Lec 1 | Brief Overview, Charged Particle on a circle surrounding a solenoid : Hamiltonian Quantization.

Chern-Simon Theory

Notes : www.physics.rutgers.edu/~gmoore/TASI-Chern Simons-studentNotes.pdf

Chern Simon Form related to Transgression (Chern, c. 1946).
→ 1974.

Anomaly descent formalism, early 1980's.
Deser, Jackiw, Templeton 1982.
1988, Witten (QFT & Jones Polynomial)

let G be a Lie Group
Consider 3d Gauge Theory with Gauge Group $= G$.
Locally $\mathcal{U} \subset M_3$, $A \in \Omega^1(\mathcal{U}, \mathfrak{g})$, $A = A_\mu^a t_a dx^\mu$

$\mathrm{Lie}(G) = \mathfrak{g}$ = \mathfrak{g}

$d + A^g = g^{-1}(d + A)g$, $F = dA + A^2 \in \Omega^2(\mathcal{U}, \mathfrak{g})$

Conjugation invariant Quadratic Polynomial $\mathcal{P}$

e.g. $\mathfrak{g}$ - Matrix Lie Algebra, $\mathcal{P}(X) = \mathrm{tr}\, X^2$
← some representation.

$\mathcal{P}(F)$ 4-form

$\mathrm{tr}(F^2) = d\, \mathrm{tr}(AdA + \frac{2}{3}A^3)$

$$\boxed{d\, CS_{\mathcal{P}}(A) = \mathcal{P}(F) = \mathrm{tr}(F^2) = d(\mathrm{tr}[AdA + \tfrac{2}{3}A^3])}$$

Some measure $[dA]$ on $\mathcal{A}$ = space of ~~Gauge fiel~~ all Gauge Fields on M_3.

$[dA]$ pushed down to $\mathcal{A}/\mathcal{G}$ = Space of all gauge-equivalent field configurations.

$\mathcal{G}$ = Group of Gauge Transformations.

$\sim \mathrm{Map}(M_3 \to G)$

$$\boxed{Z(M_3) = \int_{\mathcal{A}/\mathcal{G}} [dA]\, e^{i\int_{M_3} CS_\rho(A)}}$$

M_3 is orientable.

$\int_{M_3} CS_\rho(A)$ gauge invariant?

$$CS_\rho(A^g) = CS_\rho(A) + CS_\rho(g^{-1}dg) + d(*)$$

drop it if $\partial M_3 = \phi$

even when $\partial M_3 = \phi$ (i.e. M_3 dont have boundary)

If G has any non-trivial topology along with M_3, then it can happen that

$$\int_{M_3} CS_\rho(g^{-1}dg) \neq 0$$

OK if $e^{i\int_{M_3} CS_p(g^{-1}dg)} = 1 \quad \forall M_3,$

$g: M_3 \to G$

~~We call this "Multivalued Action Pri~~

We call this "Multivalued Action Principle" ~~and~~ or "Exponentiated Action Principle"

↪ Action can fail to be Gauge Invariant, and still be good for physics; because only thing we need in path integral is Exponentiated Action.

This will happen for Quantized values of p.

$$\begin{array}{c} P \\ \downarrow \pi \\ M_3 \end{array}$$

P is Principle Bundle.

$P \sim M_3 \times G$. If P is trivializable, then

$$\mathrm{Aut}(P) \cong \mathrm{Map}(M_3 \to G)$$

An interesting class of observables,

"line defects" or "line operator"

Oriented loop $\gamma \hookrightarrow M_3$

Take a finite dimensional representation of the gauge group of G : R

Then we can write the Wilson ~~loop~~ line defect as

$$W(R, \gamma) = \mathrm{Tr}_R \left(\mathcal{P} \exp \oint_\gamma A \right)$$

~~$\mathcal{P}$ is path ordering~~. Path ordered exponiant

This is gauge Invariant.

Think of $W(R, \gamma)$ as a function of A.

Now, we enhance the path integral little bit by imagining that there are bunch of loops (They might be linked, they might be knotted)

$$\boxed{\left\langle \prod_\alpha W(R_\alpha, \gamma_\alpha) \right\rangle = \int_{\mathcal{A}/\mathcal{G}} [dA] \prod_\alpha W(R_\alpha, \gamma_\alpha) \cdot e^{i \int_{M_3} CS_p(A)}}$$

<u>Note</u>: that we did not use metric to define the action.

↪ So, we might hope naively that this theory is going to be Topological Field Theory, i.e. Correlation Function is going to be independent of Distance, metric.

And they are just going to define Topological invariants of M_3 and loops imbedded in M_3.

We could imagine that these loops are linked or form ~~or~~ knots.

$G = SU(2)$ $\qquad \mathcal{P}(x) = \frac{-k}{8\pi^2} \, tr_{\underline{2}}(x^2) \quad k \in \mathbb{Z}$

Then, $\langle W(\underline{2}, \gamma) \rangle_{M_3 = S^3}$ = Polynomial in $q = e^{\frac{2\pi i}{k+2}}$

= Jones Polynomial.

2 dimensional representation.

A Breakthrough! (An important part for Witten's field Medal)

Jones polynomial is polynomial in q, that depends on the knot γ.

The idea was that, for every knot we would write down a polynomial, in q. And then, if we deform the knot smoothly and recompute, it's the same polynomial.

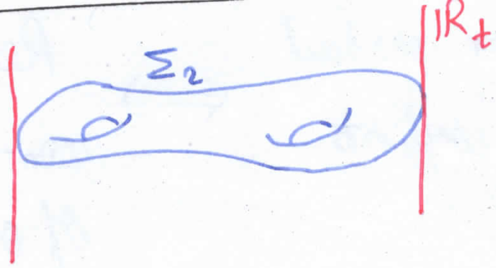
Applications

① Solvable, but non-trivial example of a field Theory. (topological field Theory)

② "Holographic" relations to 2D (Rational) Conformal field Theory.

$$\boxed{\begin{aligned} tr_R(x^2) &= C_2(R) \, tr_{adj}(x^2) \\ \text{eg} \quad tr_{\underline{2}}(x^2) &= \frac{1}{4} tr_{adj}(x^2) \end{aligned}}$$

$M_3 = \Sigma_2 \times \mathbb{R}$



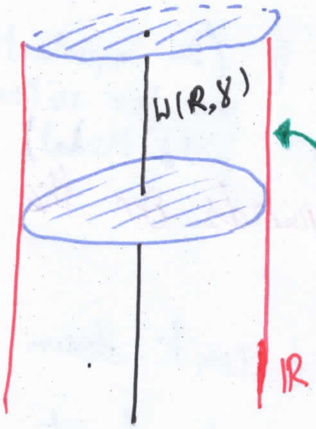
$\mathcal{H}(\Sigma_2)$

Then physics tells us that, there should be some Hilbert space

$\mathcal{H}(\Sigma_2)$ finite dimensional

$\simeq$ V.S. of "Conformal Blocks" of a corresponding 2D CFT

V.S. $\equiv$ Vector space.



$A|_\partial = ?$ (what boundary conditions to put)

With suitable boundary conditions, The Gauge modes on the boundary become Propagating fields of a CFT. (but this time, the CFT is on the boundary of the solid cylinder)

We call it the "Edge Modes"

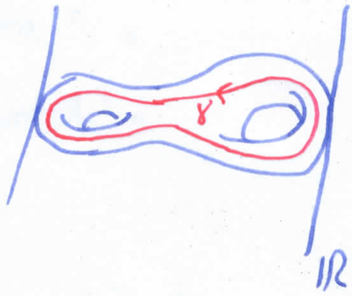
So there was 3D point of view on 2d CFT

3d view point gave a simple proof of "Verlinde Formula"

Operator product expansion coeffints $\Longleftrightarrow$ Response of correlators to non-trivial diffeomorphism. of our surface.

If we take γ to be some loop inside the surface.

$$M_3 = \Sigma_2 \times \mathbb{R}$$



Then $W(R,\gamma)$ becomes a operator $\widehat{W(R,\gamma)}$ acting on the ~~the~~ Hilbert space.

~~This is one of our~~ (We can write down the Algebra of Wilson line operators)

(3) C-S for some non-compact groups might be related to 3d quantum gravity.

B.H. (Brown & Hanaux) identified edge modes of graviton with Virasoro symmetry. (This is one of the first manifestation of AdS-CFT correspondance)

(4) Action $\dfrac{k}{8\pi^2}\int tr\left(A\,dA + \dfrac{2}{3}A^3\right)$

one derivative

Effective action for FQHE (Fractional Quantum Hall Effect) (it uses emergent gauge field)

$\hookrightarrow$ Then edge modes ~~because~~ becomes real physical mode that we can measure in lab.

$\exists$ nontrivial statistics

- "anyons".
- "Nonabelian".

(5) Dynamics of Gauge Theories.

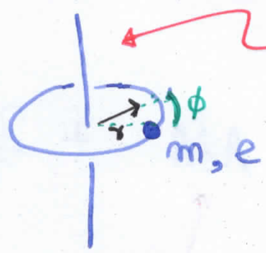
(6) Supergravity: Chern Simon terms often appear in SUGRA actions.

(7) Topological String Theory: Target spacetime theories are often C-S type.

(8) Fundamental Formulation of String Theory.

- String Field Theory: formulations often involve C-S type.

Quantum Mechanics



d.o.f of particle is the angle $\phi(t)$.

Ring of radius r.

$$\phi(t) \sim \phi(t) + 2\pi$$

(ϕ is defined modulo 2π)

So; its better to think of $h(t) = e^{i\phi(t)}$

($h(t)$ is single valued ... good)

$$h : \mathbb{R}_t \longrightarrow U(1)_{\text{Target}}$$

. h is map from time to target space (which happens to be $U(1)$)

We are really doing $0+1$ dimensional Field Theory.

$$S = \int \frac{1}{2} m r^2 \dot{\phi}^2 + \text{e.m. coupling.}$$

$$= \int -\frac{1}{2} m r^2 (h^{-1}\dot{h})^2 + \text{e.m. coupling.}$$

This is a kind of action we see in non-linear ~~~~ Sigma Model.

(we have a map from spacetime (which is just time) to a group)

Lie algebra of $U(1) = i\,\mathbb{R}$

Lie algebra of $U(N)$ = Anti-Hermitian ~~~~ matrices.

$$d + A^g = g^{-1}(d+A)g$$

$$A \rightarrow A + i\,d\epsilon \quad ; \quad g = e^{i\epsilon(x)} \in U(1)$$

(since $\epsilon(x) \in \mathbb{R}$)

Here, we talk about charged particles. So we will be talking about Electromagnetism; There we take gauge group to be $U(1)$

$U(1)$ is topologically non-trivial.

It is not ~~contract~~ contractible to a point.

$$\pi_1(U(1)) \cong \mathbb{Z}$$

Here, the gauge field is $A = \frac{B}{2\pi} d\phi$

The coupling of charged particle to e.m. field is

$$\exp\left(ie \int A_\mu (x(t)) \frac{dx^\mu}{dt} dt \right)$$

So, The action is

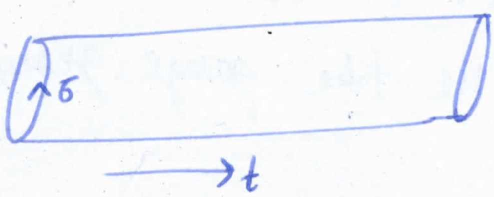
$$S = \int \frac{1}{2} I \dot{\phi}^2 + \int e\, \phi^*(A)$$

$I \Rightarrow$ moment of inertia

$$= \int \frac{1}{2} I \dot{\phi}^2 + \underbrace{\frac{eB}{2\pi} \dot{\phi} \cdot dt}_{\text{"}\theta\text{" term}}$$

(θ terms are related to interested aspects of topology in field Theory)

Consider $1+1$ dim. E.M.



