



Statistical Physics



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STATISTICAL PHYSICS

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These notes are consequence of my self study; and are mostly inspired from David Kubiznak lectures on **Statistical Physics**. These notes also includes Statistical Field Theory, and can be effectively regarded as a book on Advanced Statistical Physics. There is also a supplementary book titled **Problems & Solutions in Statistical Physics** written by me for the present book which contains tutorials, exercises and their solution which should be read in parallel with these notes.

Sr No.	Topic	Page No.
1	Review of Thermodynamics: Thermodynamic Laws, Temperature, Energy, Heat, Entropy	1-12
2	Basics of Statistical Physics: Liouville's Equation, Gibbs Distribution, Ensembles	13-24
3	Thermodynamic Approach to Phase Transitions: Ideal Gas, Van der Waals Fluid, Reentrant PT, Gravitational Meissner Effect	25-34
4	Ising Model, Order Parameter, Mean Field Theory, Critical Exponent Beta	35-44
5	Critical Exponents, Landau's theory, Hubbard-Stratonovich Transformation	45-54
6	Spin-Spin Correlation Function, Correlation Length, Landau-Ginsburg Functional	55-66
7	Wilsonian Renormalization 1: Basic idea: Integration over Fast Modes, Flowing Couplings	67-82
8	Wilsonian Renormalization 2: Explicit calculation: Cumulant expansion, RG equations	83-96
9	Wilsonian Renormalization 3: Triumph: Wilson-Fisher Fixed Point, Critical Exponents	97-108
10	Vector Model, Goldstone Modes, CMW Theorem and Lower Critical Dimension	109-120
11	Near $d=2$, Vortices, Kosterlitz-Thoules Phase Transition	121-132
12	Monte Carlo Methods, Sampling, Markov Chain, Single-Spin-Flip Metropolis Algorithm	131-138
13	Equilibration & Measurement, Monte Carlo for the 2d Ising Model, Finite-Size Scaling	139-148
14	Black Hole Thermodynamics, Hawking-Page Transition	149-166

ZEROth Law (helps to define temperature) (Pg 2)

If two systems are separately in Thermal Equilibrium with a third system, ~~then~~ they are also in thermal equilibrium with each other.

⇒ can define Empirical Temperature

Experimental Fact: State of Gas is typically characterized by two properties ~~temperature~~ ^{Pressure} (P) and Volume (V).

Typically these can be independent.

However if it is in thermal contact with another gas then only one of them is actually independent.

"If in thermal eqn with another gas; then only one out of P, V is independent."

* System 1 and 2 in Thermal Equilibrium with 3

Gas no. 3 will be used as Thermometer.

1-3. $\exists$ ~~not~~ a function $F_1(P_1, V_1, P_3, V_3) = 0$
which is actually a constraint equation
constraining P_1, V_1 in terms of P_3, V_3 .

similarly 2-3 in eqn so, $\exists$ another function $F_2(P_2, V_2, P_3, V_3) = 0$.

∴ now we can solve both equation for P_3 .

$$P_3 = P_3(P_1, V_1, V_3) \quad (\text{comes from } F_1 = 0)$$

$$P_3 = P_3(P_2, V_2, V_3) \quad (" \quad " \quad F_2 = 0)$$

Equating this we get $P_3(P_1, V_1, V_3) = P_3(P_2, V_2, V_3)$ (excl)

An algebraic relation.

Let's solve this for P_1 ; $\Rightarrow P_1 = P_1(V_1, V_2, P_2)$ - excl

1-2 by Zeroth law

(pg 3)

this means : $F_3(P_1, V_1, P_2, V_2) = 0$

$$\Rightarrow P_1 = P_1(V_1, V_2, P_2) \quad \text{--- eqn (ii)}$$

compare eqn (i) & eqn (ii)

Conclusion; V_3 in (i) has to drop out

So; in eqn (i) : $P_1 = P_1(V_1, V_2, \cancel{V_3}, P_2)$

So; in $P_3(P_1, V_1, \cancel{V_3}) = P_3(P_2, V_2, \cancel{V_3})$ also

V_3 has to drop out

So; we conclude, there is some universal function f .

ie; eq (iii) implies $f(P_1, V_1) = \hat{f}(P_2, V_2)$

$$= \theta \quad (\text{constant})$$

↑ and this is the

Empirical temperature.

So; we conclude $f(P, V) = \theta$ is our Thermal Temperature.

& we see that for each gas separately whatever P, V be : they are always tied by θ : as $\theta = f(P, V)$ in thermal equilibrium.

Now we extend it to anything.

FIRST LAW

Step 1 For Thermal Isolated System, there exists a function of state called the Internal Energy U , so that

$$dU = dW \quad \text{--- Work on the system.}$$

(isolate the system, don't allow any heat exchange.)

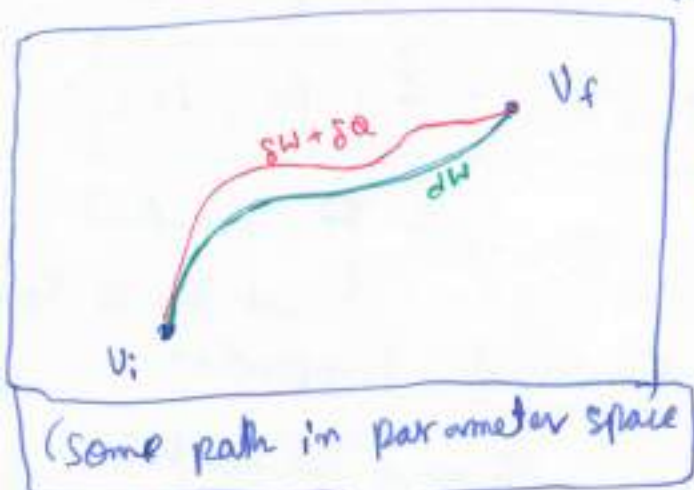
Step 2 If system not thermally isolated, one can also change the state of the system without doing work. HEAT Q Measures to which extent the change is not Adiabatic. $\rightarrow$ (don't allow anything to happen with heat)

So; the first law reads

$$dU = \delta Q + \delta W$$

δQ & δW are in general not total differentials.

$$U_f - U_i = dU = \underbrace{\delta Q}_{\substack{\text{Heat} \\ \text{transferred to} \\ \text{the system}}} + \underbrace{\delta W}_R \quad \text{Work done on the system.}$$



Only if we specify which path we are taking; we can sort of distinguish work δW from heat δQ .

Convention

* Heat flows from hotter to colder.

Examples of Work Terms (comes from fundamental laws of physics)

i) $\delta W = -P dV$

(Hydrostatic Pressure work)

iii) $\delta W = \gamma dA$

(Surface tension work when expanding or contracting area)

Heat Capacity

$$C_{B, \gamma}^{(\alpha)} = \frac{\delta Q_{\alpha, B \dots}}{d\alpha}$$

Generalized heat Capacity

Changing Variable α while keeping P & V ... variables constant. (to specify paths we are taking.) (195)

$C_V \equiv C_V^{(T)} = \left(\frac{\delta Q}{dT}\right)_V$ Heat capacity for constant volume.

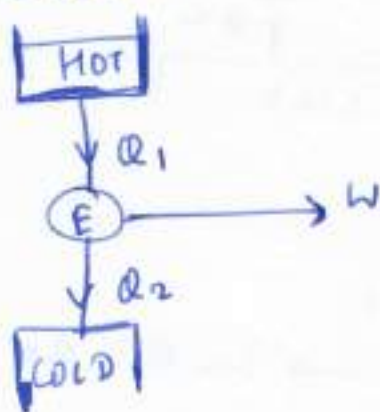
$C_P \equiv C_P^{(T)} = \left(\frac{\delta Q}{dT}\right)_P$ " " " " pressure

Second Law "Not all processes that are energetically possible actually can happen."

"Heat is a less noble cousin of work."

(you can always turn work to heat; but not always we can change heat to work)

Heat Engine (operates between two thermal boxes)
Works in cycles, and returns after each cycle to its initial state.



Heat Engine (E)

Thermal Efficiency:

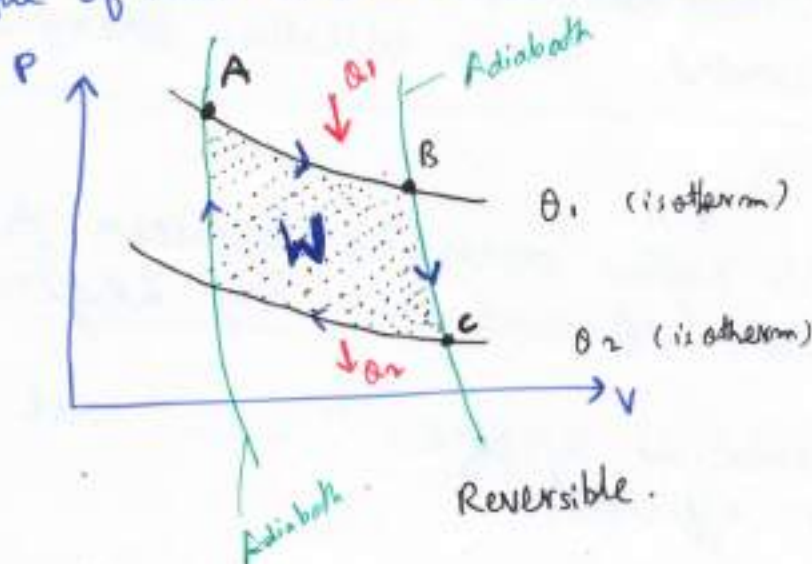
defined by

$$\eta = \frac{\text{Work Out}}{\text{Heat In}} = \frac{W}{Q_1}$$

by conservation of energy

$$\eta = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

Example of Heat Engine (CARNOT CYCLE)

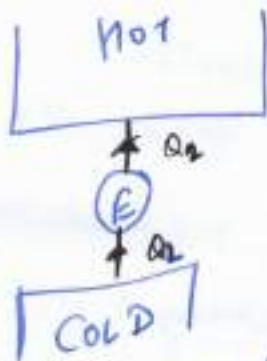


2nd Law (due to KELVIN) No process is possible whose sole result is the complete conversion of Heat into work. (pg 6)

2nd Law (due to CLAUSIUS) No process is possible whose sole result is the transfer of Heat from colder to hotter body.

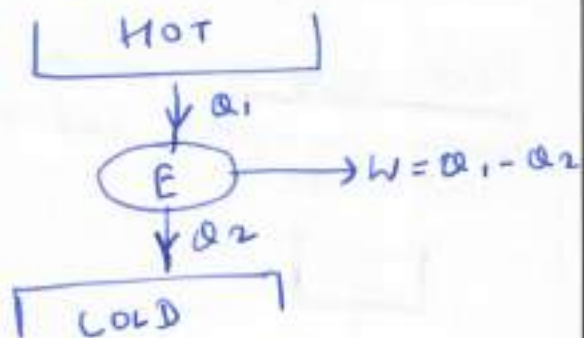
Equivalence of statements.

If Clausius is violated,

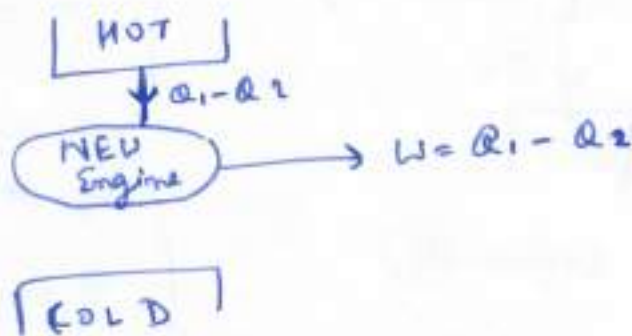


Now we could build engine like this.

