# Quantum Gravity

Shoaib Akhtar

**Lec 1:** Introduction and overview by Bianca Ditrich on the different Quantum Gravity approaches.

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For now, no experimentally accessible situations

$$\frac{F_{grav}}{F_{electmag}} \text{(proton - electron)} \sim 10^{-10}$$

$SU(3) \times SU(2) \times U(1)$ standard Model.

Quantum Mechanics Framework.

General Relativity.

Open Problems:
(i) Dark Matter
(ii) Unification
(iii) Quantum Gravity

} There may be related.

Quantum Gravity = Quantum Mechanics + General Relativity.

**G.R.** $S = \frac{1}{16\pi G} \int \sqrt{g} \cdot R$

**Q.M.** Classical description of Hydrogen atom unstable.
In Quantum description, we have levels of energy.
QM acts like repulsive force to prevent singularity.

# Planck Scale

$$L_{planck} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35} m$$

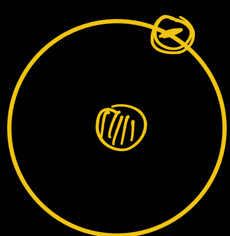
- QM, Compton wavelength : $\lambda_{min} \sim \frac{\hbar}{mc}$
- GR, Schwarzschild radius : $R \sim \frac{mG}{c^2}$

$$L_{min} \equiv \lambda_{min} = R \Rightarrow L_{min} = L_{planck}$$

from Q.M we expect something small.
from G.R " " " dense.

## Quantization

Classical



$$F = \frac{q_1 q_2}{R^2}$$

$\xrightarrow[\text{Quantization}]{\text{Canonical}}$

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 - \frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{r}$$

Electromagnetic force $\xrightarrow{\text{Feynman}}$ Standard Model.

Gravity $\xrightarrow{?}$ Quantum Gravity.

Perturbative approach : $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$
does not work !

..... String Theory .....

- Loop Quantum Gravity. (Dirac's Approach, canonical quantization)

GR (background independence, $g_{\mu\nu}$ field)

+

QM (Uncertainity Principle)

=

Theory of Quantum Gravity.

- Einstein-Hilbert formulation (1915)

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{g} \cdot R .$$

- Ashtekar Formulation (1986) (gravity looks as Yang Mill theory in this language.)

$$S = \frac{1}{8\pi G\gamma} \int d^4x \left( \tilde{E}^a_i \dot{A}^i_a + \underset{\sim}{N} \epsilon_{ijk} \tilde{E}^a_i \tilde{E}^b_j F^k_{ab} + \lambda^i (D_a \tilde{E}^a)^i \right)$$

$\gamma$ : Barbero - Immirzi parameter

$$\{A^i_a(x), \tilde{E}^b_j(y)\} = 8\pi G \gamma \cdot \delta^a_b \cdot \delta^i_j \cdot \delta^3(x-y)$$

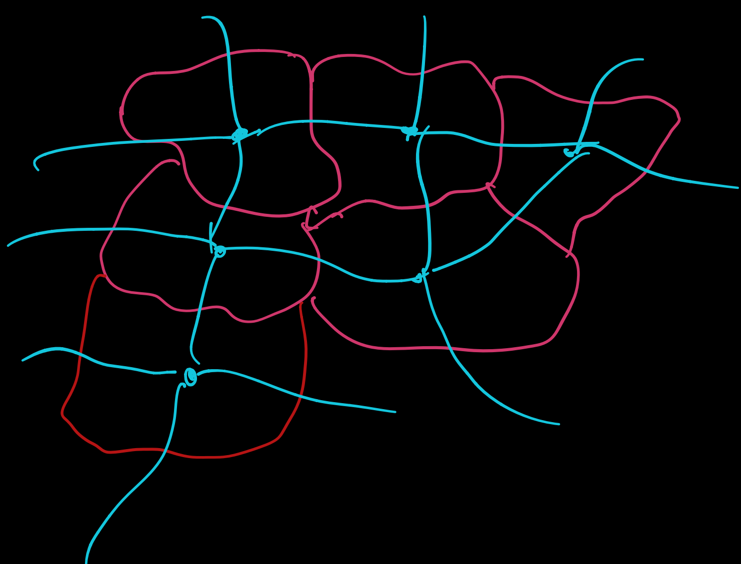
can use loops to measure curvature, so called Loop Quantum Gravity.

☆ Spin Networks : basis of the Kinematical Hilbert space of Loop Quantum Gravity.

* Carlo Roveli.

