a second system, & the second system is called the Apparatus.

In Classical mechanics, we never bother thinking too much about details we thought that a system can be described completely without ever worrying about how you detect it, measure it or anything else.

• but in Quantum mechanics, one cannot get away ignoring the concept of a Apparatus.

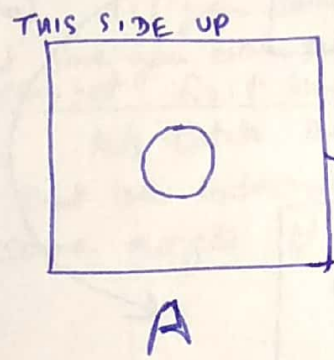
So, let's introduce an Apparatus.

... mathematically abstract apparatus ...

... its a black box. A (A for apparatus)

... and has a little window  ... a little screen & numbers appear on that screen.

... and has some sort of detectors ~~at~~ at the end of wire that senses the q-bit.



↑ : $\sigma = 1$: H
↓ : $\sigma = -1$: T

• so you start with your q-bit & you don't know what state it is in.
(?) → the experiment is to determine what state it is in.

... +1 or -1.

• so the qubit begins in unknown (?) state.

• Before we do the detection, the apparatus is in the Neutral state, the Blank state (no number here at all) ... let's call that Blank Configuration of Apparatus.

• We take the Apparatus ever closer enough to q-bit to interact with it, and if it is good apparatus, it will register a number +1 or -1.

→ That's the Active measurement.

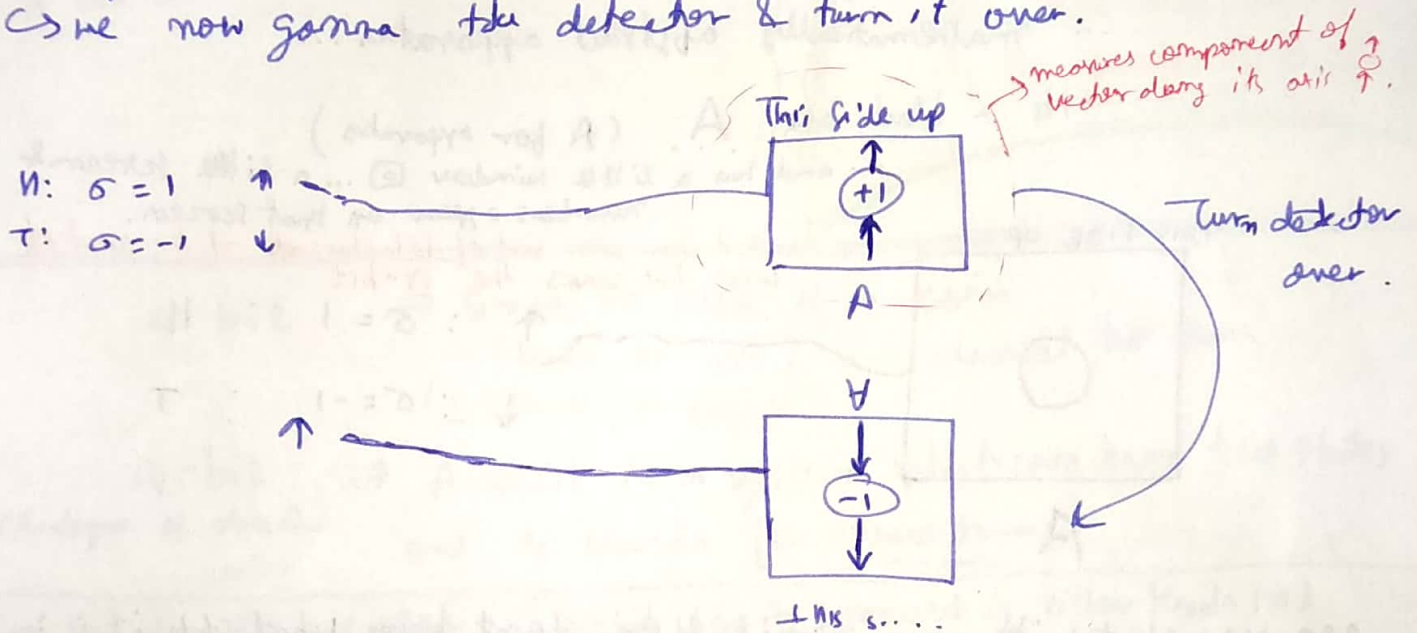
- ... we can think as for measurement in a way that, it prepared it in a state, which for example is H .

(19)

↳ A device which measures is also a device that prepares the system in a given state.

now after we determined or prepared the qubit

↳ we now gonna turn the detector & turn it over.



- by turning the detector over, we get the opposite answer.

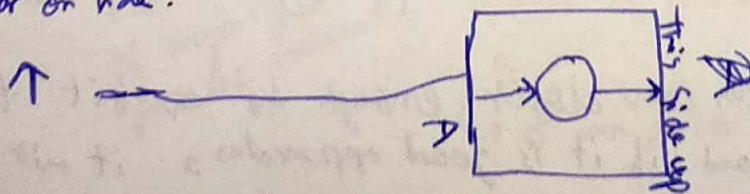
~~22~~ up & down motion

~~That's an~~

That's an indication that qubit has some directionality in space ... spatial directionality associated with it.

(not all qubits have spatial directionality associated with it)

* turn detector on side.



but in real world of Quantum Physics (• If this were classical... we would get 0.)

→ Since the box A has only two options 1 or -1 ... (it will not show 0...)

So what we get 1 or -1

where (1 or -1 seem to be located symmetric relative to axis of detector)

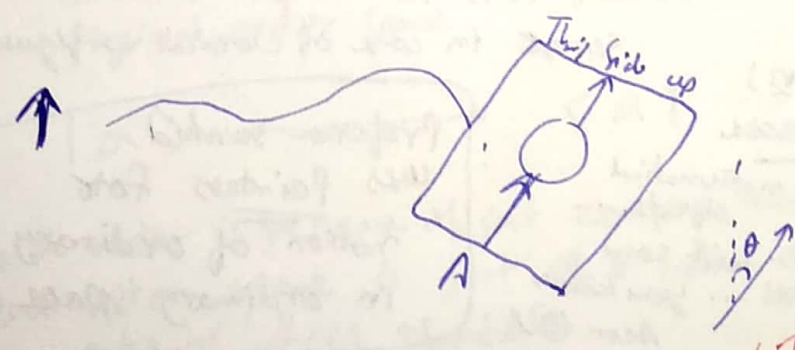
take an ensemble of qubits pointing ↑

we get 1 or -1 ... randomly
 (each time bring a new qubit which is initially confirmed to be at ↑ up...)
 but in such a way that if you do in great many times, its average becomes 0.

→ because once you measured it, you prepared it... so now you will always get same answer unless you intervene by something else.
 if you would have used the same qubit... that would have given always same value... & that would confirm the experiment.
 (if you don't disturb it)
 (first time you either get +1 or -1... but second time you get same as what you got first time)

example: let's rotate the box A by some arbitrary angle so that the internal detector orientation is not characterized by some angle θ relative to vertical.

• When you turn apparatus you get statistical distribution.



(This is real Quantum world, not classical) so we get +1 or -1

→ here also we get +1 or -1

→ but the average value of experiment will not be zero, but be equal to $1 \cdot \cos \theta$

(like component of vector along detector axis)

• it's seen that average values of σ are behaving as if σ was the component of a vector.

This all tells us that there is something funny with about the notion of the state of the system.

(P. 21)

→ it's not ~~vector~~ but ~~is~~ not as simple as state of either
→ there is something intrinsically different

↪ & the difference does trace to the big difference between the notion of state in Quantum Mechanics and state in classical mechanics.

• State of Apparatus may affect the measurement.

↪ eventually, later we have to reconcile that detectors are themselves Quantum Systems., and has to respect same laws as system themselves.

Mathematics of Space of States of Quantum System

(or State Space)

~~State Space~~

(collections of vectors)

in Quantum System.

↪ here, it is not a set in the same sense in case of classical configuration space.

* Mathematics of Vector Spaces

(Collection of mathematical objects....)

Complex vector spaces.

... with some properties ... you know them (2)

axioms

$|a\rangle$

Dirac notation for abstract vector a (whatever it is...)

Professor said
uses pointers for
notion of ordinary vector
in ordinary space.

↪ he calls it pointers to distinguish it from abstract mathematical concepts of vector.

$$\# |a\rangle + |b\rangle = |c\rangle$$

(Vector space consists of collection of objects which can be added)

$$\# z|a\rangle = |a'\rangle$$

$$; z \in \mathbb{C}$$

axioms $\langle b|a\rangle = \langle a|b\rangle^*$

ex Real numbers are real vector space.

ex Complex numbers are complex vector space.

$$\psi(n) : n \in \mathbb{R}$$

a complex function on a line

$$\psi: \mathbb{R} \rightarrow \mathbb{C}$$



so; we can add functions & get another functions $\psi(n) + \phi(n)$

* multiply $c\psi(n)$

so; functions form a vector space.

ex:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

collection of

but symbol is also a vector space

column vector

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \end{pmatrix}$$

$$z \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} z\alpha_1 \\ z\alpha_2 \end{pmatrix}$$

... so ~~not~~ form vector space.

Complex Conjugate Vector Space, (Dual Vector Space)

↳ ... its a separate vector space whose elements are in one to one correspondence with the elements of the original vector space.

$$|a\rangle \longleftrightarrow \langle a|$$

ex: if the vector is the space of just complex numbers, then the dual vector space is just a space of complex conjugates of those numbers

$$|a\rangle + |b\rangle \longleftrightarrow \langle a| + \langle b|$$

$$z|a\rangle \longleftrightarrow z^* \langle a|$$

$$z \in \mathbb{C}$$

$$\textcircled{\text{or}} \langle a| z^*$$

ex:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$|a\rangle$$

$$\longleftrightarrow \begin{pmatrix} \alpha_1^* & \alpha_2^* \end{pmatrix}$$

$$\langle a|$$

we write it in row.
↳ just a trick to remember that this object ^{belong} to dual vector space. (the rows one)
↳ 2 column ones, belong to original vector space.

Inner product of two vectors. (Generalization of Dot product) (138)

$|a\rangle$
 $|b\rangle$

$\langle b |$ multiplies $|a\rangle$
 $\equiv$
 $\langle b | a \rangle$

Inner product $V \rightarrow \mathbb{C}$

really product of a vector & dual of a vector.

Outer Product of two vectors
 is a matrix $|a\rangle\langle b|$ not just a number.

is it same as $\langle a | b \rangle$

no; $\langle b | a \rangle \neq \langle a | b \rangle$ in general.

axiom $\langle b | a \rangle$ & $\langle a | b \rangle$ are related by complex conjugation.

$$\langle b | a \rangle = (\langle a | b \rangle)^*$$

$$\alpha \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \therefore (\beta_1^*, \beta_2^*) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \beta_1^* \alpha_1 + \beta_2^* \alpha_2$$

$|a\rangle$ $\langle b|$ construction $\rightarrow$

by construction, it follows that

$$\begin{aligned}
 (\alpha_1^*, \alpha_2^*) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} &= \alpha_1^* \beta_1 + \alpha_2^* \beta_2 \\
 &= (\beta_1^* \alpha_1 + \beta_2^* \alpha_2)^*
 \end{aligned}$$

$$\# \langle a | a \rangle = \langle a | a \rangle^*$$

so; $\langle a | a \rangle \in \mathbb{R}$

$$\therefore (\alpha_1^*, \alpha_2^*) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \alpha_1^* \alpha_1 + \beta_1^* \beta_2 \geq 0$$

here in our case assume $\langle a | a \rangle$ is positive

$\sqrt{\langle a | a \rangle} :=$ defines length of vector. $\in \mathbb{R}$.

Notion of orthogonality.

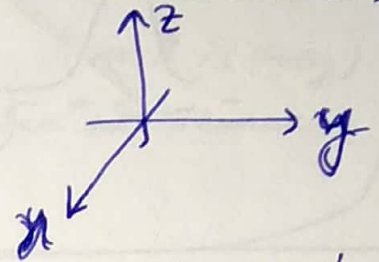
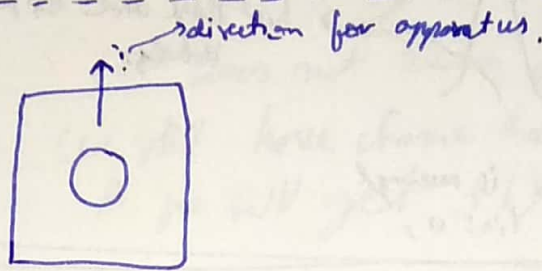
$\langle b | a \rangle = 0$ orthogonal or perpendicular.

Maximum no. of ^{non-zero mutually} orthogonal vectors that can be found is dimensionality of space.

- * change terminology for orbit to spin.
- * pointer terminology changed to 3-Meter.
- * Vector stands for elements of abstract vector space.

ex

(have some sense of directionality)



no matter in what direction we orient the apparatus, it always gives us ± 1 .

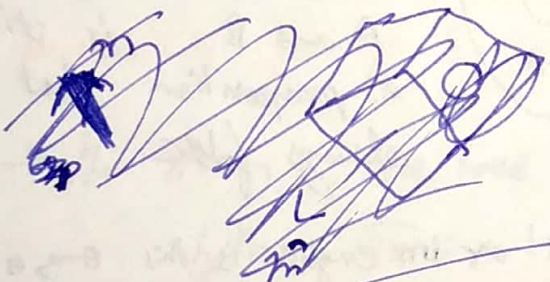
$\sigma_z :=$ component of spin along z axis

$$= \pm 1$$

$$\sigma_x = \pm 1 \quad (\text{always } \pm 1)$$

$$\sigma_y = \pm 1$$

$\hat{m} :=$ a unit three vector. along a particular axis.

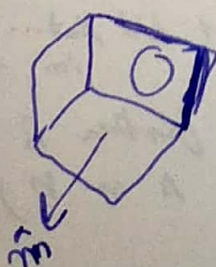
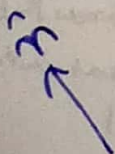


we start with detector along n axis; doing experiment & discover that along n axis, the spin comes up +!

Then we rotate apparatus in any direction & do another experiment on same spin $\rightarrow$ it will either come ± 1 . (never in between)

but the average value of spin along the n axis here will not ± 1 , but :

$\hat{n} \cdot \hat{m}$ ~~cosine~~ cosine of angle between $\hat{n}$ & $\hat{m}$.



notation for average

$$\langle \sigma_m \rangle = \hat{n} \cdot \hat{m} := \text{component of } m \text{ along } n.$$

average value of σ_m (knowing that it was originally oriented along n direction)

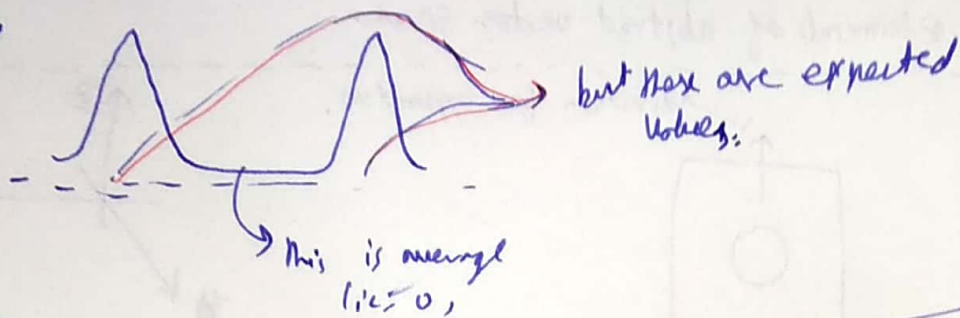
This is what a classical physicist might have expected.

* Average quantities do behave like vectors... (in this case)

15

* expected values & average values are very different in general.

example



* Logic of Quantum Mechanics.

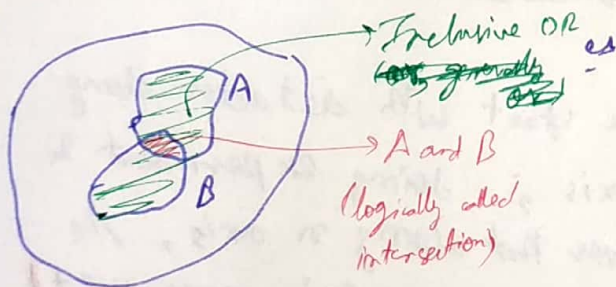
And ~~and~~ or are different in Quantum Mechanics than Classical Mechanics.

Given two proposition A, B.

we can form two combinations of them A or B ; A and B.

→ let's see what that means in terms of sets.

or



~~A & B is still~~
A and B is still a proposition but it's always false.

OR → exclusive OR (and or in english is this one; can't be both)
→ inclusive OR (this includes possibility where both is true)
(generally we mean this)

A: $\sigma_z = +1$

B: $\sigma_x = +1$

(A or B)

→ measure first A

→ you get +1 (because ghost did that... was initially in +1)

→ so now no need to check further B
(it is true because A or B)

* assume, secretly the spin is pointing in z direction.

(ghost made it...
... he don't know that)
think as if you don't know that...

our propositions
A & B.

→ Now let's change the order of measuring, B... then A. (Pg 11)

- measure B, we get $+1$ or -1 with 50% probability.

→ 50% chance that $B = +1$

if $B = -1$ (it does not mean that A or B is wrong)
(we still have chance that when we measure A we will get $+1$)

★ But the problem is,

when we measure B → we have prepared the spin along x axis (whether in $+x$ or $-x$)

→ so, now after measuring B when ~~we~~ measure A, i.e. for z -axis → we will have 50% probability for up & 50% for down.

→ this means; we have 25% probability of finding both $\sigma_x = -1$, and $\sigma_z = -1$

(so we have a net probability of 1/4 that both A & B were false) when we do the experiment in one order.

(→ when we do the experiment in other order, i.e., A then B, it is certain that A or B is ~~right~~ right (because A is right ~~prepared~~ ... because the ghost has oriented that way)

So what is true in Classical Physics that the idea of $A \text{ or } B$ is symmetrical between A and B . (Pg 1)

↳ the idea of ~~or~~ statement (or statement)

i.e. if $A \text{ or } B$ is true, then $B \text{ or } A$ is true.

→ We find that in Quantum Mechanics,
the concept of $A \text{ or } B$ at best can only be defined in a way that depends on the order in which you determine A or B .

~~$A \text{ or } B$ not necessarily equal to $B \text{ or } A$~~

i.e. "A or B in general is not equivalent to B or A."
in Quantum Mechanics.

→ This is how the logic of Quantum Mechanics is different from Classical Mechanics.

Space of States of Quantum System is linear vector space & that means given any two states you can add them in some sense & make another state. (from example of qubit, we saw they have some sense of directionality...)

(in Classical system, the space of state is set of collection of some elements → & we don't add 2 elements in set & create a new one...)

$\langle A|B \rangle =$ complex number $\equiv$ complex conjugate of $\langle B|A \rangle$

(Pg 13)

* we usually don't think about length of the vector, but the square of the length of vector. & that is $\langle A|A \rangle$

inner product with its own dual.

and $\langle A|A \rangle$ is always real

$\therefore$ hence $\langle A|A \rangle = \langle A|A \rangle^*$

& it is also positive.

better we say positive definite.

* Notion of orthogonality; $\langle A|B \rangle = 0 \Rightarrow$ orthogonal.

* Connection between Vector Space & the space of possible states of a Quantum System.

Postulate : The state of Quantum System form a complex vector space.

ex $\begin{array}{c|c} u : + \uparrow : \sigma_z = 1 \\ d : - \downarrow : \sigma_z = -1 \end{array}$ there are two states which we know about which can be prepared by turning the apparatus upward along the z-axis.

• these states are vectors in a vector space according to hypothesis. and can be called as $|u\rangle, |d\rangle$

$l \leftarrow \sigma_x = 1$; label as $|l\rangle, |r\rangle$

$r \rightarrow \sigma_x = -1$

(left & right)

$i \swarrow \sigma_y = \pm 1$; label as $|i\rangle, |o\rangle$

o

(in and out)

- That the space of all states of the simple system in concern here is two dimensional. (has only two independent state) (p.5)

ii)

$$|A\rangle = \alpha_u |u\rangle + \alpha_d |d\rangle$$

; α 's are complex numbers in general.

Next Postulate Next Postulate
~~next postulate~~: if we were to measure up & down, (if somebody prepared as an electron for spin in this state)...

→ we would find probabilities for up & down.

$$P_u = \alpha_u^* \alpha_u$$

P 's are probabilities

$$P_d = \alpha_d^* \alpha_d$$

∴ P_u is probability for being $|u\rangle$, i.e. up
 ∴ P_d " " " " " $|d\rangle$, i.e. down.

* After postulate; Question What's the relation between up and down?
Answer They are not negative of each other.

→ They are orthogonal.
 (That postulate)

The basic postulate is, ~~then~~

when things are measurably different,
 when two states are such that you can clearly tell the difference between them by an experiment, they are orthogonal.

→ Orthogonality literally means they are sufficiently different that you can, with a single experiment ~~or~~ or may be few experiments tell the difference between them uniquely.

What they corresponds to in terms of sets?

Not standard terminology

"that you might say that two sets are orthogonal if they don't share any element"



(P315)

↳ that means they are distinctly different,

there is no chance that an experiment will confuse two of them.

↳ a nice experiment to distinguish them will not lead to any ambiguity

↳ The same is true for up & down

↳ up is distinctly different from down.

if you measure up, you measure up again & again

& the version of this in Quantum mechanics (or implication of this in quantum mechanics) for vectors

is that,

The things which correspond to physically unambiguous different situations are orthogonal to each other.

↳ so; up & down are orthogonal to each other.

now, what if we have some combination of up and down with some coefficients → then there is probability to be up & probability to be down.

(and the probabilities are given by $\alpha_u^* \alpha_u$ & $\alpha_d^* \alpha_d$.)

also;

$$P_u + P_d = 1$$

(relation between probabilities)

(18)

(Sum of distinct probability should add up to 1)

↳ and this says

$$\alpha_u^* \alpha_u + \alpha_d^* \alpha_d = 1$$

& this is equivalent to the statement :

$$\langle A | A \rangle = 1$$

(prove)

(it does follow from postulate 3 & from the assumption that $|u\rangle$ & $|d\rangle$ are orthogonal unit vectors) (a postulate indeed)

↳ if they are not unit vectors, you can change them until they are unit vectors.

↳

↳ This says that a physical state corresponds to a vector in vector space of unit length. (a unit vector)

↳ unit standing for the fact that the total probability is one.

States := Unit vectors.

other term for unit vectors is Normalized Vector.

* Q1 What happened to right & left?

As They seem like different state. But according to postulate, they right & left must be linear combination of up & down.

It's not too surprising, because if you prepare right & left, & then measure up or down, you get ± 1 .

(B17)

right state must be a combination of $|u\rangle$ & $|d\rangle$
 # i) $|r\rangle = \alpha_{r,u} |u\rangle + \alpha_{r,d} |d\rangle$

in form of column vector.
 or: $\begin{pmatrix} \alpha_{r,u} \\ \alpha_{r,d} \end{pmatrix}$

(ii) Suppose electron is created pointing along right direction (the n -axis)

→ what are the probabilities to be up & down. ... $\sim 1/2$

→ this means: $\alpha_u^* \alpha_u = 1/2$

$$\alpha_d^* \alpha_d = 1/2$$

lets make a simple guess to put $\alpha_u = 1/\sqrt{2} = \alpha_d$

$$\text{ie: } |r\rangle = \frac{1}{\sqrt{2}} |u\rangle + \frac{1}{\sqrt{2}} |d\rangle$$

→ now; ~~but~~ what ambiguity is there,

... we could put $-1/\sqrt{2}$, or $i/\sqrt{2}$...
 ... something else.

→ we will come back to it, what that ambiguity means.

∴ ~~the~~ lets take $|r\rangle = \frac{1}{\sqrt{2}} |u\rangle + \frac{1}{\sqrt{2}} |d\rangle$ this as working hypothesis for what the state right is.

→ this leaves up with state left.

