

WEAK FIELD GRAVITY & HIGHER DIMENSIONS

An advanced course in Gravity

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These notes are consequence of my self study; and are mostly inspired from Prof. Ruth Gregory lectures on **Gravitational Physics**.

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Lec 1: Review of differential geometry: manifolds, tensors, differential forms.

Convention: Met signature $+ - - - \dots$

$$R^a_{bcd} = \Gamma^a_{bd,c} \text{ etc.}$$

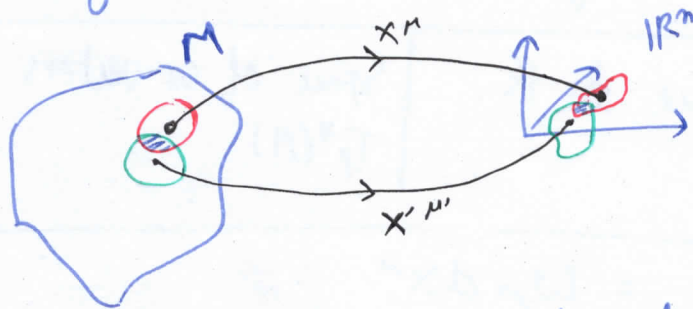
$$R_{ab} = R^c_{acb} ; \hbar = c = 1.$$

$$M_p = \sqrt{\frac{\hbar c}{8\pi G}} \sim 2 \times 10^{18} \text{ GeV}$$

$$T_p = \sqrt{\frac{8\pi G \hbar}{c^3}} \sim 5 \times 10^{-44} \text{ s.}$$

$$L_p = \sqrt{\frac{8\pi G \hbar}{c^3}} \sim 10^{-35} \text{ m}$$

Recall a manifold (spacetime) is a set of events that looks locally like $\mathbb{R}^n$. (Take ∞ ly differentiable)

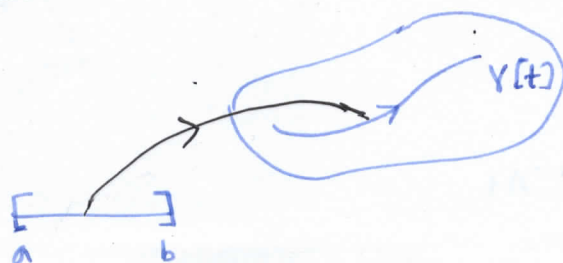


Charts map neighbourhoods of point of $\mathbb{R}^n$. on overlap of charts, transformations are ∞ ly differentiable. All charts form Atlas together. $x^\mu \rightarrow x'^\mu$ is ∞ ly differentiable.

Functions: $C^\infty(M)$ - collections of ∞ ly differentiable functions on M .

$f: M \rightarrow \mathbb{R}$.
 $x^\mu \mapsto f(x^\mu)$ (use charts to define differentiability)

Curves: These are map from $\mathbb{R}$ (or subset) to M .



$\gamma(t)$ is the curve on M .
 Can also write as $x^\mu(t)$ in local chart. (∞ ly differentiable)

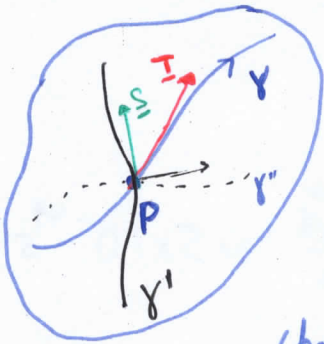
Vectors: Defined as tangent to curve at P .

(pg 2)

$$\underline{T}: C^\infty(M) \rightarrow C^\infty(M)$$

$$f \mapsto \frac{df}{dt} \quad \forall f \in C^\infty(M) \text{ at } P.$$

associated to curve $\gamma[t]$.



These operators form a vector space associated to P .

Add vectors by composing curves in chart, scale by taking $\gamma[\lambda t]: I \rightarrow \frac{I}{\lambda}$

Tangent Space $T_p(M)$

Collection of tangent spaces gives tangent bundle $T(M)$

Covectors: Maps from tangent space to $\mathbb{R}$. | Space of co-vectors $T_p^*(M)$

$$\underline{\omega}: T_p(M) \rightarrow \mathbb{R}.$$

In more familiar language $\underline{\omega} = \omega_\mu dx^\mu$

$$\underline{\omega}(\underline{V}) = \omega_\mu V^\mu.$$

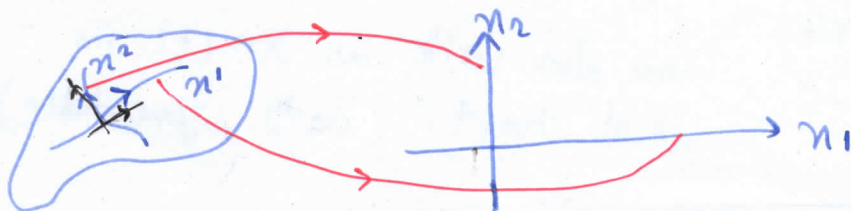
Note | Often in physics we write T^μ for a vector.

Strictly speaking $\underline{T} = T^\mu \frac{\partial}{\partial x^\mu}$ vector is an operator.

T^μ are components of $\underline{T}$ in a particular basis

Writing T^μ is called Abstract Index Notation (AIN - Penrose)

T^μ are components. $\frac{\partial}{\partial x^\mu}$ is basis; coordinate basis.



Another useful basis is orthonormal (o/n) basis.

Choose $\{e_a\}$ at points on M .

$$g(e_a, e_b) = \eta_{ab} \quad (g \text{ is an element of } T^*(M) \otimes T^*(M))$$

$$\text{In } \mathbb{R}^3, \quad e_r = \frac{\partial}{\partial r} ; e_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} ; e_\phi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

Spherical polars (r, θ, ϕ) in $\mathbb{R}^3$.

→ these are orthonormal basis.

$$\text{Coordinate basis: } \frac{\partial}{\partial x}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}$$

FORMS // A p-form is a rank p anti-symmetric co-tensor

~~Note~~ Note that we can differentiate a scalar to get a vector.

$$f \rightarrow \frac{\partial f}{\partial x^\mu} dx^\mu = df$$

dx^μ forms a basis for $T^*(M)$.

$$\left\langle dx^\mu, \frac{\partial}{\partial x^\nu} \right\rangle = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu$$

↑
dual basis

P-forms are built by taking anti-symmetric products of covectors: Wedge or " $\wedge$ ".

~~$$A, B \in T^*(M)$$~~

$$\omega, \eta \in T^*(M) ; \quad \omega \wedge \eta = \omega \otimes \eta - \eta \otimes \omega$$

$$\omega \wedge \eta = \omega \otimes \eta - \eta \otimes \omega$$

In components $[A^{(p)} \wedge B^{(q)}]_{a_1 \dots a_{p+q}}$

$$[A^{(p)} \wedge B^{(q)}]_{a_1 \dots a_{p+q}} = \frac{(p+q)!}{p! q!} A[a_1 \dots a_p B_{a_{p+1}} \dots a_{p+q}]$$

$$\underline{A}^{(p)} \wedge \underline{B}^{(q)} = (-1)^{pq} \underline{B}^{(q)} \wedge \underline{A}^{(p)}$$

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Note: cannot have a form of rank greater than dimension of manifold.
 $n = \dim(M)$.

Rank n form is unique up to a factor $\epsilon_{abcd} = \pm 1$
~~dependent~~ a, b, c, d odd or even permutation of 0123.

Coordinate Transformations

$x^\mu \rightarrow x'^\mu$ an overlapping chart.

$\underline{I}$ is geometric, so independent of chart; but components of $\underline{I}$ change.

$$\underline{I} = T^\mu \frac{\partial}{\partial x^\mu} = T'^\mu \frac{\partial}{\partial x'^\mu} = T^\mu \cdot \frac{\partial x'^\mu}{\partial x^\mu} \cdot \frac{\partial}{\partial x'^\mu}$$

$$\Rightarrow \underline{I} \quad T'^\mu = \frac{\partial x'^\mu}{\partial x^\mu} \cdot T^\mu \quad \underline{\text{covariant.}}$$

Similarly $\omega'_\mu = \frac{\partial x^\mu}{\partial x'^\mu} \omega_\mu$ covariant.

$$\frac{1}{n!} \epsilon_{\mu\nu\lambda\rho} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho = \text{circled}$$

$$(n=4) = \frac{1}{n!} \det\left(\frac{\partial x}{\partial x'}\right) \epsilon_{\mu'\nu'\lambda'\rho'} dx'^{\mu'} \wedge dx'^{\nu'} \wedge dx'^{\lambda'} \wedge dx'^{\rho'}$$

Construct n -form ??

$\hookrightarrow$ This is density because of the $\det\left(\frac{\partial x}{\partial x'}\right)$ factor.

(it is almost a tensor)

With a metric; we can define a 4 (or n)-form tensor.

$$\det g \xrightarrow{x \rightarrow x'} \det(g'_{\mu'\nu'}) = \det\left[\frac{\partial x^\mu}{\partial x'^{\mu'}} \cdot \frac{\partial x^\nu}{\partial x'^{\nu'}} \cdot g_{\mu\nu}\right]$$

$$= \det\left(\frac{\partial x}{\partial x'}\right)^2 \det g$$

Define $\epsilon_{\mu\nu\alpha\beta} = \sqrt{\det g} \epsilon_{\mu\nu\alpha\beta}$

$$= \sqrt{\det g'} \cdot \epsilon_{\mu'\nu'\alpha'\beta'} \cdot \frac{\partial x^{\mu'}}{\partial x^\mu} \cdot \frac{\partial x^{\nu'}}{\partial x^\nu} \cdot \frac{\partial x^{\alpha'}}{\partial x^\alpha} \cdot \frac{\partial x^{\beta'}}{\partial x^\beta}$$

With ϵ define Hodge dual $*$

$$* : \Lambda^{(p)} \longrightarrow \Lambda^{(n-p)}$$

$$\underline{A} \longmapsto * \underline{A}$$

In components

$$(*A)_{a_1 \dots a_{n-p}} = \frac{1}{p!} \epsilon_{a_1 \dots a_{n-p} a_{n-p+1} \dots a_n} A_{a_{n-p+1} \dots a_n}$$

Exterior Derivative

Can define differentiation on forms

$$d : \Lambda^p \longrightarrow \Lambda^{p+1}$$

1) d reduces to df on functions.

2) d is pseudo-Leibnizian

$$d(\underline{A}^{(p)} \wedge \underline{B}^{(q)}) = \underline{dA} \wedge \underline{B} + (-1)^p \underline{A} \wedge \underline{dB}$$

3) $d^2 = 0$ $\longleftrightarrow$ These three conditions are sufficient to find d .

In components.

$$(\underline{dA})_{a_1 \dots a_{p+1}} = \frac{(p+1)!}{p!} \partial_{[a_1} A_{a_2 \dots a_{p+1}]}$$

Lecture 2 | Lie Derivative & Symmetries, Killing Vectors, O/N basis.

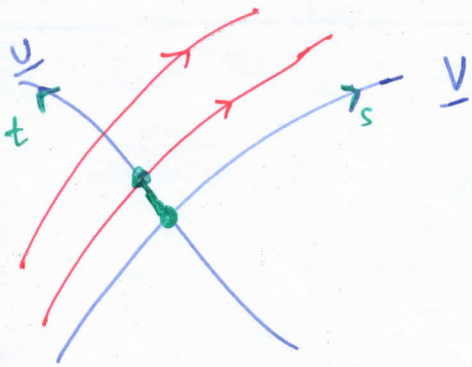
Recall, vector is an operator $f \rightarrow \frac{df}{dt}$ at a point P .

Consider

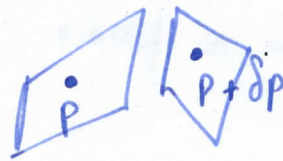
$$\begin{aligned} & (\underline{u} \underline{v} - \underline{v} \underline{u}) f \\ &= \left(u^\mu \frac{\partial}{\partial x^\mu} v^\nu \frac{\partial}{\partial x^\nu} - v^\mu \frac{\partial}{\partial x^\mu} u^\nu \frac{\partial}{\partial x^\nu} \right) f \\ &= \left(u^\mu v^\nu_{, \mu} - v^\mu u^\nu_{, \mu} \right) \frac{\partial}{\partial x^\nu} f \end{aligned}$$

→ these are components of new vector; we call it $[\underline{u}, \underline{v}]^\nu$

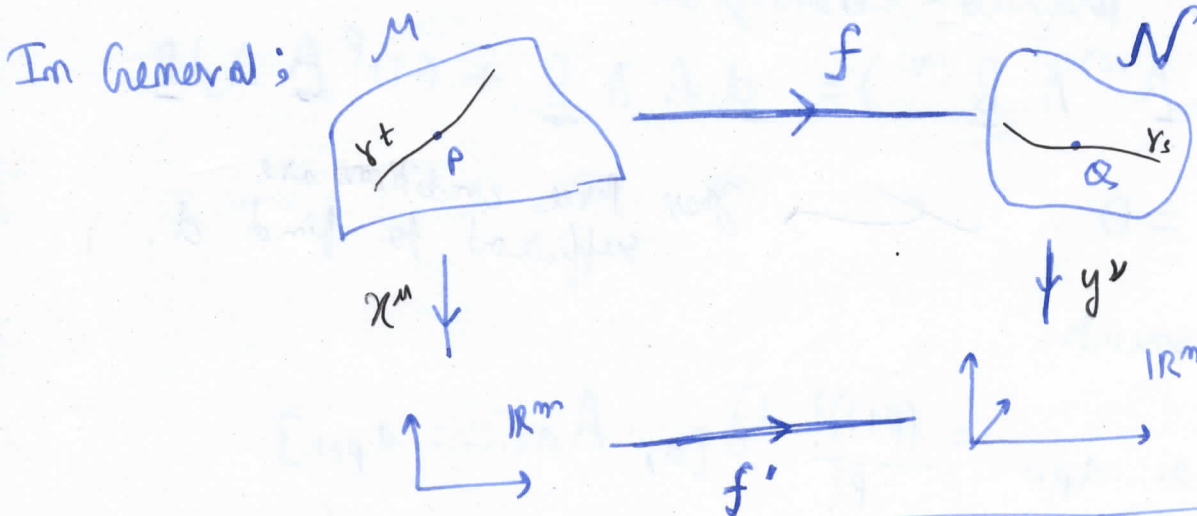
and it is Lie Bracket / commutator of $\underline{u}$ & $\underline{v}$.



Recall; $Q' = \lim_{\delta t \rightarrow 0} \frac{Q_{t+\delta t} - Q_t}{\delta t}$



? use $\underline{u}$ to transport Q 's?



If $\dim M = \dim N$; f 1-1 and onto and C^∞ etc.
Then we say f is diffeomorphism, and regard M ,
and N as the same

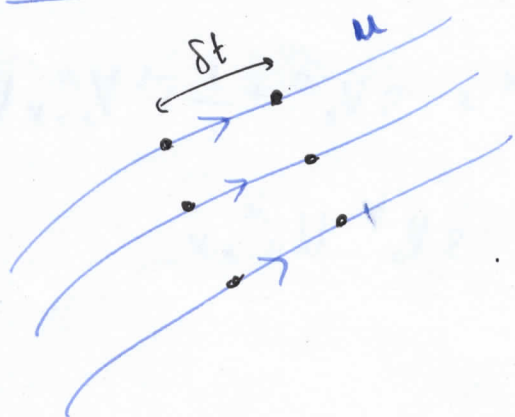
Push forward: $f_*: T_p(M) \rightarrow T_q(N)$

$$\underline{I} = \frac{d}{dt} \text{ on } \gamma_t \longmapsto \underline{S} = \frac{d}{ds} \text{ on } \gamma_s$$

Pull-back: $f^*: T_q^*(N) \rightarrow T_p^*(M)$

via preimage: via $\langle f^*(\underline{\omega}) | \underline{I} \rangle = \langle \underline{\omega} | f_*(\underline{I}) \rangle$

Return to u

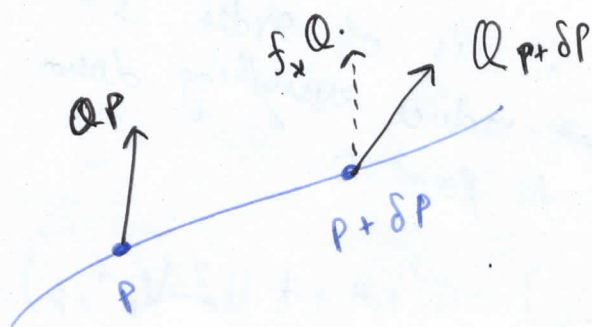


Integral curves ($\underline{u}$ tangent)

Define a coordinate transformation:

$$X_t^\mu = X_0^\mu + \delta t \cdot u^\mu$$

Can now compare at



check for vector field $\underline{V}$

$$\begin{aligned} \underline{V}_{t+\delta t} &= \underline{V}(X_0^\mu + \delta t u^\mu) \\ &= (V_0^\nu + V_0^\nu{}_{,\mu} u^\mu \delta t) \cdot \frac{\partial}{\partial X_0^\nu} \end{aligned}$$

Compare ~~to~~ ~~push~~ push forward.

$$\begin{aligned} \underline{V}_* &= \frac{\partial X_t^\mu}{\partial X_0^\nu} \cdot V_0^\nu \cdot \frac{\partial}{\partial X_0^\mu} \\ &= \left(V_0^\mu + \delta t \cdot \frac{\partial u^\mu}{\partial X^\nu} \cdot V_0^\nu \right) \frac{\partial}{\partial X_0^\mu} \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{\underline{u}} \underline{V} &= \lim_{\delta t \rightarrow 0} \frac{\underline{V}_{t+\delta t} - \underline{V}_t}{\delta t} = (u^\nu V_0^\mu{}_{,\nu} - V_0^\nu u^\mu{}_{,\nu}) \frac{\partial}{\partial X_0^\mu} \\ &= [\underline{u}, \underline{V}]^\mu \cdot \frac{\partial}{\partial X_0^\mu} \end{aligned}$$

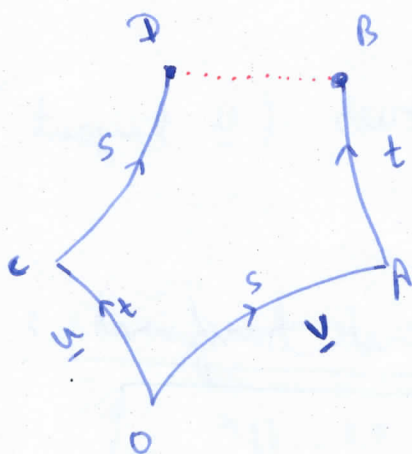
This method give $\mathcal{L}_u$ in general.

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$$\mathcal{L}_u \omega = (u^\nu \omega_{\mu,\nu} + u^\nu_{,\mu} \omega_\nu) dx^\mu$$

$$(\mathcal{L}_u g)_{\mu\nu} = g_{\mu\nu,\lambda} u^\lambda + g_{\mu\lambda} u^\lambda_{,\nu} + g_{\lambda\nu} u^\lambda_{,\mu}$$

Geometrical Significance



$$\vec{BD} \propto [u, v]$$

we expect this

$$\text{use } X_A^\mu = X_0^\mu + s V_0^\mu + \frac{1}{2} s^2 V_0^\mu{}_{,\nu} V_0^\nu$$

$$U_A^\mu = U_0^\mu + s V_0^\nu U_0^\mu{}_{,\nu}$$

find $\vec{BD} \approx X_A^\mu - X_B^\mu$

$$X_D^\mu = X_C^\mu + s V_C^\mu + \frac{1}{2} s^2 V_0^\nu V_0^\mu{}_{,\nu}$$

since it is of order s^2
we can reduce everything down
to point 0.

$$= (\underline{X_0^\mu} + \underline{t U_0^\mu} + \underline{\frac{1}{2} t^2 U_0^\nu U_0^\mu{}_{,\nu}}) + \underline{s(V_0^\mu + t U_0^\nu V_0^\mu{}_{,\nu})} + \underline{\frac{1}{2} s^2 V_0^\nu V_0^\mu{}_{,\nu}}$$

X_B^μ swaps $s \leftrightarrow t$, $u \leftrightarrow v$

$$X_D^\mu - X_B^\mu = st [U^\nu V^\mu{}_{,\nu} - V^\nu U^\mu{}_{,\nu}]$$

$$= st [\underline{u}, \underline{v}]^\mu$$

If $\underline{u}, \underline{v}$ tangent to co-ordinate lines, $[\underline{u}, \underline{v}] = 0$. (pg 9)

Another geometric application of $\mathcal{L}$ is to symmetries of a spacetime.

Definition A Killing Vector is a vector field along which the metric is ~~not~~ invariant.

$$\mathcal{L}_k g = 0$$

(It represents the symmetries of spacetime)

$$(\mathcal{L}_k g)_{\mu\nu} = g_{\mu\nu, \lambda} k^\lambda + g_{\mu\lambda} k^\lambda{}_{,\nu} + g_{\lambda\nu} k^\lambda{}_{,\mu}$$



$$\nabla_{(\mu} k_{\nu)} = 0$$

e.g. 1 $ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$

metric independent of φ .

define $k_\varphi = \frac{\partial}{\partial \varphi} : k^\varphi = 1$

$$\mathcal{L}_k g_{\mu\nu} = k^\lambda g_{\mu\nu, \lambda} = 0$$

+ 0 + 0

Aside: also have 2 more killing vectors.

$$\cos\varphi \csc\theta \cdot \partial_\theta + \sin\varphi \cdot \partial_\varphi = k_1$$

$$\sin\varphi \cdot \csc\theta \partial_\theta - \cos\varphi \partial_\varphi = k_2$$

} corresponds to rotation along different axis.

These three killing vectors obey $SO(3)$ algebra.

$\therefore$ Vectors obey $SO(3)$ algebra:

$$[k_i, k_j] = \epsilon_{ijk} k_k$$

The Algebra of Killing Vectors is reflecting the underlying symmetries of that particular geometry.

$\mathcal{L}$ works because it has a method of connecting tangent spaces.

The connection is a structure on tangent bundle connecting tangent spaces. Tells us how a basis $\{\underline{e}_a\}$ at p links to $\{\underline{e}_a\}$ at q .

e.g. $\mathbb{R}^2$ $\{x, y\} \longleftrightarrow \{r, \theta\}$
Standard Cylindrical
Cartesian Polars.

$\underline{e}_x = \frac{\partial}{\partial x}$, $\underline{e}_y = \frac{\partial}{\partial y}$ both coordinate & o/n.

$\frac{\partial}{\partial r} = \frac{x}{r} \underline{e}_x + \frac{y}{r} \underline{e}_y = \underline{e}_r$ $\|\underline{e}_r\| = 1$

$\frac{\partial}{\partial \theta} = -y \underline{e}_x + x \underline{e}_y$; but $\|\frac{\partial}{\partial \theta}\| = r$

so; ~~not~~ orthogonal, coordinate , ~~but not o/n~~.
but not o/n. ~~o/n is orthogonal~~

o/n basis useful.

o/n $\equiv$ orthonormal.

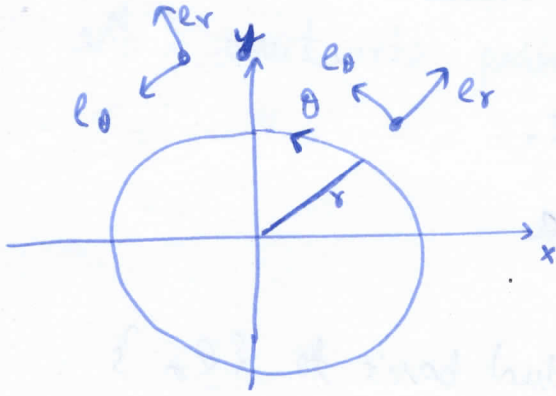
$\underline{e}_r = \frac{\partial}{\partial r}$; $\underline{e}_\theta = \frac{1}{r} \frac{\partial}{\partial \theta}$

we could write equivalently in matrix format -

$\underline{e}_a = \begin{bmatrix} 1 & 0 \\ 0 & 1/r \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix}$
↑ ↑ ↑
o/n basis $\underline{e}_a$ coordinate basis

→ this gives us transformation between o/n & coordinate basis. In some sense it is square root of the metric.

$$\eta^{\hat{a}\hat{b}} \underline{e}_{\hat{a}}^{\mu} \underline{e}_{\hat{b}}^{\nu} \sim g^{\mu\nu}$$



$\underline{e}_x, \underline{e}_y$ same throughout $\mathbb{R}^2$.
 $\underline{e}_r, \underline{e}_\theta$ not.

$$\underline{e}_r = \cos\theta \cdot \underline{e}_x + \sin\theta \cdot \underline{e}_y$$

$$\underline{e}_\theta = -\sin\theta \cdot \underline{e}_x + \cos\theta \cdot \underline{e}_y$$

these are
 giving ~~connecting~~ ~~between~~
 connection between
 tangent spaces.

$$\delta \underline{e}_r = \underline{e}_\theta \cdot \delta\theta$$

$$\delta \underline{e}_\theta = -\underline{e}_r \cdot \delta\theta$$

$\therefore$ $\rightarrow$ is connection;

$$\Gamma_{\theta\theta}^r$$

$$\Gamma_{\theta r}^\theta$$

} non zero.

Lecture 3 Cartan's formalism: connection & curvature

The covariant derivatives works by adding structure (the connection) to the tangent bundle.

$$\nabla \underline{e}_b = \Gamma_{ab}^c \underline{e}_c \otimes \underline{\omega}^a$$

$\uparrow$ derivation
 $\nwarrow$ basis vector
 $\nwarrow$ connection components - scalars (set of numbers)
 $\uparrow$ basis
 $\nwarrow$ dual basis to $\{\underline{e}_a\}$

or, $\Gamma_{bc}^a = \langle \underline{\omega}^a | \nabla_b \underline{e}_c \rangle$

$\uparrow$
 $\underline{e}_b \cdot \nabla$

Dual basis: $\langle \underline{\omega}^a | \underline{e}_b \rangle = \delta_b^a$

Definition ∇ is a derivation that commutes with contractions

- Leibnizian
- Reduces to d on functions.

We are used to $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\lambda}^\nu V^\lambda$

For a vector,

$$\begin{aligned} \nabla \underline{V} &= \nabla (V^a \underline{e}_a) \stackrel{(2)}{=} (\nabla V^a) \underline{e}_a + (\nabla \underline{e}_a) V^a \\ &\stackrel{(3)}{=} dV^a \underline{e}_a + \Gamma_{ca}^b \underline{e}_b \underline{\omega}^c V^a \end{aligned}$$

$\uparrow$
 $T^*(M) \otimes T(M)$

$$\nabla \underline{V} = (V^a_{,c} + \Gamma_{cb}^a V^b) \underline{e}_a \underline{\omega}^c$$

In G.R. we use Levi-Civita connection:
 "Symmetric", $\nabla g = 0$.

Torsion ~~$\nabla_{\underline{u}} \underline{v} - \nabla_{\underline{v}} \underline{u} = [\underline{u}, \underline{v}]$~~ $T(\underline{u}, \underline{v}) = \nabla_{\underline{u}} \underline{v} - \nabla_{\underline{v}} \underline{u} - [\underline{u}, \underline{v}]$

almost the anti-symmetric part of connection.

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} - C^a_{bc}$$

where: $C^a_{bc} = \langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle$

are the structure constants of the basis $\{\underline{e}_a\}$.

$\therefore$ Torsion is a tensor.

Definition The connection 1-forms, or spin-connection are defined as $\underline{\theta}^a_c = \Gamma^a_{bc} \underline{\omega}^b$

The indices a & c are not in general tensor indices. They are just labels.

Allows us to differentiate spinors.

Cartan relates spin connection (& torsion) to derivatives of ~~$\{\underline{\omega}^a\}$~~ $\{\underline{\omega}^a\}$

• For metric connection $d g_{ab} = \underline{\theta}_{ab} + \underline{\theta}_{ba}$

Proof $d g_{ab} = \underline{\nabla} g_{ab} = \underline{\nabla} \langle g | \underline{e}_a, \underline{e}_b \rangle$

because: g_{ab} are just scalars.

$$= \langle g | \underline{e}_a, \underline{\nabla} \underline{e}_b \rangle + \langle g | \underline{\nabla} \underline{e}_a, \underline{e}_b \rangle$$

(when it hits g : it gives zero)

$$= \Gamma^c_{da} \langle g | \underline{e}_c, \underline{e}_b \rangle \underline{\omega}^d + \Gamma^c_{db} \langle g | \underline{e}_a, \underline{e}_c \rangle \underline{\omega}^d$$

$$= g_{cb} \underline{\theta}^c_a + g_{ac} \underline{\theta}^c_b = \underline{\theta}_{ba} + \underline{\theta}_{ab}$$

This says that metric connection has symmetric spin connection.