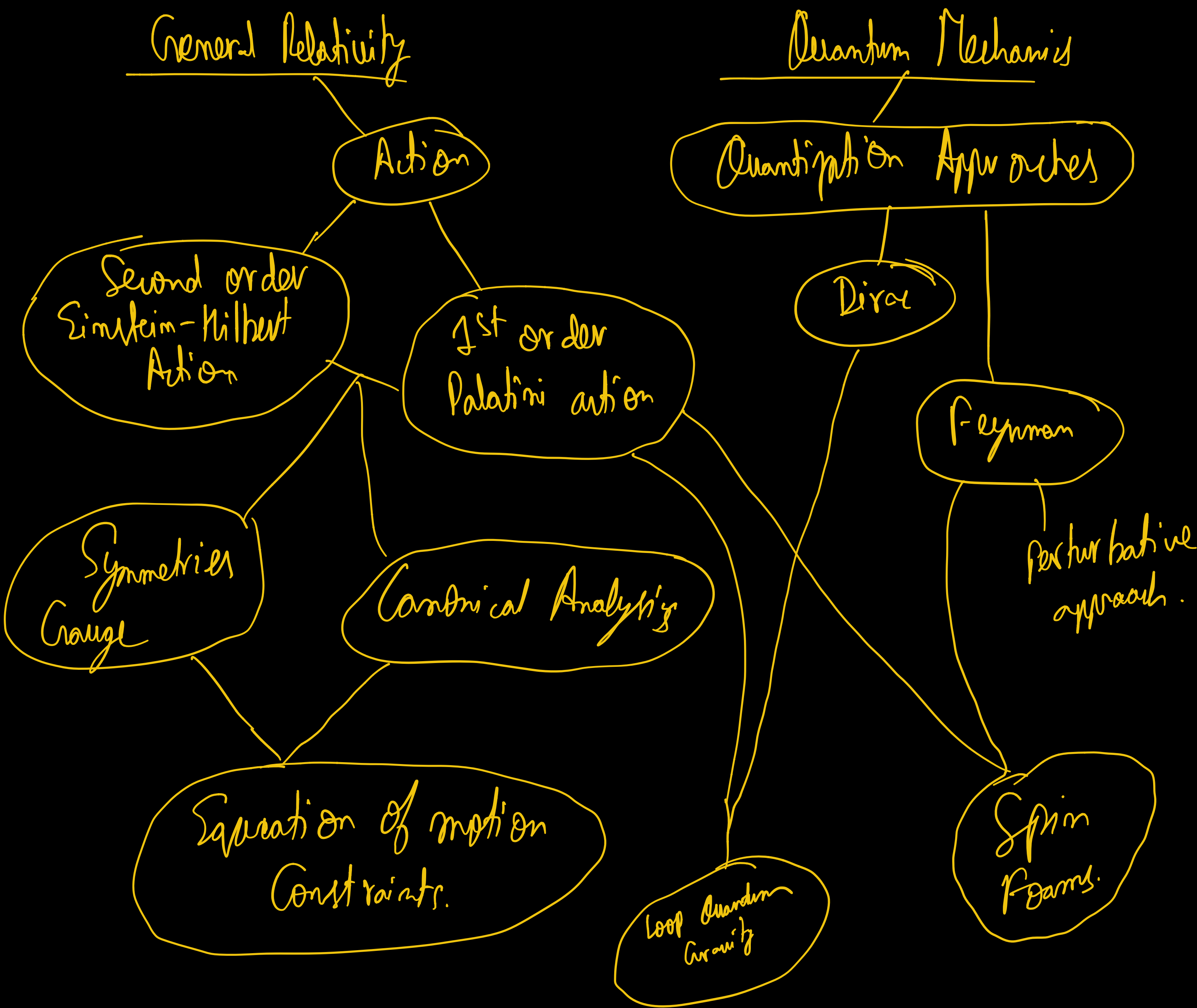
$$A|\Psi_n\rangle = 8\pi\gamma \cdot L^2_{\text{plank}} \sum_{p \in sur} \sqrt{j_p(j_p+1)}\, |\Psi_n\rangle$$

with $L_{\text{planck}} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35}$ m.



☆ Spin Foam Framework :

Transition amplitude between spin network state of LQG.




Gravity = Space Time

Quantum Gravity $\Longrightarrow$ Quantum Space Time.

Quantum space?

Quantum time?

→ Background for QFT.

Universe

LHC

Planck Scale

- **Top Down Approach**

  find theories consistent with what we know.

  (larger scale to smaller scale)

- **Bottom Up Approach.**

  make an intuition about Quantum Gravity; and then show we do get the large scale physics what we know.

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ex// Top Down approach (Asymptotic Safety)

- Takes QFT input

ex// LQG : standard tool of Quantization.

applied to Spacetime Geometry

- "Quantum Geometry"

☆ **Bottom Up Approaches.**

- Causal sets.
- Causal Triangulations.
- Non Commutative Spacetime.
- String Theory.
- 

Quantum Gravity

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# Lec 2: Constrained systems and Hamiltonian formalism: Gauge Symmetries.