→ we know right & left are physically distinguishable by a single experiment. → this means they are orthogonal.

$$\text{ie: } \langle l | r \rangle = 0$$

(incidentally we have assumed that $\langle u | d \rangle = 0$)

~~is there any vector orthogonal to it.~~

is there a vector orthogonal to $|r\rangle$, yes.

see, $|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$

now, it's ambiguous to put the minus sign with either

(i.e. $|l\rangle = -\frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$ will also work)

→ we will find out that there is no physical difference between those possibilities.

we will find out later that there is no physics in the overall sign of a vector.

how to check: $|r\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$; $|l\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

(column vector describing right)

check $\langle l|r\rangle$

see, $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{2} - \frac{1}{2} = 0 = \checkmark$

works ~~works~~ because basis are orthogonal, i.e. $\langle u|d\rangle = 0$.

• and what more, we notice that $|l\rangle$ ~~is also~~ also has a property that the probability for it to be up & down is equal to $1/2$ for each one.

we could have also written: up & down in terms of right & left.

PS 19

add:

$$|r\rangle + |l\rangle = \sqrt{2}|u\rangle \Rightarrow |u\rangle = \frac{|r\rangle + |l\rangle}{\sqrt{2}}$$

$$\Rightarrow |u\rangle = \frac{|r\rangle + |l\rangle}{\sqrt{2}} \quad \& \quad |d\rangle = \frac{|r\rangle - |l\rangle}{\sqrt{2}}$$

$$\Rightarrow |d\rangle = \frac{|r\rangle - |l\rangle}{\sqrt{2}}$$

now, in & out.

is here a third pair of vector $|i\rangle, |o\rangle$; which stand in a same relationship to $|r\rangle, |l\rangle$ & $|u\rangle, |d\rangle$

↳ a third combination, which is symmetrically defined.

$$|u\rangle, |d\rangle \quad \langle u|d\rangle = 0$$

$$|r\rangle = \frac{1}{\sqrt{2}} [|u\rangle + |d\rangle]$$

$$; \langle r|l\rangle = 0$$

$$|l\rangle = \frac{1}{\sqrt{2}} [|u\rangle - |d\rangle]$$

$$|i\rangle = \frac{1}{\sqrt{2}} |u\rangle + \frac{i}{\sqrt{2}} |d\rangle$$

$$|o\rangle = \frac{1}{\sqrt{2}} |u\rangle - \frac{i}{\sqrt{2}} |d\rangle$$

$$\text{requirement: } \langle i|o\rangle = 0$$

$$|i\rangle, |o\rangle = \alpha |u\rangle + \beta |d\rangle + \gamma |l\rangle + \delta |d\rangle$$

write various column vectors

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|r\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}; |l\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$; |i\rangle = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}; |o\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$

enough to determine coefficients with little ambiguity.

$$|I\rangle = \frac{1}{\sqrt{2}} \left(\frac{|L\rangle + |R\rangle}{\sqrt{2}} \right) + \frac{i}{\sqrt{2}} \left(\frac{|R\rangle - |L\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{2} \left((1+i)|R\rangle + (1-i)|L\rangle \right)$$

~~$$\Rightarrow |I\rangle = \frac{(1+i)}{2}|L\rangle + \frac{(1-i)}{2}|R\rangle$$~~

$$\Rightarrow |I\rangle = \left(\frac{1-i}{2} \right) |L\rangle + \left(\frac{1+i}{2} \right) |R\rangle$$

$\left(\frac{1-i}{2} \right) \left(\frac{1+i}{2} \right) = \frac{1}{2}$

 implies that if you start with state $|I\rangle$ & instead of measuring $|L\rangle$ or $|R\rangle$... you measured $|L\rangle$ or $|R\rangle$, the probability would be half.

Question: could we find other doublet $|f\rangle, |s\rangle$ such that $\langle f|s\rangle = 0$ & expansion coefficient in other three pairs is of magnitude $1/\sqrt{2}$.

$\begin{pmatrix} \alpha_u \\ \alpha_d \end{pmatrix}$: how much freedom does α_u & α_d such that it does not change the physical nature of ~~state~~ state.

constraints $\alpha_u^* \alpha_u + \alpha_d^* \alpha_d = 1$

we will later find that (the consequence of the postulates)

- Every physically observable quantity always associated with a state, always involves $\alpha^* \alpha$... (we will find that.) ... (we put this as another postulate) ... it emerges as a consequence of other postulates.

→ If you multiply both of the α 's, (i.e. α_u, α_d) by the same phase factor. (a complex number of form $e^{i\theta}$... unit length), it does not affect anything.

It means that, if we would have given just two complex numbers here $\begin{pmatrix} \alpha_u \\ \alpha_d \end{pmatrix}$, ... Pg 21
 → meaning, (no longer needed for a job and therefore out of work)

There are two ~~redundant~~ redundant pieces of information, that really don't have anything to do with the physical character of state.

redundant := not necessary or wanted.

~~if first~~ ii) The overall magnitude,

because the overall magnitude should always be such that the sums of squares of these add up to 1.

iii) If you multiply both entries by same phase factor, it also does not change anything physical.

→ Question! How many independent variables are there which characterises the state of the system.

~~lets count~~ lets count, α_u ; α_d
 ↙ ↘
 2 real numbers + 2 real numbers = 4 real numbers.

→ & because of constraint $\alpha_u^* \alpha_u + \alpha_d^* \alpha_d = 1$
 ; one does not count, so $4 - 1 = \underline{\underline{3}}$

→ but also another one does not count, because the overall phase if we would multiply everything by the same phase factor also does not count.

so; $3 - 1 = 2$

→ so; two of max of four parameters are irrelevant to the physical character of the state.

so; ~~As~~ 2 independent variables.

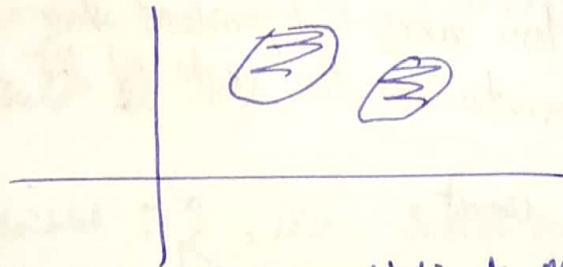
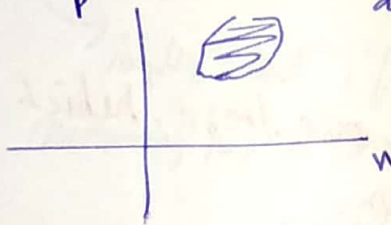
How many numbers does it take to characterize direction of unit vector. ... θ & ϕ . (Pg)

- If you add two unit vectors, you may get non-unit vectors, but then the rule for states of system are divided through an overall factor until it is normalized.

- * Note // • orthogonal does not mean perpendicular, it means physically distinguishable, (unambiguously distinguishable)
- left & right are not physically distinguishable uniquely from up & down.

Orthogonality in Space of States. ~~Don't confuse~~

Classical analogue ^{assume, biased}
^{probabil}
assume, probability in the patch is uniform $\Rightarrow$ 2 at rest place zero.



The analogue of orthogonality between two distinct probability distribution is that they don't overlap. (i.e. no ambiguity)

- up & down don't overlap. (If you are up, you are not down & vice versa)

* Don't confuse orthogonality in space of states with orthogonality in 3 dimensional space.



- There is a situation in which orthogonality is the same in space & in the space of states; it's the polarization of photons.

- There is one way to tell by a single well defined experiment whether the spin was pointing ~~up~~ up or right.

$\hookrightarrow$ If you get $|u\rangle$ in your measurement, it does not tell you whether it was initially up (or) it was initially sideways & with probability $1/2$ you get up.

So there is no unique single experiment ~~that~~ you PS 23
can do to distinguish $|L\rangle$ and $|d\rangle$.

→ There is an experiment that you can do to distinguish $|u\rangle$ & $|d\rangle$.

→ That's what makes $|u\rangle$ & $|d\rangle$ orthogonal.

→ & $|L\rangle$ & $|d\rangle$ non orthogonal. ... (they overlap)

→ and in fact, ~~that~~ the probability that if you are in $|L\rangle$, that you measure $|d\rangle$, when you measure σ_z ; that's the measure of overlap between them.

→ They overlap, which means in technical language that they are not orthogonal.

$$|A\rangle = \sum_i \alpha_i |i\rangle$$

$|i\rangle := i^{\text{th}}$ basis vector.
ket vector.

$$\Rightarrow \langle j|A\rangle = \sum_i \alpha_i \langle j|i\rangle = \sum_i \alpha_i \delta_{ji} = \alpha_j$$

$$\Rightarrow \boxed{\alpha_j = \langle j|A\rangle} \rightarrow \alpha_i = \langle i|A\rangle$$

so; rewrite; $|A\rangle = \sum_i |i\rangle \alpha_i = \sum_i |i\rangle \langle i|A\rangle$

$$\Rightarrow \boxed{|A\rangle = \sum_i |i\rangle \langle i|A\rangle}$$

expression of vectors in terms of its components.

Same true for bra vectors

$$\langle A| = \sum_i \langle A|i\rangle \langle i|$$

→ the basis bra vector

Observables (the things that we measure) are related to Linear operators in space.

Notion of Linear operator
(is a process which you apply to vectors)

let; M be a linear operator.

$$\underline{M|A\rangle = |B\rangle}$$

it is true that M applied to any vector $|A\rangle$ gives a unique reflection of $|A\rangle$ which is process M acting on $|A\rangle$.

• it ~~may or may not~~, given a $|B\rangle$,

this does not necessarily mean that there is one unique $|A\rangle$ that gives rise to that $|B\rangle$.
..... for a general operator.

Linearity

$$M[z|A\rangle] = z[M|A\rangle]$$

$$M[|A\rangle + |B\rangle] = M|A\rangle + M|B\rangle$$

#let suppose $M|A\rangle = |B\rangle$

~~We are interested~~

we are now interested in the components of output vectors.

$$\beta_i := \langle i|B\rangle = \langle i|M|A\rangle \quad \therefore B = \sum_i \beta_i |i\rangle$$

$$\sum_j \langle i|M|j\rangle \langle j|A\rangle = \sum_j \underbrace{\langle i|M|j\rangle}_{\substack{\text{vectors...} \\ \& \text{take inner product} \\ \text{with } j}} \alpha_j = \beta_i$$

$$\text{let } M_{ij} = \langle i|M|j\rangle \quad \therefore \langle i|(M|j\rangle) = \langle i|M|j\rangle$$

$\downarrow$
is a number.

$$\text{where } |M|j\rangle = M|j\rangle$$

$\hookrightarrow$ it is possible to characterise linear operators by ~~the~~ matrix elements M_{ij} .

$$\text{so; } \sum_j M_{ij} \alpha_j = \beta_i$$

M_{ij}

M_{ij} is a matrix,
square matrix

$\hookrightarrow$ if the dimensionality of space is N

$\hookrightarrow$ then M is
 N by N matrix
 $\therefore N \times N \dots$

$\hookrightarrow$ α ~~can be~~ can be regarded as column, & output is also a column.

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & \dots & M_{1N} \\ M_{21} & M_{22} & M_{23} & \dots & M_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{N1} & M_{N2} & M_{N3} & \dots & M_{NN} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix}$$

$\hookrightarrow$ the specifics of matrix elements & the column vectors will depend on your choice ~~you~~ of your basis.

▷ You ~~lose~~ something & gain something by going to matrix type representation.

What you LOSE, is the geometric idea of the relationship between two vectors that is independent of the basis choice! (p327)

What you GAIN, is a computation, actual mechanical algorithm for producing (for the machine itself...) the final answer.

• Different choice of basis leads to same physical answer.

$$* \langle A | M | B \rangle = \langle A | M | B \rangle = \langle A | [M | B] \rangle$$

• We define the action of an M on $\langle A |$ to the left such that it does not matter which way we do it

$$\text{check } \sum_{i,j} \left[\langle A | i \rangle \langle i | M | j \rangle \langle j | B \rangle \right]$$

$$\text{we had } |A\rangle = \sum \alpha_i |i\rangle$$

$$\therefore \langle A | = \sum \alpha_i^* \langle i |$$

$$\sum_{i,j} \alpha_i^* M_{ij} \beta_j$$

$$\text{or } = \sum \langle i | \alpha_i^*$$

→ So from this form we see that it does not matter, if I first sum over j & think of M acting on ket vector B

ⓐ I sum over i & think of M as first acting to the left on α .

→ you get the same answer.

→ So you can define M in such a way that it does not depend on which order you put brackets.

Supper $MIAS = IB$

Question if here a linear operator which has the property.

$$\langle A | \sigma M = \langle B |$$

$\therefore$ for M ; does
 $\exists \sigma M \dots$

The notation, that is standard, is to say,

that the operator which when acting to the left is the reflection of dual of M if you like...

so σM is called $M^\dagger$

the dagger ($\dagger$) is a operator version of complex conjugation.

(or) matrix version of complex conjugation,
 its called Hermitian Conjugation.

$\therefore$ now, find $(M^\dagger)_{ij} = ?$

$$(M^\dagger)_{ij} = \langle i | M^\dagger | j \rangle$$

$\therefore$ let suppose, $M|i\rangle = |i'\rangle$

$$\text{now, take, } \langle j | M | i \rangle = \langle j | i' \rangle$$

$$M_{ji} =$$

$$\therefore \text{w. } \langle i | M^\dagger = \langle i' |$$

$$\text{now, } (M^\dagger)_{ij} = \langle i | M^\dagger | j \rangle$$

$$= \langle i' | j \rangle = (\langle j | i' \rangle)^* = (M_{ji})^*$$

$$\therefore M_{ij}^\dagger = M_{ji}^*$$

... That is the definition of Hermitian Conjugation

$$\text{ex } M = \begin{pmatrix} M_{11} & M_{12} & \dots \\ M_{21} & M_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} ; \text{ then, } M^\dagger = \begin{pmatrix} M_{11}^* & M_{21}^* & \dots \\ M_{12}^* & M_{22}^* & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Real number is the one which is equal to its own complex conjugate.

- Corresponding notion in operators,
An operator which is its own Hermitian conjugate, i.e. $M^\dagger = M$, that is called Hermitian.
(analogy for matrix (as for reals))

Hermitian : $M = M^\dagger$; $M_{ij} = M_{ji}^*$

- diagonal elements are ~~real~~ always real.

example
$$\begin{pmatrix} 1 & i & 4-i \\ -i & 7 & 5 \\ 4+i & 5 & -2.6 \end{pmatrix}$$

* Hermitian matrix has enormous importance in Quantum Mechanics, they represent Observable Quantities (the measurable quantities).
(the things you can measure)

- Directionality is not necessarily preserved by linear operators in general.

→ There may be particular vectors, that when you apply matrix, (specific to the operator), they come out in same direction.

Eigenvectors $|I\rangle$

$$M|I\rangle = \lambda_I |I\rangle \quad ; \quad \lambda_I \in \mathbb{C}$$

• Given a matrix, it had certain eigen vectors (or eigen direction)

→ the eigenvectors are not completely unambiguous,

if you take a multiple of a given eigenvector, the result is still an eigenvector (because of linearity of operator) with the same eigen value,

→ so strictly speaking, we should talk about eigen directions.

• If a matrix is Hermitian, & dimensionality of space is N
~~it has~~
→ there will be N mutually orthogonal eigenvectors.

(ie: in other words, Eigenvectors of Hermitian matrices define a basis)

• Prove: ~~if~~ Given: $M \Rightarrow$ Hermitian. means $M = M^\dagger$
& let suppose have eigenvalue λ_i . (it is real)

now, let's prove that if we have two different eigenvalues, the corresponding eigenvectors must be orthogonal.

(remember what orthogonal means from physical point of view, it means physically distinct, $\Rightarrow$ that is an experiment you can do to distinguish the two of them unambiguously)

• let's prove that eigenvectors of M with different eigenvalue are orthogonal.

→ Different eigenvalue $\Rightarrow$ Physically Distinguishable.

let $M|I\rangle = \lambda_I |I\rangle$
 $\Rightarrow$ $M|J\rangle = \lambda_J |J\rangle$

turn it into bra equation

$\langle J|M = \lambda_J \langle J|$

$\therefore$ used $M^\dagger = M$

and $\lambda_J^* = \lambda_J$

$\langle J|M|I\rangle = \lambda_I \langle J|I\rangle$

$\langle J|M|I\rangle = \lambda_J \langle J|I\rangle$

$\Rightarrow (\lambda_I - \lambda_J) \langle J|I\rangle = 0$

by assumption $\lambda_I \neq \lambda_J$

$\Rightarrow$ so; $\langle J|I\rangle = 0$

$\Rightarrow$ so, I & J are orthogonal.

If all eigenvalues are different from each other,
then all eigenvectors are mutually orthogonal to each other.

⇔ If there are enough of them (prove this... Wk);
they form a basis.

↳ form eigen basis.

Principle of Quantum Mechanics.

① First of all there are measurable quantities, and the result of the measurement is always a real number.

- It takes at least two measurements (~~and~~ two compatible measurements) that can both be performed simultaneously to measure a complex number.

↳ The basic ingredients, i.e. the basic measurable are real numbers.

- Physical measurables are called Observables in Quantum Mechanics.
(The thing ~~you~~ which you can measure)

★ The Observables are identified with Hermitian operators.

∴ OBSERVABLES ⇔ HERMITIAN OPERATORS

ex the observables which we have previously identified. associated with apparatus.

The components of spin

∴ we measure the spin along z axis & call it σ_z

∴ we could measure σ_z .

→ by turning apparatus on side σ_x ... & similarly σ_y

so: $\sigma_x, \sigma_y, \sigma_z$

2 The Possible observables values measured in an experiment are Eigenvalues

POSSIBLE VALUES $\longleftrightarrow$ EIGENVALUES.

ex for : all the three observables : $\sigma_x, \sigma_y, \sigma_z$
the eigenvalues are ± 1

3 The eigenvectors correspond to states vectors.

$\hookrightarrow$ The eigenvectors are the states in which the corresponding quantities are definite.

(in other words, in which the measurement of them is unambiguous)

ii: The eigen-vectors with a given eigen value is the state of the system where if you measure that particular observable you ~~will~~ would definitely get the ~~of~~ eigenvalue associated with that eigenvector.

~~EIGENVECTORS~~

STATES IN WHICH OBSERVABLES HAS UNAMBIGUOUS RESULT. $\longleftrightarrow$ EIGENVECTORS

ex $|u\rangle, |d\rangle$... these are states in which σ_z had definite values.

" when you are up, $|u\rangle : \sigma_z = 1$

when " " down, $|d\rangle : \sigma_z = -1$

any other vector, with randomness, σ_z is statistically either ± 1 but (p33)

→ So $|u\rangle$ & $|d\rangle$ must be the eigenvectors of σ_z .

~~For~~ $|e\rangle$ & $|g\rangle$ must be eigenvectors of σ_x . ~~(σ_x is,~~
or $|i\rangle$ & $|o\rangle$ " " " "~~we found~~

(whatever σ_x is, we gonna try to figure out what σ_x is, knowing its eigenvalues & its eigenvectors)

4 Given a general state $|A\rangle$ ~~(not necessarily an eigenvector...)~~
(where $|A\rangle$ is not necessarily an eigenvector) ~~(not necessarily)~~

- Somebody made ~~an~~ a system of some sort and left it in the state $|A\rangle$, then we can talk about what is the probability that we measure the eigenvalues of $\hat{m}$ at different values.

→ There may be several different eigenvalues,

What is the probability that when you measure m you get λ_1 . (a particular eigenvalue)
• where; m is observable.

$$\text{Probability}(\lambda_I) \overset{\text{actually}}{\longleftrightarrow} \langle A|I \rangle \langle I|A \rangle = \|\langle I|A \rangle\|^2$$

(probability to measure λ_i) equal.
(that, the result of the experiment comes out a particular eigenvalue of the observable m)

$\therefore \langle IAS \rangle$ is in general complex number $\rightarrow$ so it can't be the probability $\rightarrow$ take square of modulus value.

(13)
The individual components ψ_i (AII) or (ZIA) before we square them, are called ~~A~~ Probability Amplitudes.

→ Probability Amplitudes are the things which you multiply by their complex conjugate to find probabilities.

NOTE : All the four postulates ~~mentioned~~ ^{mentioned} are not necessarily all independent.

→ Prof. Susskind uses ^{all of} them because they are ~~very~~ reasonably ~~intuitive~~ intuitive.

→ In fact you can probably get rid of at least two of them, & ~~prove~~ ~~derive~~ them ~~from the others~~ from the others.

★★★★

Other Postulate : When you measure a quantity, you prepare the system in a state of definite value of that quantity.

★★★★★

→ i.e.; if outcome of some experiment is particular eigenvalue, this means that you have pushed the system into the eigenstate of that quantity ~~with~~ with that eigenvalue.

from [4] there is an implication about the nature of state vectors coming from the assumption that the total probability is equal to 1.

i.e. you want:

$$\sum_I \text{Prob}(\lambda_I) = 1$$

(B25)

$$\therefore \sum_I \langle A|I \rangle \langle I|A \rangle = 1$$

now; $|I\rangle$ is basis; so; $\sum_I |I\rangle \langle I|A\rangle$

$$\text{so; } \langle A| \sum_I |I\rangle \langle I|A \rangle = 1$$

$$\Rightarrow \langle A| (|A\rangle) = 1$$

$$\Rightarrow \boxed{\langle A|A \rangle = 1}$$

$\hookrightarrow$ i.e. the total length of $|A\rangle$ should be 1.
(i.e. sum of the squares of the components of $|A\rangle$ add upto 1)

$\text{ex } |A\rangle$ is equivalent to α_s in our example

$$\dots \sum_I \alpha_I^* \alpha_I = 1$$

this is a real equation.

$\hookrightarrow$ this means ~~that there~~

that there is one real combinations of α_s
which has to be constrained.

(α_s are not completely free.... ~~constrained~~ such that $\langle A|A \rangle = 1$)

~~All probabilities are always conserved~~
★ Any probability which you want to measure is always connected with a real quantity like $\langle A \rangle \langle I \rangle$

→ The collection of such real quantities is invariant under certain operation. → the operation is multiplying the state vector by a phase i.e. $e^{i\theta}$.

- If you multiply all the components of $|A\rangle$ by the same overall phase then every probability is unchanged.

- Physical quantities don't respond to change of phase.

★ this means there are two fewer degrees of freedom or parameters in the specification of the state than the collection of complex numbers that would define a vector in the abstract space.

- one is the overall ~~phase~~ phase.

- magnitude.

Other Point: ex when we measured σ_z , we said that not only measures σ_z , but it prepares the system in a state of definite σ_z ; it leaves it over in which σ_z is definite for next measurement, i.e. that means, it leaves in an eigenstate of σ_z .

Example σ_z let's start in the basis associated with the measurement of spin along z-axis

So, we can say from experiment we learnt that σ_z has two eigenvalues & two eigenvectors $|u\rangle$ and $|d\rangle$ associated with σ_z +1 or -1.

$$\sigma_z |u\rangle = +1 |u\rangle$$

$$\sigma_z |d\rangle = -1 |d\rangle$$

→ let's see if we can compute matrix elements... ~~for~~ ... exhibit as matrix

• when we say that exhibit as a matrix, I have must picked up a Basis.

→ we are going to pick the Basis $|u\rangle$ and $|d\rangle$.

~~$\langle u|d\rangle = 0$ according to.~~

$\langle u|d\rangle = 0$ $\therefore$ $|u\rangle$ and $|d\rangle$ are orthogonal to each other,

they must be according to the principles,

because if σ_z is an observable, it correspond to a Hermitian matrix

→ if there are two different eigenvalues, the eigenvectors must be orthogonal for Hermitian matrix and so $|u\rangle$ & $|d\rangle$ are orthogonal. $\therefore \langle u|d\rangle = 0$

• The Matrix elements are just the array that we get when we consider the four possibilities. 2×2 matrix.