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If $\{\underline{e}_a\}$ is an o/n basis.

$$g_{ab} = \eta_{ab} \Rightarrow dg_{ab} = 0$$

Cartan's 1st Structural Equation

$$\text{Consider } \underline{\theta}^a_c \wedge \underline{\omega}^c = \Gamma^a_{bc} \underline{\omega}^b \wedge \underline{\omega}^c$$

$$\underline{\theta}^a_c \wedge \underline{\omega}^c = \frac{1}{2} (\Gamma^a_{bc} - \Gamma^a_{cb}) \underline{\omega}^b \wedge \underline{\omega}^c$$

$$\begin{aligned} \underline{\omega}^r &= dr \\ \underline{\omega}^\theta &= r d\theta \\ \underline{\omega}^\varphi &= r \sin\theta d\varphi \end{aligned} \quad \left\| \quad \begin{aligned} \underline{e}_r &= \frac{\partial}{\partial r} = (1, 0, 0) \text{ in coordinate basis} \\ \underline{e}_\theta &= \frac{1}{r} \frac{\partial}{\partial \theta} = (0, \frac{1}{r}, 0) \\ \underline{e}_\varphi &= \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} = (0, 0, \frac{1}{r \sin\theta}) \end{aligned}$$

$$\begin{aligned} \|\underline{e}_r\| &= 1 \\ \|\underline{e}_\theta\| &= 1 \\ \|\underline{e}_\varphi\| &= 1 \end{aligned}$$

$$g(\underline{e}_a, \underline{e}_b) = \delta_{ab}$$

$$\underline{V} = V_r \underline{e}_r + V_\theta \underline{e}_\theta$$

$$\|\underline{V}\| = \sqrt{V_r^2 + V_\theta^2}$$

$$\underline{\theta}^a_c \wedge \underline{\omega}^c = \frac{1}{2} (T^a_{bc} + C^a_{bc}) \underline{\omega}^b \wedge \underline{\omega}^c$$

$$= \underline{T}^a + \frac{1}{2} \langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle \underline{\omega}^b \wedge \underline{\omega}^c$$

↑
Torsion tensor

$$T^a = \frac{1}{2} T^a_{bc} \underline{\omega}^b \wedge \underline{\omega}^c$$

$$\underline{\theta}^a_c \wedge \underline{\omega}^c = T^a + \frac{1}{2} \langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle \underline{\omega}^b \wedge \underline{\omega}^c$$

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$$\text{but, } \langle d\omega^a | \underline{e}_b, \underline{e}_c \rangle = \underline{e}_b (\langle \omega^a | \underline{e}_c \rangle) - \underline{e}_c (\langle \omega^a | \underline{e}_b \rangle) - \langle \omega^a | [\underline{e}_b, \underline{e}_c] \rangle$$

from identity.

$$\langle d\omega^a | \underline{e}_b, \underline{e}_c \rangle = \underline{e}_b (\langle \omega^a | \underline{e}_c \rangle) - \underline{e}_c (\langle \omega^a | \underline{e}_b \rangle) - \langle \omega^a | [\underline{e}_b, \underline{e}_c] \rangle$$

$\xrightarrow{\text{SO}} \quad \quad \quad \xrightarrow{\text{SO}}$

So;

$$\langle d\omega^a | \underline{e}_b, \underline{e}_c \rangle = - \langle \omega^a | [\underline{e}_b, \underline{e}_c] \rangle$$

$$\begin{aligned} \underline{T}^a &= \underline{\theta}_c^a \wedge \underline{\omega}^c - \frac{1}{2} \langle \omega^a | [\underline{e}_b, \underline{e}_c] \rangle \underline{\omega}^b \wedge \underline{\omega}^c \\ &= \underline{\theta}_c^a \wedge \underline{\omega}^c + \frac{1}{2} \langle d\omega^a | \underline{e}_b, \underline{e}_c \rangle \underline{\omega}^b \wedge \underline{\omega}^c \end{aligned}$$

$\underline{T}^a = d\omega^a + \underline{\theta}_c^a \wedge \underline{\omega}^c$

CI (Cartan 1st Structural equation)

If torsion vanishes, then $\underline{\theta}_c^a \wedge \underline{\omega}^c = -d\omega^a$

and $\underline{\theta}_{(a b)} = 0$

Curvature // defined as commutator of derivatives...

$$\underline{R}(\underline{u}, \underline{v}) \underline{W} = [\underline{\nabla}_u \underline{\nabla}_v - \underline{\nabla}_v \underline{\nabla}_u - \underline{\nabla}_{[\underline{u}, \underline{v}]}] \underline{W}$$

$$\underline{R} : T_p(M) \times T_p(M) \times T_p(M) \rightarrow T_p(M)$$

physically $\underline{R}$ represents tidal forces.

In components

(pg 16)

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce} \Gamma^e_{db} - \Gamma^a_{de} \Gamma^e_{cb} - C^e_{cd} \Gamma^a_{eb}$$

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce} \Gamma^e_{db} - \Gamma^a_{de} \Gamma^e_{cb} - C^e_{cd} \Gamma^a_{eb}$$

Cartan's 2nd Structural equation

Curvature 2-form is.

$$\underline{R}^a_b = -\frac{1}{2} R^a_{bcd} \underline{\omega}^c \wedge \underline{\omega}^d$$

$$\textcircled{C2} \quad \underline{R}^a_b = d\theta^a_b + \theta^a_c \wedge \theta^c_b$$

e.g. $\mathbb{R}^2$ in polar coordinates

$$dr^2 + r^2 d\theta^2$$

$$\underline{\omega}^r = dr \quad ; \quad d\underline{\omega}^r = 0$$

$$\underline{\omega}^\theta = r d\theta \quad ; \quad d\underline{\omega}^\theta = dr \wedge d\theta = \frac{1}{2} \underline{e}_r \wedge \underline{e}_\theta = -\frac{1}{2} d\theta \wedge dr$$

$$\textcircled{C1}: \quad d\underline{\omega}^a = -\theta^a_b \wedge \underline{\omega}^b = -\frac{1}{r} r d\theta \wedge dr = -\frac{1}{r} \underline{\omega}^\theta \wedge \underline{\omega}^r$$

$$\theta^a_r = \frac{1}{r} \underline{\omega}^\theta \quad ; \quad \text{no } \underline{R}^a_b \quad \text{ie: } \underline{R}^a_b = 0.$$

$$S^2 = d\theta^2 + \sin^2 \theta d\varphi^2$$

$$\underline{\omega}^\theta = d\theta \quad \underline{\omega}^\varphi = \sin \theta d\varphi$$

$$d\underline{\omega}^\theta = 0 \quad d\underline{\omega}^\varphi = \cot \theta \underline{\omega}^\theta \wedge \underline{\omega}^\varphi$$

$$\underline{\theta}^\varphi_\theta = \cot \theta \cdot \underline{\omega}^\varphi = \cot \theta d\varphi \quad | \quad \underline{R}^\varphi_\theta = d\underline{\theta}^\varphi_\theta = -\sin \theta d\theta \wedge d\varphi = \underline{\omega}^\varphi \wedge \underline{\omega}^\theta$$

$$R^{\hat{\varphi}}_{\hat{\theta} \hat{\varphi} \hat{\theta}} = 1$$

Lec 4: Spherically Symmetric Spacetimes, Schwarzschild - (A)ds
black holes.

Look at Static Spherically Symmetric metric.

$$ds^2 = A^2(r)dt^2 - B^2(r)dr^2 - C^2(r) \underbrace{[d\theta^2 + \sin^2\theta d\varphi^2]}_{d\Omega_{II}^2}$$

To apply Cartan; we need to know orthonormal basis of 1-forms.

$$\begin{aligned}\omega^{\hat{t}} &= A(r)dt \\ \omega^{\hat{r}} &= B(r)dr \\ \omega^{\hat{\theta}} &= C(r)d\theta \\ \omega^{\hat{\varphi}} &= C(r)\sin\theta d\varphi\end{aligned}$$

$$d\omega^{\hat{r}} = 0$$

Spin connection

$$\begin{aligned}d\omega^{\hat{t}} &= A' dr \wedge dt \\ &= -\frac{A'}{AB} \omega^{\hat{t}} \wedge \omega^{\hat{r}} \\ &= \theta^{\hat{t}}_{\hat{r}} \wedge \omega^{\hat{r}}\end{aligned}$$

$$\Rightarrow \theta^{\hat{t}}_{\hat{r}} = \frac{A'}{AB} \omega^{\hat{t}}$$

$$d\omega^{\hat{\theta}} = C' dr \wedge d\theta \Rightarrow \theta^{\hat{\theta}}_{\hat{r}} = \frac{C'}{CB} \omega^{\hat{\theta}}$$

$$d\omega^{\hat{\varphi}} = C' \sin\theta dr \wedge d\varphi + C \cos\theta d\theta \wedge d\varphi$$

$$\begin{aligned}\Rightarrow &= -\theta^{\hat{\varphi}}_{\hat{\theta}} \wedge \omega^{\hat{\theta}} \Rightarrow \theta^{\hat{\varphi}}_{\hat{r}} = \frac{C'}{CB} \omega^{\hat{\varphi}} \\ &\theta^{\hat{\varphi}}_{\hat{\theta}} = \frac{1}{C} \cot\theta \omega^{\hat{\varphi}}\end{aligned}$$

~~Aside: $\theta^{\hat{r}}_{\hat{\theta}} = \eta^{\hat{r}\hat{\theta}}$~~

$$\begin{aligned}\text{Aside: } \theta^{\hat{r}}_{\hat{\theta}} &= \eta^{\hat{r}\hat{\theta}} \theta_{\hat{\theta}\hat{\theta}} \\ &= -\eta^{\hat{r}\hat{\theta}} \theta_{\hat{\theta}\hat{r}} = -\eta^{\hat{r}\hat{\theta}} \eta_{\hat{\theta}\hat{r}} \theta^{\hat{r}}_{\hat{r}}\end{aligned}$$

$$\Rightarrow \hat{\theta}^{\hat{r}} = -\hat{\theta}^{\hat{\theta}}$$

pg 18

Curvature 2-form.

$$\begin{aligned} \hat{R}^{\hat{t}}_{\hat{r}} &= d\hat{\theta}^{\hat{t}}_{\hat{r}} \\ &= d\left(\frac{A'}{B} dt\right) \end{aligned}$$

$$\hat{R}^a_b = d\hat{\theta}^a_b + \hat{\theta}^a_c \wedge \hat{\theta}^c_b$$

Working in orthonormal basis: $g = \eta$

$$\Rightarrow dg_{\hat{a}\hat{b}} = d\eta_{\hat{a}\hat{b}} = 0$$

$$\Rightarrow \boxed{\hat{\omega}_{\hat{a}\hat{b}} + \hat{\theta}_{\hat{b}\hat{a}} = 0}$$

$$= \left(\frac{A'}{B}\right)' dr \wedge dt = \boxed{-\frac{1}{AB} \left(\frac{A'}{B}\right)' \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}} = \hat{R}^{\hat{t}}_{\hat{r}}}$$

$$\hat{R}^{\hat{t}}_{\hat{\theta}} = \hat{\theta}^{\hat{t}}_{\hat{r}} \wedge \hat{\theta}^{\hat{r}}_{\hat{\theta}} = \frac{-A'C'}{AB^2C} \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\theta}}$$

$$\Rightarrow \boxed{\hat{R}^{\hat{t}}_{\hat{\theta}} = \frac{-A'C'}{AB^2C} \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\theta}}}$$

$$\& \hat{R}^{\hat{t}}_{\hat{\varphi}} = -\frac{A'C'}{AB^2C} \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\varphi}}$$

$$\hat{R}^{\hat{\theta}}_{\hat{r}} = d\hat{\theta}^{\hat{\theta}}_{\hat{r}} + \underbrace{\hat{\theta}^{\hat{\theta}}_{\hat{c}} \wedge \hat{\theta}^{\hat{c}}_{\hat{r}}}_{\rightarrow \text{gives zero}} = d\left(\frac{C'}{B} d\theta\right)$$

$$\boxed{\hat{R}^{\hat{\theta}}_{\hat{r}} = -\left(\frac{C'}{B}\right)' \frac{1}{CB} \underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{r}}}$$

$$\hat{R}^{\hat{\varphi}}_{\hat{r}} = d\hat{\theta}^{\hat{\varphi}}_{\hat{r}} + \hat{\theta}^{\hat{\varphi}}_{\hat{\theta}} \wedge \hat{\theta}^{\hat{\theta}}_{\hat{r}}$$

$$= d\left(\frac{C'}{B} \sin\theta d\varphi\right) + \frac{1}{C} \cos\theta \underline{\omega}^{\hat{\varphi}} \wedge \frac{C'}{CB} \underline{\omega}^{\hat{\theta}}$$

$$= -\left(\frac{C'}{B}\right)' \frac{1}{CB} \underline{\omega}^{\hat{\varphi}} \wedge \underline{\omega}^{\hat{r}} + \frac{C'}{B} \cos\theta d\theta \wedge d\varphi + \frac{C'}{B} \cos\theta d\varphi \wedge d\theta$$

$$= -\left(\frac{C'}{B}\right)' \frac{1}{CB} \underline{\omega}^{\hat{\varphi}} \wedge \underline{\omega}^{\hat{r}}$$

$$\begin{aligned} \underline{R}^{\hat{\varphi}}_{\hat{\theta}} &= \underline{d} \underline{\varphi}_{\hat{\theta}} + \underline{\varphi} = \underline{d}(\cos\theta d\varphi) + \frac{c}{cB} \underline{\omega}^{\hat{\varphi}} \wedge \left(-\frac{c'}{cB}\right) \underline{\omega}^{\hat{\theta}} \quad (1919) \\ &= -\frac{1}{c^2} \underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\varphi}} - \frac{c''}{c^2 B^2} \underline{\omega}^{\hat{\varphi}} \wedge \underline{\omega}^{\hat{\theta}} \end{aligned}$$

$$R^{\hat{t}}_{\hat{r}\hat{t}\hat{r}} = -\frac{1}{B^2} \left(\frac{A''}{A} - \frac{A'B'}{AB} \right)$$

$$R^{\hat{t}}_{\hat{\theta}\hat{t}\hat{\theta}} = -\frac{A'C'}{AB^2 c} = R^{\hat{t}}_{\hat{\varphi}\hat{t}\hat{\varphi}}$$

$$R^{\hat{\theta}}_{\hat{r}\hat{\theta}\hat{r}} = -\frac{1}{B^2} \left(\frac{c''}{c} - \frac{c'B'}{cB} \right) = R^{\hat{\varphi}}_{\hat{r}\hat{\varphi}\hat{r}}$$

$$R^{\hat{\varphi}}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = \frac{1}{c^2} - \frac{c''}{c^2 B^2}$$

→ Riemann tensor in orthonormal basis (not in coordinate basis)

For Coordinate Basis

$$\underline{R} = \underbrace{R^a_{bcd}}_{\text{components}} \underbrace{\underline{e}_a \underline{\omega}^b \underline{\omega}^c \underline{\omega}^d}_{\text{Tensorial part.}}$$

Recall: $\underline{e}_a = \left(e_a^\mu \right) \frac{\partial}{\partial x^\mu}$ components of orthonormal vector in coordinate basis.

$$\underline{\omega}^a = \omega^a_\mu dx^\mu$$

$$R^\mu_{\nu\lambda\rho} = e_a^\mu \omega^b_\nu \omega^c_\lambda \omega^d_\rho R^a_{bcd}$$

$$R^{\hat{t}}_{\hat{r}\hat{t}\hat{r}} = \underbrace{e_{\hat{t}}^\mu}_{1/A} \underbrace{\omega^{\hat{r}}_\mu}_B \underbrace{\omega^{\hat{t}}_\mu}_A \underbrace{\omega^{\hat{r}}_\mu}_B R^{\hat{t}}_{\hat{r}\hat{t}\hat{r}} = B^2 R^{\hat{t}}_{\hat{r}\hat{t}\hat{r}}$$

$$R^t_r = \frac{1}{B^2} \left(\frac{A''}{A} - \frac{A'B'}{AB} \right)$$

$$R^{\theta r}_{\theta r} = \frac{1}{B^2} \left(\frac{C''}{C} - \frac{C'B'}{CB} \right) = R^{\varphi r}_{\varphi r}$$

$$R^t_{t0} = R^{t\varphi}_{t\varphi} = \frac{A'C'}{AB^2C}$$

$$R^{\varphi\theta}_{\varphi\theta} = \frac{1}{C^2} \left(\frac{C'^2}{B^2} - 1 \right)$$

$$R^t_t = \frac{1}{B^2} \left(\frac{A''}{A} - \frac{A'B'}{AB} + 2 \frac{A'C'}{AC} \right)$$

$$R^0_0 = R^{\varphi}_{\varphi} = \frac{1}{B^2} \left(\frac{C''}{C} - \frac{C'B'}{CB} + \frac{A'C'}{AC} + \frac{C'^2}{C^2} \right) - \frac{1}{C^2}$$

$$R^r_r = \frac{1}{B^2} \left(\frac{A''}{A} + 2 \frac{C''}{C} - \frac{B'}{B} \left(\frac{A'}{A} + 2 \frac{C'}{C} \right) \right)$$

For spherically
symmetric vacuum
solution...

coming
because of
2 sphere.

Gauge: $C=r$: area gauge.
 $B=1$: proper distance is r .

$A = \frac{1}{B}$: Schwarzschild gauge.

$C=rB$: "ADM" gauge

Area Gauge $C=r, C'=1, C''=0$.

$$R^t_t = \frac{1}{r^2} - \frac{1}{r^2} \left(\frac{r}{B} \right)' = 8\pi G T^0_0$$

$$\Rightarrow B^{-2} = 1 - \frac{2G}{r} \int 4\pi r^2 T^0_0 dr$$

Energy between r & $r+dr$

The integral tells how much
energy you got inside radius
 r (in some sense).

He can then interpret it as $M(r) = \int 4\pi r^2 T^0_0 dr$

$$B^{-2} = 1 - \frac{2GM(r)}{r}$$

(Pg 21)

Vacuum Solution ; $T^0 = 0 \Rightarrow B^{-2} = 1 - \frac{2GM}{r}$

M an integration constant.

$$R^t_t - R^r_r = \frac{2}{r} \left(\frac{A'}{A} + \frac{B'}{B} \right) = 0 \text{ is vacuum}$$

$$\Rightarrow A' \propto 1/B \Rightarrow \text{Schwarzschild solution.}$$

$$A^2 = B^{-2} = 1 - \frac{2GM}{r}$$

~~What~~ what about Cosmological Constant Λ ?

Cosmological Constant. $8\pi G T_{ab} = \Lambda g_{ab}$

$$R^t_t = R^r_r \text{ even when we get cosmological constant.}$$

$$A \propto \frac{1}{B} \Rightarrow \int 4\pi r^2 T^0_0 = \frac{\Lambda}{2G} \int r^2 = \frac{\Lambda r^3}{6G}$$

$$A^2 = B^{-2} = 1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2$$

de-Sitter has $\Lambda > 0$; $\Lambda = 3/L^2$

get 2 horizons $r \approx 2GM$

$r \approx L$ (at large values of r)

Cosmological Horizon at $r \sim L$

$M=0$ gives pure d.S. ; $A^2 = 1 - \frac{r^2}{L^2}$

dS is constant curvature spacetime.

Represent ~~as 4D~~ as a 4D hyperboloid in 5D (flat) Minkowski Spacetime.



$$X^2 + Y^2 + Z^2 + U^2 - T^2 = L^2 \quad (\text{pg 22})$$

The metric on d.S. is inherited by projecting down the 5 dimensional flat metric onto this 4d hyperboloid.

For static slicing $T = L \sqrt{1 - \frac{r^2}{L^2}} \sinh(t/L)$

$$U = L \sqrt{1 - \frac{r^2}{L^2}} \cosh(t/L)$$

$$\underline{X} = r \underline{n}$$

$$r = \sqrt{X^2 + Y^2 + Z^2}$$

FRW : global

$$T = L \sinh(z/L)$$

$$U = L \cosh(z/L) \cdot \cos \chi$$

$$\underline{X} = L \cosh(z/L) \cdot \sin \chi \cdot \underline{n}$$

$K = 1$ cosmology

STATIC

$$z = t + \frac{L}{2} \log(1 \pm r^2/L^2)$$

→ FLAT

$$\rho = \frac{r e^{-t/L}}{\sqrt{1 - r^2/L^2}}$$

Lec 5: Black holes, horizons and Causal structure.

Causal structure is a key feature of black holes: how do we encode this?

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

$$r_+ = 2GM$$

Kruskals explicitly extended across horizon at r_+

① TORTOISE COORDINATES

$$r^* = \int \frac{dr}{1 - \frac{2GM}{r}} = r + 2GM \log\left(\frac{r - 2GM}{2GM}\right)$$

$$\text{or } r^* = r + r_+ \log\left(\frac{r - r_+}{r_+}\right)$$

$$\text{Then; } ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right) dr^{*2} - r^2 d\Omega^2$$

→ radial null geodesics $t = \pm r^* + \text{constant}$.

$$\text{KRUSKAL: } U = -r_+ \exp\left(-\frac{(t - r^*)}{2r_+}\right)$$

$$V = r_+ \exp\left(\frac{t + r^*}{2r_+}\right)$$

$$dU dV = r_+^2 \exp\left[\frac{t+r^*}{2r_+} - \frac{t-r^*}{2r_+}\right] \cdot \frac{(dt^2 - dr^{*2})}{4 r_+^2}$$

$$\Rightarrow dU dV = \frac{1}{4} e^{r^*/r_+} \left[dt^2 - \frac{dr^2}{\left(1 - \frac{r_+}{r}\right)^2} \right]$$

$$\downarrow$$

$$e^{\frac{r}{r_+}} \cdot \left(1 - \frac{r_+}{r}\right) \cdot \frac{r}{r_+}$$

so, $dU dV$ is first bit of Schwarzschild metric upto the exponential factor. ~~$dU dV = e^{r/r_+}$~~

$$dUdV = \frac{e^{r/r_+}}{4} \left[\left(1 - \frac{r_+}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{r_+}{r}\right)} \right]$$

pg 26

$$\Rightarrow ds^2 = \frac{r_+}{r} 4 dUdV e^{-r/r_+} - r^2 d\Omega^2_{II} \Rightarrow \text{This metric is regular at } r_+$$

$$dUdV = \frac{e^{r/r_+}}{4} \cdot \left[\left(1 - \frac{r_+}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{r_+}{r}\right)} \right]$$

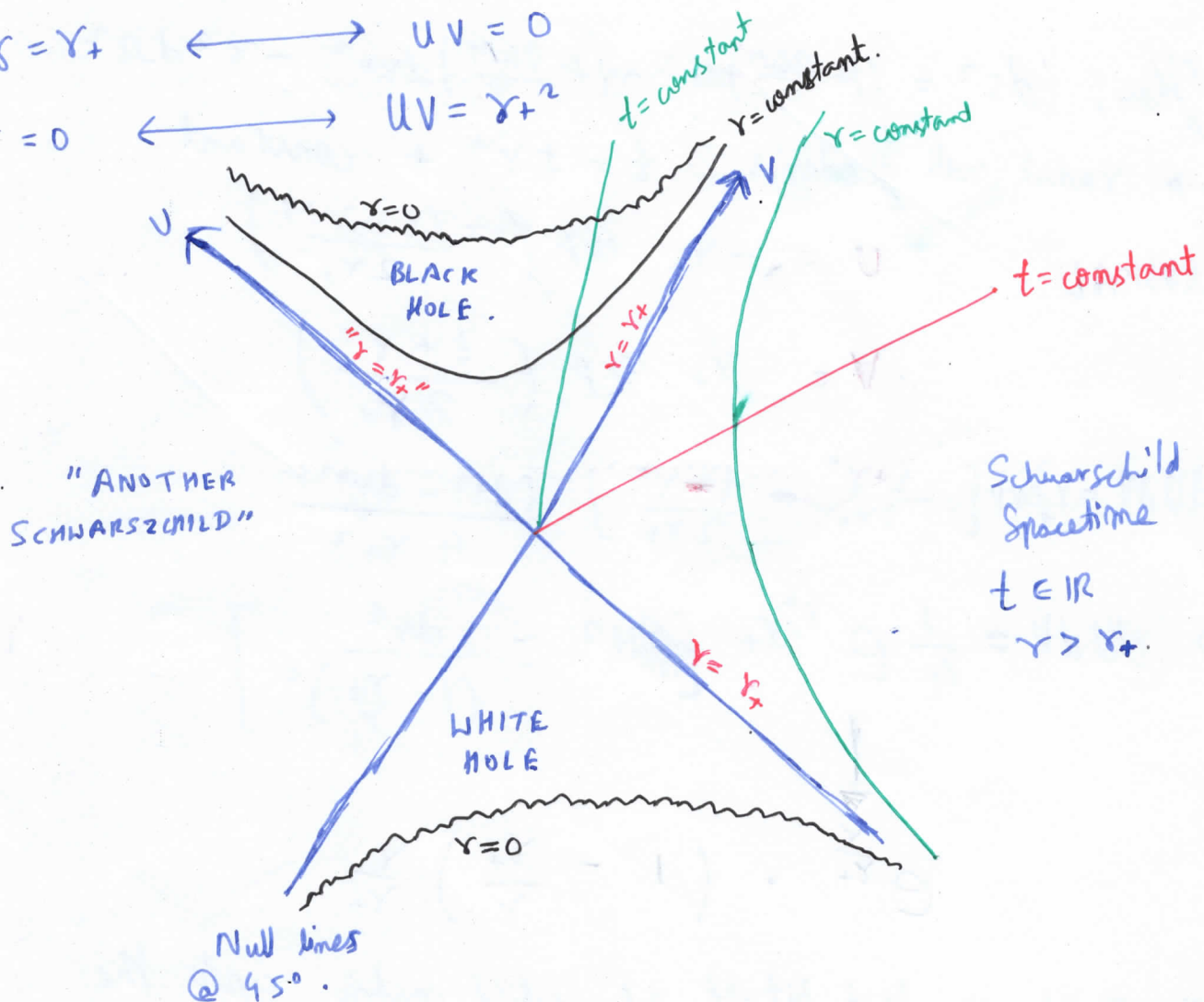
$$ds^2 = 4 \frac{r_+}{r} \cdot e^{-r/r_+} dUdV - r^2 d\Omega^2_{II}$$

$$r = \text{constant} \iff UV = \text{constant}$$

$$t = \text{constant} \iff U/V = \text{constant}$$

$$r = r_+ \iff UV = 0$$

$$r = 0 \iff UV = r_+^2$$



Penrose - Carter diagrams

pg 25

Compact representation of Black Hole.

Now define $P = \arctan V/r_+$
 $q = \arctan U/r_+$ } maps U, V to finite range in P, q

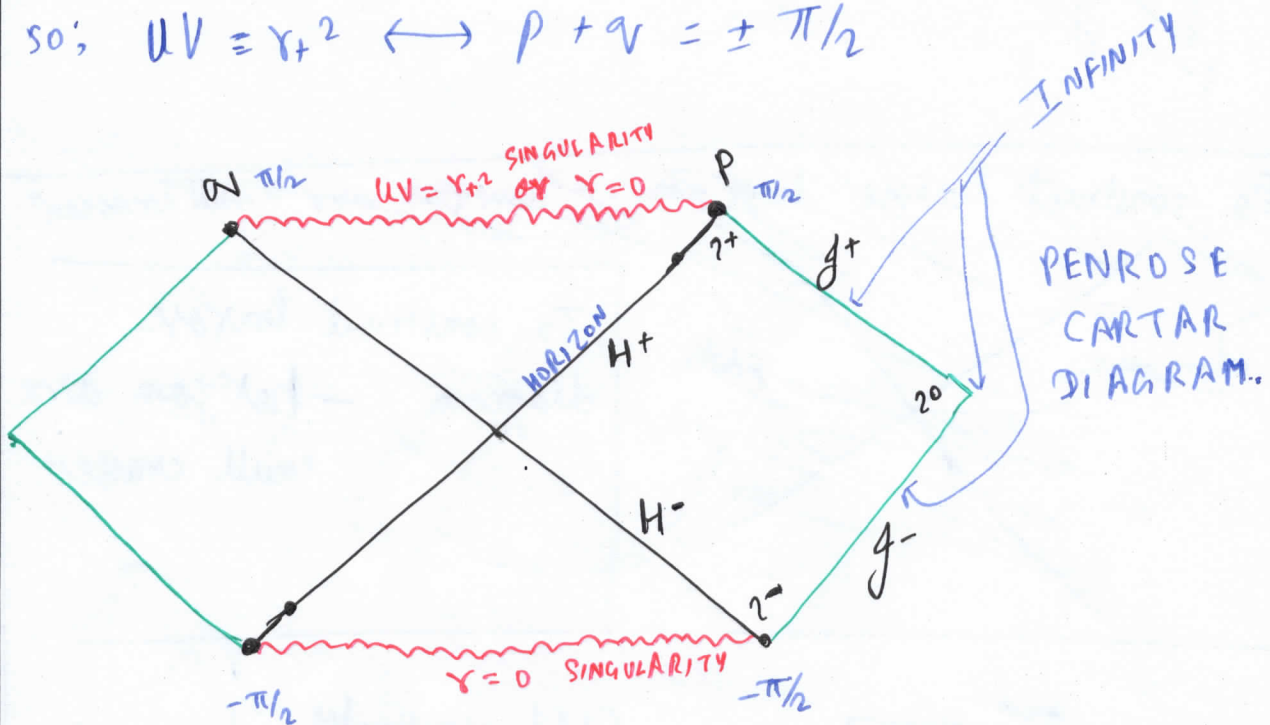
$$V=0 \longleftrightarrow P=0$$

$$V=r_+ \longleftrightarrow P=\pi/4$$

$$V \rightarrow \infty \longleftrightarrow P=\pi/2, \text{ etc.}$$

$$\tan(p+q) = \frac{\tan p + \tan q}{1 - \tan p \tan q} = \frac{U+V}{r_+^2 - UV}$$

$$\text{so; } UV = r_+^2 \longleftrightarrow p+q = \pm \pi/2$$



$H^\pm \equiv$ event horizon $(\pm)$ / (future / past)

i^+ $\equiv$ future timelike ∞

i^- $\equiv$ past " ∞

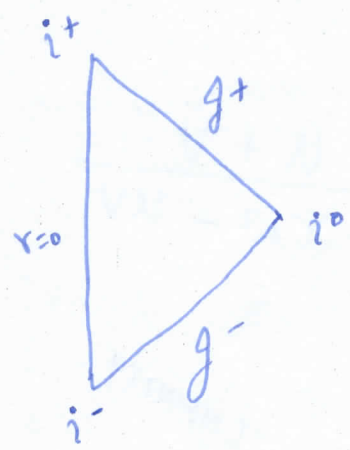
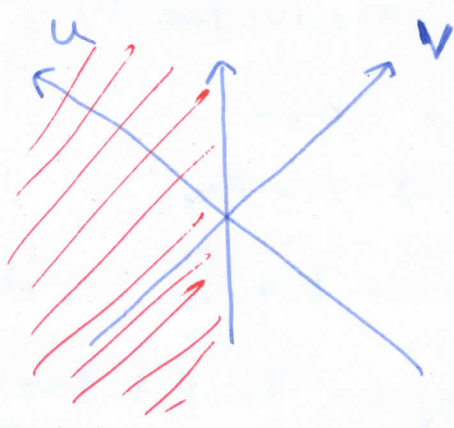
$i^0 \equiv$ spacelike ∞ (this is where you get if you go away from black hole on spatial geodesic)

$g^\pm \equiv$ future / past Null ∞ .

