

Strong Field Gravity

Shoaib Akhtar 25/7/2020

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Lec 1: Introduction ; Kerr Metric , ergoregion , symmetries.

Geometric units : $G = c = 1$

Index conventions : $a, b, c, \dots$ 4 index (spacetime)
 $i, j, k, \dots$ 3 index (spatial)

ex $V^a = (v^t, v^i)$

East Coast Metric signature : $(-, +, +, +)$

Metric : $ds^2 = g_{ab} dx^a dx^b$

Christoffel symbol : $\Gamma_{ab}^c = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})$

check notes

~~$\nabla_a T^{b_1 \dots b_n}_{c_1 \dots c_m}$~~

$$\nabla_a T^{b_1 \dots b_n}_{c_1 \dots c_m} = \partial_a T^{b_1 \dots b_n}_{c_1 \dots c_m} + \sum_j \Gamma_{a d}^{b_j} T^{b_1 \dots d \dots b_n}_{c_1 \dots c_m} - \sum_k \Gamma_{a c_k}^d T^{b_1 \dots b_n}_{c_1 \dots d \dots c_m}$$

Defining this covariant derivative...

... Then our metric is compatible with it $\nabla_a g_{bc} = 0$

Riemann Tensor : $(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = R^c_{dab} V^d$

Ricci " : $R_{ab} = R^c_{a c b}$, Ricci Scalar = $R = R_{ab} g^{ab}$

Einstein " : $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$

Einstein Equation $G_{ab} = 8\pi T_{ab}$

Perfect fluid $\Rightarrow T_{ab} = (\rho + P) v^a v^b + P g_{ab}$

energy
density as seen
by observer with four
velocity u^a
pressure

S.E. Conservation $\nabla_a T^{ab} = 0$.
(stress Energy)

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If we don't have any Energy condition, i.e. conditions of matter (i.e. on T_{ab}) Then we can get ~~any~~ ^{any} geometry.

So, we want to impose some energy conditions on T_{ab} .

Null Energy Condition: $T_{ab} k^a k^b \geq 0$

k^a is future pointing arbitrary null vector.

(for fluid, it is $(\rho + p)(k^a v_a)^2 \geq 0$)

Kerr Black Holes

- Only possible stationary, vacuum B.H. solutions.
- Perturbations: decay rapidly under perturbation (~~perturbations~~)
- Final state generically collapse: $(M, J) + \text{G.H.}$

Metric in Boyer - Lindquist Coordinates

$$ds^2 = - \left(\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \right) dt^2 - \frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma} dt d\phi$$

$$+ \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

where $\Sigma = r^2 + a^2 \cos^2 \theta$

$\Delta = r^2 + a^2 - 2Mr$

$a = \frac{J}{M}$

$J \rightarrow$ Angular momentum of spinning black hole
 $M \rightarrow$ Mass of black hole.

$\bar{a} = J/M^2$ (dimensionless number) (dimensionless spin of Black Hole)

$a=0$ Kerr reduces to SCH.

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lets consider Kerr metric ~~is~~;

~~limit for $a \rightarrow 0$~~ fix a , and let $M \rightarrow 0$.

Then metric takes the form

$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \right) dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2$$

$$\left. \begin{aligned} x &= (r^2 + a^2)^{1/2} \cdot \sin \theta \cdot \cos \phi \\ y &= (r^2 + a^2)^{1/2} \cdot \sin \theta \cdot \sin \phi \\ z &= r \cos \theta \end{aligned} \right\} \text{Ellipsoidal coordinates}$$

$\theta \Rightarrow$ polar coordinate

$\phi \Rightarrow$ azimuthal coordinate

$r=0$ gets maps to ring of radius a .

Symmetries of Kerr

Killing Vector , $\mathcal{L}_K g_{ab} = 0$ Then K is Killing vector.

$$\text{ie; } K^c \partial_c g_{ab} + (\partial_a K^c) g_{cb} + (\partial_b K^c) g_{ac} = 0$$

$\Downarrow$ (can replace ∂ with ∇ ; Γ will cancel...)

$$\mathcal{L}_K g_{ab} = 0 + \nabla_a K_b + \nabla_b K_a$$

$$= \frac{1}{2} \nabla_{(a} K_{b)}$$

$$\Rightarrow \mathcal{L}_K g_{ab} = \frac{1}{2} \nabla_{(a} K_{b)}$$

$$\text{so; } \nabla_{(a} K_{b)} = 0$$

Killing Vectors of Kerr

$\hat{\phi}$ (phi with hat denotes axis symmetric Killing vector)

Killing vectors of Kerr

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$$\hat{\phi}^a = (\partial_\phi)^a \iff \text{Axis symmetric}$$

$$\hat{t}^a = (\partial_t)^a \iff \text{Stationary}$$

$$\text{Killing Tensor } \nabla_{(a} K_{bc)} = 0$$

$$\text{Killing tensor for Kerr } K_{ab} = r^2 g_{ab} + 2 \Sigma l_{(a} n_{b)}$$

l, n are the null vectors

Horizons

There is two places where Kerr metric breaks ~~down~~ down in Boyer-Lind. coordinate.

$$\Sigma = 0, \Delta = 0 \Rightarrow \text{Coordinate's Breakdown.}$$

$R_{abcd} R^{abcd}$ blows up at $\Sigma = 0$; so $\Sigma = 0$ is singularity.

$$\text{so; } \Sigma = 0 \Rightarrow r^2 + a^2 \cos^2 \theta = 0$$

$\Delta = 0$ turns out to give Horizon of Black Hole.

A simple way to determine horizons:
look at surface of constant r , and ~~find~~ figure out where they become null.

Surface of constant r being null is equivalent to $g^{ab} \partial_a r \partial_b r = 0$

$$\text{ie: } g^{ab} \partial_a r \cdot \partial_b r = 0$$

(because $\partial_a r$ is normal to constant r surface)

$$\neq g^{rr} = 0 = \frac{\Delta}{\Sigma} \Rightarrow \Delta = 0$$

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So, we have horizon at $\Delta = 0$

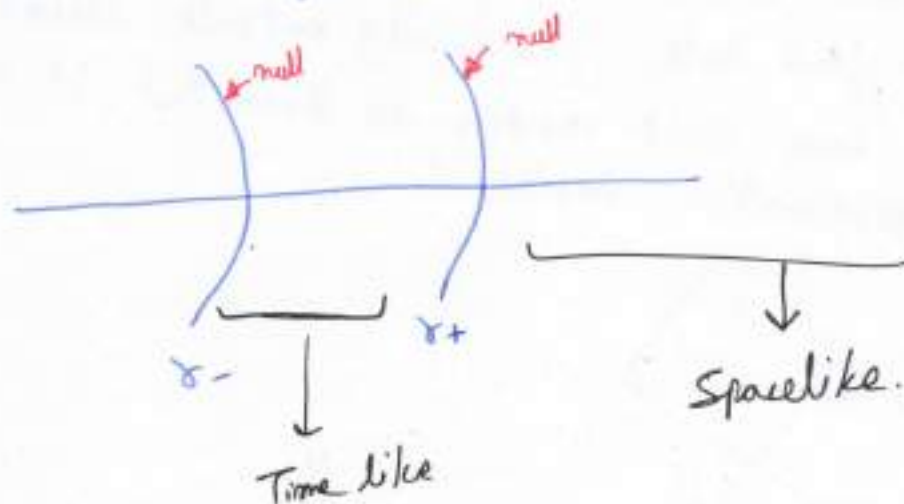
we get two solutions for r for which $\Delta = 0$.

$$\Delta = (r - r_+)(r - r_-)$$

$$\text{where; } r_{\pm} = M \pm \sqrt{M^2 - a^2}$$

$r_+ > r_-$; $r_+ \Rightarrow$ outer horizon.

$r_- \Rightarrow$ inner "



If $a > M$, then there is no horizon to our spacetime.

$\hookrightarrow$ but still we will be then getting singularity where curvature is blowing

$\hookrightarrow$ It will be then naked singularity. because there is no horizon to hide it from ~~any~~ observers far away.

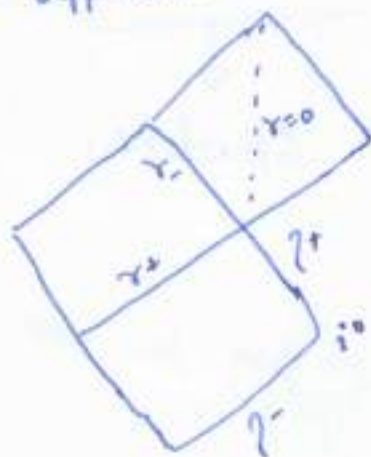
Conformal Diagram for Kerr Black Hole

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$\mathcal{I}^- \Rightarrow$ Past null infinity (light rays coming in from ∞)
 $\mathcal{I}^+ \Rightarrow$ Future null infinity (light rays escaping to ∞)
 $i_0 \Rightarrow$ spatial infinity.

(like Schwarzschild we can extend the Kerr solution; and can have copy of spacetime...)

- We know Kerr solution is very good approximation to astrophysical black hole; basically outside event horizon.
- We don't have much reason to think that it is good approximation inside



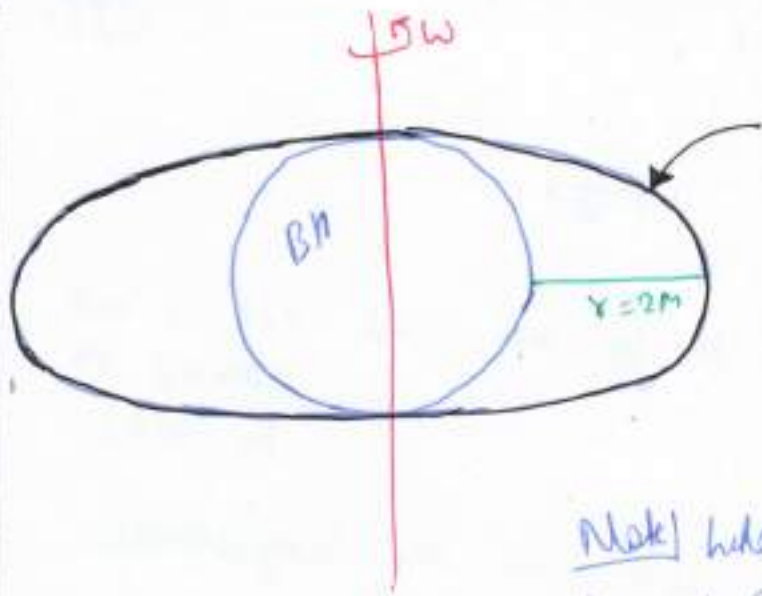
Ergosphere

$$\hat{t}^a \hat{t}_a = g_{tt} = \frac{a^2 \sin^2 \theta}{\Sigma} \geq 0 \quad \text{at horizon } r=r_+$$

$\hookrightarrow$ This means time like killing vector is spacelike at horizon except at $\theta = \pi, 0$.

$$\text{Null means } \hat{t}_a \hat{t}^a = 0 \Rightarrow g_{tt} = 0 \Rightarrow \underline{\Delta = a^2 \sin^2 \theta}$$

$$\text{Using definition of } \Delta, \quad r^2 + a^2 \cos^2 \theta - 2Mr = 0.$$



Ergosphere (meets ~~the~~ black hole at $\theta=0$ & $\theta=\pi$ and ~~the~~ the equator extends to $r=2M$)

Note when we let $a \rightarrow 0$ (we get SCH Black Hole): The Ergosphere moves in to touch the Horizon of Black Hole & disappears at $a=0$.

Top down view



The word Ergo refers to work. ... it turns out that we can use BH to do work in Ergosphere.

Ergosphere is also the region where timelike observers must rotate with Black Hole

let v^a timelike (observer) so $-1 = g_{ab} v^a v^b$

in Boyer-Lindquist metric we get

$$-1 = g_{ab} v^a v^b = g_{tt} (v^t)^2 + g_{rr} (v^r)^2 + g_{\theta\theta} (v^\theta)^2 + g_{\phi\phi} (v^\phi)^2 + 2g_{t\phi} v^t v^\phi$$

> 0 in the ergosphere

(because $g_{tt} \geq 0$ in Ergosphere; and $g_{rr}, g_{\theta\theta}, g_{\phi\phi}$ are all positive)

This has to become negative

$$g_{t\phi} < 0, \quad v^t = \frac{dt}{dz} > 0,$$

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$$\text{Then } v^\phi = \frac{d\phi}{dz} > 0$$

So; note that v^ϕ has to be non zero (it is not allowed to be zero) in the ergosphere.

