MATHS FOR QFT

Author: **Shoaib Akhtar**
Started on: 1-May-2020
Completed on: 1-May-2020

These notes are consequence of my self study; and are mostly inspired from Dr.Daniel Wohns lectures on **Maths for QFT**. This notes were made in a single blow on the midnight while waiting for Sehri in Ramazan, during the Corona Pandemic.

Tame the infinities ∞ !

— Shoaib Akhtar
1/5/2020

Maths for QFT Lec 1: Distributions, Test Functions

— Shoaib Akhtar 15/5/2020

tools + concept for QFT I, II, III

Outline : Distributions (Generalized Functions) : (2 lectures)
Asymptotic Series : (2 lectures.)
"Summation" of divergent series (1 lecture)

Distributions

Motivation: ex a familiar example; point charge :
has a charge distribution
i.e.: $\rho(x) = 0$ for $\vec{x} = \vec{0}$
$\rho(\vec{0}) = \infty$.

ex $\frac{d^n}{dx^n}|x|$ at $x = 0$?   ex $\int_{-\infty}^{+\infty} \frac{1}{x} dx$ ex $\frac{\delta F}{\delta f(x)}$

ex Greens function; they satisfy equation of the form $LG = \delta$.

ex existence of solutions.

Outline : • Definations (what they are not / what they are)
• Operations
  – Derivations
  – Multiplication
  – Composition.
• Applications
  – $\frac{\delta F}{\delta f(x)}$
  – Green's functions

lecture 2

first two lectures

Most important example : Dirac delta

$$\delta(x) \stackrel{?}{=} f(x) = \lim_{\sigma \to 0} f_\sigma(x) \quad \text{where: e.g. } f_\sigma(x) = \begin{cases} \frac{1}{\sigma} & |x| < \frac{\sigma}{2} \\ 0 & |x| \geq \frac{\sigma}{2} \end{cases}$$

$\int_{-\infty}^{+\infty} f_\sigma(x)\, dx = 1$ for all σ

$f(x) = 0$ for $x \neq 0$.

not a function : $\delta(0)$?

Defimation: Function space is a vector space whose elements are (equivalence classes) functions with a norm that satisfies.

- positivity. $\|f\| \geq 0$
- unique identity $\|f\| = 0 \Leftrightarrow f = 0$
- Triangle Inequality; $\|f+g\| \leq \|f\| + \|g\|$
- linearity + homogenity : $\|\lambda f\| = |\lambda| \|f\|$

main example ; $L^2[a,b]$ set of square integrable functions

ie $\int_a^b |f|^2 dx < \infty$. then $f \in L^2[a,b]$

note; $f_1(x) = 0$; $\|f_1\| = 0$

$$f_2(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases} \quad ; \|f_2\| = 0$$

almost everywhere.

we need to identify $f_1(x)$ & $f_2(x)$ if we want to treat $L^2[a,b]$ as vector space ; other wise we will not have unique identity ; say that they live in same equivalent class.

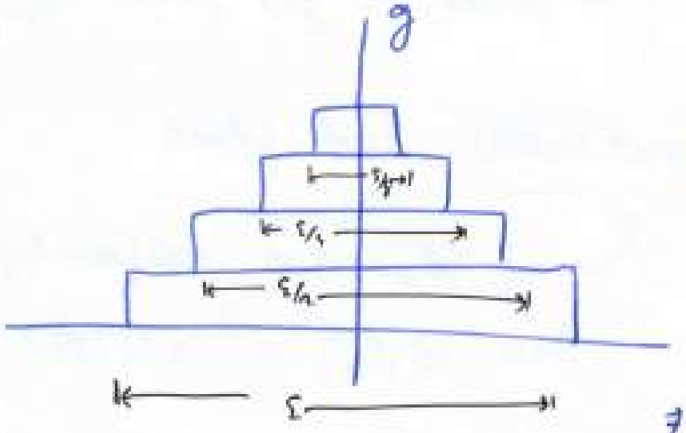
$\delta \notin L^2[a,b]$

$$\therefore \|f_\sigma\|^2 = \int_{-\infty}^{+\infty} |f_\sigma(x)|^2 dx = \int_{-\sigma/2}^{+\sigma/2} \frac{1}{\sigma^2} dx = \frac{1}{\sigma}$$

$\|f(x)\| \longrightarrow \infty$. so; $\delta \notin L^2[a,b]$

Choose some g_σ so that $\lim_{\sigma \to 0} g_\sigma(x) = 0$ for $x \neq 0$

$\int_{-\infty}^{+\infty} g_\sigma(x)\, dx = 1$ for all $\sigma > 0$



For each ϵ and each x we can find σ_0 so that $g_\sigma(x)$ lies inside the boxes for all $\sigma < \sigma_0$

$$\int_{-\infty}^{+\infty} g(x)\,dx < A = \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \dots = 2\varepsilon$$

area of boxes

$$\int_{-\infty}^{+\infty} g(x)\,dx = 0 \neq 1 = \lim_{\sigma\to 0}\int_{-\infty}^{+\infty} g_\sigma(x)\,dx$$

order of limit and integral matters.

Test functions: we will choose set $\mathcal{D}$: set of C^∞ functions with compact support

compact support: vanishes outside of some finite region

C^∞: continuous with ∞ many derivatives

(The nicer your test functions are, the wilder your distributions can be)

ex e^{-x^2} not belongs to $\mathcal{D}$; because don't have compact support for it.

ex
$$f(x) = \begin{cases} e^{-\frac{1}{x^2-1}} & ; |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$\mathcal{D}^*$ = dual vector space of linear functionals of $\mathcal{D}$ to $\mathbb{R}$.

$\therefore u \in \mathcal{D}^*$ if $u(\varphi) \in \mathbb{R}$ for any test function $\varphi \in \mathcal{D}$

linear ; $u(\alpha\varphi + \beta\psi) = \alpha u(\varphi) + \beta u(\psi)$

We will put one more restriction in order to get space of distributions.

$\mathcal{D}'$ = distributions , continuous dual space.