Normal Engine



=



(by combining Clausius violating engine with Normal engine we created Kelvin violating engine)

Entropy

Carnot's Theorem No engine "operating between two given reservoirs" can be more efficient than my engine.

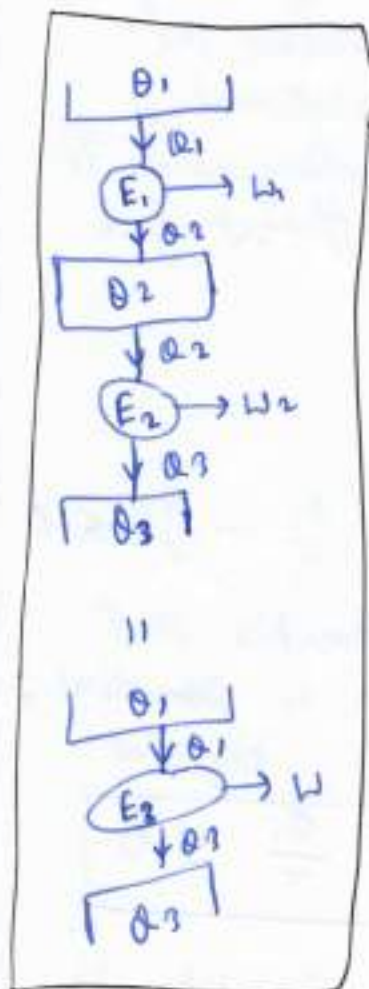
COLLORARY: All reversible engines " — " are equally efficient.

Take Reversible Engine,

$$\frac{Q_1}{Q_2} = f(\theta_1, \theta_2)$$

where θ_1 and θ_2 are empirical temperature of the two reservoirs

(Pg 7)



let E_1 & E_2 be reversible engine which are compared in ~~the~~ the way as shown in diagram.

so:

$$E_1: \quad \frac{Q_1}{Q_2} = f(\theta_1, \theta_2)$$

$$E_2: \quad \frac{Q_2}{Q_3} = f(\theta_2, \theta_3)$$

$$E_3: \quad \frac{Q_1}{Q_3} = f(\theta_1, \theta_3)$$

($\theta_1 \rightarrow \theta_3$)

$$\Rightarrow \frac{Q_1}{Q_3} = f(\theta_1, \theta_3) = \frac{Q_1}{Q_2} \frac{Q_2}{Q_3} = f(\theta_1, \theta_2) \cdot f(\theta_2, \theta_3)$$

$$\Rightarrow \boxed{f(\theta_1, \theta_3) = f(\theta_1, \theta_2) f(\theta_2, \theta_3)}$$

→ we want to look for a function which has this property.

Your mathematician friend will say that solution is

$$f(\theta_1, \theta_2) = \frac{T(\theta_1)}{T(\theta_2)}$$

$T(\theta_i)$ is some function of θ_i .

So, we conclude :

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2}$$

where

$$T_1 = T(\theta_1)$$

$$T_2 = T(\theta_2)$$

$$\boxed{\eta_{\text{rev}} = 1 - \frac{T_2}{T_1}}$$

efficiency of Reversible machine

T : Thermodynamic Temperature
~~because it must be~~

Thermodynamic Temperature must be a function of θ .

(178)

reversible

→ to find the function, we look at efficiency of a heat engine, η , and this allows us to introduce the Thermodynamic Temperature; Namely the relationship between something universal (T) and something which is an empirical thing (θ) which can be different for different substances.

Carnot's Theorem:

$$\eta \leq \eta_r$$

$$\Rightarrow 1 - \frac{Q_2}{Q_1} \leq 1 - \frac{T_2}{T_1} \Rightarrow \frac{Q_1}{T_1} - \frac{Q_2}{T_2} \leq 0$$

$\Rightarrow \frac{Q_1}{T_1} - \frac{Q_2}{T_2} \leq 0$ if you introduce a convention that a heat entering the engine is positive; & a heat leaving the engine is negative

then we can write more generally

$$\sum_i \frac{Q_i}{T_i} \leq 0$$

for closed cycle,

$$\oint \frac{\delta Q}{T} \leq 0$$

Equality is only for Reversible.

for reversible $\oint \frac{\delta Q}{T} = 0$ so; The integrand must be a total differential.

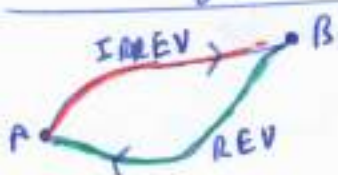
→ This motivates our definition of entropy as total differential.

DEFINITION Entropy S as

$$dS = \frac{\delta Q_r}{T}$$

$S \Rightarrow$ Function of State

(that will not depend on the path on which you go)



since S is function of state; not of the path; so define it through the Reversible path.

(Pg 9)

$$\oint \frac{\delta Q}{T} \leq 0 \Rightarrow \int_A^B \frac{\delta Q}{T} \leq \int_A^B \frac{\delta Q_{rev}}{T}$$

$$\Rightarrow \int_A^B \frac{\delta Q}{T} \leq \int_A^B dS = S_B - S_A$$

$$\Rightarrow \int_{A \gamma}^B \frac{\delta Q}{T} \leq S_B - S_A$$

γ is some path in parameter space.

→ differential version... $dS \geq \frac{\delta Q}{T}$

In particular; for thermally isolated system $\delta Q = 0$

$$\Rightarrow dS \geq 0$$

This defines sort of ...
... "ARROW OF TIME"

"Microscopic laws typically are reversible in time; it's the thermodynamics which tells you it is not".

Once we have entropy; we can write first law as

$$dU = T dS - P dV$$

$\uparrow$ δQ $\downarrow$ δW

Similarly; ~~definition~~ $C = \frac{\delta Q}{dT} = T \frac{dS}{dT}$

Third Law The Entropy of an equilibrium system at $T=0$ is exactly equal to zero.

$S=0$ at $T=0$.
on an equilibrium system.

~~$S(T, X) = \int_0^T \frac{C(X)}{T} dT + S(0, X)$~~

$$S(T, X) = \int_0^T \frac{C(X)}{T} dT + S(0, X)$$

$$S(0, X) = 0$$

X stands for different parameters & system.

by Third law; we say $S(0, X) = S(0)$ is independent of system (because X was something which was ~~distinct~~ distinguishing the system)

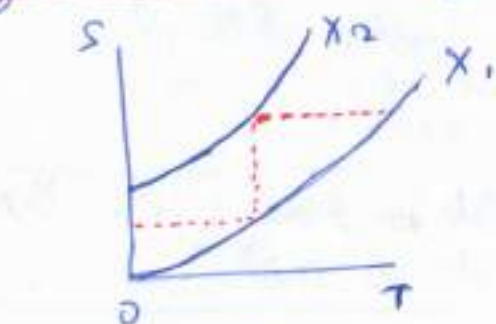
Since $S(0)$ is independent of systems, we can set it to be zero. $S(0) = 0$ ~~set~~ (pg 10)

$$S(T, x) = \int_0^T \frac{C(x)}{T} dT \quad \text{completely fixes } S.$$

Fixes $S(T)$ completely ie.

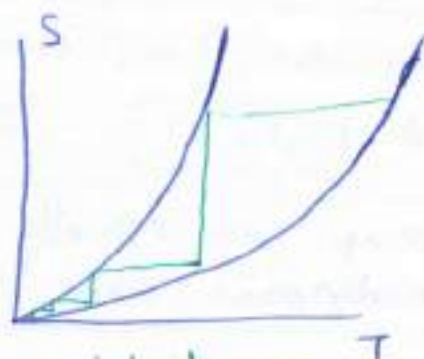
"You can think of 3rd law something which fixes entropy completely".

As a consequence, it is impossible to cool the system to $T=0$ by finite number of ~~states~~ steps.



(if $S(0) \neq 0$)

However



infinite no. of steps needed to reach zero

Thermodynamic potentials are given by MAGIC SQUARE.



Most interesting for statistical physics is free energy is

$$F = U - TS$$

$$dF = -SdT - PdV$$

Thermodynamic Potentials

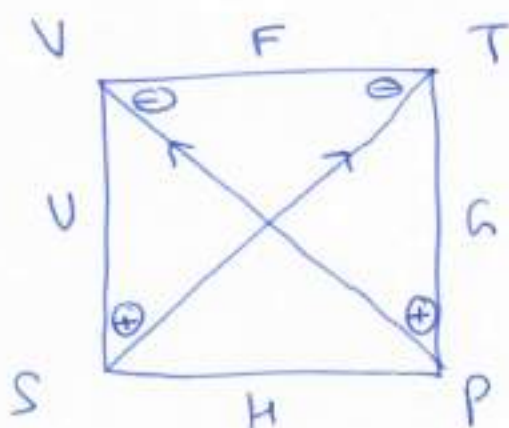
Thermodynamic potentials "determine the equilibrium state of a system under various constraints". Each potential is a function of state and has proper independent variables.

When its proper variables are held constant, stable equilibrium corresponds to the minimum of the potential.

Potential	Expression	Variables	Differential Form
Internal Energy	U	S, V	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	S, P	$dH = TdS + VdP$
Free Energy	$F = U - TS$	T, V	$dF = -SdT - PdV$
Gibbs Energy	$G = U - TS + PV$	T, P	$dG = -SdT + VdP$

pg 11

Magic Square



① TD potentials are total differentials when expressed in these variables.

$$dU = TdS - PdV$$

(sign $\oplus \ominus$)
near potential.

[$\nearrow \nwarrow$ any
carry the
same info]

② Legendre Transform

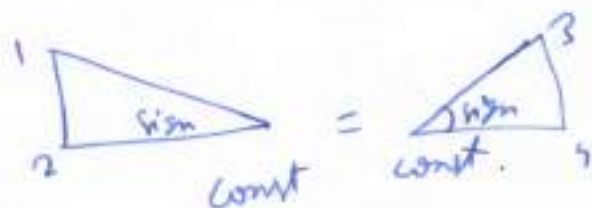
$$U(V, S) \rightarrow F(V, T)$$

$$F = U - TS, \quad T = \frac{\partial U}{\partial S}$$

(sign $\oplus \ominus$)
near potential.

③ Maxwell's Relations

$$\left(\frac{\partial V}{\partial S}\right)_P = \left(\frac{\partial T}{\partial P}\right)_S$$



proof $dH = TdS + VdP$

$$\therefore \frac{\partial^2 H}{\partial S \partial P} = \frac{\partial T}{\partial P} \Big|_S = \frac{\partial V}{\partial S} \Big|_P$$

1. $1000 \times 1000 \times 1000$

2. $1000 \times 1000 \times 1000$

3. $1000 \times 1000 \times 1000$

4. $1000 \times 1000 \times 1000$

5. $1000 \times 1000 \times 1000$

6. $1000 \times 1000 \times 1000$

7. $1000 \times 1000 \times 1000$

8. $1000 \times 1000 \times 1000$

9. $1000 \times 1000 \times 1000$

10. $1000 \times 1000 \times 1000$

11. $1000 \times 1000 \times 1000$

12. $1000 \times 1000 \times 1000$

13. $1000 \times 1000 \times 1000$

14. $1000 \times 1000 \times 1000$

15. $1000 \times 1000 \times 1000$

16. $1000 \times 1000 \times 1000$

17. $1000 \times 1000 \times 1000$

18. $1000 \times 1000 \times 1000$

19. $1000 \times 1000 \times 1000$

20. $1000 \times 1000 \times 1000$

Lecture 2: Basics of Statistical Physics; Liouville's Equation, Gibbs Distributions and Ensembles.