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$uu \quad ud$
 $du \quad dd$

$$\begin{aligned} \langle u|\sigma_z|u\rangle &= \langle u|+|u\rangle \\ &= \langle u|u\rangle \\ &= 1 \end{aligned}$$

Since $|u\rangle$ is basis vector, it is normalized.

this is the matrix associated with σ_z .

$$\begin{aligned} \langle d|\sigma_z|u\rangle &= \langle d|+|u\rangle = 0 \\ \langle d|\sigma_z|d\rangle &= \langle d|-|d\rangle \\ &= -(\langle d|d\rangle) \\ &= -1 \end{aligned}$$

what is column vector associated with $|u\rangle$ or $|d\rangle$

$$\begin{aligned} |u\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |d\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

• $\sigma_x, \sigma_y, \sigma_z$ are called Pauli matrixes.

★ σ_x : (if we use x basis ... we find exactly same thing.)

↳ but we stick with $|u\rangle, |d\rangle$ basis ; but find out what are matrix elements of σ_x are.

Now what is the property of σ_x ?

★ Is, that its eigenvectors are $|e\rangle$ & $|g\rangle$

so; for example : $\sigma_x |g\rangle = +1 |g\rangle$

$$\sigma_x |e\rangle = -1 |e\rangle$$



↳ now find matrix elements of σ_x in $|u\rangle, |d\rangle$ basis.

$$\langle u | \sigma_x | u \rangle$$

$$; \text{ but } |u\rangle = \frac{|e\rangle + |g\rangle}{\sqrt{2}}$$

$$|d\rangle = \frac{|e\rangle - |g\rangle}{\sqrt{2}}$$

$$\text{we: } \left\langle \frac{|e\rangle + |g\rangle}{\sqrt{2}} \right|$$

This is meaningless symbol

↳ but we know what we mean

$$; \left\langle \frac{|e\rangle + |g\rangle}{\sqrt{2}} \right| = \frac{1}{\sqrt{2}} (\langle e| + \langle g|)$$

$$\text{Note: } \langle \sigma_x \rangle_{ud} = \langle u | \sigma_x | d \rangle$$

()_{uu} =

$$\text{so; } \left\langle \frac{|e\rangle + |g\rangle}{\sqrt{2}} \right| \sigma_x \left| \frac{|e\rangle + |g\rangle}{\sqrt{2}} \right\rangle$$

$$\sigma_x = \begin{pmatrix} (0)_{uu} & (1)_{ud} \\ (1)_{du} & (0)_{dd} \end{pmatrix}$$

$$\text{so; } ()_{uu} = \frac{1}{2} \langle |e\rangle + |g\rangle | \sigma_x | |e\rangle + |g\rangle \rangle$$

$$= \frac{1}{2} \langle |e\rangle + |g\rangle | |e\rangle - |g\rangle \rangle = \frac{1}{2} [\langle e|e\rangle - \langle e|g\rangle - \langle g|e\rangle + \langle g|g\rangle] \\ = \frac{1}{2} [1 - 1] = 0$$

... similarly ... ()_{dd}

$$\langle |e\rangle - |g\rangle | \sigma_x | |e\rangle - |g\rangle \rangle = \langle |e\rangle - |g\rangle | |e\rangle + |g\rangle \rangle = 0$$

∴ now,

8339

$$\langle d | \sigma_x | u \rangle$$

$$= \frac{1}{2} \langle r-l | \sigma_x | r+l \rangle$$

$$= \frac{1}{2} \langle r-l | r-l \rangle$$

$$= \frac{1}{2} (\langle r|r \rangle + \langle l|l \rangle - \cancel{2\langle r|l \rangle})$$

$$= \frac{1}{2} (2) = 1$$

$$\therefore \langle u | \sigma_x | d \rangle = 1$$

~~$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$~~

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

→ now, let's see what do we have,

we have pair of linear operators, whose eigen vectors $|u\rangle, |d\rangle$ and $|r\rangle, |l\rangle$ respectively

$$\# |i\rangle = \frac{1}{\sqrt{2}} (|u\rangle + i|d\rangle)$$

$$|o\rangle = \frac{1}{\sqrt{2}} (|u\rangle - i|d\rangle)$$

can ask: can we find a matrix which has eigenvector $|i\rangle$ & $|o\rangle$, with eigenvalue $+1$ or -1 ... respectively.

∴ the matrix is unique; ~~etc.~~

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\frac{|i\rangle + |o\rangle}{\sqrt{2}} = |u\rangle$$

$$\frac{|i\rangle - |o\rangle}{\sqrt{2}i} = |d\rangle$$

$$\frac{1}{2} \langle i | \sigma_y | i \rangle = \langle o | \sigma_y | i \rangle \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2} \times 2 \times 1 = 1$$

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The three Pauli matrixes which represent the components of spin in different direction.

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Ans: If we have component of spin along arbitrary direction, & we want to calculate using the principle.
• We want to calculate what the probability is for the measurement of σ_z , +1 or -1.

Principle of Quantum Mechanics.

① Observables = L
represented by

L is Hermitian operator
 $L^\dagger = L$ here

now Question: whether every Hermitian operator is a thing which can be measured is generally a hard question

generally, the abstract rule is that,
any Hermitian operator is identified with an observable (and some day, somebody will figure out how to measure it).

② Eigenvalues of L = Possible outcome of L -measurement.
 $= \lambda$

③ Physically Distinguishable State
Meaning of saying, that two states are physically distinguishable is that there exist a measurement of some kind that you can do, which can tell you the difference between them. unambiguously.

Physically Distinct States = Orthogonality.

② where λ is real & along with ③ $\Rightarrow$ imply the the operator L is Hermitian operator

$\hookrightarrow$ whose eigenvalues are real & eigenvectors are orthogonal

necessary & sufficient condition

④ Probability principle also called Born's Rule.

BORN RULE

if system in $|A\rangle$
&
measure L

$\longrightarrow$ # Probability
$$Prob(A) = \|\langle A | I \rangle\|^2$$

notation
$$Prob(A) = \|\langle A | A \rangle\|^2$$

~~some~~ ~~sometimes~~ ~~identity~~

Sometimes incidentally, the inner product is called the Overlap $\rightarrow$ It's a measure of...

- If we say that orthogonality is complete distinguishability,

then a lack of orthogonality represents to some extent the inability to make clear distinction between two states.

Notation: every eigenvalue λ is associated with an eigenvector.

$\therefore$ the vector labelled by λ , i.e. $|\lambda\rangle$ is the eigenvector associated with the eigenvalue λ .

$\rightarrow$ In general, in complex vector space, the Overlap, i.e. the inner product is complex numbers.

$\rightarrow$ Inner product itself is called the Probability Amplitude.

- If you want ~~the~~ ^{the} probability for given value of λ , and there is more than one eigenvector with the same value of λ , then you add the sums of squares of modulus of inner product.
 $\hookrightarrow$ just Probability addition.

$\therefore |\lambda\rangle_1, |\lambda\rangle_2$

$$\therefore L|\lambda\rangle_1 = \lambda|\lambda\rangle_1, \\ L|\lambda\rangle_2 = \lambda|\lambda\rangle_2$$

$$\therefore L[\alpha|\lambda\rangle_1 + \beta|\lambda\rangle_2] = \lambda[\alpha|\lambda\rangle_1 + \beta|\lambda\rangle_2]$$

$\hookrightarrow$ now, given two distinct vectors, and all the possible linear combinations of them, $\therefore$ you can always find perpendicular $\perp$ combinations.

Rest of the theorem is,

that; if two eigenvalues are the same, you have the freedom to choose two eigenvectors which are perpendicular (orthogonal)

★ Given a Hermitian Operator, its eigen vectors can be chosen so that they form orthonormal basis.

∴ $P(\lambda) = \|\langle A|\lambda\rangle\|^2 + \|\langle A|\lambda\rangle_2\|^2 \dots$ add probabilities.

★ ~~Why~~

★ Why probability is squared.

∴ for normalized vector; we know

$$\sum_I \|\langle A|I\rangle\|^2 = 1$$

↳ so, ~~$\|\langle A|I\rangle\|^2$~~ $\|\langle A|I\rangle\|^2$ so serves as a way to represent probabilities, since they add up to 1

↳ so, the Total probability is 1 & state vectors are normalized fits together very well.

★ Suppose, somebody gives us a spin & they don't tell us anything about how they prepared it, ~~they~~

↳ Then it is not possible to find an experiment which has unambiguous answer.

5] How do states change with time?

(this we called in Classical Mechanics as the Dynamical Laws of motion)

Classical Physics

(or) Dynamical Laws of Nature

Classical The laws of physics are updating from instant to instant of a configuration, that tells you how the configuration changes.

→ & if you have a good laws in Classical Physics, they are deterministic (tell you how state changes)

→ but also the reversibility idea was there.

→ Reversibility was basically just the idea that states don't run into each other.

States which are distinguishable, namely different, #
→ two states which are ~~at~~ different will evolve mathematically and stay different.

(this was the idea of Reversibility)

→ & this comes down to exactly that, if you know the state at an instant, you not only will know what comes next, but you will also know what came before, (because there is no chance that states run together)

... & two different states giving you the same outcome, ~~then you would~~ not be able to tell where you came from.

→ Reversibility is ~~also~~ the idea we also called the Information Conservation

... / think it as -1st law, since it comes before everything else)

• ~~Quantum Mechanics also has a rule of both Reversibility and Information Conservation, (you can call it -1st Law, (2))~~

→ ~~but it comes down to something, States which are distinguishable stay distinguishable.~~
(distinguishable in the sense that there exist ~~an~~ ^{an} ~~expression...~~)

Quantum Mechanics also has a rule of both
Reversibility and Information Conservation (you can call it
- 1st Law (90))

↳ but it comes down to same thing, minus first law.

States which are distinguishable (distinguishable in the sense
that there exist an
experiment that
uniquely distinguishes
them)
Stay distinguishable.

Postulate : if you have system at some specific time (say time 0)

$|\Psi(t)\rangle$ represent state ~~which~~ which
changes with time

note (note: change notations for states, instead of calling
them A or B, we are going to start using notation
which is more standard in Quantum Mechanics, ... call it Ψ)

↳ Ψ is commonly used letter to represent state of the
system. ~~$|\Psi\rangle$~~ $|\Psi\rangle$

↳ (now, states of system do change with time,
so, you have to think of these vectors, as being functions
of time, ~~they~~ ... i.e. they evolve ... so : $|\Psi(t)\rangle$)

• But, we are following one ~~at~~ specific state of the system
that has been prepared (say at time t_0 or 0), and
then allowed to evolve, by whatever means things evolve.

∴ so: ~~Postulate~~

take $|\psi(0)\rangle$ or $|\psi(t_0)\rangle \dots$

(we could start evolving from any time)

Pg

~~Postulate~~ $|\psi(t)\rangle$ can be obtained from $|\psi(0)\rangle$ by ~~an~~ ~~op~~

Postulate $|\psi(t)\rangle$ can be obtained from $|\psi(0)\rangle$ by a unique operation on $|\psi(0)\rangle$ which is governed by the laws by the Quantum Mechanical laws for that system.

$\therefore$ let represent the operator by $U(t)$

- incidently we don't have to start with time 0, we could have started with time something else and evolve by amount t .

so; the t in $|\psi(t)\rangle = U(t) |\psi(0)\rangle$ really ~~represents~~ represents time of evolution from the initial state to the state at same time.

now two Basic Postulates

$U(t)$ is called time development operator
 $\rightarrow$ its a linear operator.

- $U(t)$ is a linear operator.
- and the same $U(t)$ for any state; whatever the initial state is; you put that $|\psi\rangle$ on right & hit from left with same U , i.e; the same operator, (by the time development operator)

$\hookrightarrow$ ~~$U(t)$ is a linear operator~~ Time Development operator it evolves the state from one instant to another.

★ You could try to ask what would happen if you made more general hypothesis,

for example, you might have hypothesis that $\hat{U}$ depends on initial state, or you might have hypothesis that $\hat{U}$ is an operation but not a linear operator.

→ You can try,

& you will very shortly run in big troubles with all kinds of things (it had been tried by earlier scientists...) and repeated every so often some physicist will try to invent something and then some other person will come along & will say look that has troubles with locality, or causality, or probability interpretation

now, the idea of conservation of information or conservation of distinguishability,

→ that if you have two different states of the same system, & evolve them, → then if Φ to begin with is orthogonal to Ψ ,

~~$$|\Psi(t)\rangle = U(t) |\Psi(0)\rangle$$

$$|\Phi(t)\rangle = U(t) |\Phi(0)\rangle$$~~

(in other words, they are physically distinguishable to begin with,)

$$|\Psi(t)\rangle = U(t) |\Psi(0)\rangle$$

$$|\Phi(t)\rangle = U(t) |\Phi(0)\rangle$$

→ then, they will remain orthogonal for all time.

so; orthogonality ~~to~~ breeds orthogonality.

↳ Physically distinct states evolve to physically distinct states.

↳ Distinguishability remains intact.

↳ Its the immediate analogue of minus First law of Classical Mechanics.

ii; if $\langle \Psi(0) | \bar{\Phi}(0) \rangle = 0$

then $\langle \Psi(t) | \bar{\Phi}(t) \rangle = 0$ for all time.

"If Ψ & Φ were initially orthogonal, ~~then~~ they remain orthogonal at all times"

★ Prove that if orthogonality is preserved,

↳ you can say that,

If you take any two vectors which may or may not be orthogonal, the inner product between them stays constant with time.

Proof: Take any two vectors, $\Rightarrow$ expand them in basis vectors
 $\Rightarrow$ and then assume basis vectors remain orthogonal with time $\rightarrow$ Prove that inner product of any two vectors is time independent.

From the Evolution of Orthogonality it follows that the Inner Product between two states remains the same

↳ i.e; the overlap between them remains the same.

↳ There is a sense in which the overlap between two states is of-course the measure of their similarity.

→ and what it says is that the degree of similarity remains the same

(p349)

* Let's see if we can figure out what that means for ψ .

to study; $\langle \Psi(t) | \Phi(t) \rangle = 0$

we have to flip the ket vector $|\Psi(t)\rangle = U(t) |\Psi(0)\rangle$
to the ~~vector~~ bra vector equation.

So: $|\Psi(t)\rangle = U(t) |\Psi(0)\rangle \rightarrow \langle \Psi(t) | = \langle \Psi(0) | U(t)^\dagger$

so from $\langle \Psi(t) | \Phi(t) \rangle = 0$

→ we get,

in general for an arbitrary states (don't have to be orthogonal)
~~whenever~~ → whatever they are, they have same inner product.

so: $\langle \Psi(t) | \Phi(t) \rangle = \langle \Psi(0) | U^\dagger(t) U(t) | \Phi(0) \rangle$

~~more more~~ ~~the product~~

(the process of repeatedly applying two operators gives you an operation which is called the Product of two operators)

∴ $U^\dagger(t) U(t)$ is some other linear operator.

→ so: it says that first you hit $|\Phi(0)\rangle$ with U , and then you hit the result with $U^\dagger$ or reverse argument in terms of $|\Psi(0)\rangle$
→ & that it says that for any $|\Phi(0)\rangle$ and $|\Psi(0)\rangle$, that when you plug in the product of $U^\dagger$ and U ,

↳ you get something which is exactly the same as had you not evolved the state,

↳ the inner product stays the same

$$\langle \Psi(0) | U^\dagger(t) U(t) | \Phi(0) \rangle = \langle \Psi(0) | \Phi(0) \rangle$$

↳ That's the principle of Conservation of Overlap.

Theorem // If you have an operator K and for any pair of vectors. Ψ & Φ .

(here K for $U^\dagger U$)
↳ but theorem is general for any operator K .

if $\langle \Psi | K | \Phi \rangle = \langle \Psi | \Phi \rangle$

↳ it follows that K is the unit operator.

↳ The Unit Operator is the operator which does nothing, and just gives you back the same vector.

~~can be proved easily by unit~~

⇒ so from the theorem it follows that $U^\dagger U$ must be the unit operator.

(That's the content of conservation of overlap)

$$U^\dagger(t) U(t) = I$$

Proof of the Theorem

it can be ~~proved by writing~~ easily proved by going into basis, that if the matrix elements.....

Now let's examine for very small time.

let: ϵ be tiny little incremental time.

What is $U(0)$?

$$U(0) = I$$

no time evolution at all.

so: for every small ϵ ,

U must be itself close to unit operator.
plus small deviation of order ϵ .

$$U(\epsilon) = I + \epsilon \{ \dots \}$$

i.e: $I + \epsilon \dots$
times another operator there.

$$U(\epsilon) = I - i\epsilon H + \epsilon^2 (\dots) + \epsilon^3 (\dots) \dots$$

now use the freedom ... & put

$$I - i\epsilon H$$

without loss of generality ... $-i$ just absorbs into definition of H .

again, there is no content yet, because H is just some operator or $-H$ is just some operator.

U is some operator,

$\rightarrow \epsilon H$ is some operator,

$\rightarrow \epsilon + i\epsilon H$ is some operator $\rightarrow$ & $-i\epsilon H$ is some operator.

now very small ϵ : $U(\epsilon)$ is close to I

& by first order approximation

(i.e. neglecting terms containing higher power of ϵ)

we get

$$U(\epsilon) = I - i\epsilon H$$

We should give a name to equation here $A^\dagger A = I$

$\rightarrow$ more specifically we should give a name to operators which satisfy this rule $A^\dagger A = I$ or saying $A^\dagger = A^{-1}$

(i.e. ~~the~~ Hermitian conjugate of A is the ~~inverse~~ inverse of A)

Such operators are called Unitary.

~~So~~ i.e. we say now, ~~U is unitary operator.~~

U is unitary operator.

$\rightarrow$ This is again, a restatement of the principle of conservation of overlap.

$\rightarrow$ Let's apply the condition to $U(\epsilon)$ and see what we find about H .

$$U(\epsilon) = I - i\epsilon H$$

$$I^\dagger = I$$

(pg 53)

$$U^\dagger(\epsilon) = I + i\epsilon H^\dagger$$

Now, multiply them together & set the result equal to I to order ϵ .

So, we have:

$$(I + i\epsilon H^\dagger)(I - i\epsilon H) = I$$

$$\Rightarrow I + i\epsilon(H^\dagger - H) + \epsilon^2 H^\dagger H = I$$

$$\Rightarrow I + i\epsilon(H^\dagger - H) = I$$

$$\Rightarrow \text{so: } i\epsilon(H^\dagger - H) = 0$$

$$\Rightarrow H^\dagger - H = 0$$

& so up to order ϵ ; H ~~has to~~ has to equal $H^\dagger$.

$$\Rightarrow \boxed{H = H^\dagger} \rightarrow \text{this is the condition that } H \text{ be Hermitian.}$$

Note: We did not call it H for Hermitian, $\rightarrow$ called it H for something else
 $\hookrightarrow$ we called it H for Hamiltonian.

$\hookrightarrow$ What it has to do with Hamiltonian of classical mechanics will eventually become clear.

• The Hamiltonian for classical mechanics also enters in to the equations for how systems evolve from one instant to the next.
 $\hookrightarrow$ (but of course that's not a good enough reason to call it Hamiltonian)

- We want to see that it enters the equations in a way very very similar to the way Hamiltonian enters into classical mechanics. ... and it does.

→ It is also the thing which is Hermitian

↳ that it could be an Observable, and it's an observable (Hamiltonian is the energy...)

it's the Hamiltonian which generates the evolution of state vectors according to the rule here.

- It could be any hermitian operator at the moment.

↳ what determines what Hermitian operator you put there.

↳ it can be any Hermitian operator ~~with~~ the associated with the system you are studying.

ex for studying the system of one spin, ~~the~~ we want to make it up at the operators that are associated with the spin.

⇒ ~~but~~ but it's experiment, or some ...

... the same thing which governs why you choose one Hamiltonian or Lagrangian in classical mechanics,

either Experiment,

some Prejudice about how things work,

Some symmetry principle,

whatever you do that might give you a clue what it is.

→ At the end of the day, you may have ~~to~~ to resort to experiment to find out whatever it is.

→ The situation with possible time dependence of H is same as in Classical physics

(PSS)

→ In Classical Physics, H can depend on time.

- If the parameters of problem are not varying with time ~~then~~ → then you say Hamiltonian is not time dependent.

→ it's exactly the same here.

- In principle H could be function of time.

→ ~~also~~ ^{case} when H is not function of time

→ is the situation where the parameters of the problem are not time dependent → where in Classical Physics we called it Time Translational Invariance.

- When H is dependent on time

→ then to conserve energy ~~the~~ normal.

you have to take into account the other part which is causing effect.

∴ $U(\epsilon) = I - i\epsilon H$ is for small amount of time

→ for finite time, apply it again & again, do it repeatedly

→ but we can use this to derive a differential equation for the way states changes with time.

~~the~~ $|\Psi(\epsilon)\rangle = (I - iH\epsilon) |\Psi(0)\rangle$

→ here this time ϵ could be thought of as time right after some arbitrary time ... which I am going to call 0 (does not matter)

$$\Rightarrow |\Psi(\epsilon)\rangle - |\Psi(0)\rangle = -i\epsilon H |\Psi(0)\rangle$$

Small incremental change in state vector

in general ; t_0 can be any time t_0

→ so; ϵ get replaced by $t_0 + \epsilon$

→ so; the modified equation looks.

$$|\Psi(t_0 + \epsilon)\rangle - |\Psi(t_0)\rangle = -i\epsilon H |\Psi(t_0)\rangle$$

↓
here we still have ^{only} small increment term ... because in the first order expansion in small terms for $U(t_0 + \epsilon)$

→ we get

$$U(t_0 + \epsilon) = I - i\epsilon H$$

∴ now divide this equation by ϵ ; we are allowed to do this (just multiplying vectors by numbers)

$$\Rightarrow \frac{1}{\epsilon} [|\Psi(\epsilon)\rangle - |\Psi(0)\rangle] = -i H |\Psi(0)\rangle$$

that just time derivative of vector.

$$\Rightarrow \frac{d}{dt} |\Psi(t)\rangle = -i H |\Psi(t)\rangle$$

→ in general; following above argument from

$$|\Psi(t + \epsilon)\rangle - |\Psi(t)\rangle = -i\epsilon H |\Psi(t)\rangle$$

∴ we have.

$$\boxed{\frac{d}{dt} |\Psi(t)\rangle = -i H |\Psi(t)\rangle}$$

; at any given time.

the equation $\frac{d}{dt} |\psi\rangle = -iH|\psi\rangle$ only tells you how states change, because knowing the state vector is not the same as knowing the values of experimental output. (pg 57)

→ You can know the state, and still there can be ambiguity & uncertainty in the value of an experiment.

→ This tells you how states change, but does not tell you how results of experiment change.

→ Result of experiment are still Statistical but with a continuously updated state vector.

Generalized Time Dependent Schrodinger Equation.

$$\frac{d}{dt} |\psi\rangle = -iH|\psi\rangle$$

name of this equation

but in order to use this, you have to know what H is.

• Schrodinger wrote down a very specific version of this.

→ This is the way a system evolves if it's isolated and completely not in contact with anything else.

• How do you solve the problem if it is long time interval (not short)

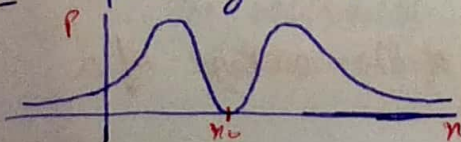
→ you basically solve the ~~diff~~ equation. $\frac{d}{dt} |\psi\rangle = -iH|\psi\rangle$

→ So, expectation value is poorly chosen terminology.
→ it's actually the average value.

~~Expectation value of observable~~

Expectation value of an observable.

ex a probability distribution



∴ here, from graph least likely answer you get is n_0
→ it's also the expectation value by definition.

Expectation value has nothing to do with expected value of an experiment (what we expect to get out from experiment) $\rightarrow$
 $\hookrightarrow$ but nevertheless, it is called Expectation value or expected value. (but in real sense, it is average ~~value~~ value)

Law of large numbers

(not official,

nobody understands it)

- that if you do a thing, enough (large no. of times) time the average gotten by averaging the result of your experiment will be equal to the average to find mathematically to with the margin of error.

----- $\rightarrow$ its a specific number.

$\hookrightarrow$ does it really mean every time you do it, the answer will come out within the margin of error.