3D Euclidean gravity with zero Cosmological Constant.

- Canonical formulation of Constrained systems.
- Gauge Transformation.

* Why are we interested in "Constrained System" ≡ Q1
* Can we relate constraints / gauge symmetries. ≡ Q2

Q1 $g_{\mu\nu}$ : 4D : $10 \longrightarrow 2$

3D : $6 \longrightarrow 0$ (Topological Theory)

$e^i_\mu$

$\mu \Rightarrow$ space time indices

$i \Rightarrow$ internal (Triad) indices)

Triad / connection.

Constraint on phase space : $\phi(q_i, p_i) = 0$

* constraints hold at every instant of time in the evolution of the system.

This means; constraint is conserved quantity.

Q2 Conserved Quantities $\overset{?}{\longleftrightarrow}$ Symmetries.

Lagrangian formalism :

Symmetry $\Longrightarrow$ Conserved quantities (Noether Theorem)

Symmetries $\begin{cases} \text{global} \\ \\ \text{gauge} \end{cases}$ : local and non-trivial transformation of the fields.

local $\Rightarrow$ gauge transformation do not affect any boundary value.

Non-trivial $\Rightarrow$ act non-trivially on extrema of action.


These two evolution are physically equivalent when they are related by gauge transformations.

☆ Read on Gauge Transformations

## Review of the Hamiltonian formalism

$$S = S[q_i] = \int L[q_i, \dot{q}_i]\, dt$$

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \quad , \; H(q_i, p_i) \equiv p_i \cdot \dot{q}_i - L$$

$$\dot{\rho} = \{\rho, H\}.$$

$\{q_i\} \to$ Configuration space

$\{(q_i, p_i)\} \to$ Phase space.

$$S = \int \left[ \sum p_i(q_i, \dot{q}_i)\, \dot{q}_i - H(p_i, q_i) \right] dt$$

Associate with this symplectic structure in terms of conjugate variables.

$$\{q_i, p_j\} = \delta_{ij}$$

Time Evaluation : $\dot{f} = \{f, H\}$ : Flow of Hamiltonian.

Any phase space function can generate flow

<u>Hamiltonian</u> formulation for a constrained system.

If is not invertible, then

$\Rightarrow \Psi_\alpha(q_i, p_i) = 0$ Primary Constraints.

Primary Constraint surface. (All the physical content of theory lie here)

$$H = H_0 + u^\alpha \Psi_\alpha$$

Add constraints in terms of Lagrange Multiplies $u^\alpha$.

$$\{f, H\} = \{f, H_0\} + \{f, u^\alpha \Psi_\alpha\}$$
$$= \{f, H_0\} + \underbrace{\{f, u^\alpha\} \Psi_\alpha}_{\text{zero on constraint surface because of } \Psi_\alpha = 0} + \{f, \Psi_\alpha\} u^\alpha$$

So; $\approx$ } Weak Equality (equal under constraint surface)

$$\boxed{\{g, H\} \approx \{g, H_0\} + \{g, \Psi_\alpha\} U^\alpha} \Rightarrow$$

Weak equality; i.e; it holds true on constraint surface.

$U^\alpha$: free or fixed?

Example: $L = \frac{1}{2}m(v_1^2 + v_2^2) - \frac{1}{3}a_3(a_1^2 + a_2^2 - r^2)$

$\Psi_1(a, p) \equiv p_3 = 0$

dim (phase space) = 6

$\downarrow$

dim (physical) = 2

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{3}a_3(a_1^2 + a_2^2 - r^2) + u\Psi_1$$

$$\{\Psi_1, H\} \approx 0 = \Psi_2$$

$\Longrightarrow$ If not zero; we impose this secondary constraint.

$\Psi_1 \longrightarrow \Psi_2$

$\Psi_2 \longrightarrow \Psi_3$

$\Psi_3 \longrightarrow \Psi_4$

$\dot{\Psi}_4 \rightarrow \underline{\underline{u = 0}}$

Ex2|| $L = \frac{1}{2} m (v_1 + v_2)^2 - V(q_1 + q_2)$

$\Psi(q,p) \equiv p_1 - p_2 \Rightarrow$ Primary Constraint.

$\dot{\Psi}(q,p) = \{\Psi, H\} = 0$ so done with constraints.

How to interpret these constraints?

Change of variable: $Q_1 = q_1 + q_2$ $\quad Q_2 = q_1 - q_2$

Lagrangian only acts on $Q_1$, not on $Q_2$.

Symmetry? $\quad \delta q_1 = u\{q_1, \Psi\} = u$ $\quad u \Rightarrow$ lagrange multiplier.