If $\phi_n \to \phi$ uniformly.
then $U(\phi_n) \longrightarrow U(\phi)$

so; $\boxed{u \in \mathcal{D}' \quad \text{if} \quad u \text{ is } \bullet \text{ finite} \quad \bullet \text{ linear} \quad \bullet \text{ continuous}}$

---

ex $\int \delta(x)\, \phi(x)\, dx = \phi(0) \longrightarrow$ real number, finite & continuous i.e; $\phi_n \to \phi$ then $\int \delta(x)\, \phi_n(x) \to \phi(0)$

Actually this is defination of delta function $\int \delta(x)\, \phi(x)\, dx = \phi(0)$; this way we don't have to define its value at any point.

We can define delta to be $\delta(\phi) = \phi(0)$:)

takes the input — output

In this sense; it is a functional.

$(\delta, \phi) = \phi(0)$:(

---

ex $u(\phi) = \int (\phi(x))^2\, dx$ — not linear ; $\underline{V(\phi)} = \phi(0) - \phi'(\pi) + 37\phi''(e)$

$w(\phi) = \begin{cases} 1 & ; \text{ if } \phi(0) \geq 0 \\ 0 & \text{otherwise} \end{cases}$ also; $w(\phi)$ is not linear

not continuous ; $\phi_n \to 0^-$;

$t(\phi) = \int \frac{\phi(x)}{x^2}\, dx$.

not finite.

= : yes, it is distribution

~ : no, it is not distribution.

functions that are locally integrable are distributions (pg 5)

$\int_a^b f(x)\,dx < \infty$ for any $a, b \in \mathbb{R}$

$$f(\phi) \equiv \int f(x)\,\phi(x)\,dx$$

~~"locally integrable functions induce distribution~~

~~locally ... integrable fs~~

"locally integrable functions induce distribution"

Lec 2 : ~~[illegible]~~ Operations on distributions, Applications of distributions. -Shoaib Akhtar 1/5/2020

Distribution maps test functions to $\mathbb{R}$.

Outline : Operations
- Derivation
- Multiplication.
- Composition.

} first part of lecture.

- Application to
  - $\frac{\delta F}{\delta(f)}$ + Greens functions ($LG = \delta$)

} second part of lecture

Derivation

$$\cancel{u'(x) \stackrel{?}{=} \lim_{\varepsilon \to 0} \frac{u(x+\varepsilon) - u(x)}{\varepsilon}}$$ dead end.

$$u'(\phi) = \int_{-\infty}^{+\infty} u'(x)\,\phi(x)\,dx$$

lets take some inspiration from locally integrable function & the induced distribution by them.

compact support of $\phi(x)$

$$= u(x)\,\phi(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u(x)\,\phi'(x)\,dx$$

$$= 0 - \int_{-\infty}^{+\infty} u(x)\,\phi'(x)\,dx = -\int_{-\infty}^{+\infty} u(x)\,\phi'(x)\,dx$$

$$= -u(\phi')$$

$$\Rightarrow \underline{u'(\phi) = -u(\phi')}$$

even we derived it assuming u is a function; but we can take it as a defination even if u is a distribution and not an ordinary function.

$$\boxed{u'(\phi) \equiv -u(\phi')}$$ weak or distributional derivatives

$$\boxed{u^{(n)}(\phi) \equiv (-1)^n\, u(\phi^{(n)})}$$

example!! Heaviside $\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

its locally integrable so it induces a distribution.

$$\Theta'(\phi) = -\Theta(\phi') = -\int_{-\infty}^{+\infty} \Theta(x)\, \phi'(x)\, dx = -\int_0^{\infty} \phi'(x)\, dx$$

$$= -(\phi(\infty) - \phi(0))$$
$$= \phi(0) = \delta(\phi)$$

we can take weak derivative.

$$\Rightarrow \Theta'(\phi) = \delta(\phi) \quad \forall\, \phi \in \mathcal{D}$$

$$\Rightarrow \boxed{\Theta' = \delta}$$

Example: $(\ln|x|)'$

$\int_0^a \ln|x|\, dx \to \infty$ (not locally integrable)

$$(\ln|x|)(\phi) \equiv \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \phi(x) \ln|x|\, dx$$

~~with~~ with this defination of distribution $\ln|x|$; we can now take weak derivative.

$$(\ln|x|)'(\phi) = -\lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \phi'(x) \ln|x|\, dx$$

$$= \lim_{\varepsilon \to 0} \Bigg[\underbrace{\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{1}{x}\, \phi(x)\, dx}_{(PV\, \frac{1}{x})(\phi)} + \underbrace{\phi(\varepsilon) \ln|\varepsilon| - \phi(-\varepsilon) \ln|\varepsilon|}_{\simeq\, 2\phi'(0)\varepsilon \ln|\varepsilon| \,\to\, 0 \text{ as } \varepsilon \to 0} \Bigg]$$

principle value distribution acting on ϕ

$$\Rightarrow (\ln|x|)' = \left(PV\, \frac{1}{x}\right).$$

Multiplication by a function.

lets develop the defination from ordinary functions

~~so~~ suppose ψ ordinary function; & u distribution; $\phi \in \mathcal{D}$

$$(\psi u)(\phi) = \int_{-\infty}^{+\infty} \psi(x)\, u(x)\, \phi(x)\, dx \quad \text{if } u \text{ is a function.}$$

$$= \int_{-\infty}^{+\infty} u(x)\, (\psi(x)\phi(x))\, dx$$

so; we can define $\boxed{(\psi u)(\phi) \equiv u((\psi)(\phi))}$

we need to check $\psi(\phi) \in \mathcal{D}$.

so; ψ should be continuous with ∞ derivatives C^∞.

so; ψ should be C^∞ for our test function space.

but in general we need $\psi\phi$ to be a test function.

---

example 1) simplify $x\delta'$

1) $-\delta$
2) $-\delta - x\delta$
3) 0
4) other.