(b) Basics of Statistical Physics

(i) Quantum v/s Classical

In Quantum System a Microstate is described by Pure State which lives in a Hilbert Space

Pure State $\in$ Hilbert Space.

A Macrostate $\Rightarrow$ Density Matrix ρ

$$\rho = \sum_n p_n |\psi_n\rangle \langle \psi_n|$$

Probability that corresponding state $|\psi_n\rangle$ happens.
 $\rightarrow$ It describes ensemble of pure state.

~~ρ describes~~ ρ describes a Statistical Ensemble of Quantum States.

• $\text{Tr } \rho = 1$ (ρ is appropriately normalized)

• Time Evolution $\frac{\partial \rho}{\partial t} - \frac{1}{i\hbar} [H, \rho] = 0$; VON-NEUMANN EQUATION

ρ is an operator.

Observable is an operator A

$\therefore$ Expectation Value $a = \langle A \rangle = \text{Tr}(\rho A)$

ESTIMATE for Validity of Classical Physics

given a gas; we ask momenta of molecule; and what ~~are~~ is the volume each molecule has.

Thermal Fluctuations can be estimated as follows.

$$\frac{(\Delta p)^2}{2m} \approx k \cdot T, \quad \Delta x = \left(\frac{V}{N}\right)^{1/3}$$

$\rightarrow$ is like $\sqrt[3]{V}$ root of volume per molecule.

& we require $\Delta x \Delta p \gg \hbar$; then we are in classical regime

$$\left(\frac{V}{N}\right)^{1/3} \sqrt{2mk \cdot T} \gg \hbar$$

(3/5)

$$n \equiv \frac{N}{V}$$

$\Rightarrow$

$$kT \gg m \frac{\hbar^2}{2m}$$

This is ~~also~~ condition for our Classical physics to be valid.

* Classical physics is OK if

- T is very high
- n is very small.

In Classical Physics

- A Microstate describing N particles of our gas corresponds to a state in $\mathbb{R}^{6N}$ phase space. Time Evolution of this point in phase space is given by Hamilton's Equation.

- A Macrostate is described by Probability distribution W (it will be a function of phase space) on phase space.

With measure $\mu = W(\vec{q}_1, \vec{q}_2, \dots, \vec{q}_N, \vec{p}_1, \dots, \vec{p}_N) d\tau_N$

$$d\tau_N = \frac{1}{N! (2\pi \hbar)^{3N}} d^3p_1 d^3p_2 \dots d^3p_N d^3q_1 \dots d^3q_N$$

integration measure on phase space

This factor can be thought of as convenient chosen pre-factor so that integrating W over whole phase is 1 (for normalization)

$$\int W d\tau_N = 1$$

(ie: we are saying that the probability of finding the dot somewhere in phase space is 1)

$2\pi\hbar$ comes from Quantum Mechanics.

Statistical Ensemble $\approx$ Having a large number of boxes with a given ~~macrostate~~ macrostate. If randomly open one box, we find a microstate with probability μ .

Observable is a function on the phase space

Average Value of $A = \langle A \rangle = \int A(p, q) w(p, q, t) d\tau_N$

	QUANTUM	CLASSICAL
Stage	Hilbert Space $\mathcal{H}$	Phase Space Σ_N
Calculation	$\text{Tr}(\cdot)$	$\int \cdot d\tau_N$
Statistical Ensemble	Density Matrix: ρ Normalized $\text{Tr} \rho = 1$	Probability distribution w $\int w d\tau_N = 1$
Observable	Operator A	Phase Space Function $A(p, q)$
Average	$\langle A \rangle = \text{Tr}(\rho A)$	$\langle A \rangle = \int A(p, q) w(p, q, t) d\tau_N$

The corresponding Classical Limit of Neumann Equation is Liouville's Equation.

iii) LIOUVILLE'S EQUATION

Von-Neumann : QM is Unitary

Classical limit: $\frac{1}{i\hbar} [\hat{a}, \hat{p}] = 1 \longleftrightarrow \{q, p\} = 1$

$$[\hat{f}, \hat{g}] \longleftrightarrow i\hbar \{f, g\}$$

$$\frac{\partial w}{\partial t} = \frac{1}{i\hbar} i\hbar \{H, w\}$$

$$\Rightarrow \frac{\partial w}{\partial t} + \{w, H\} = 0$$

$$\neq \frac{dw}{dt} = 0$$

$$\Rightarrow \boxed{\frac{\partial w}{\partial t} = \{H, w\}}$$

Liouville's Equation,

$$\left[\frac{dW}{dt} = \frac{\partial W}{\partial t} + \{W, H\} = 0 \right] \text{ Liouville's Equation. } \quad \textcircled{\text{Pg 16}}$$

* Consequence of Statistical Physics

since $\frac{dW}{dt} = 0 \Rightarrow$ Thus W is an integral of motion along trajectory.

• we can solve by taking $W = \text{constant}$ (say)

If you take probability distribution as constant; in other words that you are finding ~~microstates~~ microstates with equal probability;

Then this is a statistical ensemble which you can use and is called ~~MICRO CANONICAL MICRO~~ MICRO CANONICAL ENSEMBLE.

(pain for calculation)

• $W = W(\text{Integrals of motion})$

$$= W(E, \vec{p}, \vec{M})$$

; comes from symmetry of space...

$\vec{p}$ linear momentum

$\vec{M}$ angular momentum (not interesting because he can go to rotating frame)

... not very interesting because he can go to comoving frame.

$$= W(E)$$

$$W = W(E)$$

Two Systems

$$W_{12} = W_1 W_2$$

(if they are non-interacting: then it is just product)

$$\Rightarrow \log W_{12} = \log W_1 + \log W_2$$

$E \Rightarrow$ Total energy of the ~~system~~ system.

(pg 17)

$E_1 \Rightarrow$ Energy of system 1

$$\log W_{12} = \log W_1 + \log W_2$$

$$\Rightarrow \log W_{12}(E) = \log(W_1(E_1)) + \log(W_2(E_2))$$

looks like sum of energies.

$$\beta E = \beta E_1 + \beta E_2$$

$$\log W_{12} \quad \log W_1 \quad \log W_2$$

(looks like sum of energy)

$$\Rightarrow \log W = \beta E \Rightarrow \boxed{W = e^{\beta E}} \quad \text{First consequence}$$

Gibbs Distribution for Canonical Ensemble.

Second Consequence of Liouville's Equation.

is Liouville's Theorem

$$\int W d\gamma_N = 1 \Rightarrow \frac{d}{dt} \left(\int W d\gamma_N = 1 \right)$$

$$\Rightarrow 0 = \int \left(\frac{dW}{dt} d\gamma_N + W \frac{d}{dt} (d\gamma_N) \right)$$

$\hookrightarrow$ This is equal to zero

$$\Rightarrow 0 = \int W \frac{d}{dt} (d\gamma_N) \Rightarrow \boxed{\frac{d}{dt} (d\gamma_N) = 0}$$

Behind ~~the~~ Hamilton vector field; we can think there is ~~unitary~~ Unitary Origin

So volume element does not change as we evolve in time.
CAT PICTURE.

Here we have been deriving everything ~~the~~ thing as a consequence of QM being unitary.

Definition: Statistical (Von Neumann) Entropy is defined as $S = -k \text{Tr}(\rho \log \rho)$.

set $k=1 \Rightarrow \boxed{S = -\text{Tr}(\rho \log \rho)}$

Assume system has some observables A_i have expectation values a_i : $a_i = \langle A \rangle = \text{Tr}(\rho A_i)$; $\text{Tr}(\rho) = 1$

Principle of Maximal Mess: The density matrix ρ representing the equilibrium state of a microsystem is such that it maximizes the Statistical Entropy, provided the ~~macro~~ (Macroscopical) Constraints.

This leads to Variational principle.

So, we want to Extremize functional I given by $I = S$ subject to same constraints.

so we add lagrange multipliers.

$$I = S - \sum_i \lambda_i (\text{Tr}(\rho A_i) - a_i) - \lambda_0 (\text{Tr} \rho - 1)$$

$$\boxed{I = S - \sum_i \lambda_i (\text{Tr}(\rho A_i) - a_i) - \lambda_0 (\text{Tr} \rho - 1)}$$

→ We want to maximize this functional.

The thing to vary in this functional is density matrix ρ .

$$\delta I = \delta S - \sum_i \lambda_i (\text{Tr}(A_i \delta \rho)) - \lambda_0 \text{Tr}(\delta \rho)$$

$$\delta S = -\text{Tr}(\delta \rho \log \rho) - \text{Tr}(\delta \rho) = -\text{Tr}(\rho \log \rho + \delta \rho)$$

$$\Rightarrow \delta I = \text{Tr} [(-\log \rho - 1 - \lambda_i A_i - \lambda_0) \delta \rho] \quad (\text{pg 19})$$

we know $\delta I = 0 \quad \therefore B := (-\log \rho - 1 - \lambda_i A_i - \lambda_0)$

$$\Rightarrow \delta I = \text{Tr} (B \delta \rho) = 0$$

true for any $\delta \rho$.

However $\delta \rho$ is not any matrix; if it was any matrix then we could conclude $B = 0$.

Now However ρ is Hermitial matrix

$\hookrightarrow$ there is some subtle things to prove but still the conclusion is same; $B = 0$.

$$\Rightarrow \log \rho = - \sum_i \lambda_i A_i - (1 + \lambda_0)$$

$$\Rightarrow \boxed{\rho = e^{-(1+\lambda_0)} \cdot e^{-\sum_i \lambda_i A_i}}$$

Now let's impose constraint.

$$\text{Tr} \rho = \text{Tr} (e^{-(1+\lambda_0)} e^{-\sum_i \lambda_i A_i})$$

$$1 = e^{-(1+\lambda_0)} \text{Tr} (e^{-\sum_i \lambda_i A_i})$$

$$\Rightarrow 1 = e^{-(1+\lambda_0)} \cancel{Z} \quad \text{where } Z = \text{Tr} (e^{-\sum_i \lambda_i A_i})$$

$$\Rightarrow \boxed{e^{-(1+\lambda_0)} = \frac{1}{Z}}$$

$$\cancel{Z} = \text{Tr} (e^{-\sum_i \lambda_i A_i})$$

$$\boxed{\rho = \frac{1}{Z} e^{-\sum_i \lambda_i A_i}}$$

This is the density matrix consistent with all the constraints, and which maximizes the corresponding function.

$\hookrightarrow$ Called Generalized Gibbs Distribution.

$$Z = \text{Tr}(e^{-\sum_i \lambda_i A_i}) \quad \text{called Partition Function.}$$

pg 20

Immediate Properties

Z does not depend on a_i
but they are function of λ_i

$$a_m = - \frac{\partial}{\partial \lambda_m} \log Z$$

$$S = \sum \lambda_m a_m + \log Z$$

Homework.

$$\lambda_i = \frac{\partial S}{\partial a_i}$$

fast

Property 1 $-\frac{\partial}{\partial \lambda_i} \log Z = -\frac{1}{Z} \frac{\partial Z}{\partial \lambda_i} = -\frac{1}{Z} \text{Tr} \left(\frac{\partial}{\partial \lambda_i} e^{-\sum_j \lambda_j A_j} \right)$

$$= -\frac{1}{Z} \text{Tr}(-A_i e^{-\sum_j \lambda_j A_j})$$

$$\Rightarrow -\frac{\partial}{\partial \lambda_i} \log Z = \text{Tr} \left(A_i \frac{e^{-\sum_j \lambda_j A_j}}{Z} \right) = \text{Tr}(A_i \rho) = a_i$$

$$\Rightarrow \boxed{a_i = -\frac{\partial}{\partial \lambda_i} \log Z}$$

(iv) CANONICAL ENSEMBLE

The only observable is H : $U = \langle H \rangle = \text{Tr}(\rho H)$
(only energy can fluctuate)

(The only A_i have is H (Hamiltonian))

The corresponding ensemble is called Canonical Ensemble.