$\delta q_2 = u\{q_2, \Psi\} = -u$

$\delta(q_1 + q_2) = \delta q_1 + \delta q_2 = 0.$

$\delta Q_1 = 0$, but $\delta Q_2 = 2u$

So; $Q_1$ is physical quantity.

$Q_2$ is "internal" degree of freedom.

Constraints: primary, secondary, .....;
first class, second class.

Distinction between 1st class / 2nd class:

Consider a system with constraints $\Psi_\alpha$ (all constraints; primary, secondary...)

* **First Class**: If there is a subset of constraints $\varphi_a$ whose bracket with all constraints weakly vanishes,

i.e; $\{\varphi_a, \Psi_\alpha\} \approx 0$

Then the $\varphi_a$ are called first class constraints.

$\Rightarrow$ No restriction on associated Lagrange multiplier.

$\Rightarrow$ Gauge symmetries are generated by first class constraints.

* **Second class**: Remaining constraints denoted by $X_m$, then the sub-matrix defined as

$$\Delta_{mn} = \{X_m, X_n\} \text{ is invertible.}$$

$X_n$ are called 2nd class constraints.

$\Rightarrow$ Corresponding Lagrange multipliers are (weakly) fixed.

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let $\mathcal{C}$ be one of the constraint.

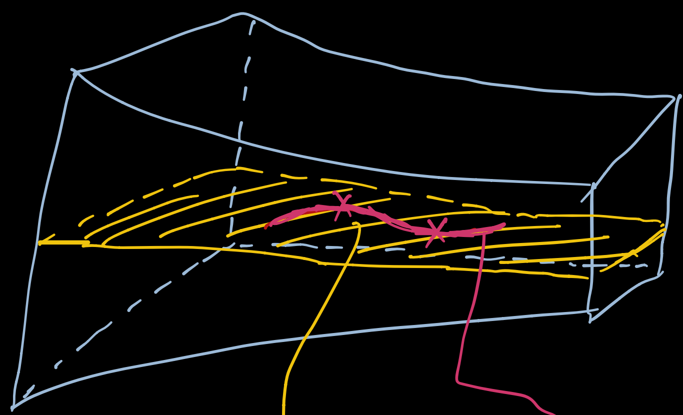
so $\{\mathcal{C}, H\} \approx 0$ : The flow generated by $H$ of $\mathcal{C}$ is zero (at least weakly)

or read as $\{H, \mathcal{C}\} \approx 0$

The flow of $H$ generated by $\mathcal{C}$ is zero.

..... and this is what we call Symmetry

# Constraints & Physical degrees of Freedom.



Constraint surface.

Gauge Orbits.

Kinematical Phase space.

$\downarrow$

Physical or Reduced Phase space

$$\dim(\text{Kinematical phase space}) - \#(1^{st}\text{ class}) \times 2 = \dim(\text{physical})$$

because of dividing by Gauge orbits.

# Quantum Gravity

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## Lec3 : Examples of parametrized particle : Canonical analysis, Physical phase space & Dirac observables. The Dirac program : case of the parametrized particle.

A totally constrained system – example of the parametrized particle.

Free non-relativistic particle in 1d.

$$S[q] = \int_{t_1}^{t_2} dt \cdot \frac{1}{2} m \dot{q}^2 \quad ; \dot{q} = \frac{dq}{dt} \quad \rightarrow q(t) = q(\tau) + \frac{P(\tau)}{m}(t-\tau)$$

$t$ independent variable

$$S_0[q(s), t(s)] = \int_{s_1}^{s_2} \left( \frac{1}{2} m \cdot \frac{q'^2}{t'} \right) \cdot ds$$

$q' = \frac{dq}{ds}$    $'$ denotes derivative w.r.t. $s$.

$S_0$ is invariant under $s \longrightarrow \tilde{s} = f(s)$.

**Canonical Analysis**    $P_t = \frac{\delta S}{\delta t'} = -m \cdot \frac{q'^2}{2t'^2}$ , $P_q = \frac{\delta S}{\delta q'} = m \cdot \frac{q'}{t'}$

We have a primary constraint $C$

$$C = P_t + \frac{P_q^2}{2m} = 0$$

Hamiltonian of the system : $H = P_t \cdot t' + P_q \cdot q' - L$

$\Rightarrow \underline{\underline{H = t' \cdot C}}$ Totally constrained system;
Hamiltonian is proportional to constraints.

$\underline{\underline{H = N(s) \cdot C}}$

flow generated by H?