The only gauge invariant degree of freedom here is,

The holonomy of our gauge field around the circle.

$$\cancel{h(t) = \exp\left(i\int d\sigma\, A_\sigma\right.}$$

$$h(t) = \exp\left(i\int d\sigma\, A_\sigma\right)(t) = e^{i\phi(t)}$$

$$S = \int \frac{1}{e^2} F * F + \int \frac{\theta}{2\pi} F$$

$$\boxed{\frac{\theta}{\pi} = 2B}$$

In 4D

$$S = \int \frac{1}{2e^2} tr(F * F) + \frac{\theta}{(2\pi)^2} tr(F \wedge F)$$

θ terms are Total derivatives, and so don't affect the equations of motion;

never the less, They matter in Quantum Mechanics.

<u>Quantize</u>

$$L = \frac{\delta S}{\delta \dot\phi} = I\dot\phi + \frac{eB}{2\pi} \quad \text{(Angular momentum)}$$

$$\Downarrow$$

$$-i\hbar \frac{\partial}{\partial \phi}$$

$$H = \frac{1}{2I}\left(L - \frac{eB}{2\pi}\right)^2 \quad \cancel{H = -\frac{\hbar^2}{2I}\left(-i\frac{\partial}{\partial\phi}\right.}$$

$$H = \frac{1}{2I}\left(L - \frac{eB}{2\pi}\right)^2 ,$$

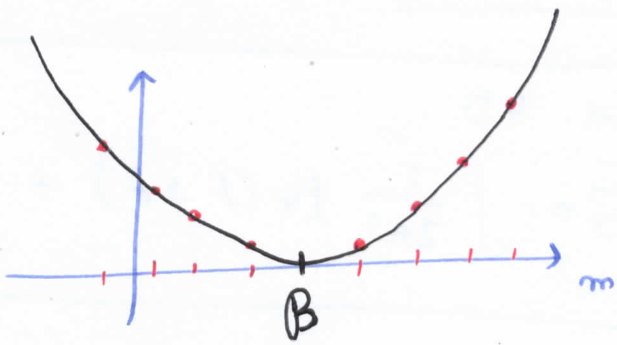
$$\Rightarrow \quad H = \frac{-\hbar^2}{2I}\left(-i\frac{\partial}{\partial\phi} - \beta\right)^2 \quad \text{where } \beta = \frac{eB}{2\pi}$$

So, we really get a family of hamiltonian parametrized by a real number β

$$H_\beta = \frac{-\hbar^2}{2I}\left(-i\frac{\partial}{\partial\phi} - \beta\right)^2$$

$\boxed{\text{Diagonalize}}$ $\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} \cdot e^{im\phi}$, $m \in \mathbb{Z}$

Then $E_m = \frac{\hbar^2}{2I}(m-\beta)^2$



m is quantized;
so β really affects the spectrum

"β matters" (ie: it affects the spectrum of the hamiltonian)

Summetry under $m \longrightarrow 2\beta - m$

m is quantized ~~to~~ to be an integer.

So, only exists if $2\beta = \frac{\theta}{2\pi} \in \mathbb{Z}$

- ~~If 2β is~~ • If 2β is odd, ground state is doubly degenerate.
- If 2β is even, " " simply degenerate. (ie: unique)

Spectrum is periodic in β.

$\exists$ U unitary on $L^2(S^1)$

such that $$U H_\beta U^{-1} = H_{\beta+1}$$

<u>Classical Symmetry</u> : Symmetrys of eqs of motion (so they dont know about θ-term)

$\ddot{\phi} = 0$ $\qquad R(\alpha) : \phi \to \phi + \alpha , \quad e^{i\alpha} \in U(1) \cong SO(2)$

$P : \phi \to -\phi$

$h(t) : \mathbb{R}_t \to U(1)_T$

so: P is really a charge conjugation.

$P = C$, because $P\, h(t) = h(t)^*$ ← complex conjugate.

ie; $h(t) \xrightarrow{P} h(t)^*$

$P R(\alpha) P = R(-\alpha) \implies \underline{O(2)} \longrightarrow$ The group we get.

<u>Wigner's Theorem</u> There is going to be Unitary or Ant-unitary operators for all symmetries of our Classical system. However, the relations that are satisfied classically, will in general only be satisfied up to ~~phs~~ phase in the Quantum System.

$g \xrightarrow{\text{Wigner Theorem}} U(g)$

symmetry of Classical system $\qquad$ (unitary or ant-unitary operator)

If $g_1 \cdot g_2 = g_3$ classically.

Then in Quantum Mechanics $U(g_1)U(g_2) = \underbrace{c(g_1, g_2)}_{\text{phase term.}} U(g_1 g_2)$

For our system in concern here,

Quantum: $R(\alpha) \cdot \psi_m = e^{im\alpha} \psi_m$

When $2B \in \mathbb{Z}$,

Then $\mathcal{C} \cdot \psi_m = \psi_{2B-m}$

$$\mathcal{C} R(\alpha) \mathcal{C} = (e^{i\alpha})^{2B} R(-\alpha)$$

① $2B = \frac{\theta}{2\pi} \notin \mathbb{Z}$ then $O(2) \Rightarrow SO(2)$

(O(2) is quantum mechanically broken down to SO(2))

② $2B = \frac{\theta}{2\pi} \in 2\mathbb{Z}$ then $O(2) \Rightarrow O(2)$

(O(2) remains quantum symmetry)

③ $2B = \frac{\theta}{2\pi} \in 2\mathbb{Z}+1$, then $O(2) \Rightarrow Pin^+(2)$

$Pin^+(2)$ is double cover (central extension) $O(2)$

$$Pin^+(2) \xrightarrow{2:1} O(2)$$

- Group Theory notes.
- Appendix D.

Chern - Simon Theory   Shoaib Akhtar   26/8/2020.

Lec 2 | Charged Particle on a circle surrounding Solenoid : Path Integral : Gauging the global SO(2) symmetry & Chern Simon Terms, U(1) C-S theory, Quantization of Phase space & Hamiltonian Reduction.

$\hbar = 1$

consider $\langle \phi_2 | e^{-\beta H_B} | \phi_1 \rangle$ matrix element of (The euclidean time propagator)

H_B is hamiltonian , β is euclidean.

(we diagonalize the hamiltonian and call it H_B)

By sticking in the complete sets of eigenstates we get

$$\langle \phi_2 | e^{-\beta H_B} | \phi_1 \rangle = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{-\frac{\beta}{2I}(m - B)^2 + im(\phi_1 - \phi_2)}$$

$$\parallel$$

$$Z(\phi_2, \phi_1 | \beta) = \int [d\phi(t)]_{\phi(0)=\phi_1}^{\phi(\beta)=\phi_2} \cdot \exp\left(-\int_0^\beta \left(\frac{1}{2} I \dot{\phi}^2 + i B \dot{\phi}\right) dt\right)$$

θ term (does not affect equation of motion)

Euclidean Path integral.

Wick Rotated Action.

This path integral is quadratic

$\Rightarrow$ So, semiclassical approximation is exact.

(So, we sum over the accessible solutions of the Euclidean equations of motion, and multiply by one loop determinant)

Equation of motion : $\ddot{\phi} = 0$

$$\phi(t) = \phi_1 + \left(\frac{\phi_2 - \phi_1 + 2\pi w}{\beta}\right) t \quad , w \in \mathbb{Z}$$

ϕ is defined modulo 2π, hence we can have the w piece.

$$\phi(t) = \phi_1 + \left(\frac{\phi_2 - \phi_1 + 2\pi w}{\beta}\right) t \quad , w \in \mathbb{Z}$$

$h(t) = e^{i\phi(t)}$ } $\Rightarrow$ This has to be single valued.

$$Z = \sqrt{\frac{I}{2\pi\beta}} \cdot \sum_{w \in \mathbb{Z}} e^{-\frac{2\pi^2}{\beta}\left(w + \frac{\phi_2 - \phi_1}{2\pi}\right)^2 - 2\pi i B\left(w + \frac{\phi_2 - \phi_1}{2\pi}\right)}$$

RHS of $\langle \phi_2 | e^{-\beta H_B} | \phi_1 \rangle$ terms are exponentially convergent for $\beta \to +\infty$

" " Z " " " " " $\beta \to 0$

But They are same

(Physical explanation: They are same because, they are same quantity)

(Mathematical explanation: Both are special cases of Reimann's ϑ function)

ϑ function plays an important role in discussing the spaces of states in Chern Simon Theory

$$\vartheta\begin{bmatrix}\theta\\ \phi\end{bmatrix}(z|\tau) \equiv \sum_{n\in\mathbb{Z}} e^{i\pi\tau(n+\theta)^2 + 2\pi i(n+\theta)(z+\phi)}$$

Absolutely convergent holomorphic function for $\mathrm{Im}\,\tau > 0$.
Entire function for z.

Properties of ϑ function

① Modular Property

$$\vartheta\begin{bmatrix}\theta\\ \phi\end{bmatrix}\left(\frac{-z}{\tau}\,\middle|\,\frac{-1}{\tau}\right) = (-i\tau)^{1/2}\cdot e^{2\pi i\theta\phi}\cdot e^{\frac{i\pi z^2}{\tau}}\cdot\vartheta\begin{bmatrix}-\phi\\ \theta\end{bmatrix}(z|\tau)$$

(Can prove this using Poisson Summation Formula (P.S.F.))
(sum of plane waves is a periodic Sum of Delta functions)

If 2β is odd,
then as $\beta \to +\infty$

$$\langle\phi_2|e^{-\beta H}|\phi_1\rangle = 2\,e^{\frac{i}{2}(\phi_1-\phi_2)}\cdot\cos\left(\frac{\phi_1-\phi_2}{2}\right)\cdot e^{-\beta E_g}\cdot(1+\text{exp. small})$$

E_g is ground state energy.

~~2nd Remark~~

General Remark: $\mathcal{T}$ theory with G-symmetry, then we can gauge it, "Gauging" (i) Make the theory $\mathcal{T}$ by equivariantizing by coupling to external G-gauge fields.

(ii) Sum over gauge inequivalent G-gauge fields.

~~If we~~ W

If we view n-dimensional field theory as a functor.

$$\mathcal{F} : \mathcal{B}^{str}_{\langle n-1, n\rangle} \longrightarrow \mathcal{T}$$

(i) $\mathcal{F}^{equiv} : \mathcal{B}^{str} + \left\{ \begin{matrix} \text{Principle G- bundles} \\ \text{with connection} \end{matrix} \right\} \longrightarrow \mathcal{T}$

(ii) Summing over isomorphism classes of ~~prin~~ principle G bundles with connection.

$$\phi(t) \longrightarrow \phi(t) + \alpha(t) \qquad g = e^{i\alpha} \in U(1)$$

To make equivariant, we allow α to depend on time. and we introduce an external gauge field $A^{(e)}$

$$A^{(e)} \longrightarrow A^{(e)} + i g^{-1}(t) \cdot d_t \cdot g(t)$$

not maxwell field.

$$S_E = \int \frac{1}{2} I (\dot{\phi} + A^{(e)})^2 + i \mathcal{B} (\dot{\phi} + A^{(e)}_t)\, dt$$

Euclidean action.

In Euclidean space; The θ term remains purely imaginary.