No.

we will now assume standard probability theory.

$\hookrightarrow$ that if you do a thing ~~the~~ enough no. of times, the average of your data will be ~~the~~ the mathematical average.

$P(\lambda_i)$

λ_i

Probability for λ_i

so; the definition of average

$$\sum \lambda_i P(\lambda_i)$$

..... & the symbol for average is $\langle \lambda \rangle$.

$$\langle \lambda \rangle = \sum_i \lambda_i P(\lambda_i)$$

(bracket symbol)
... will come it later.

standard probability theory definition of the average of a quantity.

What is the average in the state $|A\rangle$ of the observable A .

(pg 59)

$$|A\rangle = \sum_i \alpha_i |\lambda_i\rangle$$

~~that~~
(let $|A\rangle$ be expanded in a basis

$\hookrightarrow$ the basis of eigenvectors of L
 $\hookrightarrow$ operator measuring λ .)

$\therefore |\lambda_i\rangle$ are orthonormal basis.

Probability of λ_i in state i .

$$P(\lambda_i) = \alpha_i^* \alpha_i$$

$$\langle A \rangle = \sum_i \lambda_i \alpha_i^* \alpha_i$$

$$= \langle A | L | A \rangle$$

prove

where operator L is the operator whose eigen values are λ_i .

plugging in the values...

$$\langle A | L | A \rangle = \sum_{i,j} \langle \lambda_j | \alpha_j^* L \alpha_i | \lambda_i \rangle$$

$$= \sum_{i,j} \langle \lambda_j | \alpha_j^* \alpha_i (L | \lambda_i \rangle$$

$$= \sum_{i,j} \langle \lambda_j | \alpha_j^* \alpha_i \lambda_i | \lambda_i \rangle$$

$\rightarrow$ not all this is just a number for each i & j

$$= \sum_{i,j} (\alpha_j^* \alpha_i \lambda_i) \langle \lambda_j | \lambda_i \rangle$$

$$= \sum_{i,j} \lambda_i \alpha_j^* \alpha_i \delta_{ji} = \sum_i \alpha_i^* \alpha_i \lambda_i = \langle A \rangle$$

So, we proved: $\langle A \rangle = \langle A | L | A \rangle$

so, the average of any observable is obtained by sandwiching the observable between bra and ket ~~of the system~~
(the same bra & ket associated with the system)

Now we will try to find equations governing the time evolution of these average values.

The idea is : by calculating the time evolution of averages you are doing something pretty close to what classical physics would have instructed us to do about calculating the equations of motion of classical system.

at any given time the average of $\langle \lambda \rangle$ is given by

ii;

$$\langle \lambda \rangle = \langle \Psi(t) | L | \Psi(t) \rangle$$

where $|\Psi(t)\rangle$ is state vector at time t , & operator L representing the observable.

lets call it average of L as function of time
i.e. the average of observable

so; $\langle A \rangle \rightarrow \langle L \rangle$
change to

for function of time; use $\langle L \rangle(t)$

but this is somewhat non standard notation.

..... or $\langle L(t) \rangle$

remember;

or we

$\langle L \rangle_t$

this is average of L as a function of time.

average of L as ~~a function of time~~
a function of time.

or call it $\bar{L}$

(another notation for average)

Notations

$$\langle L \rangle(t) := \langle A \rangle(t) := \langle L(t) \rangle := \langle L \rangle_t := \bar{L}(t)$$

and call it $\bar{L}(t)$

meaning average as a function of time

Notations: $\langle \lambda \rangle(t) \equiv \langle \lambda(t) \rangle \equiv \langle L \rangle(t) \equiv \langle L(t) \rangle \equiv \langle L \rangle_t$
 $\equiv \bar{L}(t)$

pg 60

now we would like to find equations of motion for it.

→ in other words we would like to find out how it changes with time by differentiating it with respect to time.

$$\dot{\bar{L}}(t) := \frac{d\bar{L}(t)}{dt}$$

note: L operator is just some fixed operator. (fixed definite operator)
 → all of the time dependence is in state vectors.

$$\text{so: } \dot{\bar{L}}(t) = \left\langle \frac{d\psi}{dt} | L | \psi \right\rangle + \left\langle \psi | L | \frac{d\psi}{dt} \right\rangle$$

∴ ~~we~~ ^{we} know: $\frac{d|\psi\rangle}{dt} = -iH|\psi\rangle$

→ so now we begin to see that Hamiltonians gonna tell us something about how average values change with time.

(the Hamiltonian of classical physics tells us how the corresponding quantities change with time)

∴ we also need bra version

∴ which is: $\frac{d\langle\psi(t)|}{dt} = +i\langle\psi(t)|H^\dagger$

so: $\frac{d\langle\psi(t)|}{dt} = +i\langle\psi(t)|H$

∴ and since H is Hermitian
 i.e. $H^\dagger = H$

we better write it (or know it... keep it in mind)

$$\dot{\bar{L}}(t) = \frac{d}{dt} \langle\psi(t)| (L |\psi(t)\rangle) + \left(\langle\psi(t)| L \right) \frac{d}{dt} |\psi(t)\rangle$$

now; just plug in

$$\begin{aligned}\dot{L}(t) &= i \langle \psi | H L | \psi \rangle - i \langle \psi | L H | \psi \rangle \\ &= i \langle \psi(t) | (H L - L H) | \psi(t) \rangle\end{aligned}$$

$\therefore$ now; operators don't necessarily ~~commute~~ commute.

in general $HL \neq LH$; it may be in some cases
ex; when you take

L to H ~~itself~~
itself.

$$\therefore HH = HH$$

Definition given any two operators,
 A and B

$$AB - BA = [A, B] \quad \text{it's called the commutator.}$$

$$\therefore \text{ here; } (HL - LH) = [H, L]$$

$$\text{so; we have; } \dot{L}(t) = i \langle \psi | [H, L] | \psi \rangle$$

time derivative
of average is equal to ψ times commutator
 $[H, L]$ times ψ .

but; $\langle \psi | [H, L] | \psi \rangle$ is just the expectation value of
the ~~comm~~ commutator of H with L .

$$\dot{L}(t) = i [H, L]$$

so we have equations of motion now, for averages in terms of averages.

we can think of this as an equation relating time derivative of certain average to the average of another quantity. (P863)

Note • If the probability distribution for everything are nice & peaked, then ~~the~~

this basically says that the time derivative of the classical or the time derivative of the approximate $L(t)$ is equal to the approximate average of commutator of H with L .

nice & peaked

→ so that averages are close to the highest point on the curve

→ & the curve is reasonably narrow.

(so there is not enormous amount of uncertainty)

→ & so one writes.

(this of course is not quite right, but write it anyway)

$$\dot{L}(t) = i [H, L]$$

now this has to be taken a little thought.

→ what it means is, in that equations among averages

notice; $[H, L] = -[L, H]$

so; $\dot{L}(t) = -i [L, H]$

→ (but it is often just written in this form)

~~Equations of motion~~

Q // What happened to $\hbar$.

→ Actually till here we have been working in units where $\hbar$ is 1.

∴ now; if we want to re insert planck's constant

→ where would it go in these equations.

~~first goes in~~

★ first see Schrodinger equations.

(thinking $\hbar$ has unit of energy...)

→ and if we follow it through equations we finally get

$$\dot{L}(t) = i \left\langle \psi \left| \frac{(H_L - L_H)}{\hbar} \right| \psi \right\rangle$$

$$\dot{L}(t) = -\frac{i}{\hbar} [L, H]$$

$$\hbar \frac{d|\psi\rangle}{dt} = -i H |\psi\rangle$$

$$\frac{d|\psi\rangle}{dt} = -\frac{i}{\hbar} H |\psi\rangle$$

now direct replacement

↓
wherever you got $\hbar$: replace it with $\frac{\hbar}{\hbar}$.

now; ~~remember~~ ~~reinsert~~ a classical mechanics equations of motion.

→ equations of motion represented in terms of poisson brackets.

∴ let; L be any classical functions of q s & p s. (not necessarily for Longrington or Angular momentum)

$$\frac{d}{dt} L(q, p) = \{L, H\}$$

→ although L is standard motion for Rem; but here we are not considering them.

now see the similarity with

$$\frac{dL}{dt} = -\frac{i}{\hbar} [L, H]$$

now $\hbar$ is very small number, so $\frac{dL}{dt}$ looks very big (pg 65)

→ but on the other hand, $[L, H]$

∴ in Classical Physics; $[L, H] = LH - HL$ would be zero.

→ so; whatever $[L, H]$ this is,

Classically its zero, & Quantum Mechanically it must be a small correction to the classical physics.
(to the extent that Quantum Mechanics is small correction to Classical Physics)

→ & so commutator is itself something typically small & of order $\hbar$.

→ that allow us to guess,
~~→ an identity~~

→ to guess an ~~identity~~ identification between commutators & poisson brackets.

→ this is non inductive at the moment, just looking at two equations,
→ & say there's gonna be any connection between Quantum Mechanics & Classical Mechanics → then there must be a connection between poisson bracket & the commutator.

namely : $\{ , \} = \frac{-i}{\hbar} [,]$

or usually written as. $[,] = i\hbar \{ , \}$

note Commutators are small things because there is to there.
↳ Poisson bracket in classical physics is not a small thing

history it was Dirac who recognized this connection & built his Quantum Mechanics out of it, out of making an identification of Poisson Brackets with Commutators.

NOTE • but of course, the Foundation of the subject Quantum Mechanics is a consistent subject without having to make any reference to Classical Physics.

↳ The right way to think about things is that Quantum Mechanics comes before classical mechanics (~~not~~ not historically, ... but in fundamentalism)

↳ Commutators before Poisson Brackets.

↳ i.e.; you have to derive Classical Physics as an approximation to Quantum Mechanics.

(so, at this point here, we have no idea why this commutator should have features in common with Poisson Brackets)

Proposition

$$\{L, H\} = -\{H, L\} \quad ; \quad [L, H] = -[H, L]$$

$$\{LM, H\} = \{L, H\}M + L\{H, M\} \quad ; \quad [LM, H] = L[M, H] + [L, H]M$$

In classical physics order of multiplication doesn't matter

Now ask how the average of energy changes with time.

↳ The energy I claim is the Hamiltonian.

(or definition if you like, ... what is energy, ~~not~~ it is Hamiltonian)

$$\frac{dH}{dt} = \frac{-i}{\hbar} [H, H] = \frac{-i}{\hbar} (HH - HH) = \frac{-i}{\hbar} (0) = 0$$

$$\Rightarrow \frac{dH}{dt} = 0$$

∴ so ; whenever the hamiltonian is
 , its expectation value with time
 is constant.... & so conserved in
 that sense.

→ This tells us that we are on some kind of right track to be able to
 compare Quantum Mechanics & Classical Mechanics.

$\langle \lambda \rangle$:= dirac notation for average.
 $\bar{\lambda}$:= bar notation.

• H is hermitian was from the condition of Unitarity.

from going $\frac{d}{dt} |\psi\rangle$ to $\frac{d}{dt} \langle \psi |$

→ think as if you are taking difference of two vectors
 do limit thing & get it.

$$\frac{d}{dt} \langle \psi | = \lim_{\Delta t \rightarrow 0} \frac{\langle \psi |_{\Delta t} - \langle \psi |}{\Delta t} \xrightarrow{\text{to}} \lim_{\Delta t \rightarrow 0} \frac{\langle \psi |_{\Delta t} - \langle \psi |}{\Delta t} = \frac{d}{dt} \langle \psi |$$

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Uncertainty Principle

The fact that there are pairs of observables which cannot simultaneously be measured.

↳ The basic principle is that $\Rightarrow$ if you measure something, then by definition of a good measurement you will leave the system with the same value that you measured. (otherwise you couldn't confirm in the next round... what you measured)

• So it's important that you leave the system at least momentarily or instantaneously having the same value that you measured.

⊗ The measurable quantities of a system are the eigenvalues of the observable operators.

⊙ And the states in which the observables are definite, ~~are~~ ^{are the} eigenvectors of the observables.

* Suppose we have two observables, which could be simultaneously measured.

↳ you make two measurements simultaneously of two different things & you leave both of them in eigenvectors of the quantity that were measured.

↳ That must mean, $\Rightarrow$ that the states that you leave things in are simultaneous eigenvectors of two distinct Observable operators.

↳ The eigenvectors of an observable can be made to be a complete orthonormal basis.

↳ And if you have two distinct operators which can be simultaneously measured & therefore simultaneously left in eigenvectors of those quantities $\rightarrow$ it must mean that there is a basis of vectors which are simultaneous eigenvectors of both quantities.

let: L, M be two observable.

if L, M are simultaneously measurable.

then there exist a basis $|i\rangle$ (complete basis) which are both simultaneous eigenvectors of L & M , i.e: $L|i\rangle = l_i|i\rangle$ and $M|i\rangle = m_i|i\rangle$

now let's apply L & M in sequence, first L, then M.
 $\rightarrow$ & then try the other way.

$$\therefore ML|i\rangle = M Li|i\rangle = LiMi|i\rangle = Li mi|i\rangle$$

means first
 measure L, then
 M.

$$\Rightarrow ML|i\rangle = Li mi|i\rangle$$

$$\therefore \text{also: } LM|i\rangle = m_i Li|i\rangle = Li mi|i\rangle$$

same thing.

$\rightarrow$ now, ^{surprisingly} ~~so~~ ~~if~~ M and L are things which can both be measured simultaneously, & if we multiply two of them together as matrices or operators, they become another observable whose eigenvalues are just the products.

$\rightarrow$ and when we do in opposite order, we exactly get the same thing.

* so; Now suppose you have two operators, which have exactly the same action on every member of a basis

$\rightarrow$ this means they have same action on superposition of vectors (formed out of superposition of basis vectors)

$\rightarrow$ they have same action on any vector.

* ML & LM have exactly the same action on any vector & therefore ML = LM as operators.

So we proved the following Theorem.

If two things are simultaneously measurable, the two observables or the operators representing those observables commute.

$$\text{If } LM = ML \Rightarrow LM - ML = [L, M] = 0$$

Theorem

(1971)

If you have two operators which commute, it means you can find complete basis of states ~~in which both of them are~~ simultaneous eigenvectors of the two operators.

so; the condition for things to be simultaneously measurable is that they commute.

i.e; their commutator is 0.

What if they don't commute?

ex G_x, G_y, G_z ; no two of them commute with each other.

& therefore they don't have simultaneous eigen vectors.

(not surprising; because we know that eigen vectors

of G_x are $(1, 0, 0)$; G_y are $(0, 1, 0)$; G_z are $(0, 0, 1)$

(and they are different)

→ & that an indication that you can't measure them simultaneously.
(This is the idea of Uncertainty,

The idea of Uncertainty is when you have two observables, each of which you could decide to measure & get an answer; but you can measure them & after you measured them be certain that you can repeat the measurement, and find consistency each of them, but not both of them.

... in Classical Physics, there is no such thing; if you can measure A & you can measure B, then you can measure both A & B.
(You can be certain about both of them, ~~then you can be certain about both of them~~ if you can be certain about either of them)

he wants quantitative measure of the necessary degree of uncertainty in measuring two things simultaneously ... & that's the Uncertainty principle.

(but for the moment, he has the idea or an explanation of why there are uncertainty between observables in the first place.

* The Schrodinger equation is the equation which governs the time evolution of systems.

$U(0) = I$; A kind of continuity

~~that~~ $\hookrightarrow$ that U of small time is close to identity. (in other words, in a very small period of time, the states does not change much)

$\Rightarrow$ so same as saying.

whether or not we include $\hbar$,
or i or $\epsilon \rightarrow$ is simply
a definition of H
(simply a matter of convention)

$$U(\epsilon) = I + \text{O}(\epsilon)$$

where: O
is big O
notation.

$$\therefore \hookrightarrow U(\epsilon) = I - \frac{i\epsilon}{\hbar} H$$

$$\text{or } U^\dagger(\epsilon) = I + \frac{i\epsilon}{\hbar} H^\dagger$$

$\hookrightarrow$ found from $U^\dagger U = I$

$\hookrightarrow$ get, $H = H^\dagger$

(for the moment; H is called Quantum Mechanical Hamiltonian)
 $\hookrightarrow$ its a thing which in a clear way generates time evolution.

$$\text{so } |\psi(\epsilon)\rangle = |\psi(0)\rangle - \frac{i\epsilon}{\hbar} H |\psi(0)\rangle \Rightarrow \frac{|\psi(\epsilon)\rangle - |\psi(0)\rangle}{\epsilon} = -\frac{i}{\hbar} H |\psi(0)\rangle$$

$$\Rightarrow \frac{d}{dt} |\psi(0)\rangle = -\frac{i}{\hbar} H |\psi(0)\rangle$$

$\hookrightarrow$ equivalently

$$\frac{d}{dt} |\psi\rangle = -\frac{i}{\hbar} H |\psi\rangle$$

$\hookrightarrow$ so; H is a rule for how you update states of system.

It's a ^{machine} ~~machine~~ which tells you to how to update the state of system (P373)

→ This sounds awfully deterministic → it sounds awfully ~~much~~ much like classical determinism

(Classical determinism is also a set of rules for ~~updating~~ updating the state of system)

~~that~~ In what sense are things not deterministic in Quantum Physics?

~~is~~ Not deterministic in the sense that even if you know the state of the system, you simply don't know result of every measurement.

• The knowledge of state of system in Quantum Mechanics is ~~not~~ equivalent to the knowledge of the outcome of every ~~even~~ experiment.

→ The state of system tells us probabilities.

→ so; $\frac{d|\psi\rangle}{dt} = \frac{-i}{\hbar} |\psi\rangle$ is basically a rule for updating probabilities. (it's not a rule for updating complete knowledge of the set of experiments that you could do)

★ State vector evolves in a known & definite way. (this is true only when you don't interfere with system)

so; $\frac{d|\psi\rangle}{dt} = \frac{-i}{\hbar} |\psi\rangle$ is the evolution of system of an undisturbed system that is not in the process of being measured.

now $|E_i\rangle$ represent eigenstates of the energy.
(of Hamiltonian indeed)

$$H|E_i\rangle = E_i|E_i\rangle$$

$|E_i\rangle$ represents ~~eigen~~ eigenvector associated with energy E_i .

~~There is complete~~

Whatever the dimensionality of the system is (the dimension of space of state)
There is that many eigenvectors of energy. ~~not~~

→ and now as a physicist find these eigenvectors.

... That's the time Independent Schrodinger Equation.

expectation value of L observable
~~(average value of L)~~

$$= \langle \Psi(t) | L | \Psi(t) \rangle$$

Statistical average value of observable.

$$\therefore \frac{d}{dt} \langle \Psi(t) | L | \Psi(t) \rangle = \frac{-i}{\hbar} \langle \Psi | (LH - HL) | \Psi \rangle$$

$$\Rightarrow \frac{d}{dt} \langle L \rangle = \frac{-i}{\hbar} \langle [L, H] \rangle$$

∴ now; we will find out that commutator of real things are imaginary....

↳ short hand notation

→ can be proved.

(In general true)

$$\frac{dL}{dt} = \frac{-i}{\hbar} [L, H]$$

∴ here we have what are called Heisenberg Equations of motion.

$$\{A, B\} = -\{B, A\}$$

$$\{AB, C\} = A\{B, C\} + \{A, C\}B$$

here order does not matter

$$[A, B] = -[B, A]$$

$$[AB, C] = A[B, C] + [A, C]B$$

but here; it matters.

Commutators satisfy some algebraic rule as Poisson bracket.

Of course this is not an accident....

Quantum certainty

Certainly the commutator is more basic.

Quantum mechanics is more basic than Classical Mechanics.

(Poisson brackets can be thought of as ~~Classical~~ Classical Approximation of commutators....)

∴ when we go from Quantum Mechanics to classical mechanics Poisson brackets becomes ...

$$\{A, B\} \Leftrightarrow \frac{-i}{\hbar} [A, B]$$

(don't put equality...) → Difference of units is actually soaked up in Planck's constant.

$$\frac{\partial A}{\partial x} \frac{\partial B}{\partial p}$$

$$\text{let write; } \frac{1}{\hbar} \{A, B\} \Leftrightarrow -i [A, B]$$

(Classically we expect $AB=BA$ & hence $AB-BA=0$)

• In classical limit the commutator is something which is just zero.

→ The $\hbar$ on left side which is very very small ~~and~~ with ordinary units

→ so; In classical approximation commutators are negligibly small.

(but keeping track of Quantum Mechanics, the commutator is not zero)

$$\frac{dL}{dt} = \frac{-i}{\hbar} [L, H] \text{ is Quantum Analogue for } \frac{dL}{dt} = \{L, H\}$$

∴ (L is anything)

$$\frac{dH}{dt} = -i [H, H] = 0$$

Energy is conserved in Quantum Mechanics

(At least the average of energy is conserved)

• In fact whenever we build a system out of two systems

→ ~~the~~ the observables for one of the system & the observables for other system commute with each other.

⇒ That's why you can measure position of Shoaib & Sushant simultaneously. (because our position commutes)

Suppose that you have eigen vectors and eigenvalues (by solving time independent Schrodinger Equation) ~~(from calculating the x.c)~~

→ let's find general solution of Schrodinger Equation.
(how things evolve with time)

• adding a constant to Hamiltonian does not do anything (because constant commutes with everything.)

$$[L, H+c] = [L, H]$$

let: $|i\rangle$ be basis of eigenvectors; ... same as $|E_i\rangle$

Since $|i\rangle$ or $|E_i\rangle$ basis of states we can take any state $|\psi(t)\rangle$

(notation $|i\rangle = |E_i\rangle$)

∴ ~~this~~ $|\psi\rangle$ can be represented as sum over all eigenstates of some set of coefficients.

$$|\psi(t)\rangle = \sum_i \alpha_i |i\rangle$$

(any state can be written as superposition of eigen vectors of energy) ∵ since the eigenvectors of a Hermitian operator are a complete ~~basis~~ basis.

How does state change with time?

* states $|i\rangle$ are just fixed set of states.

∴ it's the α s which change, the coefficients here change with time.

$$\text{so; } |\Psi(t)\rangle = \sum_j \alpha_j(t) |j\rangle$$

so; The coefficients in the expansion of an arbitrary state are typically time dependent coefficients.

That ~~what~~ gives rise to time dependence in expectation values & other things.

$$\text{now; } \frac{d|\Psi\rangle}{dt} = \sum_j \frac{d\alpha_j(t)}{dt} |j\rangle$$

from Schrödinger equation:

$$\text{so; } \sum_j \frac{d\alpha_j(t)}{dt} |j\rangle = \frac{-i}{\hbar} \sum_j H (\alpha_j(t) |j\rangle)$$

$$= \frac{-i}{\hbar} \sum_j \alpha_j(t) H |j\rangle$$

$$= \frac{-i}{\hbar} \sum_j \alpha_j(t) E_j |j\rangle$$

$$\Rightarrow \sum_j \frac{d\alpha_j(t)}{dt} |j\rangle = \frac{-i}{\hbar} \sum_j \alpha_j(t) E_j |j\rangle$$

∴ multiply by $\langle i|$ ~~get~~ $\Rightarrow \frac{d\alpha_j(t)}{dt} = \frac{-i}{\hbar} \alpha_j(t) E_j$

$$\boxed{\frac{d\alpha_j}{dt} = \frac{-i}{\hbar} \alpha_j E_j}$$

It tells us how α s change with time.

• For each j , this is differential equation.

∴ solution here is exponential clearly...

$$\alpha_j(t) = \alpha_j(0) e^{-\frac{i}{\hbar} E_j t}$$

$$\alpha_j(t) = \alpha_j(0) \cdot e^{-\frac{i}{\hbar} E_j t}$$

now we can write
General solution of
Schrödinger equation.

→ we solved it for each individual eigenvector

→ This the way, each individual ~~eigen~~ coefficients of α s change with time.