" " lagrange multiplier to H is $\lambda_H = \beta$
(rename to β)

Corresponding lagrange multiplier.

So, Gibbs distribution reduces to

$$\rho = \frac{1}{Z} e^{-\beta H} \quad \text{where } Z = \text{Tr}(e^{-\beta H})$$

we define $\delta W \equiv \text{Tr}(\rho dH)$
 $\delta Q \equiv \text{Tr}(H d\rho)$

so: $dU = d(\text{Tr} \rho H) = \text{Tr}(\rho dH) + \text{Tr}(H d\rho)$
 $dU = \text{Tr}(d\rho H) + \text{Tr}(\rho dH)$
 $= \delta Q + \delta W$

$\boxed{dU = \delta Q + \delta W}$ This is the first law provided we
 call $\delta W = \text{Tr}(\rho dH)$ work
 & $\delta Q = \text{Tr}(d\rho H)$ Heat.

Why? (we can motivate the definitions)

take $\rho = \sum p_m |m\rangle\langle m| \Rightarrow \delta \rho = \sum \delta p_m |m\rangle\langle m|$

take $H = \sum \epsilon_m |m\rangle\langle m|$
 (in same basis)

$\delta Q = \text{Tr}(H d\rho) = \text{Tr}(\sum \delta p_m |m\rangle\langle m| \sum \epsilon_n |n\rangle\langle n|)$

in orthogonal basis; $\langle m | n \rangle = \delta_{mn}$

$\Rightarrow \boxed{\delta Q = \sum_m \delta p_m \cdot \epsilon_m}$

Similarly

$\boxed{\delta W = \sum_m p_m \cdot \delta \epsilon_m}$

Now we can see why
 we should call one
 as Work & the other
 as Heat.

keeping the probability ~~distribution~~
 distribution fixed; we are changing

the energy level.

or It can easily be done by some conservative
 forces like Electric field.

Shift Energy levels, which can be
 done by conservative ~~force~~ fields.
 $\hookrightarrow$ so we better call it Work (Motivation)

$\delta Q \rightarrow$ we keep energy level fixed. (pg 22)
 $\hookrightarrow$ but we change probabilities
 $\dots$ we somehow redistribute probabilities.
 $\hookrightarrow$ because we change probabilities...
 $\dots$ because it is so dirty it is natural to
 call it heat (??) Hah...)

Now, to define Temperature & Entropy we need second law.

$$\delta S = -\delta \text{Tr}(\rho \log \rho) = -\text{Tr}(\delta \rho \log \rho + \delta \rho)$$

if $\text{Tr}(\rho) = 1 \Rightarrow \text{Tr}(\delta \rho) = 0$ (because of normalization of density matrix.)

$$\Rightarrow \boxed{\delta S = -\text{Tr}(\delta \rho \log \rho)}$$

$\therefore$ In canonical Ensemble.

$$\Rightarrow \delta S = -\text{Tr}(\delta \rho (-\beta H - \log Z))$$

$$\Rightarrow \delta S = +\text{Tr}(\delta \rho \beta H)$$

$$\text{Tr}(\delta \rho \log Z) = \log Z (\text{Tr}(\delta \rho))$$

$\downarrow$
Just a number
 $\rightarrow 0$
 $= 0$

$$\Rightarrow \boxed{\delta S = \beta \text{Tr}(\delta \rho H)}$$

$$\Rightarrow \boxed{\delta S = \beta \delta Q}$$

variational
of Statistical Entropy

a Lagrange multiplier.

Remind Clausius; $\delta S = \frac{\delta Q}{T}$

where T is
thermodynamic
temperature.

So, we have to identify

that the Lagrange multiplier β is given by $\frac{1}{T}$.

So 2nd law really provides us
so we have the total description of
Canonical Ensemble

$$\rho = \frac{e^{-\beta H}}{Z} \quad ; \quad \beta = \frac{1}{T}$$

We can make another definition,

$F \equiv -T \log Z$

 $\rightarrow$ FREE ENERGY

$\hookrightarrow$ It would be nice if this free energy is the free energy
we know from thermodynamics.

Is it?

looking at one of the properties $S = \sum \lambda_i a_i + \log Z$

$$S = \beta U + \log Z \quad \cancel{+ T \beta U} = T \beta U + T \log Z$$

$$\Rightarrow \cancel{T \beta U} = T \beta U + T \log Z$$

$$\cancel{+ T \beta U} \quad TS = T \beta U + T \log Z$$

$TS = U + T \log Z$

$-T \log Z = U - TS$

from the formula ; we find $F \equiv U - TS$ as in TD.

So; $-T \log Z$ is good definition of FREE ENERGY

GRAND CANONICAL

Its not only energy ; but no. of particles are also allowed
to fluctuate.

so; corresponding partition function is given by

$$Z_G = \text{Tr} (e^{-\beta H + \beta \mu N})$$

N is the operator for particle number.

μ is the chemical potential.

(Pg 24)

$$Z_G = \text{Tr} (e^{-\beta H + \beta \cdot \mu \cdot N})$$

Lecture 3: Thermodynamic Approach to Phase Transitions: Ideal Gas, Van der Waals Fluid, Reentrant P.T.

Recall The equilibrium state of a system subject to some constraints $\text{Tr}(P A_i) = a_i$, $\text{Tr}(P) = 1$

Maximizing Stat. Entropy $S = -\text{Tr}(P \log P)$

$$P = \frac{1}{Z} e^{-\sum_i \lambda_i A_i} \quad ; \quad Z = \text{Tr}(e^{-\sum_i \lambda_i A_i}) \quad \text{Gibbs Distribution}$$

(c) THERMODYNAMIC APPROACH TO PHASE TRANSITIONS

(i) Statistical Machinery at Work. ... Ideal Gas.

- Given Hamiltonian H which characterizes the system.
- First calculate partition function Z .

use canonical ensemble: $Z = \text{Tr}(e^{-\beta H})$; $\beta = \frac{1}{T}$

- Calculate Free Energy to know Z

$$\therefore F = -T \log Z$$

↑ (Thermodynamic potential (free energy))

From T.D., we know; $F = F(T, V, N)$

$$dF = -SdT - PdV + \mu dN.$$

$$S = - \left. \frac{\partial F}{\partial T} \right|_{V, N} ; \quad P = - \left. \frac{\partial F}{\partial V} \right|_{T, N} ; \quad \mu = \left. \frac{\partial F}{\partial N} \right|_{T, V}$$

(This is how we derive properties of system starting from Statistical physics)

Example 1) Ideal Gas

N non-interacting particles.

Each of them characterized by position & momenta

$$\vec{r}_i, \vec{p}_i, \quad i=1, \dots, N$$

Say all of them have same mass

(pg 26)

So, Hamiltonian is $H = \frac{p_i^2}{2m}$

$$Z = \text{Tr}(e^{-\beta H}) = \int d\vec{r}_1 \dots d\vec{r}_N d\vec{p}_1 \dots d\vec{p}_N e^{-\beta H}$$

Quantum Mechanical
Prescription

Classical prescription.

$$\int d\vec{r}_1 \dots d\vec{r}_N = (V)^N \quad V \neq \text{volume of gas.}$$

$$\Rightarrow Z = V^N \int d\vec{p}_1 e^{-\beta \frac{\vec{p}_1^2}{2m}} \cdot \int \dots \cdot \int \dots$$

$$\Rightarrow Z = V^N \prod_{i=1}^N \int d\vec{p}_i \cdot \exp\left(-\beta \frac{\vec{p}_i^2}{2m}\right)$$

→ gaussian integral.

$$\cancel{Z = V^N \prod_{i=1}^N \left(\sqrt{\frac{2\pi m}{\beta}} \right)} \Rightarrow Z = \left(\sqrt{\frac{2\pi m}{\beta}} \right)^N$$

(for each momentum on one x...)

In d dimensions:

$$d \dots \text{spacetime dimensions} \Rightarrow \int d\vec{p}_i \cdot \exp\left(-\beta \frac{\vec{p}_i^2}{2m}\right) = \left(\sqrt{\frac{2\pi m}{\beta}} \right)^d$$

$$Z = V^N \left(\frac{2\pi m}{\beta} \right)^{\frac{N(d-1)}{2}}$$

d = space-time dimension.

• Free Energy:

$$F = -T \log Z = -TN \log V - \frac{(d-1)TN}{2} \log(2\pi m T)$$

now; the most interesting thing is to get the equation of state for ideal gas.

$$P = - \left. \frac{\partial F}{\partial V} \right|_{T, N} = \frac{TN}{V} \Rightarrow$$

$$P = \frac{TN}{V}$$

$$PV = TN$$

$$RV = NRT$$

Eq 2.7

$$V = \frac{V}{N} \neq \text{specific volume}$$

$$PV = T$$

↳ Ideal Gas Law
(Equation of State)

Entropy:

$$S = - \left. \frac{\partial F}{\partial T} \right|_{V, N} = N \log V + \frac{(d-1)}{2} \cdot N \log(2\pi m T) + \frac{(d-1) \cdot N}{2} \cdot \frac{1}{2\pi m}$$

$$S = N \log V + \frac{(d-1)}{2} N \log(2\pi m T) + \frac{N \cdot (d-1)}{2}$$

Entropy

Internal Energy

$$U = F + TS = \frac{(d-1)}{2} NT \Rightarrow U = \frac{(d-1)}{2} NT$$

So; U only depends on T
(it is independent of volume V)

From where does the pressure comes from?

mainly we can think it comes from F; & F depends on U & S $\neq$ so; we can think there is some contribution of U to P $\neq$ so really the particles are hitting the ~~walls~~ walls. There might be some contribution from S.

$\therefore$ but it is quite puzzling that contribution from one of them is zero!!

$$P = - \frac{\partial F}{\partial V} = - \frac{\partial}{\partial V} (U - TS)$$

$\therefore$ U depends on T
not on V

$\Rightarrow$ so; only contribution in P is from entropy

$$\Rightarrow P = T \frac{\partial S}{\partial V} \text{ Only contribution from Entropy}$$

So; its really that the pressure comes from how you redistribute the particles in your room; rather than particles are actually hitting the walls. because they have some internal energy.

→ Origin of Pressure in Ideal gas is Entropic.

$$C_v = \left. \frac{\delta Q}{\delta T} \right|_v = T \cdot \left. \frac{\partial S}{\partial T} \right|_v = \frac{(d-1)N}{2}$$

$$C_p = T \left. \frac{\partial S}{\partial T} \right|_p = \frac{(d+1)}{2} N$$

People are obsessed that
(Maha...)