$2B = \frac{\theta}{\pi}$

$A_t^{(e)}$ spoils periodicity

We can restore it partially by ~~[illegible]~~ adding a one dimensional Cherns Simon term.

$$e^{-S_E} \cdot e^{ik\int A_t^{(e)}\,dt}$$ with some constant k.

So, it should be manifest now that

$$\underbrace{\mathcal{T}(B,k)}_{\text{Theory with } B \text{ \& } k} = \underbrace{\mathcal{T}(B+1,k+1)}_{\text{Theory with } B+1 \text{ \& } k+1}$$

Under gauge transformation.

$$e^{ik\int_{t_1}^{t_2} A_t^{(e)}\cdot dt} \longrightarrow e^{-i\alpha(t_2)k} \cdot e^{ik\int_{t_1}^{t_2} A_t^{(e)}\cdot dt} \cdot e^{ik\alpha(t_1)}$$

(depends on gauge transformation at the end points)

- If $\mathbb{R}_t$, and $\alpha(t) \longrightarrow 0$, Then its gauge invariant.
- If $G^{ext} = \mathbb{R}$, yes, even on S^1_t (euclidean circle)

$$\therefore e^{ik\oint_{S^1_t} A_t^{(e)}\cdot dt}$$

- If $G^{ext} = U(1)$, Then $\alpha(t+\beta) = \alpha(t) + 2\pi w$, $w \in \mathbb{Z}$

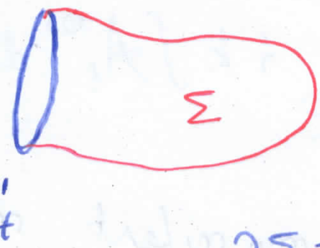
because only $e^{i\alpha(t)}$ is periodic.

$\Rightarrow \quad (\; " \;) \; e^{i 2\pi w k}$

↳ for this to be 1
$\Rightarrow k \in \mathbb{Z}$

Another way to try to define:

$$e^{ik \oint_{S^1_t} A^{(e)}_t \, dt}$$



$\partial \Sigma = S^1_t$

Imagine that we could extend the gauge field to the rest of the surface.

Then $$e^{ik \oint_{S^1_t} A^{(e)}_t \, dt} := e^{ik \int_\Sigma F}$$

LHS is gauge invariant for $k \in \mathbb{Z}$
but RHS seems to be gauge invariant for any k (if we fix Σ)
Whats going on?

F is locally dA
$F = dA$ locally.

Now, it is nice because F is gauge invariant.

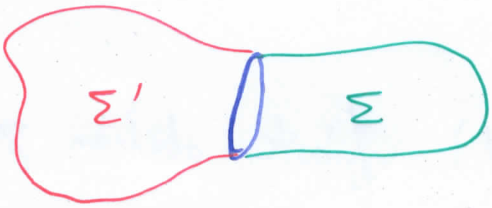
" but we will have to make a choice for Σ "

$\Rightarrow$ If we make two different choices of Σ, we might ~~exten~~ get different answers; so would actually depend on higher dimensional theory (which from the point of view we are doing here is fictitious)

" Extension is independent of Σ only if $k \in \mathbb{Z}$ "

because $F \in \Omega^2_{2\pi\mathbb{Z}}$

F is closed 2 form, with 2π period.



Two different extensions Σ & Σ'!

The difference of the integral, over the closed manifold$_3$ of F is $2\pi \times$ (Intger).

(because $\frac{F}{2\pi}$ represents first chern class)

What data goes into a C-S theory?

① Lie group G

② Level. Topology gives $G \rightsquigarrow$ $k \in$ $H^4(BG, \mathbb{Z})$ satisfies a non-degenaracy condition. finite dimensional Abelian group

~~M_3 oriente~~

- M_3 oriented (+ P_1 - structure)
- line defects becomes ribbons

$\boxed{\text{Best understood for } G \text{ compact (Theory is best understood..)}}$

G compact Lie group.

Then we ask about connected components

We have G_0 = connected component with 1_G (identity)

$G_1 \triangleleft G \Rightarrow G/G_1 \cong \pi_0(G)$

↖ any finite group.

$G_1 \simeq \left((G_1^{ss} \times U(1)^d) \right) / Z$

$Z \subset Z(G_1^{ss} \times U(1)^d)$ finite abelian group.

$G_1^{ss} \Rightarrow$ (G_i semi-simple)

$G_1^{ss} = \prod G_i^s$, G_i^s Cartan : $A \to G$.

$G = U(1)$, $H^4(BU(1), \mathbb{Z}) \cong \mathbb{Z} \Rightarrow k \neq 0$.

$BU(1) = \mathbb{C}P^\infty$

$A^g = A - i g^{-1} d g$; $g : M_3 \to U(1)$

Small gauge transformation: $g(x) = e^{i\epsilon(x)}$ single valued.
$\mathbb{R}$ - valued function on M_3.

Large gauge transformation: $\exists$ loop in M_3 $\oint_\gamma g^{-1} dg \neq 0$
(period not zero)

$A \to A + \omega$; ω is closed 1-form with $2\pi\mathbb{Z}$ period.

$$\exp\left(i\frac{k}{2\pi}\int_{M_3} AF\right)$$

looking at Gauge Invariance of U(1) Chern Simons.

gauge invariant iff $k \in \mathbb{Z}$

$$F = dA$$

$$AF \longrightarrow AF + \omega F$$

$2\pi\mathbb{Z}$ $\quad$ $2\pi\mathbb{Z}$

$$\int_{\Sigma_2} F \in 2\pi\mathbb{Z}$$

(closed 2 cycle)

$\frac{F}{2\pi}$ is called Chern-Weil (C-W) representative of $c_1(L)$.

$\exp\left(i\frac{k}{2\pi}\int_{M_3} AF\right)$ is gauge invariant iff $k \in \mathbb{Z}$,

because $\exists\ M_3$, $P \longrightarrow M_3$ s.t. $\int AF \in (2\pi)^2 \times$ (Any integer)

(principle U(1) bundle over ~~to~~ the 3-manifold M_3).

i.e. $\int AF \in (2\pi)^2 \times$ (Any integer)

This fact forces k to be an integer for gauge invariance.

$$e^{iS_{Lorentz}} = e^{i\int \frac{-1}{8\pi^2 e^2} F * F}$$

$$e^{-S_{Euclidean}} = e^{-\int \frac{1}{8\pi^2 e^2} F * F}$$

Maxwell Theory.

Now we can add AdA, The Chern-Simons term.

(we can only do it when K is integer)

$$e^{iS_{Lorentz}} = e^{i\left(\int_{M_3} \frac{-1}{8\pi^2 e^2} F * F + \frac{k}{2\pi} A dA\right)}$$

$$e^{-S_{Euclidean}} = e^{-\int_{M_3} \frac{1}{8\pi^2 e^2} F * F + i\frac{k}{2\pi}\int_{M_3} A dA}$$

depend on a metric; not on orientation

$$d^3x \cdot \frac{1}{e^2}\sqrt{\det g}\cdot g^{\mu_1\nu_1} g^{\mu_2\nu_2} F_{\mu_1\mu_2} F_{\nu_1\nu_2}$$

independent of metric; and depend on orientation.

Consider (1, 2) Minkowski Space $\mathbb{M}^{1,2}$.

Equations of motion $\quad d * F - m F = 0$

where $m = 4\pi k e^2$

(in 2+1 dimensions; electromagnetic coupling constant square, has the units of mass) looking at plane waves solution.

$$A_i \sim \epsilon_i(p)\, e^{ip\cdot x}$$

plugging this in equation of motion, leads to the following dispersion relation $\quad p_0^2 \cdot (p_0^2 - \vec{p}^{\,2} - m^2) = 0.$

$p_0^2 = 0$ or $\boxed{p_0^2 - \vec{p}^{\,2} - m^2 = 0}$

Dispersion relation of Massive scalar.

$\Rightarrow p_0 = 0$

$\Longleftrightarrow F_{12}(p) = 0$

$\Rightarrow$ flat gauge field.

Solutions of equations of motion factorize.

$$\mathcal{S}_{flat} \times \mathcal{S}_{non\text{-}flat}$$

Space of flat solution.

Space of non-flat solution.
"describes the degrees of freedom of a massive scalar particle in 2+1 dimensions"

Correlation Functions

$$\langle F_{\mu_1\nu_1}(x_1) \ldots F_{\mu_n\nu_n}(x_n) \rangle$$

$$\langle W(n_1, \gamma_1) \ldots W(n_s, \gamma_s) \rangle$$

Take the metric dependent term

$$\underbrace{d^3x}_{\Omega^3 \text{ (volume)}} \frac{1}{e^2} \sqrt{\det g} \cdot \underbrace{g^{\mu_1\nu_1} g^{\mu_2\nu_2}}_{\frac{1}{\Omega^4}} F_{\mu_1\mu_2} F_{\nu_1\nu_2}$$

When we look at longer distance scale $g_{\mu\nu} \longrightarrow \Omega^2 g_{\mu\nu}$

(Ω constant : just rescalling distance by factor of Ω)

So; scalling distance by factor of Ω $\equiv$ scalling coupling constant by factor of Ω.

$$\boxed{g_{\mu\nu} \to \Omega^2 g_{\mu\nu}} \longleftrightarrow \boxed{e^2 \to \Omega e^2}$$

$\Rightarrow$ but ~~m~~ $m \propto e^2$ ← mass of particle.

So; if $\Omega \longrightarrow \infty$

$\Rightarrow m \to \infty$ (Then mass can't propagate)

~~At large~~ At large distances or large masses

$$\langle F_{\mu_1\nu_1}(x_1) \dots F_{\mu_n\nu_n}(x_n) \rangle \sim e^{-|m| \cdot \min\{|x_i - x_j|\}} \longrightarrow 0$$

for $\langle W(m_1, \gamma_1) \ldots W(m_s, \gamma_s) \rangle$ this correlator

↪ nothing can propagate from here to there.

⇒ So we expect to get topological invariants.

In this limit, the action degenerates into Chern Simons action.

Remark

① Formal metric independence.

② In C-S, $k \sim 1/\hbar$ so $k \to \infty$ is Semi-classical (S.C.) limit.

③ Theory is Gaussian : No excuse, have to understand everything about the theory.

Take $M_3 = \Sigma_2 \times \mathbb{R}$, $G = U(1)$

There should be a Hilbert space.

$$\mathcal{H}(\Sigma_2, \mathcal{O}(\Sigma_2), k) = ?$$

(The theory depends on orientation of M_3 but $\mathcal{O}(M_3) \cong \mathcal{O}(\Sigma_2)$ equivalent)

When we have spacetime decomposition like $\Sigma_2 \times \mathbb{R}$

Then we can write A as

$$A = A_s + A_0 dx^0$$

$$\Rightarrow \int A\,dA = -\int dx^0 \underbrace{\left(\int_{\Sigma_2} A_s\, \partial_0 A_s\right)}_{\text{first order in time derivatives}} - \int dx^0 \left(\int_{\Sigma_2} A_0\, F_s\right)$$

A_0 does not appear with derivative

$\hookrightarrow$ so ; its functioning as Lagrange multiplier

$\hookrightarrow$ so $F_s = 0$ flat gauge field on Σ_2.

General lesson $S = \int \lambda_i(\phi) \frac{d\phi^i}{dt}\, dt$

$\Longleftrightarrow$ The interprete path integral ~~as~~ as, paths in phase space

(so path integrals over the spaces of p_s & q_s, (not just q))

(We can hope that path integral is going to localize to the space of flat gauge fields on Σ_2 (generalizing the plane waves in minkowski space)

Phase space = Symplectic manifold

$$(P\ , \ \omega)$$

phase space

2 form which is closed & non degenerate.