$~~x\delta'(\phi) = \int_{-\infty}^{+\infty} \delta' (x(\phi)) dx~~$

$$x\delta'(\phi) = \delta'(x\phi)$$

$$= -\delta((x\phi)') = -\delta(\phi) - \delta(x\phi') = -\delta(\phi) - \underbrace{\delta(x\phi')}_{0}$$

$$= -\delta(\phi)$$

$$\Rightarrow x\delta'(\phi) = -\delta(\phi) \Rightarrow \boxed{x\delta' = -\delta}$$

$x\delta(\phi) = \delta(x\phi) = 0$ so; $1 \equiv 2$ (haha )

Composition

$$(u \circ f)(\phi) = \int_{-\infty}^{+\infty} u(f(x))\, \phi(x)\, dx \quad \text{if } u \text{ is a function.}$$

$y = f(x)$.

$g(y) = x$

$$\Rightarrow (u \circ f)(\phi) = \int_{-\infty}^{+\infty} u(y)\, \underbrace{\phi(g(y))\, |g'(y)|}_{\text{if test function;}}\, dy$$

$\phi(g(y))\,|g'(y)|$ → test function { if f is C^∞

{ $y = f(x)$ has unique solution.

→ sometimes possible to relax restriction.

So: f has to be a function, which goes from $-\infty$ to $+\infty$; so; limit of $\int$: $-\infty$ to $+\infty$

$$\boxed{(U \circ f)(\phi) \equiv U((\phi \circ g)|g'|)}$$

---

eg $(\delta \circ f)(\phi) \equiv \sum_i \frac{\delta_{x_i}}{|f'(x_i)|}(\phi)$

$f(x_i) = 0$

$\delta_{x_i}(\phi) = \phi(x_i)$

works if f has only simple roots (otherwise denominator becomes zero)

$f(x) = x^2 - 1$; $\delta \circ f = \frac{\delta_1}{2} - \frac{\delta_{-1}}{2}$

$f(x) = x^2$; $\delta \circ f$ → undefined.

Application : $\frac{\delta F}{\delta f(x)}$

$$\delta S = \int dt \, \underbrace{\frac{\delta S}{\delta q(t)}}_{\text{functional derivative}} \delta q$$

δS ← functional

(its the generalization of partial derivative)

$$\delta S = \sum_i \frac{\partial S}{\partial q_i} \delta q_i$$

← function

~~close~~ } ⇒ close analogy

$\Sigma \rightarrow \int dt$

$\delta q(t) = \epsilon \, \phi(t)$

ϵ : some small parameter. $\phi(t) \in \mathcal{D}$

$$\int \frac{\delta F}{\delta f(x)} \phi(x)\, dx = \lim_{\varepsilon \to 0} \frac{F[f+\epsilon\phi] - F[\cancel{f}]}{\epsilon}$$

functional derivative acting on a test function.

$$\boxed{\int \frac{\delta F}{\delta f(x)} \phi(x)\, dx = \lim_{\varepsilon \to 0} \frac{F[f+\epsilon\phi] - F[f]}{\epsilon}}$$

Example !! $F[f] = f^2(x_0)$

$$\int \boxed{\frac{\delta F}{\delta f(x)}} \phi(x)\, dx = \lim_{\varepsilon \to 0} \frac{\cancel{f^2(x_0)} + 2\varepsilon f(x_0)\phi(x_0) + \varepsilon^2\phi^2(x_0) - \cancel{f^2(x_0)}}{\epsilon}$$

$$= 2f(x_0)\phi(x_0)$$

$$= \int \boxed{2f(x_0)\delta(x-x_0)}\, \phi(x)\, dx$$

works for any ϕ

so: $\frac{\delta F}{\delta f(x)} = 2f(x_0)\delta(x-x_0)$

because of this property;

- linear
- product rule
- chain rule.

similar to partial derivatives.

Application to Greens Function

$Ly = f$ ← function

solve for y.

linear differential operator

$y = L^{-1} f$

Analogy -

Finite Dimension	Infinite dimensions
$\vec{v}$	~~fn~~ function f
components v_i	Value ~~of~~ at a point $f(x)$
matrix M	operator O
matrix element M_{ij}	Integral Kernel $(K(x,y))$

$$(M\vec{v})_i = \sum_j M_{ij} v_j$$

$$O\, f(x) = \int K(x,y)\, f(y)\, dy$$

Integral kernel of operator ; & is often a distribution.

density matrix	Dirac delta δ some subtilities here

So; Greens function can be thought about integral Kernel of the inverse operator L^{-1}.

$\therefore$ $L L^{-1} = \mathbb{1}$ (identity; which is ~~dirac~~ dirac delta here.

$$L L^{-1} = \delta$$

$$\Rightarrow \quad L_x G(x,\xi) = \delta(x-\xi)$$

(Greens function equation)

~~so the~~

Maths for QFT   Lec 3: Asymptotic Series and Stirling's Approximation.   (pg 9)

- [illegible] 1/5/2020

Asymptotics

- Gamma function
- Definations of Asymptotic series
- Stirling's Approximation.
- Saddle point approximation
  - Stoke's Phenomenon.

} lec 3 and lec 4

Gamma Function

Motivation : * example for ~~Sadd~~ Saddle point approximation (SPA)

* dimensional regularization (in tutorial)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \qquad Re(z) > 0 .$$

generalization of factorial ; $n! = n(n-1)!$ (functional equation)

$$\frac{d}{dt}(t^z e^{-t}) = z t^{z-1} e^{-t} - t^z e^{-t}$$

$$\underbrace{t^z e^{-t}\Big|_0^\infty}_{0} = z \int_0^\infty t^{z-1} e^{-t} dt - \int_0^\infty e^{-t} t^z dt$$

$$\Rightarrow \quad 0 = z\Gamma(z) - \Gamma(z+1)$$

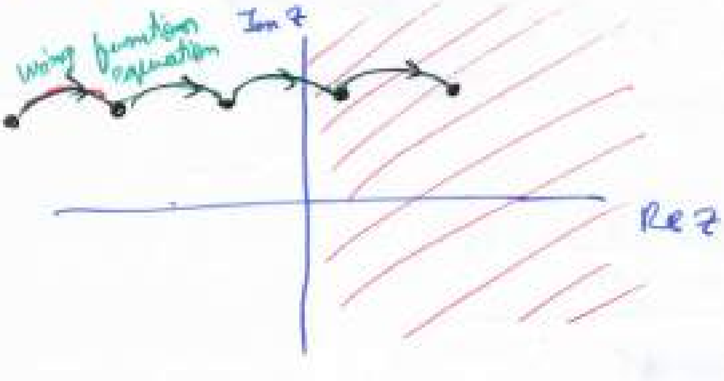
$$\Rightarrow \boxed{\Gamma(z+1) = z\Gamma(z)}$$ Functional Equation.