MYER'S LAW

$$K = \frac{C_p}{C_v} = \frac{d+1}{d-1}$$

$$d=4 ; K = \frac{5}{3}$$

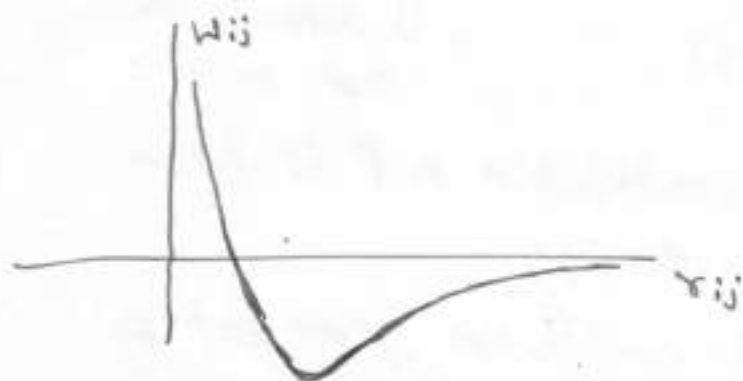
(ii) VANDERWAALS EQUATION

If you want an interacting gas, then it is quite complicated: Interacting Gas

$$H = \sum_i \frac{p_i^2}{2m} + \sum_{i < j} W_{ij}$$

$$W_{ij}(r_{ij}) : r_{ij} = |\vec{r}_i - \vec{r}_j|$$

example we can use the model: $W_{ij} = \frac{A}{r_{ij}^{12}} - \frac{B}{r_{ij}^6}$
LENNARD-JONES Potential.



... doing this we recover

$$P = nT(1 + nB_2(T) + n^2 B_3(T) + \dots)$$

$$n = \frac{N}{V}$$

Van Der Waals Equation

$$\left(P + \frac{a}{V^2}\right)(V-b) = T$$

$a > 0$; measures attraction between particles.

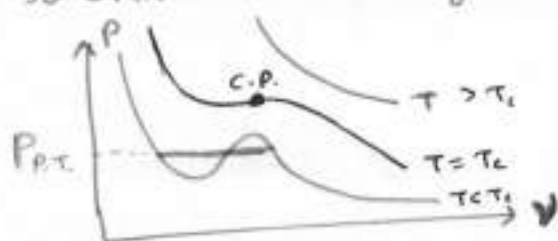
P is effectively bigger due to attraction of particle.

b : related to excluded volume of particles.

$V-b \Rightarrow$ effective volume.

Isotherms in P.V. Diagram.

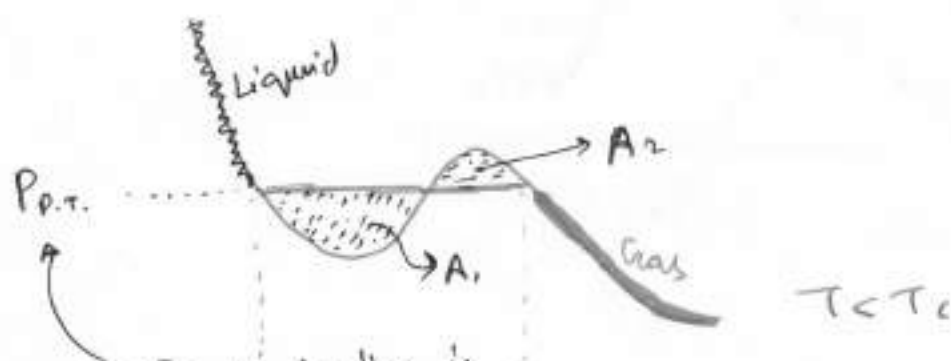
c.p. $\equiv$ Critical Point



$T_c \equiv T_{\text{critical}}$.

We don't see a gas doing this; but see a phase transition ~~at~~ for $T < T_c$

P.P.T. $\Rightarrow$ P Phase Transition



The construction is due to Maxwell; called Maxwell's Equal Area Law.

$$A_1 = A_2$$

$$\oint V dP = 0$$

$$\iff$$

$$P_{P.T.} (V_g - V_l) = \int_{V_l}^{V_g} P dV$$

Maxwell Constructions

(iii) Phase Transitions (Thermodynamic Perspective)

pg 30

Characterized by Gibbs Free Energy $G = G(P, T)$

Surface $G = G(P, T)$ is continuous; but may intersect

..... then we have Phase Transition.



This part is also possible but is not thermodynamically preferred.
 $P = \text{constant}$ slice.

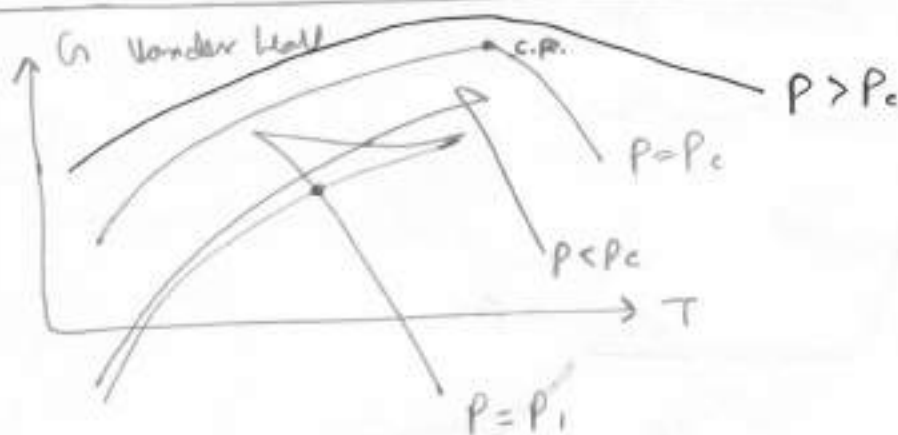
This is where phase transition happens.

Global Minimum of G corresponds to the preferred phase.

First Order Phase Transition

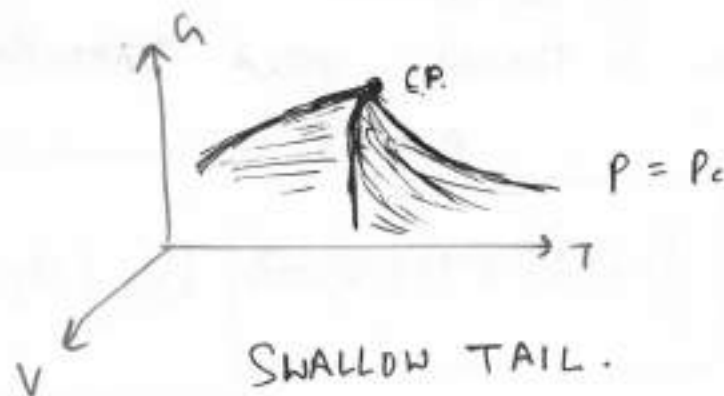
G_{Min} Continuous, G'_{Min} (derivative of G) is not continuous.

Second order P.T.: G_{Min} continuous, G'_{Min} continuous, G''_{Min} is not.

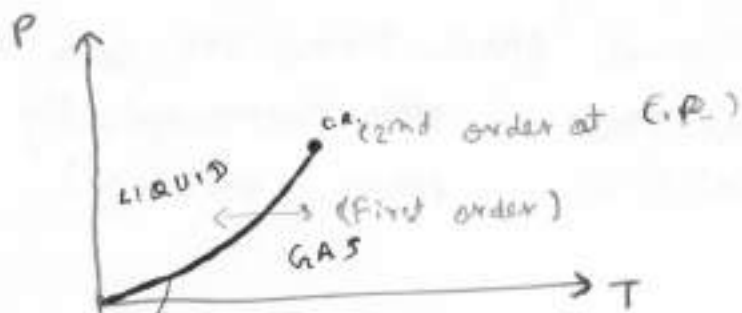


~~3-D picture~~

~~3-D picture~~



Phase Diagram.



→ here two phases
co-exist

Co-existence line of the phases. Shape determined from Clausius-Clapeyron Equation.

Applies to
1st order
Phase Transition.

$$\left. \frac{dP}{dT} \right|_{\text{Co-existence}} = \frac{\Delta S}{\Delta V}$$

$$= \frac{S_g - S_l}{V_g - V_l}$$

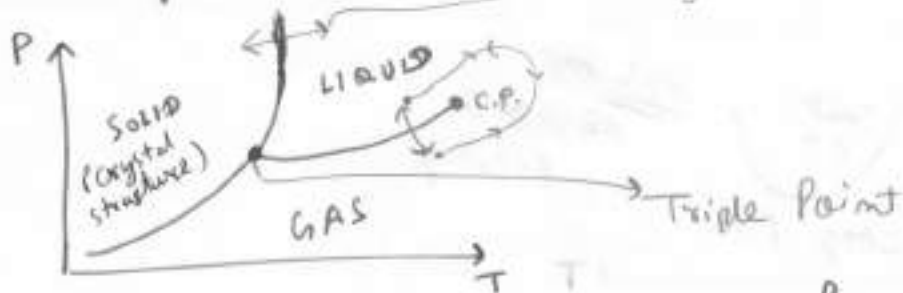
For 2nd order; the analogue of Clausius-Clapeyron equation is ENRNFEST Equation.

→ they have Universality.

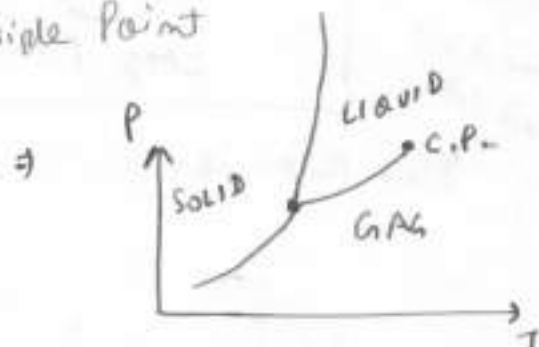
Other examples of Phase Transitions.

- Solid-Liquid-Gas

→ always Phase Transition



Liquid & gas are not
so very different because
we can go around C.P.
without going through phase
transition