ϕ has to be changing with time in Ergosphere.

~~Now, let's consider observer~~

Now; let's consider observer with following four velocity

$$\chi^a = \alpha (\hat{t}^a + \Omega \hat{\phi}^a)$$

↑
some normalization

constant (find via $\chi^a \chi_a = -1$)

(we discussed that; we can't set $\Omega = 0$ in Ergosphere)

So, instead choose an observer which has zero angular momentum,

$$\text{ie: } \hat{\phi}_a \chi^a = 0 \quad : \quad \text{ZAMO} \\ (\text{Zero Angular Momentum Observer})$$

$$\Rightarrow g_{\phi t} + g_{\phi\phi} \Omega = 0$$

$$\Rightarrow \Omega = \frac{d\phi}{dt} = - \frac{g_{\phi t}}{g_{\phi\phi}}$$

↙ new derivative w.r.t
* for ZMO

} This is
coordinate frequency
of ZAMO's

as we approach the Horizon of Black hole

(19.9)

for
$$\frac{-g_{\phi t}}{g_{\phi\phi}} \rightarrow \frac{\bar{a}}{(2r_+)}$$

so: we can interpret $\frac{\bar{a}}{2r_+} \equiv \Omega_H$

Ω_H can be interpreted as Horizon frequency of B.H.

$\chi^a \equiv \hat{t}^a + \Omega_H \hat{\phi}^a$ } This is new Killing vector because just a linear ~~combination~~ combination of Killing vectors.

↙ This is the killing vector that has event horizon as ~~for each~~ Killing horizon

↘ i.e. $\chi^a \chi_a \rightarrow 0$ as $r \rightarrow r_+$

so: χ^a is basically generator of horizon for Kerr B.H.

For $\bar{a} \ll 1$, $r_+ \approx (2 - \frac{1}{2}\bar{a}^2)M$
 $\Omega_H \approx \bar{a}/4M$

for $\bar{a} = 1 - \epsilon$; $\epsilon \gg 1$

$$r_+ = (1 + \sqrt{2\epsilon})M$$

$$\Omega_H \approx \frac{1 - \sqrt{2\epsilon}}{2M}$$

Lec 2: Geodesics on the Kerr SpacetimeGeodesic Equation given by $\nabla_a \nabla_a U^b = 0$ where U^a is four ~~vector~~ velocity, $U^a = \frac{dx^a}{d\tau}$

$$U^a U_a = -1 \quad (\text{for null geodesic RHS is } 0)$$

In generic setting $\nabla_a \nabla_a U^b = 0$ gives 2nd order eqn for position coordinates.

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0$$

For Kerr Spacetime we have conserved quantities because we have killing vector. These conserved quantities will be constant along geodesics.

So, Constants: $\tilde{E} = -\hat{t}^a U_a = -U_t$

remember $\hat{t}^a = (1, 0, 0, 0)$ In Boyer Lindquist

$$\hat{\phi}^a = (0, 0, 0, 1)$$

Minus sign so that $\tilde{E}$ will be positive.

$\tilde{E}$ is actually $E / (\text{rest mass of particle})$

because contracted with four velocity (not momentum)

$$E = m \tilde{E} = -\hat{t}^a p_a$$

$$\tilde{J} = \hat{\phi}^a U_a = U_\phi \Rightarrow \text{Angular momentum per unit mass}$$

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$$\tilde{E} = -(g_{tt} V^t + g_{t\phi} U^\phi)$$

$$= \left(1 - \frac{2Mr}{\Sigma}\right) \frac{dt}{dz} + \frac{2Mar \sin^2 \theta}{\Sigma} \frac{d\phi}{dz}$$

$$\tilde{J} = \hat{\beta}^a U_a = U_\phi = \frac{(r^2 + a^2)^2}{\Sigma} \sin^2 \theta \frac{d\phi}{dz} = \frac{2Mar \sin^2 \theta}{\Sigma} \frac{dt}{dz}$$

recall: $\Delta = r^2 + a^2 - 2Mr$
 $\Sigma = r^2 + a^2 \cos^2 \theta$

$\tilde{E} = 1$ at large r (rest mass contribution of particle
 $\tilde{E} = \frac{E_0}{m} = \frac{r_m}{m} = 1$)

$C = K_{ab} U^a U^b$ ← This is another constant

C allows us to integrate generic non-equatorial orbit.
 Killing Tensor

Equatorial case $\theta = \frac{\pi}{2}$

$U^\theta = 0 = U_\theta$

$-1 = g_{ab} U^a U^b = g^{ab} U_a U_b$

$= g^{tt} (\tilde{E})^2 + g^{rr} (U_r)^2 + g^{\phi\phi} (\tilde{J})^2 - 2g^{t\phi} \tilde{E} \tilde{J}$

(*) if we start with no velocity in θ direction for $\theta = \pi/2$ plane; Then we stay in $\theta = \pi/2$ plane.

(replace lower U_t & U_ϕ with constants)

~~scribble~~ → This gives U_r

and we know: $U_r = g_{rr} U^r = \frac{r^2}{\Delta} \cdot \frac{dr}{dt}$

$$g_{tt} = -\frac{(r^2 + a^2) - a^2 \Delta}{\Delta \cdot r^2}$$

(1912)

$$g_{rr} = \frac{\Delta}{r^2}, \quad g^{\theta\theta} = \frac{\Delta}{r^2}, \quad g^{\phi\phi} = \frac{\Delta - a^2}{\Delta \cdot r^2}$$

$$g_{t\phi} = -\frac{2Ma}{\Delta \cdot r}$$

Equation (x) can now be written as

$$-1 = \frac{\Delta}{r^2} \cdot u_r^2 + \frac{a^2 \Delta - (r^2 + a^2)^2}{\Delta \cdot r^2} \tilde{E}^2 + \left(\frac{\Delta - a^2}{\Delta \cdot r^2} \right) \tilde{J}^2 + \frac{4M \cdot a}{\Delta r} \tilde{E} \tilde{J}$$

doing algebraic calculation.

$$-\Delta^2 \cdot u_r^2 = r^2 \Delta + (a^2 \Delta - (r^2 + a^2)^2) \tilde{E}^2 + (\Delta - a^2) \tilde{J}^2 + 4M \cdot a \cdot r \tilde{E} \tilde{J} \quad (x x)$$

$$= (1 - \tilde{E}^2) r^4 - 2Mr^3 + [a^2(1 - \tilde{E}^2) + \tilde{J}^2] r^2$$

$$- 2M(a\tilde{E} - \tilde{J})^2 r$$

$$:= V(\tilde{E}, \tilde{J}, r)$$

$$\Rightarrow \boxed{-\Delta^2 \cdot u_r^2 = V(\tilde{E}, \tilde{J}, r)}$$

This is essentially effective potential for radial motion.

Now, find where $V=0$; This will tell turning points of orbit.

Turning points $\left(\frac{dr}{d\tau} = 0 \right)$, when $V=0$.

Bound Orbit $\tilde{E} < 1$

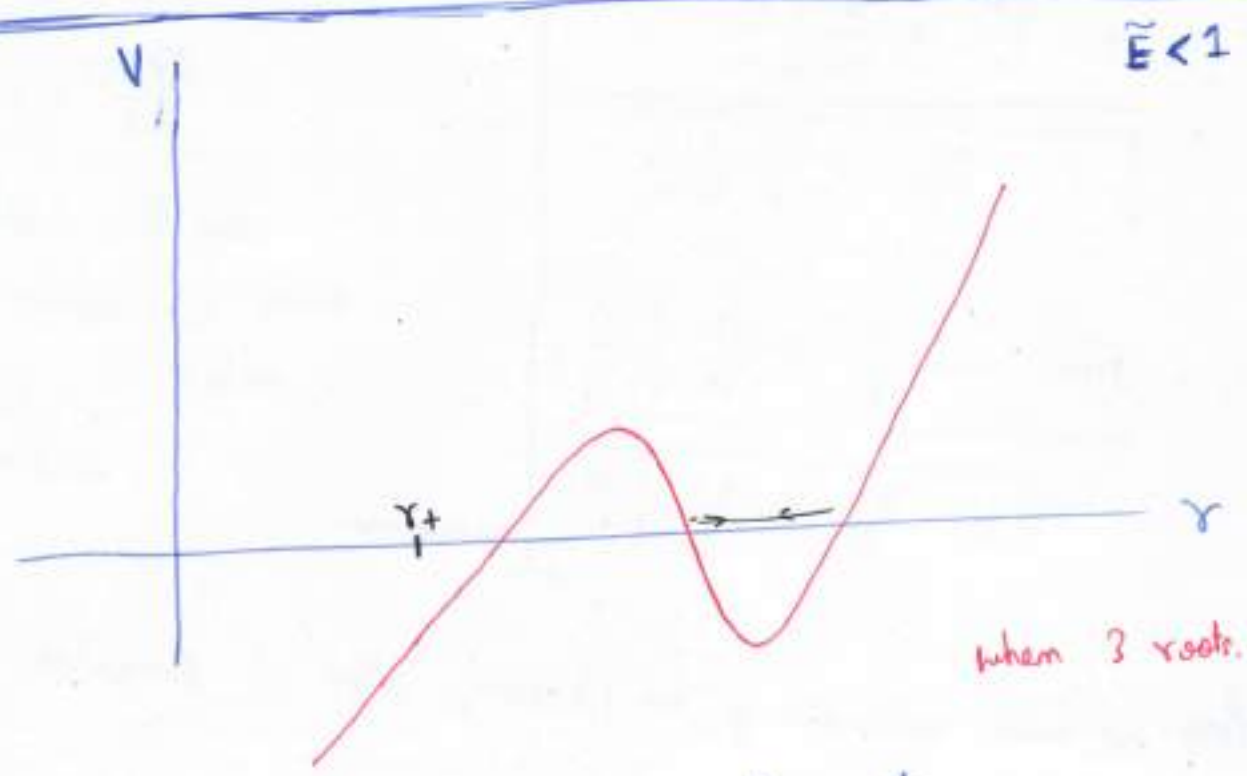
Then $r \rightarrow \infty$, Then r^4 term dominates in V

$$\text{i.e., } V(\tilde{E}, \tilde{J}, r) \rightarrow (1 - \tilde{E}^2) r^4 > 0$$

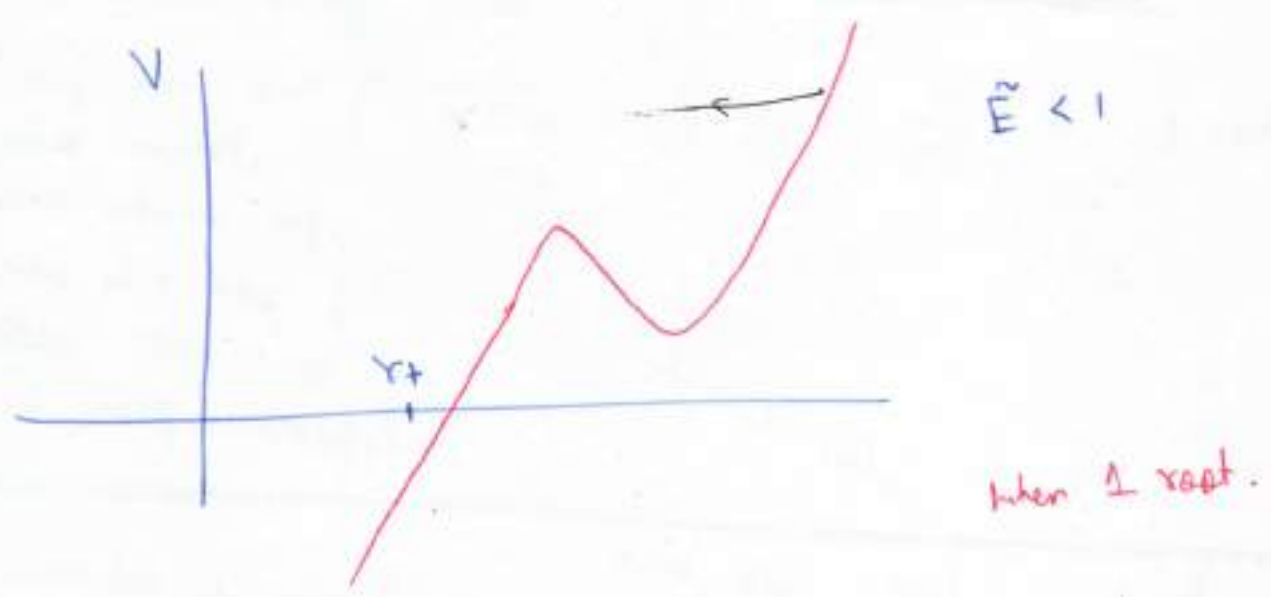
For $r=r_+$ (At Horizon)

$$(\ast \ast) \Rightarrow V(r_+) = -(2M r_+ \tilde{E})^2 - (a \tilde{J})^2 + 4 M a r_+ \tilde{E} \tilde{J}$$
$$= -(2M r_+ \tilde{E} - a \tilde{J})^2$$