together with the fact $\Gamma(1) = 1$ (easy to prove)

$$\Rightarrow \quad \Gamma(n+1) = n!$$

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\dots(z+n-1)}$$

if $Re(z) + n > 0$



$\Gamma(z)$ has no zeroes; poles at $z=-n$; for $n \geq 0$ integer.

$$z = -(n-1) + \epsilon$$

$$\Gamma(z) = \frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)\cdots(\epsilon-n+1)} \approx \frac{(-1)^{n-1}}{\epsilon(n-1)!} \quad \text{as } \epsilon \to 0$$

so; poles are simple poles and ~~red~~ residue

$$\boxed{\text{residue} = \frac{(-1)^n}{n!} \text{ at } z=-n}$$

Asymptotics

Motivation: understand behavior when parameter is large or small

Toy Model: $Z(\lambda) = \int_{-\infty}^{+\infty} e^{-x^2 - \lambda x^4}\, dx$ (zero dimensional path integral)

$$= \int_{-\infty}^{+\infty} e^{-x^2} \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n x^{4n}}{n!}\, dx \overset{?}{=} \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \frac{(-1)^n \lambda^n x^{4n}}{n!}\, dx$$

Asymptotic series.

$$= \sum \int_0^{\infty} \frac{(-1)^n \lambda^n}{n!} e^{-u} u^{2n-1/2}\, du$$

$$= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \Gamma(2n+1/2) \quad > \underbrace{\sum_{n=0}^{\infty} (-\lambda)^n n!}_{\text{for large } n.}$$

~~does not converge~~ does not converge.

$$\therefore \Gamma(2n + \tfrac{1}{2}) \simeq (2n)! > (n!)^2$$

$$\Rightarrow \Gamma(2n + \tfrac{1}{2}) \simeq (2n)! > (n!)^2, \text{ for large } n.$$

Definitions

① $f(x) \underset{x\to x_0}{<<} g(x)$: if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$

example: $x \underset{x\to 0}{<<} \frac{1}{x}$: ~~[illegible]~~

True (T) or False $\boxed{(F)}$: If $f(x) \underset{x \to x_0}{\ll} g(x)$; then $f(x) < g(x)$ for all x.

→ False.

Counter example

$x^2 \underset{x \to 0}{\ll} -1$.

$A \sim B$ asymptotic to

∴ A asymptotic to B.

$\boxed{2}$ $f(x) \underset{x \to x_0}{\sim} g(x)$

if $f(x) - g(x) \underset{x \to x_0}{\ll} g(x)$

or $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$

examples: $\cos x \underset{x \to \pi}{\sim} -1$; ~~For f~~ :

~~If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$~~,

(i) T or $\boxed{F}$: If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$, then $f(x) \underset{x \to x_0}{\sim} g(x)$

(ii) T or $\boxed{F}$: If $\lim_{x \to x_0} f(x) = y$; then $f(x) \underset{x \to x_0}{\sim} y$

~~Counter example~~ Counter examples

(ii) $\lim_{x \to 0} e^{-1/x} = 0$. : but; nothing can be asymptotic to zero

∵ because it comes in denominator in the definition

$\boxed{3}$ Asymptotic Series $y \underset{x \to x_0}{\sim} \sum_{n=0}^{\infty} a_n (x - x_0)^n$

if $y(x) - \sum_{n=0}^{N} a_n (x - x_0)^n \underset{x \to x_0}{\ll} (x - x_0)^N \ \forall N$.

(this defination does not says that series converges.

Comments

- convergent series gets better with more terms.
- Asymptotic series better as $x \to x_0$ with a fixed no. of terms (taking more terms can be better and also bad)

If $F(\lambda) \underset{\lambda \to 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$

then $F(\lambda) + \underline{b\, e^{-c/\lambda^2}} \underset{\lambda \to 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$

$$e^{-c/\lambda^2} \underset{\lambda \to 0}{<<} \lambda^n \quad \text{for all } n$$

↑ non-perturbative function → QFT !!!

This term is invisible to asymptotic series and is known as non perturbative function.

- $f(z) \underset{z \to z_0}{\sim} g(z)$ generally valid in wedge

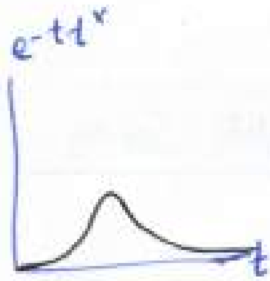
$a < \arg(z - z_0) < b$ in complex plane.

(example $e^{-c/\lambda^2} \underset{\substack{\lambda \to 0 \\ (\text{imaginary})}}{<<} \lambda^n$ will not be true)

Example: Stirling's Approximation.

Want $n! = \Gamma(n+1)$ for large real n.

we have: $\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt$



$t = wx$

$\Gamma(x+1) = x^{x+1} \int_0^\infty e^{-x(w - \ln w)} dw$

$$= x^{x+1} \int_0^\infty e^{-x f(w)} \, dw$$

→ most of the integrals can be put in this form

expand ; $f(w)$ around minimum.