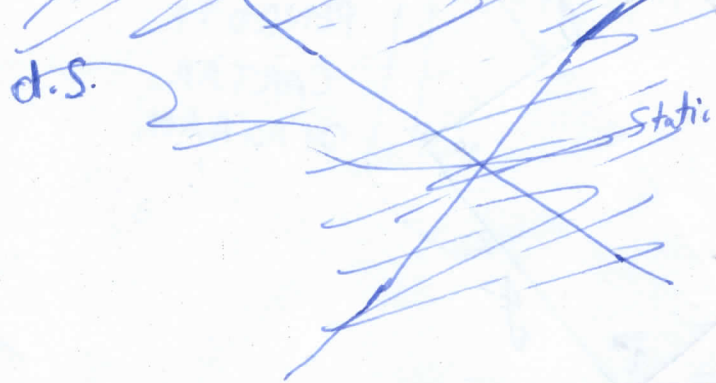
Event Horizon is boundary of the causal past of future timelike infinity.

MINKOWSKI $dt^2 - dr^2 - r^2 d\Omega^2$

$u = t - r, \quad v = t + r$
 $r > 0, \quad v > u$

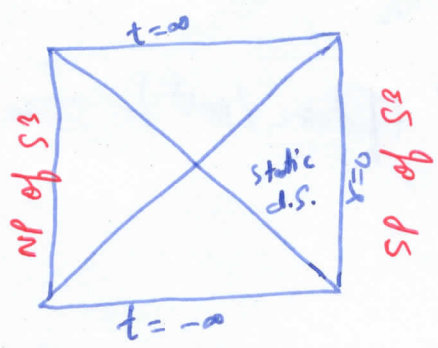


~~To construct Penrose diagram - horizon are "null crosses"~~



To construct Penrose diagram - horizon are "null crosses"

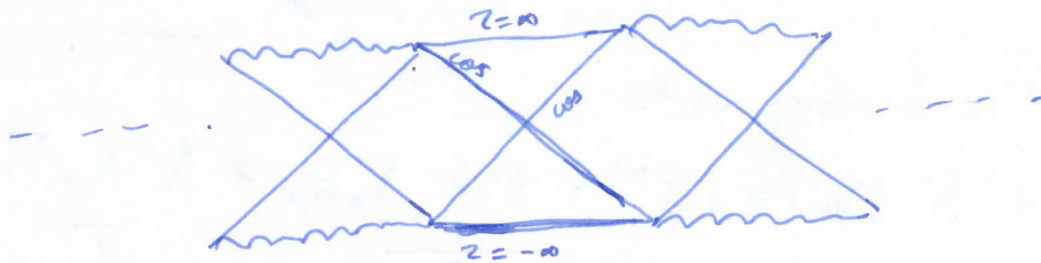
d.S.



North Pole of 3 sphere.

Global coordinates

$ds^2 = dr^2 + \cosh^2(r/2) d\Omega^2$



→ analytically extend

Kerr Black Hole

Astro black holes rotate.

$$ds^2 = \left(1 - \frac{2GM\gamma}{\Sigma}\right) dt^2 + 4GMa\gamma \sin\theta dt d\varphi$$

$$- \left[(\gamma^2 + a^2)^2 - \Delta a^2 \sin^2\theta \right] \frac{d\varphi^2}{\Sigma} - \frac{\Sigma}{\Delta} d\gamma^2 - \Sigma d\theta^2$$

$$\Sigma = \gamma^2 + a^2 \cos^2\theta$$

$$\Delta = \gamma^2 + a^2 - 2GM\gamma$$

$$a = J/M$$

Boyer-Lindquist coordinates.

$$ds^2 = \frac{\Delta}{\Sigma} [dt - a \sin^2\theta d\varphi]^2 - \frac{\sin^2\theta}{\Sigma} [a dt - (\gamma^2 + a^2) d\varphi]^2$$

$$- \frac{\Sigma}{\Delta} d\gamma^2 - \Sigma d\theta^2$$

$$\Delta = 0 \iff \gamma = \gamma_{\pm} = GM \pm \sqrt{G^2 M^2 - a^2}$$

notice: $\gamma_+ < 2GM$

$$\left\| \frac{\partial}{\partial t} \right\|^2 = \gamma^2 + a^2 \cos^2\theta - 2GM\gamma$$

$$\left\| \frac{\partial}{\partial t} \right\|^2 = 0 \iff \gamma_e = GM \pm \sqrt{G^2 M^2 - a^2 \cos^2\theta} \geq \gamma_+$$

If event horizon is $\Delta = 0$

then; for $\gamma_+ < \gamma \leq \gamma_e$ it is impossible to remain at rest (w.r.t. ∞) : ERGOSPHERE

Explore geodesics for $\theta = \pi/2$ (looking at equatorial geodesics) pg 28

Note, if k^μ is a killing vector.

$$\frac{d}{dt} [k_\mu \dot{x}^\mu] = \dot{x}^\nu \nabla_\nu (k_\mu \dot{x}^\mu) = \dot{x}^\nu \dot{x}^\mu \nabla_\nu k_\mu + \dot{x}^\nu k_\mu \nabla_\nu \dot{x}^\mu$$

anti - killing vector has symmetric covariant derivative.

zero because geodesics.

$$= 0$$

$\Rightarrow k_\mu \dot{x}^\mu$ is conserved quantity.

Kerr has 2 killing vectors $\partial_t, \partial_\varphi$

$$k_t = \frac{\partial}{\partial t} \leftrightarrow E = g_{t\mu} \dot{x}^\mu = (1 - \frac{2GM}{r}) \dot{t} + \frac{2GMa}{r} \dot{\varphi} \quad \text{at } \theta = \frac{\pi}{2}$$

$$k_\varphi = \frac{\partial}{\partial \varphi} \leftrightarrow h = -g_{\varphi\mu} \dot{x}^\mu = (r^2 + a^2 + \frac{2GMa^2}{r}) \dot{\varphi} - \frac{2GMa}{r} \dot{t} \quad \text{at } \theta = \pi/2$$

angular momentum conserved quantity.

Now define; $P_\alpha = (E, -h) = g_{\alpha\beta} \dot{x}^\beta \quad (\alpha, \beta \in \{t, \varphi\})$

then, $P_\alpha P_\beta g^{\alpha\beta} = g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$

$$= E^2 g_{tt} - 2Eh g_{t\varphi} + h^2 g_{\varphi\varphi}$$

$$\rightarrow \frac{1}{\Delta} \left[E^2 (r^2 + a^2) + \frac{2GM}{r} (h - aE)^2 - h^2 \right]$$

Geodesic equation: $p^2 + g_{rr} \dot{r}^2 = \eta$ — 1 for timelike (pg 29)
 — 0 for null

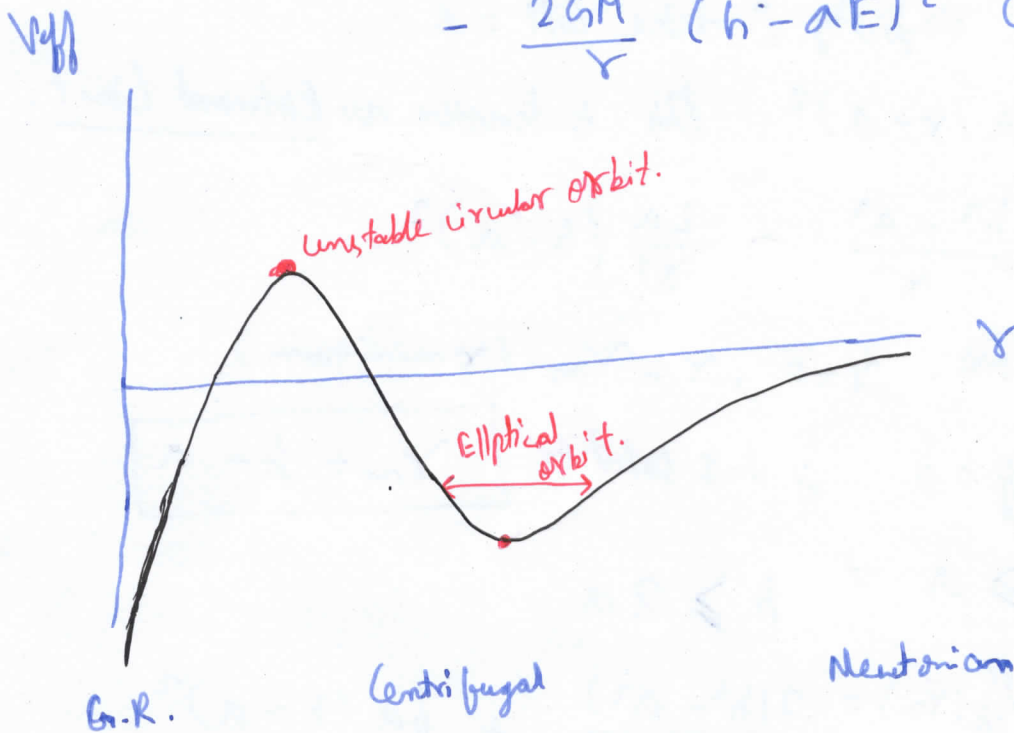
has form $\dot{r}^2 + V_{\text{eff}}(r) = 0$

$$\hookrightarrow V_{\text{eff}} = \eta \frac{\Delta}{\Sigma} - \frac{p^2 \Delta}{\Sigma}$$

$$= \eta^2 - E^2 - \frac{2GM\eta}{r} \quad \text{Newton}$$

$$+ \frac{a^2\eta + h^2 - a^2E^2}{r^2} \quad \text{Centrifugal}$$

$$- \frac{2GM}{r} (h - aE)^2 \quad \text{GR/KERR}$$



Lec 6: Rotating black holes & black hole thermodynamics

More on black holes.

Recall $\dot{r}^2 + V_{\text{eff}}(r) = 0$

$\theta = \frac{\pi}{2}$: $\eta^2 - E^2 - \frac{2GM\eta}{r} + \frac{a^2(\eta^2 - E^2) + h^2}{r^2} - \frac{2GM}{r^3} (h - aE)^2$

as $\Delta = r^2 + a^2 - 2GM r \rightarrow 0$

Look at Null geodesics

$\eta = 0$, $E = 1$ (by adjusting affine parameters)

for algebraic simplicity, * take $GM = a$.

i.e; $\Delta = (r - a)^2$ This is known as Extremal Limit.

$1 + V_{\text{eff}} = \frac{h^2 - a^2}{r^2} - \frac{2a}{r^3} (h - a)^2$

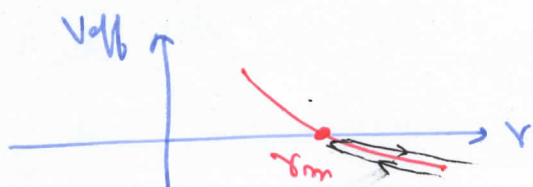
Initial conditions: $\dot{r} = 0$, $r = r_m$ (r minimum)

$\dot{r} = 0 \Rightarrow V_{\text{eff}} = 0$; has solution $r_m = h - a$

Since $r_m \geq a \Rightarrow h \geq 2a$

Now check $V'_{\text{eff}}(r_m) = \frac{2(h^2 - a^2)}{r_m^3} + \frac{6a(h - a)^2}{r_m^4}$
 $= \frac{4a - 2h}{r_m^2} \leq 0$

$\Rightarrow$ hence r increases..



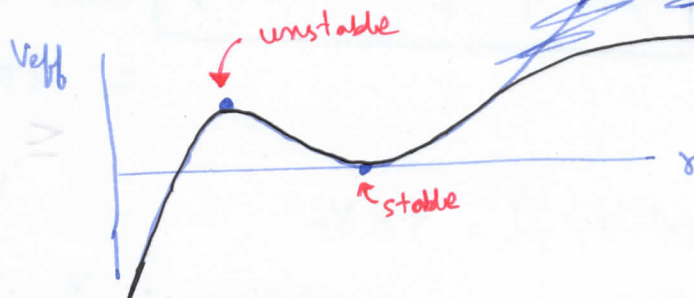
Limit at $h=2a$, $r_m = a$.

Hence $\Delta = 0$ is event horizon.

Black Holes do not exist in vacuum. Often have an accretion disc - modeled by concentric circular orbits.

Circular Orbit has $r = \text{constant}$.

$$V = V' = 0$$



Coordinate Picture

For simplicity take $a=0$, $\eta=1$, $\theta=\pi/2$

$$V = 1 - E^2 - \frac{2GM}{r} + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3}$$

$$V' = \frac{2GM}{r^2} - \frac{2h^2}{r^3} + \frac{6GMh^2}{r^4}$$

$$V'' = -\frac{4GM}{r^3} + \frac{6h^2}{r^4} - \frac{24GMh^2}{r^5}$$

$V = V' = 0$ determines h , E at given r .

$$h^2 = \frac{GM r^2}{r - 3GM} ; E^2 = \frac{r - 2GM}{r} + \frac{GM(r - 2GM)}{r(r - 3GM)}$$

$$V'' = \frac{2GM}{r^3} \cdot \frac{r - 6GM}{r - 3GM} > 0 \text{ for } r > 6GM$$

$r = 6GM$ is the limit of stable circular orbits.

Innermost Stable Circular Orbit - ISCO

For Kerr $r_{\text{ISCO}} < 6GM$

pg 32

Extremal Kerr: $r_{\text{ISCO}} = a = r_H$

Change track!

For Kerr Black Hole; event horizon at $r_+ = GM + \sqrt{G^2 M^2 - a^2}$

Its area is $A = 4\pi(r_+^2 + a^2) = 4\pi \cdot 2GM r_+ = 8\pi GM r_+$

$\leq$ Area of Schwarzschild Black Hole.

If it was Schwarzschild; $A = 4\pi r_+^2$

"Once a Black Hole starts to rotate; its horizon shrink"

$$\delta A = 8\pi(r_+ \delta r_+ + a \delta a)$$

$$\Rightarrow \delta A = 8\pi(r_+ \delta r_+ + a \delta a) = 8\pi\left(r_+ \left[GM + \frac{G^2 M \delta M - a \delta a}{r_+ - GM}\right] + a \delta a\right)$$

$$\Rightarrow \delta A = 8\pi\left[\frac{r_+^2 GM}{r_+ - GM} - \frac{GMa \delta a}{r_+ - GM}\right]$$

recall, $a = J/M \Rightarrow \delta a = \frac{\delta J}{M} - \frac{J}{M^2} \delta M$

$$\delta a = \frac{\delta J}{M} - a \frac{\delta M}{M}$$

$$\Rightarrow \frac{\delta A}{8\pi} = \frac{r_+^2 GM \delta M + G \cdot a^2 \delta M}{r_+ - GM} - \frac{Ga \delta J}{r_+ - GM}$$

$$\Rightarrow \frac{\delta A}{8\pi} = \frac{G(r_+^2 + a^2) \delta M}{r_+ - GM} - \frac{G(r_+^2 + a^2)}{r_+ - GM} \cdot \frac{a}{r_+^2 + a^2} \delta J$$

let $\Omega = \lim_{r \rightarrow r_+} - \frac{g_{\phi t}}{g_{\phi\phi}}$

Angular Velocity of Event Horizon.

ANGULAR
VELOCITY
OF Black
Hole

$$= \frac{a}{r_+^2 + a^2}$$

Rewrite the differential equations as.

$$\Rightarrow \delta M = \frac{r_+ - 2GM}{2\pi(r_+^2 + a^2)} \cdot \frac{\delta A}{4G} + \Omega \delta J$$

Compare it, to $dU = T dS - p dV + \mu dQ$
First law of Thermodynamics.

Suggest thermodynamic interpretation with

$M \longleftrightarrow \text{Energy}^*$

$$T = \frac{r_+ - 2GM}{2\pi(r_+^2 + a^2)} \quad ; \quad S = \frac{A}{4G}$$

Then $\boxed{\delta M = T \delta S + \Omega \delta J}$

For Schwarzschild; $T = \frac{1}{8\pi GM}$

Explore a bit more...

Take Schwarzschild solution, $t \rightarrow i\tau$

$$|ds^2| = \left(1 - \frac{2GM}{r}\right) d\tau^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

This is Ricci flat Euclidean solution.

What happens at $2GM$?

Try to find proper distance from $2GM$.

1939

$$\rho^2 = \lambda(r - 2GM)$$

$$2\rho d\rho = \lambda dr$$

$$1 - \frac{2GM}{r} \approx \frac{\rho^2}{\lambda \cdot 2GM} \quad | \quad r \rightarrow 2GM$$

$$ds_{E, \rho}^2 = \frac{4\rho^2}{\lambda^2} \frac{d\rho^2}{\rho^2/2GM\lambda} + \frac{\rho^2}{2GM\lambda} dz^2$$

$$\Rightarrow \lambda = 8GM \quad ; \quad \text{then}$$

$$\Rightarrow dS_E^2 = d\rho^2 + \rho^2 d\left(\frac{z}{4GM}\right)^2$$

$$" \theta " = \frac{z}{4GM}$$

Origin of IR^2 in polars. And suggest z is periodic
with periodicity $8\pi GM$.

- Signal of field theory: at limit T , $\Delta z = \beta = \frac{1}{T}$

$$T = \frac{1}{8\pi GM}$$

Other charges? $J \longleftrightarrow \Omega$ Ker

for Reissner Nordström: RN; $Q \longleftrightarrow \Phi = -Q/r_+$

What about PdV ?

Pressure could be cosmological constant Λ ?

$$g_{tt} = 1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2 = f(r)$$

Ex) Check Euclidean argument ($\rho^2 \propto (r - r_+)$) gives $T = \frac{f'_+}{4\pi}$

Horizon defined as $f(r_+) = 0$

$$(f + \delta f)(r_+ + \delta r_+) = 0$$

$$\Rightarrow \delta r_+ \cdot f'_+ - \frac{2G \delta M}{r_+} - \frac{\delta \Lambda}{3} \cdot r_+^2 = 0$$

he will define pressure

$$P = \frac{\Lambda}{8\pi G}$$

cosmological constant tells about pressure.

$$\Rightarrow \delta M = T \cdot \frac{\delta A}{4G} + \frac{r_+^3}{6G} \delta \Lambda = T \delta S + \underbrace{\frac{4\pi}{3} r_+^3}_{V} \delta P$$

$$V = \frac{4\pi}{3} r_+^3$$

$$\Rightarrow \delta M = T \delta S + V \delta P$$

may not be actual volume; but it is geometric volume inside event horizon.

This suggest M is enthalpy.

Now Λ can vary; it was a constant.

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} |\partial \phi|^2 - V \right)$$

If potential $V \gg |\partial \phi|^2$

ie; if fields move slowly; then $T_{\mu\nu}$ is potential dominated.

$$T_{\mu\nu} \approx V g_{\mu\nu} \quad \text{but if } \phi \text{ rolls slowly}$$

then V changes $\Rightarrow \Lambda$ changing.



This is Inflation.

P changing.

This is how we can get varying Λ .

Lecture 7: Gravitational Action: Palatini Lemma, Gauss-Codazzi formalism, Gibbons - Hawking term.

Have seen the Einstein - Hilbert action.

$$S_{EH} = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \cdot R$$

review how this works

Vary using inverse metric: (use δg^{ab}).

$$\det(g) = \exp(\text{tr}(\log g))$$

$$\delta \det(g) = \det(g) \cdot \underbrace{\text{tr} g^{-1} \delta g}_{-\text{tr} g_{ab} \delta g^{ab}}$$

$$\Rightarrow \delta \det(g) = -\det(g) \cdot \text{tr}(g_{ab} \cdot \delta g^{ab})$$

$$R = R_{ab} g^{ab}$$

$$\Rightarrow \delta R = R_{ab} \delta g^{ab} + g^{ab} \delta R_{ab}$$

In coordinate basis: $\delta R_{ab} = \delta R^c{}_{a c b}$

$$= \delta [\partial_c \Gamma^c{}_{ab} - \partial_b \Gamma^c{}_{ac} + \Gamma^c{}_{ce} \Gamma^e{}_{ba} - \Gamma^c{}_{be} \Gamma^e{}_{ca}]$$

$$= \underbrace{\partial_c \delta \Gamma^c{}_{ab}} - \underbrace{\partial_b \delta \Gamma^c{}_{ac}} + \underbrace{\delta \Gamma^c{}_{ce} \Gamma^e{}_{ba}} + \underbrace{\Gamma^c{}_{ce} \delta \Gamma^e{}_{ba}} - \underbrace{\delta \Gamma^c{}_{be} \Gamma^e{}_{ca}} - \underbrace{\Gamma^c{}_{be} \delta \Gamma^e{}_{ca}}$$

$$\Rightarrow \boxed{\delta R_{ab} = \nabla_c \delta \Gamma^c{}_{ab} - \nabla_b \delta \Gamma^c{}_{ac}}$$

Palatini's Lemma

To find $\delta \Gamma$, choose Normal Coordinates (which correspond to local inertial frame)

(local inertial frame) these set Γ , but not $\partial \Gamma = 0$ at a point: i.e. $g_{ab,c} = 0$, not $\underline{g_{ab,cd} = 0}$

$$\delta \Gamma^a_{bc} = \frac{1}{2} \delta g^{ae} (\cancel{g_{eb,c} + g_{ec,b} - g_{bc,e}}) + \frac{1}{2} g^{ae} (\delta g_{eb,c} + \delta g_{ec,b} - \delta g_{bc,e})$$

$$= \frac{1}{2} g^{ae} (\delta g_{eb,c} + \delta g_{ec,b} - \delta g_{bc,e})$$

In N.C. $\partial_a \equiv \nabla_a$

So; we can re-write.

$$\delta \Gamma^a_{bc} = \frac{1}{2} g^{ae} (\nabla_c \delta g_{eb} + \nabla_b \delta g_{ec} - \nabla_e \delta g_{bc})$$

(This is covariant geometric expressing)

→ Since it is true in one coordinate frame, i.e. N.C.

and it is a tensorial expression.

So; it is true in general.

$$\delta \Gamma^a_{bc} = \frac{1}{2} g^{ae} (\nabla_c \delta g_{eb} + \nabla_b \delta g_{ec} - \nabla_e \delta g_{bc})$$

$$\delta \Gamma^a_{bc} = \frac{1}{2} (\nabla_c \delta (g^{-1})^a_b + \nabla_b \delta (g^{-1})^a_c - \nabla^a \delta (g^{-1})_{bc})$$

$$\Rightarrow \delta R_{ab} = \frac{1}{2} \square$$

$$\delta R_{ab} = \frac{1}{2} (\square \delta (g^{-1})_{ab} + \nabla_a \nabla_b \delta (g^{-1}) - \nabla_c \nabla_b \delta (g^{-1})^c_a - \nabla_c \nabla_a \delta (g^{-1})^c_b)$$

$$\delta R = \square (g_{ab} \delta g^{ab}) - \nabla_a \nabla_b \delta g^{ab}$$

$$= \nabla_c (\nabla^c (g_{ab} \delta g^{ab}) - \nabla_a \delta g^{ac})$$

Integrating up the variation of S_{EH} gives

$$\delta S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{g} \cdot (R_{ab} - \frac{1}{2} R g_{ab}) \delta g^{ab} - \frac{1}{16\pi G} \int_{\partial M} d^3x \sqrt{g} \eta_a (\nabla^a \delta g^{11} - \eta^a \delta g^a_b)$$

normal

$$\delta S_{EH} = \frac{-1}{16\pi G} \int_M d^4x \sqrt{g} \left(R_{ab} - \frac{1}{2} R g_{ab} \right) \delta g^{ab} - \frac{1}{16\pi G} \int_{\partial M} d^3x \sqrt{g} \cdot n_a \cdot (\nabla^a \delta g^1 - \nabla_b \delta g^{ab})$$

Boundary depends on $\nabla \delta g$.

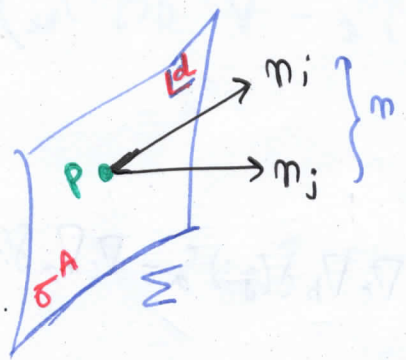
Usual E.L. rules would set $\delta g = 0$ on ∂M .

How to deal with a boundary?



Need to be able to describe Submanifolds.

Gauss - Codazzi Formalism



Let $\Sigma \subset M$, be a manifold, $\dim d$.

$$\dim(M) = D = d + n.$$

σ^A coordinates on Σ
 x^M " " " M

$$\mu = \{0, 1, 2, \dots, (D-1)\}$$

$$A = \{0, 1, 2, \dots, (d-1)\}$$

Definition The co-dimension of Σ is $n = D - d$, & there exists n linearly independent normal vectors n_i to Σ .

$$\{n_i\}_{i=1, \dots, n} \in T_p(M)$$

s.t. $n_i(\sigma^A) = 0 \quad \forall$ coordinate functions on Σ

Alternatively

$$\eta_{i,\mu} \frac{\partial X^\mu}{\partial \sigma^A} = 0$$

$i = 1, \dots, n$ (pg 39)

Choose $\eta_{i,\mu} \eta_j{}^\mu = \pm \delta_{ij}$ (he can always choose these by choosing orthogonal; and then re-scaling to get orthonormal).

η_{μ} timelike: $\eta^2 = +1$ (ADM if $n=1$)

Assume $\eta_{i,\mu}$ spacelike for $i > 1$; then $\eta_i^2 = -1$.

i.e. ~~$\eta_{i,\mu} \eta_j{}^\mu$~~

i.e.

$$\eta_{i,\mu} \eta_j{}^\mu = -\delta_{ij}$$

Definition The 1st Fundamental Form or induced metric on

$$\Sigma \text{ is } h_{ab} = g_{ab} + \eta_{ia} \eta_{ib}$$

↑ This is the metric which Σ inherits from M .

note: $h_{ab} \in T^*(M) \otimes T^*(M)$

Can also express as element of $T^*(\Sigma) \otimes T^*(\Sigma)$ via

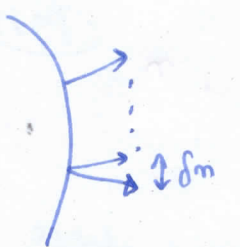
$$\frac{\partial X^\mu}{\partial \sigma^A} : T^*(M) \rightarrow T^*(\Sigma) \quad \text{Pull Back.}$$

$$\gamma_{AB} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B}$$

Definition The 2nd Fundamental form or Extrinsic curvature measures how Σ curves in M .

$$K_{i,\mu\nu} = h_{(\mu}{}^\sigma h_{\nu)}{}^\eta \nabla_\sigma \eta_{i,\eta}$$

~~$K_{i,\mu\nu} = h_{(\mu}{}^\sigma h_{\nu)}{}^\eta \nabla_\sigma \eta_{i,\eta}$~~



$$\textcircled{a} K_{iAB} = \frac{\partial X^A}{\partial \sigma^A} \cdot \frac{\partial X^B}{\partial \sigma^B} \nabla_{(\mu} n_{i\nu)}$$

$$= -n_{i\mu} D_A \cdot \frac{\partial X^A}{\partial \sigma^B}$$

where D is the connection inherited by Σ .

$$D_\mu V_\nu = h_\mu{}^\lambda h_\nu{}^\sigma \nabla_\lambda V_\sigma \quad (V_\mu n^\mu = 0)$$

Definition The Normal Fundamental Form

$$\beta_{\mu ij} = n_{i\nu} \nabla_\mu n_j{}^\nu$$

Can decompose the curvature of Σ in terms of curvature of M + extrinsic curvature.

Gauss Equation

$$R_{(d)}{}^a{}_{bcd} = h^a{}_{a'} h^b{}_{b'} h^c{}_{c'} h^d{}_{d'} R_{(d)}{}^{a'}{}_{b'c'd'}$$

$$+ K_i{}^a{}_c K_i{}_{bd} - K_i{}^a{}_d K_i{}_{bc}$$

Gauss Equation

$$R_{(d)}{}^a{}_{bcd} = h^a{}_{a'} h^b{}_{b'} h^c{}_{c'} h^d{}_{d'} R_{(d)}{}^{a'}{}_{b'c'd'} + K_i{}^a{}_c K_i{}_{bd} - K_i{}^a{}_d K_i{}_{bc}$$

Prove using Reimann identity $(D_a D_b - D_b D_a) \dots$

Now consider $n=1$. Σ is a "wall".