$q' = \{q, H\} = N \cdot \frac{p_q}{m}$ | $t' = \{t, H\} = N.$

$p'_q = 0$ | $p'_q = 0$

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## Physical Phase Space: Dirac Observables

Gauge orbits of phase space variables generated by C.

$$\frac{dq}{ds} = \{q, C\} = \frac{p_q}{m} \Rightarrow q(s) = q + \frac{p_q}{m} \cdot s$$

$$\frac{dt}{ds} = \{t, C\} = 1 \rightarrow t(s) = s + t$$

Treating q & t on same footing.

$$\frac{dp_q}{ds} = \{p_q, c\} = 0 \Rightarrow p_q = p_{q_0}$$

$$\frac{dp_t}{ds} = \{p_t, C\} = 0 \Rightarrow p_t = -\frac{p_q^2}{2m}$$

2 independent <u>dirac observables</u>:
$\hookrightarrow \{F, C\} \approx 0$ F is dirac observable

To <u>construct a physical observables</u>:

- As many gauge fixing as constraints.
  $t(s) = \tau$

* Good gauge fixing. $\hookrightarrow$ J. Tambornino. } Read.

- $f(s) \longrightarrow F(\tau)$

phase space function — Associated physical observable.

ex) $q(s) \longrightarrow \left\{ \begin{array}{l} F_q^{\tau} = q_0 + \frac{p_0}{m}(\tau - t_0) \\ F_{p_q} = p_{q_0} \end{array} \right\}$

$\hookrightarrow$ These form complete set of Dirac Observables.

* At Kinematical level constraints are not imposed.

So; we form Poisson Bracket : $\{F^{\tau}_{q}, F^{\tau}_{p}\} = 1$

## Dirac Program

a) Find a representation of the phase space variable as operators acting in the Kinematical Hilbert space $\mathcal{H}_{Kin}$ satisfying the standard commutation relation:

$$\{\cdot,\cdot\} \longrightarrow -i/\hbar \, [\cdot,\cdot]$$

b) Promote the constraints to (self-adjoint) operators in $\mathcal{H}_{Kin}$

c) Characterize the phase space of solutions of the constraints

$\longrightarrow$ Inner product $\mathcal{H}_{phys}$.

d) Find a complete set of gauge invariant observables.

# Case of Parametrized Particle

(a) $\mathcal{H}_{kin} = \mathcal{L}^2[\mathbb{R}^2]$ functions depend on $q$ & $t$.

$\psi(q,t)$, $\langle \psi, \varphi \rangle = \int dq\, dt \cdot \overline{\psi}(q,t)\, \varphi(q,t)$

Kinematical Inner product.

$$q \longrightarrow \hat{q} \qquad \hat{q}\,\psi(q,t) = q \cdot \psi(q,t)$$

$$t \longrightarrow \hat{t} \qquad \hat{t}\,\psi(q,t) = t \cdot \psi(q,t)$$

$$p_q \longrightarrow \hat{p}_q \qquad \hat{p}_q\,\psi(q,t) = -i\hbar \cdot \frac{\partial}{\partial q}\psi(q,t)$$

$$p_t \longrightarrow \hat{p}_t \qquad \hat{p}_t\,\psi(q,t) = -i\hbar \frac{\partial}{\partial t}\psi(q,t)$$

(b) $$C = p_t + \frac{p_q^2}{2m} \longrightarrow \hat{C} = -i\hbar \cdot \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial q^2}$$

So, the equation becomes $\underline{\hat{C}\,|\psi\rangle = 0}$.

This is just Schrodinger's Equation.

(c) $$\psi_{phys}(q,t) = \exp\left(-\frac{i}{\hbar}\hat{h}t\right)\psi(q)$$

$\psi(q)$ initial wave function $\psi(q) = \psi(q, t = t_0)$

$$\hat{h} = -\frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial q^2}$$

<u>Inner Product</u>

$$\langle \psi^{(1)}_{phys} \,|\, \psi^{(2)}_{phys} \rangle_{kin} = \int \left( \int \overline{\psi}^{(1)}(q)\, \psi^{(2)}(q)\, dq \right) dt$$

$\mathcal{H}_{phy} = L^2(\mathbb{R})$

Physical Inner product.

$$\langle \Psi^{(1)}_{phy} | \Psi^{(2)}_{phy} \rangle_{phy} = \int \bar{\Psi}^{(1)}(t_{fixed}, q)\, \Psi^{(2)}(t_{fixed}, q)\, dq$$

Here we dont integrate over $t$ variable.