$f'(w) = 1 - \frac{1}{w}$; min at $w = 1$

$f(w) = 1 + \frac{1}{2}(w-1)^2 + \dots \infty$

$$\Gamma(x+1) = x^{x+1} e^{-x} \int_0^\infty e^{-\frac{x}{2}(w-1)^2 + \dots} \, dw$$

→ This is not Gaussian integral if we forget higher order terms

$$\Rightarrow \Gamma(x+1) \approx x^{x+1} \cdot e^{-x} \int_{-\infty}^{+\infty} e^{-\frac{x}{2}(w-1)^2} \, dw$$

$$= x^{x+1} e^{-x} \sqrt{\frac{2\pi}{x}}$$

$\int_0^\infty (\,\ \,) \, dw$ ← suppressed after a large deviation from some w_0
$\therefore$ $w \to 0 \Rightarrow$ it is suppressed to zero

& if we go negative ; it will be suppressed more.

so; $\int_{-\infty}^0 (\,\ \,) \, dw$ will not contribute much.

$$\boxed{\frac{\Gamma(x+1)}{\sqrt{2\pi}\, e^{x+\frac{1}{2}} e^{-x}} \underset{x \to \infty}{\sim} 1 + \frac{1}{12x}}$$

↑ if we kept + . . .

Next lecture: Saddle-point approximation.

$$I(z) = \int_C g(w)\, e^{z f(w)}\, dw \quad \text{as } z \to \infty$$

contour

If this complex integral, + phase changes ⇒ destructive interference.

Lec 4: Saddle-point Approximation, Stokes Phenomenon

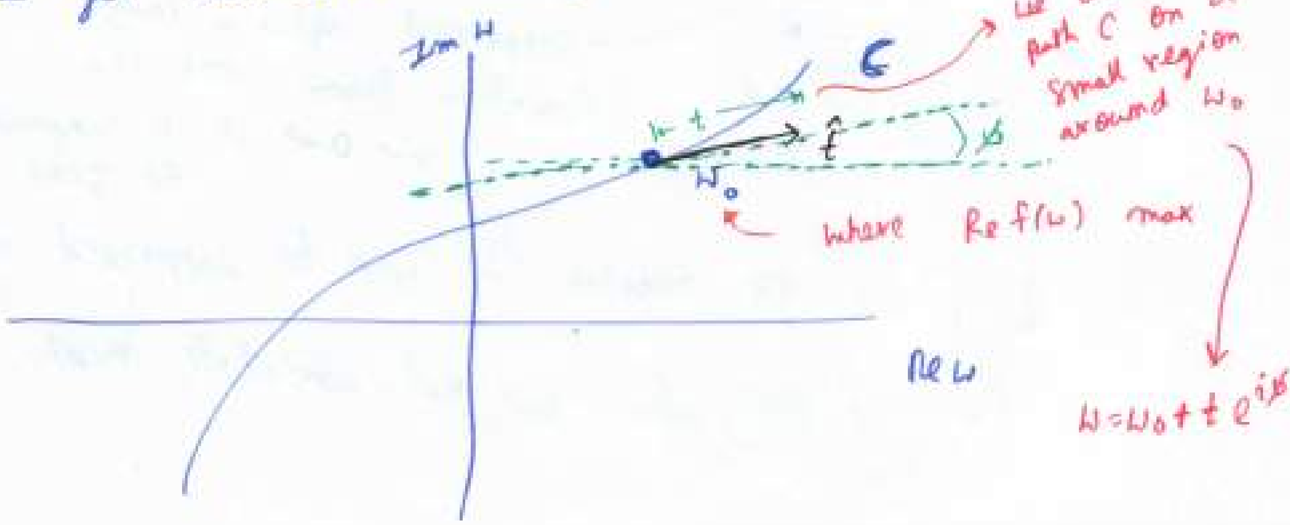
— Shoaib Akhtar; 2/5/2020

We want to approximate integral of the form

$$I(z) = \int_C g(w)\, e^{z f(w)}\, dw$$

∴ we want to understand its behavior as $z \to \infty$.

* We might expect dominant contributions to come from regions where $\mathrm{Re}(f(w))$ is maximum.
  ↳ works if C is a constant phase contour; otherwise we get destructive interference.



$u = \mathrm{Re}(f(w))$

$v = \mathrm{Im}(f(w))$

If we want constant phase curve

then $\nabla V = 0$ along the curve

(because phase is determined by V (imaginary part of $f(\omega)$))

also; we know; at ω_0 ; $\underline{\underline{\nabla U = 0}}$ at ω_0

$\nabla V = 0$ along curve C

$\nabla U = 0$ at ω_0

$\Rightarrow f'(\omega_0) = 0 \Rightarrow \omega_0$ is critical point of $f(\omega)$.

$\therefore$ we know complex function dont have any maximum.

no local maximum for holomorphic function.

↪ told by Cauchy Riemann equations.

must change in this direction.

thats why : it is named Saddle point approximation (SPA)

because ω_0 is actually ~~a~~ saddle point.

$$f(\omega) = f(\omega_0) + \frac{1}{2}(\omega-\omega_0)^2 \underbrace{f''(\omega_0)}_{\text{in general a complex number; } e^{i\delta_0} \| f''(\omega_0) \|} + \dots \infty$$

locally; we should have that

$$f(\omega) = f(\omega_0) - \frac{1}{2} t^2 \| f''(\omega_0) \|$$

$$f(\omega) = f(\omega_0) - \frac{1}{2}t^2\,|f''(\omega_0)|$$

comparing we have $(\omega-\omega_0)^2 e^{i\delta_0} = -t^2$

$$\Rightarrow \quad e^{2i\phi}\cdot e^{i\delta_0}\, t^2 = -t^2$$

$$\Rightarrow \quad \boxed{e^{2i\phi}\, e^{i\delta_0} = e^{\pm i\pi}}$$

↳ determines angle of constant phase contour.

$$I(z) \approx e^{zf(\omega_0)} \int_{-t_1}^{t_1} g(\omega_0 + e^{i\phi} t)\, e^{-\frac{z}{2}t^2|f''(\omega_0)|} \left.\frac{d\omega}{dt}\right|_{t_0} dt$$

linear for $-t_1 < t < t_1$

due to change of variable

$$\cancel{\approx e^{zf(\omega_0)} \int_{-t_1}^{t_1} \left[g(\omega_0) + \frac{g''(\omega_0)}{2} e^{2i\phi} t^2 + \dots\right] e^{-\frac{z}{2}t^2}}$$

now; ... we do similar to what done in Stirling approximation.

$$\approx e^{zf(\omega_0)} \int_{-\infty}^{+\infty} \left[g(\omega_0) + \frac{g''}{2}(\omega_0)\, e^{2i\phi} t^2 + \dots\right] e^{-\frac{z}{2}t^2|f''(\omega_0)|}\, \underbrace{e^{i\phi}}_{\left.\frac{d\omega}{dt}\right|_{t_0}} dt$$

↳ we can extend limit of integration $-t_1$ to t_1 to $-\infty$ to $+\infty$; because only for small region the integrand is large; because of $e^{-(z)}$.