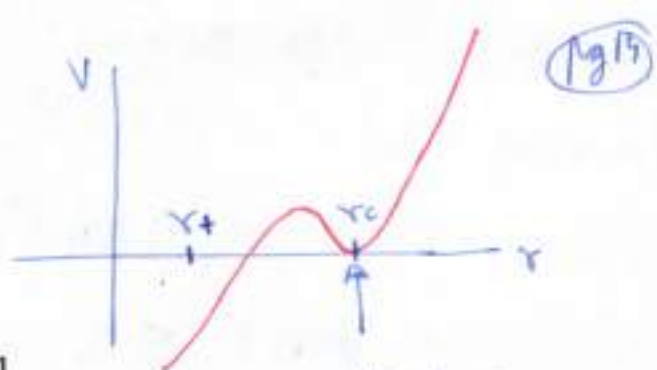
$$\Rightarrow V(r_+) \leq 0$$



$\frac{V(r)}{r} = \text{Cubic polynomial} \Rightarrow$ Will have 1 root or 3 roots.



Marginal case for $\tilde{E} < 1$
when 2 roots repeated



$$V(r_c) = 0$$

$$\frac{dV(r_c)}{dr} = 0$$

↳ This would mean Circular orbit...
 $r = r_c$ constant.

$$\tilde{E}_c = \frac{1 - \frac{2M}{r_c} + \frac{a}{r_c} \sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c} \sqrt{\frac{M}{r_c}}}}$$

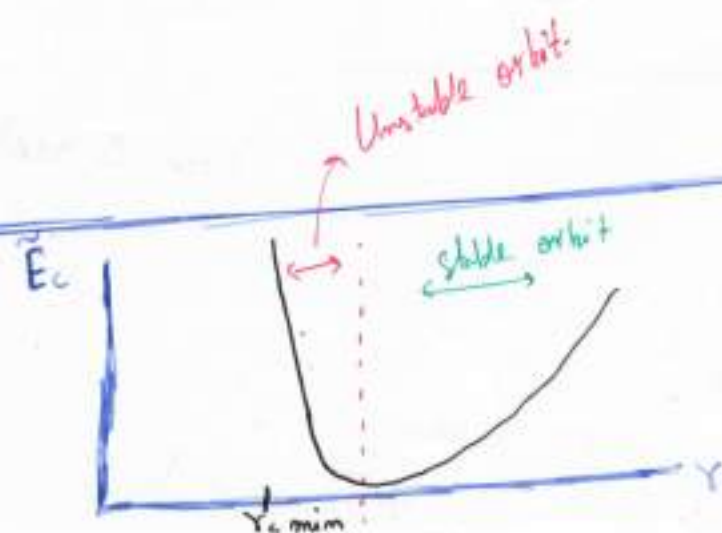
$$\tilde{J}_c = \frac{\sqrt{Mr_c} - \frac{2aM}{r_c} + \frac{a^2}{r_c} \sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c} \sqrt{\frac{M}{r_c}}}}$$

Now we have redundancy in choosing sign of angular momentum

For $\underline{r_c / M} \gg 1$ (Non relativistic limit; far away from black hole)

Then $\tilde{E} = 1 - \frac{M}{2r_c}$, $\tilde{J}_c = \sqrt{Mr_c}$ } These are familiar Newtonian values for circular orbit.

(if you take away rest mass ~~constant~~ contribution 1 from $\tilde{E}$)



r_{min} is determined by making denominator of $\tilde{E}$, $\tilde{J}$ to be zero.

$$\tilde{J}_c, \tilde{E}_c \rightarrow \infty \quad \text{when} \quad r_c^{3/2} - 3Mr_c^{1/2} + 2a\sqrt{M} = 0 \quad (1915)$$

Once we reach this lower value of radius,
we no longer have geodesics with finite energy
and will ~~approach~~ actually approach to limit of moving
arbitrarily close to light.

$$a=0; \quad r_c > 3M \quad \rightarrow \text{lower bound for circular orbit.}$$

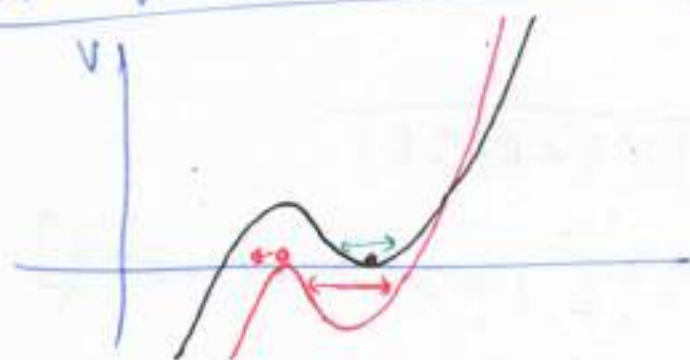
$$\bar{a} = 1 \quad (\text{orbital angular momentum \& spin point in same direction}) ; \quad r_c > M$$

$$\bar{a} = -1 ; \quad r_c > 4M$$

(opposite direction)

when radius have these r_{min} value in
any case; It will actually be photon orbit
& will be followed by null geodesics.

However; not all these circular orbits are stable.
It might be true that with small perturbation it can
either fall into black hole or go to ∞ .

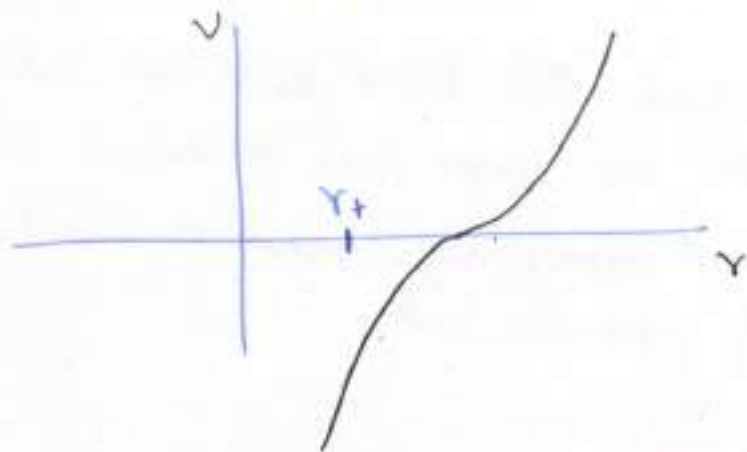


Some circular orbits are unstable.

marginal case; when these 2 pieces merge

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then we have



ISCO (Innermost Stable Circular Orbit)

we can determine value of ISCO

by $\frac{d\tilde{E}_c}{dr_c} (r_{ISCO}) = 0$

or

by solving $V = \frac{dV}{dr} = \frac{d^2V}{dr^2} = 0$

Solving we get $r_{ISCO}^2 - 6Mr_{ISCO} + 8a\sqrt{M} \cdot r_{ISCO}^{1/2} - 3a^2 = 0$

$\hookrightarrow$ quartic in $r_{ISCO}^{1/2}$

for $a=0$; $r_{ISCO} = 6M$

for $a=M$ (extremal case) ; $r_{ISCO} = M$

for $a=-M$ (retrograde case) ; $r_{ISCO} = 9M$

$$\frac{r_{ISCO}}{M} = 3 + B \mp \sqrt{(3-A)(3+A+2B)}$$

where ; $A = 1 + (1-\bar{a}^2)^{1/3} [(1+\bar{a})^{1/3} + (1-\bar{a})^{1/3}]$

$$B = \sqrt{3a^2} + A^2$$

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we can also check ; r_{ISCO} is ~~monotonically~~ monotonically decreasing function of a .

In between

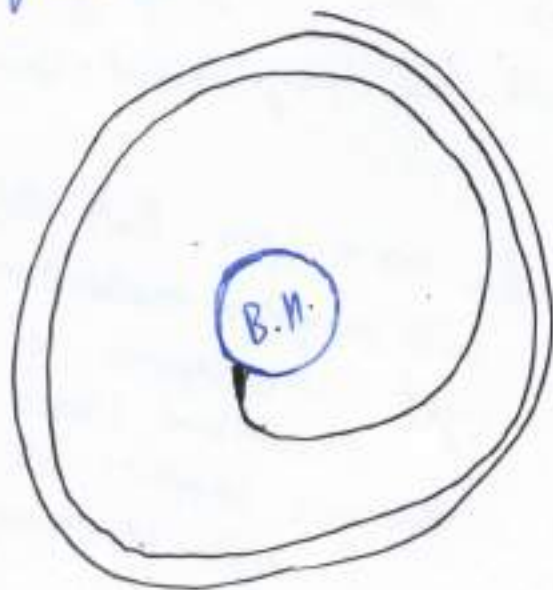
$$r_c^{min} < r < r_{ISCO}$$

we have circular ~~orbits~~ orbits,
but they are unstable.

Solve perturbation around unstable circular orbits.

Imagine we have small object, and its ~~orbit~~ⁱⁿ a circular orbit around a Black Hole ; And its emitting gravitational wave ; sort of losing energy $\Rightarrow$ Radius of orbit will be shrinking.

Assume, object to be small ; so that lost energy ~~to~~ due to gravitational wave ~~are very~~ is happening very slowly $\Rightarrow$ So can think of orbit being quasi circular.



At some point it will start undergoing plunge ;
very rapidly fall into black hole.

$\Leftarrow$ At what radius we expect plunge to happen for small objects ... It is r_{ISCO} .

When black hole spin is anti-aligned with
~~angular~~ angular momentum of particle $\Rightarrow$ Will plunge happen
 sooner or later.

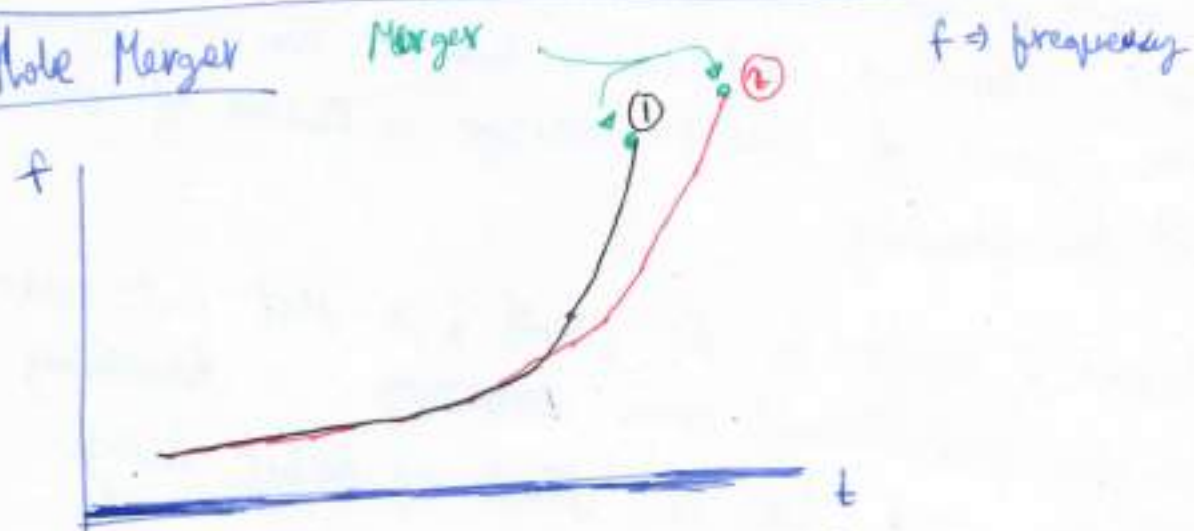
... occurs sooner compared to aligned case.

known $(\gamma_{\text{eff}})_{a=-1} > (\gamma_{\text{ISCO}})_{a=1}$



Here we get
 more cycles.