∴ now go back in $|\Psi\rangle = \sum \alpha_j |j\rangle$ & plug in what we know about α .
lets suppose we know it at time 0; i.e., $|\Psi(0)\rangle = \sum \alpha_j(0) |j\rangle$

$$|\Psi(t)\rangle = \sum_j \alpha_j(0) \cdot e^{-\frac{i}{\hbar} E_j t} |j\rangle$$

→ all that happens is, each α_j picks up a time dependance,
& the time dependance is just governed by the energy of the state.

$$|\Psi(t)\rangle = \sum_j \alpha_j(0) \cdot e^{-\frac{i}{\hbar} E_j t} |j\rangle$$

This is the connection between energies & oscillations,
→ oscillations of wave functions (not oscillations of real things)
→ i.e., oscillations of the coefficients of state vectors.

lets see how expectation value of arbitrary observable changes L.
 $\langle \Psi(t) | L | \Psi(t) \rangle = \sum_j \alpha_j e^{-\frac{i}{\hbar} E_j t}$

$$\langle \Psi(t) | = \sum_k \alpha_k^*(0) e^{\frac{i}{\hbar} E_k t} \langle k |$$

Let's see how expectation value of an arbitrary observable L changes. (37)

~~$\langle \psi(t) | L | \psi(t) \rangle$~~

$$\langle \psi(t) | L | \psi(t) \rangle = \left(\sum_k \alpha_k^*(0) e^{\frac{iE_k t}{\hbar}} \langle k | \right) L \left(\sum_j \alpha_j(0) e^{\frac{-iE_j t}{\hbar}} | j \rangle \right)$$

let: $\underline{L_{kj} := \langle k | L | j \rangle}$ (this is actually the kind of matrix elements ~~that~~ we computed when we talked about matrix representation of observables)

$$\Rightarrow \langle \psi(t) | L | \psi(t) \rangle = \sum_k \sum_j \alpha_k^*(0) \alpha_j(0) L_{kj} e^{\frac{i}{\hbar} (E_k - E_j) t}$$

↪ It is Heisenberg matrix formulation of quantum mechanics.
• Heisenberg's formulations of ~~the~~ way averages change with time.

→ If you go back to Heisenberg's first paper on Quantum Mechanics, he guessed this formula.

• every observable has two frequencies associated with it.

↳ That was weird.

• It was common for people to understand that an observable might be an oscillating observable & would have frequency associated with it.

↪ But Heisenberg said, there are two frequencies associated with every matrix element of an observable
(This came out of Heisenberg's head)

→ This is the way Dirac understood what Heisenberg was saying through this ~~math~~ machinery.

$$\langle \Psi(t) | L | \Psi(t) \rangle = \sum_{k,j} \underbrace{\alpha_k^*(0) \alpha_j(0)}_{\text{these things have to do with initial states}} \underbrace{L_{kj}}_{\text{this represents the observable (its matrix representation)}} \underbrace{e^{\frac{i(E_k - E_j)t}{\hbar}}}_{\text{these things have to do with the way things change with time}}$$

these things have to do with initial states

this represents the observable (its matrix representation)

these things have to do with the way things change with time
 ↓
 • Everything changes with time two separate frequencies.

• Magnitude of α s remains fixed.

→ It's the relative phases between them.

→ and then you sum it all together & that tells you how average of L changes with time.

So all of time evolution has to do with this relative phases.

Note

* It's very brittle to change Special Relativity.

→ it is harder than changing General Relativity.
 (G.R. is relatively easier to change)

* Quantum Mechanics is absolutely brittle.
 (very very hard to change anything.... till now nothing yet changed)

ex single spin.

(put it in magnetic field)

$$E \propto (\text{spin}) \cdot (\text{Magnetic field})$$

↑ B_z

→ spin; ~~normalization~~

guess or postulate

$$H = \frac{\omega}{2} \sigma_z$$

$\frac{\omega}{2}$ is like proportionality constant.
 (right combination of magnetic field, charge..., etc.)

∴ calculate;

time derivative of the averages of all the components of spin.

σ_x
 σ_y
 σ_z

how they vary with time; $\left. \begin{matrix} \frac{d\sigma_x}{dt} \\ \frac{d\sigma_y}{dt} \\ \frac{d\sigma_z}{dt} \end{matrix} \right\}$ calculate this

convention

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~~$\dot{\sigma}_x$ means the average of σ_x & its derivative~~
 $\dot{\sigma}_x$ means the derivative of average of σ_x

$$\therefore \dot{\sigma}_x := \frac{d}{dt} \langle \psi | \sigma_x | \psi \rangle \quad ; \quad \sigma_x := \langle \psi | \sigma_x | \psi \rangle$$

$$\dot{\sigma}_x = -i [\sigma_x, H] = -i [\sigma_x, \sigma_z] \frac{\omega}{2}$$

$$\dot{\sigma}_y = -i [\sigma_y, \sigma_z] \frac{\omega}{2}$$

$$\dot{\sigma}_z = -i [\sigma_z, \sigma_z] \frac{\omega}{2} = 0 \quad \left. \begin{array}{l} \text{tells that the average} \\ \text{spin along magnetic field} \\ \text{does not change with time.} \end{array} \right\}$$

$$\boxed{\dot{\sigma}_z = 0}$$

lets workout commutator relations of σ s.

$$[\sigma_z, \sigma_x] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we will do by matrices
 (we remember they have matrix representation ... & use it)

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 2i \sigma_y$$

$$\text{so; } [\sigma_z, \sigma_x] = 2i \sigma_y$$

$$\text{similarly, } [\sigma_x, \sigma_y] = 2i \sigma_z$$

$$[\sigma_y, \sigma_z] = 2i \sigma_x$$

(or use cyclic permutations)

$$\therefore \text{so; } \dot{\sigma}_x = -i \frac{\omega}{2} (-2i \sigma_y) = -\omega \sigma_y$$

$$\Rightarrow \dot{\sigma}_x = -\omega \sigma_y$$

$$\text{similarly } \dot{\sigma}_y = +\omega \sigma_x \quad \& \quad \dot{\sigma}_z = 0$$

These are equations for things moving in circular motion.

• we can do it without matrices just by applying the operators to the vectors $|u\rangle$ & $|d\rangle$... etc.

but matrices are very convenient tool for manipulating operators.

(operator product is same as matrix product)

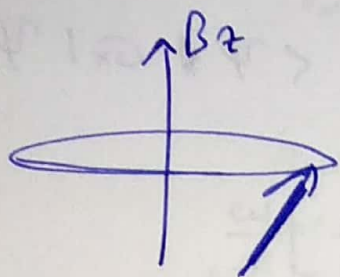
$$\text{we; } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Rightarrow i \sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\rightarrow$ In the sense of expectation values (they are ~~not numbers~~ real numbers)

(Pg. 2)

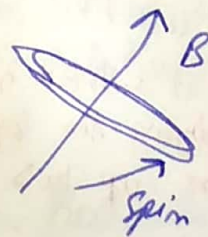


In the sense of expectation values, the spin vector is precessing around the z axis.

$\rightarrow$ This is same kind of formula which we found in Classical Mechanics. (its ~~also the~~ basically the gyroscopic ~~precession~~ ~~or like or like classical vector~~ precession)

$\Rightarrow$ Quantum Mechanical spin, in the sense of expectation values (it does not make any sense to say that Quantum Mechanical ~~oper~~ operators themselves precess... that does not mean anything) precesses like a Classical Gyroscope.

~~Question~~ ★ You could put spin along some peculiar axis
 $\hookrightarrow$ but what you find after you worked out the whole thing out is that it just precesses around the peculiar axis of the magnetic field.



★ ★ The most general Hermitian operator in two dimensions is a linear combination of $I, \sigma_x, \sigma_y, \sigma_z$.

$\hookrightarrow$ Putting a unit operator into a Hamiltonian is like adding constant to the energy
(Unit operators (I) commute with everything)

$\hookrightarrow$ so, putting unit operators does nothing physically important
 $\hookrightarrow$ so we get rid of it.

$\rightarrow$ now, the next possibility is general linear combination of $\sigma_x, \sigma_y, \sigma_z$

$$m_x \sigma_x + m_y \sigma_y + m_z \sigma_z$$

This just corresponds to putting the magnetic field at some angle.

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→ This will just correspond to components of spin along n -axis.

→ and you would do exactly the same calculation ~~except~~ except you might work instead in the basis of eigen vectors of $\underline{\sigma} \cdot \underline{n}$.

→ So, this is really the most general Hamiltonian that you can write down.

Remark • For every direction that you can think of, there is a state for which that component of the spin is definite.

We get to think about things deterministically when it's so heavy & so many degrees of freedom, that when we calculate the Quantum Mechanics of it, we find that the fluctuations are really small. (ex: here the magnet)

→ so far we have divided the world into the Quantum & the heavy Apparatuses which we don't have to deal with Quantum Mechanics)

(that of course has to be a temporary thing)

... real apparatuses are made up of same things as the system we study, & so at some point we have to come back & say what was the justification for saying the Apparatus was allowed ~~to be~~ to be considered just classically, and the system in concern to be dealt Quantum Mechanically).

In general for spins:

Hamiltonian H is of the form

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$$H = \frac{\omega}{2} \vec{\sigma} \cdot \hat{n} \quad ; \quad \vec{\sigma} \cdot \hat{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z$$

$$\therefore \vec{\sigma} \cdot \hat{n} = \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix} \quad \therefore H = \frac{\omega}{2} \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$$

$\vec{\sigma} \cdot \hat{n}$ is the matrix

where the matrix elements are calculated by taking components of $\hat{n}$ & multiplying them by corresponding components of $\vec{\sigma}$, ...

now we have to calculate commutators of various components of spin like $[\dots, \vec{\sigma} \cdot \hat{n}]$

$$\text{ex } [\vec{\sigma} \cdot \hat{n}, \vec{\sigma} \cdot \hat{n}] = 0$$

so component of spin along the $\hat{n}$ -axis will be the thing that does not change.

ex what about the component of spin perpendicular to that axis.

we can work that out and we will discover that the spin precesses about the $\hat{n}$ -axis.

$$\star \vec{\sigma} \cdot \hat{n} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z \quad \text{exactly as if } \vec{\sigma} \text{ was a classical vector.}$$

here it's just a collection of three matrices

but of course it's not

so $\vec{\sigma} \cdot \hat{n}$ stands for a single matrix which happens to be some of three particular matrices.

ex if the magnetic field was along the z -axis, then the only component of $\hat{n}$ which is non zero be n_z ..

$$\vec{B} = |\vec{B}| \hat{n}$$

Relationship between Magnetic field vector & $\hat{n}$.

Figure Pg 85

→ Stands for magnetic field pointing along $\hat{n}$ axis with magnitude $|\vec{B}|$.

$\vec{\sigma} \cdot \hat{n}$; the eigen values of $\vec{\sigma} \cdot \hat{n}$ does not depend on $\hat{n}$
 → it is ± 1 in any direction.

exercise $\vec{\sigma} \cdot \hat{n} = \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$

∴ what are eigenvalues of this matrix for $\hat{n}$ as long as $\hat{n}$ is unit vector, i.e. $n_x^2 + n_y^2 + n_z^2 = 1$.

→ the eigenvalues of this matrix is always $+1, -1$.

→ This is simply the statement; "no matter what direction you point your detector (apparatus), when the apparatus acts it gives you $+1$ or -1 ."

$\text{Tr}(\vec{\sigma} \cdot \hat{n}) = \sum \text{eigen value} \quad \therefore n_z - n_z = \lambda_1 + \lambda_2$

$\Rightarrow 0 = \lambda_1 + \lambda_2$

∴ now; the $|\vec{\sigma} \cdot \hat{n}| = (\text{Product of eigen values}) \Rightarrow \underline{\lambda_1 = -\lambda_2}$

determinant of matrix

$\therefore \therefore |\vec{\sigma} \cdot \hat{n}| = -(n_x^2 + n_y^2 + n_z^2) = -1$

$= \lambda_1 \lambda_2$

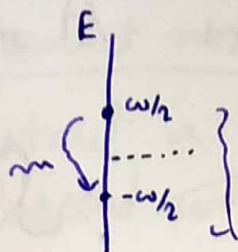
→ gives; λ s are $1, -1$.

$$\hat{H} = \frac{\omega}{2} \hat{\sigma}_z$$

sed; $\hbar=1$

$$E_- = -\frac{\omega}{2}$$

$$E_+ = +\frac{\omega}{2}$$



this much energy photon emitted!
(actually; $\frac{\hbar\omega}{2}$)

let; initial state be

$$\alpha|u\rangle + \beta|d\rangle$$

* if $\alpha = \beta \Rightarrow$ that corresponds orientated along x-axis

* if $\alpha = 1; \beta = 0 \Rightarrow$ pointing upwards.

it emits with probability $\alpha^* \alpha$.

does not emit with probability $\beta^* \beta$.

What is average energy?

comes out to be same as classical energy.

(but each individual event ~~emits~~ either emits no photon, or photon of energy $\hbar\omega$)

Collapse of Wave Function.

$$L : \sum \alpha_i |\lambda_i\rangle \rightarrow \text{measure } \hat{B} \text{ say } |\lambda_7\rangle$$

* ~~Here~~ hereafter, or right after measuring λ_7 , the system is in state $|\lambda_7\rangle$

* The course of operation of measurement is ~~governed~~ governed by the Schrodinger Equation for the system.

its governed by the nature of measuring process.

Terminology $\therefore$ The wavefunction has collapsed to ~~eigenstate~~ one of the eigenvectors,

and only one states pops up in existence.

Combined Systems

let there be two systems A and I

$\therefore$ notations

$$S_A$$

denotes space of
quantum states of system A

$$S_I$$

denotes space of
quantum states of
system I .

$$|a\rangle$$

$$|i\rangle$$

system A has states
which are ~~labelled~~ labelled
by a basis of states $|a\rangle$
 $\therefore a$ ranges from 1
to whatever the no.
of states of system A is
(whatever the dimensionality
of the space of states of A is).

* Any state of system A , (if you did not have I) would
be a superposition of these basis states $|a\rangle$.

$\rightarrow$ same true for I ...

Q) How do we put systems together to single quantum system?

~~but~~ * build a space of system AI i.e. S_{AI}
(means A & I both together)

$\rightarrow$ that space of state is ~~called~~ mathematically called
the Tensor Product of the two space of states.

$$S_{AI} := S_A \otimes S_I$$

$$S_{AI} = S_A \otimes S_I$$

* What does a Tensor Product mean?

A) It means you construct a new mathematical state whose vectors are labelled ~~with~~ by a value of "i" & ~~at~~ a value of "j".

(in other words, you pair off the basis vectors in all possible pairings ... one from A & one from I)

↳ & you construct a basis which is labelled $|a i\rangle$

$$S_A, S_I \rightarrow S_{AI} = S_A \otimes S_I$$

$$\text{basis } |a\rangle, |i\rangle \longrightarrow |a i\rangle$$

Dimensionality
 D_A

D_I

$D_A D_I$

(multiply two numbers D_A & D_I)

* Now, think of $|a i\rangle$ as orthonormal basis.

take two such vectors $|a i\rangle$ & $|b j\rangle$

then: ~~their inner product~~

(take them to be normalized...)

Then the Rule for Tensor Product is

↳ that the inner product between $|a i\rangle$ & $|b j\rangle$ is 0 unless $a = b$ and $i = j$. (unless they match)

• If they don't match, they might be observably different.

$$\therefore \boxed{\langle b j | a i \rangle = \delta_{ab} \delta_{ij}}$$

... by construction or by definition,

the vectors $|a_i\rangle$ form an orthonormal basis for the combined system

* So, the general vector in the tensor product is of the form

$$\sum_{i,a} \alpha_{ia} |i\rangle |a\rangle$$

, i.e.: a general linear combination with coefficients that depends on ~~state~~ i, a times the basis vector.

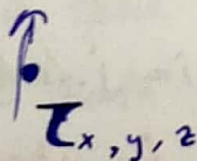
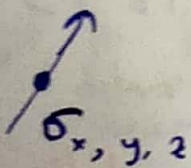
$\therefore$ (if you would have chosen other basis, you would have got the exactly same vector, (except they would have been written differently) $\rightarrow$ they would have been written in

different basis of combined system.

* Changing basis for the individual system corresponds to a basis change of combined system.

ex Two spins.

(two spins have four basis vectors in Tensor Product space)
 $2 \times 2 = 4$.



• suppose we chose for the basis vectors for each of these the up & down basis vectors.

$\rightarrow$ in other words,

the state in which σ_z is up or down, & the states in which τ_z are up or down.

* Basis vector for G : $|u\rangle, |d\rangle$
" " " Z : $|u\rangle, |d\rangle$

$|u\rangle$ just a symbol notation in track that
 $|u_0\rangle$ & $|u_z\rangle$ are different
to avoid indices ... we use $|u\rangle$ for $|u_0\rangle$
& $|d\rangle$ for $|u_z\rangle$

Now what is the basis vector for the combined system.

$|uu\rangle, |u, d\rangle, |d, u\rangle, |dd\rangle$

first entry correspond to G
second entry correspond to Z .

* so any state in the space can be written as

$$|\Psi\rangle = \alpha_{uu}|uu\rangle + \alpha_{ud}|ud\rangle + \alpha_{du}|du\rangle + \alpha_{dd}|dd\rangle$$

- suppose you start from initial state $|uu\rangle$; ~~you~~ both spins up, and you let them interact,
2) Then in general what happens is that the coefficients change. (There will not be a change what basis vectors are, it would be change in what ~~the state~~ α 's are,

~~space of states~~ # Space of states.

- We have two operators, one one apply to G & other to Z .
to We now take our one of operators, σ -operator, we point along the x axis & the other σ -operator along z axis.
to & we are kind up discovering that σ was along x axis with $+1$, & σ was along z axis with $+1$.
2) this means G was up & Z was along right.

$$\frac{1}{\sqrt{2}}(|u u\rangle + |u d\rangle)$$

think as product

$$|u\rangle |x\rangle$$

what is $|x\rangle$

$$= |u\rangle \left[\frac{|u\rangle + |d\rangle}{\sqrt{2}} \right]$$

it is up + down.

This kind of state is called product state in which you know exactly what's going on with σ (namely it is in up direction) & you know exactly what's going along z (it is in right direction)

• Product State

$$[\alpha_1 |u\rangle + \alpha_2 |d\rangle][\beta_1 |u\rangle + \beta_2 |d\rangle]$$

This is actually an element of tensor product.

most general product state

(it is vector drawn in composite system as a tensor product of a vector describing one system & a vector describing the other system).

↳ this means that σ was prepared in this particular $(\alpha_1 |u\rangle + \alpha_2 |d\rangle)$ - linear combination

here (it might have been by holding your detector in some funny angle)

↳ $\beta_1 |u\rangle + \beta_2 |d\rangle$ & this one is response to holding the z detector at some other angle.

↳ we can also write as:

$$\alpha_1 \beta_1 |u u\rangle + \alpha_1 \beta_2 |u d\rangle + \alpha_2 \beta_1 |d u\rangle + \alpha_2 \beta_2 |d d\rangle$$

↳ now of course this is of the form of a general vector, but there is something special about the coefficients,

↳ the coefficients factorize.

(or special in some sense.)

Product space is not general, & ~~se~~ to see how non-general it is, we count parameters.

(PS 91)

Given a general state vector

$$\alpha_u |u\rangle + \alpha_d |d\rangle + \dots + \dots$$

↳ how many independent parameters does it take to describe it?

• Answer is Four complex numbers.

↳ i.e. 8 real numbers.

- but state is normalized (1 constraint)
- and also the overall phase is not important

↳ So when you count parameters you always subtract 2, (1 for normalization & 1 for the phase)

↳ so; $8 - 2 = \underline{6}$: so 6 real parameters describing a general state of two spins.

* now how many in $[\alpha_u |u\rangle + \alpha_d |d\rangle] [\beta_u |u\rangle + \beta_d |d\rangle]$ here.

↳ well, the state describing the 1 particle, that has 2 real parameters ($4 - 2 = 2$)

↳ and same thing for 2 ... ($4 - 2 = 2$)

↳ so; overall $(2 + 2) = \underline{4}$ real parameters.

↳ it is the same 4 real parameters that you might have guessed if you ~~just~~ just said I have 2 spins.

* So the difference in no. of parameters in General state & Product state simply proves that not all states are product states.

* In fact most states are not product states.

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Product States: the product state we can really think of separate states for each of the particles or each of the subsystem. (product states are the one for which you can separately say there is state of one system & state of other system ... end of story)

but for more General state, you can't really divide it into a state for one subsystem & a state for the other subsystem.

They are entangled.

Observables

* What are the observables for the composite system?

Well, they must be the same observables as for the separate system by themselves.

* let's see how this * let's figure out how these operators ought to operate on the state of the combined system.

$$\therefore G_x |u\rangle \rightarrow |d\rangle$$

$$G_x |d\rangle \rightarrow |u\rangle$$

check ~~this~~ using the fact that $G_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

* the off diagonal character takes $|d\rangle$ to $|u\rangle$, & $|u\rangle$ to $|d\rangle$.

this is what G_x does ordinarily if we only had one subsystem.

$$i.e.: G_x |u\rangle = |d\rangle$$

$$G_x |d\rangle = |u\rangle$$

$$\sigma_x |u\rangle = |d\rangle$$

$\therefore$ when σ_x acts on $|u\rangle$, $\Rightarrow$ all it sees the first thing u here, and it is completely passive w.r.t.

second index. $\rightarrow$ in other words, it does not even notice second index & leaves it unchanged but it does whatever it should do on the first entry

$$\text{likewise; } \sigma_x |d\rangle = |u\rangle$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_z |u\rangle = |u\rangle$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z |d\rangle = -|d\rangle$$

$\hookrightarrow \sigma_y$ also flips u to d & d to u but it throws in i here and there.

* now, how about z .

~~(only first)~~ (the first entry becomes the spectator & only the second entry gets acted on)

$$\text{so; } \tau_x |a u\rangle = |a d\rangle$$

$$\tau_x |a d\rangle = |a u\rangle$$

$$\tau_z |a u\rangle = |a u\rangle$$

$$\tau_z |a d\rangle = -|a d\rangle$$

* If you are talking about observable associated with G , then the 2 degree of freedom are completely passive & nothing happens to them.

$\hookrightarrow$ The eigen vectors of components of G are simply the product states of the eigen vectors of G times any vector for z .

13) σ_y

ordinarily
eigenvector of σ_y are
 $\begin{pmatrix} 1 \\ i \end{pmatrix}$ or $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

so; $|u\rangle + i|d\rangle$ is eigen vector of σ_y .

→ now if you want the eigenvectors of σ_y when we are talking about composite system.

$$|u_j\rangle + i|d_j\rangle$$

* From these definitions it is pretty clear that the separate subsystems behave ~~independently~~ quite independently as long as you don't let them interact & feel each other,
i.e.; as long as they are kept separated.

(which should be of course,

→ i.e.; if they don't interact, then the result is independent of the fact that whether you treat the subsystem ~~together~~ as composite system or ~~independently~~ independently.)

$$\# \sigma_x \tau_x |u u\rangle = \sigma_x (\tau_x |u u\rangle) = \sigma_x |u d\rangle = |d d\rangle$$

$$\text{* let's check: } \tau_x \sigma_x |u u\rangle = \tau_x |d u\rangle = |d d\rangle$$

$$\text{so; } \underline{[\sigma_x \tau_x] = 0} \quad (\text{no difference between order of operations})$$

* let's see $\sigma_x \tau_x$

→ not yet proved, here just showed for one particular state.

→ in order to argue $\sigma_x \tau_x = \tau_x \sigma_x$ you have to show they give same thing when acting on any basis vectors. (enough to show this)

Ex: $G \times Z_2 |d u\rangle = G \times |d u\rangle = |u u\rangle$

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* $Z_2 G \times |d u\rangle = Z_2 |u u\rangle = |u u\rangle$

$$[G_m, Z_m] = 0$$

→ if you have two ~~separate~~ separate subsystems which are combined in this way, the observables for one & the observables for other can be simultaneously measured.

• In particular, if you took the systems ~~for~~ apart from each other & you treat them as a single quantum system, you would expect that measuring one of them does not interfere with any measurement of the other one and that's what this says that measuring a G does not preclude measuring a Z & vice versa for any components of them.