$\times$ no linear term; because integral of odd function trivially vanishes.

$$I(z) \underset{z\to\infty}{\sim} \sqrt{\frac{2\pi}{z}} \frac{e^{z f(w_0)}}{\sqrt{-f''(w_0)}} \left[g(w_0) - \frac{1}{z} \frac{g''(w_0)}{1!\, 2f'(w_0)} + \dots \right]$$

phase drops out.

at some points higher derivatives of f : $f^{(n)}$ becomes more important than $g^{(n)}$

$$\cancel{I(z) \underset{z\to\infty}{\sim} \sqrt{\frac{2\pi}{z}}\, e^{z f(w_0)}}$$

$$I(z) \underset{z\to\infty}{\sim} \sqrt{\frac{2\pi}{z}} \frac{e^{z f(w_0)}}{\sqrt{-f''(w_0)}} \left[g(w_0) - \frac{1}{z} \cdot \frac{g''(w_0)}{1!\, 2f'(w_0)} + \dots \right]$$

Stokes Phenomenon

- abrupt change in asymptotic relations as phase of z changes

Example)

$$I(z) = \int_0^1 dt\, e^{-4zt^2} \cos(5zt - zt^3) \qquad z \to \infty \text{ (for now}$$

Naive attempt : neglect cos( )

$$I(z) \underset{z\to\infty}{\sim} \int_0^1 dt\, e^{-4zt^2} = \sqrt{\frac{\pi}{16z}}$$

critical when ; $t \underset{z\to\infty}{\sim} \frac{1}{\sqrt{z}}$ arg of cos is not small.

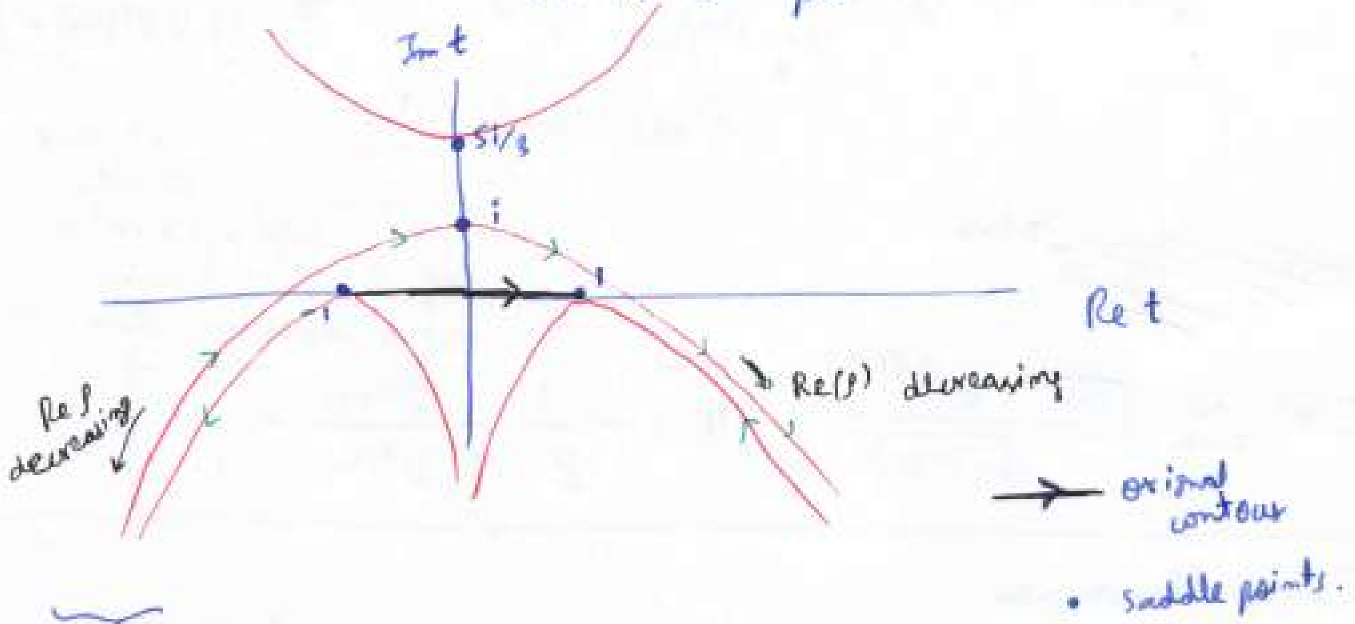
↳ destructive interference.

$$I(z) = \mathrm{Re} \int_0^1 dt\, e^{-4zt^2 + 5izt - izt^3}$$

$$= \frac{1}{2} \int_{-1}^1 dt\, e^{-4zt^2 + 5izt - izt^3}$$

$$= \frac{1}{2} e^{-2z} \int_{-1}^1 dt\, e^{z\rho(t)} \qquad ; \rho(t) = -(t-i)^2 - i(t-i)^3$$

$\rho'(t) = 0$ if $\left.\begin{matrix} t = i \\ t = 5i/3 \end{matrix}\right\}$ saddle points.