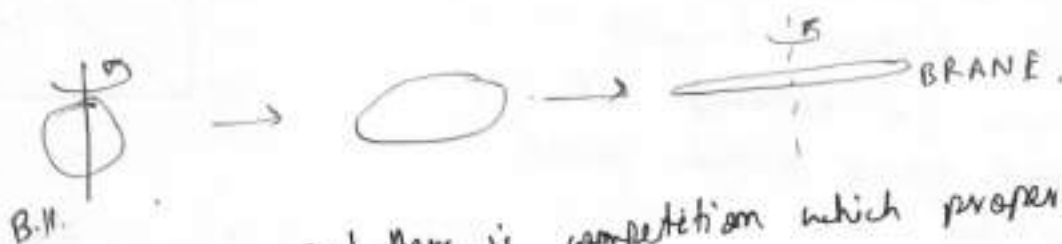
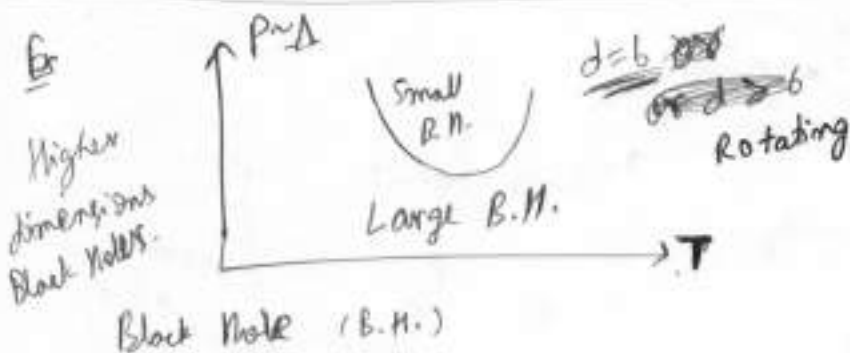
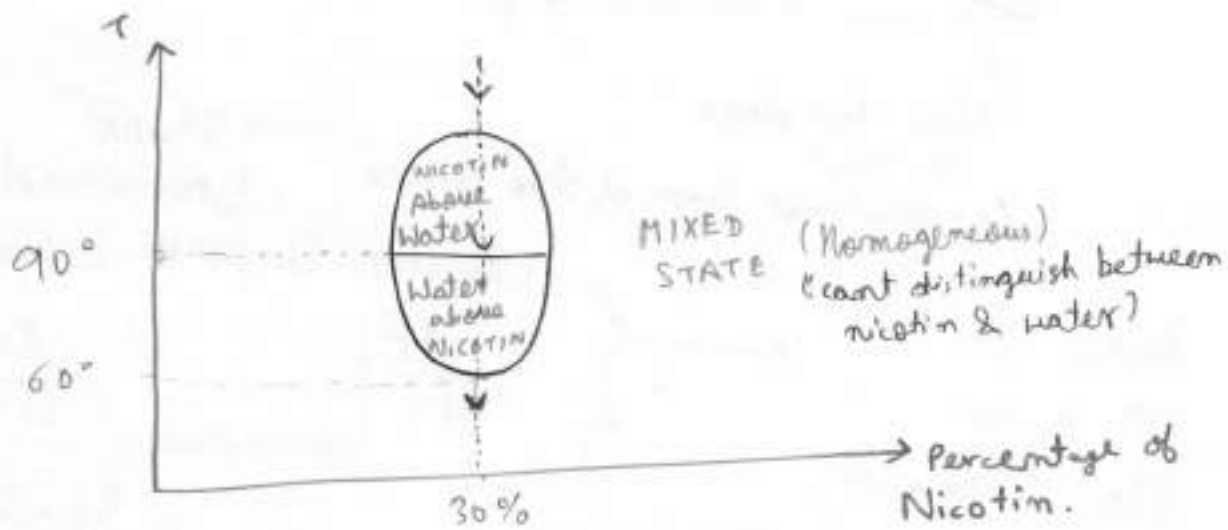
→ But solid is very different from them.

~~2nd order~~

★ RE-ENTRANT Phase Transition.

(Pg 32)

If monotonic variation of one thermodynamic variable results in several phase transitions such that the initial and final state is ~~the~~ macroscopically ~~same~~ similar to initial one. (1904 HUDSON)



and there is competition which property is winning.

* Black Hole property of D.H. (or) Brane property of B.H.

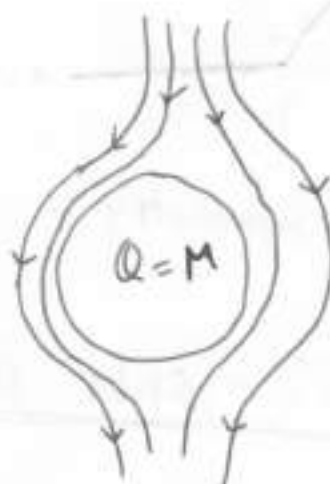
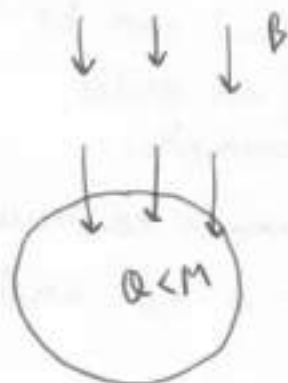
Gravitational Meissner Effect.

Known from 1960s.

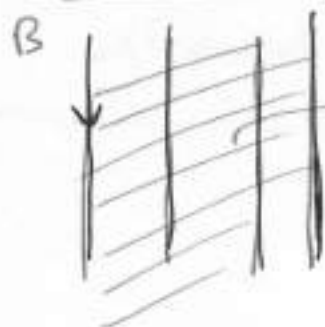
- External B.H. ($Q = M$)

(Also for maximally rotating B.H.)

Magnetic field gets expelled from B.H.

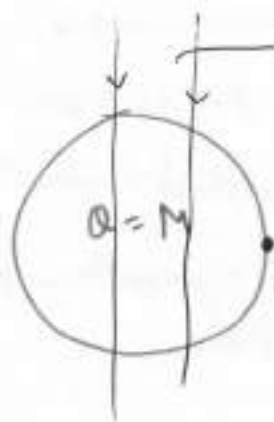


• Cosmic String



ϕ (a scalar field) holds together magnetic field

↳ so that cosmic string does not stretch too far & has finite size of cosmic string.

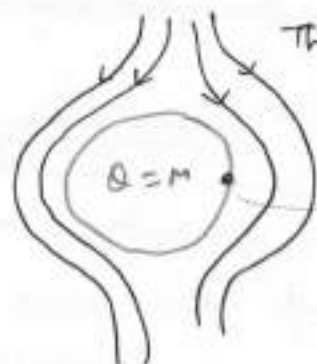


Thin Cosmic String (penetrates even extremal B.H.)

$\phi(\pi/2)$ is order parameter

∴ If cosmic string becomes wider than a critical radius

∴ Then it goes around.



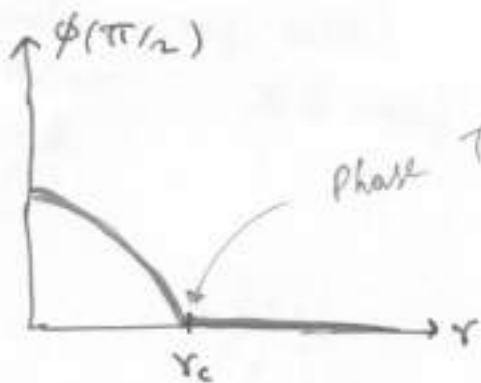
Thick cosmic string.

$\phi(\pi/2) = 0$

So; There is a phase transition. There is a critical radius r_c

$r < r_c \Rightarrow$ penetrate
 $r > r_c \Rightarrow$ expell.

(pg 34)



$\phi(\pi/2)$ can be thought of as order parameter.

Spherical Symmetry is here

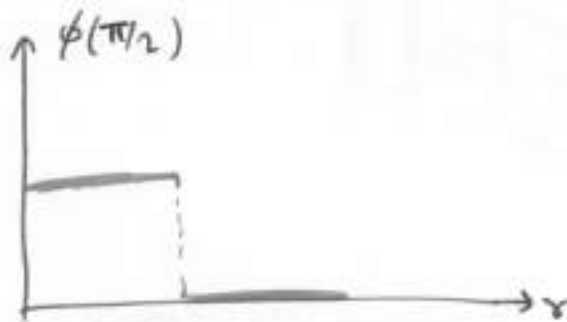
It changes continuously but it is not analytic

* 2nd - order phase transition here.

* Just magnetic field $\Rightarrow$ no phase transition.

* Magnetic field + scalar $\Rightarrow$ we see phase transition

$M = J$



1st order ~~phase~~ phase transition.

(order parameter jumps at the critical radius)

~~(not spherically symm)~~

Spherical symmetry is not here.

~~more~~ ~~more~~ ~~symt~~

more symmetry in system you have
 $\hookrightarrow$ you have nicer phase transition;
 meaning higher order phase transition)

Lecture 4: Ising Model, Order Parameter, Mean Field Theory, Critical Exponent Beta.

2) Microscopic Theory of Phase Transition.

a) Introduction

Experimental fact: Matter organizes into phases characterized by different macroscopic properties.

(still described by same Hamiltonian)

examples Solid / Liquid / Gas:

Which macroscopic property characterize each state?

• Solid has "Rigidity to shear".

• Liquid has well defined surface, but not rigidity.

• Gas does not even has well defined surface.

example Ferromagnetic / Paramagnetic:

• Ferromagnetic: has macroscopic magnetization.

• Paramagnet: does not.

example fluid / Super fluid ~~(conductor / superconductor)~~

example Superconductor / Normal matter.

Superconductor has zero resistance (in the same way Superfluid has zero viscosity); and expels magnetic field. known as Meissner effect.

"The phases however have to emerge from one Hamiltonian." ... same Hamiltonian works.

Different ~~phas~~ macroscopic properties are consequence of ordering.

• Spins in ferromagnet; atoms in crystal.. solid..

Canonical example: ISING MODEL

(simple mathematical model to model ferromagnet & paramagnetic phase)

The model was first developed by LENZ (1920), but ISING was the one first gave the solution in 1D (1925) ; hence called ISING MODEL.

pg 36

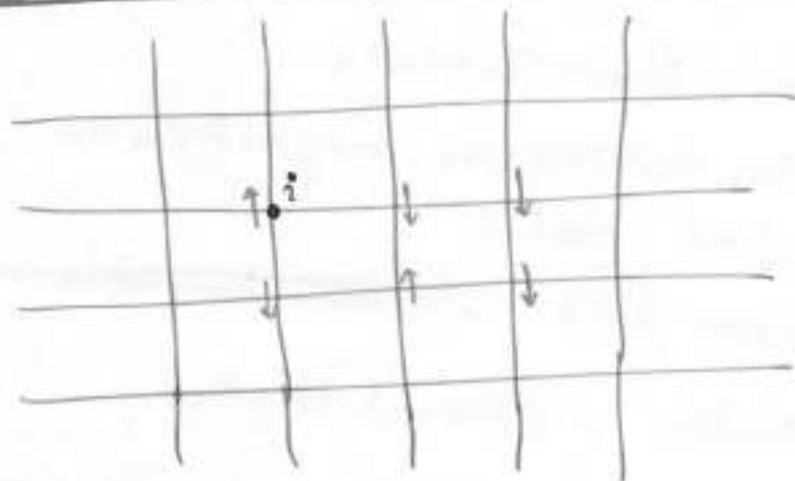
ONSAGER (1944) solution in 2d.

nobody knows how to solve in 3d. ... ongoing problem

- 4d & higher it is simple ; and the solution is given by Mean Field Theory approximation is exact.

As you increase dimensions ; effect of fluctuations become small ... & you get answers.

Model for FERROMAGNET



d-Dimensional lattice.

at each point of lattice you can either have spin up or down.

Spins are discrete variables. σ_i

i denotes where we are in the lattice.

$$\sigma_i = \pm 1$$

$$H = -\frac{1}{2} \sum_{i,j} J_{ij} \sigma_i \sigma_j$$

(you try to align spins in same direction)

we have two spin interaction & interaction strength is described by the coupling constant J_{ij} .

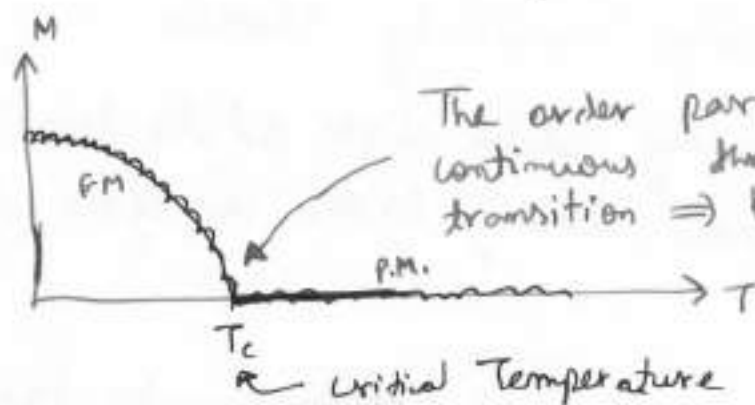
(we also see, the Ising model belongs to the same class as transition for water)

- 2 Phases : F.M. & P.M.
(Ferromagnetic) (Paramagnetic)

(pg 37)

distinguished by Order Parameter, which here is Magnetization which is given by

$$M = \langle \sigma_i \rangle = \begin{cases} 0 & \text{P.M.} \\ \neq 0 & \text{F.M.} \end{cases}$$

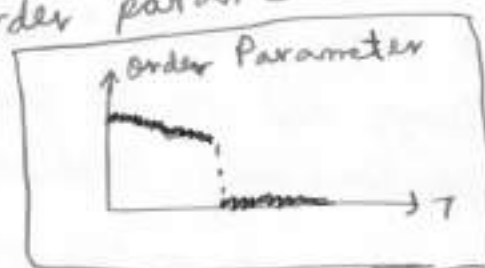


The order parameter M is continuous through the phase transition $\Rightarrow$ hence we call it Continuous Phase Transition.