↳ in that ~~sense~~ sense, they behave like independent systems; you can make any measurement on G or Z .

• If the combined system is in product state, so that each component has its own state vector, then Quantum Mechanics of each one of them is separate from the other. They don't interfere with each other. They don't know about each other in any way.

• The states which are not product states, are the states which have to varying degrees of degree of entanglement. (i.e. more & more than product states)

~~Not~~ # Entangled state.

ex $\frac{1}{\sqrt{2}} (|ud\rangle - |du\rangle)$

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~~this~~ is called Singlet State.

~~now this state has the property~~

* this has interesting property

→ it has correlation.

i.e. if you know z is d then σ must be up.

on the other hand if you know z is u then σ must be down.

→ This is not a product state

• (There is no way that this can be written as product of one state & the other)

$\langle \Psi | \sigma_x | \Psi \rangle$

$\langle \Psi | \sigma_y | \Psi \rangle$

$\langle \Psi | \sigma_z | \Psi \rangle$

They cannot all be zero.
Reason,

for single spin.

Given any state, there is always

a direction in three dimensional space,
a direction for which the component of σ in
that direction is definite, & is equal to ± 1 .

Given any $|\Psi\rangle$, there is some linear combinations of σ 's
which is definitely equal to ± 1 .

→ "Given any state, there is always some direction

such that: $\langle \Psi | \sigma \cdot n | \Psi \rangle = \pm 1$ "

* if $\sigma \cdot n$ is definite, if it is known & there is no
ambiguity about it, then the average is just equal to
value it's known to be.

∴ for any Ψ , $\exists n : \langle \Psi | \sigma \cdot n | \Psi \rangle = \pm 1$

Claim: All three components of G , have zero expectation value in the state $(\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle))$

pg 99

$\Rightarrow$ that means, the state cannot possibly be a product state.

- "A product state is one, in which each component behaves as if it were just state of subsystem by itself".

$\therefore$ let's work with

$$|ud\rangle - |du\rangle$$

$$\therefore (\langle ud| - \langle du|) G_z (|ud\rangle - |du\rangle)$$

$$= (\langle ud| - \langle du|) (|ud\rangle + |du\rangle)$$

$$= 1 + 0 - 0 - 1 = 0$$

Says $\langle G_z \rangle = 0$ (says that expectation value of G_z in this state is 0)

$$\therefore \langle G_x \rangle = (\langle ud| - \langle du|) G_x (|ud\rangle - |du\rangle)$$

$$= (\langle ud| - \langle du|) (|dd\rangle - |uu\rangle) = 0$$

$$\langle G_y \rangle = (\langle ud| - \langle du|) G_y (|ud\rangle - |du\rangle)$$

$$= (\langle ud| - \langle du|) (i|dd\rangle + i|uu\rangle)$$

$$= 0$$

$$\Rightarrow \langle G_x \rangle = \langle G_y \rangle = \langle G_z \rangle = 0$$

ie: All three components have zero expectation value.

★ whether the notion of Entanglement is invariant notion?

Ans Yes

There is a very definite quantity which can be calculated called the Entanglement Entropy which measures the strength of Entanglement which does not dependent on any particular basis.

$$|u\rangle \rightarrow \frac{|u\rangle + |d\rangle}{\sqrt{2}}$$

then

(P)

$$|d\rangle \rightarrow \dots$$

$$\{|ud\rangle - |du\rangle\} \Rightarrow |ur\rangle - |rl\rangle$$

$$\xrightarrow{\text{schreib in terms of in & out}} |i0\rangle - |0i\rangle$$

$$\# \langle \sigma_z \tau_z \rangle = \left(\frac{\langle ud| - \langle du|}{\sqrt{2}} \right) (\sigma_z \tau_z) \left(\frac{|ud\rangle - |du\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{2} (\langle ud| - \langle du|) \sigma_z (-|ud\rangle - |du\rangle)$$

$$= \frac{1}{2} (\langle ud| - \langle du|) (-|ud\rangle + |du\rangle)$$

$$= \frac{1}{2} (-1 - 1) = -1 \Rightarrow \boxed{\langle \sigma_z \tau_z \rangle = -1}$$

actually, $\frac{|ud\rangle - |du\rangle}{\sqrt{2}}$ is an eigen state for $\sigma_z \tau_z$.

Note-Here; $\langle \sigma_x \rangle = 0$; $\langle \tau_x \rangle = 0$

$$\hookrightarrow \text{but } \langle \sigma_z \tau_z \rangle = -1$$



* Entangled States behaves different ... must have seen till here a bit.

$$\# \langle \sigma_x \tau_x \rangle = \frac{1}{2} (\langle ud| - \langle du|) \sigma_x \tau_x (|ud\rangle - |du\rangle)$$

$$= \frac{1}{2} (\langle ud| - \langle du|) (-|ud\rangle + |du\rangle) = -\frac{2}{2} = -1$$

$$\frac{\langle \sigma_x \tau_x \rangle = -1}{\langle \sigma_y \tau_y \rangle = -1} \quad \boxed{\langle \sigma_\alpha \tau_\alpha \rangle = -1 ; \forall \alpha \in \{x, y, z\}}$$

$$\# \langle \sigma_x \tau_z \rangle = \frac{1}{2} (\langle ud| - \langle du|) \sigma_x \tau_z (|ud\rangle - |du\rangle)$$

$$= \frac{1}{2} (\langle ud| - \langle du|) \sigma_x (-|ud\rangle - |du\rangle)$$

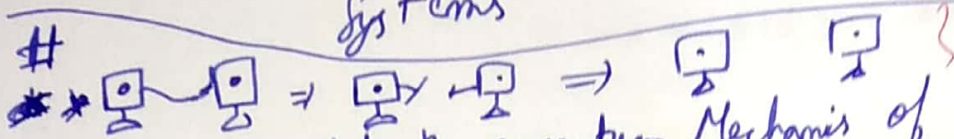
$$= \frac{1}{2} (\langle ud| - \langle du|) (-|dd\rangle - |uu\rangle) = 0$$

$\hookrightarrow$ It tells you that if you measure the same component of either of spins, they will come out to be opposite.

Einstein Rosen Podolsky correlation. or Entanglement Pg 101
(and it does not violate Special Relativity)

~~John Bell~~ John Bell said,

"You can't apply classical reasoning to Quantum Systems"



* Can I simulate the quantum Mechanics of two particle system (simulating on a Classical Computer) in such a way that one of the computer contains one particle & the other computer contains the other particle.

initially connected, the data of different particle stored in different computer → & then cut.

The only way to do it would be to have wires connecting the computers at all time.
(you cannot do it, unless computers are capable of interacting directly at a distance)

→ This does not mean that the system is Non-Local.

Non-local means that signals can be propagated from one place to other.

What it tells you that you simply can't simulate Quantum Mechanics in a local way by a Classical Computer. • The logic of Quantum Mechanics is different.

• It simply does not allow itself to be faked by a computer, unless that computer is connected over large distance to everything in world.
• Quantum Mechanics manages to do same thing, without having wires connecting things.
→ It does not constitute in any way the ability to send message from one place to other.
→ It does not constitute non-locality.

[The page contains extremely faint, illegible handwritten text, likely bleed-through from the reverse side. The text is organized into several paragraphs and includes some mathematical notations such as $\frac{1}{2}$ and $\frac{1}{3}$.]

Wave Function (may or may not have ^{to} do anything with waves.)

The concept grew historically out of the Quantum Mechanics of particles & the probability description of them in terms of waves.

↳ The logical concept of the wavefunction, is the one which you can apply to any Quantum System (and in general, it does not have to do anything with wave motion of particular kind)

$|\Psi\rangle$

* Also I have a basis system,

The basis system is typically labelled by a collection of eigenvalues of something.

↳ It might be labelled by one or more variables.

↳ If there are several commuting operators in a system, you can specify simultaneously the eigen value of more than one thing.

↳ & in many cases you want to do that, and the basis vectors in these cases are labelled by a collection of observables, or collection of eigenvalues of perhaps several quantities.

or If you are ~~talking~~ talking about motion of particle, then x, y, z ~~component~~ coordinates of the particle are commuting variables. (They can be specified simultaneously.)

* Things which don't commute are components of momentum with component of position (will be seen later....)

• "A useful basis for studying Quantum Mechanics of particle is basis which is labelled by three coordinates of the particle."

So we have a basis, and basis is labelled by collection of eigen values.

$|a b c \dots\rangle$

↳ However many commuting variables there are, which need to be specified in order to specify a system.

* and the inner product of the basis vector with the state of the system itself is called the Wave Function.

(Pg 104)

∴ since $|a b c \dots\rangle$ form basis

$$\dots \text{so: } |\Psi\rangle = \sum_{a, b, \dots} \langle a b c \dots | \Psi \rangle |a b c \dots\rangle$$

define
 $\Psi := \langle a b c \dots | \Psi \rangle$

Inner Product
 (and it is function of $a, b, c, \dots$)

↓
 this is called wave function
 (and a function of $a, b, c, \dots$)

lets call it $\Psi(a, b, c, \dots)$

define it in such a way that $\therefore \Psi(a, b, c, \dots) := \langle a b c \dots | \Psi \rangle$

$$|\Psi\rangle = \sum_{a, b, \dots} \Psi(a, b, c, \dots) |a b c \dots\rangle$$

ex single spin: here Ψ will just be probability amplitude for spin to be up or down.
 ~~$|\Psi\rangle = \dots$~~

ex Two spin: ... we take two labels to label state vectors,
 (up & down for one spin, up & down for the other)
 ∴ the two labels would be the labels of the two spins.

Wavefunction
of the system

Think in two ways -

(i) Set of coefficients that you use to expand an arbitrary vector in terms of the basis vector

(or)

(ii) Inner products of the state vector with the basis vectors themselves.

~~Star~~ Note // Schrodinger wavefunction is a special case of (PS105) this.

* If you know wavefunctions, you can compute all sorts of things, ex, expectation value.

~~Density Matrix~~

Expectation Value.

Suppose we have some observable L , and it has matrix elements.

$$\langle a' | L | a \rangle$$

Matrix elements of observables in a given basis.

→ don't write multiple labels. For simplicity they are absorbed in this.

$$\therefore L_{a'a} = \langle a' | L | a \rangle$$

^{note}
~~Sketch~~ $\langle \Psi | = \sum_{a', b', \dots} \langle a', b', \dots | \Psi^*(a', b', \dots)$

$\therefore$ now calculate $\langle L \rangle = \langle \Psi | L | \Psi \rangle = \left(\sum_{a'} \langle a' | \Psi^* \right) L \left(\sum_a \Psi(a) | a \rangle \right)$

Expectation value.

~~WTF~~

$$= \sum_{a', a} \langle a' | \Psi^* L \Psi(a) | a \rangle$$

$$\langle L \rangle = \sum_{a', a} \Psi^*(a') \Psi(a) L_{a'a}$$

This reduces every expectation value to calculations in terms of wavefunctions

→ basic formula for expectation value of anything that we might be interested in.

Suppose we have composite system.
Composed of two subsystem, Alice & Bob subsystem.

(p9106)

∴ ~~let use~~ let use a to represent complete sets of observables for Alice's subsystem.

& b ... ~~for Bob~~ set of variables for Bob system.

∴ $|a\rangle \rightarrow$ Alice's state for her subsystem

$|b\rangle \Rightarrow$ Bob's " " his " "

∴ now, there is a composite vector space.

$\hookrightarrow$ has basis $|a, b\rangle$

★ Wavefunction of Composite system.

$$|\Psi\rangle \neq ; \langle a, b | \Psi \rangle = \Psi(a, b)$$

↓
State vector
in composite,
Tensor
Product space.

Wavefunction.

★ Observables of Composite system.

Alice Bob
 L M

Matrix elements $L_{a'a}$ $M_{b'b}$

How do we represent observables acting on the composite space of states.

Ans The operator associated with Alice subsystem acts on Alice's piece of state vector, & leaves Bob piece of state vector unchanged.

★ Alice's Observables are matrices, which can be written as.

$$L_{a'b', ab} = \langle a' b' | L | a b \rangle$$

L only belongs to Alice

What do he say about matrix $\mathbb{I}$ here, (Pg 107)
to indicate that it has no effect on Bob's variable.

Answer: We make it proportional to Identity matrix in Bob's Variables.

$$L_{a'b', ab} = \sum_b \delta_{bb'} L_{a'a}$$

The Identity operator in a space, is the operator whose matrix element are the Kronecker Delta symbol.
 → The observables that is supposed to be associated with Alice's half of the subsystem, is represented by the same matrix that she would have used, had she not ~~known~~ even known about Bob. i.e. $L_{a'a}$.

~~$M_{a'b', ab} = \sum_a \delta_{aa'} M_{b'b}$~~

→ that she describes in

Bigger Hilbert space?

→ She says that it does nothing to Bob's variables, and so it's just Identity matrix in Bob's half of the subsystem or space.

∴ Similarly, $M_{a'b', ab} = \sum_a \delta_{aa'} M_{b'b}$

Let's calculate expectation value of Alice's variables in product state. (A product state is one in which two systems are not entangled with each other)

→ In the language of wave function, it means that wave function is the product of two wave functions, one for Alice & one for Bob. (it factorises...)

$$\Psi(a, b) = \Psi_A(a) \Psi_B(b)$$

~~$\Psi_A^*(a) \Psi_B^*(b)$~~

$$\sum_{a', a, b', b} \Psi_A^*(a') \Phi_B^*(b') L_{a'a} \delta_{bb'} \Psi_A(a) \Phi_B(b)$$

$$= \sum_{a, a', b} \Psi_A^*(a') \Phi_B^*(b) L_{a'a} \Psi_A(a) \Phi_B(b)$$

$$= \sum_{a, a'} \Psi_A^*(a') L_{a'a} \Psi_A(a) \left(\sum_b \Phi_B^*(b) \Phi_B(b) \right)$$

$$= \sum_{a, a'} \Psi_A^*(a') L_{a'a} \Psi_A(a)$$

→ This sum is 1
(when the wave functions are normalized)

* The product state

- is just that ~~any~~ party does not know about each other.
- there is no correlation.
- No information in one of them about the other.

Expectation value of LM (in the product state)

$$\sum_{a', a, b', b} \Psi_A^*(a') \Phi_B^*(b') M_{bb'} L_{a'a} \Psi_A(a) \Phi_B(b)$$

$$= \left(\sum_{b, b'} \Phi_B^*(b') M_{bb'} \Phi_B(b) \right) \left(\sum_{a, a'} \Psi_A^*(a') L_{a'a} \Psi_A(a) \right)$$

→ we get product of expectation value of L & product of expectation value of M.

For Product States

$$\langle LM \rangle = \langle L \rangle \langle M \rangle$$

→ expectation value of L times M

When probability distribution does not factorize,

$$\langle LM \rangle - \langle L \rangle \langle M \rangle$$

→ This is called the Correlation between L & M.

& is measure of how much these variables are correlated.

→ Whenever, there is anything in the system, an Alice variable & the Bob variable, whose correlation is not zero, that's an indication of Entanglement.

ex $\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$

$$\begin{aligned} \langle \sigma_z \rangle &= 0 \\ \langle \tau_z \rangle &= 0 \\ \langle \sigma_z \tau_z \rangle &= -1 \end{aligned}$$

→ up appears as much as down for Bob.

also $\langle \sigma_x \rangle = 0$
 $\langle \tau_x \rangle = 0$
 $\langle \sigma_x \tau_x \rangle = -1$

also $\langle \sigma_y \rangle = 0$
 $\langle \tau_y \rangle = 0$
 $\langle \sigma_y \tau_y \rangle = -1$

~~σ_z~~
 ~~τ_z~~
 σ_i & τ_i are correlated
 $i \in \{x, y, z\}$

These are examples of highly entangled or correlated states.

general

All operators associated with one system commutes with operators of other system.

note ← note; $\langle LM \rangle = \langle L \rangle \langle M \rangle = \langle M \rangle \langle L \rangle = \langle ML \rangle$

$$\langle \Psi | LM | \Psi \rangle = \langle \Psi | L | \Psi \rangle \langle \Psi | M | \Psi \rangle$$

When we say things are correlated with, we mean it.

→ We mean that we can learn something about one variable by looking at the other.

in the example $\frac{|ud\rangle - |du\rangle}{\sqrt{2}}$

→ you don't know much about what Alice will measure (it's just $\frac{1}{2}$ probability) ... same for Bob.

→ but if Bob does his measurement & Alice does her measurement & you compare them afterwards, ~~everytime~~ every single time, the product will be minus one, $= -1$.

→ This is an example of odd thing in Quantum Mechanics, that.

You can know everything ~~about the~~ about the Composite system and know nothing about pieces.

(when we say you know everything, we ~~mean~~ mean you know as much as quantum mechanics allows you to know)

★ "Given any quantum state whatever of a spin, there exist a direction for which the spin is definite"

→ but when you combine the system together into an entangled state, there is no direction in which spin is definite."

~~In fact there is no direction for one of the spins.~~

$\frac{|ud\rangle - |du\rangle}{\sqrt{2}}$ is an eigen state for

$\sigma_x \otimes \sigma_x$, $\sigma_y \otimes \sigma_y$, $\sigma_z \otimes \sigma_z$.

(each have the eigen value -1)

→ it's also an eigen vector for their linear combination.

~~or~~ $\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$.

→ Alice & Bob can't easily measure this whole quantity.

because Alice can't measure σ_x & σ_y simultaneously.

→ It would be hard to measure by separate measurement. *actually not possible, because no separate measurement can measure σ_x , σ_y & σ_z at the same time.

But " $\sigma_x z_x + \sigma_y z_y + \sigma_z z_z$ " is a perfectly good Hermitian operator.

$\Rightarrow$ so there must exist a way of measuring it.

$\Rightarrow$ but more importantly now, there ~~must~~ is a way of preparing the state which is an eigen vector of this.

What state?

is: $\vec{\sigma} \cdot \vec{z}$.

with an eigen value -3.

can also be written

$$\vec{\sigma} \cdot \vec{z}$$

$\Rightarrow$ the singlet state

$\hookrightarrow$ it is an eigen vector of $\sigma_x z_x, \sigma_y z_y, \sigma_z z_z$

with eigenvalue 2

$\hookrightarrow$ so, it is also eigen vector for any linear combination of them.

$\hookrightarrow \vec{\sigma} \cdot \vec{z}$ is a piece of H. Hamiltonian of two spin system.

Density Matrix

Alice's Density matrix, is a quantity which even in an entangled state is as complete description of Alice's system for purposes of Alice's measurements as Alice can ever get.

* It's ~~not~~ a wave function.

(If it was un-entangled, the wave function would be the best description you could had)

$\hookrightarrow$ But, for an entangled system, you cannot extract out a wave-function for the either piece of the entangled system.

$\hookrightarrow$ The best you can do is something called the Density Matrix.

1. L is an observable ... without Bob noticing.

(PS112)

$$\langle L \rangle = \sum_{a, a', b, b'} \psi_{a'b'}^* \int_{bb'} L a'a \psi_{ab}$$

$$= \sum_{a, a', b} \psi_{a'b}^* L a'a \psi_{ab}$$

(with respect to Bob it is just unit operator)

$$\langle L \rangle = \left(\sum_{a, a'} L a'a \left(\sum_b \psi_{a'b}^* \psi_{ab} \right) \right)$$

→ this portion is only function of a & a' (not function of b)

→ Of course to calculate it, you may have to know more about wavefunction.

→ but, once you have calculated it, that's all that Alice ever needs to know to calculate anything about her half of the system. → let's give it a name.

$$\sum_b \psi_{a'b}^* \psi_{ab} = \rho_{a'a}$$

It does not depend on b 's degree of freedom.

called Density Matrix.

* If we know Density matrix $[\rho]$; it does not matter where it came from

→ then we know

$$\langle L \rangle = \sum_{a, a'} \rho_{a'a} L a'a$$

Density matrix for ~~pure state~~ product state.

* what if Ψ is a ~~pure state~~ product state, means the wave function factorises.

~~Ψ_{ab}~~ $\Psi_a \phi_b$

$$\rho_{a'a} = \sum_b \Psi_{a'b} \Psi_{ab}$$

$$= \sum_b \Psi_{a'}^x \phi_b^x \cancel{\phi_b} \Psi_a = \Psi_{a'}^x \Psi_a \underbrace{\sum_b \phi_b^x \phi_b}_{\text{this is 1}}$$

$$\Rightarrow \boxed{\rho_{a'a} = \Psi_{a'}^x \Psi_a} \Rightarrow \text{we find, that for product state, (only for product state)}$$

that ρ is simply made up of Alice's half of the wavefunction without any regard to Bob (she doesn't need to know anything about Bob).

$$\rho_{aa'}$$

↑ process of multiplying a column vector by a matrix

$$\begin{pmatrix} \rho_{11} & \rho_{12} & \dots \\ \rho_{21} & \rho_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$\sum_{a'} \rho_{aa'} \Psi_{a'}$$

→ This is action of ρ on $\Psi_{a'}$

$$(P \Psi_a)_{ax1} = \left(\sum_{a'} \left(\Psi_{a'}^x \Psi_a \right) \Psi_{a'} \right)_{ax1} = \left(\Psi_a \sum_{a'} \Psi_{a'}^x \Psi_{a'} \right)_{ax1} = \Psi_a$$

ρ

This tells that original wavefunction is an eigen vector of density matrix with eigen value +1.

$$\therefore \boxed{\rho \Psi_A = +1 \cdot \Psi_A}$$

★ let's take $\tilde{\Psi}$ (which is orthogonal to original Alice's vector)

$$\text{so: so: } \sum_{a'} (\Psi_{a'}^* \Psi_a) \tilde{\Psi}_{a'} = \Psi_a \left(\sum_{a'} \Psi_{a'}^* \tilde{\Psi}_{a'} \right)$$

→ now this inner product is zero

$$\Rightarrow \sum_{a'} (\Psi_{a'}^* \Psi_a \tilde{\Psi}_{a'}) = 0$$

so: $\rho \tilde{\Psi} = 0 \Rightarrow$ This means any vector perpendicular to Alice's vector, is an eigenvector

with eigen value 0.

→ The original state of Alice's spin is an eigen vector with eigen value +1.

↳ Conclusion: The ~~den~~ density matrix for pure state has one and only one non zero eigen value and its equal to +1.

What if not product state?

- Then you find a collection of eigenvalues.
- In fact, in general there will be all eigen values non zero of a density matrix.

★ so we learn: the information in the density matrix, in particular the eigen values of density matrix which tell you whether it is great deal of entanglement or not.

* The more eigen vectors which are non zero (eigen value), the more entangled it is.

• If there is only one eigen vector which is non zero, it's not entangled at all. ... and Alice's half of wavefunction is just that eigenvector.

• If they are entangled; one finds,
and the more they are entangled, the more
eigen values you find; and maximum entanglement,
when all the eigen ~~values~~ ^{values} are equal.

PS 115

maximum entanglement

$$\begin{pmatrix} 1/n & 1/n & 1/n & \dots \end{pmatrix}$$

where n is dimensionality of the space.

$$\text{diag} (1/n, 1/n, 1/n, \dots) \quad \text{or} \quad \frac{1}{n} I_{n \times n}$$

i.e. $P \sim \frac{1}{n} I_{n \times n}$

i.e. P is similar to $\frac{1}{n} I_{n \times n}$.

Measurement & Entanglement.

Measurement is the process of entangling
of starting with a product state

(a product state of
system & the apparatus;

→ the apparatus is in definite state to begin with
& the system is " " " ")

* What state is the apparatus in?

Ans Well, Apparatus might be something which detects
something rather, and if its not detecting you
might put it in something neutral state.

↳ which is starting state
or ~~und~~ undetected state.
Now, you bring the system you are
interested in; right upto apparatus ⇒ it does something to apparatus;
something happens to system itself. ⇒ and what is left over is an

$$A = X^{-1} B X$$

A & B
are
similar
if there exist
such X ...

i.e. diagonalization
in this case.

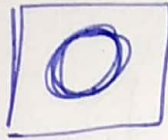
Entangled state between two of them.

PS 116

example think of apparatus as a very simple system of only two states.

and my system is a spin.