Black Hole Merger

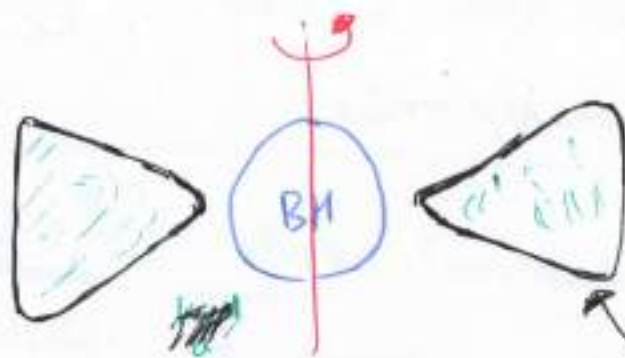


Case ① & Case ②

If masses of the black holes are roughly
 same. Then what ~~can~~ could we say about spin
 of BH's in these two cases

Case ② $\Rightarrow$ More cycle $\Rightarrow$ Has more spin than aligned
 than case ①. (get more cycles of
 gravitational waves &
 merger happens at
 higher frequency)

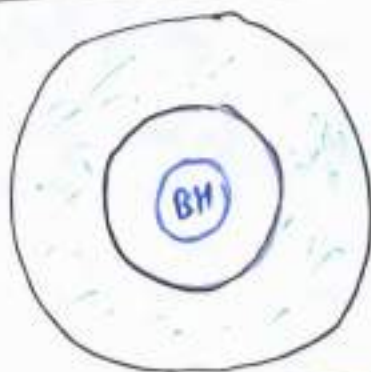
$\rightarrow$ This is called Orbital Hang up Effect.



We can see X-rays
coming from Accretion
disk.

Side Angle View

Accretion Disk



Top View

Lec 3: Review of Black Hole Thermodynamics.

Marginally bound (parabolic in Newtonian) orbits | $\tilde{E} = 1$

(These are dividing line which can escape to infinite & those which can't)

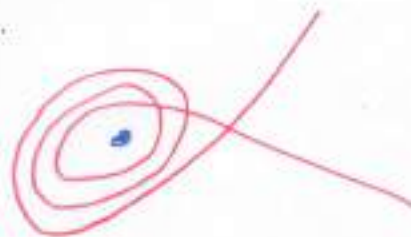
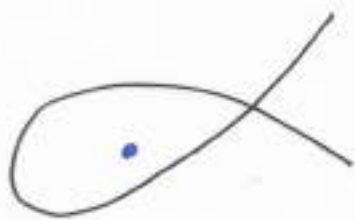
or those who come in from ∞ , & those who have to come in from finite radius)

$\tilde{E} = 1$ (normalized energy)



Newtonian Theory.

In Kerr Spacetime; we can have precession.



$$V = -2Mr \left[r^2 - \frac{\tilde{J}^2}{2M} r + (a - \tilde{J})^2 \right] \Big|_{\tilde{E}=1}$$

Again we are interested in turning points; $V = 0$

→ just solve the quadratic.

$$r_p = \frac{\tilde{J}^2}{2M} \pm \sqrt{\frac{\tilde{J}^2}{2M} - 4(a - \tilde{J})^2}$$

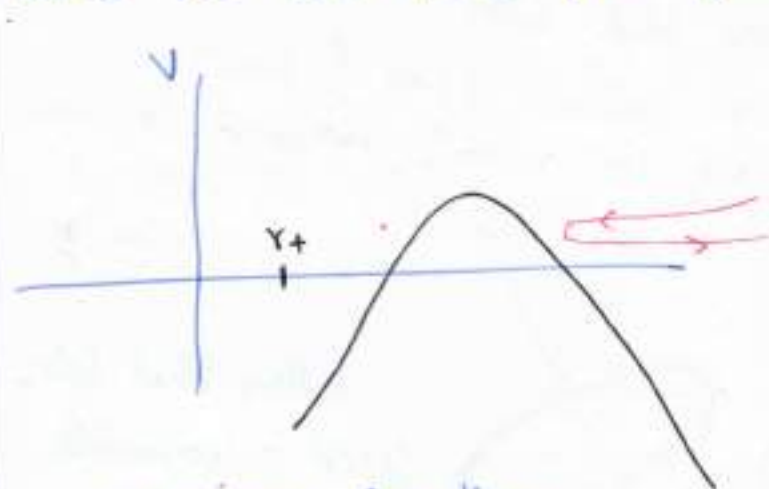
Pericaps Radius

(1921)

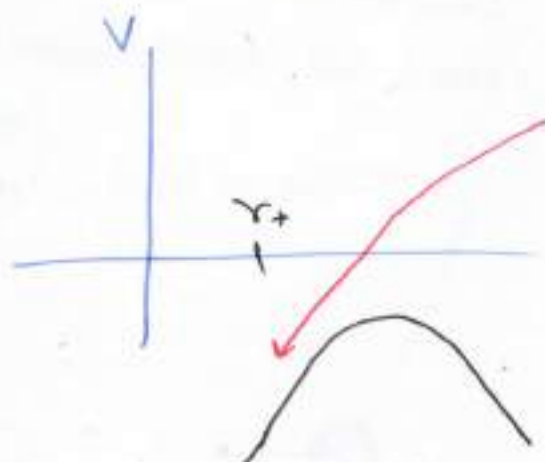
Recall; we showed $V(r_+) < 0$

also ~~have~~ $V < 0$ for large r

Now; we can have two zeroes or no zeroes



Come from ∞ , And then
again go to ∞ .



Come from ∞ &
fall into black hole.

lets find the dividing line;

ie; when discriminant $= 0 \Rightarrow \tilde{J}_m = 2M(1 + \sqrt{1 - \bar{a}})$



~~graph of V~~ ($\tilde{J}_m > 0$)

$$\text{Then } r_p = \frac{\tilde{J}^2}{4M} = M(2 - \bar{a} + 2\sqrt{1 - \bar{a}})$$

for anything $r < r_p \Rightarrow$ It will go to the black hole.

For $\bar{a} = -1$; Retrograde case ; $r_p = M(3 + 2\sqrt{2})$
 $\approx 5.8M$

For $\bar{a} = 0$; Non spinning ; $r_p = 4M$

For $\bar{a} = 1$; Prograde case ; $r_p = M$

Tidal disruption Events

here, we consider super massive black holes

(mass = 1 million) (mass of Sun)

⇒ Think of the stars having very low velocity compared to speed of light at infinity



Ignore ~~the~~ the velocity at ∞ ; & treat it as marginally bound orbit.

When star gets close to BH; it experiences tidal forces.

(parts of star closer to BH; experience more gravity as compared to part further away from BH)

If this tidal forces become significant to overwhelm self gravity of star $\Rightarrow$ You then have what we call Tidal disruption events.

Star will get pulled apart & ~~moves~~ then
moves ~~more~~ more like cloud of ~~so~~ dust or
something like that.

↳ Then motion of stars fluid is determined by BH's
gravity

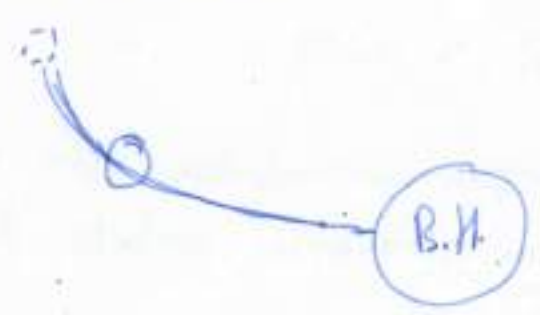
↳ & then this material can fall into BH &
can form accretion disc; ~~as~~ and power the
great flares that astronomer see.

In order for this to happen

- The star should get close enough to black
hole; where tidal forces are strong enough to
pull apart the star.

(~~depends on~~ depends on relative size of star
compared to BH).

If star goes directly into BH,
then it will not have time to
form accretion disc



What would be effect of B.H. spin on whether or pg 24
not we get Tidal Disruption event for given star?

If star is too small compared to B.H; it won't be tidally
~~disrupt~~ disrupted unless it gets very close to B.H.

But if we treat star spinning vs non spinning;
how would this enhance or change the size of
stars which can get tidally disrupted.

(ie; given the stars; we can disrupt without having them
fall into black hole)

★ When ^{spin} aligned; Then favour disruption

parabolic orbit can basically go closer to BH &
therefore can experience greater tidal force
without actually falling into B.H.

Recall; Ergosphere: $r_p < 2M$

Then $\bar{a} > 2\sqrt{1-\bar{a}}$

$$\Rightarrow \bar{a} > 2 - \frac{1}{2} \approx 0.91$$

$$\Rightarrow \bar{a} > 0.91$$

For black holes with ~~spin~~ dimensionless spin greater
than 0.91, we can have parabolic orbits that
can enter the B.H. ergosphere.

(A) Penrose Process

(pg 25)

$$p^a = m u^a$$

four momentum ← m ← rest mass
four velocity ← u^a

Observer with 4-velocity n^a ,

Then energy that n^a will see is $E_n = -n_a p^a > 0$
as long as n^a is timelike.

E_n is not conserved

The conserved one is the one we get by contracting with the killing vector that's timelike at ∞ .

$$E_k = -\hat{t}_a p^a \text{ } \} \text{ Conserved.}$$

$\hat{t}_a$ becomes spacelike in Ergosphere.

→ So; it's possible to arrange E_k to be negative in Ergosphere.

$$E_k = -\hat{t}_a p^a \text{ can be negative in Ergosphere}$$

Whenever we see that we can have something whose energy is negative $\Rightarrow$ This might lead us to think that somehow gain energy out of this system by still by conserving energy.

→ We can do this only in Ergosphere

Let's suppose; we ~~are~~ are in a space ship, and running out of fuel. We see a rotating black hole nearby & we want to use it to gain some energy.

→ Say, we point our spaceship towards the Black hole with some initial 4-momentum p^a

Initial 4-momentum : p^a

We chose this p^a ~~so that~~ such that we could enter Black hole's Ergosphere.

Then take all your garbage, and toss them out of the spaceship into the B.H.

→ Still we need to conserve momentum-energy.

$$p^a = p_s^a + p_R^a$$

$\uparrow$ escape B.H. piece $\nwarrow$ fall in B.H. piece.

In particular, we want to arrange $E_R = -\tilde{\chi}_a p_R^a < 0$

So: $E_s = E - E_R > E$ (because $E_R < 0 \Rightarrow -E_R > 0$)

Let's find out what is maximum efficiency we can get out of this process.

Let's focus on parabolic orbits; i.e. assume $\tilde{E} = 1$, $m = 1$

set $m = 1$ for convenience.

lets assume spin of B.H. is sufficiently large such that ~~these~~ these parabolic orbits can actually enter the Ergosphere Pg 27

Incoming 4-momentum given by

$$p^a = \frac{dt}{d\tau} (1, 0, 0, \Omega) \quad (\text{consider this parabolic orbit at its turning point})$$

$$\Omega = \frac{d\phi/d\tau}{dt/d\tau} = \frac{d\phi}{dt}$$

ii; where $\dot{r}=0$.

We know $-1 = p^a p_a$
(due to normalization)

$$\text{hence we get; } -1 = \left(\frac{dt}{d\tau}\right)^2 (g_{tt} + g_{\phi\phi} \Omega + 2\Omega g_{t\phi}) \quad (*)$$

$$\text{also; } E = \tilde{E} = 1 \quad (\text{because } m=1)$$

$$\text{Then } \frac{d\tau}{dt} = - (g_{tt} + g_{t\phi} \Omega) \quad \left. \vphantom{\frac{d\tau}{dt}} \right\} \Rightarrow \text{after dividing expression for energy by } dt/d\tau$$

(**)

We can combine (*) and (**) to get an expression for Ω

$$\text{ie; } -(g_{tt} + g_{\phi\phi} \Omega^2 + 2\Omega g_{t\phi}) = (g_{tt} + g_{t\phi} \Omega)^2$$

$$\Rightarrow (g_{\phi\phi} + g_{t\phi}^2) \Omega^2 + 2(g_{t\phi})(1 + g_{tt})\Omega + g_{tt} \cdot (1 + g_{tt}) = 0$$

can solve for Ω from this quadratic equation.

$$p_R^a = C_R (1, 0, 0, \Omega_R)$$

$$p_S^a = C_S (1, 0, 0, \Omega_S)$$

} split p^a like pg 28
this during ~~turn~~ turning point.

p_R^a & p_S^a can have radial component
such that the sum has zero radial component

↔ It turns out that adding radial component to either will reduce the efficiency.