Extend n^μ geodesically into bulk $n^\mu n^\nu g_{\mu\nu} = 1$, $D_\mu n^\mu = 0$

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

Then $\delta(\eta^\mu \eta^\nu g_{\mu\nu}) = 0$

$$\Rightarrow \boxed{2\eta_\mu \delta\eta^\mu = \eta_\mu \eta_\nu \delta g^{\mu\nu}}$$

$\Rightarrow$ Relation between $\delta\eta$ projected along η & normal variation of $g_{\mu\nu}$

$\delta(\eta^\mu h_{\mu\nu}) = 0$ (because we need η^μ to be normal

i.e.; $\eta^\mu h_{\mu\nu} = 0$

$\Rightarrow \delta(\) = 0$)

$$\Rightarrow \boxed{h_{\mu\nu} \delta\eta^\mu = \eta_\mu \delta h^{\mu\sigma} g_{\sigma\nu}}$$

$\Rightarrow$ gives Relation in terms of parallel

component of $\delta\eta^\mu$ & δh

but; $h^{\mu\nu} = g^{\mu\nu} + \eta^\mu \eta^\nu$

$$\Rightarrow (g_{\mu\nu} + \eta_\mu \eta_\nu) \delta\eta^\mu = \eta_\mu (\delta g^{\mu\sigma} + \delta\eta^\mu \eta^\sigma + \eta^\mu \delta\eta^\sigma) g_{\sigma\nu}$$

$$\Rightarrow \boxed{\delta\eta^\mu = \frac{1}{2} \eta_\nu \delta g^{\mu\nu}} \Rightarrow \text{Variation of normal is related to variation of metric.}$$

Consider $\delta K = \delta(\nabla_\mu \eta^\mu) = \nabla_\mu \delta\eta^\mu + \delta\Gamma^\mu_{\mu\lambda} \eta^\lambda$

$$= \frac{1}{2} \nabla_\mu (\eta_\nu \delta g^{\mu\nu}) - \frac{1}{2} \eta^\lambda \nabla_\lambda \delta g^\mu_\mu$$

$$= \frac{1}{2} [\eta_\nu \nabla_\mu \delta g^{\mu\nu} - \nabla_\mu \delta g^\mu_\mu] + \frac{1}{2} \nabla_\mu \eta_\nu \delta g^{\mu\nu}$$

But $\nabla_\mu \eta_\nu = K_{\mu\nu}$

(because of our coordinate choice)

~~$$\delta g^{\mu\nu} \nabla_\mu \eta_\nu = \delta h^{\mu\nu} \nabla_\mu \eta_\nu$$~~

$$\boxed{\delta g^{\mu\nu} \nabla_\mu \eta_\nu = \delta h^{\mu\nu} K_{\mu\nu}}$$

Boundary E-L Term

$$= \frac{-1}{16\pi G} \int d^3x \sqrt{g} [2\delta K - K_{ab} \delta h^{ab}]$$

Finally $\det h = \det g$; because $\eta^2 = -1$

Boundary term is then

(pg 92)

$$= -\frac{1}{8\pi G} \delta \int d^3x (\sqrt{h} K) + \frac{1}{16\pi G} \int d^3x \sqrt{h} \delta h^{ab} (K_{ab} - K h_{ab})$$

→ GIBBONS - HAWKING BOUNDARY TERM.

no problem with this
 .. we have δh^{ab} .. this
 we can set to zero on
 boundary.

$$S = -\frac{1}{16\pi G} \int_M d^4x \sqrt{g} \cdot R - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{h} K$$

→ gives

Einstein
 Equations.

Boundary
 (they cancel
 each other)

$K_{ab} - K h_{ab}$

ISRAEL EQUATION
 (if we keep the boundary
 dynamical)

Bonus Lecture 1: Gravity and field Theory Action principles, and area Domain walls and gravity.

In field theory, we use a Lagrangian formulation on the manifold; we need to be able to integrate.

$d^4x \longleftrightarrow$ coordinate volume

but $d^4x \longrightarrow \left| \frac{\partial x}{\partial \tilde{x}} \right| d^4\tilde{x}$

However, $\sqrt{|g|} d^4x$ is invariant.

To vary this volume element;

use the identity $\det M = \exp(\text{tr} \log M)$

$$\delta(\det M) = \det M \cdot \text{tr}(M^{-1} \delta M)$$

$$\Rightarrow \delta(\sqrt{-g}) = \frac{1}{2} \frac{(-1)}{\sqrt{-g}} \cdot g \cdot \underbrace{g^{ab} \delta g_{ab}}_{\text{This is } \text{tr}(g^{-1} \delta g)}$$

$$= -\frac{\sqrt{-g}}{2} g_{ab} \delta g^{ab}$$

e.g. Massless Scalar field: $\mathcal{L}_\phi = \frac{1}{2} (\partial\phi)^2$

$$S_\phi = \int d^4x \sqrt{-g} \cdot \frac{1}{2} (\partial\phi)^2 = \int d^4x \sqrt{-g} \frac{1}{2} \phi_{,n} \phi_{,r} g^{nr}$$

$$\delta S_\phi = \int d^4x \left[\sqrt{-g} \cdot \partial_n \phi \partial_r \delta\phi g^{nr} + \frac{1}{2} \sqrt{-g} \left(\phi_{,n} \phi_{,n} - \frac{1}{2} (\partial\phi)^2 g_{nn} \right) \delta g^{nn} \right]$$

Now; integrate by parts

~~$$\delta S_\phi = \int d^4x \sqrt{-g} \left[\eta^{\mu\nu} \partial_\mu \phi \right] + \int d^4x \sqrt{-g} \left[\frac{1}{\sqrt{-g}} \partial_\mu \right]$$~~

Now integrate by parts

$\sqrt{g}$ corresponds to $\sqrt{-g}$. (1944)

$$\frac{\delta S_\phi}{\delta \phi} = \int d^3x \sqrt{g} \cdot (m^2 \phi) + \int d^3x \sqrt{g} \left[\frac{-1}{\sqrt{g}} \partial_\mu (\sqrt{g} \partial^\mu \phi) \right] + \frac{1}{2} [\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial \phi)^2 g_{\mu\nu}] \delta g^{\mu\nu}$$

$$\frac{\delta S_\phi}{\delta \phi} = \frac{-1}{\sqrt{g}} \partial_\mu \sqrt{g} \cdot \partial^\mu \phi = -\square \phi$$

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{2} [\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial \phi)^2 g_{\mu\nu}] = \frac{1}{2} T_{\phi\mu\nu}$$

Symmetrized
Energy Momentum
tensor of ϕ .

$$\delta S_\phi = \underbrace{\int d^4x \sqrt{-g}}_{\text{"dV"}} \cdot \frac{\delta S}{\delta \phi} + \dots$$

For gravitational action; we need scalar.

Normally when we write down Lagrangian; we write $T - V$ (Kinetic - Potential)

~~so we can imagine that~~

For gravity; we would like to write down kinetic term involving derivatives of g

$\iff$ but there is a problem;

whatever we ~~now~~ write; must be covariant.

And we saw that; when writing covariant objects

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on manifold that:
we don't have anything with first derivative of g
which is covariant.

(The only thing we know is Levi-Civita connection &
that is not a tensor)

So, we can't construct first order kinetic term for
gravity.

But; we go forward & take second derivatives of g

ie: terms involving $\partial\partial g \dots$ like
Rici...

Take R (note this contains $\partial\partial g$)

$$R = R_{\mu\nu} g^{\mu\nu}$$

$$\delta R = \delta R_{\mu\nu} g^{\mu\nu} + R_{\mu\nu} \delta g^{\mu\nu}$$

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$$

Compute $\delta R_{\mu\nu}$ in "normal coordinates"

(local Inertial frame) $\partial g = 0$; $\partial \Gamma \neq 0$

$$\delta R_{\mu\nu} = \delta \Gamma^\lambda{}_{\mu\nu, \lambda} - \delta \Gamma^\lambda{}_{\mu\lambda, \nu}$$

but $\stackrel{\text{n.c.}}{=} \delta \Gamma^\lambda{}_{\mu\nu; \lambda} - \delta \Gamma^\lambda{}_{\mu\lambda; \nu}$

→ Note this is tensorial
equation because difference between
connection is Tensor.

Recall δP is a tensor

$$\delta R_{ab} = \nabla_c \delta \Gamma_{ab}^c - \nabla_b \delta \Gamma_{ac}^c$$

Remember $\partial g = 0$ in N.C.

$$\delta \Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\sigma} (\delta g_{\sigma\nu, \lambda} + \delta g_{\sigma\lambda, \nu} - \delta g_{\nu\lambda, \sigma})$$

hence:

$$\delta \Gamma_{bc}^a = \frac{1}{2} [\nabla_b \delta g_{ac} + \nabla_c \delta g_{ab} - \nabla^a \delta g_{bc}]$$

$$\delta \Gamma_{bc}^a = \frac{1}{2} [\nabla_b \delta g_{ac} + \nabla_c \delta g_{ab} - \nabla^a \delta g_{bc}]$$

$$\text{Hence } \delta R_{ab} = \frac{1}{2} [\nabla_c \nabla_a \delta g_{bc} + \nabla_c \nabla_b \delta g_{ac} - \nabla_c \nabla^c \delta g_{ab} - \nabla_b \nabla_a \delta g_{cc}]$$

Change to δg^{-1} .

$$\text{Then } g^{\mu\nu} \delta R_{\mu\nu} = -\nabla_\mu \nabla_\nu \delta g^{\mu\nu} + g_{\mu\nu} \square \delta g^{\mu\nu}$$

► This is a total derivative; but

The boundary term we get on integrating is

$$\int d^3x \, n_\mu \nabla_\nu \delta g^{\mu\nu} - n^\mu \nabla_\mu \delta g$$

we get derivatives of δg ; not δg .

So; we need to set normal derivatives of δg to be zero.

Convention

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- using greek indices in Normal Coordinates (N.C.)
- using latin indices for general coordinates.

he therefore write

(pg 57)

$$S_{EH} = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \cdot R$$

$$\& \frac{\delta S}{\delta g^{\mu\nu}} = -\frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right)$$

Combine with interacting scalar.

$$\mathcal{L}_\phi = \frac{1}{2} (\partial\phi)^2 - V(\phi)$$

$$\text{take } V(\phi) = \frac{\lambda}{2} (\phi^2 - \eta^2)^2$$

$$S_{\text{Tot}} = S_{EH} + S_\phi$$

$V(\phi)$ has two vacua $\phi = \pm\eta$



φ e.o.m.

$$\square \phi + 2\lambda (\phi^2 - \eta^2) \phi = 0$$

Ask that $\phi \rightarrow \pm\eta$ as $z \rightarrow \pm\infty$ in the Minkowski;
in absence of G.R.

$$\phi = \phi(z)$$

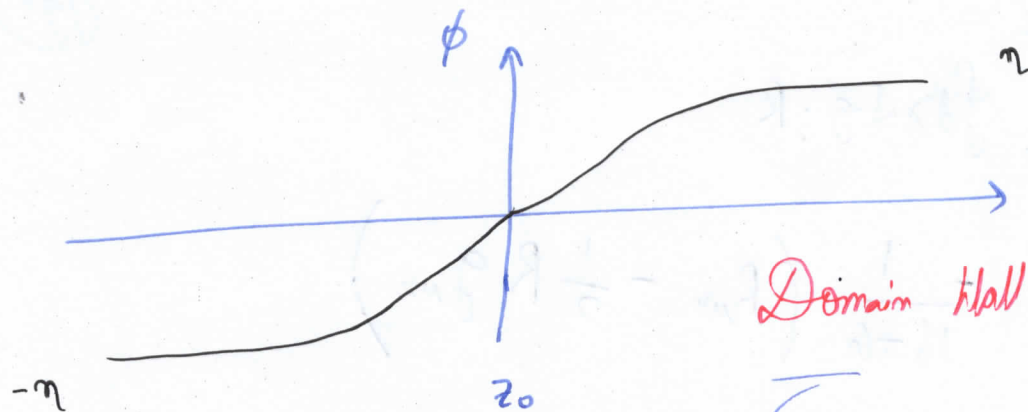
$$\text{Then } \phi = \eta \tanh[\sqrt{\lambda} \cdot \eta (z - z_0)] \quad \rightarrow \text{solution.}$$

$$\phi' = \sqrt{\lambda} \eta^2 \text{sech}^2[\dots]$$

$$\phi'' = -2\lambda \eta^2 \text{sech}^2[\dots] \tanh[\dots]$$

$$= -2\lambda \eta^2 (1 - \phi^2/\eta^2) \phi \quad \text{E.o.M.}$$





Domain Wall / Kink

↳ (its an example of topological defect:

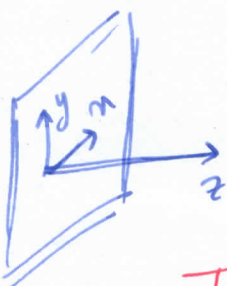
Its a non-perturbative solution to field Theory)

↳ Even if you try doing QFT; you

cant : If you put the solution in background; you cant transition from this solution to either vacuum.

In order to go to the vacuum, the field will have to go say to $+\eta$ vacuum from $-\eta$, over an infinite amount of space; and this will cost infinite amount of energy.

So; This is Topologically stable.



$$T_{\mu\nu} = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} (\phi')^2 g_{\mu\nu}$$

$$= \phi^2 \delta_{\mu\nu} = \phi^2 \delta_{\mu\nu} + g_{\mu\nu} \left(\frac{1}{2} \phi'^2 + V \right)$$

$$T_{\mu\nu} = \phi^2 \delta_{\mu\nu} + g_{\mu\nu} \left(\frac{1}{2} \phi'^2 + V \right)$$

$$\phi'^2 = \lambda \eta^4 \operatorname{sech}^4 [\sqrt{\lambda} \eta (z - z_0)]$$

1949

strongly localized ;

Actually $\phi'^2 = 2V$ (for that solution)

$$T^0_0 = T^x_x = T^y_y = 2V$$

$$T^z_z = 0.$$

so; It looks something three dimensional.
(nothing much going on in z direction)

Mostly a vacuum, but wall has ∞ ~~area~~ ~~area~~ area

$$\int \int_0 dz = \frac{4}{3} \sqrt{\lambda} \eta^3$$

Adding Curvature

T^0_0 , etc looks like Λ in 3D space at z_0 .

Try $ds^2 = A^2(z) \gamma_{\mu\nu} dx^\mu dx^\nu - dz^2$

warning: μ indicates $t, x \dots$; not 4D!

Now, calculate Curvature.

use Cartan.

$\gamma_{\mu\nu}(x)$ arbitrary $(D-1)$ dimensional metric.

$$\underline{\omega}^{\hat{z}} = dz \quad ; \quad \underline{\omega}^{\hat{0}} = A(z) \underline{\omega}^{\hat{0}}_0$$

$$\uparrow \quad \partial^{\hat{0}}_\mu dx^\mu$$

$$d\underline{\omega}^{\hat{z}} = 0 \quad ; \quad d\underline{\omega}^{\hat{0}} = \frac{A'}{A} \underline{\omega}^{\hat{z}} \wedge \underline{\omega}^{\hat{0}} - \partial_0^{\hat{0}} \underline{\omega}^{\hat{0}}$$

$$\Rightarrow \partial^{\hat{a}} \hat{b} = \partial_a \hat{a} \hat{b} \quad \& \quad \partial^{\hat{a}} \hat{b} = \frac{N}{A} \omega^{\hat{a}}$$

~~$$R^{\hat{a}}_{\hat{b}} = \partial_a \hat{a} \hat{b}$$~~

~~$$R^{\hat{a}}_{\hat{b}} = \partial_a \hat{a} \hat{b} + \partial_a \hat{a} \hat{b} + \partial_a \hat{a} \hat{b} + \partial_a \hat{a} \hat{b}$$~~

$$R^{\hat{a}}_{\hat{b}} = \partial_a \hat{a} \hat{b} + \partial_a \hat{a} \hat{b} \wedge \partial_a \hat{a} \hat{b} + \partial_a \hat{a} \hat{b} \wedge \partial_a \hat{a} \hat{b} \dots$$

$$\Rightarrow R^{\hat{a}}_{\hat{b}} = R^{\hat{a}}_{\hat{b}} - \left(\frac{N}{A}\right)^2 \omega^{\hat{a}} \wedge \omega^{\hat{a}} \eta^{\hat{c}}_{\hat{b}}$$

Bonus Lecture 2: Gauss Codazzi Formalism & the Gibbon-Hawking boundary TermGeometry of Submanifold. (Gauss Codazzi Formalism)

Recall the variational principle for S_{EH} .

~~$$\delta S_{EH} = \delta \int d^4x \sqrt{-g} R$$~~

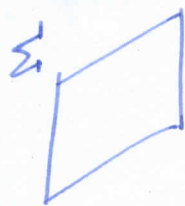
$$\delta S_{EH} = \delta \int d^4x \sqrt{-g} R$$

$$= \int d^4x \sqrt{-g} G_{ab} \delta g^{ab} + \int_{\partial M} d^3x \sqrt{-g} (\nabla_n \delta g - n_a \nabla_b \delta g^{ab})$$

$$\delta S_{EH} = \int_M d^4x \sqrt{-g} G_{ab} \delta g^{ab} + \int_{\partial M} d^3x \sqrt{-g} (\nabla_n \delta g - n_a \nabla_b \delta g^{ab})$$

we will add boundary term to action.

$$S_{\partial M} = \int_{\partial M} (\text{boundary terms}) \delta g^{ab}$$

Geometry of Submanifold (Gauss-Codazzi Formalism)

let $\Sigma \subset M$ be a submanifold of dim n
(dim $M = D$)

The codimension of Σ is $D-n$; & is the no. of linearly independent normals, i ; vectors $n \in T_p(M)$

s.t. $n \cdot \sigma^A = 0$ $\forall$ coordinate functions σ^A on Σ .

or $\eta_{i\mu} \frac{\partial x^\mu}{\partial x^A} = 0$ (will take $\eta_{i\mu} \eta_j^\mu = \pm \delta_{ij}$)

The 1st Fundamental form of Σ is

$$h_{ab} = g_{ab} + \sum_{i=1}^{D-n} (-1)^{\#i} n_{ia} n_{ib}$$

↑
- timelike
+ spacelike.

(The first fundamental form;
what it does is: takes the metric and kills off
any piece of metric normal to Σ .)

(Subtracting off all the normal components)

h_{ab} projects out the components of g_{ab} normal to Σ ,
gives a local distance on Σ .

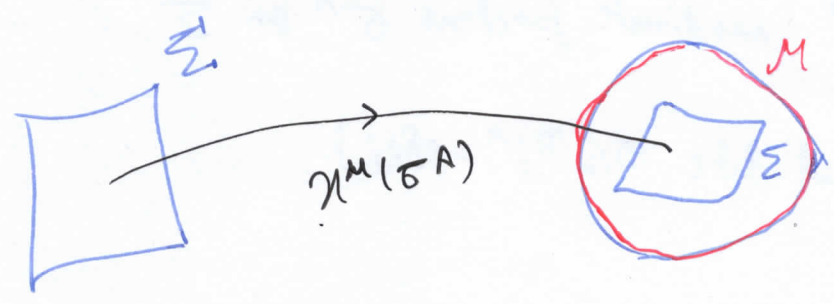
Since Σ is a manifold, we can also consider
intrinsic quantities. living in $T(\Sigma)$

$$(h_{ab} \in T^*(M))$$

$$\frac{\partial x^M}{\partial \sigma^A} : T_p^*(M) \rightarrow T_p^*(\Sigma) \quad \text{pull back map}$$

$$\omega_M \longmapsto \omega_A = \omega_M \frac{\partial x^M}{\partial \sigma^A}$$

$\star$ is a projection operator on to $T_p^*(\Sigma)$



$g_{AB} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^A} \frac{\partial x^\nu}{\partial \sigma^B}$ is the intrinsic metric. (pg 52)

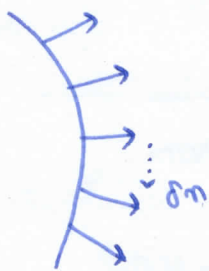
The 2nd Fundamental Form or Extrinsic curvature of Σ measures the "curvature" of the embedding in M .

$$K_{i,\mu\nu} = h_{(\mu}^{\sigma} h_{\nu)}^{\lambda} \nabla_{\sigma} \eta_{i\lambda}$$

(or) $K_{iAB} = \frac{\partial x^\mu}{\partial \sigma^A} \frac{\partial x^\nu}{\partial \sigma^B} \nabla_{\mu} \eta_{i\nu}$

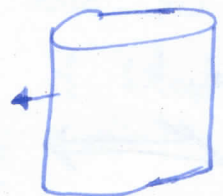
$$= -\eta_{i\nu} {}^{(3)}\nabla_A \frac{\partial x^\nu}{\partial \sigma^B}$$

where ${}^{(3)}\nabla_A$ is the connection Σ inherits from M .



e.g. Cylinder embedded in $\mathbb{R}^3$
 $\{x^2 + y^2 = a^2\} \subset \mathbb{R}^3$

$$h_{ab} = \delta_{ab} - \eta_a \eta_b$$



so $h_{ab} = \begin{pmatrix} \sin^2 \theta & -\sin \theta \cos \theta & 0 \\ -\sin \theta \cos \theta & \cos^2 \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

in cartesian.

$$\eta = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}$$

(or) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in polars

Extrinsic curvature :

(in polar coordinates)

$$K_{ab} = -\Gamma_{ab}^r \quad \text{in polars}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} r \\ \theta \\ z \end{matrix}$$

Intrinsic picture; : $\sigma^A = \{0, z\}$

(pg 54)

~~$\delta_{AB} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}$~~

$$K_{AB} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

For Codimension > 1

; normals could vary...

→ The normals can vary across Σ

(we can choose different set of normals)

ex: line embedded in 3d.



So; we also ~~have~~ have a fundamental form associated to it

↔ So; it is like telling us ; gauge potential ~~associated~~ associated to that choice of normal.

The Normal Fundamental forms are the connection on the normal bundle of Σ

$$P_{\mu ij} = \eta_{i\nu} \nabla_{\mu} \eta_j^{\nu}$$

(Normal fundamental forms actually take the variation of normal & project it down transverse to Σ)

so; choose $\nabla_{\eta_i} \eta_j = 0$ (ie; we extend normal fields geodesically off the manifold)

But; what we can't do is that;
we can't get rid of this any twisting of my
normal forms although that is potentially a gauge
choice.

(what isn't a gauge choice is of course the curvature of
this connection)

The curvature of Σ is now related to the
curvature of M & extrinsic curvature.

Gauss Equation:

$${}^{(m)}R^a_{bcd} = {}^{(D)}R^{a'}_{b'c'd'} h^{a'}_{a'} h^{b'}_{b'} h^{c'}_{c'} h^{d'}_{d'} \\ - \sum_{i=1}^{D-m} (-)^{\#i} [K_i^a{}_c K_i^b{}_d - K_i^a{}_d K_i^b{}_c]$$

Proof for codimension 1.

uses Riemann identity.

$$\left({}^{(m)}\nabla_a {}^{(m)}\nabla_b - {}^{(m)}\nabla_b {}^{(m)}\nabla_a \right) V^c = {}^{(m)}R^c{}_{dab} V^d$$

$${}^{(m)}R^a{}_{bcd} V^b = {}^{(m)}\nabla_c {}^{(m)}\nabla_d V^a - c \leftrightarrow d$$

$$= h^e{}_c h^f{}_d h^a{}_g \nabla_e ({}^{(m)}\nabla_f V^g) - c \leftrightarrow d$$

Taken higher dimensional derivative; and projected
down.

$$= h^e{}_c h^f{}_d h^a{}_g \nabla_e (h^p{}_f h^q{}_a \nabla_p V^q) - c \leftrightarrow d$$

do the same thing.

$$= h_c^e h_a^p h^a_q \nabla_e \nabla_p V^q$$

$$+ h_c^e h_d^f h^a_q \nabla_e (h_f^p) \nabla_p V^q$$

because $\nabla_n V = 0$

$$+ h_c^e h_d^p h^a_g \nabla_e (h_g^q) \nabla_p V^q \quad - \leftrightarrow d$$

$$h_{ab} = g_{ab} + n_a n_b$$

; take n to be spacelike
for the sake of argument

$$\Rightarrow \nabla_c h_{ab} = n_a K_{bc} + n_b K_{ac}$$

$$(m) R^a_{bcd} V^b = h_c^e h_d^p h^a_q \nabla_e \nabla_p V^q$$

$$+ h_c^e h_d^p h^a_g K_c^g \underbrace{n_{av} \nabla_p V^q}_{- \nabla_p n_{av} = - \nabla_p K_{av}} \quad - \leftrightarrow d$$

$$- \nabla_p n_{av} = - \nabla_p K_{av}$$

(because n & V are
orthogonal)

$$= \nabla^a R^a_{b'c'd'} h^{c'}_c h^{d'}_d h^a_{a'} V^{b'}$$

$$- V^{b'} K_{b'd} K^a_c + V^{b'} K_{b'd} K^a_d$$

(we proved for n spacelike; and codimension 1)

Now let's look at,

how we can add a boundary term to cancel off
the normal derivatives of g .

Back to Boundary Term:

1957

Hold boundary metric fixed;

only n^a varies

(take spacelike for simplicity)

$$\left. \begin{aligned} \delta n^a &= -n^a n_b \delta n^b \\ \delta g_{ab} &= -n^a \delta n_b - n_b \delta n^a \end{aligned} \right\} \Rightarrow \delta n^a = -\frac{1}{2} n^a n_b n_c \delta g^{bc}$$

$$\boxed{\delta n^a = -\frac{1}{2} n^a n_b n_c \delta g^{bc}}$$

Now consider $\delta(D_a n^a) = P_a \delta n^a + \delta P_a n^a$

$$= -\frac{1}{2} P_a (n^a n_b n_c \delta g^{bc}) - \frac{1}{2} P_n \delta g$$

$K = P_a n^a$

$$\delta(K) = -\frac{1}{2} K n_b n_c \delta g^{bc} - \frac{1}{2} n_b n^c n^a P_a \delta g^{bc} - \frac{1}{2} P_n \delta g$$

$$= -\frac{1}{2} K n_b n_c \delta g^{bc} - \frac{1}{2} P_n \delta g - \frac{1}{2} P_a n^a$$

$$- \frac{n_b}{2} h^{ac} P_a \delta g^{bc} + \frac{1}{2} n_b P_a \delta g^{ba}$$

$$\delta K = \frac{1}{2} [n_b P_a \delta g^{ab} - P_n \delta g] - \frac{1}{2} K n_b n_c \delta g^{bc} - \frac{1}{2} {}^{(3)}\nabla_c (n_b \delta g^{bc}) + \frac{1}{2} K_{ab} \delta g^{ab}$$

These are the terms which we want

$({}^{(3)}\nabla_c (n_b \delta g^{bc}))$ gives zero contribution

because it is total derivative

because on boundary; boundary is zero.

(There is nothing to my boundary)

Hence;

$$\delta(k\sqrt{g}) = \frac{1}{2} [m_b \nabla_a \delta g^{ab} - \nabla_m \delta g] + \frac{1}{2} [K_{ab} - K_{hab}] \delta g^{ab}$$

This vanishes
on boundary

So, the boundary term

$$S_{GH} = \frac{-1}{8\pi G} \int K \sqrt{g} \quad \text{is Gibbons-Hawking term.}$$

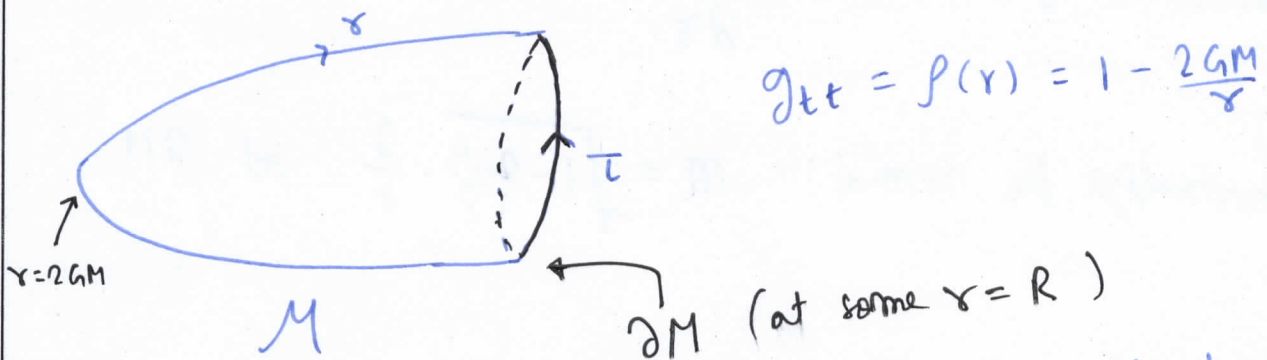
Lec 8: Calculation of Euclidean Action & Black-hole entropy, Israel Equations.

$$S_{G.H.} + S_{E.H.} = S$$

we finally have

$$S = \underbrace{\frac{-1}{16\pi G} \int_M d^4x \sqrt{-g} R}_{S_{E.H.}} - \underbrace{\frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{-h} K}_{S_{G.H.}}$$

Recall, Euclidean Schwarzschild has periodic τ : $\Delta\tau = 8\pi GM$.



for Schwarzschild; $S_{EH} \equiv 0$ because it has zero Ricci Curvature

i.e. $S_{EH}|_{\text{Schwarzschild}} \equiv 0$

Boundary at large $r=R$

$$dS^2_{\partial M} = \left(1 - \frac{2GM}{R}\right) d\tau^2 + R^2 d\Omega^2_{\pi}$$

$\uparrow$
 $d\theta^2 + \sin^2\theta d\phi^2$

~~has curvature because~~

∂M does not have curvature because of $r=R$ in an intrinsic sense (because R is just a constant)

The ~~sub~~ submanifold ∂M has curvature because of the $d\Omega_{\mathbb{R}^2}$ piece.

So; ∂M has curvature because ds^2 is curved (constant positive curvature space)

↳ it does not have curvature dependent on R in an intrinsic sense because R is constant.

∂M does have extrinsic curvature.....

normal in M $n \propto \frac{\partial}{\partial r}$

lets normalize the normal; $n = \sqrt{1 - \frac{2GM}{r}} \cdot \frac{\partial}{\partial r}$ at ∂M

~~Extrinsic curvature~~

Extrinsic Curvature:

$$K = \nabla_a n^a = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} n^a) = \frac{1}{r^2} (n^2 n^r)'$$

$$= \frac{1}{r^2} (r^2 n^r)' \Big|_{r=R} = \frac{2}{r} \sqrt{1 - \frac{2GM}{r}} + \frac{GM}{r^2 \sqrt{1 - \frac{2GM}{r}}} \Big|_{r=R}$$

$$K = \frac{2}{r} \sqrt{1 - \frac{2GM}{r}} + \frac{GM}{r^2 \sqrt{1 - \frac{2GM}{r}}} \Big|_{r=R}$$

volume factor; $\sqrt{h} d^3x = \sqrt{1 - \frac{2GM}{r}} \cdot R^2 \sin\theta d\theta d\phi dz$

Put together:

$$\int K \sqrt{h} d^3 x = \underbrace{4\pi\beta}_{\int dt, d\theta, d\phi} \cdot R^2 \left(\frac{2}{R} \left(1 - \frac{2GM}{R} \right) + \frac{GM}{R^2} \right) = 4\pi\beta [2R - 3GM]$$

S_{GR} blows up as $R \rightarrow \infty$.

ie; ~~This~~ This diverges as $R \rightarrow \infty$!!