$d\omega = 0$

non-degenerate.

$\omega = \frac{1}{2!}\omega_{ij}(\phi)\, d\phi^i d\phi^j$

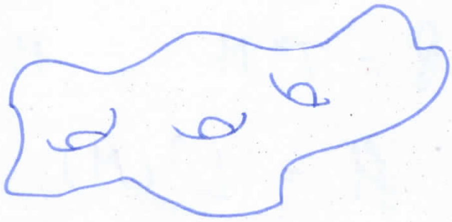
always ~~inv~~ invertible

$$\delta S = \int \underbrace{(\partial_i \lambda_j - \partial_j \lambda_i)}_{\omega_{ij}} \delta\phi^j \frac{d\phi^i}{dt}$$

In C.S. theory,

Phase Space = Space of Gauge Fields on Σ_2

Σ_2 is some topological surface, with orientations and with some holes.

$\Sigma_2 =$ 

$$\mathcal{P} = \mathcal{A}(P \to \Sigma_2)$$

Phase space $\mathcal{P}$ is equal to (here) space of all connections on principle $U(1)$ bundles P over Σ_2.

(we got rid of third dimension, by taking Hamiltonian point of view)

What does it meant to quantize $(\mathcal{P}, \omega)$ (phase space) ?

- $\omega^{(k)} = k\,\omega \quad , \; k \in \mathbb{Z}$
- $\mathcal{H}_k$ = Hilber space (for each k)
- $f \in \mathrm{Fun}(\mathcal{P} \to \mathbb{C}) \rightsquigarrow Q^{(k)}(f)$ $\mathbb{C}$-linear operator on $\mathcal{H}_k$.

We need some conditions on $Q^{(k)}$.

① $\lim\limits_{k \to \infty} k \, \| Q^{(k)}(f)\, Q^{(k)}(g) - Q^{(k)}(fg) \| < \infty$

② $[Q^{(k)}(f), Q^{(k)}(g)] = -\frac{i}{k} Q^{(k)}(\{f,g\}) + \mathcal{O}\left(\frac{1}{k^2}\right)$

③ $(Q^{(k)}(f))^\dagger = Q^{(k)}(f^*)$

$\exists$ formal procedures (Fedosov, Kontsevich (Cattaneo-Felder))

gives formal series in $1/k$.

① Global seperation of Darboux coordinates,

$\mathcal{P} = T^* M$, M Riemanian,

$\mathcal{H} = L^2(M)$, $p_i = -i\hbar \left(\frac{\partial}{\partial q^i} + \cdots \right)$

② <u>Kähler Quantization</u>

$\mathcal{P}$ has a complex structure.

$J_\mu{}^\nu(\phi)$, $J^2 = -1$

suppose the complex structure is compatible with symplectic structure.

$$J_\lambda{}^\mu(\phi)\, \omega_{\mu\nu}(\phi) = g_{\lambda\nu}(\phi) > 0$$

$$\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$$

$$\Rightarrow g(v_1, v_2) := \omega(Jv_1, v_2) > 0$$

$\Rightarrow$ Kähler metric.

With complex structure, we can choose local complex coordinates dz^i, $d\bar{z}^{\bar{j}}$, and can form Kähler form

$$\omega = \frac{i}{2} g_{i\bar{j}} \, dz^i d\bar{z}^{\bar{j}}$$

$\rightarrow$ here we gave an extra piece of data; The data about Complex Structure (One extra data)

We have to assume that there is Complex structure.

(other piece of extra data)

$L_i \longrightarrow \mathcal{P}$ holomorphic line bundle with a positive curvature.

$\xleftarrow{s}$ (section, not necessarily holomorphic)

If we have a section s; Then, we have a Hermitian metric on our line bundle $h(s)$

Then $$\frac{i}{2\pi} \partial \cdot \bar{\partial} \log (h(s)) = \omega = \frac{i}{2\pi} \partial \bar{\partial} K$$

Kähler potential

(locally defined real analytic function)

Given those extra data.

$$\mathcal{H}_k := \mathrm{Hol}(L^{\otimes k})$$ (Holomorphic section of k^{th} tensor power of L)

Inner product is,

Kähler potential

$$\langle \psi_1, \psi_2 \rangle_{\mathcal{H}_k} = \int_{\mathcal{P}} e^{-kK} \cdot \underline{\psi_1^*(z,\bar{z}) \psi_2(z,\bar{z})} \frac{(\omega/2\pi)^n}{n!}$$

using local trivialization of our line bundle (Taking it as local holo. function of z and $\bar{z}$)

where $\dim_{\mathbb{R}} \mathcal{P} = 2n$

$\dim_{\mathbb{C}} \mathcal{P} = n$

$$\langle \psi_1 | Q^{(k)}(f) | \psi_2 \rangle = \int e^{-kK} \psi_1^* f(z,\bar{z}) \psi_2(z) \ldots$$

Ex 1) $P = \mathbb{R}^2 \cong \mathbb{C}$

$$\frac{\omega}{2\pi} = \frac{dp \wedge dq}{2\pi} = \frac{i}{2\pi} dz \wedge d\bar{z}$$

$L \longrightarrow \mathbb{C}$, $L^{\otimes k} = \mathbb{C} \times \mathbb{C}$

$\mathrm{Hol}(L^{\otimes k})$ = suitably normalizable, entire functions.

$$\langle \psi_1, \psi_2 \rangle = \int e^{-k|z|^2} \psi_1^* \psi_2 \cdot \frac{\omega}{2\pi}$$

define $a^+ := \sqrt{k}\, \underbrace{Q^{(k)}(z)}_{\text{quantization of } z}$, $a = \sqrt{k} \cdot \underbrace{Q^{(k)}(\bar{z})}_{\text{quantization of } \bar{z}}$

Bergman representation.

We can check a^+ & a satisfies usual harmonic oscillator commutation relation.

Ex 2) $P = \mathbb{C}P^1 = S^2$. $L \to \mathbb{C}P^1$

$$\begin{array}{c} S^2 \\ \big\downarrow \text{stereograph.} \\ \mathbb{C} \end{array}$$

Hopf $c_1(L) = +1$

Magnetic monopole bundle

(This is the space where the wave function of an electron in the presence of a magnetic monopole which is sitting inside S^2, is valued) so, $\mathcal{H}_k = \text{Hol}(L^{\otimes k})$ = Holomorphic section of $L^{\otimes k}$.

$$\langle \psi_1, \psi_2 \rangle = \frac{i}{2\pi} \int e^{-kK} \cdot \psi_1{}^* \psi_2 \cdot \frac{dz\, d\bar{z}}{(1+|z|^2)^2}$$

$$= \frac{i}{2\pi} \int_{\mathbb{C}} \psi_1^* \psi_2 \cdot \frac{dz\, d\bar{z}}{(1+|z|^2)^{k+2}} \quad , \; k > 0.$$

so; $1, z, z^2, \ldots, z^k$ normalizable. (but not beyond)

(We get different set of Holomorphic functions which are normalizable depending on k)

different k is corresponding to the different line bundles, gives very different spaces of states.

$k+1$ dimensional spaces of states.

(In fancy cases, we could not just stereographically project to the complex plane; and ~~so we need~~ so we need to work with Line Bundles)

Define $J^i := (j+1)\, Q^{(k)}(x^i)$, where $j = k/2$

Think of $S^2 \subset \mathbb{R}^3 \ni (x^1, x^2, x^3)$

Then $[J^i, J^j] = \sqrt{-1} \cdot \epsilon^{ijk} J^k$

$\Longleftrightarrow$ so $\mathcal{H}_k$ is representation of $SU(2)$.

$1, z, z^2, \ldots, z^k$ are actually eigenstates of J^3.

$\mathcal{H}_k \cong$ irrep of $SU(2)$ of spin $j = k/2$

$$\dim_{\mathbb{C}} \mathcal{H}_k = k+1 \quad = \quad k(1+\hbar) \qquad \hbar = 1/k$$

Bohr Summerfold rule, for each unit of phase ~~space~~ space area we should get a quantum state in the semi-classical limit.

$$\int_{\mathbb{C}P^1} \frac{\omega^{(k)}}{2\pi} = k$$

but we got $k+1$ → This plus 1 is quantum correction of what we expect from Bohr Summerfold rule.

In general, if L is sufficiently positive, then,

$$\dim_{\mathbb{C}} H^0(P, L^{\otimes k}) = \sum_{i=0} (-1)^i \dim H^i(P, L^{\otimes k})$$

$$= \operatorname{Index}(\bar{\partial}_L) = \int_P e^{k\, c_1(L)}\, Td(T^{1,0}P)$$

$$= k^{\dim P/2} \cdot \text{Symplectic volume} \left(1 + O(1/k)\right)$$

$\boxed{\text{Borel-Weil-Bott Theorem}}$

(so we see, The curvature either adds or substracts quantum states)

Gauge Theory

Shoaib Akhtar 24/7/2020

Lec 1

$\mathcal{C}$ manifold, $\dim \mathcal{C} = \infty$

$\circlearrowleft$

$\mathcal{G}$ Lie group, $\dim \mathcal{G} = \infty$

$F : \mathcal{C} \longrightarrow V$; $\mathcal{G}$: representation.

F is $\mathcal{G}$ - equivariant $F(m \cdot g) = g^{-1} F(m)$

$$F^{-1}(0) \circlearrowleft \mathcal{G} \qquad \mathcal{M} = F^{-1}(0)/\mathcal{G}$$

$$\mathcal{M}^* = \{ m \in F^{-1}(0) \mid \mathrm{Stab}(m) = \{1\} \} / \mathcal{G}$$

assume $\mathcal{M}^*$ is smooth, compact, oriented manifold of $\dim = d < \infty$.

$[\mathcal{M}^*] \in H_d(\mathcal{B}^* ; \mathbb{Z})$ (Homology class)

where $\mathcal{B}^* = \mathcal{C}^*/\mathcal{G}$;

Now we want to extract numbers out of it.

$$0 \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow \{1\}$$

finite dimensional Lie group.

$$\hat{\mathcal{M}}^* = F^{-1}(0)/\mathcal{G}_0 \qquad \begin{array}{c} G \hookrightarrow \hat{\mathcal{M}}^* \\ \downarrow \\ \mathcal{M} \end{array}$$

$\rightsquigarrow$ Characteristic classes $\eta \in H^d(\mathcal{M}^*, \mathbb{Z})$ cohomology class.

$\int_{M^*} \eta$ "invariant"

Lec 1+2: Bundles, Connection, Curvature.

Lec 3+4: Tools from analysis.
(Fredholm maps, Elliptic PDEs, ...)

Lec 5+6: Seiberg-Witten gauge Theory

<u>Vector Bundles</u> family of vector spaces E_m, parametrized by points in manifold M (more generally in topological space)

$\{E_m \mid m \in M\}$
assumed to be locally trivial.

ie; $\forall m \in M$, $\exists U$; $\psi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$

$\{E_m \mid m \in M\} \xrightarrow{\pi} M$ (Natural projection map)

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "pr_1"] \\
& U &
\end{tikzcd}$$

This diagram commutes.

SECTION MAP

$\underbrace{s : M \longrightarrow E}_{\text{smooth.}}$ <u>sections</u>, if $\pi \circ s = id_M$.