So; consider here that ~~before~~ at a turning point before physical splitting both components have zero radial component.

it turns out that $C_R = \left(\frac{dt}{d\tau} \right)_R m_R$ rest mass for that component

$$C_S = \left(\frac{dt}{d\tau} \right)_S m_S$$

so; we get (by conservation of p^a)

$$\left\{ \begin{array}{l} \frac{dt}{d\tau} = C_R + C_S \quad (\text{from time component}) \\ \Omega = C_R \Omega_R + C_S \Omega_S \quad (\text{from radial component}) \end{array} \right.$$

→ Can solve for C_S

$$\frac{dt}{d\tau} \Omega = \left(\frac{dt}{d\tau} - C_S \right) \Omega_R + C_S \Omega_S$$

$$\Rightarrow C_S = \frac{\frac{dt}{d\tau} (\Omega - \Omega_R)}{(\Omega_S - \Omega_R)}$$

→ This $\frac{dt}{d\tau}$ is for original incoming.

So, the energy is just $E_s = g_{ab} \hat{t}^a p_s^b$

$$= -C_s (g_{tt} + g_{t\phi} \Omega_s)$$

$$\Rightarrow E_s = \left(\frac{g_{tt} + g_{t\phi} \Omega_s}{g_{tt} + g_{t\phi} \Omega} \right) \left(\frac{\Omega - \Omega_R}{\Omega_s - \Omega_R} \right)$$

$$E_s = \left(\frac{g_{tt} + g_{t\phi} \cdot \Omega_s}{g_{tt} + g_{t\phi} \cdot \Omega} \right) \left(\frac{\Omega - \Omega_R}{\Omega_s - \Omega_R} \right)$$

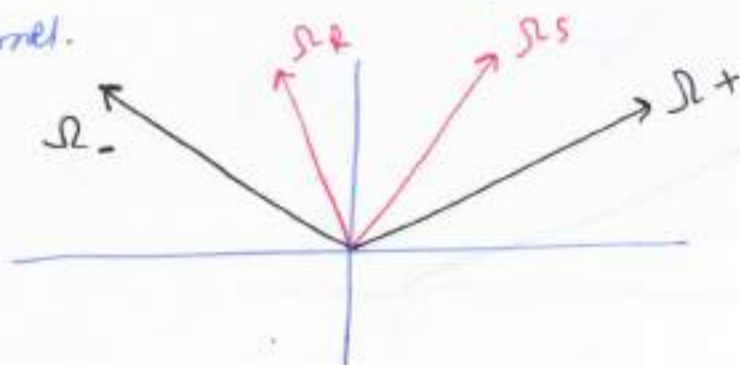
$$\Omega_{\pm} = \frac{-2g_{t\phi} \pm \sqrt{4 \cdot g_{t\phi}^2 - 4 g_{tt}}}{2 g_{\phi\phi}}$$

$$\Rightarrow \Omega_{\pm} = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}}}{g_{\phi\phi}}$$

$\Omega_- < \Omega_R < \Omega_s < \Omega_+$

This should be true to give time like four momentum.

These Ω_- , Ω_+ are like light cones.



If we want to ~~maximize~~ ^{maximize} efficiency of process; we take the limit, where Ω_R goes arbitrarily close Ω_- & Ω_s " " " Ω_+ .

$$\Omega_S \rightarrow \Omega_+ , \Omega_R \rightarrow \Omega_S ; \bar{a} \rightarrow 1$$

(pg 30)

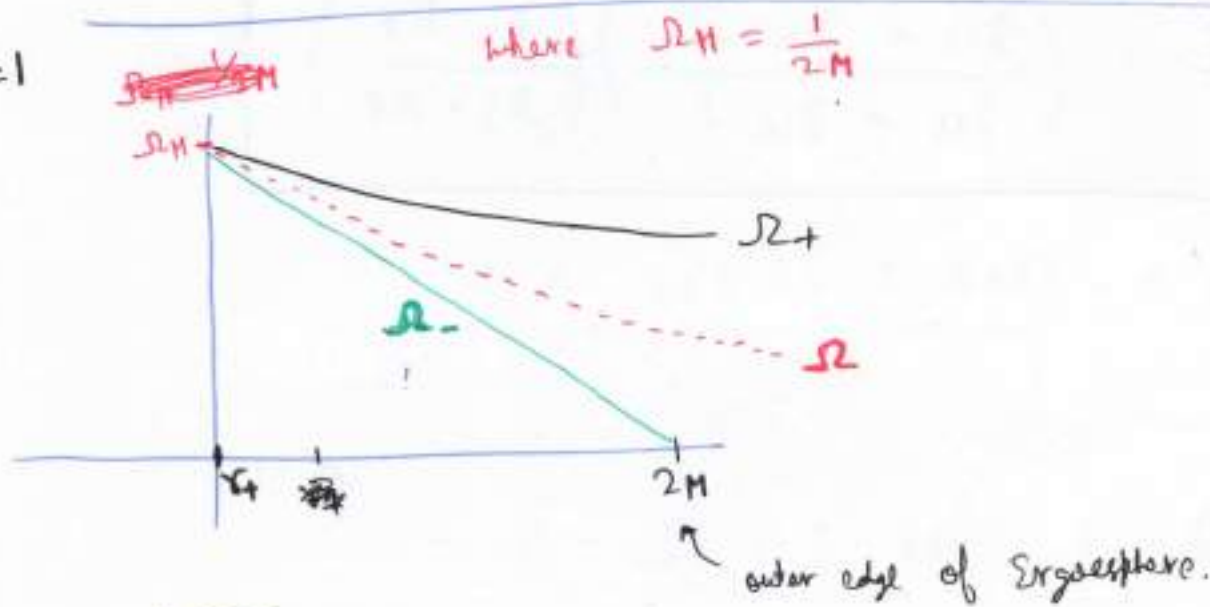
$$\eta = \frac{E_S - E}{E} = \frac{E_S - 1}{1} = E_S - 1$$

Efficiency

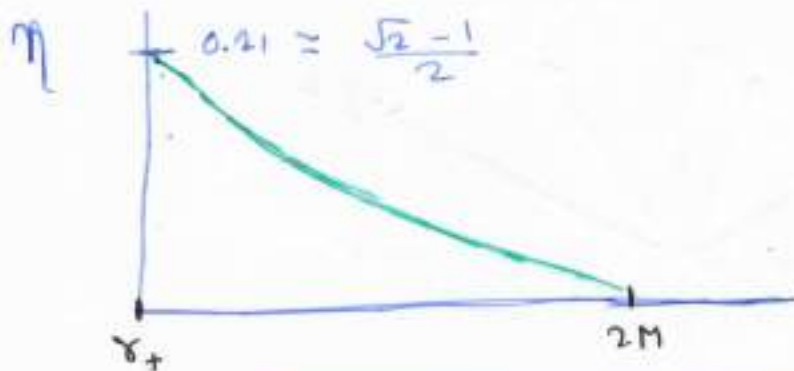
$$\Rightarrow \boxed{\eta = E_S - 1}$$

$\bar{a}=1$

$$\text{where } \Omega_H = \frac{1}{2M}$$



We have some sort of singular limit
as $r \rightarrow r_+$
(at horizon all three meet)



We call $\chi^a = \hat{t}^a + \Omega_H \hat{\phi}^a$ (This killing vector is timelike outside r_+ ; Ω becomes null at r_+)

~~So for any p^a timelike T_+~~

give p_a outside horizon r_+

$$p^a \chi_a < 0$$

by our energy conditions.

$$\text{Then } -\delta E + \Omega_H \delta J < 0$$

$\delta E \rightarrow$ energy the particle carries
(conserved killing energy)

$\delta J \rightarrow$ conserved angular momentum

So: $-\delta E + \Omega_H \delta J < 0$ for any particle
~~enters~~ that enters the Black Hole.

Penrose Process :

$$\delta E < 0$$

This means $\delta J \cdot \Omega_H < \delta E < 0$

$$\Rightarrow \Omega_H \delta J < 0 \Rightarrow \boxed{\Omega_H \delta J < 0}$$

$\hookrightarrow$ So, The particle is not only carrying negative killing energy; but also negative angular momentum.
So it is spinning in opposite direction of B.H.

$\hookrightarrow$ This will cause B.H. to spin less rapidly (because of negative δE & δJ).

$\hookrightarrow$ So, B.H. is losing rotational energy; and it will ~~lose~~ be gained by the particle ~~as~~ p_s going out of B.H.

(New mass of B.H. is $M + \delta E$)

Lec 4: ^{Laws of BH Mechanics} General Penrose type processes, including superradiance

B.H. Area Theorem + Thermodynamics.

Theorem due to Hawking (1971): $\delta A_{BH} \geq 0$ with time

assumption:

- (i) Null Energy Conditions.
- (ii) ~~cosmic~~ Cosmic Censorship Conjecture.

Black Hole area

$$A_{BH} = \int \sqrt{g_{\theta\theta} g_{\phi\phi}} \Big|_{r=r_+} d\theta d\phi$$

$$g_{\theta\theta} g_{\phi\phi} = (r_+^2 + a^2) \sin^2 \theta$$

$$\Rightarrow \boxed{A_{BH} = 4\pi (r_+^2 + a^2)}$$

recall $r_+ = M + \sqrt{M^2 - a^2}$

$$A_{BH} = 4\pi ((M + \sqrt{M^2 - a^2})^2 + a^2)$$

$$\boxed{A_{BH} = 8\pi (M^2 + \sqrt{M^4 - J^2})}$$

$J \Rightarrow$ B.H. Angular momentum

$$\frac{\delta A_{BH}}{8\pi} = 2M \delta M + \frac{1}{2} (M^4 - J^2)^{-1/2} (4M^3 \delta M - 2J \delta J)$$

$$= \frac{J}{\sqrt{M^4 - J^2}} \left[\left(\frac{2M \sqrt{M^4 - J^2}}{J} + \frac{2M^3}{J} \right) \delta M - \delta J \right]$$

$$\frac{2M}{J} (M^2 + \sqrt{M^4 - J^2}) = \frac{2r_+}{a} = \Omega_{H}^{-1}$$

$$\frac{\delta A_{BH}}{8\pi} = \frac{\bar{a}}{\sqrt{1-\bar{a}^2}} [\Omega_H^{-1} \delta M - \delta J]$$

(1938)

$$\frac{\delta A_{BH}}{8\pi} = \frac{\bar{a}}{\sqrt{1-\bar{a}^2}} [\Omega_H^{-1} \delta M - \delta J]$$

if we want $\delta A_{BH} \geq 0$

$$\begin{aligned} \text{Then } \Omega_H^{-1} \delta M - \delta J &\geq 0 \Rightarrow \delta M - \Omega_H \delta J \geq 0 \\ &\Rightarrow \delta M \geq \Omega_H \delta J \end{aligned}$$

↙ This is same relation for particles crossing B.H.

Define $M_{ir} = \sqrt{\frac{A_{BH}}{16\pi}}$

↑
called Irreducible mass

Since $\delta A_{BH} \geq 0 \Rightarrow \delta M_{ir} \geq 0$

for $a=0$ (SCH); we have $M_{ir} = \sqrt{\frac{4\pi (2M)^2}{16\pi}} = M$

so; for SCH, its total mass can't decrease.

for extremal BH; $a=M$, $M_{ir} = \sqrt{\frac{4\pi (M^2 + M^2)}{16\pi}} = \frac{1}{\sqrt{2}} M$

we it to define; the total mass of B.H. or say energy which is rotational.

$$E_{rot} = M - M_{ir}$$

for $a=M$, $E_{\text{rot}} = (1 - \chi_{\text{H}})M \approx 0.29 M$

(1939)

If we can find any astrophysical process to extract this $E_{\text{rot}} \Rightarrow$ we can power very energetic events.

$\delta A_{\text{BH}} \geq 0$ reminds us of 2nd law of Thermodynamics.