Note: even if $M=0$; it is still divergent

so; This means, flat space has divergent action.

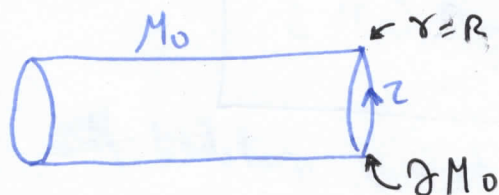
Divergent also for $M=0$; ie; flat space has divergent action.

Flat spacetime has diverging action in this particular way of writing.

Then it is just some intrinsic divergent; that we normalize.

So we take it away.

Subtract this background divergence



$$ds_0^2 = \Lambda^2 dt^2 + dr^2 + r^2 d\Omega_{\text{II}}^2$$

we kept a constant Λ^2 at front of dt^2 .

because ~~the~~ ~~the~~, If we going to subtract off the background action; then the two boundary manifolds have to agree. And therefore

periodicity of time has to be the same.

(1962)

$$\Delta z = \beta$$

Choose Λ to make sure that manifolds match up at $r=R$.

$$ds^2_{0, 2M_0} = \Lambda^2 dz^2 + R^2 d\Omega_{\pi}^2$$

To match $2M_0$ and $2M$ (the time circles)

$$\Lambda^2 = 1 - \frac{2GM}{R}$$

For M_0 (flat spacetime) $K_0 = \frac{2}{R}$ (because no mass term)

$$\sqrt{h_0} d^3x \longrightarrow 4\pi\beta \cdot \sqrt{1 - \frac{2GM}{R}} R^2$$

$$\text{so; } \int \sqrt{h_0} K_0 d^3x = 4\pi\beta \times \frac{2}{R} \sqrt{1 - \frac{2GM}{R}}$$

Taylor expand
 $1 - \frac{GM}{R} + \mathcal{O}(R^{-1})$

$$\Rightarrow S_{GH}^{(0)} = -\frac{1}{8\pi G} \times 4\pi\beta [2R - 2GM]$$

has the piece GM in it because matched the circles at $r=R$.

Hence, $S_{Schwarzschild} = S_{GH} - S_{GH}^{(0)}$

(1962)

$$S_{\text{Schwarzschild}} = S_{\text{GH}} - S_{\text{GH}}^{(0)}$$

$$= -\frac{1}{8\pi G} (4\pi\beta) \times (-GM)$$

$$= \frac{\beta M}{2}$$

$$S_{\text{Schwarzschild}} = \frac{\beta M}{2}$$

$$= 4\pi G M^2$$

This is nice & finite;
dependent on periodicity
of Euclidean time & Mass

Consider the partition function.

$$Z \sim \text{tr } e^{-\beta H}$$

$$H \sim \int d^3x \mathcal{H} = \frac{1}{\beta} \int d^4x \mathcal{L}_E$$

Euclidean Lagrange
density.

$$= \frac{I_E}{\beta} \quad \leftarrow I_E \text{ is Euclidean action.}$$

So;

$$Z \sim \text{tr } e^{-I_E}$$

At $T = \frac{1}{\beta} = 8\pi G M$ has Schwarzschild solution as
a saddle point.

Entropy; $S = \beta^2 \frac{\partial}{\partial \beta} [-\beta^{-1} \ln Z]$

$$= \beta^2 \frac{\partial}{\partial \beta} [I_E / \beta] = \beta^2 \frac{\partial}{\partial \beta} \left[\frac{M}{2} \right]$$

$$M = \frac{\beta}{8\pi G}$$

$$\Rightarrow S = \frac{\beta^2}{16\pi G} = 4\pi G M^2 = \frac{4\pi (2GM)^2}{4G}$$

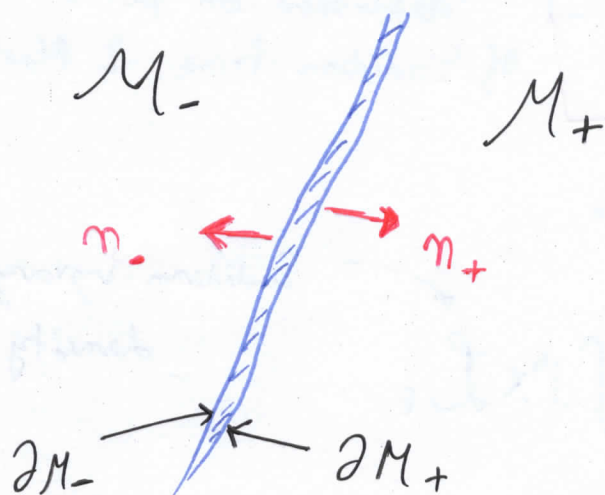
$$S = \frac{4\pi(2GM)^2}{4G}$$

$$S = \frac{4\pi(2GM)^2}{4G}$$

(Pg 63)

Note; In varying δS_{can} we ~~held a term~~ had a term $(K_{ab} - K h_{ab}) \delta h^{ab}$

A "shell" in spacetime has 2 sides



~~The shell~~

The very thin shell will be a combination of the ~~two~~ boundaries of M_- & M_+

So; If we think about this from Gauss Codazzi perspective: looking at the variation of action there

Get a term, $(K_{+ab} - K_+ h_{ab}) - (K_{-ab} - K_- h_{ab})$

↑
flip normal n_-
to get this minus sign.

(now normals point in same direction)

→ ISRAEL EQUATION

$$\Delta K_{ab} - \Delta K \cdot h_{ab} = 8\pi G S_{ab}$$

where
$$S_{ab} = \int_{\Sigma_-}^{\Sigma_+} T_{ab}$$

ISRAEL
EQUATION.

where we take;

Energy momentum tensor for the surface (the shell)
is strongly localized.

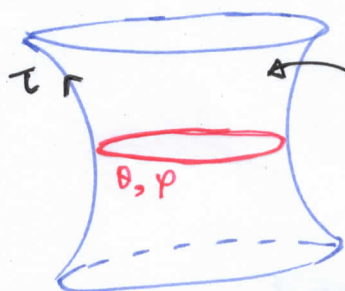
→ Integrate Energy momentum Tensor through the
shell surface.

and it gives you S_{ab} .

i.e: $T_{ab} \sim S_{ab} \delta(z)$ where, z is coordinate
through wall

Can take δ -function limit for codomain 1 only.

Example 1



Consider $x^2 - \tau^2 = L^2$ in
Minkowski spacetime.

To get normal, we look at

$$d(x^2 - \tau^2) = 2(x dx - \tau d\tau) \propto n$$

To make calculations easy, we put coordinates on this
hyperboloid: τ, θ, ϕ .

Σ is $X^M(\sigma^A) = (L \sinh \tau/L, L \cosh \tau/L, \theta, \phi)$

i.e: $X^M(\sigma^A) = (L \cdot \sinh \frac{\tau}{L}, L \cdot \cosh \frac{\tau}{L}, \theta, \phi)$ in polars.

$$\eta_\mu = (-\sinh(\frac{T}{L}), \cosh \frac{T}{L}, 0, 0) \quad (1966)$$

This gives $K_{00} = -\Gamma_{00}^\rho \eta_\rho$

$$= R \eta_R$$

$$= L \cdot \cosh T/L$$

$$= -\frac{g_{00}}{L}$$

$$\Rightarrow K_{00} = -\frac{g_{00}}{L}$$

similarly:

$$K_{\phi\phi} = -g_{\phi\phi}/L$$

$$K_{zz} = -g_{zz}/L$$

Writing curvature in terms of intrinsic components.

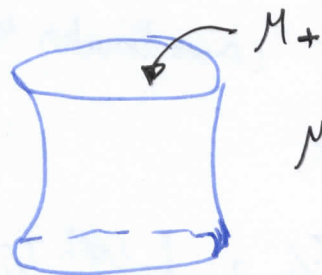
Then extrinsic curvature is proportional to intrinsic ~~or~~ or induced metric.

$$K_{AB} = -\frac{\gamma_{AB}}{L}$$

$$\Rightarrow K_{AB} = K \gamma_{AB} = \frac{2}{L} \gamma_{AB}$$

But if we take inside & outside of ~~hyperboloid~~

Hyperboloid



M_-

$$\text{so: } K_{-AB} = K_{+AB}$$

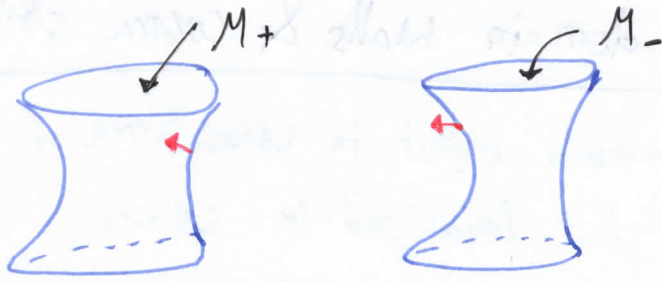
$\Downarrow$

Hence you get zero

energy momentum tensor.

This was expected

But if we take 2 ~~the~~ interiors.



Take two copies of same setup.
(like mirror image)

~~Then~~

Then $K_{-AB} = -K_{+AB}$

~~equal & opposite.~~

extrinsic curvatures are equal & opposite

we get $\Delta K_{AB} - \Delta K h_{AB} = \frac{4}{L} \gamma_{AB} = 8\pi G S_{AB}$

ISRAEL Equation.

Surface with energy momentum

$$T_{ab} = \frac{1}{2\pi G L} h_{ab} \delta(z)$$

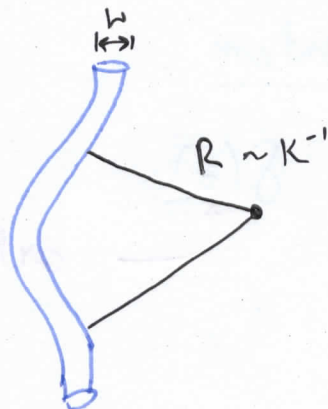
or normal coordinate.

Lec 9: Physical submanifolds: domain walls & cosmic strings.

- A point particle ~~is~~ is 1 dimensional object in spacetime, because it exists in time but localized in space.
- We could have extended objects in 3d.
like, ~~str~~ strings, membranes or walls.

Another way submanifolds appear is as appropriate ways of describing extended objects that are localized in some direction.

Example String



$w \Rightarrow$ width
 $R \Rightarrow$ radius of curvature
 (... extrinsic curvature)

w is small; i.e.; $w \ll R$

Wall



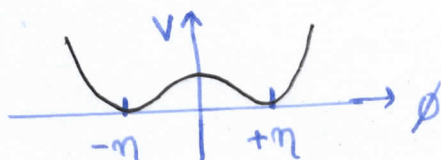
These objects, have a realization as non-perturbative solutions in QFT.

Domain Wall

consider scalar field ϕ .

$$\mathcal{L}_\phi = \frac{1}{2}(\partial\phi)^2 - V(\phi) \quad ; \quad V(\phi) = \frac{\eta}{2}(\phi^2 - \eta^2)^2$$

$$\phi \in \mathbb{R}$$



potential with symmetry breaking.

$V(\phi)$ has $\mathbb{Z}_2$ symmetry.

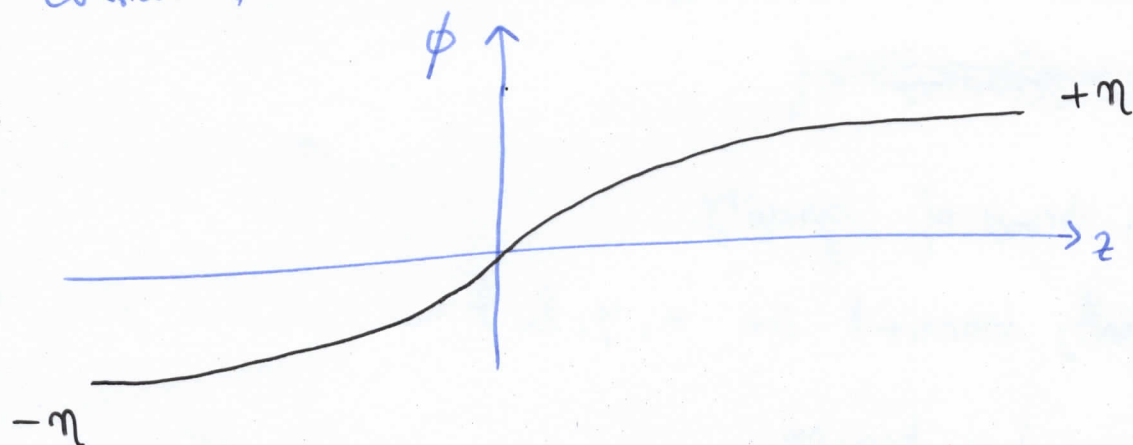
(1969)

Once we choose a vacuum, we broke our $\mathbb{Z}_2$ symmetry.

(Normally; we pick on vacu & quantize around it)

But, instead

consider the solution $\phi = \eta \tanh(\sqrt{\lambda} \cdot \eta z)$



This solution interpolates from one vacuum to the other vacuum.

first of all; lets check $\phi = \eta \tanh(\sqrt{\lambda} \cdot \eta z)$ is a solution.

$$\phi' = \sqrt{\lambda} \eta^2 \text{sech}^2(\sqrt{\lambda} \eta z)$$

$$\phi'' = -2\lambda \eta^3 \text{sech}^2(\sqrt{\lambda} \eta z) \tanh(\sqrt{\lambda} \eta z)$$

for the moment; if we ignore gravity;

our equation of motion is

~~Ignore~~

~~e.g.~~ Ignore gravity;

e.o.m. $\square\phi + \frac{\partial V}{\partial\phi} = 0$

$$-\phi'' + 2\lambda\phi \cdot (\phi^2 - \eta^2) = 0$$

$$\phi'' = -2\lambda\eta^3 \cdot (1 - \phi^2/\eta^2) \cdot \phi/\eta$$

(pg 70)

→ satisfies equation of motion.
in the absence of any sort
of gravitational field.

✓... good.

"Solution in absence of gravity"

For the moment, let's proceed.

~~Solution in absence of~~

Solution in absence of gravity.

Translationally invariant in $x, y, & t$.

Energy momentum tensor:

$$T_{\mu\nu\phi} = \phi_{,\mu} \phi_{,\nu} - g_{\mu\nu} \left(\frac{1}{2} (\partial\phi)^2 - V \right)$$

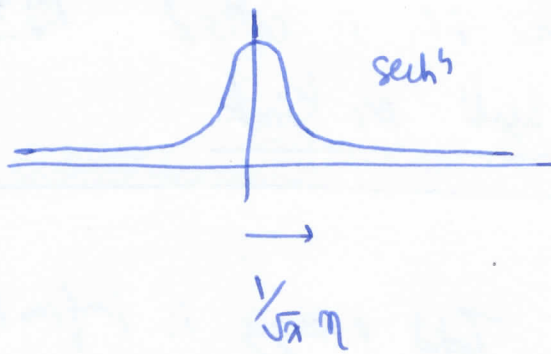
The scalar field only depends on z

so: $\phi_{,\mu} \propto \delta^z_\mu$

$$\text{so: } T_{\phi\mu\nu} = \lambda\eta^4 \text{sech}^4(\sqrt{\lambda}\eta z) \delta^z_\mu \delta^z_\nu - g_{\mu\nu} (-\lambda\eta^4 \text{sech}^4(\sqrt{\lambda}\eta z))$$

$$T_{\phi\mu\nu} = 2V \cdot (g_{\mu\nu} + \overset{\uparrow}{\eta_\mu} \delta^z_\mu \overset{\uparrow}{\eta_\nu} \delta^z_\nu) = 2V h_{\mu\nu}$$

$$\boxed{T_{\phi\mu\nu} = 2V h_{\mu\nu}}$$



T_ϕ is sharply localized with some finite width.

Now; the width is small compared to the scale we are interested in.

to hence; localized around $z=0$.

$$\int T_0 dz = \frac{4}{3} \sqrt{2} \cdot \eta^3 = 6$$

energy per unit area

⊗ Tension.

It is non-perturbative; because

we cannot get to complete vacuum everywhere without an infinite amount of energy.

i.e; if we wanted to make the true vacuum either $+\eta$ or $-\eta$ $\therefore$ we will have to lift ϕ over the barrier in field space (one well to other) over infinite region of space

$\Rightarrow$ This will cost us infinite amount of energy $\Rightarrow$ So we can't get rid of this by any sort of finite process.

~~The energy~~ So; This is topologically stable solution.

Topologically stable ... sometime it is called Domain Wall or Kink Pg 72

Add gravity

Energy per unit area is finite; so Total energy is infinite because here space extends in all x, y, z direction.

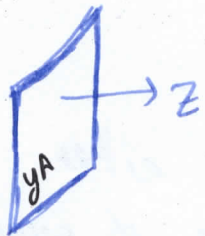
? But the total energy of wall is infinite.

Is that going to be a problem?

lets ~~just~~ just try it any-way

$$ds^2 = \underbrace{A^2(z)}_{\text{wall coordinate}} \gamma_{AB} dy^A dy^B - dz^2$$

$\hookrightarrow z$ distance from the wall.



$$ds^2 = A^2(z) \gamma_{AB} dy^A dy^B - dz^2$$

$z \neq$ spacelike distance from the wall
(signature $- + + + \dots$)

Use Cartan Method to calculate curvature

$$\underline{\omega}^z = \underline{dz} \quad \text{(crossed out)} \quad \underline{\omega}^A = A(z) \underline{dy}^A$$

$$\underline{\omega}^A = A(z) \cdot e^{\hat{A}}_A \underline{dy}^A$$

$\hookrightarrow$ orthonormal basis for γ
(o/n basis for γ)
(dreibein)

$$\underline{\omega}^z = dz, \quad \underline{\omega}^{\hat{A}} = A(z) e^{\hat{A}}_A dz^A$$

pg 73

we have $d\underline{\omega}^z = 0$

$$d\underline{\omega}^{\hat{A}} = \frac{A'(z)}{A(z)} \underline{\omega}^z \wedge \underline{\omega}^{\hat{A}} - A(z) \vartheta^{\hat{A}}_{\hat{B}} \wedge \underline{\omega}^{\hat{B}}$$

connection
on γ

$$e^{\hat{B}}_B dy^B$$

Now, we can read off our connection one forms.

$$\vartheta^{\hat{A}}_{\hat{B}} = \vartheta^{\hat{A}}_{\hat{B}0} \quad ; \quad \vartheta^{\hat{A}}_z = \frac{A'(z)}{A(z)} \underline{\omega}^{\hat{A}}$$

Now, we want to calculate curvature two forms

$$\text{compute } R \quad ; \quad R^{\hat{A}}_{\hat{B}} = d\vartheta^{\hat{A}}_{\hat{B}} + \vartheta^{\hat{A}}_{\hat{C}} \wedge \vartheta^{\hat{C}}_{\hat{B}} + \vartheta^{\hat{A}}_z \wedge \vartheta^z_{\hat{B}}$$

$$\Rightarrow R^{\hat{A}}_{\hat{B}} = R^{\hat{A}}_{\hat{B}0} + \left(\frac{A'(z)}{A(z)} \right)^2 \underline{\omega}^{\hat{A}} \wedge \underline{\omega}^{\hat{B}}$$

$$\text{Similarly: } R^{\hat{A}}_z = d\vartheta^{\hat{A}}_z + \vartheta^{\hat{A}}_{\hat{B}} \wedge \vartheta^{\hat{B}}_z$$

$$= \frac{A''}{A}$$

$$\Rightarrow R^{\hat{A}}_z = \frac{A''(z)}{A(z)} \underline{\omega}^z \wedge \underline{\omega}^{\hat{A}} + A'(z) d\underline{\omega}^{\hat{A}} + A'(z) \vartheta^{\hat{A}}_{\hat{B}} \wedge \underline{\omega}^{\hat{B}}$$

$$A'(z) [d\underline{\omega}^{\hat{A}} + \vartheta^{\hat{A}}_{\hat{B}} \wedge \underline{\omega}^{\hat{B}}]$$

→ 0 by Cartan's first equation

$$\Rightarrow R^{\hat{A}}_z = \frac{A''(z)}{A(z)} \underline{\omega}^z \wedge \underline{\omega}^{\hat{A}}$$

Till now; no where we invoked dimensions being 4.

(1974)

→ so; It is true for D dimensions.

We have only assumed that $\gamma_{AB}^{(y)}$ depends on y .

(has not many assumption of the y^A part; ...
other than it has $O(n)$ basis)

Note: Calculation independent of dimensionality & form of K .

$$\Rightarrow R^{\hat{A}\hat{B}}_{\hat{C}\hat{D}} = \frac{1}{A^2} R_0^{\hat{A}\hat{B}}_{\hat{C}\hat{D}} + \left(\frac{A'}{A}\right)^2 \left(\delta^{\hat{A}}_{\hat{C}} \delta^{\hat{B}}_{\hat{D}} - \delta^{\hat{A}}_{\hat{D}} \delta^{\hat{B}}_{\hat{C}} \right)$$

$$R^{\hat{A}\hat{Z}}_{\hat{B}\hat{Z}} = \frac{A''}{A} \delta^{\hat{A}}_{\hat{B}}$$

So; Our Ricci Tensor

$$R^A_B = \frac{1}{A^2} R_0^A_B + \left((D-2) \left(\frac{A'}{A}\right)^2 + \frac{A''}{A} \right) \delta^A_B$$

$$R^Z_Z = (D-1) \frac{A''}{A}$$

Now $D=4$; and take Y to be constant curvature space

ie; $D=4$, and take γ_{AB} constant curvature K/L^2

* $K=0$ flat

* $K=1$ dS

* $K=-1$ AdS

$$\hat{\omega}_A = \omega_A + \frac{1}{L^2} \gamma_{AB} \hat{x}^B = \hat{\omega}_A + \frac{1}{L^2} \hat{x}^B \gamma_{AB}$$

Einstein Equation & ϕ equation.

(1975)

$$G^2_2 = \frac{3K}{L^2 A^2} - \frac{3A'^2}{A^2} = 8\pi G (V - \frac{1}{2} \phi'^2)$$

$$G^0_0 = \frac{K}{L^2 A^2} - \frac{A'^2}{A^2} - \frac{2A''}{A} = 8\pi G (V + \frac{1}{2} \phi'^2)$$

} 2 Einstein equation

$$\phi'' + \frac{3A'}{A} \phi' = 2\lambda \phi (\phi^2 - \eta^2) \quad \} 1 \phi \text{ equation.}$$

Analytically; consider $8\pi G \eta^2 = \epsilon \ll 1$

~~mag~~ magnitude of RHS $8\pi G \lambda (\eta^2 - \phi^2)^2 = 0$ $(8\pi G \eta^2) \times \lambda \eta^2$

~~$\sqrt{\lambda} \eta$ was setting scale for ϕ~~

$\sqrt{\lambda} \eta$ was setting scale for $\tanh(\dots)$ solution.

tells about
Gravitational
Strength of
the wall

gradient
... how wide
it ... how
fast things
vary

$m^2 \phi$

If solution is weakly gravitating ($\eta/M_P \ll 1$) (ie: well below plank scale)

(Then we can solve it in iterative way)

Linearization in G.R.

$\lambda \eta^2$ is actually $m^2 \phi$
: if we were thinking
about it in terms of spontaneous
symmetry breaking; it would
just be the mass of ϕ
particle around the vacuum.

We solve for our background

field in flat space $\rightarrow$ calculate energy momentum $\rightarrow$
(for $\Sigma = 0$)

→ plug in Einstein Equation → it then gives correction to geometry to order ϵ (1376)

this can feedback in.....

so; $A = A_0 + \epsilon A_1 + \dots$

$\phi = \phi_0 + \epsilon \phi_1 + \dots$

doing series expansion & solving order by order in ϵ .

$O(\epsilon^0)$: $A_0 = 1$, $\phi_0 = \eta \tanh(\sqrt{\lambda} \cdot \eta z)$

Now we use the energy momentum we calculated to find solution for A : so the back reaction.

$O(\epsilon^1)$: $A'' \sim \frac{K}{L^2} \Rightarrow K=1$, $L = O(1/\epsilon)$

$A'' = -8\pi G V$: $O(\epsilon)$

Solved by $A_1 = -\frac{1}{3} \log[\cosh(\sqrt{\lambda} \cdot \eta z)] - \frac{1}{3} \ln z \approx -\frac{1}{12} \text{sech}^2(\sqrt{\lambda} \eta z)$

so; very roughly A is. ; $A \approx 1 - \frac{|z|}{L}$, $L = \frac{3}{\sqrt{\lambda} \eta \epsilon}$

→ smooths off around the wall..



→ away from wall.

A function graph.

So the A function we get is

$$ds^2 = \left(1 - \frac{z^2}{L^2}\right)^2 \left[dt^2 - L^2 \cosh^2(t/L) d\Omega_{II}^2 \right] - dz^2$$

(*)

→ This is what we get for Geometry.

L is large if ϵ is small.

Now does the geometry look like

$$\left. \begin{aligned} \rho &= (L-z) \cosh(t/L) \\ z &= (L-z) \sinh(t/L) \end{aligned} \right\} (*)$$

So, if we move away from wall; and sitting roughly at $\phi = +\eta$ or $-\eta$.

My energy momentum is vanished.

So; from the perspective of Einsteins equation, we are in vacuum; so we should have vacuum solution of Einstein Gravity.

In this case; if we have vacuum solution of Einstein gravity that looked like z^2 times something $\Rightarrow$ Its kind of not clear there should be any curvature sitting outside away from wall.

lets do the coordinate transformation (*)

$$\rho^2 - \tau^2 = (L - z)^2$$

(pg 78)

$$ds^2 = d\tau^2 - d\rho^2 - \rho^2 d\Omega^2 \quad (\text{Minkowski in polar coordinates})$$

$$\text{but; } z=0 \longleftrightarrow \rho^2 - \tau^2 = L^2 \quad (\text{hyperboloid})$$

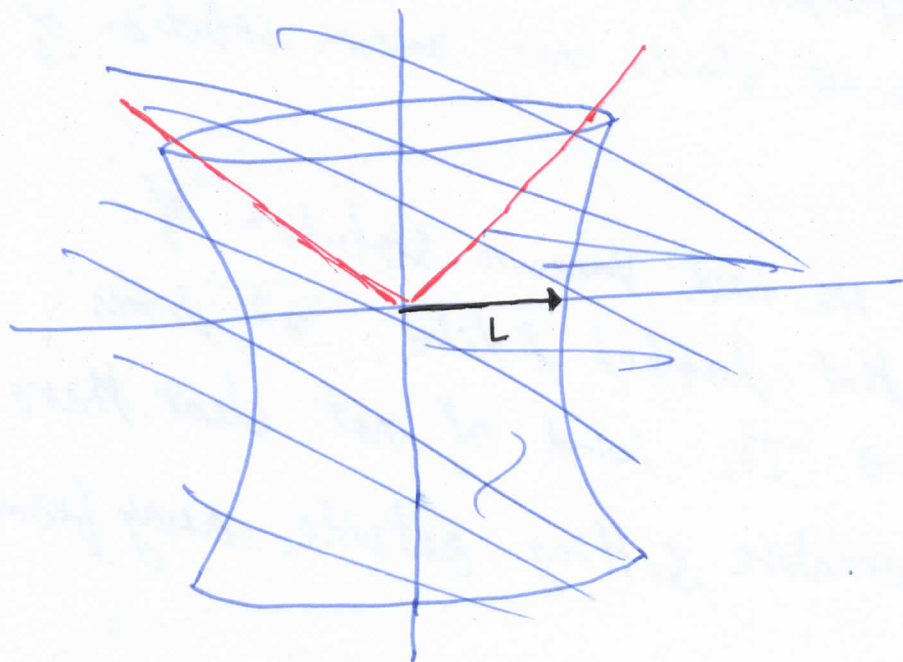
So; we got a curious ~~so~~ situation ~~but~~ where we might think there is problem at $z = \pm L$ (from equation $\star$)

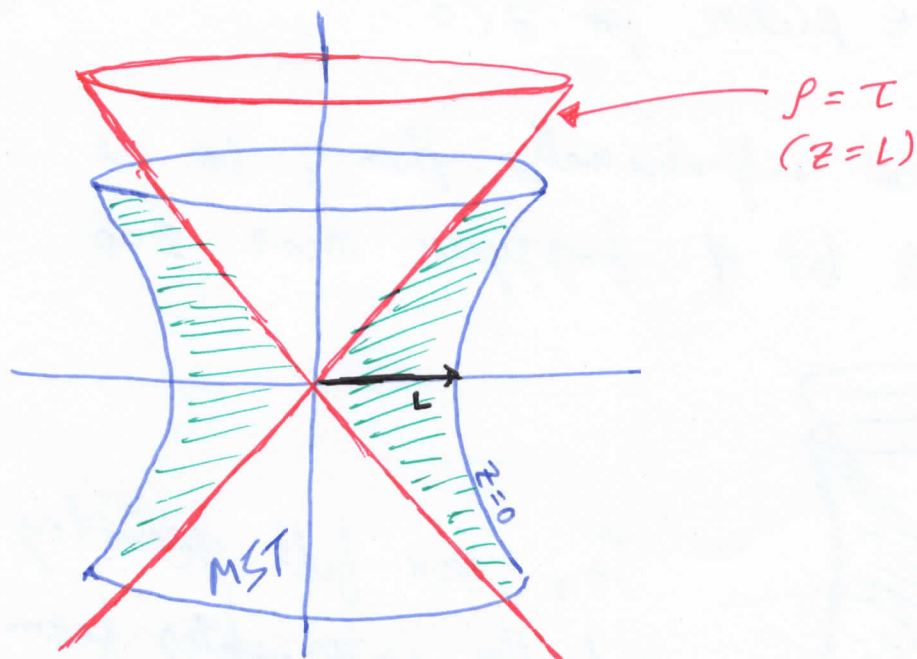
but doing coordinate transformation $(\star)$; we see that $z=0$ looks like hyperboloid in new coordinates.