Constant phase contours are determined by $\; ; u = \mathrm{Re}(t), \; v = \mathrm{Im}(t)$

$$\mathrm{Im}\,(\rho(t)) = \text{constant}$$
$$= 3uv^2 - 8uv + 5u - u^3$$

$$\mathrm{Im}\,(\rho(-1)) = -4$$
$$\mathrm{Im}\,(\rho(1)) = 4$$

This analysis shows that we should do SPA around i saddle point; not at $5i/3$

endpoints contribute

~~$I(x) = \int f(t)\, e^{x\psi(t}$~~

$$I(x) = \int f(t)\, e^{ix\psi(t)}\, dt \underset{\substack{x\to\infty \\ x = \text{Real}}}{\sim} \frac{f(t)}{ix\psi'(t)} \cdot e^{ix\psi(t)} \Big|_{t=a}^{t=b}$$

works when
- RHS is non zero
- everything is C'
- $\int_a^b |f(t)|\, dt < \infty$
- ψ is not constant anywhere

no saddle point in particular

Saddle point $\quad \frac{1}{2} e^{-2z} \cdot \sqrt{\frac{\pi}{z}}$ $\quad$ dominates if $|\arg z| < \arctan\left(\frac{1}{2}\right)$

$t = \pm 1$ (endpoints) $\quad \mp \frac{i \pm 4}{68z} e^{-4z \pm 4iz}$ $\quad$ dominates is $|\arg z| > \arctan\left(\frac{1}{2}\right)$

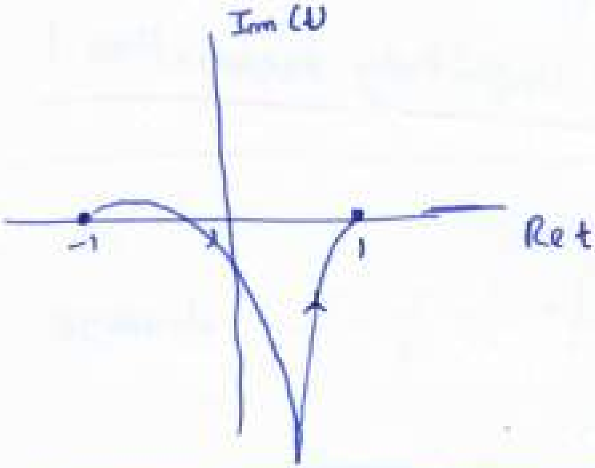
$$I(z) \underset{\substack{z \to \infty \\ \arg(z) = 0}}{\sim} e^{-2z} \sqrt{\frac{\pi}{z}}$$

$$e^{-2z} \underset{\substack{z \to \infty \\ \arg(z) \neq 0}}{>>} e^{-4z \pm 4iz} \; ?$$

Not true in general.

$$\lim_{z \to \infty} \frac{e^{-4z \pm 4iz}}{e^{-2z}} = \lim_{z \to \infty} e^{-2(\pm i2 - 1)z} = 0$$

if $|\arg z| < \arctan\left(\frac{1}{2}\right)$



$\arg z = 135°$

for large $|\arg z| \Rightarrow$ no saddles.

Summary: Approximate integrals using Gaussians
- results will be asymptotic.
- use constant phase contours.

Next lecture: Divergent Series

— Shoaib Akhtar : 1/5/2020.

Motivation: - perturbative series are usually asymptotic, not convergent.

- divergent series in QFT, ie: Casimir Force.

Outline: How to "sum" a divergent series.

Perturbative series.

Examples: ① $1 + 1 + 1 + \dots$ ← someones series

② $1 - 1 + 1 - 1 + \dots$ ← Grandi series (does not converge ⇒ divergent)

different series.

③ $1 + 0 - 1 + 1 + 0 - 1 + \dots$

$1 + (-1+1) + (-1+1) + \dots = 1 + 0 + 0 + \dots = 1$

$(1-1) + (1-1) + (1-1) + \dots = 0$

} assuming we are allowed to use associated property of addition, ~~we get~~ we get different answers.

↳ Conclusion: Addition is not infinitely associative!

$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

has divergent sub series: $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$ diverge.

assume commutativity of addition;

∴ we can then create any number

$$\text{eg: } \pi = \underbrace{\underbrace{1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \dots}_{> \pi} - \frac{1}{2} - \frac{1}{4} - \dots}_{< \pi} + \dots - \dots$$

Conclusion: Addition is not infinitely commutative

Euler Summation:

If $y = \sum_{n=0}^{\infty} a_n$ and a_n grows like n^k.

then $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge for $|x| < 1$

$$E = \lim_{x \to 1^-} f(x)$$

factor of x^n helps the series to converge

Euler sum

ex compute $E(1-1+1-\dots)$

$$E = \lim_{x\to 1^-} \sum_{n=0}^{\infty} (-x)^n = \lim_{x \to 1^-} \frac{1}{1+x} = \frac{1}{2}$$

$f(x) = \frac{1}{1+x} \rightarrow E = \frac{1}{2}$

Borel Summation

$n! = \Gamma(n+1) = \int_0^\infty dt\, e^{-t} t^n$

$1 = \int_0^\infty dt\, e^{-t} \frac{t^n}{n!}$

dividing by $n!$ helps to converge
(this is more powerful)
↳ will give finite answer even when Euler summation fails sometimes

$$B(\emptyset) \equiv \int_0^\infty dt\, e^{-t} \sum \frac{a_n t^n}{n!}$$

Borel sum

↳ actually a special case ; it is actually $(B(\emptyset)(1))$

... there can be one more parameter here.

ex $B(1-1+1-\dots) = \int_0^\infty dt\, e^{-t} \sum \frac{(-t)^n}{n!} = \int_0^\infty dt\, e^{-t} e^{-t} = \frac{1}{2}$

$$(B(x)(1)) = E(x)$$

whenever they both exist.