Non analytic behavior at T_c .

NOTE: 1st order phase transition;

In the language of Order parameter there will be a jump.



When you differentiate Gibbs energy you get order ~~parameter~~

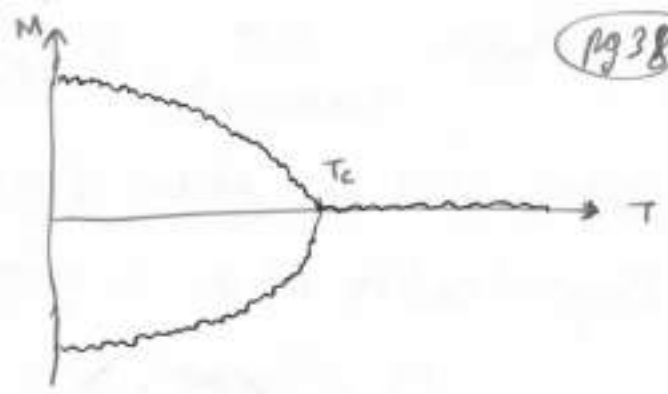
Parameter; so for first order P.T. there will be a jump in order parameter during phase transition because the first derivative of Gibbs energy is discontinuous

A beautiful symmetry in Hamiltonian

H is invariant under $\sigma_i \rightarrow -\sigma_i$

Z_2 ~~symmetry~~ symmetry.

~~actually we have~~
actually we have other
solution: pointing down.
(the negative part of
 M)



Somehow the system chooses one of the state as we
reduce the temperature: either pointing up or down
; and is basically randomly chosen.

So: the symmetry which exists at the level of
Hamiltonian is spontaneously broken as you decrease
the temperature.

$\therefore$ FM phase spontaneously ~~breaks~~ breaks this
symmetry. (Randomly choose up or down)

$\leadsto$ More generally; the existence of order parameter is
always tied to the fact that some symmetry is
spontaneously broken.

Existence of order parameter is tied to
spontaneous symmetry breaking.

• Exercise Add magnetic field to this system B ;
and consider finite system with N sites.

Magnetization $M = \frac{1}{N} \sum_{i=1}^N \sigma_i$

find probability of magnetization up & down.

$\therefore$ lets find the ratio : $m = \frac{P_{M>0}}{P_{M<0}}$

$$H = -\frac{1}{2} \sum_{i,j} J_{ij} \sigma_i \sigma_j - B \sum_i \sigma_i$$

(1339)

$$m = \frac{P_{m>0}}{P_{m<0}} = \frac{e^{-NMB/K}}{e^{+NMB/K}} \Rightarrow \boxed{m = e^{-2NMB/T}}$$

$M = |m|$

Limits ① N -Finite and set $B \rightarrow 0$

$m=1$; over long periods of time both up & down states equally probable.

then ~~average~~ average magnetization M has to be zero ; $M=0$

② Thermodynamic Limit $N \rightarrow \infty$ (Then we can take $B \rightarrow 0$)

Then $\boxed{m=0} \Rightarrow$ means one of them is preferred.

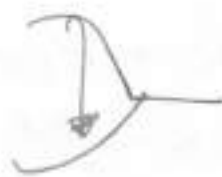
or ∞
(depending on value & direction of B)

$\therefore$ System remains in a state defined by previous B .

and M is now - non trivial.

If we have infinite no. of ~~states~~ sites; it will never happen that all of them swap.

Its infinitely improbable.



So; In thermodynamic limit we have phase transition & the new phase stays there forever (only in the thermodynamic limit)

General Conclusion Phases and Phase Transitions are defined sharply only for infinite systems.

$\therefore$ In practice ; if the system is sufficiently big; we can approx. it to infinite system.

(Macroscopic system (which in practice is never infinite) ... has long time scale for which they stay in that phase) (9.40)

(b) Mean Field Theory (MFT) (independent of dimensions)

• Idea: Replace all multi-body interactions with an effective one-body interaction with an average Mean Field. (molecular)

The idea is sort of equivalent to neglecting fluctuations.

• Effectively this amounts to ~~the~~ expanding Hamiltonian H to zeroth order in fluctuations.

(4D and higher; it is exact)

Application to ISING

$$\sigma_i = \sigma_i - \langle \sigma_i \rangle + \langle \sigma_i \rangle = \sigma_i - M + M$$

Then $\sigma_i \sigma_j = ((\sigma_i - M) + M)((\sigma_j - M) + M)$

$$= (\sigma_i - M)(\sigma_j - M) + M(\sigma_i + \sigma_j) - M^2$$

$\underbrace{(\sigma_i - M)(\sigma_j - M)}$ This is actually the fluctuation.

$\therefore$ if this difference is small

if ~~this~~

$(\sigma_i - M)(\sigma_j - M)$ is of $O(\epsilon^2)$ if fluctuation is small

$\hookrightarrow$ so neglect this.

||; if $(\sigma_i - M) \sim O(\epsilon)$

$$J_{ij} = J_{ij}$$

Then $H = -\frac{1}{2} \sum_{ij} J_{ij} (\sigma_i + \sigma_j) M + \frac{1}{2} \sum_{ij} J_{ij} M^2$

notation

$$J \equiv \sum_i J_{ij}$$

"Assuming Finite Range of Interaction".

$\hookrightarrow$ for a given j ; sum over all i .

In principle this should depend on j ; but you assume lattice is uniform; so each site is exactly same as any other site.

so; J is independent of j , because lattice is uniform.

Then $H = -MJ \sum_i \sigma_i + \frac{1}{2} NJM^2$

(pg 41)

Mean Field Theory, Ising Hamiltonian.

→ We replaced 2 point particle interaction with 1 spin interaction with mean field M .

We replace many-body interaction with one body interaction in some external field.

Applying Statistical physics machinery

$$Z = \text{Tr}(e^{-\beta H}) = \sum_{\sigma_1 = \pm 1} \dots \sum_{\sigma_N = \pm 1} e^{-\beta H}$$

→ sum over all possibilities (over all sites & all possible spin on those sites)

$$= \left(\prod_{i=1}^N \sum_{\sigma_i = \pm 1} \right) e^{-\beta H}$$

$$\Rightarrow Z = \left(\prod_{i=1}^N \sum_{\sigma_i = \pm 1} \right) e^{\beta MJ \sum_i \sigma_i - \frac{1}{2} \beta NJM^2}$$

$$= e^{-\frac{1}{2} \beta NJM^2} \prod_{i=1}^N \left(\sum_{\sigma_i = \pm 1} e^{\frac{\beta MJ \sigma_i}{T}} \right)$$

$$\Rightarrow Z = e^{-\frac{1}{2} \beta NJM^2} \prod_{i=1}^N 2 \cosh\left(\frac{MJ}{T}\right)$$

$$\Rightarrow Z = e^{-\frac{1}{2} \frac{NJM^2}{T}} \times \left(2 \cosh\left(\frac{MJ}{T}\right) \right)^N$$

Partition function.

$$F = -T \log Z = \frac{NJM^2}{2} - NT \log \left(2 \cosh\left(\frac{MJ}{T}\right) \right)$$

Free Energy

Equilibrium $\Rightarrow$ Minimize F w.r.t. M .

$$\frac{\partial F}{\partial M} = 0 \Rightarrow NJM - \frac{NT}{\cosh\left(\frac{MJ}{T}\right)} \sinh\left(\frac{MJ}{T}\right) \times \frac{J}{T} = 0$$

$$\Rightarrow NJM - NJ \tanh\left(\frac{MJ}{T}\right) = 0 \Rightarrow M = \tanh\left(\frac{MJ}{T}\right)$$

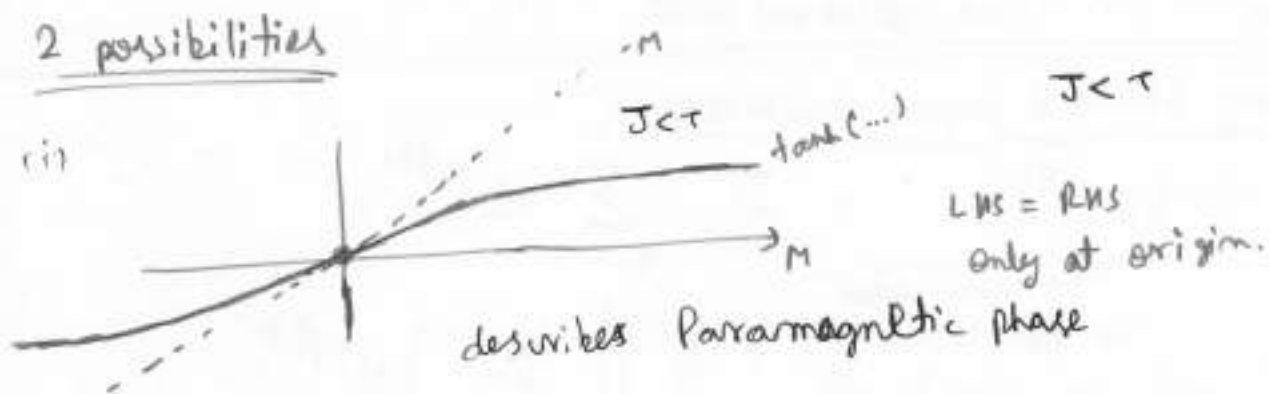
$M = \tanh\left(\frac{MJ}{T}\right) \Rightarrow$ Tells how M looks as a function of temperature: $M = M(T)$