$z=L$ looks like is just $\rho^2 = \tau^2$

~~what if this thing has an area?~~

~~We argue; that it does not have an area.~~





The coordinates cover the shaded green region.

This shows that $z=L$ is horizon (not a singularity)

$z=L$ is a horizon.

by writing down the coordinate transformation (*) we have explicitly analytically continued across this horizon.

~~Inside~~ Inside here we just have
Minkowski Space Time (MST)

This is only half the picture for $z > 0$

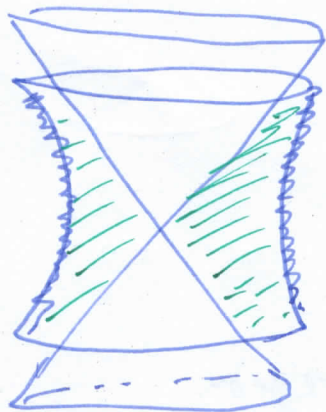
Infact if we look at $z < 0$; we would flip the sign in the definition of ρ and τ .

and it would now be $(L+z) \cosh(-)$
 $(L+z) \sinh(-)$

So; its the same picture for $z < 0$

(1780)

The wall is not infinitesimally thin; so we get little bit of buzziness near $z=0$.



So; our full geometry actually interpolates from interior of one hyperboloid to the interior of other.

If we look at $\tau = t = 0$

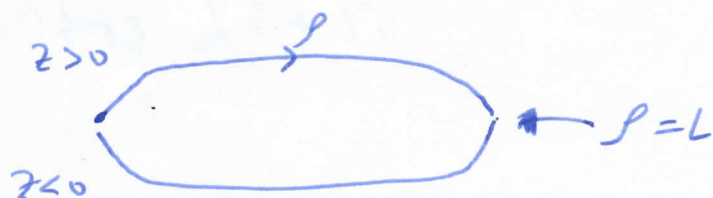


This is what happens along $t = \text{constant}$ surface if we glue these things together & smudge out due to wall thickness

Has topology S^3

i.e; it is compact.

So, the total energy of my wall is finite because the wall no longer itself has ∞ area.



"Gravity has solved infinite energy by spontaneously compactifying space."

Check, Israel Equation $8\pi G\sigma = 4/L$

$$\Rightarrow 8\pi G \cdot \frac{4}{3} \sqrt{\Lambda} \eta^3 = \frac{4 \sqrt{\Lambda} \cdot \eta}{3} \epsilon$$

✓ fits together $\epsilon = 8\pi G \eta^2$

This says that; The linearized calculation we did here is totally ~~ap~~ consistent with Israel Approach.

In many ways, the wall behaves like a universe itself: The ~~wall~~ wall behaves like dS space and its energy momentum is Cosmological Constant.

If we go beyond; and look at perturbations around the wall spacetime: we find is that these perturbations behaves like standard lower dimensional Einstein gravity perturbation within that wall space.

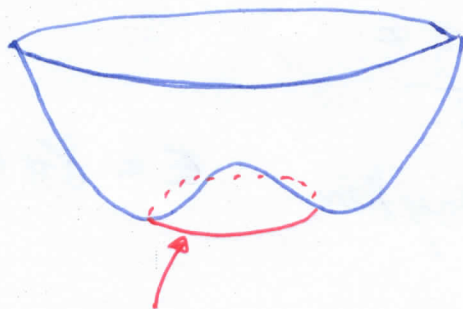
We begin to see that, although we start off with higher dimensional gravity solution; we constructed something which is lower dimensional, and its behaving almost as if it is our little spacetime with its own little energy momentum & its own gravity.

↪ This leads naturally to extra dimensions in gravity.

In 4D

pg 82

we can have strings, these also corresponds to solutions of QFT.



vacuum manifold: S^1

$$\phi \in \mathbb{C}$$

and ϕ seeing a sort of potential.

String: $\phi \sim \eta f(r) e^{i\theta}$

$$A_n = \frac{1}{e} (P(r) - 1) \partial_n \theta$$



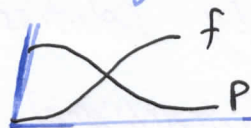
we kind of wind ~~around~~ around the vacuum ~~to~~ manifold. In order to remove this winding number, we have to lift over the potential barrier along an infinite length

So, we have String being topologically stable.

This can happen in $U(1)$ Abelian Higgs model

Here, The functions giving us the profile are no longer analytic like ~~tanh~~ $\tanh()$ function.

we have sort of



The gauge field kind of components for the winding; but since at $r=0$ we

have polar coordinate singularity: we can't unwind this gauge field. It is like, localized magnetic flux of z particle

(Magnetic flux is kind of confined in sort of Higgs core)



Lec 10 : Kaluza - Klein Theory, K K Black holes

4 is an interesting no. of dimension for gravity. ~~4 is the~~ 4 is the lowest dimension in which gravity freely propagates. We can think of it as for gravitational waves. There are no gravitational waves in 3 dimensions.

4 is the lowest dimension in which spacetime can be Ricci flat, but not Riemann flat:

which are tidal forces essentially.

In 3d dimensions; we still have gravity but its trivial. We only get spacetime curvature where you have energy momentum: So its all localized where you have got matter.

But in 4 dimensions; we have tidal forces which is like the free squeezing or deforming of space.

If you go up from 4 dimensions: We find that from geometry side things are very similar; but not the physics.

In 4d; we get centrifugal repulsion looking like $1/r^2$ & gravitational attraction looking like $1/r^2$.

So; we can have these two balancing out and can have orbits.

In 5 or higher dimensions; gravitational

attraction goes like $\frac{1}{r^3}, \frac{1}{r^4}, \dots$ (pg 89)

& so you no longer get this balance; and you don't have solar system for example.

(Also ~~the~~ our shoe laces will become undone in higher dimensions )

If extra dimensions; then must explain why we "see" 4.

KK Theory does this by having small extra dimensions and geometry independent of these.

We have something like $ds_5^2 = ds_4^2 - \underbrace{L^2 dX^2}_{\text{size } L}$

Set X has periodicity 2π .

(The extra dimension is ~~not~~ curled up on a scale of L)

(for many years it was assumed that this scale was plank scale; ~~but~~ but it does not have to be)

Wavefunctions that depend on X have large masses.

$$\Psi(x, X) \sim \Psi(x) e^{iX}$$

Now if we look at ~~the~~ 5 dimensional wave operator acting on Ψ ; and peel off the bit which depends on 4 dimensions $\Rightarrow$ It then looks like the equation of motion for massive particle.

pg 85

$$\nabla_5^2 \Psi \sim (\square \Psi + \underbrace{L^{-2}}_{\rightarrow m^2} \Psi) e^{i\chi_m}$$

If we think L to be small; The mass is very big.

So; We have the following understanding:

Yes things could depend on extra dimensions but they will have very high masses. So; if the extra dimension were something like plank scale, we would be talking about plank mass excitations which is way beyond any energy scale we are interested in.

But that we are ~~interest~~ interested in something which happens at low energy, and ~~therefore we d~~
~~therefore~~ therefore we don't have anything dependent on χ .

We could also have $\Psi(n, \chi) \sim \psi(n) e^{in\chi}$

Then $\nabla_5^2 \Psi \sim (\square \Psi + m^2 L^{-2} \Psi) e^{in\chi}$

So; we get a tower of massive modes.

If L is small (very small); These modes will have high mass.

For us; it is relevant because it can describe low energy physics: below the energy scale ~~a~~ set by size of internal manifold.

What about gravity?

(pg 86)

$$ds_5^2 = \hat{g}_{\mu\nu}^{(4)} dx^\mu dx^\nu - e^{2\sigma} [dz + A_\mu dx^\mu]^2$$

$$A_\mu = A_\mu(x)$$

size of internal direction.

→ We split up our general metric in this way.

(It is ~~is~~ general for now...)

we have

$$\Rightarrow \sqrt{g_5} = e^\sigma \sqrt{\hat{g}_4}$$

} ⇒ This is why we splitted like that to get this beautiful relation

$\hat{g}_{\mu\nu}^{(4)}$: Tensor

A_μ : vector

σ : scalar

(sets ~~scene~~ scene for computing curvature)

→ So, with ~~respect~~ respect to 4 dimensional spacetime ; 5 dimensional metric decomposes into 3 natural degrees of freedom: a tensor, a vector & a scalar.

Use Cartan (hand-out) ... (handout printed by Prof. David)

example with off-diagonal terms

eq 87

$$\underline{\omega}^a = e^a_\mu dx^\mu \quad ; \quad \underline{\omega}^5 = e^\sigma [dz + A_\mu dx^\mu]$$

$$\left. \begin{aligned} d\underline{\omega}^a &= -\underline{\Theta}_0^a{}_b \wedge \underline{\omega}^b \\ d\underline{\omega}^5 &= \sigma_{,a} \underline{\omega}^a \wedge \underline{\omega}^5 + e^\sigma \underline{F} \end{aligned} \right\} \text{Algebraic Relations}$$

$$\leadsto \underline{\Theta}_a^5 = \sigma_{,a} \underline{\omega}^5 + \frac{1}{2} e^\sigma F_{ab} \underline{\omega}^b$$

F lies in four dimensional subspace of 2 forms.

also; we get

$$\underline{\Theta}_b^a = \underline{\Theta}_0^a{}_b + \frac{1}{2} e^\sigma F^a{}_b \underline{\omega}^5$$

Computation of Riemann is straightforward.

$$R^a{}_{bcd} = \hat{R}_0^a{}_{bcd} + \frac{1}{4} e^{2\sigma} (F^a{}_c F_{bd} - F^a{}_d F_{bc} + 2F^a{}_b F_{cd})$$

$$R^5{}_{a5b} = -\nabla_a \nabla_b \sigma - \nabla_b \sigma \nabla_a \sigma - \frac{1}{4} e^{2\sigma} F_{ac} F_b{}^c$$

$\leadsto$ These are the ones in which we are interested.

$$\Rightarrow R_5 = \hat{R}_4 + \frac{1}{4} e^{2\sigma} F^2 - 2e^{-\sigma} \square e^\sigma$$

$\leadsto$ like
maxwell piece

giving the Einstein action S_5

$$S_{5, EH} = \frac{-1}{16\pi G_5} \int d^4x dz \sqrt{g_4} \left[e^\sigma \hat{R}_4 + \frac{1}{4} e^{3\sigma} F^2 - 2\Box e^\sigma \right] \quad (pg 88)$$

$$S_{5, EH} = \frac{-1}{16\pi G_5} \int d^4x dz \sqrt{g_4} \left[e^\sigma \hat{R}_4 + \frac{1}{4} e^{3\sigma} F^2 - 2\Box e^\sigma \right]$$

$\int dz$ gives length scale L

This is total derivative piece ; so no kinetic term for scalar

(?)

It looks like Einstein gravity

but there are few key differences

- ① Scalar fields : we can't just throw them away ; It is multiplying $\hat{R}_4$; also hiding with the max well piece. #

When you do variational principle to find new Einstein equation ; we will find that derivative of σ will come up naturally as part of the integration by parts

So ; it is misleading to say : " Just because ~~there~~ there is no explicit kinetic term for scalar ; scalar does not have equations of motion "

This is an example of Scalar - Tensor gravity
(The nature of gravitational interaction is not just spin 2 ; it has got scalar field in it as well)

The scalar multiplies the Ricci terms, & derivatives ⁽¹⁹⁸⁹⁾ appear in the "Einstein equations" through integration by parts.

$\hat{g}_{ab}$ is said to be the "Jordan frame".
(It is a choice of metric in which gravitational ~~action~~ action has scalar multiplied)

We can conformally transform our metric

$$\hat{g}_{ab} = \Omega^2(x) g_{ab}$$

$$\text{Then } \hat{R} = \Omega^{-2} (R - 6\Omega^{-1}\square\Omega)$$

Choose $\Omega = e^{-\sigma/2}$ (a particular value of ~~conformal~~ conformal factor related to scalar field)

$$e^{\sigma} \cdot \sqrt{\hat{g}} \cdot \hat{R} = e^{\sigma} \underbrace{\Omega^4 \sqrt{g}} \cdot \underbrace{\Omega^{-2} (R - 6\Omega^{-1}\square\Omega)}$$

$$= \underbrace{\sqrt{g} (R + 3\square\sigma)}_{\text{Standard Einstein Action}} - \underbrace{\frac{3}{2} (\nabla\sigma)^2}_{\text{Kinetic term}}$$

"Conformal transformations should not change degrees of freedom"

We got gradient term ~~for Riemann~~ in our E.O.M. by integration by parts on the Jordan frame; we will

Still be getting gradients on the new frame,
which we call Einstein Frame.

Final step of getting K.K. action: redefine your scalars.

Redefine $\phi = \frac{\sigma}{\sqrt{3}}$ to canonically normalize

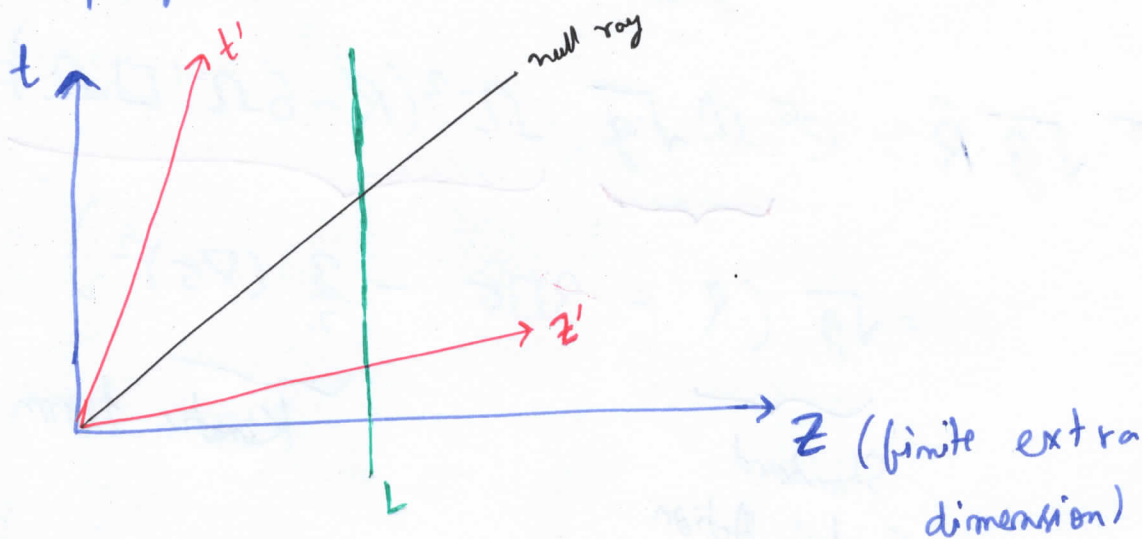
Kinetic terms

$$S = \frac{1}{16\pi G_5/L} \int d^4x \sqrt{g} \left(-R + \frac{1}{2} (\partial\phi)^2 - \frac{e^{\sqrt{3}\phi}}{4} F^2 \right)$$

$$ds^2 = e^{-\sqrt{3}\phi} \hat{g}_{\mu\nu}^{(4)} dx^\mu dx^\nu - e^{2\sqrt{3}\phi} [dz + A_\mu dx^\mu]^2$$

KK Black Holes

Note: Compactification picks a rest frame.



$$z \sim z + L \quad (\text{at constant } t)$$

If we do Lorentz transformations,

$$\left. \begin{aligned} t' &= \gamma(t - vz) \\ z' &= \gamma(z - vt) \end{aligned} \right\} \text{ Then } z \sim z + L \Leftrightarrow z' \sim z' + \gamma L$$

$$t' \sim t - v\gamma L$$

So; we are identifying points at different times.

What does this do to our black hole solutions?

lets start by getting Black hole solution; rather Black Brane solution,

because we started off by saying that geometry could not depend on extra dimension.

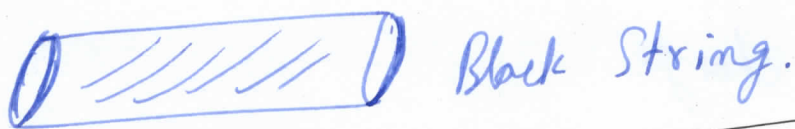
So; we cant have a Black Hole in 3d; it got to be translationally invariant in z -direction.

& fortunately it turns out that,

SCH $\times$ IR is solution of 5D gravity
(Black String...)

abbreviation.
(SCH $\equiv$ Schwarzschild)

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} - r^2 d\Omega_{II}^2 - dz^2$$



→ This is simplest KK Black hole solution; uncharged; nothing happening in z direction.

Now lets do boost on that;

and then identify (and see what we have)

Boost before identifying.

(1992)

$$ds^2 = \left(1 - \frac{2GM}{r}\right) \frac{(dt + v dz)^2}{1 - v^2} - \frac{dr^2}{1 - \frac{2GM}{r}} - \frac{(dz + v dt)^2}{1 - v^2} - r^2 d\Omega_{II}^2$$

Now identify, $z \sim z + L$, and do the compactification.

and rearrange to get.

(boost in z direction,
and then do identification)

$$ds^2 = \left(1 - \frac{2GM}{(1-v^2)r + 2GMv^2}\right) dt^2 - r^2 d\Omega_{II}^2 - \frac{dr^2}{1 - \frac{2GM}{r}} - \left(1 + \frac{2GMv^2}{(1-v^2)r}\right) \left[dz + \frac{2GMv dt}{(1-v^2)r + 2GMv^2} \right]^2$$

When we rearrange; and complete the square for the dz part of the metric: we know that the gauge field $A_\mu dx^\mu$ is just going to have a t component.

This says that; boost in z direction has given us effective electric charge.

This shows; we can identify velocities in extra direction with electric charge in the sense that it is Kaluza Klein electric charge.

Velocity in extra dimension $\longleftrightarrow$ Electric KK charge

Now; its natural to redefine coordinates the new metric we got after boost to make it look like old one say.

(93)

Note The black string with no velocity in extra dimension; and the black string which is ~~going~~ moving in extra ~~dimension~~ dimension are two distinct solutions.

So; The velocity in the extra direction is showing up as an electric charge from 4d perspective (because we got A_t coming from Lorentz transformation)

Write $\hat{\gamma} = \gamma + \frac{2GMv^2}{1-v^2}$; $\alpha = \frac{2GMv}{1-v^2}$

$$\hat{M} = \frac{2GM}{1-v^2}$$

Transformation from Jordan frame to Einstein frame.

Then; get the following for Einstein frame metric

~~$$g_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2GM}{\hat{\gamma}}\right) \left(1 - \frac{v\alpha}{\hat{\gamma}}\right)^{-1/2} dt^2 - \left(1 - \frac{v\alpha}{\hat{\gamma}}\right)^{3/2} \hat{\gamma}^2 d\Omega_{II}^2$$~~
~~$$- \left(1 - \frac{2GM}{\hat{\gamma}}\right)^{-1} \left(1 - \frac{v\alpha}{\hat{\gamma}}\right)^{1/2} d\hat{\gamma}^2$$~~

$$g_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2GM}{\hat{\gamma}}\right) \left(1 - \frac{v\alpha}{\hat{\gamma}}\right)^{-1/2} dt^2 - \left(1 - \frac{v\alpha}{\hat{\gamma}}\right)^{3/2} \hat{\gamma}^2 d\Omega_{II}^2$$

$$- \left(1 - \frac{2GM}{\hat{\gamma}}\right)^{-1} \left(1 - \frac{v\alpha}{\hat{\gamma}}\right)^{1/2} d\hat{\gamma}^2$$

→ K.K. electric Black hole.

This look similar to Reimer-Nordstrom black hole;

(pg 95)

but has some differences.

Just like R-N. ; we can have extremal limit for
K.K. electric black hole $\Rightarrow$ But the extremal limit here
where q goes to M ; Then this is not
singular

... There are differences in terms of
properties of Black Hole

For a magnetic black hole.

~~$$\underline{F} = Q \sin \theta \, d\theta \wedge d\varphi$$~~

$$\underline{F} = Q \sin \theta \, d\theta \wedge d\varphi$$

Take the field strength; and make it proportional to
the area form on the sphere surrounding your
object. (in this case it is 2 sphere)

$$\underline{dF} = 0$$

but $\underline{F} \neq \underline{d\omega}$ except locally.

This is an example of form which is closed, ie $\underline{dF} = 0$

but not exact ie; not expressible as ~~$\underline{F} = \underline{d\omega}$~~
 $\underline{F} = \underline{d\omega}$

When you have closed forms, which are not exact
 $\iff$ It is ~~also~~ usually a sign of something
happening with the topology of the ~~space~~ space.

In this case, our topology is non-trivial.
We got non-trivial 2-sphere; we got event horizon;
if we ~~try~~ try to shrink 2-spheres we hit the
event horizon.

~~So~~ We can of course write $F = d\omega$ locally.

ie; locally, ~~A~~

N, S $\rightarrow$ stands for
North & South.

$$\begin{aligned}\tilde{A}_N &= Q(1 - \cos \theta) d\varphi \\ \tilde{A}_S &= -Q(1 + \cos \theta) d\varphi \\ \underline{A_S} &= \underline{A_N} - 2Q d\varphi\end{aligned}$$

we can write;
magnetic solution as
this
... gauge part

A_N works everywhere
around the north pole;
but the south pole axis

& A_S works everywhere, but the north pole axis.

And the two are related by gauge transformations

$$\tilde{A}_S = \tilde{A}_N - 2Q d\varphi$$

With this Ansatz;

find the solution

$$ds^2 = \left(\frac{r-r_+}{r-r_-} \right) dt^2 - \frac{dr^2}{\left(1 - \frac{r_+}{r} \right)} - r(r-r_-) d\Omega_{II}^2$$

$$- \left(1 - \frac{r_-}{r} \right) [dz_N + Q(1-\cos\theta)d\varphi]^2$$

$$\downarrow$$

$$\sqrt{r_+ r_-}$$



A_N

we have two horizon; inner & outer.

we could also have the south pole patch.

$$Z_S = Z_N + 2Q \cdot \varphi \quad \text{for S pole patch.}$$

$$= Z_N + 2Q(\varphi + 2\pi) \sim Z_S + L \cdot \pi; m \in \mathbb{Z}$$

If we go round in the φ direction (round the circle)

we go to $\varphi + 2\pi$.

(The two periodicity has to be in alignment!)

This tells us that;

$$4\pi Q \in L \cdot \mathbb{Z} \quad (\text{because } z \sim z + L)$$

This means that charge is quantized.

(similar to Dirac quantization of electric charge...)

Extremal Limit

1997

$$r_+ = r_- \quad ; \quad Q = r_+ = \frac{L}{4\pi}$$

$$\text{let } \chi = \frac{4\pi z}{L}$$

so; the periodicity of χ is now 4π

$$(\Delta\chi = 4\pi)$$

$$ds^2 = dt^2 - \frac{dr^2}{1 - Q/r} - \left(1 - \frac{Q}{r}\right) \left[r^2 d\Omega_{II}^2 + Q^2 (d\chi + A_N)^2 \right]$$

↪ so; if $r_+ = r_-$ is our time coordinate now becomes an extra coordinate tacked on to the geometry; and we have some non-trivial spatial manifold which a kind of just exists in time.

so; when $r \rightarrow Q$ (what happens)

$$\rho^2 = 4Q(r - Q)$$

$$ds^2 \sim dt^2 - d\rho^2 - \frac{\rho^2}{4} \left[d\theta^2 + \sin^2\theta d\varphi^2 + (d\chi + (1 - \cos\theta) d\varphi)^2 \right]$$

Origin of $\mathbb{R}^4$ in Euler Angles

(used Euler angles to describe three spheres)

$$x + iy = \rho \cos \frac{\theta}{2} \cdot e^{i\chi/2}$$

$$z + iw = \rho \sin \frac{\theta}{2} \cdot e^{i(\varphi + \chi/2)}$$

} (*)

While doing the transformation (*):

pg 98

fibring S^2 by S^1 with a "twist"

(when we take KK compactification; P adds a little manifold;
here its a little circle)

and this "lifts" S^2 to an S^3 (instead of $S^2 \times S^1$)
(because of twist)

This is an example of HOPF FIBRATION.

Symmetries of S^3 , $SO(4) \sim SO(3) \times SO(3)$
or $SU(2) \times SU(2)$

locally
algebra is
the same.

We see this;

Transformations acting Left & Right.

This gives two $SO(3)$ Killing Algebra.

Lec 11: Black Branes : higher-dimensional Schwarzschild, P-form charges, D3 branes, black string instability

Have seen black string $SCH \times \mathbb{R}$

 $ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} - r^2 d\Omega_{II}^2 - dz^2$

$f(r) \longrightarrow K - \frac{2GM}{r} - \frac{\Lambda}{3} r^2$

K is curvature of space $d\Omega_{II}^2$ surrounding the black at constant distance from Black Hole.

In 4d we have black hole.

In higher dimension; say 5d we have black string.

In " " we have more possibility of "black hole" kind of thing.

We can add more extra dimension (sort of flat space)

$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} - r^2 d\Omega_{II}^2 - dz_1^2 - \dots - dz_m^2$

$\hookrightarrow$ also a solution of Einstein equation.

Can add extra dimensions to sort of end ~~with~~ up with Black String, Black membrane, etc.



Recall the domain wall metric

$ds^2 = A^2(z) \gamma_{\mu\nu} dx^\mu dx^\nu - dz^2$

Using Cartan we found the connection.

(pg 100)

$$\left. \begin{aligned} \tilde{\Theta}^a_b &= \tilde{\Theta}_0^a_b \\ \tilde{\Theta}^a_z &= \frac{A'}{A} \tilde{\omega}^a \end{aligned} \right\} \rightarrow \begin{aligned} R^\mu_\nu &= \frac{1}{A^2} R_0^\mu_\nu + \left((D-2) \frac{A'^2}{A^2} + \frac{A''}{A} \right) \delta^\mu_\nu \\ R^z_z &= (D-1) \frac{A''}{A} \end{aligned}$$

This suggest that we can Generalize to higher dimensional thing (by simply bolting together ~~which have their own~~ different subspaces that have their own warp factors)

$$ds^2 = \underbrace{A^2(z) g_{\mu\nu} dx^\mu dx^\nu}_{\substack{\text{first space} \\ \text{containing time}}} - dz^2 - \underbrace{c^2(z) \gamma_{\alpha\beta} dy^\alpha dy^\beta}_{\substack{n \text{ dimensional} \\ + \dots + \\ (\text{Euclidean.})}}$$

(1 + P) dimensional; 0, 1, ... Pth
(Lorentzian)

$$ds^2 = A^2(z) g_{\mu\nu} dx^\mu dx^\nu - dz^2 - c^2(z) \gamma_{\alpha\beta} dy^\alpha dy^\alpha$$

Its quite clear; from the way it is set up:

At the connection level; the μ & α part of the metric don't mix.

g and γ don't mix at connection level.

The non-zero pieces are as follows:

$$\tilde{\Theta}^{\hat{\mu}}_{\hat{\nu}} = \tilde{\Theta}_0^{\hat{\mu}}_{\hat{\nu}} \quad \tilde{\Theta}^{\hat{\alpha}}_{\hat{\beta}} = \tilde{\Theta}_0^{\hat{\alpha}}_{\hat{\beta}}$$

$$\tilde{\Theta}^{\hat{\mu}}_{\hat{z}} = \frac{A'}{A} \tilde{\omega}^{\hat{\mu}} \quad \tilde{\Theta}^{\hat{\alpha}}_{\hat{z}} = \frac{C'}{C} \tilde{\omega}^{\hat{\alpha}}$$

2 forms are largely the same,
but have

$$\tilde{R}^{\hat{\mu}}_{\hat{\alpha}} = \tilde{\omega}^{\hat{\mu}}_{\hat{z}} \wedge \tilde{\omega}^{\hat{z}}_{\hat{\alpha}}$$

$$\tilde{R}^{\hat{\mu}}_{\hat{\alpha}} = \frac{A'C'}{Ac} \tilde{\omega}^{\hat{\mu}} \wedge \tilde{\omega}_{\hat{\alpha}}$$

After calculations we finally get

$$R^{\mu}_{\nu} = A^{-2} R_0^{\mu}_{\nu}(g) + \left(\frac{A''}{A} + p \frac{A'^2}{A^2} \right) \delta^{\mu}_{\nu} + n \frac{A'C'}{Ac}$$

$$R^{\alpha}_{\beta} = C^{-2} R_0^{\alpha}_{\beta}(\gamma) + \left(\frac{C''}{C} + (n-1) \frac{C'^2}{C^2} \right) \delta^{\alpha}_{\beta} + (p+1) \frac{A'C'}{Ac}$$

$$R^z_z = (p+1) \frac{A''}{A} + n \frac{C''}{C}$$

building blocks for finding
(simple) brane solutions

Riemann tensor tend to split in two parts;

The curvature is splitted; (the curvature is inherited if
g or γ metric does ; and pieces due to the
warp factor A or C &

→ This is what you get ; (say the g space of
 γ space changing as we move
in z-direction)

& ; and also the last piece & mixing $\frac{A'C'}{Ac}$
(cross talk between two subspaces)

Note: for the moment we are not doing rotations...

pg 102

e.g. ~~SCH~~ SCH_D (Schwarzschild solution in D dimensions)

$$\underline{A^2 dt^2 - dz^2 - C^2 d\Omega_{D-2}^2}$$

here $n = D-2$, $p = 0$

So now; we change gauge and get it in following form

$$\cancel{dz} \quad C = r, \quad dz = \frac{dr}{B} \quad (*)$$

This gauge $(*)$ splits the solution in more familiar form.

$$C' = \frac{dC}{dz} \longrightarrow B; \quad C'' \longrightarrow \frac{B dB}{dr}$$

~~C''~~

From Ricci's

$$G^0_0 = -(D-2) \frac{B}{r} \frac{dB}{dr} - \frac{(D-2)(D-3)}{2r^2} (B^2 - 1)$$

$$\text{but } G^0_0 = 8\pi G_D T^0_0 + \Lambda \quad (\text{by Einstein's equation})$$

(we get possibly the same thing what we got for normal SCH solution)

$$\Rightarrow B^2 = 1 - \frac{16\pi G}{(D-2)r^{D-3}} \int r^{D-2} T^0_0 dr + \left(\frac{-2}{(D-2)r^{D-3}} \int r^{D-2} \Lambda \right)$$

→ This is total energy divided by area A_{D-2} of $D-2$ sphere (at least in linearized sense) M/A_{D-2}

for total mass $\int T_0$ (volume)

(19/10/23)

$$\text{volume} = dr r^{D-2} \cdot (\text{Angles})$$

The integral over (Angles) gives A_{D-2} .