Apparatus



state

0, 1

spin



always starts in state zero.

(if it is not in state zero 0; give it a shove and put it into state 0)

supposing the spin is in up state, and the apparatus of course begins in 0 state.

so; we represent that by a combined ket vector

$$|u \ 0\rangle$$

it is in Tensor product of spin Hilbert space & the Detector Hilbert space.

~~or if~~

or if spin comes in down state;

$$|d \ 0\rangle$$

★ Now, we will be making a rule which would be a good detector

namely: • If combined system starts in the state $|u \ 0\rangle$, then the detector feels the spin and jumps to state 1: $|u \ 0\rangle \rightarrow |u \ 1\rangle$

(nothing happens to spin ... it just activates the detector)

- on the other hand, if the spin is down, it does not activate the detector.

(PS 114)

$$|d0\rangle \rightarrow |d0\rangle$$

now;

lets take;

$$\alpha_u |u0\rangle + \alpha_d |d0\rangle \xrightarrow[\text{after detector acts}]{\text{after state}} \alpha_u |u1\rangle + \alpha_d |d0\rangle$$

This all of a sudden becomes an entangled state

- if spin is up, detector is 1
- " " "down, " " 0.

↳ that is an entangled state.

→ Why it is an Entangled state?

→ because by looking at detector you can tell what the spin is doing (in fact that's what a measurement is)

A Measurement is an interaction between system & an apparatus that leaves the apparatus in a state that depends on the whatever it is measuring about the system.

~~Another point~~ From another point of view; its physics of a combined system, (combined system being spin & apparatus) which leaves them in a highly entangled state. $\Rightarrow$ so Entangled that by looking at the Detector, you can tell what the spin is.

So: Entanglement & Measurement are really the same thing.

* In fact, to measure a spin you ~~only need~~ really only need another spin. ~~the detector~~ (the detector here could be spin, & 1 or 0 could be up or down)

★ It is a hierarchy here

(PS 119)

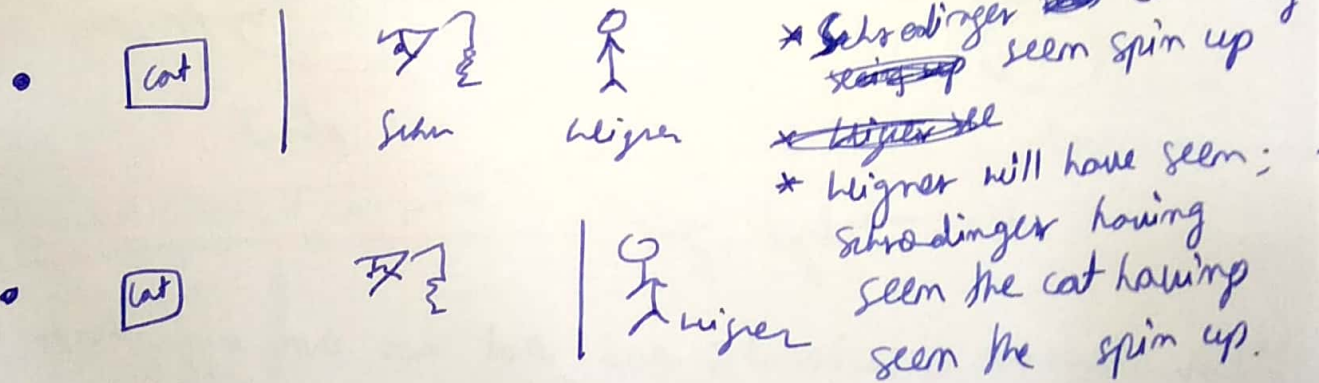
detector $\rightarrow$ eye looks $\rightarrow$ some other person looks $\rightarrow$ some other ...

⊙ where you decide to break the hierarchy?

★ determines at what stage you will say the wave function collapsed. (if you like)

Prof Leonard Susskind: "I suppose the wavefunction collapse when the last person looks at it, ... but you can always include that last person in the system, and then no collapse happens. ; but then in order to make sense out of whole thing, ~~you have again~~ again you have to imagine some other person $\Rightarrow$ so you always have to imagine hierarchy of observation"

$\hookrightarrow$ What is satisfying is that, it does not matter where you draw the line (the collapse thing...)



$\hookrightarrow$ That's what the nature of state is.

$\hookrightarrow$ This is satisfying $\Rightarrow$ You don't have to say, where you draw the line $\Rightarrow$ & until you draw the line (you can put the line as far as you like) everything is described by the rules of conventional quantum mechanics.

Locality and Entanglement

Pg 120

Einstein Question 1933

* Einstein when said
"Spooky Action at a distance"
he didn't mean the literal
meaning of Action at a distance
↳ & this got everybody
then confused.

Real meaning of Action at a distance

- forces between system at a distance.
- The ability to affect the outcome of an experiment at one place by doing something at other place.

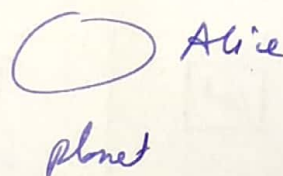
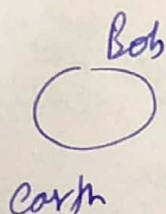
suppose you have

$\Psi(a, b) \rightarrow$ Palice.

this could be an
entangled state.

↳ this density matrix
contains every possible
statistical prediction that
Alice can ever make
for her subsystem A.

now let.



← light year →

* Bob does an experiment; and that has an effect on his
half of system here; but it has no effect whatever on
Alices density matrix. (he can also prove this)

↳ i.e. any physically allowable change in the wavefunction $\Psi(a, b)$
which only involves Bob half of it, & it will not have
any effect on Alices density matrix.

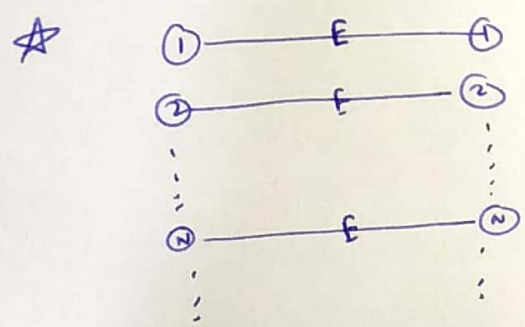
↳ That's the way, Quantum Mechanics is really put together.

→ And the implication is that ,

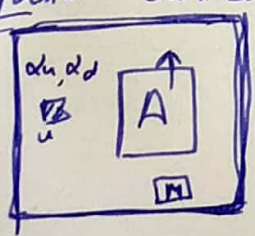
anything that Bob does suddenly at a fixed instant of time , will not have ^{any} affect on any prediction or success of any prediction ; it will have no effect on the statistical probabilities of Alice measuring anything about A. It just will not.

So in that sense , it was impossible for Bob to send the message to Alice suddenly.

~~The mere~~ fact that when Bob measures his variable , he instantly knows something about Alices is not source of anxiety for locality.



Computer (Classical Computer)



-Shoaib Akhtar

Note: There is no rule which says Entanglement is irreversible.
i.e. Entanglement is not inevitably irreversible.

* If the combined wave function for Alice & Bob is $\Psi(a, b)$

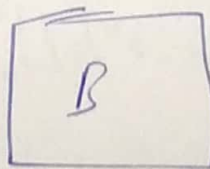
Then: Alice's density matrix is defined to be

$$\rho_{aa'}^A = \sum_b \Psi^*(a', b) \Psi(a, b)$$

→ This is only function of Alice's variable.
→ It contains all possible statistical information that Alice can ever know about her half of system unless Bob comes in & interact with her system.

* of course: if Bob comes from wherever he is, and has system A short of some sort, that will change her density matrix.
→ But as long as Bob stays away from her system (as long as Bob does not interact directly with her system): her density matrix does not change (no matter what Bob does).

Small Proof



for get Alice for a minute; and look what Bob can do to his ~~sys~~ sub-system.

$$\Psi(b) \longrightarrow \sum_{b'} U_{bb'} \Psi(b')$$

Bob going to manipulate his system somehow, and send it to a new wave function.

The Rule is: The new wave function of his ~~sys~~ subsystem must be

related to his old wavefunction by a Unitary Operator. (p 124)

→ Describing same thing in composite language. $\Psi(a, b)$

$$\Psi(a, b) \longrightarrow \sum_{b'} U_{bb'} \Psi(a, b')$$

→ Now; look at the manipulated wavefunction.

$$\Psi_M(a, b) = \sum_{b'} U_{bb'} \Psi(a, b')$$

now; calculate Alice's density matrix. ~~$\rho_{MM}^{AA} = \sum_{b, b'} \Psi_M^*(a, b) \Psi_M(a, b')$~~

$$\Psi_M^*(a, b) = \sum_{b''} \Psi^*(a, b'') U_{bb''}^*$$

$$\Psi_M^*(a, b) = \sum_{b''} \Psi^*(a, b'') U_{b''b}^\dagger$$

$$U_{b''b}^\dagger = U_{bb''}^*$$

unitary operator $UU^\dagger = I \Rightarrow U^\dagger = U^{-1}$

$$(U^\dagger)_{mm} = (U_{mm})^* \Rightarrow U_{b''b}^\dagger = U_{bb''}^*$$

$$\rho_{aa'}^A = \sum_b \sum_{b''} \sum_{b'} \Psi^*(a, b'') U_{b''b}^\dagger U_{bb'} \Psi(a, b')$$

$$\Rightarrow \rho_{aa'}^A = \sum_{b''} \sum_{b'} \Psi^*(a, b'') \Psi(a, b') \left(\sum_b U_{b''b}^\dagger U_{bb'} \right)$$

$$= \sum_{b''} \sum_{b'} \Psi^*(a, b'') \Psi(a, b') \delta_{b''b'}$$

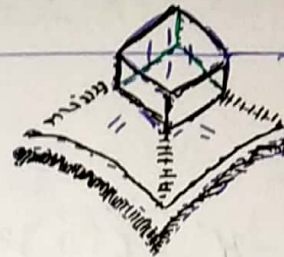
$$= \sum_{b'} \Psi^*(a, b') \Psi(a, b')$$

$$= \sum_b \Psi^*(a, b) \Psi(a, b)$$

The Inability of Bob to influence any statistical prediction of Alices by doing manipulation on his own degrees of freedom.
 → This is what Physicist called **Locality**.

* This form of Locality is not violated in any way by phenomenon of Entanglement.

One dimension



$|n\rangle$

∞ line labelled by coordinate x .

* The states of the particle are also labelled by position of particle.
 (Here we are departing from classical logic: in classical logic states of particle are labelled by a position and a momentum)

$|n\rangle \Rightarrow$ ket vector n is the state of particle when it is located at position x .

$|\Psi\rangle$

↓ General state of particle
 (vector in some vector space) $\Rightarrow$ The vector space now has continuously infinite number of basis vectors.

now, the wavefunction is

$$\Psi(x) = \langle x | \Psi \rangle$$

probability ~~for the outcome~~ of to find particle at x is

$$P(x) = \Psi^*(x) \Psi(x)$$

$$\int \Psi^*(x) \Psi(x) dx = 1$$

we talk about Probability located within range of x ... or better yet we talk about Density of probability.

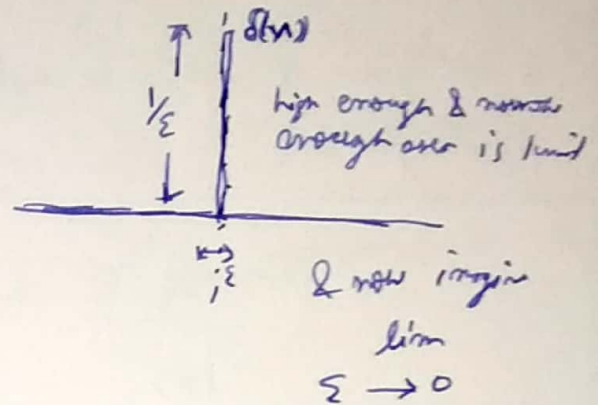
$$\begin{aligned} \overline{\Psi^*(a) \Psi(a)} & \quad \Psi(a) = \langle a | \Psi \rangle \\ \sum \Psi^*(a) \Psi(a) &= 1 \\ \langle a | a' \rangle &= \delta_{a a'} \end{aligned}$$

* when probability is referred to continuous variable, we don't talk about probability that the particle is located exactly at point x_0 ... its too sharp ... ; it is actually zero.

* We talk about probability density and we integrate $P(n)$ over small range of n to find probability that you are in that range of n . (Pg 126)

$\langle n/n' \rangle = 0 \quad \because n \neq n'$

actually: $\langle n/n' \rangle = \delta(n-n')$



$$|\Psi\rangle = \int dn' \Psi(n') |n'\rangle$$

$$\begin{aligned} \langle n|\Psi\rangle &= \int dn' \Psi(n') \langle n|n'\rangle \\ &= \int dn' \Psi(n') \delta(n-n') = \Psi(n) \end{aligned}$$

(we find consistency)

$\int \delta(n-n') F(n') = F(n)$

$|\Psi\rangle = \sum_{a'} \Psi(a') |a'\rangle$

Remember

$$\langle \Psi | \phi \rangle = \sum_{a,a'} \Psi^*(a') \langle a' | \phi(a) | a \rangle$$

$$\begin{aligned} \langle \Psi | \phi \rangle &= \sum_{a,a'} \Psi^*(a') \langle a' | \phi(a) | a \rangle \\ &= \sum_{a,a'} \Psi^*(a') \phi(a) \delta_{aa'} = \sum_a \Psi^*(a) \phi(a) \end{aligned}$$

* Inner product for two state vectors for particle on a line.

$$\langle \Psi | \phi \rangle = \int_{-\infty}^{+\infty} dn \Psi^*(n) \phi(n)$$

$$\langle \phi | \Psi \rangle = \int_{-\infty}^{+\infty} dn \phi^*(n) \Psi(n) = \left(\int_{-\infty}^{+\infty} dn \Psi^*(n) \phi(n) \right)^* = \langle \Psi | \phi \rangle^*$$

Observables.

M = hermit linear operator.

(M used for machine)

$$M|\Psi\rangle \rightarrow |\Psi'\rangle$$

$$\text{hermit}; \langle \phi | M | \Psi \rangle = \langle \Psi | M^\dagger | \phi \rangle^*$$

$$M\Psi(n) \rightarrow \Psi'(n)$$

* Hermitian Linear Operator (L)

$$\langle \phi | L | \Psi \rangle = \langle \Psi | L | \phi \rangle^*$$

Trick: Hermitian is analogue for real.

$$\Psi(u) \rightarrow n \Psi(u)$$

$$: n(\Psi(u) + \phi(u)) = n\Psi(u) + n\phi(u)$$

$$\langle \phi | n | \psi \rangle = \int du \phi^*(u) (n \Psi(u))$$

$$\langle \psi | n | \phi \rangle = \int du \Psi^*(u) n \phi(u)$$

note: $n = n^*$

they are
complex conjugate
of each other.

n is an observable, and it corresponds to position of the particle.

D for derivative operator:

$$* D|\psi\rangle \rightarrow |\psi'\rangle$$

$$* \text{what about wave functions: } D\Psi(u) = \frac{d\Psi(u)}{du}$$

$$\left. \begin{aligned} \int \phi^*(u) \frac{d\Psi(u)}{du} \cdot du \\ \int \Psi^*(u) \frac{d\phi(u)}{du} \cdot du \end{aligned} \right\} \begin{aligned} &\text{are they complex} \\ &\text{conjugate of each} \\ &\text{other?} \end{aligned}$$

$$\left[\int \Psi^*(u) \frac{d\phi(u)}{du} \cdot du \right]^* = \int \Psi(u) \frac{d\phi^*(u)}{du} \cdot du$$

we integrate by parts ... we are assuming that at far away the function goes to zero.

$$\int \Psi^* \frac{d\phi}{du} du = (\Psi \phi^*) \Big|_{-\infty}^{+\infty} - \int \phi^* \frac{d\Psi}{du} du$$

not Hermitian operator ... actually comes out to be anti-Hermitian

$$P = -i \frac{d}{du}$$

} This is now Hermitian operator.

$$\# \langle \psi | L | \psi \rangle$$

* let take expectation value of momentum.

(B128)

$$\langle \Psi | p | \Psi \rangle = \int_{-\infty}^{+\infty} \Psi^*(u) \left(-i \frac{d}{du} \Psi(u) \right) du$$

$$= -i \left(\int_{-\infty}^{+\infty} \Psi^*(u) \frac{d}{du} \Psi(u) du \right) = -i \left(\Psi^* \Psi \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{d\Psi^*}{du} \Psi du \right)$$

$$= +i \int_{-\infty}^{+\infty} \frac{d\Psi^*}{du} \Psi du = \left(-i \int_{-\infty}^{+\infty} \frac{d\Psi}{du} \cdot \Psi^* du \right)^*$$

This proves; $\langle \Psi | p | \Psi \rangle$ is real.

so; we know $p = -i\hbar \frac{d}{dx}$ is a hermitian operator... & therefore an observable in Quantum Mechanics.

$[n, p]$: $\left(-i\hbar \frac{d}{dx} \right) (n) \Psi(x) = p n \Psi(x)$

$\downarrow$

$-i\hbar \left[\Psi(x) + n \frac{d\Psi}{dx} \right]$

$\left\{ \begin{array}{l} (n) \left(-i\hbar \frac{d}{dx} \right) \Psi(x) = n p \Psi(x) \\ -i\hbar n \frac{d\Psi}{dx} \end{array} \right.$

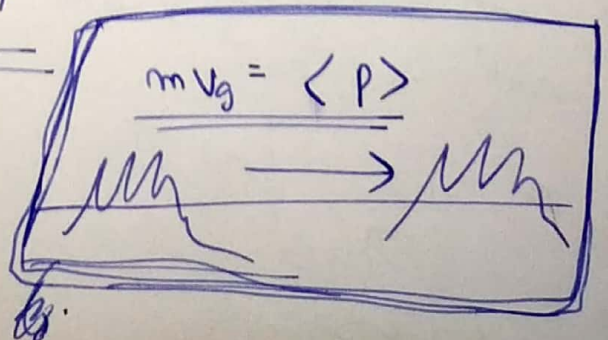
$[n, p] \Psi(x) = +i\hbar \Psi(x) \quad \forall \Psi \Rightarrow \{ [n, p] - i\hbar \} \Psi(x) = 0 \quad \forall \Psi$

$\Rightarrow [n, p] = i\hbar$

$\Rightarrow [n, p] - i\hbar = 0$
 $\Rightarrow [n, p] = i\hbar$

$\therefore p = -i\hbar \frac{d}{dx} \Rightarrow [n, p] = i\hbar$

In classical mechanics: $\{n, p\} = 1$



Lec - 9 Quantum Mechanics: Fourier Analysis applied to Quantum mechanics and the Uncertainty principle

P3129

Fourier Transform.

$$e^{ipx}$$

we call it plane wave.

wave like.

$$\psi(x)$$

Decomposing $\psi(x)$ into plane waves is called Fourier Analysis.

$$\psi(x) = \int_{-\infty}^{+\infty} \frac{dp}{\sqrt{2\pi}} \tilde{\psi}(p) \cdot e^{ipx}$$

large p
small p

this is sort of convention

... it may have observed in $\tilde{\psi}(p)$.

Amplitude of that wave of type 'p' contained in $\psi(x)$

Theorem of Fourier Analysis now states that

$$\tilde{\psi}(p) = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} \psi(x) e^{-ipx}$$

area under the bump is just equal to 1; does not matter where you put it.

Delta Function.

representation of Delta function.

$$\textcircled{1} \delta(x-x') F(x) = \delta(x-x') F(x')$$

$$\textcircled{2} \int_{-\infty}^{+\infty} \delta(x-x') dx = 1$$

$$\textcircled{3} \int_{-\infty}^{+\infty} \delta(x-x') F(x) dx = F(x')$$

$\int \delta(x-x') F(x) dx = F(x')$ this is how mathematicians define Delta function.

$$\# \text{ so; } \psi(x) = \int_{-\infty}^{+\infty} \frac{dp}{\sqrt{2\pi}} \cdot e^{ipx} \left[\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} \cdot \psi(x) \cdot e^{-ipx} \right]$$

$$\Psi(n') = \int dn \int \frac{dp}{2\pi} \cdot e^{ip(n'-n)} \Psi(n)$$

now look at

$$\int \frac{dp}{2\pi} \cdot e^{ip(n'-n)} \cdot \Psi(n) \Rightarrow \text{some function of } \underline{n'-n}.$$

$$:= \Delta(n'-n)$$

$$\text{now: } \Psi(n') = \int_{-\infty}^{+\infty} dn \Delta(n'-n) \cdot \Psi(n)$$

$$\int \frac{dp}{2\pi} \cdot e^{ip(n-n')} = \delta(n-n')$$

$$\boxed{\int \frac{dp}{2\pi} \cdot e^{ip(n-n')} = \delta(n-n')}$$

$$\text{also: } \boxed{\int \frac{dn}{2\pi} \cdot e^{in(p-p')} = \delta(p-p')}$$

* classically (x, p)

A particle on a line.

Quantum mechanically; how we represent state of particle.

$$\text{or } |n\rangle \text{ or } |p\rangle \text{ or } |x, p\rangle$$

but we know; in Quantum mechanics, knowing n & p is too much information. We cannot know both of them simultaneously (Heisenberg uncertainty does not allow)

so; we can't have n and p ... rules out $|n, p\rangle$

we can have n or p so; we don't describe by both, we describe by one or the other.

so; we suppose ; there is an eigen vector of an observable position for every value of the observable position.

(Pg 131)

$|n\rangle$

→ This vector represents a state in which the position of the particle is definite and at position n .

Point of any good basis is to write any wave function $|\Psi\rangle$ as superposition, of sum of the linear combination of basis vectors.

$$|\Psi\rangle = \int dn \underbrace{\Psi(n)}_{\text{coefficients here}} |n\rangle$$

→ These are the coefficients here, and we call these things as wave function.

$\langle n | \Psi \rangle = \Psi(n)$; $\underbrace{\Psi^*(n) \Psi(n)}_{\text{Probability of particle being found at } n} = P(n)$

→ Strictly speaking, we have to think it as probability density (Probability per unit interval of x)

$\langle n' | n \rangle = \delta(n - n')$ } ⇒ It is a continuum analog of Kronecker delta δ_{mn} .

also; KETS $\Rightarrow \Psi(n)$
BRAS $\Rightarrow \Psi^*(n)$

$$\langle \Psi | \Phi \rangle = \int \psi^*(u) \phi(u) \cdot du$$

described by.
 $\Phi \Rightarrow \phi$
 $\Psi \Rightarrow \psi$

proof
 actually; $\langle \Psi | \Phi \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} du du' \cdot \psi^*(u') \phi(u) \langle u' | u \rangle$
 $= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} du du' \cdot \psi^*(u') \phi(u) \delta(u' - u)$
 $= \int_{-\infty}^{+\infty} du \psi^*(u) \phi(u)$

~~X~~ let this be an operator or the observable whose eigen values are the position of particles.

~~X~~ $|\psi\rangle$

$\therefore$ ~~X~~ $\psi(u) = x \psi(u)$

* Wave function and state vectors, can be thought of as one being representation of the other

~~now, let's prove that~~
~~what is likely to be the eigenvector and eigenvalue~~
~~of X.~~

$\hookrightarrow$ well, it should be a state where the particle is localized at position x .



* in particular let it be x_0

$\therefore$ what would we expect eigen vector to be.

$\hookrightarrow$ it should be function which is zero everywhere except at x_0 .

$\hookrightarrow$ so, let's try $\delta(u - x_0)$

$$\therefore \lambda \delta(n-n_0) = \lambda_0 \delta(n-n_0)$$

↳ but this is exactly the equation for eigenvector & eigenvalue for operator X .
 ↳ it multiplies by a constant number λ_0 .

so; $\delta(n-n_0)$ is eigenvector of X with eigenvalue λ_0 .