(d) Two independent observables:

$$\hat{O}_1 = \hat{q} - \frac{\hat{p}_q}{m}(\hat{t} - t_0)$$

$$\hat{O}_2 = \hat{P}_q$$

we can show $[\hat{O}_1, C] = 0$ , $[\hat{O}_2, C] = 0$

# Quantum Gravity

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## Lec 4: Actions of gravity; The Einstein-Hilbert action - 3D gravity - first order formalism of 3D gravity (triads & connection)

### Actions for Gravity.

Einstein-Hilbert action.

$$S = \frac{1}{K}\int R \cdot \sqrt{g} \cdot d^3x$$

$$\Rightarrow G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

$$\Longleftrightarrow R_{\mu\nu} = 0 \quad (\text{for } \dim(\text{spacetime}) \neq 2)$$

Proof of equivalence.

$$R^{\mu}{}_{\nu} - \frac{1}{2} R \cdot g^{\mu}{}_{\nu} = 0 \qquad \text{let } D = \dim(\text{spacetime})$$

$$R - \frac{1}{2} \cdot R \cdot D = 0 \quad \Rightarrow \left(1 - \frac{D}{2}\right) R = 0$$

$$\text{for } D \neq 2 \quad \Rightarrow \quad R = 0$$

$$\Rightarrow \quad G_{\mu\nu} = 0 \quad \Longrightarrow \quad R_{\mu\nu} = 0 \quad \text{for } D \neq 2.$$

$$\boxed{R_{\mu\nu} = 0 \Longrightarrow R_{\mu\nu\gamma\delta} = 0 \text{ in 3d spacetime}}$$

(flat solution)

because in 3d; $R_{\mu\nu}$ & $R_{\mu\nu\gamma\delta}$ have same no. of degrees of freedom.

$R_{\mu\nu\gamma\delta}$
- antisymmetric in 1st two indices.

$$\Rightarrow \left.\frac{n(n-1)}{2}\right|_{n=3} = 3 \text{ pairs.}$$

- antisymmetric in last two indices

$$\Rightarrow \left.\frac{n(n-1)}{2}\right|_{n=3} = 3 \text{ pairs.}$$

- Symmetric under $1^{st}$ pair & $2^{nd}$ pair.

so; $\left.\frac{n(n+1)}{2}\right|_{n=3} = 6$ independent components in 3D.

$R_{\mu\nu}$ : 6 independent components in 3D.

3D gravity = Topological Theory (no local degrees of freedom)

* We dont expect graviton field in 3D. because we dont expect local degrees of freedom.

* Topology of the manifold is characterized by the Fundamental group.

In 3D, $\mathcal{M} = \mathbb{R} \times \Sigma$

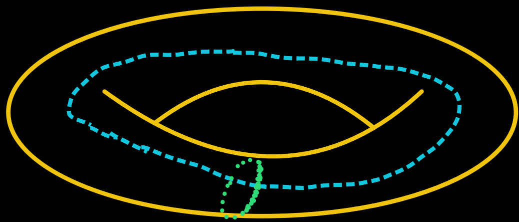
$\Sigma$ is 2 d surface.

The topology of 2d surface is very well classified.

Orientable surfaces are holomorphic to sphere $S^2$ or to connected sum of $g$ tori.

$g$ = genus of the surface.

$g(S^2) = 0$ ; $\pi_1(S^2) = 0$.



## First order formalism of 3D gravity.

$$g_{\mu\nu}(x) = e^i_{\mu}(x) \cdot e^j_{\nu}(x) \cdot \delta_{ij}$$

$\delta_{ij} \Rightarrow$ Minkowski Metric $\qquad i, j \in \{1, 2, \ldots, D\}$

At each point we raise orthonormal frame

In 3 dimension $e^i_{\mu}(x)$ are called Tetrad.

In 4 " $e^i_{\mu}(x)$ " " Quatriad.

In 3D $\qquad g_{\mu\nu} = e^i_{\mu} \cdot e^j_{\nu} \cdot \delta_{ij}$

$g_{\mu\nu}$ has 6 degrees of freedom.

for $e^i_{\mu}$ we get 9 degrees of freedom.

So we see that from going ; from metric to triad. we had introduced some irrelevant informations.

So we Additional Gauge Symmetry.

$e^i_\mu(x) = R^i_j(x) \cdot e^j_\mu(x)$ ; $R \in SO(3)$

so; we get addition $SO(3)$ gauge symmetry

This transformation keeps the metric unchanged.

Instead of the metric, we can work with triad.

Spin Connection // transport objects with internal indices and also to ensure the proper transformation behavior under internal rotation for the covariant derivative of objects with internal indices.

$\hookrightarrow$ Spin Connection.

$\omega_\mu{}^i{}_k$ $\quad D_\mu$

$i, k$ internal indices.

$\mu$ spacetime index.