So; we end up with.

$$B^2 = 1 - \frac{16\pi G}{(D-2) A_{D-2}} \cdot \frac{M}{r^{D-3}} - \frac{2\Lambda r^2}{(D-2)(D-3)}$$

$A_{D-2} \Rightarrow$ area of S_{D-2} . ($D-2$ sphere)

If $\Lambda=0$, this gives us the sch radius in terms of mass

$$r_s = \left(\frac{16\pi G M}{(D-2) A_{D-2}} \right)^{1/(D-3)}$$

Note; Λ piece remains proportional to ~~the~~ r^2

Λ is kind of not changing so much.

The only thing that is changing is specific relation between Λ and Ads or ds length scale.

Λ remains an r^2 dependent, dimension shows up in relation to radius of curvature.

$$\frac{1}{l^2} = \frac{2|\Lambda|}{(D-2)(D-3)}$$

(Ads or ds)

$$B^2 = 1 - \frac{16\pi G}{(D-2)r^{D-2}} \int r^{D-2} T_0 dr + \left(\frac{-2}{(D-2)r^{D-3}} \int r^{D-2} \Lambda \right)$$

Charged Black Hole

(Pg 104)

Charge ?

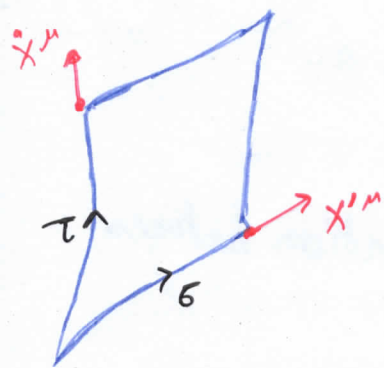
In 4D, a point particle comes electric (and magnetic) charge

$$L_0 \sim q \int d\tau \dot{x}^\mu A_\mu \quad \int x^{\mu(2)}$$

Thinking classically;

if we want point particle with charge; we add an extra piece to Lagrangian which would be coupling the tangent vector to gauge field A_μ (and this will tell how to modify our geodesic equations)

How to construct Charged String:



string worldsheet.

We get two vectors $\dot{x}^\mu$ & x'^μ that tell us about the string: They are parallel to string worldsheet.

This suggests that.

we have two index object (one index for each tangent vector)

Now could that be gauge field & how could it make sense:

Suggests : ~~$\int d\tau d\sigma \dot{x}^\mu \dot{x}^\nu B_{\mu\nu}$~~ $\int d\tau d\sigma \dot{x}^\mu x'^\nu B_{\mu\nu}$

(19/10)

If we make B a 2-form

Then our gauge transformation

is $\underline{B} \longrightarrow \underline{B} + \underline{dA}$

(The 2 form has a gauge transformation just like the gauge transformation for electromagnetism)

$$A \rightarrow A + d\phi$$

where ϕ is scalar)

& our field strength is going to be $\underline{H} = \underline{dB}$

$$\left. \begin{array}{l} \underline{B} \longrightarrow \underline{B} + \underline{dA} \\ \underline{H} = \underline{dB} \end{array} \right\} \begin{array}{l} \text{(Analogue of } A \rightarrow A + d\phi \\ \underline{F} = \underline{dA} \end{array}$$

Now, we see how to construct general electrically charged objects by simply instead of having 2 forms, we have $(p+1)$ forms.

In general $\underline{A}^{(p+1)}$ gives an electrically charged p -brane.

(This is one way of extending charges; ... or say $U(1)$ gauge group to higher dimensions by simply extending the notion of what we mean by gauge potential)

What do we mean by Magnetic Solution?

Magnetic Solution

In 4D electromagnetism.

$$F_{\theta\phi} \sim Q \sin\theta$$

so: $\underline{F} \sim Q \underline{E}_2$ $\underline{E}_2 \Rightarrow$ Unit S^2 are form.

In general, $\underline{H} \propto \underline{E}_{p+2}$; ~~area form~~

where $\underline{E}_{p+2}$ is area form on $(p+2)$ -sphere.

(Now think of $(p+2)$ ~~there~~ sphere surrounding our black-brane; and therefore this means that we get

$(p+2)$ dimension surrounding the brane; one dimension which is z or r coordinate

$$\text{so, } (p+2) + 1 = (p+3)$$

So, The dimensionality of the object is $D - (p+3)$

which is $D - (p+4)$ brane.

Low Energy string supergravity has several form fields

$$S = - \int d^{10}x \sqrt{g} \cdot e^{-2\phi} \left[R + (\nabla\phi)^2 - \frac{1}{12} H_{abc}^2 \right]$$

When we take an effective low energy gravity theory from string theory we ~~at~~ always get this initial piece & which is coming from basic excitation of basic fields that live on the string world sheet)

It is of course the field strength of the B

So, the string always has $B_{\mu\nu}$ living on its world sheet.

But depend on other supersymmetries we ask for we get the following schematic for.

$$(-)^{P+1} \frac{2(d A_P)^2}{(P+1)!}$$

~~odd~~ II A
~~even~~ II B

P odd II A

P even II B

If we think again going back to electric & magnetic solutions in 4D,

point objects (0-branes) have both e, Q charges

e $\Rightarrow$ electric charge

Q $\Rightarrow$ magnetic charge.

In general D dimensions, to have same dimensionality for electric & magnetic charges

$$P-1 = D - P - 3$$

$$\Rightarrow \boxed{P = \frac{D-4}{2}}$$

$$\Rightarrow \underline{\underline{P=3 \text{ in } 10 \text{ D}}}$$

$\rightarrow$ This gives ~~self~~ self-dual ~~3-brane~~ 3-brane
in II B

$$ds^2 = \left(1 - \frac{r_+^4}{r^4}\right)^{1/2} (dt^2 - d\underline{x}^2) - \frac{dr^2}{\left(1 - \frac{r_+^4}{r^4}\right)^2} - r^2 d\Omega^2_{\text{II}} \quad (\text{pg 108})$$

$$\underline{F} = Q (\underline{E}_S + * \underline{E}_S) \quad (\text{it is self dual})$$

$$\underline{F} = * \underline{F}$$

$$r_+^4 = 4\pi g_s N \alpha'^2$$

↑
string
coupling

no. of unit charged D3's.

In 4 dimensions; we have point particle, and it can be electrically or magnetically charged.

If we were in 5 dimensions; our electric charged object in Einstein-Maxwell theory will still have to be a point particle because we still got that A_μ coupled to $\dot{x}^\mu$.

But, the magnetic object would now be a string.

$$F_{\mu\nu} \propto 2 \text{ sphere.}$$

a 2-sphere in 4 spatial dimension surrounds an extended object.

So; If we change gauge

$$\rho^4 = r^4 - r_+^4$$

we get a kind of usual expression.

$$ds^2 = \left(\frac{\rho^4 + 4\pi g_s N \alpha'^2}{\rho^4} \right)^{+1/2} (dt^2 - d\underline{x}^2) - \left(\frac{\rho^4 + 4\pi g_s N \alpha'^2}{\rho^4} \right)^{+1/2} d\rho^2 - (\rho^4 + 4\pi g_s N \alpha'^2)^{1/2} d\Omega^2_{\text{II}}$$

~~$$ds^2 = \left(\frac{\rho^4 + 4\pi g_s N \alpha'^2}{\rho^4} \right)^{-1/2} (dt^2 - dx^2) - \left(\frac{\rho^4 + 4\pi g_s N \alpha'^2}{\rho^4} \right)^{-1/2} d\rho^2 - \left(\frac{\rho^4 + 4\pi g_s N \alpha'^2}{\rho^4} \right)^{1/2} d\Omega_{II}^2$$~~

$$ds^2 = \left(\frac{\rho^4 + 4\pi g_s N \alpha'^2}{\rho^4} \right)^{+1/2} (dt^2 - dx^2) - \left(\frac{\rho^4 + 4\pi g_s N \alpha'^2}{\rho^4} \right)^{-1/2} d\rho^2 - \left(\frac{\rho^4 + 4\pi g_s N \alpha'^2}{\rho^4} \right)^{1/2} d\Omega_{II}^2$$

Here we got an offset from $\rho = 0$

(what happens as we go to $\rho = 0$? (which is event horizon or would be event horizon of this object))

lets suppose N is large.

$$4\pi g_s N \alpha'^2 \equiv R^4$$

and now look at $\rho \ll R$.

so; we have

$$ds^2 \approx R^2 \left[\frac{(dt^2 - dx^2)}{\rho^2} - \frac{\rho^2 d\rho^2}{R^4} \right] - R^2 d\Omega_{II}^2$$

$\rightarrow$ This is $AdS_5 \times S_5$

$R^2 \left[\frac{dt^2 - dx^2}{\rho^2} - \frac{\rho^2 d\rho^2}{R^4} \right]$ looks like AdS_5

$R^2 d\Omega_{II}^2$ is like S_5 .

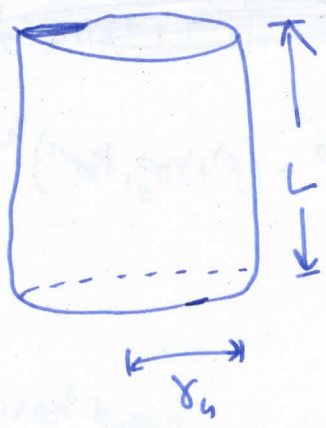
Stability?

look at KK black string.

SCH $\times$ IR



Think about Black string in KK sense ; and look at length L of Black String.

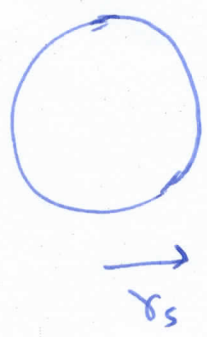


mass $M = \frac{r_h L}{2G_5}$

Entropy $S_{BS} = \frac{\pi r_h^2 L}{G_5}$

↑
entropy of Black String.

Now, let's look at 5d Black Hole solution SCH5



Black Hole
with radius r_s

mass: $M = \frac{3 \times 2\pi^2 \times r_s^2}{16\pi G_5}$
 $= \frac{3\pi r_s^2}{8G_5}$

$\Rightarrow M = \frac{3\pi r_s^2}{8G_5}$

Entropy: $S_{BH} = \frac{\pi^2 r_s^3}{2G_5} = \frac{\pi^2}{2G_5} \left(\frac{8G_5 M}{3\pi} \right)^{3/2}$

Now let's compare the entropies.

$\frac{(S_{BH})_{SCH5}}{(S_{BH})_{SCH11.2}} = \frac{\frac{\pi^2}{2G_5} \left(\frac{8G_5 M}{3\pi} \right)^{3/2}}{\frac{4\pi M^2 G_5}{L}} = \frac{\sqrt{\frac{8}{27\pi}} \cdot L}{\sqrt{G_5 M}}$

$$\frac{(S_{BH})_{SCHs}}{(S_{BH})_{SCH \times 12}} = \sqrt{\frac{8L}{27\pi G_5 M}}$$

~~If $L > \frac{27}{8} \pi G_5 M$~~

~~The black hole will have bigger entropy as compare.~~

If $L > \frac{27}{8} \pi G_5 M$: Then Black Hole SCHs will have bigger entropy as compared to Black String SCH $\times 12$.
Black Hole is preferred.

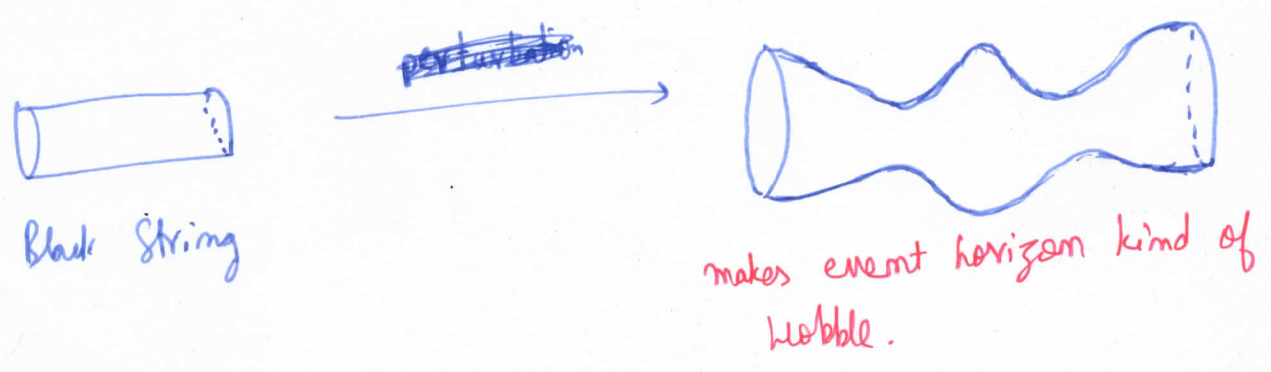
From entropy thermodynamic perspective; If we try to make the string too long and thin $\Rightarrow$ it will ~~try~~ prefer to clump into a black hole.

This entropy argument suggests that Black String is unstable.

Suggests an instability of black string.

Confirmed by Linearized analysis. (RG + Laflamme)
and full non-linear numerical ~~Lehner~~ (Pretorius, Lehner)

There is an instability:



(pg 122)

So, Black String is definitely unstable in 5 dimensions,
but it seems like at the end points of instability,
that the Black String fragments into Black holes;
and if this is the case, then at the moment of
fragmentations we should get sort of instantaneous
naked singularities

(looks like cosmic censorship is violated in higher
dimensions)



Lec 12: Warped compactifications: Rubakov - Shaposhnikov
& Randall - Sundrum toy models.

Warped Compactification

Alternative way of hiding extra dimensions is to confine particle physics to a submanifold.

(an alternative way to hide extra dimensions)
by doing confinement; rather than the problem of things not depending on extra dimensions which we did in KK theory.

Toy model (Rubakov + Shaposhnikov)

Domain wall $\phi \sim \eta \tanh(\sqrt{\lambda} \cdot \eta z)$

Couple the scalar to fermion

$$\mathcal{L}_\Psi = i \bar{\Psi} \Gamma^a \nabla_a \Psi - g \phi \bar{\Psi} \Psi$$

Ψ is 5 dimensional fermion.

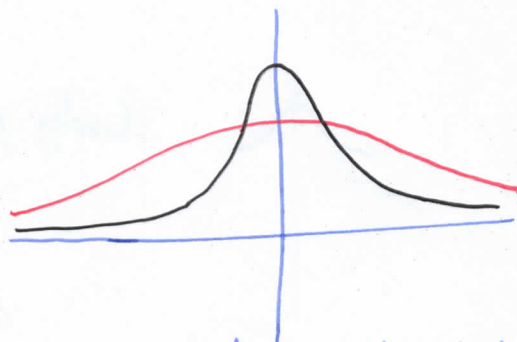
Γ^a is " " Γ matrices.

~~Normally we expect~~ because ϕ profile is non-trivial the lowest energy profile for Ψ is non-trivial - a condensate.

$$i \gamma^5 \Psi' = g \eta \tanh(\sqrt{\lambda} \cdot \eta z) \Psi$$

$$\rightarrow \Psi = \Psi_0 [\text{sech}(\sqrt{\lambda} \eta z)]^{2/g}$$

$$\text{if } i \gamma^5 \Psi_0 = -\Psi_0 \text{ chiral.}$$



how strongly localized will depend on coupling g .

ϕ shifting in background $\Rightarrow$ provides shift in mass for spinor.

(pg 114)

Also if $\partial \Psi_0 = 0$, this solves 5D equations.

(The Background of domain wall provided the shift in, what we might think is lowest energy is for ψ profile)

$\Leftrightarrow$ We get non trivial profile around the wall, and call it Condensate.

ie; Also, if $\partial \Psi_0 = 0$, this solves 5D equations. A massless chiral fermion "confined" to brane.

$\rightarrow$ Provides notion for localizing particle physics $\checkmark$

What about Gravity?

Toy model (Randall - Sundrum)

domain wall in AdS space.

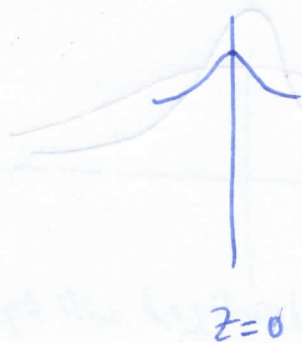
$$ds^2 = A^2(z) \eta_{\mu\nu} dx^\mu dx^\nu - dz^2$$

$$\uparrow e^{-2\beta_c |z|}$$

$$\text{or } e^{-2|z|/l}$$

Brane is flat

∞ extra dimension



sharply peaked warp factor around $z=0$.

∞ extra dimension; geometry depends only on z .
(completely opposite to KK compactification)

Recall Gauss Equations:

(Pg 115)

$$(4) R^a_{bcd} = h^a_{a'} h^b_{b'} h^c_{c'} h^d_{d'} (5) R^{a'}_{b'c'd'} - K^a_c K_{bd} + K^a_d K_{bc}$$

Israel Equation : $\Delta K_{ab} - \Delta K \cdot h_{ab} = 8\pi G S_{ab}$

Now; we will show in a general way; how we might get 4D gravity on this sharply localized solution in spite of the fact there is ∞ extra dimension.

S_{ab} will be $(\underbrace{\sigma}_{\text{wall part}} h_{ab} + \underbrace{T_{ab}}_{\text{matter on Brane}})$

Assume that, the bulk remains 5D pure AdS.

$D=5$, $(5) R_{abcd} = \frac{1}{l^2} [g_{ac} g_{bd} - g_{ad} g_{bc}]$

where $\Lambda = -6/l^2$

RS has spacetime $\mathbb{Z}_2$ -symmetry around brane.

If $T_{ab} = 0$,

Then $K^+_{ab} = -\Gamma^z_{ab} n^+_z = \frac{1}{2} g_{ab,z} = -\frac{1}{\ell} h_{ab}$

$K^-_{ab} = -K^+_{ab}$ ($\mathbb{Z}_2$ symmetry)

Israel Equation for $\mathbb{Z}_2$ symmetry is

$K^+_{ab} - K^+ h_{ab} = (\frac{3}{\ell} h_{ab}) = 6\pi G \sigma h_{ab}$

Now add T_{ab} :

(pg 116)

$$K^+_{ab} = 4\pi G (S_{ab} - \frac{1}{3} S h_{ab}) \quad (\text{By taking trace of Israel Equation...})$$

$$\Rightarrow K^+_{ab} = 4\pi G \left(-\frac{5}{3} h_{ab} + T_{ab} - \frac{1}{3} T h_{ab} \right)$$

$$= -\frac{1}{2} h_{ab} + 4\pi G \left(T_{ab} - \frac{1}{3} T h_{ab} \right)$$

Einstein 4D gravity; $G_{ab} = 8\pi G_{(4)} T_{ab}$
 or $R_{ab} = 8\pi G_{(4)} \left(T_{ab} - \frac{1}{2} T h_{ab} \right)$

Contract Gauss Equation: (& average over brane)

(If we contract the Gauss eqⁿ; we still kind of get hidden $\delta\delta$ function in there because 5d geometry has discontinuity in derivative of warp factor)

We are kind of taking average; rather than difference across the brane.

$$(4) R_{ab} = \frac{3}{l^2} h_{ab} + K^+_{ac} K^{+c}_b - K^+ K^+_{ab}$$

~~K^+_{ab}~~ use Israel equation to replace K^+ ..

$$(4) R_{ab} = \frac{3}{l^2} h_{ab} + \left(\frac{1}{2} h_{ac} - 4\pi G [T_{ac} - \frac{1}{3} T h_{ac}] \right) \left(\frac{1}{2} h^c_b - 4\pi G [T^c_b - \frac{1}{3} T h^c_b] \right)$$

$$- \left(\frac{4}{2} + \frac{4\pi G}{3} T \right) \left(\frac{1}{2} h_{ab} - 4\pi G [T_{ab} - \frac{1}{3} T h_{ab}] \right)$$

$$^{(4)}R_{ab} = \frac{8\pi G_5}{l} T_{ab} - \frac{4\pi G_5}{l} T \cdot h_{ab}$$

$$+ (4\pi G_5)^2 \left[T_{ac} T^c_b - \frac{1}{3} T T_{ab} \right]$$

$$= 8\pi G_4 \left(T_{ab} - \frac{1}{2} T h_{ab} \right) + \underbrace{(4\pi G_5)^2 T_{ab}}_{O(T^2_{ab})}$$

$$\text{so: } G_4 = \frac{G_5}{l}$$

if doing perturbation expansion.

To leading order, we get 4D Einstein gravity.
living on the brane.

Note: G_4 is a derived quantity
↖ Newton's constant.

l gives us a shift in Planck scale.

$$\text{Heuristically: } M_P^2 = M_D^{D-2} L^{D-4}$$

If we integrate over some extra dimension give us effective cutoff because of the warping (this is what AdS is doing; Negative curvature is kind of putting us in a box)

⊙
because in KK theory & we have finite size manifold
Plank mass Higher dimensional plank scale

$$\text{Then we could say } M_P^2 = M_D^{D-2} L^{D-4}$$

$L \Rightarrow$ length scale which represents volume of internal manifold.

Turn around;

19118

$$L \sim \left(\frac{M_P}{M_D} \right)^{\frac{D-2}{D-4}} \cdot \underbrace{M_P^{-1}}_{\sim L_P}$$

↑
true quantum gravity scale.

putting in numbers:

$$L \sim \left(\frac{10^{16} \text{ TeV}}{M_D} \right)^{1 + \frac{2}{n}} \cdot L_P \quad (n = D-4) \text{ (no. of extra dimensions)}$$

$$L \sim \left(\frac{\text{TeV}}{M_D} \right)^{1 + \frac{2}{n}} \cdot 10^{\frac{32}{n} - 17} \text{ cm}$$

as $n \uparrow \Rightarrow L$ decreases

$n = 6 ; L \sim 10^{-13} \text{ cm} , M_D \sim 10 \text{ TeV}$

Instead of having KK theory; we have some kind of confinement mechanism. We then have much larger mechanism than we originally thought. That larger internal manifold gives us the volume factor which takes us between fundamental plank scale & our 4d plank scale.

↪ This alters ~~the~~ ~~pr~~ magnitude of hierarchy problem in field theory.

(This is why people originally did it)

Randel - Sundrum (RS)

(pg 119)

Shiromizu - Maeda - Sasaki

4D effective Einstein's equation in the form

$$G_{ab} = 8\pi G_{(4)} T_{ab} + O(T_{ab}^2) + \mathcal{E}_{ab}$$

projection of 5D Weyl tensor.

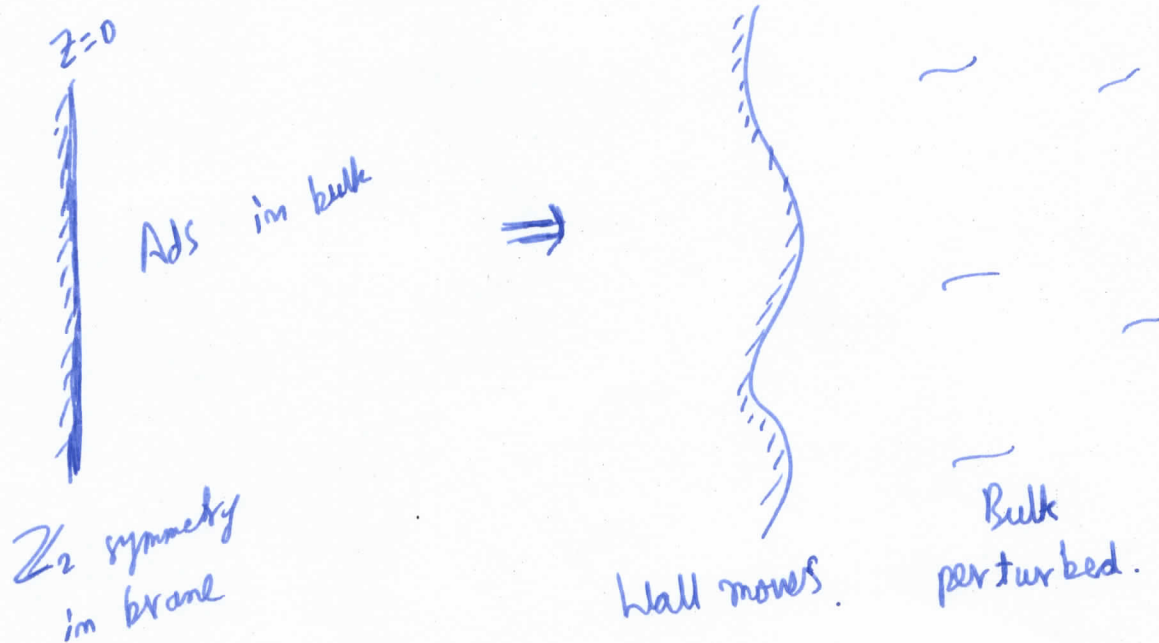
$$\mathcal{E}_{ab} = C_{acbd} n^c n^d$$

↑ Bulk perturbations appear here.

$\mathcal{E}_{ab}$ is unknown here.

So, It is a non-perturbative relation

In general, can perturb both brane and bulk



General Perturbation.

But ~~that~~ we can always write

(pg 120)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu - dz^2$$

} Gaussian
Normal

within some range moving away, near the wall $z=0$

Lec 13: Gravitational Perturbation theory, perturbing Randall - Sundrum

Because exact solutions hard to find, perturbation theory is useful in GR

- Solar system tests.
- Cosmological P.T.
- Gravitational waves.

Idea: perturb around background

$$g_{ab} = g_{0ab} + h_{ab}$$

→ now δg_{ab} !

Reminder: we have seen

$$\delta R^a_{bc} = \frac{1}{2} (\nabla_c h^a_b + \nabla_b h^a_c - \nabla^a h_{bc})$$

$$\delta R_{ab} = \frac{1}{2} \nabla_c \nabla_a h^c_b + \frac{1}{2} \nabla_c \nabla_b h^c_a - \frac{1}{2} \square h_{ab} - \frac{1}{2} \nabla_a \nabla_b h$$



we want to swap up the order

because we would like to have $\nabla_a (\nabla \cdot h_b)$

use Riemann Identity to swap.

$$\begin{aligned} \delta R_{ab} = & \frac{1}{2} \nabla_a \nabla_c h^a_b + \frac{1}{2} R^c_{aca} h^d_b + \frac{1}{2} R_{bdca} h^{cd} \\ & + \frac{1}{2} \nabla_b \nabla_c h^c_a + \frac{1}{2} R^c_{dcb} h^d_a + \frac{1}{2} R_{adcb} h^{cd} \\ & - \frac{1}{2} \square h_{ab} - \frac{1}{2} \nabla_a \nabla_b h. \end{aligned}$$

$$\delta R_{ab} = -\frac{1}{2} \square h_{ab} - R_{acbd} h^{cd} + R_d (a h^d_b) + \nabla_{(a} \nabla^c \bar{h}_{b)c}$$

where $\bar{h}_{ab} = h_{ab} - \frac{1}{2} h g_{ab}$

we write this in the following way;

$$\delta R_{ab} = -\frac{1}{2} \Delta_L h_{ab}$$

where Δ_L is the Lichnerowicz operator

$$\Delta_L h_{ab} = \square h_{ab} + 2R_{acbd} h^{cd} - 2R_{(a}{}^c h_{b)c} - 2\nabla_{(a} \nabla^c \bar{h}_{b)c}$$

$$\delta R_{ab} = -\frac{1}{2} \Delta_L h_{ab}$$

$$\Delta_L h_{ab} = \square h_{ab} + 2R_{acbd} h^{cd} - 2R_{(a}{}^c h_{b)c} - 2\nabla_{(a} \nabla^c \bar{h}_{b)c}$$

Lichnerowicz operator Δ_L is spin 2 wave operator in curved background.

This is wave equation for gravity.

Now let's decide ; how much it represents physical degree of freedom & how much of it represents gauge degree of freedom

Riemann, and Ricci piece are physical

We have to think about $\bar{h}$ piece.

Coordinate transformations are generated by vector fields. (pg 123)

Now consider gauge transformation.

$$x^a \longrightarrow x^a + \xi^a$$

ξ^a is small.

$$g_{ab} \longmapsto g_{ab} + \underbrace{L_{\xi} g_{ab}}_{h_{\xi ab}}$$

$\mapsto$ This is gauge perturbation (not physical)

but it is perturbation in components of g_{ab} .

changing coordinates produces a "perturbation".

$$\text{lets check: } \bar{h}_{\xi ab} = 2 \nabla_{(a} \xi_{b)} - (\nabla \cdot \xi) \cdot g_{ab}$$

(we will use g_{ab} instead of g_{0ab} when meaning is clear)

$$\Rightarrow \nabla^a \bar{h}_{\xi ab} = \square \xi_b + \nabla^a \nabla_b \xi_a - \nabla_b \nabla^a \xi_a$$

use Reimann Identity to reduce it to Ricci term.

$$\nabla^a \bar{h}_{\xi ab} = \square \xi_b + R^a_b \xi_a$$

Well posed PDE ; so can solve given initial conditions.

we can solve for ξ satisfying

$$\nabla^a \bar{h}_{\xi ab} = \square \xi_b + R^a_b \xi_a = -(\nabla^a \bar{h}_{\xi ab})$$

Can choose a gauge in which $\nabla_a \bar{h}^a_b = 0$

(19/12/19)

This is De Donder gauge.