We can construct $\Lambda^p E$, E^*, $End(E)$, ...

sections of TM are called vector fields.
" " $\Lambda^p T^* M$ " " differential p-forms

Def^n $\Gamma(E)$ space of all sections

then $\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$ is an $\mathbb{R}$ linear map $\nabla(f \cdot s) = df \otimes s + f \nabla s$

---

given "∇" connection,

we can construct another connection ∇'

$$\nabla' = \nabla + a \quad , \text{ where } a \in \Omega^1(M, \mathrm{End}(E))$$

a is one form in M, with values in $\mathrm{End}(E)$.

Lemma let $A : \Gamma(E) \to \Omega^p(F)$ be an $\mathbb{R}$-linear map, which is also $C^\infty(M)$-linear i.e, $A(fs) = f A(s)$ $\forall f \in C^\infty(M)$ and $\forall s \in \Gamma(E)$
Then $\exists\, a \in \Omega^p(\mathrm{Hom}(E,F))$ s.t. $A(s) = a \cdot s$

Theorem The space of all connections $\mathcal{A}(E)$ is an affine space modelled on $\Omega^1(\mathrm{End}(E))$

$$0 \to \Omega^0(E) \xrightarrow{\nabla = d_\nabla} \Omega^1(E) \xrightarrow{d_\nabla} \Omega^2(E) \xrightarrow{d_\nabla} \cdots$$

$$\Omega^0(E) = \Gamma(E)$$

$\alpha \in \Omega^1(E)$

α can be locally written as $\alpha = \omega \otimes s$

k form on m

a section on E.

define $\quad d_\nabla \alpha = d\omega \otimes s + (-1)^{\deg \omega} \omega \wedge \nabla s$

$$d_\nabla \circ d_\nabla = F_\nabla \in \Omega^2(\mathrm{End}(E))$$

Curvature of ∇

If we choose local coordinates on M, then we can ask what is $\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\partial/\partial x_j} s - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\partial/\partial x_i}$

$(x_1, \dots, x_n)$ local coordinates on M

$$\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} s - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} s = F_\nabla\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

so; Curvature is measure of non ~~comp~~ commutivity of ~~covariant~~ partial covariant derivatives.

locally, $\nabla = d + A$, where $A \in \Omega^1(M, \mathfrak{gl}_k(\mathbb{R}))$

Then curvature form F_∇ is locally given as

$$F_\nabla = dA + A \wedge A$$

$$\boxed{F_{\nabla + a} = F_\nabla + d_\nabla a + a \wedge a}$$

~~Example~~ let $\mathcal{G}(E)$ denote

$$\mathcal{G}(E) = \{ g \in \mathrm{End}(E) \mid g(m) \in GL(E_m) \}$$

It is called Gauge Group.

so; $\nabla^g s := g^{-1} \nabla (g \cdot s)$

∇ and ∇^g are gauge equivalent.

Exercise] Compute F_{∇^g}

Principle Bundle

$P \xrightarrow{\pi} M$ (projection) — manifold P, base manifold M.

G lie group, $G \circlearrowright P$ (G acts on P)

- $\pi(p \circ g) = \pi(p)$ (π is g invariant)
- $\forall\, m \in M$, $\pi^{-1}(m) \circlearrowleft G$ free transitive
- locally trivial

(Here we have family of Lie groups parametrized by point of Base Manifold)

If we have Vector Bundle, then there is canonical way to construct Principle Bundle.

$\cancel{\text{Ex}}$ $Fr(E_m) = \{$set of all bases of $E_m\}$

~~D.~~ E is vector bundle.

Def^n $Fr(E) = \bigsqcup_{m \in M} Fr(E_m) = \bigsqcup_{m \in M}$ set of all isomorphism from $\mathbb{R}^k$ to E_m

Frame bundle of E

$$Fr(E_m) = \{\mathbb{R}^k \xrightarrow{\cong} E_m\}$$

P principle G bundle. associated vector Bundle

$\rho : G \longrightarrow GL_k(\mathbb{R})$

Then $P \times \mathbb{R}^k / G$: $(p, x) \cdot g = (p \cdot g, \rho(g^{-1})x)$

$E := P \times \mathbb{R}^k / G$ is a vector bundle; There is natural projection to M; $E \longrightarrow M$.

Def^n A connection on P is $A \in \Omega^1(P; \mathfrak{g})$ (A is 1-form on P, whose value is in $\mathfrak{g}$)

$\mathfrak{g} = \text{Lie}(G)$ s.t.

- A is G equivariant,

$R_g^* \omega = ad_{g^{-1}} \omega$; ω takes values in lie Algebra $\mathfrak{g}$

P86

- $A(K_\xi) = \xi \quad \forall \xi \in \mathfrak{g}$

  ↑ infinitisimal action of G on P.

Vector Bundle	Principle bundles
vector Bundle E	$Fr(E)$
$E := P \times_{G,\rho} \mathbb{R}^k$	P
S section	$\hat{S} : P \to \mathbb{R}^k$ G-equivariant.
∇ connection	$A \in \Omega^1(P, \mathfrak{g})$
$\cancel{S(p,x) = [\hat{S}(p), x]}$	$S(p,x) = [\hat{S}(p), x]$
$\mathcal{G}(E)$ gauge group	$\mathcal{G}(P) = \{\varphi : P \to P \mid \varphi(pg) = \varphi(p)\cdot g\}$ $\mathcal{G}(P) = \{\psi : P \to G \mid \psi(p\cdot g) = g^{-1}\psi(p) g\}$
F_∇ Curvature	$F_A = dA + \frac{1}{2}[A \wedge A]$

Theorem (Bianchi Identity)

$$d_\nabla(F_\nabla) = 0 \iff dF_A = -[A \wedge F_A]$$

Ex) $P \to M$ $\quad U(1)$ - Bundle

Propagators

Shoaib Akhtar 2nd/9/2020

$$\langle F \rangle = \frac{\int \mathcal{D}[\phi] \, F(\phi) \exp\left(\frac{i}{\hbar} S[\phi]\right)}{\int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)}$$

$$Z = \int \mathcal{D}\phi \cdot \exp\left(\frac{i}{\hbar} S[\phi]\right)$$ Partition Function.

We have to Study Propagators.

$$\Delta_F(x-y) = \Theta(x^0-y^0)\langle 0|\varphi(x)\varphi(y)|0\rangle + \Theta(y^0-x^0)\langle 0|\varphi(y)\varphi(x)|0\rangle$$
$$= \langle 0|T(\varphi(x)\varphi(y))|0\rangle$$

Feynman Propagator.

For simplicity, we also write as $G(x,y)$

(usually $G(x,y) = G(x-y)$)

Generality

Suppose $S[\phi] = \frac{1}{2}\int d^{d+1}x \; \phi \, L \, \phi$

note path order formalism takes into account ordering of operators.

~~Then~~

~~$\langle \phi(x_1)\phi(x_2)\rangle = \int \mathcal{D}[\phi] F(\phi)$~~ L is an operator.

Then

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{\int \mathcal{D}[\phi] \, \phi(x_1)\phi(x_2) \cdot \exp\left(\frac{-i}{\hbar}\int d^{d+1}x \, \frac{1}{2}\phi(x) L \phi(x)\right)}{\int \mathcal{D}[\phi] \exp\left(\frac{-i}{\hbar}\int d^{d+1}x \, \frac{1}{2}\phi(x) L \phi(x)\right)}$$

$$= \left(L^{-1}\right)_{x_1 x_2}$$ $\Rightarrow$ This is a continuum limit for N dimensional Gaussian integral.

L^{-1} is the inverse of the operator $\frac{iL}{\hbar}$

~~FOR BOSONIC FIELD VARIABLES.~~

where $L^{-1}(x_1, x_2)$ satisfies

$$\boxed{\frac{iL}{\hbar}\left(L^{-1}(x_1,x_2)\right) = \delta^{d+1}(x_1 - x_2)}$$

↪ This is the meaning of $L^{-1}(x_1,x_2)$ being inverse of $\frac{iL}{\hbar}$ (RHS is continuum analogue of identity)

$$\boxed{L\left(L^{-1}(x_1,x_2)\right) = -i\hbar\, \delta^{d+1}(x_1 - x_2)}$$

$L^{-1}(x_1,x_2) = L^{-1}(x_1 - x_2)$

This is the equation which is needed to be solved.

---

lets call $L^{-1}(x_1 - x_2)$ to be $G(x_1 - x_2)$

So, we have

$$\langle \phi(x_1)\phi(x_2)\rangle = G(x_1 - x_2)$$

This is to be solved to get propogator.

where $G(x_1 - x_2)$ satisfies $\boxed{L(G(x_1 - x_2)) = -i\hbar\,\delta^{d+1}(x_1 - x_2)}$

$\boxed{1}$ <u>Klein Gordon Field.</u> $\quad \eta_{\mu\nu} = \text{diag}(+1, \underbrace{-1, -1, \ldots, -1}_{d \text{ times}})$

Action for Klein Gordon Field is $\quad \Box \equiv \partial_\mu \partial^\mu$

$$S = \int d^{d+1}x \left[+\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2\right]$$

$$= \int d^{d+1}x\left[-\frac{\phi}{2}(\partial_\mu\partial^\mu\phi) - \frac{1}{2}m^2\phi^2\right] + \int d^{d+1}x\left[+\frac{1}{2}\partial_\mu(\phi\,\partial^\mu\phi)\right]$$

use $\partial_\mu(\phi\,\partial^\mu\phi) = \partial_\mu\phi\,\partial^\mu\phi + \phi\,\partial_\mu\partial^\mu\phi$

$$\Rightarrow (\partial_\mu\phi)(\partial^\mu\phi) = \partial_\mu(\phi\,\partial^\mu\phi) + \phi\,\Box\phi$$

$$S = \int d^{d+1}x \, -\frac{1}{2} \phi \left[+\Box + m^2 \right] \phi + \int d^{d+1}x \, \frac{1}{2} \partial_\mu (\phi \partial^\mu \phi)$$

This is surface integral term.
(Taking the fields to vanish at boundary (at ∞); This term drops out)

$$\Rightarrow \quad S = \int d^{d+1}x \, -\frac{1}{2} \phi \left[+\Box + m^2 \right] \phi$$

~~$L_{K.G.} = -\Box + m^2$~~ so, $\boxed{L_{K.G.} = \Box + m^2}$

let $G_{K.G.}(x_1 - x_2)$ be the Greens function or better say Propagator for K.G field.