$\phi^j$ is vector for internal indices, and scalar for spacetime index

$$D_\mu \phi^j = \partial_\mu \phi^i + \omega_\mu{}^i{}_k \phi^k$$

$$D_\mu \phi_j = \partial_\mu \phi_j - \omega_\mu{}^k{}_j \phi_k$$

$$D_\mu V^j{}_\nu = \partial_\mu V^j{}_\nu - \Gamma^\rho_{\mu\nu} V^j_\rho + \omega^j{}_{\mu k} \cdot V^k{}_\nu$$

Check the derivative of a scalar; $D_\mu \phi = \partial_\mu \phi$

It comes out to be ordinary partial derivative

Covariant Derivative, $\Gamma^{\mu}_{\nu\gamma}$ ; notation used $\nabla_\mu$

(levi civita connection)

Compatibility Condition : $\boxed{D_\mu e^i{}_\nu = 0}$

↪ This condition makes sure that

- Derivative commutes with the contraction.
- $$\omega^j{}_{\mu k} = -e^\nu{}_k \nabla_\mu e^j{}_\nu$$
  $$= e^\nu{}_k \Gamma^\rho_{\mu\nu} e^j_\rho - e^\nu{}_k \partial_\mu e^j{}_\nu \qquad (*)$$

Spin connection compatibity determined by $\Gamma$, $e$.

Proof!!

Conversely, we can also define a connection using (*) because its invertible. $\tilde{\Gamma}$

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = e^{\rho}{}_{j}\, \omega^{j}{}_{\mu k}\, e^{k}{}_{\nu} + e^{\rho}{}_{j}\, \partial_\mu e^{j}{}_{\nu}$$

Condition on $\omega$ / $\tilde{\Gamma} = \Gamma$ ?

We impose metric compatibility & torsion free on $\tilde{\Gamma}$.

$$\omega_{\mu j k} := \omega_{\mu}{}^{l}{}_{k} \cdot \delta_{lj}$$

Metric Compatible : $\tilde{\nabla}_\mu g_{\rho\nu} = 0$.

We know $\tilde{\nabla}_\mu (\delta_{ij}) = 0$

so; $\tilde{\nabla}_\mu (e^{\nu}{}_{i}\, e^{\rho}{}_{j}\, g_{\nu\rho}) = 0$

Now we use metric compatibility.

$$\tilde{\nabla}_\mu (e^{\nu}{}_{i}\, e^{\rho}{}_{j}\, g_{\nu\rho}) \underset{\substack{\uparrow \\ \text{metric combatible}}}{=\!=} -\omega_{\mu j i} - \omega_{\mu i j}$$

$\Rightarrow$ $\omega_{\mu j i} = -\omega_{\mu i j}$ Antisymmetric

So; we show that $D_\nu(\delta_{jk}) = 0$

$\Rightarrow$ Also metric compatible.

Torsion Free Condition of $\tilde{\Gamma}$

$$\Updownarrow$$

$\omega$ is also Torsion free.

ie; $$T^{j}_{\mu\nu} = \partial_\mu e^j{}_\nu - \partial_\nu e^j{}_\mu + \omega^j_{\mu k} e^k{}_\nu - \omega^j_{\nu k} e^k{}_\mu$$

<u>Remark</u>|| In $1^{st}$ order formulation, we will use $\omega_{\nu jk}$ (antisymmetric in $j$ and $k$)

but where Torsion does not vanish apriori; but vanishes as Equation of Motion.

Quantum Gravity Shoaib Akhtar

**Lec 5:** First order formalism of 3D gravity : action & symmetries, Canonical Analysis.

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Assume $\omega$ is antisymmetric.

We want to write Rieman Tensor as a function of anti-symmetric spin connection.

$$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v_\rho$$

$$= e^j_\rho (D_\mu D_\nu - D_\nu D_\mu) v_j$$

$$v_\rho = e^j_\rho v_j$$

so;

$$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = e^j_\rho \cdot \underbrace{F_{\mu\nu jk}}_{\text{Curvature 2-form of the connection } \omega.} \cdot e^{\sigma k} \cdot v_\sigma$$

$$F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu jl} \omega_\nu{}^l{}_k - \omega_{\nu jl} \cdot \omega_\mu{}^l{}_k$$

**In 3D**

enforce the antisymmetry of $\omega_{\nu jk}$, and of $F_{\mu\nu jk}$ by defining.

$$\omega_{\mu jk} = \varepsilon_{jlk} \omega_\mu{}^l$$

$$F_{\mu\nu jk} = \varepsilon_{jlk} F^l_{\mu\nu}$$

$\varepsilon_{ijk}$ totally antisymmetric ; $\varepsilon_{123} = 1$

$\varepsilon_{ijk}$ are also the structure constant for $SU(2)$, $SO(3)$ Lie Algebra.