Generic Summation (desirable properties which we would like to our summation procedure have)

1. $\mathcal{S}(a_0 + a_1 + \dots) = S = a_0 + \mathcal{S}(a_1 + a_2 + \dots)$
2. $\mathcal{S}(\Sigma(\alpha a_n + \beta b_n)) = \alpha\, \mathcal{S}(\Sigma a_n) + \beta\, \mathcal{S}(\Sigma b_n)$: linearity

---

example

$$S = \mathcal{S}(1 - 1 + 1 - 1 + \cdots)$$

$$S = 1 + \mathcal{S}(-1 + 1 - 1 + 1 - \dots)$$

$$\Rightarrow S = 1 - \mathcal{S}(1 - 1 + 1 - 1 + \cdots) \Rightarrow S = 1 - S \Rightarrow S = \frac{1}{2}$$

$$\boxed{\mathcal{S}(1 - 1 + 1 - 1 + \dots) = \frac{1}{2}}$$

"Broad class of summation method gives same value"

Perturbation Theory

$$f(g) \underset{g \to 0}{\sim} \sum_{n \geq 0} a_n g^n \equiv \phi(g)$$

f: exact quantity of interest; g: perturbation parameter

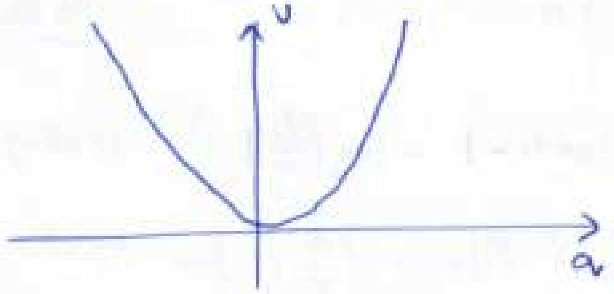
$$f(g) \xrightarrow{\text{perturbation theory}} \phi(g)$$

$$\phi(g) \xrightarrow{?} f(g)$$

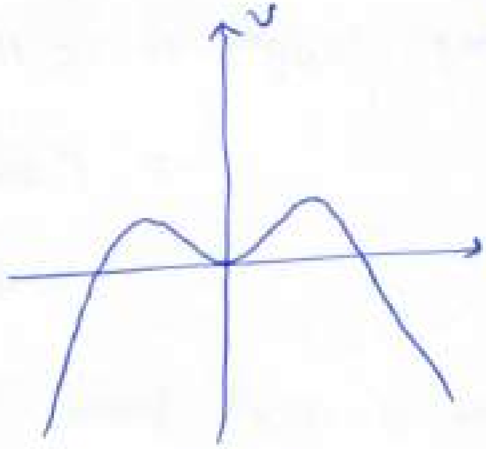
Dyson's Argument (why asymptotic series are usually divergent from physical viewpoint?)

Imagine Quantum Mechanical (QM) particle

In $V(q) = \frac{q^2}{2} + \frac{g}{4} q^4$



$g > 0$
stable
$Im(E_0) = 0$.



$g < 0$
unstable

quantity of interest $f(g) = E_0(g)$ ← ground state energy

non-perturbative effect. → tunnelling.

$Im(E_0)$ FD

So: if $\phi(g)$ has finite radius of convergence
$\Rightarrow$ describes $g > 0$ and $g < 0$.

contradiction.

So: $\phi(g)$ must have zero radius of convergence.

Non Perturbative physics ⟶ divergence of ϕ.

Optimal Truncation

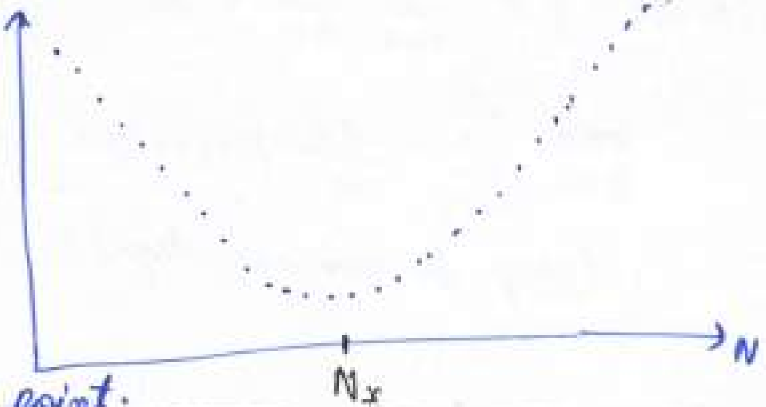
Typically in QM or QFT

$$a_n \underset{n \to \infty}{\sim} A^{-n}\, n!$$

NB
Optimal truncation is usually to keep terms up to the smallest term

$|f - \sum_{n=0}^{N} a_n g^n|$

Nature of graph is so, because usually the series is diverging ∴ so has minima at some point.



minimizing $|a_N g^N| \approx C N! \left|\frac{g}{A}\right|^N$

$$\simeq C \exp\left(N\left(\log N - 1 - \log\left|\frac{A}{g}\right|\right)\right) \quad \text{stirling}$$

minimized at $N_* \simeq \left|\frac{A}{g}\right|$ for $N >> 1$

error $\simeq$ next term

$$\simeq e^{-|A/g|}$$

non-perturbative : ambiguity

For many purposes; "divergent series converge faster than convergent series"

(because they dont have to converge  haha)

<u>Borel Summation</u> $\qquad \phi(g) = \sum a_n g^n$

<u>Borel Transform</u>: $\hat{\phi}(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$

<u>Borel Sum</u> : $B(\phi)(z) = \int_0^{\infty} e^{-t} \hat{\phi}(zt)\, dt$

The reason Borel summation procedure is interesting; is because Borel Sum of a series has same asymptotic expansion as the original series when both exist.

ie; Matches with orignal series when both exist.

proof:
$$B(\phi)(z) = \int_0^{\infty} e^{-t} \sum_{n=0}^{\infty} \frac{a_n t^n}{n!} z^n \, dt$$

$$\underset{z\to 0}{\sim} \sum_{n=0}^{\infty} \frac{a_n}{n!} \Gamma(n+1) z^n = \phi(z)$$

(Swap the sum and integral)

$$f(g) \underset{\text{Borel Sum}}{\overset{\text{perturbation theory}}{\rightleftarrows}} \phi(g)$$

$\hat{\phi}(t)$ has singularities on real line, whenever there are non-perturbative contributions to f.

"If orignal series has finite radius of convergence, then Borel sum matches the function in that region"

↳ We can often use $B(\phi)/z)$ to analytically extend it beyond that.

↳ The case of interest for us is when our orignal series is divergent and has zero radius of convergence.

Summary :
- Distributions
- Asymptotics
- Divergent series.

Tame the beast ;
give meaning to
infinities !!

— Shoaib Akhtar