Then $\boxed{(+\Box + m^2) \, G_{K.G.}(x_1 - x_2) = -i\hbar \, \delta^{d+1}(x_1 - x_2)}$

$\rightarrow$ To solve this; we resort back to Fourier Transforms.

$$\delta^{d+1}(x_1 - x_2) = \frac{1}{(2\pi)^{d+1}} \int d^{d+1}k \cdot e^{ik^\mu (x_\mu - x_{2,\mu})}$$

$$\cancel{G_{K.G.}(x_1 - x_2) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{ik^\mu \cdot x_\mu} \hat{G}_{K.G.}(k}$$

$$G_{K.G.}(x_1 - x_2) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{ik^\mu \cdot (x_1 - x_2)_\mu} \hat{G}_{K.G.}(k^\mu)$$

Substituting This in the equation gives.

$$\int \frac{d^{d+1}}{(2\pi)^{d+1}} e^{ik^\mu \cdot (x_1-x_2)_\mu} \cdot [-k_\mu k^\mu + m^2] \hat{G}_{K.G.}(k^\mu) = \int \frac{d^{d+1}}{(2\pi)^{d+1}} \cdot (-i\hbar) e^{ik^\mu (x_1-x_2)_\mu}$$

$$\Rightarrow \quad (k_\mu k^\mu - m^2) \hat{G}_{K.G.}(k) = i\hbar$$

$$\Rightarrow \quad \hat{G}_{K.G.}(k) = \frac{i\hbar}{(k^2 - m^2)}$$

$$\boxed{\langle \phi(x_1)\phi(x_2)\rangle\Big|_{K.G.} = G_{K.G.}(x_1 - x_2) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{i\hbar \cdot e^{ik^\mu \cdot (x_1-x_2)_\mu}}{(k^\mu k_\mu - m^2)}}$$

---

[2] Dirac Field (Spin 1/2)

$$\mathcal{L}_D = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$$

$$\left(\mathcal{L}_D = \bar{\Psi}_a (i\gamma^\mu_{ab} \partial_\mu - m \mathbb{I}_{ab})\Psi_b \right)$$

Hence the action is

$$S_D = \int d^{d+1}x \cdot \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$$

Feynman Slash Notation

$$\gamma^\mu F_\mu = \not{F}$$

$$\Longleftrightarrow \text{Dirac Equation} : \boxed{(i\not{\partial} - m)\Psi(x) = 0}$$

let $G_D(x_1, x_2)$ be the propagator for fermionic field

~~Note~~ $\cancel{\langle 0 | T \Psi(y) \bar{\Psi}(x) | 0\rangle = G_D(y,}$

Note that $\Psi(y)$, $\bar{\Psi}(x)$ are Fermionic Grassman field variables.

$$\langle 0 | T \bar{\Psi}(x_1)\Psi(x_2) | 0\rangle = G_D(x_1, x_2)$$

$(G_D(x_1 - x_2))_{ab}$.

Hence the equation for the propagator is

$$\frac{i}{\hbar}[i\not{\partial} - m]\, G_D(x_1 - x_2) = \delta^{d+1}(x_1 - x_2)$$

$$\Rightarrow \boxed{[i\not{\partial} - m]\, G_D(x_1 - x_2) = -i\hbar\, \delta^{d+1}(x_1 - x_2)}$$

$$G_D(x_1 - x_2) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot e^{-ik^\mu \cdot (x_1 - x_2)_\mu} \cdot \hat{G}_D(k)$$

$$\delta^{d+1}(x_1 - x_2) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}}\, e^{-ik^\mu \cdot (x_1 - x_2)_\mu}$$

$$\Rightarrow \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot e^{-ik^\mu (x_1 - x_2)_\mu} \left[\gamma^\mu k_\mu - m\right] \hat{G}_D(k) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot e^{-ik^\mu (x_1 - x_2)_\mu} (-i\hbar)$$

$$\Rightarrow \quad [\not{k} - m]\, \hat{G}_D(k) = -i\hbar$$

$$\Rightarrow \quad \hat{G}_D(k) = \frac{-i\hbar}{(\not{k} - m)}$$

$$\Rightarrow \boxed{G_D(x_1 - x_2) = -i\hbar \int \frac{d^{d+1}}{(2\pi)^{d+1}} \cdot \frac{e^{-ik \cdot (x_1 - x_2)}}{(\not{k} - m)}}$$

<u>Note</u> The propagator $\Delta_{\text{Fermion}}(x_1 - x_2)$ is defined as

$$\Delta_{\text{Fermion}} = \langle 0 | T\, \psi(x_1)\, \bar{\psi}(x_2) \rangle$$

from anticommutation of Grassman field variables ψ & $\bar{\psi}$

$$\Rightarrow \Delta_{\text{Fermion}}(x_1 - x_2) = -\langle 0 | T\, \bar{\psi}(x_2)\, \psi(x_1) | 0 \rangle$$

but $\langle 0 | T \bar{\Psi}(x_2) \psi(x_1) | 0 \rangle = G_D(x_2 - x_1)$

$$\Rightarrow \quad \Delta_{\text{fermion}} = - G_D(x_2 - x_1)$$

$$= - \left(-i\hbar \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot \frac{e^{-ik^\mu (x_2 - x_1)_\mu}}{\not{k} - m} \right)$$

$$\Delta_{\text{Fermion}}(x_1 - x_2) = i\hbar \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot \frac{e^{ik^\mu \cdot (x_1 - x_2)}}{\not{k} - m}$$

Note) $\not{k}\not{k} = \frac{1}{2}(\not{k}\not{k} + \not{k}\not{k}) = \frac{1}{2}\left(\gamma_\mu k^\mu \gamma_\nu k^\nu + \gamma_\nu k^\nu \gamma_\mu k^\mu\right)$

$$= \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) k^\mu k^\nu$$

we know $\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu}$

$$\Rightarrow \quad \not{k}\not{k} = \frac{1}{2} \cdot 2 g_{\mu\nu} k^\mu k^\nu = k^\mu k_\mu = k^2$$

also: $(\not{k} - m)(\not{k} + m) = k^2 + \not{k} m - \not{k} m - m^2$

$$= k^2 - m^2$$

Multiply denominator & neumerator of the integrand of $\Delta_{\text{Fermion}}(x_1 - x_2)$ by $(\not{k} + m)$

we get

$$\boxed{\Delta_{\text{Fermion}}(x_1 - x_2) = i\hbar \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot \frac{(\not{k} + m)}{(k^2 - m^2)} \cdot e^{ik^\mu \cdot (x_1 - x_2)_\mu}}$$

3 U(1) Chern Simon Theory.

Action $$S = \frac{k}{8\pi}\int_{M_3} d^3x \, \epsilon^{\lambda\mu\nu} A_\lambda \partial_\mu A_\nu$$

finding eqn of motion

$$\Rightarrow \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \frac{\partial \mathcal{L}}{\partial A_\nu} \Rightarrow \partial_\mu(\epsilon^{\lambda\mu\nu}A_\lambda) = \epsilon^{\lambda\mu\sigma}\cdot\delta_{\lambda\nu}\cdot\partial_\mu A_\sigma$$

$$\Rightarrow \partial_\mu(\epsilon^{\lambda\mu\nu}A_\lambda) = \epsilon^{\nu\mu\sigma}\partial_\mu A_\sigma = \epsilon^{\nu\mu\lambda}\partial_\mu A_\lambda = \epsilon^{\nu\lambda\mu}\partial_\lambda A_\mu$$
$$= -\epsilon^{\lambda\nu\mu}\partial_\lambda A_\mu$$
$$= \epsilon^{\lambda\mu\nu}\partial_\lambda A_\mu$$

$$\Rightarrow \epsilon^{\lambda\mu\nu}(\partial_\mu A_\lambda - \partial_\lambda A_\mu) = 0$$

$$\boxed{\epsilon^{\lambda\mu\nu}F_{\mu\lambda} = 0}$$

$F_{\mu\mu} = 0$ due to antisymmetry

$$\boxed{F_{\mu\lambda} = 0}$$

↳ Equation of Motion corresponds to Flat Gauge Field.

Now, we compute Propagators.

$$\langle A_\mu(x) A_\nu(y)\rangle = \frac{\int \mathcal{D}[A] \exp\left(\frac{ik}{8\pi\hbar}\int_{M_3} d^3x \, \epsilon^{\lambda\mu\nu} A_\lambda\partial_\mu A_\nu\right) A_\mu(x) A_\nu(y)}{\int \mathcal{D}[A]\exp\left(\frac{ik}{8\pi\hbar}\int_{M_3} d^3x\,\epsilon^{\lambda\mu\nu}A_\lambda\partial_\mu A_\nu\right)}$$

$$\equiv G_{\mu\nu}(x-y)$$

$$\frac{iS}{\hbar} = \frac{1}{2}\int_{M_3} d^3x \, A_\lambda\left[\frac{ik}{4\pi\hbar}\epsilon^{\lambda\mu\nu}\partial_\mu\right]A_\nu$$

Hence $[G]$ is inverse of the operator

$$[L] = \frac{ik}{4\pi\hbar}\left[\epsilon^{\mu\sigma\nu}\partial_\sigma\right]$$ $[\cdot]_{\text{matrix}}$ represents

$$\Rightarrow \quad [L][G(x-y)] = \mathbb{1}_{3\times 3}\cdot\delta^3(x-y)$$

$$\Rightarrow \quad \frac{ik}{2\pi\hbar}\cdot\epsilon^{\mu\sigma\nu}\partial_\sigma\left[G_{\nu\lambda}(x-y)\right] = g^\mu{}_\lambda\cdot\delta^3(x-y)$$

$$\Rightarrow \quad \boxed{\epsilon^{\mu\sigma\nu}\partial_\sigma G_{\nu\lambda}(x-y) = -\frac{i\,\cancel{4}2\pi\hbar}{k}\,g^\mu{}_\lambda\cdot\delta^3(x-y)} \quad (*)$$

→ Solving this equation gives our propagator $G_{\nu\lambda}$.

$$\delta^3(x-y) = \int\frac{d^3p}{(2\pi)^3}e^{ip\cdot(x-y)}.$$

$$G_{\mu\nu}(x-y) = \int\frac{d^3p}{(2\pi)^3}e^{ip\cdot(x-y)}\,\hat{G}_{\mu\nu}(p)$$

Hence we get

$$\Rightarrow \quad \epsilon^{\mu\sigma\nu}\cdot ip_\sigma\cdot\hat{G}_{\nu\lambda}(p) = -i\frac{2\pi\hbar}{k}g^\mu{}_\lambda$$

~~$\Rightarrow \epsilon^{\mu\sigma\nu} G_{\nu\lambda}$~~ $$\Rightarrow \quad \boxed{\epsilon^{\mu\sigma\nu}\cdot p_\sigma\cdot\hat{G}_{\nu\lambda}(p) = -\frac{\cancel{4}2\pi\hbar}{k}g^\mu{}_\lambda}$$

multiply this equation by $\epsilon_{\mu\alpha\beta}$

using $\epsilon_{\mu\alpha\beta}\,\epsilon^{\mu\sigma\nu} = \delta_\alpha^\sigma\delta_\beta^\nu - \delta_\alpha^\nu\delta_\beta^\sigma$

$$\Rightarrow \left(\delta_\alpha^\sigma\delta_\beta^\nu - \delta_\alpha^\nu\delta_\beta^\sigma\right) P_\sigma\, \hat{G}_{\nu\lambda}(p) = -\frac{4\pi\hbar}{k}\,\epsilon_{\mu\alpha\beta}\, g^\mu{}_\lambda$$

$$\Rightarrow P_\alpha \hat{G}_{\beta\lambda} - P_\beta \hat{G}_{\alpha\lambda} = -\frac{4\pi\hbar}{k}\,\epsilon_{\lambda\alpha\beta}$$

$$\boxed{P_\alpha \hat{G}_{\beta\lambda} - P_\beta \hat{G}_{\alpha\lambda} = -\frac{4\pi\hbar}{k}\,\epsilon_{\lambda\alpha\beta}}$$

holds trivially for $\alpha=\beta$.