$$D_\mu \phi_j = \partial_\mu \phi_j + \varepsilon_{jkl} \cdot \omega^k_\mu \phi^l$$

$$F^l_{\mu\nu} = \partial_\mu \omega^l_\nu - \partial_\nu \omega^l_\mu + \varepsilon^{ljk} \omega_{\mu j} \omega_{\nu k}$$ Curvature form

$$T^l_{\mu\nu} = \partial_\mu e^l_\nu - \partial_\nu e^l_\mu + \varepsilon^{ljk} \omega_{\mu j} e_{\nu k} - \varepsilon^{ljk} \omega_{\nu j} e_{\mu k}$$

Torsion form

Einstein-Hilbert Action: in terms of $(e, \omega)$

Start with $S = \frac{1}{k} \int \sqrt{g} \cdot R \cdot d^3x$

$$R = R_{\mu\nu} g^{\mu\nu} = R_{\mu\sigma\nu}{}^{\sigma} \cdot g^{\mu\nu}$$

$$\sqrt{\det g} = |\det(e_i)| = |\det(e^i)|^{-1}$$

$\det(e_i)$ → det of cotriad.

$\det(e^i)$ is determinant of triad.

$$\boxed{S = -\int e_{\sigma l} F^l_{\mu\nu}(\omega) \cdot \tilde{\varepsilon}^{\sigma\mu\nu} \cdot d^3x}$$

→ Invariant under
- Rotion in internal indices.
- Translation
- diffeomorphism.

$\tilde{\varepsilon}$ → tensor density with weight 1.

In this formulation we can couple fermions with gravity.

→ "BF" action. (Topological Theory in any dimensions)

Equation of Motion : Vanishing of curvature $F^{l}_{\mu\nu} = 0$

Vanishing of Torsion $T^{l}_{\mu\nu} = 0$

Diffeo can be written as translation & rotation in 3D gravity.

Symmetries:

- Rotation in the internal index.
- Diffeomorphism.
- Translation

Translation characterised by scalar field with one internal index $N^i$

$$\delta^{T}_{N} e^{j}_{\mu} = D_{\mu} N^{j}$$

$$\delta^{T}_{N} \omega^{j}_{\mu} = 0$$

We can check that the Action is indeed invariant.

## Canonical Analysis

Hamiltonian analysis.

* chose a slicing of the manifold.

$\mathcal{M} = \mathbb{R} \times \Sigma$ $\Sigma$ is 2d surface.

$\mu \to (0, a)$ (spacial coordinate)

(Time component)

$\Sigma$ is equal time surface.

$$S = \frac{1}{2}\int e_{\sigma l} F^{l}_{\mu\nu} \tilde{\varepsilon}^{\sigma\mu\nu} d^3x$$

Notation: Use $A$ instead of $\omega$ to denote Spin Connection.

$$S = \int e^{j}_{b}\, \partial_a A_{0j}\, \tilde{\varepsilon}^{ab} + \underbrace{e_{0j}}\, \frac{1}{2} F^{j}_{ab}\, \tilde{\varepsilon}^{ab}$$

$$+ \underbrace{A_0^{j}} \left(\partial_a e_{bj} + \varepsilon_{jlm} A^{l}_{a} e^{m}_{b}\right) \tilde{\varepsilon}^{ab}\; d^2x\, dt$$

Lagrange multiplier (because dont have time derivative term)

Canonical pair of variable

$$\{A^{j}_{a}(x), E^{b}_{k}(y)\} = \delta^{j}_{k}\, \delta^{b}_{a} \cdot \delta(x,y)$$

$$E^{a}_{j} = \frac{\delta S}{\delta(\partial_0 A^{j}_{a})} = \tilde{\varepsilon}^{ab} e^{j}_{b} = \text{Densitized Triad.}$$

So the constraints are;

$$\mathcal{F}^{j} = \frac{1}{2} F^{j}_{ab}\, \tilde{\varepsilon}^{ab}$$

$$\mathcal{G}^{j} = \frac{1}{2} T_{ab}{}^{j}\, \tilde{\varepsilon}^{ab}$$

$$\mathcal{F}^{j} = \tilde{\varepsilon}_{ab}\left(\partial_0 A^{j}_{0} + \frac{1}{2}\varepsilon^{j}{}_{kl} A^{k}_{a} A^{l}_{b}\right) \Big\}\ \text{Flatness constraint}$$

generate translational symmetry

$$\mathcal{G}^{j} = \partial_a E^{a}_{j} + \varepsilon_{jml} \cdot A^{l}_{a} E^{am} \Big\}\ \text{Gauss Constraint.}$$

Hamiltonian :

$$H = -\int d^2x \left( N^j \mathcal{F}_j + \Lambda^j \mathcal{G}_j \right)$$

Totally constrained system.

<u>Reference</u> ; ROMANOS article (canonical anyalsis)