2nd Law of B.H. mechanics : $\delta A_{\text{BH}} \geq 0$

Now what about 0th law... (have to define something like temperature)

$$\chi_a = \hat{t}_a + \Omega_H \hat{\phi}_a \quad (\text{recall this})$$

$$\chi^a \chi_a = 0 \quad \text{at Horizon.}$$

This means $\nabla_a (\chi^b \chi_b)$ should be normal to horizon

$$\Leftrightarrow \text{So this means } \underline{\chi_a \nabla_a (\chi^b \chi_b) = 0}$$

So; we can write $\nabla_a (\chi^b \chi_b)$ as

$$\boxed{\nabla_a (\chi^b \chi_b) = -2K \chi_a} \quad \text{for some } K$$

~~by taking~~ by taking Lie derivative w.r.t respect to χ_a

$$\text{But } \underline{\mathcal{L}_{\chi_a} K = 0}$$

for Kerr B.H.

$$K = \frac{\sqrt{M^2 - a^2}}{2M(M + \sqrt{M^2 - a^2})} = \frac{\sqrt{M^2 - a^2}}{2Mr_+}$$

$$= \left(\frac{\Omega_H}{\bar{a}} \right) \sqrt{1 - \bar{a}^2}$$

K is constant on Horizon

K is called Surface Gravity; and plays the role of B.H. temperature.

~~0th Law of B.H. Thermodynamics~~

~~Surface gravity K is constant on B.H.~~

0th Law of B.H. Thermodynamics

Surface Gravity K is constant on Stationary B.H. horizon.

for $a=0$ (SCH) ; $K = \frac{1}{4M}$

This justifies we call it Surface Gravity.

SCH $a=0$

$V^a = ((1 - \frac{2M}{r})^{-1/2}, 0, 0, 0)$ $\Rightarrow$ four velocity for static observer in SCH

$\hookrightarrow$ has only time component.

$$V^a = (1 - \frac{2M}{r})^{-1/2} ; \text{ where } V \text{ is redshift.}$$

$$\Rightarrow V^{-1} = (1 - \frac{2M}{r})^{-1/2}$$

$V \rightarrow \infty$ at horizon

Acceleration : $A_a = V^b \nabla_b V_a = \cancel{V^t \partial_t V_a} - \Gamma_{ta}^t (V^t V^t)^{-1}$

$$= -\Gamma_{ta}^t \delta^a_r = \frac{M}{r(r-2M)} \delta^r_a$$

Combining redshift & acceleration gives finite quantity

$$A^a A_a \equiv a = \frac{M}{r^2 \sqrt{1 - \frac{2M}{r}}}$$

$$V_a = \left(1 - \frac{2M}{r}\right)^{1/2} \cdot \frac{M}{r^2 \left(1 - \frac{2M}{r}\right)^{1/2}} = \frac{M}{r^2}$$

$$K = V_a|_{r=2M} = \frac{1}{4M} \quad \left\{ \Rightarrow \text{Think of taking long thin string;}\right.$$

and attach it to object which is hovering down very close to horizon; and see how much tension string has to exert to keep the object ~~hovering~~ (hovering over horizon) static as measured by some one who is very far away from black hole holding the string.

Now, 1st Law of B.H. Thermodynamics

$$\delta M = \frac{K}{8\pi} \delta A_{BH} + \Omega_H \delta J$$

compare it to 1st Law of ordinary Thermodynamics
c.f. $\delta E = T \delta S - P \delta V.$

3rd Law of B.H. Thermodynamics

$$K \geq 0$$

(Looking at expression for K for Kerr; we see that as $K \rightarrow 0$; $\bar{a} \rightarrow 1$; reach Extremal case)

~~How to liberate rotational Energy from B.H.~~

1937

How to liberate rotational Energy from B.H.

ex) Mechanical Penrose Processes (as discussed before)

now; we discuss more general Penrose processes.

General Penrose Process

Matter Tab (ignore back reaction) (geometry will still be Kerr)

$$J_a = -T_a^b \hat{t}_b \quad (\text{since, still we are on Kerr spacetime, we can use } \hat{t}_b \text{ to define conserved energy current})$$

$$\nabla_a J^a = -(\nabla_a T^a_b) \hat{t}^b - T^a_b (\nabla_a \hat{t}^b) = 0$$

$$\text{since } \nabla_a \hat{t}_b = 0$$

Zero because
 T^a_b is stress
Energy Tensor
(conservation of T_{ab})

Zero because of
Killing equation

$$\boxed{\nabla_a J^a = 0}$$

Now we can use Stokes Theorem to integrate this current on some surface

$$\text{Stokes Theorem: } \int_M \nabla_a J^a \sqrt{g} d^n x = \int_{\partial M} n_a J^a \sqrt{\gamma} d^{n-1} x$$

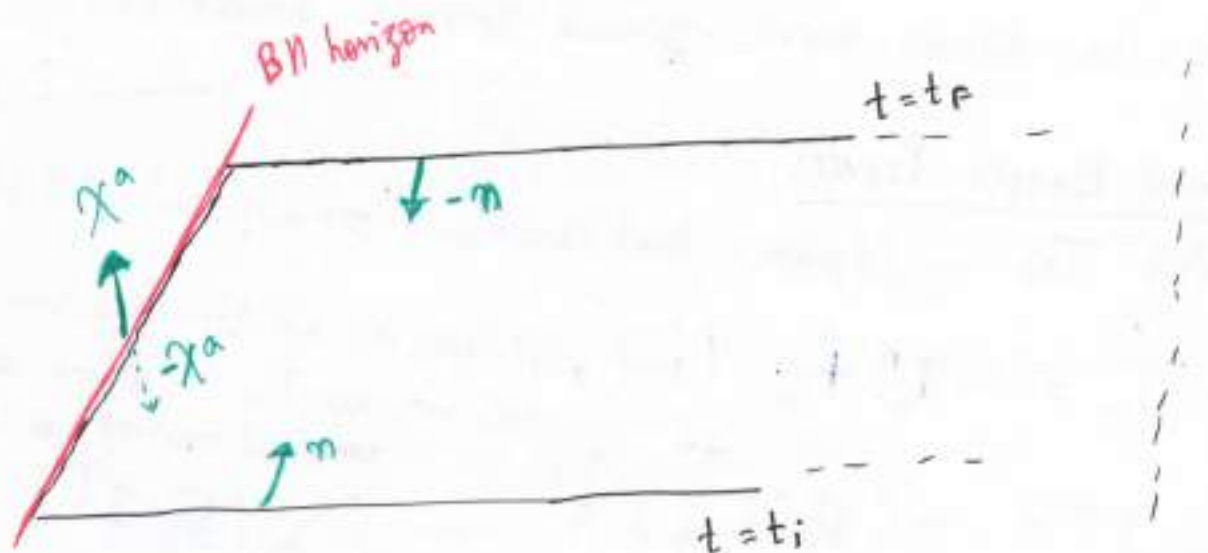
$\uparrow$ unit normal

$$\text{here; } \nabla_a J^a = 0 \Rightarrow \int_{\partial M} n_a J^a \sqrt{\gamma} d^{n-1} x = 0$$

$$\int_{\partial M} n_a T^a \sqrt{\gamma} d^{n-1}x = 0$$

1938

Choose M as follows:



(all the sources go to zero at $r=\infty$; basically ignore contributions from $r=\infty$)

we χ^a are normal to BH horizon.

$\hookrightarrow$ we will use negative of χ^a ; i.e. $-\chi^a$.

doing the calculation $\int_{\partial M} n_a T^a \sqrt{\gamma} d^{n-1}x = 0$

$$\int_{t=t_i}^{t=t_f} n_a T^a \sqrt{\gamma} d^3x - \int_{t=t_i}^{t=t_f} n_a T^a \sqrt{\gamma} d^3x = - \int_{t_i}^{t_f} dt \int_{r=r_+} \chi_a T^a \sqrt{\gamma_{\text{BH}}} d\Omega$$

$\underbrace{\hspace{10em}}_{T_t^t \sqrt{-g}} \quad \underbrace{\hspace{10em}}_{T_t^t \sqrt{-g}} \quad \underbrace{\hspace{10em}}_{DE_{\text{BH}}}$

$$T_t^t \sqrt{-g}$$

$\hookrightarrow$ like energy density of stress energy tensor

$$E(t_f) = \int_{t=t_i}^{t=t_f} m_a T^a \sqrt{r} d\theta$$

1939

$$E(t_f) - E(t_i) = -\Delta E_{BH}$$

↑ flux of energy into B.H.

If we can arrange for flux of negative energy into B.H.; Then we can have more energy than what we had before initially.

Repeat the same argument for Angular Momentum.

$$\text{Also } J_a^\phi = T_{ab} \phi^b \text{ conserved.}$$

do the same argument, we get

$$J^\phi(t_f) - J^\phi(t_i) = -\Delta J_{BH}$$

Assumption NEC (Null Energy Condition)

$$\text{Then } T_{ab} \chi_a \chi_b \geq 0 \quad \text{here } \chi_a \text{ is null vector.}$$

$$\text{we } \chi^a = \hat{t}^a + \Omega_n \hat{\phi}^a$$

$$\text{Then } (-T_b + \Omega_n J^\phi_b) \chi^b \geq 0$$

↪ integrat it over BH & horizon.

$$\boxed{\Delta E_{BH} - \Omega_n \Delta J_{BH} \geq 0}$$

↪ ie; we again get the same condition that B.H. area has to increase.

(This analysis is independent of type of matter)

Lee 5: Superradiant instability or Blandford-Znajek mechanism

Generalized Penrose Processes: we get $0 \leq \Delta E_{BH} - \Omega_H \Delta J_{BH}$.

Superradiance (wave version of Penrose process)

Scalar Field (massless) Ψ

Satisfies massless K.G. equation:

$$\square \Psi = \nabla_a \nabla^a \Psi = 0$$

Stress Energy Tensor: $T_{ab} = \nabla_a \Psi \nabla_b \Psi - \frac{1}{2} g_{ab} \nabla_c \Psi \nabla^c \Psi$

Imagine solution of the form ~~wave, wave, wave~~

$$\Psi = \bar{\Psi}(r, \theta) \operatorname{Re}[e^{-i(\omega t - m \phi)}]$$

wave of pure frequency ω

azimuthal number (determines... axial symmetry)

i.e; $\Psi = \bar{\Psi}(r, \theta) \cos(\omega t - m \phi)$

plug it in $\Delta E_{BH} = \int_{r=r_+} dt \int \chi^a T_{ab} \hat{t}^b dS_{BH}$

~~integrate over surface~~

~~integrate over surface of B.H. horizon~~

~~$$\Delta E_{BH} = \int_{r=r_+} dt \int [\chi^a \nabla_a \Psi] \nabla_b \Psi$$~~

$$\Delta E_{BH} = \int dt \int_{r=r_+} [\chi^a (P_a \Psi) (P_b \Psi) \hat{t}^b - \frac{1}{2} \chi_a \hat{t}^a P_c \Psi D^c \Psi] dS_{BH} \quad (19.41)$$

$$\chi_a \hat{t}^a = 0 \quad \text{because} \quad \chi^a = \hat{t}^a + \Omega_H \hat{\phi}^a$$

$$\Rightarrow \chi_a \hat{t}^a = \chi^a / \chi^a - \Omega_H \hat{\phi}^a$$

$$\Rightarrow \chi_a \hat{t}^a = -\Omega_H \chi^a \hat{\phi}^a$$

we define Ω_H when $\chi^a \hat{\phi}_a$ goes to zero.

$\Rightarrow$

~~$$\Delta E_{BH} = (\omega - m \Omega_H) \omega$$~~

is:

$$\Delta E_{BH} = \int dt \int_{r=r_+} dS_{BH} \cdot [\partial_t \Psi + \Omega_H \partial_{\phi} \Psi] \partial_t \Psi$$

$$= \int dt \int_{r=r_+} dS_{BH} (\omega - m \Omega_H) \omega \underbrace{|\bar{\Psi}|^2 \sin^2(\omega t - m \phi)}_{\geq 0}$$

~~so~~ ~~sig~~ so; sign of ΔE_{BH} depends on sign of $(\omega - m \Omega_H)$

taking the convention $\omega > 0$

$$\text{Then} \quad \underline{0 < \omega < m \Omega_H} \Rightarrow \Delta E_{BH} < 0$$

$\hookrightarrow$ Superradiance Condition

heuristicly; send a wave of some frequency towards the B.H.; some will be absorbed & some will be scattered.