Is there any remaining gauge freedom.

$$\chi^a \rightarrow \chi^a + \chi^a$$

$$\text{s.t. } \square \chi^a + R^a_b \chi^b = 0$$

(satisfying massless wave equation)

D linearly independent solutions.

lets count, ~~left~~ degrees of freedom.

$$h_{ab} \longleftrightarrow \frac{D(D+1)}{2} \text{ components} \quad (\text{because } h_{ab} = h_{ba} \text{ symmetric})$$

However, we chose gauge $\nabla_a \bar{h}^{ab} = 0 \longleftrightarrow D$ constraints.

(use these constraints to write some of these components in terms of others)

finally we have got χ^a remaining d.o.f $\longleftrightarrow D$

So; what is left over now is

$$\frac{D(D+1)}{2} - 2D \text{ physical d.o.f.}$$

$$\frac{D^2 + D - 4D}{2} = \frac{D(D-3)}{2} \text{ physical d.o.f.}$$

~~The~~ ~~physi~~ The no. of physical solutions of Lichnerowicz operator Δ_L is given by $\frac{D(D-3)}{2}$ in D dimensions.

$D = 4 \rightarrow$ 2 degrees of freedom $\leftarrow$ 2 polarization
+ , x

$D = 5 \rightarrow$ 5 " " "

$D = 10 \rightarrow$ 35 " " "

$$D = 4: 2 \text{ d.o.f}$$

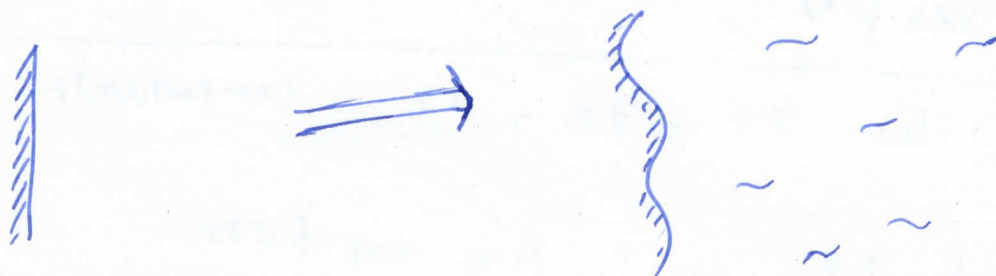

Return to R.S.

we could always choose the gauge

we could always choose the gauge where extra dimension is orthogonal to measuring proper distance.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu - dz^2 \quad (\text{G.N.})$$

(gambian normal)



To set wall at $z=0$, must gauge transform RNS picture.

R.S. background (g.)

K.S. Background (20)

$$ds^2 = e^{-2|z|/l} \cdot \eta_{\mu\nu} dx^\mu dx^\nu - dz^2$$

to keep in G.N. gauge.



make the wall flat again by doing translation in z -directions

$\xi^z = f(z^u)$ and add ξ^u .

19/26

$$\delta g_{z\mu} = \xi_{z,\mu} + \xi_{\mu,z} = 2 \Gamma_{\mu}^{\nu} z \xi_{\nu} = 0$$

Solved by $\xi_{\mu} = 2 \ell f_{,\mu} + \chi_{\mu}(x)$

we could also add another field depending on x .

$$\delta g_{\mu\nu} = 2 \xi_{(\mu,\nu)} + \Gamma_{\mu\nu}^z f$$

Then

$$h_{\mu\nu} = \underbrace{h_{\mu\nu}^{TT}} + \ell f_{,\mu\nu} - \frac{2}{\ell} f \cdot g_{\mu\nu}$$

Transverse trace free ($h^{\lambda}_{\lambda} = 0$, $\nabla_{\mu} h^{\mu}_{\lambda} = 0$)

Lich. op. ∇_L has zz , $z\mu$, $\mu\nu$ components.

Scalar component zz

A is warp factor

$$A^{-2} [A^2 (A^{-2} h^{\lambda}_{\lambda})']' = 4 \delta(z) \partial^2 f = -\frac{2}{3} \delta(z) \cdot 8 \pi G T^{\lambda}_{\lambda}$$

Matter on brane with non-zero T induces an f - "Brane Bending"

Now equating these we get

$$\Rightarrow \partial^2 f = -\frac{8 \pi G T^{\lambda}_{\lambda}}{6}$$

This gives what f is.

pg 127

$$\mu\nu : A^{-2} \partial^2 h_{\mu\nu}^{TT} - A^{-2} [A^4 (A^{-2} h_{\mu\nu}^{TT})']']$$

$$= -16\pi G \delta(z) \left[T_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} \right]$$

-2 $\delta R_{\mu\nu}$
(after doing algebra)

This comes from $\delta R_{\mu\nu}$ } comes because we are in 5D.

Construct solutions.

(the fourier transform)

$$h_{\mu\nu}^{TT} = u_m(z) e^{ipx} \epsilon_{\mu\nu}$$

eigen functions because of warping in z direction

polarization tensor

This is just standard momentum decomposition of the graviton.

$$\partial^2 h_{\mu\nu}^{TT} = -m^2 h_{\mu\nu}^{TT} \quad (p^2 = m^2)$$

tells something about mass in 4D dimensions.

We solved the homogeneous problem, i.e. set
(set $T_{\mu\nu} = 0$ and Israel Boundary conditions)

$$e^{2|z|/l} [m^2 \cdot u_m + \{e^{-4|z|/l} \cdot (e^{2|z|/l} u_m)'\}'] = 0$$

This is now a ~~straight forward~~ Sturm-Liouville problem.

change variables ~~$\psi = l e^{2|z|/l}$~~

now, change variable $\psi = l \cdot e^{2|z|/l}$

$$\frac{\partial}{\partial z} \rightarrow (\pm) \frac{\xi}{l} \frac{\partial}{\partial \xi} \quad \text{transforms this}$$

(p.128)

$$\text{to } u_m'' + \frac{u_m'}{\xi} + \left(m^2 - \frac{4}{\xi^2} \right) u_m = 0$$

a Bessel Equation.

Eigenfunctions are

$$u_m = \sqrt{\frac{m l}{2}} \cdot \frac{[J_1(m l) J_2(m \xi) - N_1(m l) J_2(m \xi)]}{\sqrt{J_1^2(m l) + N_1^2(m l)}}$$

at $z=0 \quad \xi = l$: Boundary condition.

Now we can construct propagator from eigenfunctions

$$\cancel{G_R(x, x') = \int \frac{d^4 p}{(2\pi)^4} e^{i p \cdot (x - x')} \left[\frac{A^2(z) A^2(z')}{l(p^2 - (\omega + i\varepsilon)^2)} + \int_0^\infty dm \cdot u_m(z) \right]}$$

$$G_R(x, x') = - \int \frac{d^4 p}{(2\pi)^4} e^{i p \cdot (x - x')} \left[\frac{A^2(z) A^2(z')}{l(p^2 - (\omega + i\varepsilon)^2)} + \int_0^\infty dm \frac{u_m(z) u_m(z')}{m^2 + p^2 - (\omega + i\varepsilon)^2} \right]$$

apart from $A^2(z) A^2(z')$
looks like something
independent of independent
dimensions ... depends
on 4 D..

Perturbation in the brane due to matter on the brane

(Pg 129)

$$G_R(\eta^\mu, 0, \eta^{\mu'}, 0) = \frac{1}{e} D_0(x-x') + \int_0^\infty dm \, U(m, 0)^2 D_m(x-x')$$

↓
massless propagator.

↘ integral over
continuum of
massive states.

Hence we can find out $h^{\tau\tau}_{\mu\nu}$

$$h^{\tau\tau}_{\mu\nu} = \frac{-16\pi G}{e} \int D_0(x-x') \left[T_{\mu\nu} - \frac{1}{3} T \eta_{\mu\nu} \right] - 16\pi G \int dm \dots$$

↙ zero mode piece

↓
dependent on massive
mode

This does
not look
correct
because
it is not
full perturbation

⇒ full perturbation have to take into account f

$$h_{\mu\nu} = h^{\tau\tau}_{\mu\nu} - \frac{2}{e} f \eta_{\mu\nu} + e f_{,\mu\nu} \quad \text{gauge.}$$

and this f has to satisfy another massless wave equation

we end up with

$$h_{\mu\nu} = \frac{-16\pi G}{e} \int D_0(x-x') \left[T_{\mu\nu} - \frac{1}{2} T \eta_{\mu\nu} \right] + \int dm \dots$$

↘ has right structure in
leading order piece

→ This is another way of accessing the fact that gravity is 4 dimensional on the brane.

⇒ it also shows how leakage in extra dimension is occurring.

The eigenfunctions tells how things can leak off
the ~~brane~~ brane.

~~For a source on~~

For a source $\sim M \delta^{(3)}(\mathbf{x})$ on brane

(for a localized source)

$$g_{tt} \sim 1 - \frac{2G_N M}{r} \left(1 + \frac{2l^2}{3r^2} + \dots \right)$$

The leading piece we get; the leading order at large r .
I find correction to my Newtonian potential
is now power law.

→ often when you see extra dimension or extra interaction,
we will ~~see the~~ find that correction is more like Yukawa
like correction to the gravitational interaction. with exponential.

But here we find its parallel ~~the reason is;~~
power law.

This illustrates example of using perturbation theory.

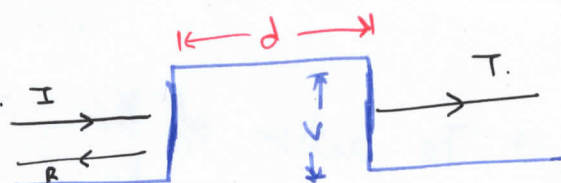
Lec 14: Gravitational instantons: false vacuum decay

OR: Decay of the false vacuum

Instanton = solution to Euclidean field equations

Perturbative $\sim O(\hbar)$ Non-perturbative $\sim O(1/\hbar)$

Paper: Coleman PRD 15 2929 • 77
 + de Lucca PRD 21 3305 • 80

Tunnelling

we solve Schrodinger's Equation

$$\Psi \sim e^{-iEt/\hbar} e^{i\psi/\hbar}$$

solve at fixed energy

and use boundary condition at barrier.

This gives probability of ~~transmission~~ transmission

$$\text{as } |T|^2 = \frac{1}{1 + \frac{V_0^2 \sinh^2 \Omega d}{4E(V_0 - E)}} \sim e^{-\Omega \cdot d}$$

$$\Omega^2 = 2m(V_0 - E)/\hbar = \frac{1}{\hbar} \int_0^d \sqrt{2m(V_0 - E)}$$

Transmission probability dominated by exponential.

Now if we input wavefunction into Schrodinger Operator.

$$\begin{aligned} -\frac{\hbar^2}{2m} \Psi'' + V\Psi &= \left[-\frac{i\hbar}{2m} \Psi'' + \frac{\Psi'^2}{2m} + V \right] \Psi \\ &= E\Psi \end{aligned}$$

The equation of motion looks like

(pg 132)

$$V + \frac{\psi'^2}{2m} = E, \quad \psi' = \pm \sqrt{2m(E-V)} \cdot x$$

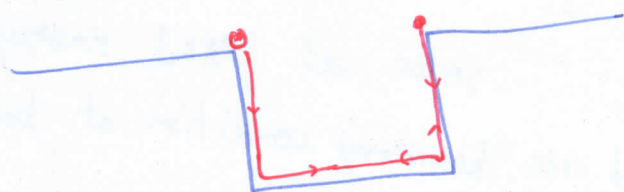
Consider a particle moving in a potential well (classical)

$$\frac{1}{2} \dot{x}^2 = \Delta V \quad \Downarrow \text{ use it}$$

$$\int \sqrt{2\Delta V} dx = \int 2\Delta V d\tau = \int \left(\Delta V + \frac{1}{2} \dot{x}^2 \right) d\tau$$

looks like standard Lagrangian (has kinetic & potential term)

Our exponent integral can be interpreted as motion in an inverted potential. (since ΔV comes with plus sign) in Euclidean Time.



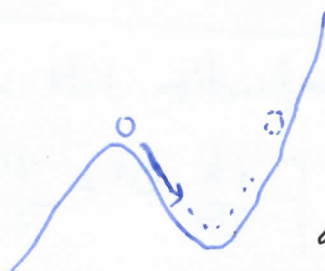
} $\Rightarrow$ The action of this process happens to agree with the ~~process~~ exponent we want

In this case it is fictitious process; but its kind of linking the notion of analytic continuation looking at some Classical Trajectory; computing the action of the trajectory & identifying the exponent we want.

For a general potential, amplitude for decay given by solving classical motion in inverted potential (to leading order)



$\Rightarrow$
Invert
the
potential



$\Rightarrow$ look at
Euclidean solution;
compute Euclidean
action & take its
~~exponent~~ exponential

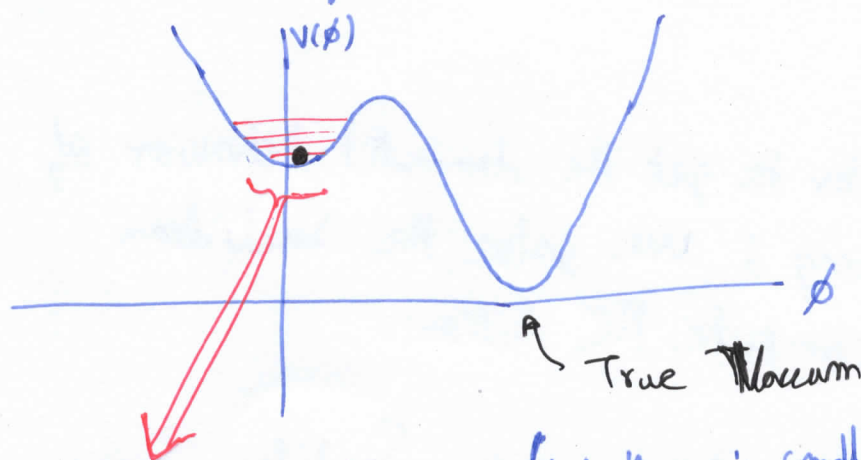


→ And this gives us the probability of decaying

(pg 133)

False Vacuum Decay

Take a scalar field



- two minimums,
- one global
 - one local.

looks like local vacuum (but there is small probability of exiting & going to the true vacuum (by quantum fluctuations))

↔ we call it False Vacuum (The local minima one)

The global minima one is called True Vacuum

→ false because it looks like vacuum but is not

Colson ... considered $V(\phi) = \frac{\lambda}{2}(\phi^2 - \eta^2)^2 - \frac{\epsilon}{2\eta}(\phi - \eta)$

double well potential with tilt
(tilting with the linear term)



ϵ has notion of difference in vacuum energy.

he kind of worked this so that

$\phi \approx -\eta$ is False vacuum F.V.

& $\phi \approx \eta$ is True " T.V.

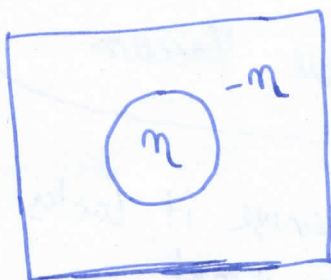
F.V. is unstable.

The idea is: In order to get the dominant behavior of this probability for decay; we solve the Euclidean field equations & compute the Action.

To compute exponent for decay, solve Euclidean eqns.

$$\frac{d^2 \phi}{dz^2} + \nabla^2 \phi = \frac{\partial V}{\partial \phi} = 2\lambda(\phi^2 - \eta^2) - \frac{\epsilon}{2\eta}$$

using (x,x)
image page
 $\phi'' + \frac{3}{\rho} \phi'$



Solving for ϕ with just this on RHS gives $\tanh(\sqrt{\lambda} \eta z)$ profile & this was Domain wall.

By symmetry; look for a hyperspherical solution $\phi(\sqrt{\tau^2 + x^2})$

And we guess its going to be close with the wall form $\tanh(\sqrt{\lambda} \eta z)$

$$\rho = \sqrt{\tau^2 + \chi^2}$$

Cross $\phi \sim -\eta \tanh(\sqrt{\lambda} \eta (\rho - \rho_0))$

$\rho \rightarrow \infty \quad \therefore \phi = -\eta \quad \text{F.V.}$

$\rho = 0 \quad \therefore \phi \approx +\eta \quad \text{T.V.}$

~~Coleman's argument is:~~

Coleman's Argument: wall is thin, so approximate by a wall of tension at ~~$\rho = R$~~ R .

~~however, the wall is not thin~~

To form the bubble, there is cost which is making the wall.

Cost: $\sigma \cdot (\text{Area})$
 $= \sigma \cdot 2\pi^2 R^3$

Area of sphere is $2\pi^2 R^3$

Gain in energy: $E \cdot (\text{Volume})$
 $= E \cdot \frac{\pi^2}{2} R^4$

Solution is stationary w.r.t. R

$$6\pi^2 \sigma R^2 = 2\pi^2 E R^3$$

$$\Rightarrow \boxed{R = \frac{3\sigma}{E}}$$

(***)

* if E small, then R is big.

Action, $B = \frac{\pi^2}{2} (4\sigma R^3 - E R^4)$

Bubble Action. $\nearrow$ $= 27 \frac{\sigma^4}{E^3} \pi^2 \underbrace{(4-3)}_1 = 27 \frac{\sigma^4}{E^3} \pi^2$

Why we expect bubble to be spherical?

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Because what we are doing is actually have competition between Surface Area & Volume.

And we kind of know spheres are most efficient in generating best $\frac{\text{Surface Area}}{\text{Volume}}$ ratio

The process will be dominated by spherical bubble

... Coleman first paper.

Coleman Second Paper

Coleman + De Luca: False vacuum energy gravitates.

Add gravity - use Israel equations for thin wall.
To calculate Euclidean action, subtract the FV manifold action from bubble action.

Euclidean dS
constant true curvature



⁵

radius of this
is proportional to $\frac{1}{\sqrt{8\pi G \epsilon}}$

→ so size of sphere tells about vacuum energy

If tunnelling to $\Lambda = 0$ (flat space)



The analogue of $\square \otimes^n$

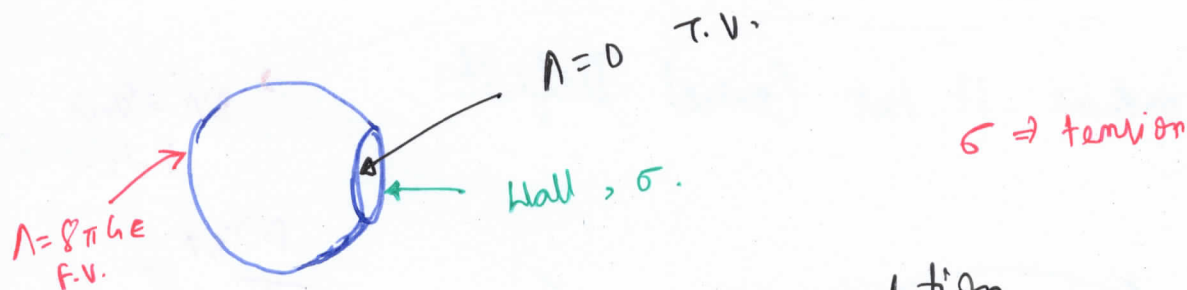
19/37

we have: inside the bubble flat
outside " " sphere.

~~The~~



Take sphere, and cut it.



This is geometry ... the bubble solution.

Calculate action,

$$S_B = \frac{\pi L^2}{G} \frac{(4\pi G \sigma L)^4}{[1 + (4\pi G \sigma L)^2]^2}$$

Check: $G \rightarrow 0$ (should match with previous result)

note: $8\pi G \sigma = 3/L^2$

$$\text{so as } G \rightarrow 0 \Rightarrow S_B \rightarrow (4\pi)^4 \cdot \left(\frac{3}{8\pi G \sigma}\right)^3 \cdot \pi \sigma^4$$

$$= \frac{27}{2} \pi^2 \sigma^4 / \epsilon^3$$

Adding an impurity makes phase transition easier to complete.

Obvious impurity in Gravity is Black Hole.

Euclidean b.h. (for schwarzschild)



$$\Delta\tau \sim \frac{1}{8\pi\hbar M}$$

If we have cosmological constant

$\Lambda > 0$, SdS
(Schwarzschild dS spacetime)

Then $\Delta\tau_{BH} \neq \Delta\tau_{CEH}$ $CEH \Rightarrow$ Cosmological Event Horizon

$\Rightarrow$ This means it has Conical Deficit.

$BH \Rightarrow$ Black Hole

$$\Delta\tau_{BH}$$

$\downarrow$
periodicity of τ

coming from B.H.

So, we have mismatch between natural periodicity

that makes event horizon ~~sing~~ non-singular

& natural periodicity that makes cosmological event horizon non-singular

$\iff$ Euclidean space has Conical Deficit

Cost: $\sim \sigma \cdot R^2 \cdot \beta \rightarrow \propto \text{area}$

Grain: $\sim \epsilon \cdot R^3 \cdot \beta \rightarrow \propto \text{volume}$

we still get $R_c \propto 2 \frac{\sigma}{\epsilon}$
 $\uparrow$ critical radius.

so; Action now goes as

$$B \sim \frac{\sigma^3}{\epsilon^2} \times \beta$$

~~But now~~

(Pg 139)

Probability for decay $\sim e^{-B}$

→ For black holes with $M < 10^7 M_p$
decay dominates over evaporation & half life is
 $\sim 10^{-22} \text{ s}$

Seed Black Hole M_+
Remnant M_-

$$B \propto M_+ \Delta M$$

Lec 15 Beyond Einstein gravity, Modifying gravity
with scalars Chameleons. Gravitational wave.

Why try other theories?

— Cosmology

Λ CDM
↑
Cosmological
constant
~ 70%

Dark Matter ~ 28%

SM ~ 5%
(standard
model model)

— String Theory

- Extra dimensions
- Module

> Beyond Einstein.

Dark Energy

Cosmology: FRW metric

$$ds^2 = dt^2 - a^2(t) d\mathbf{x}_k^2$$

↑ ↑ ↑
 cosmological Scale constant curvature
 time factor space

constant curvature
space

$K=1$ closed S^3
 0 Flat $\mathbb{R}^3$
 -1 open $\mathbb{H}^3$

Friedman equations $\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G \rho}{3}$

$$\ddot{a} + 3\frac{\dot{a}}{a} (p + \rho) = 0$$

Perfect fluid $P = w\rho$

$\rho_m \propto a^{-3}$ Dust $P=0$
 $\rho_r \propto a^{-4}$ $w=1/3$
 $\rho_\Lambda \propto a^0$ $w=-1$

$$H = \frac{\dot{a}}{a}$$

~~$H^2 = H_0^2 \sum \Omega_i$~~

$$H^2 = H_0^2 \sum \Omega_i (1+z)^{3(1+w_i)}$$

↓
fractional density

$$\Omega_i = \frac{8\pi G \rho_i}{3 H_0^2} \quad \text{at } t_0 \text{ (now)}$$

and $\Omega_k = \frac{-K}{a_0^2 H_0^2}$ (curvature contribution)

and z is redshift: $(1+z) = \frac{a_0}{a}$

~~SN Surveys~~

SN Surveys
(Super Nova)

$$d_L = (1+z) d_{\text{phys}} \\ = (1+z) \int_0^z \frac{dz}{H}$$

To leading order $d_L = \frac{z}{H_0}$

but to next orders, dependence of H on z comes in.
SN fainter than expected. So H dominated by an Ω
with $w \approx -1$

Can have $w < -1$

? Modify gravity or ~~particle~~ particle sector.

Modify gravity with scalars.

(pg 142)

$$S_{JBD} = \frac{1}{16\pi} \int \sqrt{g} \left[-\frac{\Phi}{\Phi} R + \omega \frac{(\nabla \Phi)^2}{\Phi} \right]$$

Φ ... gives background Newtons Constant.

Brans Dickey

$$\rightarrow G_{ab} = \frac{8\pi}{\Phi} T_{ab} + T_{\Phi ab}$$

$$T_{\Phi ab} = \Phi^{-1} \nabla_a \nabla_b \Phi + \omega \frac{\nabla_a \Phi \nabla_b \Phi}{\Phi^2} - g_{ab} \left(\Phi^{-1} \square \Phi + \frac{\omega}{2} \cdot \frac{(\nabla \Phi)^2}{\Phi^2} \right)$$

$$R + 2\omega \frac{\square \Phi}{\Phi} - \omega \frac{(\nabla \Phi)^2}{\Phi^2} = 0$$

$$\text{so } \boxed{\frac{\square \Phi}{\Phi} = \frac{8\pi T}{3 + 2\omega}}$$

$\Rightarrow$ scalar field coupled to Trace of Energy momentum tensor; the more matter there is, stronger

the scalar field reacts.

$\hookrightarrow$ So, Newton's constant depends on what else is around.

If scalar field is reacting to matter, so it means in our Solar system its going to react to local density of environment.

Putting in particle physics way; we get scalar forces .. we have fifth forces.

Screening BD like theory, and replace Φ with..

$$\Phi \rightarrow A^{-2}(\phi)$$

and Einstein frame

Pg/93

In Einstein frame,

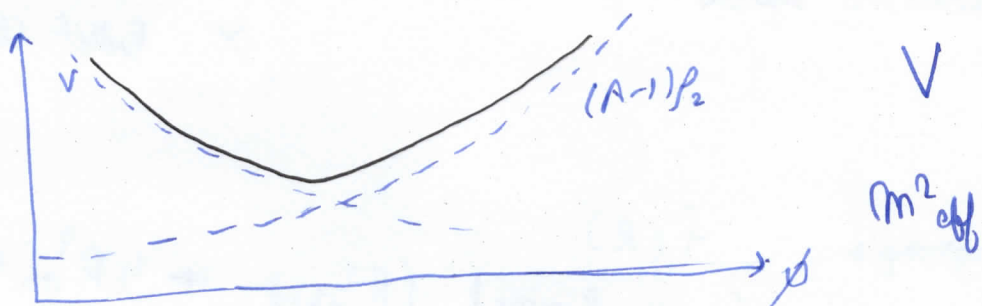
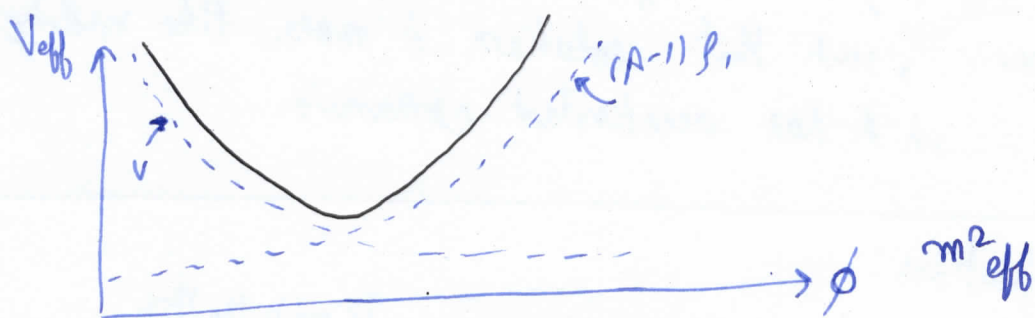
$$S = \int \underbrace{-\frac{M_P^2}{2} R + \frac{1}{2} (\partial\phi)^2 - V}_{\text{Gravity sector}} + \mathcal{L}_{\text{em}}[\Psi, A^2 g_{\mu\nu}]$$

In Einstein frame
matter couples ϕ .

Scalar equation, $\square\phi + \frac{\partial V}{\partial\phi} + \frac{\partial A}{\partial\phi} \rho = 0$

cosmological density ρ_{m}/A
of fluid (or matter)

$$\frac{\partial}{\partial\phi} [V + (A-1)\rho] = \frac{\partial V_{\text{eff}}}{\partial\phi}$$



So, denser the region, the stronger the lift is in effective potential.

"Chameleon"

This is simple example where we come across notion of screening mechanism;
because you have some sort of scalar which comes in the gravity sector as a scalar tensor theory; so we have

conformal factor dependent on our scalar.

(9/19)

If we look at physics in Einstein frame; our scalar field, while it still couples to the trace of energy momentum.

But Trace term gives something which depends on density of environment.

So; it is possible, that we have an effective potential such that if we are in the galaxy or the solar system, the ambient density is sufficiently ρ to not to disturb solar system test.

However on cosmological density scale; The scalar mass becomes lower, such that solution is more like rolling scalar field which has accelerated expansion.

Other modifications:

• Modify Newtonian ~~Dynamics~~ Dynamic — Dark Matter
X Bullet Cluster.

• Modify Lagrangian — $f(R)$
→ Gauss Bonnet $(R^2_{abcd} \neq 4R^2_{ab} + R)\Xi$
→ $R^2_{\mu\nu\lambda}$ —

careful to
get 2nd
order equations.

• Massive gravity, Galileons, Lorentz rotation

Test against:

Cosmology: Structure formation

~~matter~~ - matter clumps under gravity

+ Numerical simulation

Large scale structure survey.

- Evolution of perturbation spectrum ~ CMB

+ Numerical simulation
CMB FAST.Solar system tests

Screened scalars +

- main sequence stars

- Neutron stars.

- Pulsars

- Black holes.

- Atom interferometry

Black Holes - Imaging a black hole?

Extreme environment from accretion.

Light can have an interesting path from black hole.

Recall for Schwarzschild, γ

$$V_{\text{eff}} = -1 + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3}$$

$$\dot{r}^2 + V_{\text{eff}} = 0$$

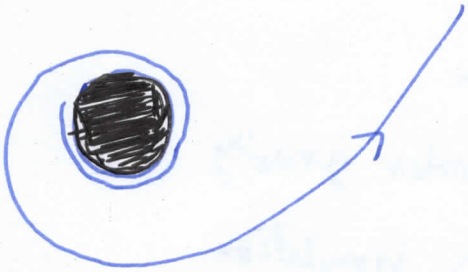
Photon initially tangent

$$\frac{h^2}{r^2} \left(1 - \frac{2GM}{r}\right) = 1$$

$$h^2 \sim (2GM)^3/\epsilon$$

$$\text{if } r_0 \sim 2GM + \epsilon$$

$$\text{and } \dot{\phi} = h/r^2 \sim 1/\sqrt{2GM\epsilon}$$



photon "hugs" black hole
before escaping

... Strong Lensing

~~leads to interesting~~

leads to interesting images

GRAVITY IS BEAUTIFUL

THANK YOU

Shoaib Akhtar