Momentum ... it's an observable. ... (it is conserved).

$$P = -i\hbar \frac{d}{dx}$$

$$\therefore \text{now; } P \psi(x) = -i\hbar \frac{d\psi}{dx}$$

$\therefore$ eigen vectors of P satisfy.

$$-i\hbar \frac{d\psi}{dx} = p_0 \psi \Rightarrow \frac{d\psi}{dx} = \frac{i p_0}{\hbar} \psi$$

$$\Rightarrow \psi_{p_0}(x) = e^{\frac{i p_0}{\hbar} x}$$

$$\frac{p_0}{\hbar} \cdot \lambda = 2\pi$$

↳ wavelength

$$\Rightarrow \lambda = \frac{2\pi\hbar}{p_0}$$

... ~~$\psi_p(x) = e^{\frac{i p x}{\hbar}}$~~

$\therefore$ note; $\psi_p(x) = e^{i p x}$ is a representation for $|p\rangle$

$$\therefore \langle p' | p \rangle = \int_{-\infty}^{+\infty} e^{-i p' x} \cdot e^{i p x} \cdot dx = \int_{-\infty}^{+\infty} e^{i(p-p')x} \cdot dx$$

$\because p \neq p' \Rightarrow$ the integral oscillates itself to zero

at $p = p' \Rightarrow$ it is ∞ infinite

B/34

$\leadsto$ something like Delta function.

↳ but we don't know the coefficient.

but ; in order to normalize, we divide by 2π

$$\psi_p(x) = \frac{e^{ipx}}{\sqrt{2\pi}}$$

$\Rightarrow$ note: here, particle is localized in momentum; it is completely delocalized in position. ... the wave is spread all over.

Wave function in momentum basis.

$$\langle x | \Psi \rangle = \psi(x)$$

$$\langle p | \Psi \rangle = \tilde{\psi}(p)$$

$\hookrightarrow$ wave function in the momentum basis.

~~$$\langle \Psi | \Phi \rangle = \int \psi^* \phi dx$$~~

$$\int dx \cdot \frac{e^{-ipx}}{\sqrt{2\pi}} \cdot \psi(x) dx$$

but this actually $\langle p | \Psi \rangle$.

$$\langle \Psi | \Phi \rangle = \int \psi^* \phi dx$$

using this.

$$\text{so: } \tilde{\psi}(p) = \int dx \cdot \frac{\psi(x)}{\sqrt{2\pi}} \cdot e^{-ipx}$$

so: $\tilde{\psi}(p)$ is exactly the Fourier transform of position wave function.

$$P(p) = \tilde{\psi}^*(p) \tilde{\psi}(p)$$

~~$$\# \quad \begin{aligned} P &= -i\hbar \frac{d}{dx} \\ X &= +i\hbar \frac{d}{dp} \end{aligned}$$~~

$$P = -i\hbar \frac{d}{dx}$$

$$X = +i\hbar \frac{d}{dp}$$

Hamiltonian for Particles.

$$i\hbar \frac{\partial \Psi}{\partial t} = H \Psi$$

Time dependent
Schrödinger Equation.

$$H = c p$$

c is for constant

lets see what happens with ~~things~~ this
and for which kind of things,
this Hamiltonian is for.

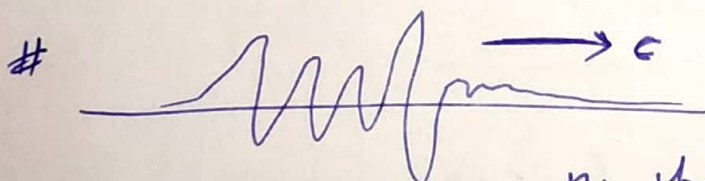
Now this is represented in terms of wave functions.

$$i\hbar \frac{\partial \Psi(n,t)}{\partial t} = c \left(-i\hbar \frac{\partial \Psi(n,t)}{\partial n} \right)$$

$$\Rightarrow \frac{\partial \Psi(n,t)}{\partial t} = -c \cdot \frac{\partial \Psi(n,t)}{\partial n}$$

$$\Rightarrow \frac{\partial \Psi}{\partial t} + c \cdot \frac{\partial \Psi}{\partial n} = 0$$

solution is ~~of the form~~
any function of the
form $\Psi(n - ct)$



as time runs ... the shape remain same ... just moves
rigidly.

note

→ this is not the correct description of photons.

→ but it is pretty close to the description of neutrinos, which
move with the speed of light (c)

(real neutrinos ~~are~~ probably move little slower than speed of
light; unmeasurably slow. ... so, this will be a
good description of one dimensional neutrinos, except we
would have to add possibility, pure right moving neutrinos,

* expectation value of x also moves with velocity c .

★ let's look at hamilton's equations.

$$\frac{\partial H}{\partial p} = \dot{x} = c \quad ; \quad \frac{\partial H}{\partial x} = -\dot{p} = 0$$

* what this says classically
that the momentum is conserved
* and position of particle moves with velocity c .

#1 Other Hamiltonian $T = \frac{1}{2}mv^2 \quad ; \quad p = mv \quad ; \quad T = \frac{p^2}{2m}$

$\therefore$ free particle; $H = \frac{p^2}{2m} \Rightarrow$ Hamiltonian for classical particle

notice; this hamiltonian has a feature that particles can move only to right.
(there is no left moving motion)
 $\rightarrow$ for; $H = c|p|$.
energy is +ve or -ve; depending on whether p is positive or negative.

~~$H = \frac{p^2}{2m}$~~

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} (-i\hbar)^2 \frac{\partial^2 \psi}{\partial x^2}$$

$$\Rightarrow i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \Rightarrow \boxed{\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2}}$$

$\rightarrow$ This is incidentally a wave equation. $\downarrow$ Schrodinger eqⁿ

& ψ is a wave function

it has property that waves of different wavelength move with different velocities. $\rightarrow$ because of this wavefunction does not maintain its shape. for ordinary non relativistic particle.

Lec-10 Quantum Mechanics: The Uncertainty Principle & Classical Analogs. — Shoab Akhtar.

(Pg 137)

$H = cP$ P can be +ve or -ve.

$\psi = e^{ipx}$

 $p > 0 \rightarrow$ has +ve energy
 $p < 0 \rightarrow$ has -ve energy.

★ Negative energy is not such a good thing in real world.

→ because if there were particles of -ve energy; the world would be unstable.

→ The empty state would not be the state of lowest energy.

⇒ you can take empty space (which we can say has zero energy) and then just start putting in particles of -ve energy

→ they will lower the energy, ... and they will be favoured more and more → Then you will have to get +ve energy from some place; example, you could radiate photons and thus produce more & more these negative particles.

→ So if there were particles of -ve energy, the empty space would be unstable ⇒ they would be unstable with respect to creating more and more +ve and -ve energy particles.

☆☆ But if there is only positive energy particles, there is no way that energy conservation will allow you to make them.

If: • only +ve energy particle.
• and vacuum has zero energy.

→ Then empty space is lowest energy state and hence nothing can happen to create particles.

~~★ So Quantum~~

66 So, Good stable world requires all particles has positive energy relative the vacuum"

Pg 138

Dirac Explained for fermions. (particles which you can't put more than one in same state)

"Just imagine that empty space or vacuum is really a state in which all the particles of -ve energy are present simultaneously filling up the vacuum"

→ you can't simply put more in, because you are not allowed to put particles in same state but everything else is empty.

• it is good theory for neutrinos (one dimensional neutrinos)

• but not a good theory for photons, or bosons.

$\langle x | \Psi \rangle$ → ~~Momentum~~ Wave function
 $\langle p | \Psi \rangle$

$\langle x | \Psi \rangle \Rightarrow$ Position Wave function.
 $\langle p | \Psi \rangle \Rightarrow$ Momentum " " } and they are related by Fourier Transform.

* if particle moves in two dimension.

→ they you could characterize, position by the coordinates.

• The two coordinates commute.

→ you can measure both x & y position of particle simultaneously.

$$\langle x y | \Psi \rangle$$

* if we have several particles, then wave function becomes a function of coordinates of all the particles.

Uncertainty Principle.

comes from the fact that position & momentum don't commute.

→ These eigenvectors are different.

Eigenvector of position.

Eigenvector of momentum.

$$\psi(n) = \int \frac{dp}{\sqrt{2\pi}} \tilde{\psi}(p) \cdot e^{ipn}$$

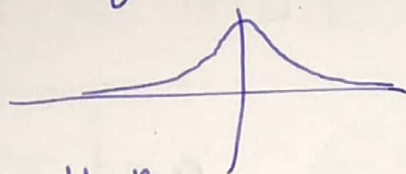
$$\tilde{\psi}(p) = \int \frac{dn}{\sqrt{2\pi}} \psi(n) \cdot e^{-ipn}$$

Narrower the $\psi(n)$ is,
the broader is $\tilde{\psi}(p)$
& vice versa.

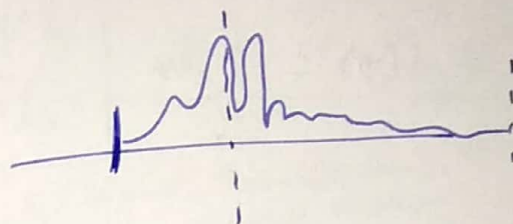
↳ This is just here property
of Fourier Analysis.
... but it is more
general.

What is meant by uncertainty in a variable.

Δn



x



- you can shift the axis, so that average can be made zero (shifting coordinates)

$$\text{so: } \langle n \rangle = 0$$

definition of uncertainty is actually $\langle n^2 \rangle$.

$$\therefore (\Delta n)^2 = \langle n^2 \rangle$$

(when $\langle n \rangle \neq 0$; then $(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$)

$$\langle n^2 \rangle = \int \psi^*(n) \psi(n) n^2 \cdot dn$$

$$\therefore \langle n^2 \rangle := (\Delta n)^2$$

a sort of definition of Δn
i.e. $\Delta n = \sqrt{\langle n^2 \rangle}$

$$(\Delta n)^2 = \int \psi^*(n) \psi(n) \cdot n^2 \cdot dn$$

$$(\Delta n)^2 = \langle \Psi | X^2 | \Psi \rangle$$

$$(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$$

What about uncertainty in momentum.

$$(\Delta p)^2 = \int \tilde{\psi}^*(p) \tilde{\psi}(p) p^2 dp = \langle \tilde{\Psi} | p^2 | \tilde{\Psi} \rangle$$

$\langle \Psi | P^2 | \Psi \rangle$ can also be calculated in x representation
 & in that context we also know how P acts i.e. $-i\hbar \frac{\partial}{\partial x}$

$$(Dp)^2 = - \int_{-\infty}^{+\infty} dx \Psi^*(x) \cdot \frac{d^2 \Psi(x)}{dx^2}$$

... by integration by parts $\Rightarrow (Dp)^2 = \int_{-\infty}^{+\infty} dx \frac{d\Psi^*(x)}{dx} \cdot \frac{d\Psi(x)}{dx}$

$$\Rightarrow (Dp)^2 = \int_{-\infty}^{+\infty} dx \left| \frac{d\Psi(x)}{dx} \right|^2$$

• You can take a wave function, and you can shift it both in x and in p until both of them have average equal to zero.

* Triangle Inequality: $|A| + |B| \geq |A+B|$
 $|A|^2 + |B|^2 + 2|A||B| \geq |A+B|^2 = |A|^2 + |B|^2 + 2(A \cdot B)$
 $\Rightarrow |A||B| \geq A \cdot B \quad (|A||B| \geq |A||B|\cos\theta)$
 $\hookrightarrow |A|^2 |B|^2 \geq (A \cdot B)^2 \quad \text{or} \quad \langle A|A \rangle \langle B|B \rangle \geq \langle A|B \rangle \langle B|A \rangle$

* for simplicity; assume Ψ is real. (can be done for general Ψ)

$|A\rangle = x \Psi(x)$
 $|B\rangle = -i\hbar \frac{d\Psi}{dx}$
 ; by definition here:
 $|A|^2 = (\Delta x)^2$
 $|B|^2 = (\Delta p)^2$

$$A \cdot B = \langle A|B \rangle = \int \Psi(x) \cdot x \cdot \left(-i\hbar \frac{d\Psi}{dx}\right) dx$$

$$|A \cdot B|^2 = \left(\int \Psi(x) \cdot x \cdot \frac{d\Psi}{dx} \cdot dx \right) \left(\int \Psi(y) \cdot y \cdot \frac{d\Psi}{dy} \cdot dy \right)$$

$$\frac{1}{2} \frac{d\Psi^2}{dx} = \Psi \frac{d\Psi}{dx} \Rightarrow |A \cdot B|^2 = \frac{1}{2} \left(\int x \frac{d^2 \Psi^2}{dx^2} \cdot dx \right) \frac{1}{2} \left(\int y \cdot \frac{d^2 \Psi^2}{dy^2} \cdot dy \right)$$

$$\Rightarrow |A \cdot B|^2 = \frac{1}{4} \left(\int x \cdot \frac{d^2 \Psi^2}{dx^2} \cdot dx \right) \left(\int y \cdot \frac{d^2 \Psi^2}{dy^2} \cdot dy \right)$$

$$|A \cdot B|^2 = \frac{1}{4} \left(-\int \frac{dn}{dn} \cdot \psi^2 dn \right) \left(-\int \frac{dy}{dy} \cdot \psi^2 dy \right)$$

$$= \frac{1}{4} \left(\int \psi^2 dn \right) \left(\int \psi^2 dy \right) = \frac{1}{4} \cdot 1 \cdot 1 = \frac{1}{4}$$

$$\Rightarrow (\Delta x)^2 (\Delta p)^2 \geq \frac{1}{4} \quad \textcircled{80}$$

$$\boxed{(\Delta x)(\Delta p) \geq \frac{\hbar}{2}}$$

Uncertainty Principle.

$$\# |\psi(t)\rangle = \sum e^{-iEt} \phi(E) |E\rangle$$

$$\# i \frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

$$i.e.; H = \frac{p^2}{2m} + V(x)$$

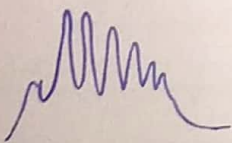
Classical Hamiltonian for a particle moving in a potential $V(x)$

$$\Rightarrow i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \cdot \frac{\partial^2 \psi}{\partial x^2} + V(x) \cdot \psi$$

$$\Rightarrow \frac{\partial \psi}{\partial t} = \frac{i}{2m} \cdot \frac{\partial^2 \psi}{\partial x^2} - i V(x) \cdot \psi$$

$$\frac{\partial \psi^*}{\partial t} = -\frac{i}{2m} \cdot \frac{\partial^2 \psi^*}{\partial x^2} + i V(x) \psi^*$$

note $V(x)$ is a real quantity.



$$\star \frac{d}{dt} \langle n \rangle = \frac{d}{dt} \int \psi^* \cdot n \cdot \psi \cdot dn = \int \frac{\partial \psi^*}{\partial t} \cdot n \psi + \int \psi^* \cdot n \frac{\partial \psi}{\partial t} + \int \psi^* \cdot \frac{dn}{dt} \psi$$

$$\psi(x, t)$$

$\Rightarrow$ so, actually;

$$\frac{dx}{dt} = 0$$

x & t are considered as independent part of.

$$\Rightarrow \frac{d}{dt} \langle n \rangle = 2 \operatorname{Re} \left(\int \psi^* n \frac{\partial \psi}{\partial t} \cdot dn \right)$$

$$= 2 \operatorname{Re} \left(\int \psi^* n \cdot \left(\frac{i}{2m} \cdot \frac{\partial^2 \psi}{\partial x^2} - i V(x) \psi \right) dn \right)$$

$V(x)$ not contribute $\Rightarrow$ that term will give imaginary.

$$\frac{d\langle x \rangle}{dt} = 2 \operatorname{Re} \left(\int \psi^* \cdot n \cdot \left(\frac{i}{2m} \cdot \frac{\partial^2 \psi}{\partial x^2} \right) dx \right)$$

$$= 2 \operatorname{Re} \left(\underbrace{-\frac{i}{2m} \left\{ \int \psi^* \frac{\partial \psi}{\partial x} + \int \frac{\partial \psi^*}{\partial x} \cdot \frac{\partial \psi}{\partial x} \cdot n \, dx \right\}}_{\substack{\hookrightarrow \text{real} \\ \text{but multiplied by} \\ i/2m}} \right) \frac{\partial x}{\partial t} = 0$$

∴ but; $\frac{dx}{dt}$ may not be 0.

∴ so; it will not contribute.

$$\Rightarrow \frac{d\langle n \rangle}{dt} = 2 \operatorname{Re} \left(\frac{-i}{2m} \int \psi^* \cdot \frac{\partial \psi}{\partial x} \cdot dx \right) = 2 \operatorname{Re} \left(\frac{1}{2m} \int \psi^* \cdot \left(-i \frac{\partial \psi}{\partial x} \right) dx \right)$$

$$= \operatorname{Re} \left(\int \psi^* \left(\frac{1}{m} \right) \left(-i \frac{\partial \psi}{\partial x} \right) dx \right)$$

$$= \operatorname{Re} \left(\langle \Psi | \frac{p}{m} | \Psi \rangle \right) = \langle \Psi | p/m | \Psi \rangle$$

~~$\Rightarrow \frac{d\langle x \rangle}{dt} = \frac{\langle p \rangle}{m}$~~

$\Rightarrow \frac{d\langle x \rangle}{dt} = \frac{\langle p \rangle}{m}$

#

$$\therefore \frac{d\langle p \rangle}{dt} = \frac{d}{dt} \int \psi^*(x,t) p \psi(x,t) dx = \frac{d}{dt} \cdot \left(-i \int \psi^* \frac{\partial \psi}{\partial x} dx \right)$$

actually; we want to show; $\dot{p} = F = -\frac{\partial V}{\partial x}$

$$\frac{d}{dt} \langle p \rangle = - \left\langle \frac{\partial V}{\partial x} \right\rangle$$

$$\Rightarrow \frac{d}{dt} \langle p \rangle = -i \left[\int \dot{\psi}^* \frac{\partial \psi}{\partial x} + \int \psi^* \cdot \frac{\partial}{\partial x} \dot{\psi} \right]$$

$$= -i \int \dot{\psi}^* \frac{\partial \psi}{\partial x} + i \int \frac{\partial \psi^*}{\partial x} \cdot \dot{\psi}$$

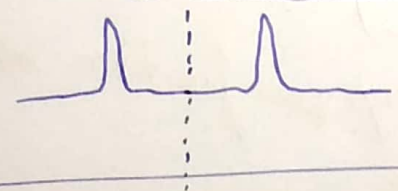
$$= 2 \operatorname{Re} \left(i \int \frac{\partial \psi^*}{\partial x} \cdot \dot{\psi} dx \right)$$

$$\begin{aligned}
 \frac{d\langle p \rangle}{dt} &= 2 \operatorname{Re} \left[i \int \frac{\partial \psi^*}{\partial x} \left[\frac{i}{2m} \cdot \frac{\partial^2 \psi}{\partial x^2} - i V(x) \psi \right] \right] \\
 &= 2 \operatorname{Re} \left[\int \frac{\partial \psi^*}{\partial x} \left\{ \frac{-1}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi \right\} \right] \\
 &= 2 \operatorname{Re} \left[\int \frac{\partial \psi^*}{\partial x} \cdot V(x) \psi \right] \quad \text{will result pure imaginary} \\
 &= \int \frac{\partial \psi^*}{\partial x} \cdot V(x) \psi + \int \frac{\partial \psi}{\partial x} \cdot V(x) \psi^* \\
 &= \int V(x) \left\{ \frac{\partial \psi^*}{\partial x} \psi + \psi^* \frac{\partial \psi}{\partial x} \right\} dx \\
 &= \int V(x) \frac{\partial}{\partial x} (\psi^* \psi) dx \\
 &= - \int \frac{\partial V}{\partial x} \cdot \psi^* \psi dx = \left[\int \psi^* \left(-\frac{\partial V}{\partial x} \right) \psi dx \right] \\
 &= \left\langle \psi \left| -\frac{\partial V}{\partial x} \right| \psi \right\rangle = - \left\langle \frac{\partial V}{\partial x} \right\rangle
 \end{aligned}$$

$$\Rightarrow \boxed{\frac{d\langle p \rangle}{dt} = - \left\langle \frac{\partial V}{\partial x} \right\rangle}$$

Note: $\langle F(n) \rangle$ not necessarily is $F(\langle n \rangle)$

ex $F = x^2$
Supposing

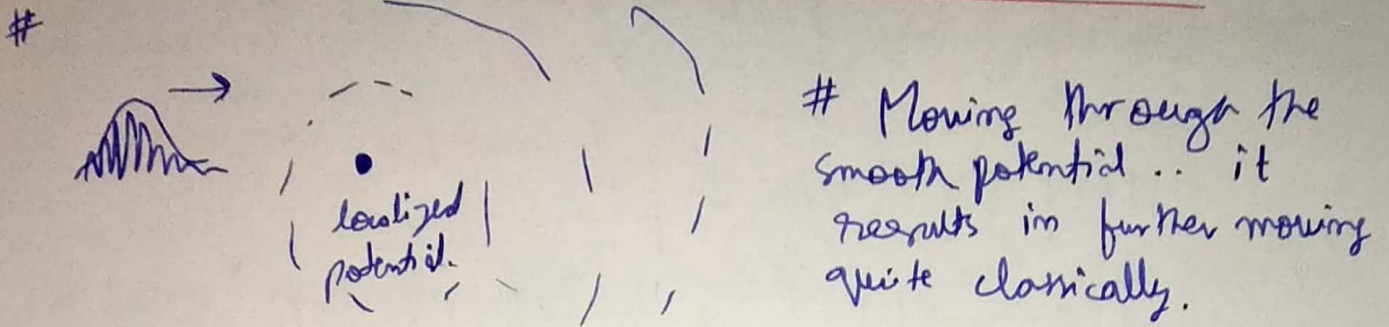


$$\begin{aligned}
 F(\langle n \rangle) &= 0 \\
 \therefore \text{but } \langle F \rangle &= \langle n^2 \rangle \neq 0
 \end{aligned}$$

★ If wave packets are concentrated over reasonably narrow range then $\langle F(n) \rangle = F(\langle n \rangle)$

What are the circumstances under which function will remain nicely localized.

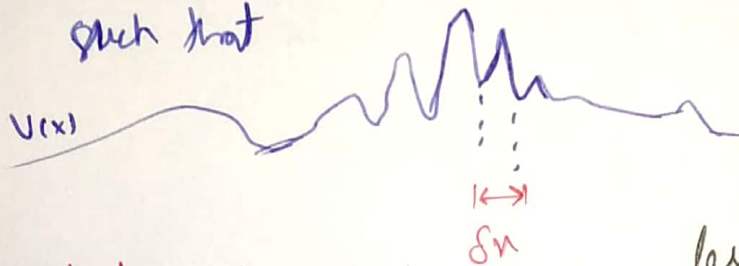
- The circumstances are when the particle is heavy and potential energy not having too many spikes.



When wave packet hits localized potential it spreads.

* by the way; no way Quantum Mechanics should produce Classical Result; since classical result is not just the reality.

Bad situations; when structure of potential is such that



$$\Delta x < \Delta n$$

(~~sharpness~~ (sharpness width less than uncertainty in position))

ie; Sharp feature of Potential function takes place on a scale which is much smaller than the size of incoming wave packet

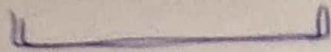
→ Then it will break up wave function into lot of pieces ... ie; scatters them.

$\Delta p \Delta x \geq \frac{\hbar}{2}$; in many reasonable cases

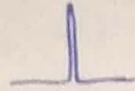
$\Delta p \Delta n$ is of order $\hbar$; ie; $\Delta p \Delta n \sim \hbar$

$\Rightarrow m \Delta v \Delta x \sim \hbar \Rightarrow \Delta v \Delta x \sim \frac{\hbar}{m}$

example for bowling ball; Δv is not much big; here $\Delta n \sim \frac{\hbar}{m \Delta v}$

$$\Delta x \sim \frac{\hbar}{m \Delta v}$$


if m is large Δv gets small
and thus wave packet gets concentrated.



- * Large m and smooth potential is limit of Classical Physics.
- * Light " & abrupt " is more Quantum Mechanical.
- Its all interplay between the m and sharpness of potential.

#