If we have spinning B.H. with the Superradiance Condition; it is called Superradiating Scattered.

It can actually be reflected with more energy (1992)
than the incoming wave.

for wave packet $\frac{\delta E}{\delta t} = \frac{\omega}{m}$

ω ← frequency.
 m ← azimuthal number.

~~Supper~~ Superradiance turns off in two limits

(i) $\omega \rightarrow m \Omega_H$

(ii) $\omega \rightarrow 0$

~~as $\omega \rightarrow 0$~~

He derived Superradiance for scalar waves here;

It is quite general phenomenon.

Can occur for

- Electromagnetic waves.
- Gravitational waves.

→ Think of this limit as the limit where basically wavelength of the wave is very large as compared to size of B.H. horizon

→ So wave sort of does not feel the B.H.

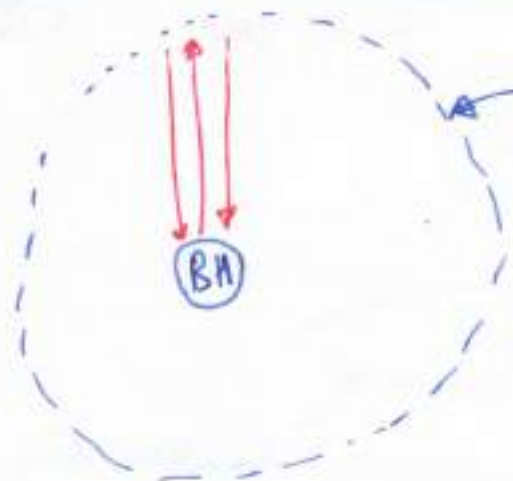
→ and ~~so~~ scatters without much flux across the horizon

Superradiance as a function of spin; it maximizes for maximally spinning B.H.

$$(I') = (1 + \text{Amplification})(I)$$

Maximum amplification.

spin zero	$S=0$	: Scalar	: 0.3 %
	$S=1$	: Electromagnetic	: 4.4 %
	$S=2$	: Gravitational waves	: 138 %



Surround Black hole by mirror

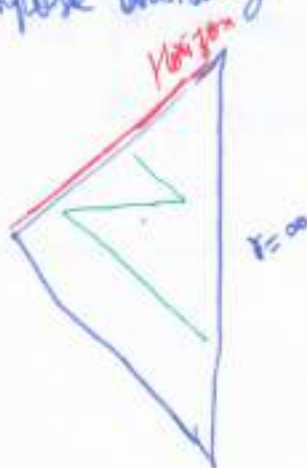
The idea is due to
Press & Teukolsky
~~Black Hole Bomb~~

B.H. Bomb

Consider AdS space

$r=\infty$ corresponds to
time like boundary

impose boundary conditions: like reflecting boundary conditions



Here we can have waves, that
can bounce back & forth.

And it turns out that we can get
Superradiant instability in AdS
this space.

So, Black Holes with ergoregion, and AdS
are generically unstable

Generalized Ergo region for Charged BHs

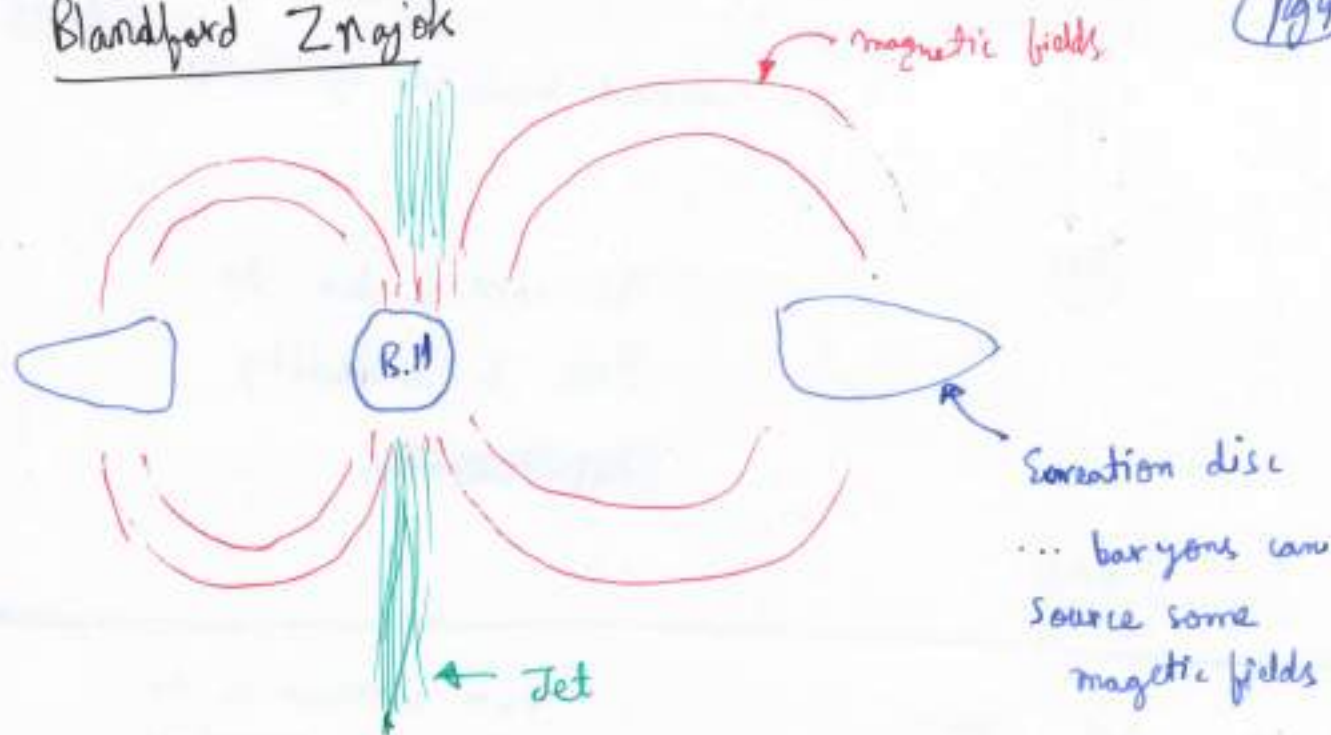
$$E = -\hat{t}^a p_a = -m \hat{t}^a u_a - q \hat{t}^a A_a$$

If we can arrange this to be
negative, then we can define effective
Ergosphere for that ~~the~~ charge.

... we get similar things...

we also have this
part due to charge.

where; A_a is vector
potential associated
with B.H.



Black holes rotation will cause these field lines to rotate, ... and therefore gives a pointy flux of energy that will carry away rotational energy of B.H.
 → This is what we think powers jets.

Superradiance for massive boson

Consider massive scalar field ϕ

then we have K.G equation: $\square \phi = \mu^2 \phi$

where $\mu^2 \sim$ (mass of boson upto factor of $\hbar$)

Concentrate on non-relativistic limit of K.G.

effective Compton wavelength μ^{-1} of field

Consider $\mu^{-1} \gg M$; $r \gg M$ } Non-relativistic limit
 (Compton wavelength) $\gg$ (size of Black Hole) (field is far away from B.H.)

$$\square \phi = \nabla_a (g^{ab} \partial_b \phi) = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} \cdot g^{ab} \partial_b \phi)$$

1/19 45

Since ~~we~~ we are far away from B.H.; we can ignore the spin of B.H.

(When you are far away from B.H. you only feel its mass; and spin falls off more rapidly)

for Sch: $\sqrt{g} = r \sin \theta$

then $g^{tt} \partial_t^2 \phi + \partial_i \partial^i \phi - \mu^2 \phi = 0 \quad (*)$

~~where $g^{tt} \approx - (1 + \frac{2M}{r})$~~ where $g^{tt} \approx - (1 + \frac{2M}{r})$

(Assume: this scalar field is non-relativistic, so its momentum is much less than frequency)

~~the ansatz: $\phi =$~~ $\iff$ so, we only need to keep gravitational potential for time components.

Use ansatz: $\phi = \frac{1}{\sqrt{2\mu}} [\psi(x) e^{-i\omega t} + \psi(x) e^{i\omega t}]$

$\omega \approx \mu$ (because of non relativistic limit)

plugging it in (*) gives

$$[-(1 + \frac{2M}{r})(-i\omega)^2 + \partial_i \partial^i - \mu^2] \psi = 0$$

$$\Rightarrow (\omega^2 - \mu^2) \psi = (-\partial_i \partial^i - \frac{2M\omega^2}{r}) \psi$$

* take $\omega + \mu \approx 2\mu$

then $(\omega - \mu) \psi = [-\frac{1}{2\mu} \partial_i \partial^i - \frac{\mu M}{r}] \psi$

(This equation is similar to Schrodinger eqⁿ for Hydrogen atom)

Making identification:

(Pg 46)

Time structure constant will be $\therefore \alpha = \mu M$

Energy : $E = \omega - \mu$

Bohr radius : Bohr radius $= \mu^{-1} \alpha \equiv r_B$

Note: we did not use any quantum mechanics.

we can label solutions by so called quantum numbers
(n, l, m)

If we look at $l=m=1$

we get $\phi \sim e^{-r/2r_B} \cdot Y_m^l(\theta, \phi) \cdot e^{-i(\omega t)}$

ω complex.

ω here will be complex no. in general
frequency of bound state

Magnitude of Re part $\gg$ Magnitude of Imaginary part.

$$\text{Re}(\omega) = \mu \left(1 - \frac{\alpha^2}{2m^2} \right)$$

$\rightarrow$ some correction, because the particle is bound to B.H.
 $\rightarrow$ so some sort of negative potential energy

also has imaginary

$|\text{Im}(\omega)|$ because $\Rightarrow$

Remember we have B.H. horizon in middle $\Rightarrow$ so there has to be some flux of energy across the B.H. horizon.

$$|\text{Im}(\omega)| \ll |\text{Re}(\omega)|$$

If frequency ω of bound state is less than $m\Omega_H$ (pg 97)
~~not~~, then we can again have superradiance.

To make this ~~instability~~ significant; we don't want α to be very small;

ie; we want ω to be comparable to Ω_H .

$$\alpha = 0.1 \left[\frac{\mu}{10^{-12} \text{ eV}} \right] \left(\frac{M_{BH}}{10 M_\odot} \right)$$

- we have BH of sort of 10 Solar mass.
- we want the mass of boson to be somewhere in the range of 10^{-12} eV (These are very ultra light bosons; that would be relevant for having superradiance instability around Astrophysical B.H.).

We don't know of any such ultralight boson in Standard Model. (They are all much more massive)

BH Candidates for extensions of Standard Model: Things like like axion, dark photon ... Things like that, where we would have this very ultralight bosons, and they may be very weakly coupled to ~~the~~ Standard Model.

Superradiance does not rely on any sort of coupling to ordinary matter. All it relies on is that they gravitate

Energy density seen by an observer with

Fig 48

four velocity ~~is~~ n^a is ~~not~~ $n^a n_b T^{ab}$

(we take observer to be perpendicular to slices of constant time) (This is not conserved)

~~The conserved one is $n^a u^b T_{ab}$~~

The conserved current is given by $T^a_b \hat{t}^b$

So; we can calculate conserved energy $n^b T^a_b \hat{t}^a$

that observer perpendicular to slices of constant time observes)

Lec 6: 3+1 formalism for decomposing spacetime

(M, g_{ab}) : Manifold ~~endowed~~ endowed with metric
satisfying Einstein Equation $G_{ab} = 8\pi T_{ab}$

$\Leftarrow$ We are looking forward to have some way of reinterpreting this equation in terms of something that tells us how things change in time.