~~NOT Able to solve; because the operator has non trivial Kernel. (Not easy to invert)~~

→ Seems problematic to solve

Get back to the eqn without fourier transformation; but multiplied with $\epsilon_{\mu\alpha\beta}$.

$$\partial_\alpha G_{\beta\lambda} - \partial_\beta G_{\alpha\lambda} = \frac{-i4\pi\hbar}{k}\,\epsilon_{\lambda\alpha\beta}\,\delta^3(x-y)$$

Note that $\epsilon_{\alpha\sigma\lambda}\,\partial^\sigma\left(\epsilon^{\alpha\beta\lambda}\partial_\beta\right) = 2\,\partial^\alpha\partial_\alpha$ ~~$= 2\partial^2$~~

~~$\epsilon_{\alpha\sigma\lambda}\partial^\sigma(\epsilon^{\alpha\beta\lambda}\partial_\beta) = 2\partial^2$ Laplacian~~

Laplacian.

$$\Rightarrow \partial^2 = \frac{1}{2}\left(\epsilon_{\alpha\sigma\lambda}\,\partial^\sigma\left(\epsilon^{\alpha\beta\lambda}\partial_\beta\right)\right)$$

∂^2 is a kind of square of our operator whose greens function we want to evaluate.

$G_{\partial^2} \sim L^{-2}$.   L is our operator in concern.

Then $L(L^{-2}) \sim L^{-1}$

$\Rightarrow L^{-1}$ is our greens function.

Normalization of L^{-1} can be adjusted from (*)

Greens function for ∂^2 is proportional to $\frac{1}{|x|}$

$$G_{\partial^2} \propto \frac{1}{|x-y|}$$

~~$G_{L^{-1}}(x-y) = \xi \cdot \epsilon_{\mu\lambda\nu} \partial^\lambda$~~

$$G_{L^{-1}}(x-y) = \xi L\left(\frac{1}{|x-y|}\right) = \xi \cdot \epsilon_{\mu\sigma\nu} \partial^\sigma \left(\frac{1}{|x-y|}\right)$$

Normalization constant.

(to be found by putting this $G_{L^{-1}}(x-y)$ in (*))

$$\Rightarrow G_{L^{-1}}(x-y) = \xi \cdot \epsilon_{\mu\sigma\nu} \partial^\sigma \left(\frac{1}{\sqrt{(x-y)\cdot(x-y)}}\right)$$

$$= -\frac{1}{2}\xi \cdot \epsilon_{\mu\sigma\nu} \frac{(x-y)^\sigma}{\left(\sqrt{(x-y)\cdot(x-y)}\right)^3}$$

$$\Rightarrow \boxed{G_{L^{-1}}(x-y)_{\mu\nu} = -\xi \, \epsilon_{\mu\sigma\nu} \frac{(x-y)^\sigma}{|x-y|^3}}$$

$$\epsilon^{\mu\sigma\nu}\partial_\sigma\left(-\xi\,\epsilon_{\nu\theta\lambda}\frac{(x-y)^\theta}{|x-y|^3}\right)$$

$$= -\xi\left(\epsilon^{\mu\sigma\nu}\epsilon_{\nu\theta\lambda}\right)\partial_\sigma\left(\frac{(x-y)^\theta}{|x-y|^3}\right)$$

$$= -\xi\left(\delta_\theta^{\ \mu}\delta^\sigma_{\ \lambda} - \delta^\sigma_{\ \theta}\delta^\mu_{\ \lambda}\right)\partial_\sigma\left(\frac{(x-y)^\theta}{|x-y|^3}\right) \qquad \delta^\mu_{\ \nu} = g^\mu_{\ \nu}$$

$$= -\xi\,\underbrace{\partial_\lambda\left(\frac{(x-y)^\mu}{|x-y|^3}\right)}_{} + \xi\cdot\underbrace{\partial_\theta\left(\frac{(x-y)^\theta}{|x-y|^3}\right)}_{}\cdot\delta^\mu_{\ \nu}$$

No need to worry about this ; Can be adjusted by Gauge Fixing.

This is laplacian of $\frac{1}{|x-y|}$ and hence give the delta function piece.

$$\partial^\theta\left(\frac{1}{|x-y|}\right) = \frac{-1}{2}\frac{\times 2\cdot(x-y)^\theta}{|x-y|^3}$$

$$\Rightarrow \underbrace{\partial_\theta\partial^\theta\left(\frac{1}{|x-y|}\right)}_{} = -\partial_\theta\left(\frac{(x-y)^\theta}{|x-y|^3}\right)$$

This is $-4\pi\delta^3(x-y)$ (famous result)

Hence equating the dirac delta terms.

$$\xi\cdot 4\pi\,\delta^3(x-y)\,\delta^\mu_{\ \nu} = \frac{-i\,4\pi\hbar}{k}\,\delta^\mu_{\ \nu}\cdot\delta^3(x-y)$$

$$\Rightarrow \boxed{\xi = \frac{-i\hbar}{\cancel{2}\,k}}$$

Hence

$$G_{L^{-1}}(x-y)_{\mu\nu} = \frac{i\hbar}{k} \cdot \epsilon_{\mu\sigma\nu} \frac{(x-y)^\sigma}{|x-y|^3}$$

$\underbrace{G_{L^{-1}}(x-y)}$ → This is the required Propagator $\langle A_\mu(x) A_\nu(x) \rangle$

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{i\hbar}{k} \epsilon_{\mu\sigma\nu} \frac{(x-y)^\sigma}{|x-y|^3}$$

$$\boxed{G_{\mu\nu}(x-y) = -\frac{i\hbar}{k} \epsilon_{\mu\nu\sigma} \frac{(x-y)^\sigma}{|x-y|^3}}$$

Consider the Wilson loop variables

$$W_n(\gamma) = \exp\left(in \oint_\gamma A\right) \qquad A = A_\mu(x)\, dx^\mu$$

(take the irreducible representation of $U(1)$)

$$W_n(\gamma) = \sum_{m=0}^{\infty} \frac{(-in)^m}{m!} \left(\oint_\gamma A_\mu(x)\, dx^\mu\right)$$

$$= \sum_{m=0}^{\infty} \frac{(-in)^m}{m!} \underbrace{\oint_\gamma \oint_\gamma \cdots \oint_\gamma}_{m \text{ times}} A_{\mu_1}(x_1) \ldots A_{\mu_m}(x_m)\, dx_1^{\mu_1} \ldots dx_m^{\mu_m}$$

$$\langle W_n(\gamma) \rangle = \sum_{m=0}^{\infty} \frac{(-in)^m}{m!} \underbrace{\oint_\gamma \cdots \oint_\gamma}_{m \text{ times}} \langle A_{\mu_1}(x_1) \ldots A_{\mu_m}(x_m) \rangle\, dx_1^{\mu_1} \ldots dx_m^{\mu_m}$$

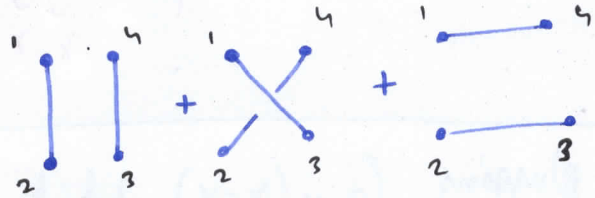
In this theory; we have propagator of the kind

$\langle A_\mu(x) A_\nu(y) \rangle = G_{\mu\nu}(x-y)$.

Use Wicks Theorem.

$\langle A_\mu(x) A_\nu(y) \rangle =$ 

note $\boxed{\langle A_{\mu_1}(x_1) A_{\mu_2}(x_2) \; \dots \; A_{\mu_{2m+1}}(x_{2m+1}) \rangle = 0}$

ex) $\langle A_{\mu_1}(x_1) A_{\mu_2}(x_2) A_{\mu_3}(x_3) A_{\mu_4}(x_4) \rangle =$ 

We see that

$$\oint_\gamma \dots \oint_\gamma \langle A_{\mu_1}(x_1) \dots A_{\mu_{2m}}(x_{2m}) \rangle dx_1^{\mu_1} \dots dx_{2m}^{\mu_{2m}}$$

$$= C_{2m} \left(\oint_\gamma \oint_\gamma \langle A_\mu(x) A_\nu(y) \rangle dx^\mu dx^\nu \right)^m$$

where C_{2m} is the associated Combinatorial factor.

C_{2m} is the no of ways of joining $2m$ elements into pairs

$$C_{2m} = \frac{(2m)!}{(2^m m!)}$$

$$\Rightarrow \langle W_n(\gamma) \rangle = \sum_{m=0}^{\infty} \underbrace{\frac{(-in)^{2m}}{(2m)!} C_{2m}}_{\frac{(-n^2)^m}{(2m)!} \cdot \frac{(2m!)}{2^m m!}} \left(\oint_\gamma \oint_\gamma \langle A_\mu(x) A_\nu(y) \rangle dx^\mu dx^\nu \right)^m$$

$$\langle W_n(\gamma) \rangle = \sum_{m=0}^{\infty} \frac{(-n^2/2)^m}{m!} \left(\oint_\gamma \oint_\gamma \langle A_\mu(x) A_\nu(x) \rangle dx^\mu dx^\nu \right)^m$$

$$\langle W_n(\gamma)\rangle = \exp\left(-\frac{n^2}{2}\oint_\gamma\oint_\gamma \langle A_\mu(x)A_\nu(y)\rangle\, dx^\mu dx^\nu\right)$$

$$\Rightarrow \langle W_n(\gamma)\rangle = \exp\left(\frac{-n^2}{2}\oint_\gamma\oint_\gamma G_{\mu\nu}(x-y)\, dx^\mu dx^\nu\right)$$

plugging $G_{\mu\nu}(x-y)$ which we found.

$$\langle W_n(\gamma)\rangle = \exp\left(\frac{n^2}{2}\frac{i\hbar}{k}\oint_\gamma\oint_\gamma \epsilon_{\mu\nu\sigma}\frac{(x-y)^\sigma}{|x-y|^3}\, dx^\mu dx^\nu\right)$$

Now we use the Gauss Linking Number Formula.

$$lk(\gamma_\alpha,\gamma_\beta) = \frac{1}{4\pi}\int_{\gamma_\alpha} dx^\mu \int_{\gamma_\beta} dx^\nu \cdot \epsilon_{\mu\nu\sigma}\cdot\frac{(x-y)^\sigma}{|x-y|^3}$$

$$\Rightarrow \boxed{\langle W_n(\gamma)\rangle = \exp\left(\frac{2\pi i\hbar}{k}\cdot n^2\cdot l.k.(\gamma,\gamma)\right)}$$

$lk(\gamma,\gamma)$ is self linking number.

$$\prod_\alpha W_{n_\alpha}(\gamma_\alpha) = \prod_\alpha \exp\left(i n_\alpha \oint_{\gamma_\alpha} A\right)$$

$$= \exp\left(i\sum_\alpha n_\alpha \oint_{\gamma_\alpha} A\right)$$

$$\exp\Big(i\sum_\alpha n_\alpha \oint_{\gamma_\alpha} A_\mu(x)\,dx^\mu\Big)$$

$$= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!}\Big(\sum_\alpha n_\alpha \oint_{\gamma_\alpha} A_\mu(x)\,dx^\mu\Big)^m$$

$$= \sum_{m=0}^{\infty} \frac{(-i)^m}{m}\Big(\sum_{\alpha_1\dots\alpha_m} n_{\alpha_1}\cdots n_{\alpha_m}\oint_{\gamma_{\alpha_1}}\cdots\oint_{\gamma_{\alpha_m}} A_{\mu_1}(x_1)\cdots A_{\mu_m}(x_m)\,dx_1^{\mu_1}\cdots dx_m^{\mu_m}\Big)$$

define $$W(\alpha_1^N) = \prod_{\alpha=1}^{N} W_{n_\alpha}(\gamma_\alpha)$$

Then by taking expectation value; and using the fact that expectation value of odd no. of fields is zero gives:

$$\langle W(\alpha_1^N)\rangle = \sum_{m=0}^{\infty}\frac{(-1)^m}{2m!}\Big(\sum_{\alpha_1\dots\alpha_{2m}} n_{\alpha_1}\cdots n_{\alpha_{2m}}\oint_{\gamma_{\alpha_1}}\cdots\oint_{\gamma_{\alpha_{2m}}}\langle A_{\mu_1}(x_1)\dots A_{\mu_{2m}}(x_{2m})\rangle\,dx_1^{\mu_1}\cdots dx_{2m}^{\mu_{2m}}\Big)$$

$$\langle A_{\mu_1}(x_1)\cdots A_{\mu_{2m}}(x_{2m})\rangle = C_{2m}\,\langle A_{\mu_1}(x_1)A_{\mu_2}(x_2)\rangle\cdots\langle A_{\mu_{2m-1}}(x)A_{\mu_{2m}}(x)\rangle$$

$$\langle W(\alpha_1^N)\rangle = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,2^m}\Big(\sum_{\alpha_1\dots\alpha_{2m}}\Big\{n_{\alpha_1}n_{\alpha_2}\oint_{\gamma_{\alpha_1}}\oint_{\gamma_{\alpha_2}}\langle A_{\mu_1}(x_1)A_{\mu_2}(x_2)\rangle\,dx_1^{\mu_1}dx_2^{\mu_2}\Big\}\times\cdots$$

$$\times\cdots\times\Big\{n_{\alpha_{2m-1}}n_{\alpha_{2m}}\oint_{\gamma_{\alpha_{2m-1}}}\oint_{\gamma_{\alpha_{2m}}}\langle A\,A\rangle\,dx\,dx\Big\}\Big)$$

$\Rightarrow$

$$\langle W(\alpha_1^N)\rangle = \sum_{m=0}^{\infty}\frac{(-1/2)^m}{m!}\Big[\sum_{\alpha_1\dots\alpha_{2m}}\Big\{\prod_{k=1}^{m} n_{\alpha_{2k-1}}n_{\alpha_{2k}}\oint_{\gamma_{\alpha_{2k-1}}}\oint_{\gamma_{\alpha_{2k}}}\langle A_\mu(x)A_\nu(x)\rangle\,dx^\mu dx^\nu\Big\}\Big]$$

Hence closing up ; and we get exponential

$$\langle W(\alpha_1^N)\rangle = \exp\left[\frac{-1}{2}\sum_{\alpha,\beta} n_\alpha n_\beta \oint_{\gamma_\alpha}\oint_{\gamma_\beta}\langle A_\mu(x)A_\nu(y)\rangle dx^\mu dx^\nu\right]$$

$$\langle W(\alpha_1^N)\rangle = \exp\left[\frac{i\hbar}{2k}\sum_{\alpha,\beta} n_\alpha n_\beta \oint_{\gamma_\alpha}\oint_{\gamma_\beta}\epsilon_{\mu\nu\sigma}\frac{(x-y)^\sigma}{|x-y|^3}dx^\mu dx^\nu\right]$$

U(1) CHERN SIMON

$$\boxed{\left\langle W(\{\alpha_n\}_{n=1}^N)\right\rangle = \exp\left[\frac{2\pi i\hbar}{k}\sum_{\alpha,\beta} n_\alpha n_\beta\, l.k.(\gamma_\alpha,\gamma_\beta)\right]}$$

☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆

Shoaib Akhtar

An intense mathematical calculation regarding propagators ; especially for U(1) Chern Simon in complete generality is potrait.

2nd September, 2020

(1+2) dimensions

$F = dA \quad , \; F \wedge F = d(A\, dA)$

Considering a 4d path integral with $\partial M_4 = M_3$,

Then we worry about $\int_{M_3} A\, dA$.

$A \wedge F \longrightarrow A \wedge F + d(\epsilon F)$ } ⇒ Small gauge tranformation.

$A \wedge F \longrightarrow A \wedge F + \omega \wedge F$ } ⇒ large gauge transformation.

$$S = \int_{M_3} \frac{-1}{8\pi^2 e^2}\, dA \wedge * dA + \frac{k}{2\pi} A \wedge dA$$

Chern Simon Action : $S_{cs} = \frac{k}{2\pi} \int_{M_3} A \wedge dA$

not gauge invariant.

$\boxed{F = dA}$

But, The thing which matters is $e^{iS_{cs}}$

$$e^{iS_{cs}} = e^{\frac{ik}{2\pi} \int_{M_3} A \wedge dA} \longrightarrow e^{\frac{ik}{2\pi} \int A \wedge dA + \frac{ik}{2\pi} \int \omega \wedge F}$$

$$\Rightarrow \boxed{e^{\frac{ik}{2\pi} \int \omega \wedge F} = 1} \Longrightarrow$$ This must hold, for path integral measure to be gauge invariant.

This leads to quantization condition, & sets $k \in \mathbb{Z}$

Let R be a finite-dimensional representation of the gauge group $U(1)$, and $\gamma \subset M_3$ is a closed oriented loop; Then the classical observable is

$$W(R,\gamma) := \mathrm{Tr}_R \exp\left(i\oint_\gamma A\right)$$

$A = A_\mu dx^\mu$.

"For $U(1)$, every finite dimensional representation is fully reducible to a sum of one-dimensional irreducibles & the irreducible representations are $\rho_n(z) = z^n$ for $n \in \mathbb{Z}$."

↳ Using
- Peter Weyl Theorem
- Shur's Lemma.

Hence, in general we can consider

$$\boxed{W(n,\gamma) := \exp\left(in\oint_\gamma A\right)}$$

The $e^2 \to \infty$ limit of the correlators will only depend on γ up to "isotopy."

So; we have to compute the following.

$$\boxed{\left\langle \prod_\alpha W(n_\alpha,\gamma_\alpha)\right\rangle = \int_{\mathcal{A}/U(1)} [dA]\cdot e^{\frac{ik}{2\pi}\int_{M_3} A\wedge dA} \cdot \prod_\alpha W(n_\alpha,\gamma_\alpha)}$$

U(1) Chern Simon Theory

Shoaib Akhtar

Chern Simon Action

$$S_{c.s.} = \frac{k}{4\pi} \int_M tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

Here we consider M to be a 3 dimensional manifold. (denote the dimensionality by the symbol M_3)
(Because in higher dimensions, every knot is an unkont, and in lower dimensions we can't form a knot (just unknot))

$$S_{c.s.} = \frac{k}{4\pi} \int_{M_3} tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

Consider the wilson loop $W_R(K)$

$$W_R(K) = Tr_R \, \mathcal{P} \exp\left(i \oint_K A \right)$$

$K \subset M_3$ be a closed loop (a knot)

R be a finite dimensional representation of U(1) gauge group.

Here we are dealing with Abelian gauge group, so, we can remove path ordering in the defination of $W_R(K)$ $\quad W_R(K) = Tr_R \exp\left(i \oint_K A \right)$

$A = A_\mu dx^\mu$ one form

$A_\mu = A^a_\mu T_a$, with $a = 1, 2, \ldots, n^2-1$

Here A_μ is $n \times n$ matrix

Concerning the Representation.

We have to compute the correlators:

$$\left\langle \prod_{\alpha} W_{R_\alpha}(K_\alpha) \right\rangle = \int_{\mathcal{A}/U(1)} [dA] \cdot e^{\frac{ik}{4\pi}\int_{M_3} tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)} \cdot \prod_{\alpha} W_{R_\alpha}(K_\alpha)$$

Since $U(1)$ is compact, its (continuous, complex, finite-dimensional) representations are unitary and thus the direct sum of irreps by the Peter Weyl Theorem.

By Schur Lemma, such irreps are all 1-dimensional.

i.e; They are given by $\chi(t) = t^n$ (identifying $U(1)$ with the unit circle in $\mathbb{C}$) for some integer n.

The representations of $U(1)$ are thus given by

$t \longrightarrow (t^{n_1}, \dots, t^{n_k})$ over some basis of $\mathbb{C}^k$

for $n_i \in \mathbb{Z}$.

In general, we can consider one-dimensional irreducible representations $\chi(t) = t^n \quad \forall\, n \in \mathbb{Z}$

and find following correlators.

$$\left\langle \prod_{\alpha} W_{n_\alpha}(K_\alpha) \right\rangle = \int_{\mathcal{A}/U(1)} [dA]\, e^{\frac{ik}{4\pi}\int_{M_3} tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)} \cdot \prod_{\alpha} W_{n_\alpha}(K_\alpha)$$

$n_\alpha \in \mathbb{Z}$

where, $W_{n_\alpha}(K_\alpha) = \exp\left(i\oint_{K_\alpha} n_\alpha A\right) = \exp\left(i n_\alpha \oint_{K_\alpha} A\right)$

$g \in U(1)$, Then $g = e^{i\theta}$, $\theta \in \mathbb{R}$
$\theta \in [0, 2\pi)$

Then in the one-dimensional irreducible representation

$g \longmapsto g^n$ it becomes g^n.

ie; $\chi_n(g) = g^n = e^{i\theta n}$

Hence the generator is n

so; $A_\mu dx^\mu$ in representation $\chi_n(g)$

becomes $A_\mu n \, dx^\mu$

$\exp\left(in \oint_K A_\mu dx^\mu\right)$ becomes the thing which we need.

$Tr_{\chi_n(g)}\left(\exp i\oint_K A\right) = \exp\left(in\oint_K A\right)$ because one dimensional representation has only one element (a scalar; so taking trace is a trivial operation)

$$\Rightarrow \boxed{W_{n_\alpha}(K_\alpha) = \exp\left(i n_\alpha \oint_{K_\alpha} A\right)}$$

The U(1) Chern - Simons action over three dimensional Manifold is

$$S_{c.s.} = \frac{-k}{4\pi}\int_{M_3} d^3x \, \epsilon^{\lambda\mu\nu} A_\lambda \partial_\mu A_\nu$$

$\mu, \lambda, \nu \in \{0, 1, 2\}$

The $A \wedge A \wedge A$ piece is zero because; Wedge product of one form with itself is zero.

Hence we simplify our problem; and need to evaluate following loop correlators.

$$\left\langle \prod_{\alpha} W_{n_\alpha}(K_\alpha) \right\rangle = \int_{\mathcal{A}/U(1)} [dA] \cdot e^{\frac{ik}{4\pi}\int_{M_3} d^3x \, \epsilon^{\lambda\mu\nu} \cdot A_\lambda \partial_\mu A_\nu} \cdot \prod_{\alpha} W_{n_\alpha}(K_\alpha)$$

Quoting answer from Edward Witten's Paper "Quantum Field Theory and the Jones Polynomial)

$$\left\langle \prod_{\alpha} W_{n_\alpha}(K_\alpha) \right\rangle = \exp\left(\frac{i}{2k} \sum_{\alpha,\beta} n_\alpha n_\beta \cdot \int_{K_\alpha} dx^\mu \int_{K_\beta} dy^\nu \cdot \epsilon_{\mu\nu\lambda} \cdot \frac{(x-y)^\lambda}{|x-y|^3} \right)$$

Now, we use the gauss linking Number formula.

$$lk(K_\alpha, K_\beta) = \frac{1}{4\pi} \int_{K_\alpha} dx^\mu \int_{K_\beta} dx^\nu \cdot \epsilon_{\mu\nu\lambda} \cdot \frac{(x-y)^\lambda}{|x-y|^3}$$

Then we get

$$\left\langle \prod_{\alpha} W_{n_\alpha}(K_\alpha) \right\rangle = \exp\left(\frac{2\pi i}{k} \sum_{\alpha,\beta} n_\alpha n_\beta \cdot l.k.(K_\alpha, K_\beta) \right)$$