So, we need to introduce some sort of time in our spacetime

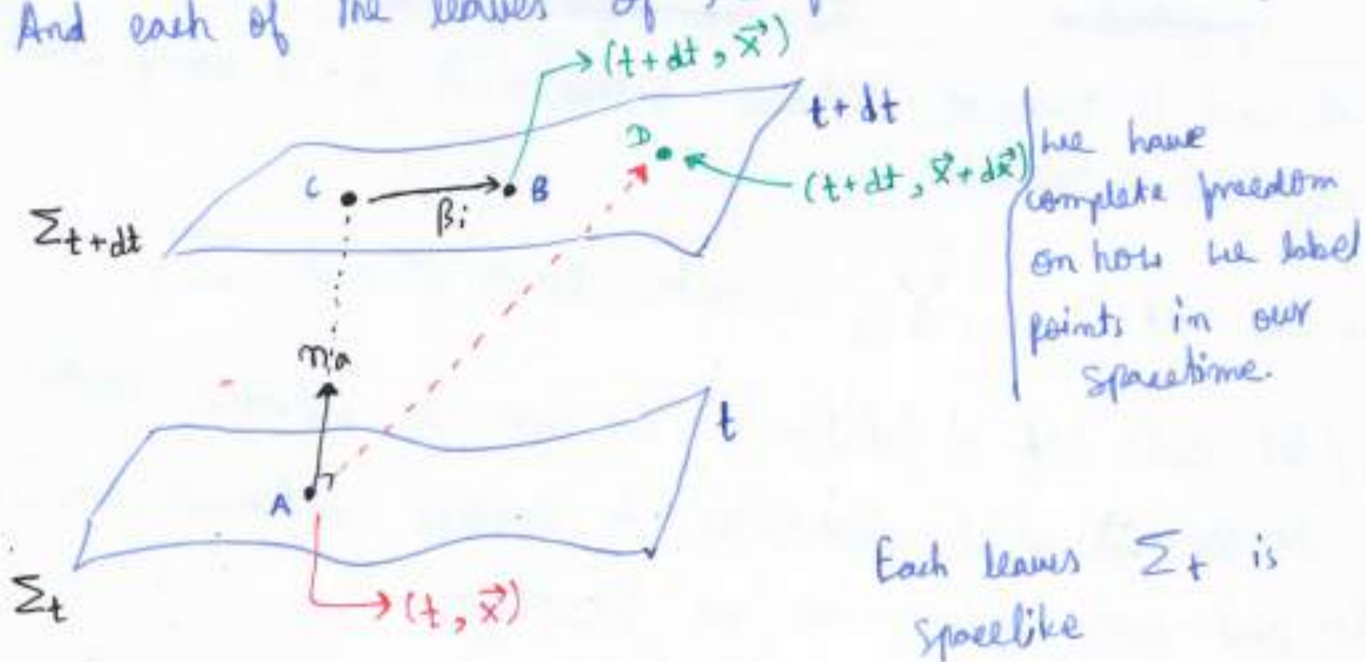
(We know it is artificial; so this will imply some freedom that we can get.)

~~So, we need to~~

So, reinterpret (M, g_{ab}) in different way

Suppose we introduce some foliation labelled by some coordinate t .

And each of the leaves of the foliation is labelled by Σ_t .



Each leaves Σ_t is spacelike,
then its normal is timelike.

Time difference between A & C is obvious dt
because they are on leaves Σ_t & Σ_{t+dt} respectively.

~~he~~ He can also ask what is proper time
if he go along that normal.
lets call it by ΔT

$\Delta T = \alpha \cdot dt$

α depends on whatever point
we are thinking about
 α : lapse function.

α : depends on each point, and it allows us to connect the
time difference as determined by foliation with the
proper time.

He would also like to relate C and B
we introduce a vector B_i :: called Shift Vector
~~(proper time) ∴ The total shift is $B_i dt$~~

The total shift vector is $B_i dt$

we want to measure distance between A & D using what
he have introduced.

we also need: γ_{ij} = metric tensor at each Σ_t

(At each leaf of foliation; we have an intrinsic metric
tensor $\Rightarrow$ which allows us to measure distances on
the each hypersurfaces as we go along)

(pg 51)

$$\begin{aligned}
 ds^2 &= \underbrace{\text{(from A to C)}}_{\text{(from A to D)}} + \underbrace{\text{(from C to B)}}_{\text{(from C to D)}} \\
 &= -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt)
 \end{aligned}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt)$$

$$= g_{ab} dx^a dx^b$$

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \gamma_{ij} \beta^i \beta^j & \beta^i \\ \beta_j & \gamma_{ij} \end{pmatrix}$$

$$g^{ab} = \begin{pmatrix} -\frac{1}{\alpha^2} & \beta^i / \alpha^2 \\ \beta_j / \alpha^2 & \gamma_{ij} - \frac{\beta_i \beta_j}{\alpha^2} \end{pmatrix}$$

Our Goal: To turn our understanding of spacetime (M, g_{ab}) & equations $G_{ab} = 8\pi T_{ab}$

in terms on leaves which are naturally occurring w.r.t. given initial data; also in ~~to~~ terms of ~~the~~ free functions α, β_i which we are introducing.

Projector: $\gamma_{ab} = g_{ab} + n_a n_b$

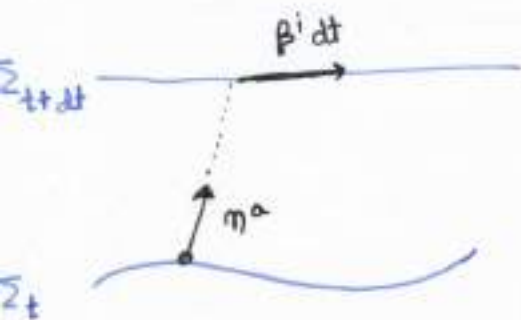
where n_a is unit normal orthogonal to given hypersurface

given our signatures η_a satisfies $\eta_a \eta^a = -1$
 $(-, +, +, +)$

(pg 52)

$$\eta^a \gamma_{ab} = \eta^a g_{ab} + \eta_a \eta^a \eta_b = \eta_b - \eta_b = 0$$

$$\Rightarrow \boxed{\eta^a \gamma_{ab} = 0}$$



$$\eta_a = (-\alpha, 0) \quad \text{normalized } \eta_a.$$

choose minus sign because we want it to be future pointing

$$\eta^a = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha} \right)$$

$$\Rightarrow \boxed{\left(\frac{\partial}{\partial t} \right)^a = \alpha \eta^a + \beta^a}$$

Extrinsic curvature

parallel transport η^a on Σ_t



$$\boxed{K_{ab} \equiv -\gamma^c_a \nabla_c \eta_b}$$

Extrinsic Curvature

γ_{ac} is actually a projector.

we are using γ_{ac} to project result of taking gradient of vector η_b to the hypersurface.

$$K_{ab} = -(g^c_a + \eta^c \eta_a) \nabla_c \eta_b$$

$$\Rightarrow K_{ab} = -(\nabla_a m_b + \underbrace{m_a m^c \nabla_c m_b}_{\substack{\text{Acceleration of vector } m \\ \text{call it } a_b}})$$

(1953)

Then

$$K_{ab} = -(\nabla_a m_b + m_a a_b)$$

if m_a happens to be geodesic

then a_b is zero.

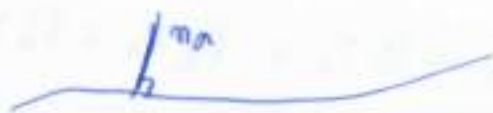
$$\text{then } K_{ab} = -(\nabla_a m_b)$$

if not, then there is component due to contribution of acceleration.

$$K_{ab} = K_{ba} \quad (K_{ab} \text{ is symmetric})$$

proof

$(\frac{\partial}{\partial t})^a \Rightarrow$ vector tangent to line at the point...



$$\tilde{t} \text{ s.t. } \nabla_a \tilde{t} = m_a \equiv \xi_a$$

(define such $\tilde{t}$)

relation between $(\frac{\partial}{\partial t})^a$ & m_a

$$K_{ab} = \gamma_a^c \nabla_c m_b = \gamma_a^c \nabla_c \xi_b$$

$$\xi_b = \nabla_a \tilde{t} \Rightarrow \text{then } \nabla_c \xi_b = \nabla_c (\nabla_b \tilde{t})$$

since $\tilde{t}$ is scalar function

then,

$$K_{ab} = \gamma_a^c \nabla_b m_c = K_{ba}$$

$$\text{so: } \nabla_c (\nabla_b \tilde{t}) = \nabla_b (\nabla_c \tilde{t})$$

$$\mathcal{L}_\eta \gamma_{ab} = \eta^c \nabla_c \gamma_{ab} + \gamma_{ac} \nabla_b \eta^c + \gamma_{bc} \nabla_a \eta^c \quad (\text{pg 59})$$

$$= \eta^c \nabla_c (g_{ab} + n_a n_b) + (g_{ac} + n_a n_c) \nabla_b \eta^c + (g_{bc} + n_b n_c) \nabla_a \eta^c$$

$$= \cancel{\eta^c (0 + \nabla_c (n_a n_b))} + \cancel{(g_{ac} + n_a n_c) \nabla_b \eta^c} + \cancel{(g_{bc} + n_b n_c) \nabla_a \eta^c}$$

$$\nabla_c (g_{ab}) = 0$$

$$\text{and } \eta_c \nabla_b \eta^c = 0 \quad \text{because } \eta_c \eta^c = -1$$

$$\text{So: } \mathcal{L}_\eta \gamma_{ab} = \eta^c \nabla_c (n_a n_b) + g_{ac} \nabla_b \eta^c + g_{bc} \nabla_a \eta^c$$

$$= (\eta^c \nabla_c n_a) n_b + (\eta^c \nabla_c n_b) n_a + \cancel{g_{ac} \nabla_b \eta^c} + \nabla_b n_a + \nabla_a n_b$$

$$\Rightarrow \mathcal{L}_\eta \gamma_{ab} = (\eta^c \nabla_c n_a) n_b + (\eta^c \nabla_c n_b) n_a + \nabla_b n_a + \nabla_a n_b$$

$$\gamma_{ab} = g_{ab} + n_a n_b \Rightarrow n_a n_b = \gamma_{ab} - g_{ab}$$

$$\text{So: } \mathcal{L}_\eta \gamma_{ab} = (\gamma^c_b - g^c_b) \nabla_c n_a + (\gamma^c_a - g^c_a) \nabla_c n_b + \nabla_b n_a + \nabla_a n_b$$

$$= \gamma^c_b \nabla_c n_a + \gamma^c_a \nabla_c n_b - \underbrace{\nabla_b n_a - \nabla_a n_b}_0 + \nabla_b n_a + \nabla_a n_b$$

$$\Rightarrow \boxed{\mathcal{L}_\eta \gamma_{ab} = \gamma^c_b \nabla_c n_a + \gamma^c_a \nabla_c n_b}$$

$$\Rightarrow \boxed{\mathcal{L}_\eta \gamma_{ab} = -2K_{ab}}$$

Property: $\mathcal{L}_\eta \gamma_{ab} = \frac{1}{\phi} \mathcal{L}(\phi \eta) \gamma_{ab} \quad \forall \text{ function } \phi.$

using it $\mathcal{L}_{ab} = -\frac{1}{2\alpha} \mathcal{L}_{\alpha \eta} \gamma_{ab} = -\frac{1}{2\alpha} [\mathcal{L}_{\vec{t}} - \mathcal{L}_{\vec{\beta}}] \gamma_{ab}$

$$\mathcal{L}_t \gamma_{ab} = \partial_t \gamma_{ab}$$

$i, j \Rightarrow$ intrinsic to hypersurface (pg 55)
 $a, b \Rightarrow$ run from $(t, \text{space coordinates})$

Then

$$\partial_t \gamma_{ij} - \mathcal{L}_\beta \gamma_{ij} = -2\alpha K_{ij}$$

(only spacial component survive)

$$\Rightarrow \boxed{\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i}$$

D is covariant derivative associated to γ_{jk} matrix

ie: $D \gamma_{jk} = 0$

... till now we just did kinematics.

now we will bring Einstein's equations.

$D_a T_{ab} =$ Projection of all components of derivatives

So: $D_a^{(3)} T_{ab} = \gamma_a^{a'} \gamma_b^{b'} \gamma_d^{d'} \nabla_{a'} T_{b'd'}$

Recall Gauss - Codazzi Equation:

$$\gamma_a^{a'} \gamma_b^{b'} \gamma_c^{c'} \gamma_d^{d'} R_{a'b'c'd'} \equiv \underbrace{{}^{(3)}R_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc}}_{\text{purely spacial.}}$$

~~Codazzi-Mainardi~~

Codazzi-Mainardi Equation:

~~$\gamma_a^{a'} \gamma_b^{b'} \gamma_c^{c'} \gamma_d^{d'} R_{a'b'c'd'} \equiv D_a K_{bc}$~~