pg 42

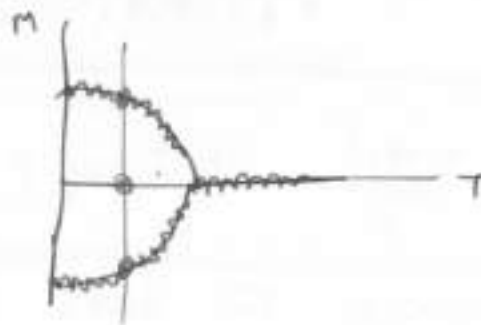
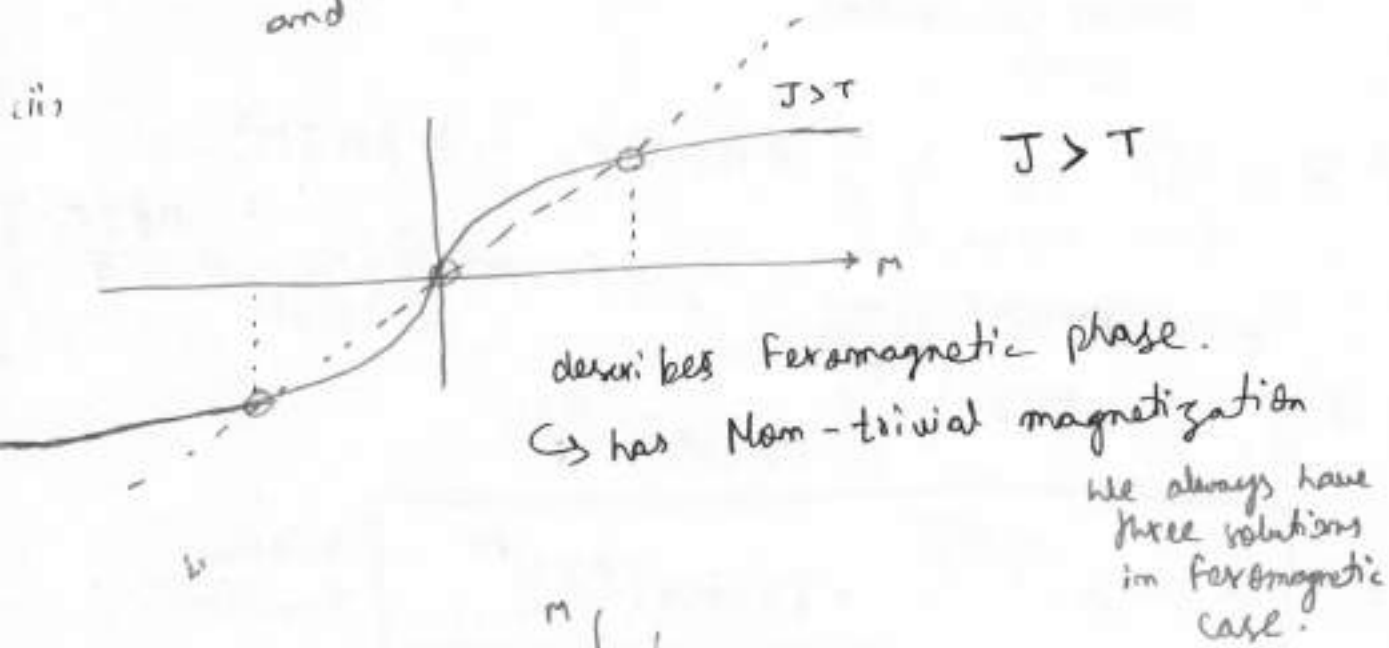


plotting $M = M(T)$ you recover this graph.

2 possibilities



and



$$\tanh(x) \approx x + \dots$$

$$\text{here} = \frac{MJ}{T}$$

if $\frac{J}{T} > 1 \Rightarrow$ Then ferro magnetic case.

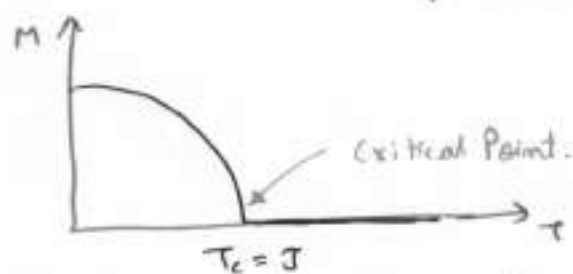
so: $T = T_c = J$

$$T_c = J$$

J describes how strongly the spins interact

(1943)

& critical temperature will depend on strength of interaction.



... continuous phase transition ... they are simple because they are Universal.

Universality = "Same Critical Exponents".

Critical exponents are just numbers which describes behavior of physical quantities near Critical Point.

Exponent β : ... describes behavior of Magnetization near critical point.

β is: describes behavior of Order Parameter near Critical point.

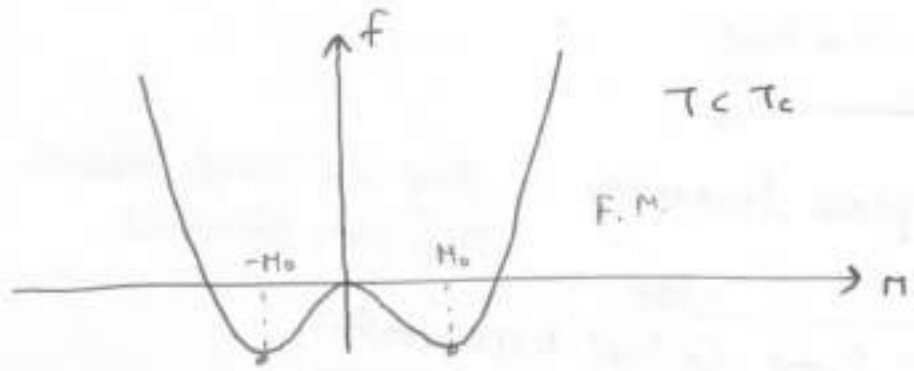
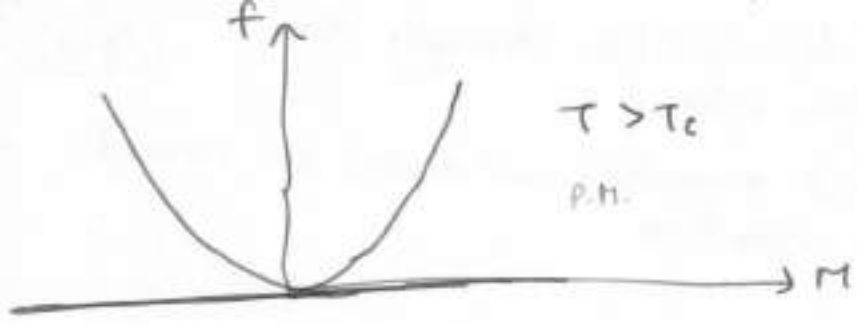
M is small near critical point, and hence we can do Taylor expansion there.

~~$F = \frac{F}{N}$~~ $f = \frac{F}{N}$ (free energy per lattice site)

$$\Rightarrow f = \frac{JM^2}{2} - T \log \left(2 \cosh \frac{MJ}{T} \right)$$

$$f \approx \frac{JM^2}{2} - \frac{1}{2} T \left(\frac{MJ}{T} \right)^2 + \frac{1}{12} T \left(\frac{MJ}{T} \right)^4 + O(M^6)$$

$$f \approx \frac{T_c}{2} \left(1 - \frac{T_c}{T} \right) M^2 + \frac{T_c^4}{12T^3} M^4 + \dots$$



$$\frac{\partial f}{\partial M} = 0 = T_c \left(1 - \frac{T_c}{T}\right) M + \frac{T_c^4}{3T^3} M^3$$

Set to zero
to find
 M_0 ...

$\Rightarrow$ ~~$M = 0$~~ ~~$M = \pm \sqrt{3(1 - \frac{T}{T_c})}$~~

$M = \pm \sqrt{3 \left(1 - \frac{T}{T_c}\right)}$

This is how M behaves in the vicinity of critical point.

$t = \frac{T - T_c}{T_c}$ define

↳ its a parameter which goes to zero as critical point is approached.

$M = \pm \sqrt{3t}$

$\therefore$ ~~$M \propto (-t)^\beta$~~

$M \propto (-t)^\beta$

$\sqrt{3}$ is not universal about continuous phase transition.
 $\rightarrow$ This will be universal.

$\beta \rightarrow$ critical exponent

$M \propto (-t)^\beta$

near critical point

$\beta = 1/2$

Critical exponent.
(prediction of MFT approximation)

(independent of d. dimensionality of ISING Model.)

Lecture 5: Critical Exponents, Landau's Theory, Hubbard-Stratonovich Transformation.

MFT Approximation $M = \langle \sigma_i \rangle$... order parameter.

$\epsilon = \sigma_i - M$... treated as small.

$$H = -MJ \sum_i \sigma_i + \frac{1}{2} M^2 NJ \quad \text{Ising MFT Hamiltonian.}$$

Interactions of spins with average field.

$$f = \frac{F}{N} = \frac{JM^2}{2} - T \ln(2 \cosh(\frac{MJ}{T}))$$

Minimum w.r.t. M : $\frac{\partial f}{\partial M} = 0 \Rightarrow M = \tanh\left(\frac{MJ}{T}\right)$

implicit relation for $M = M(T)$



Continuous phase transition
(because order parameter through phase transition is continuous)

Note that f is non-analytic as a function of T
 $\Rightarrow$ phase transition. (irrespective of d)

Compare 1d Ising: exact solution (tutorial)

$$f = -T \ln[2 \cosh(J/T)]$$

Analytic & so no phase transition.

MFT approximation breaks down here.

Reason as to why MFT breaks in 1d.
is that the fluctuations are huge.

1d: $\uparrow \uparrow \uparrow \rightarrow \text{flip} \Rightarrow \uparrow \uparrow \downarrow$

so: ϵ cannot be treated as small.

$M \rightarrow$ changes drastically if we flip one of them

2d



if you flip one of them
 M changes; but
 not changes much

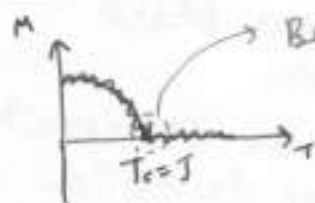
$\therefore$ Fluctuations are little
 bit less important.

So; ~~actually~~ actually in 2d ~~there~~ there is phase
 transition. (which is correctly predicted by MFT)

... of course critical exponent predicted by MFT is
 wrong.

$d=4$ upper critical dimension

($d > 4$: we get "exact" result by MFT)



Behavior of physical quantities near
 critical point is described by Critical
Exponents

$\rightarrow$ tells whether something
 diverges or diminishes to zero
 around critical point.

$$t = \frac{T - T_c}{T_c}$$

if $M \propto \frac{1}{(t)^{\text{positive power}}} \Rightarrow$ diverge

if $M \propto (t)^{\text{positive power}} \Rightarrow$ ~~converge~~ converge
 to zero.

Expansion in small M

$$F = \frac{T_c^2}{2T} t M^2 + \frac{T_c^4}{12T^3} M^4 + \dots \infty$$

This coefficient
 changes sign
 as critical point
 is crossed.

$\rightarrow$ positive definite.

$$\frac{\partial F}{\partial M} = 0$$

$$\Rightarrow M \sim \sqrt{-3t} \sim (-t)^{\beta}$$

$\beta \Rightarrow$ Critical Exponent for order parameter.

(Pg 97)

here, we found $\beta = 1/2$ for 1-d ising model.

Other Critical Exponents

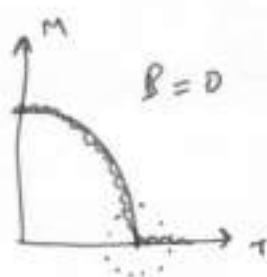
Susceptibility:

$$\chi = \left(\frac{\partial M}{\partial B} \right)_T \Big|_{B=0}$$

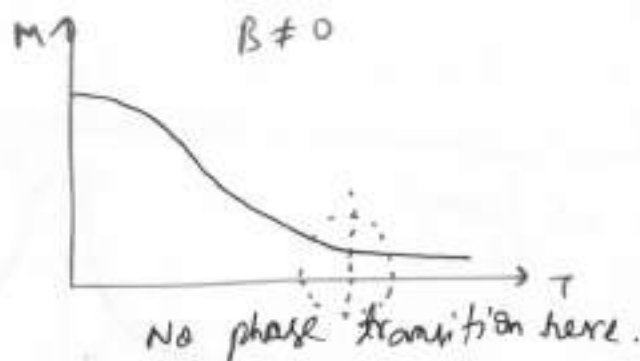
Hamiltonian MFT $H = -(MJ + B) \sum_i \sigma_i + \frac{1}{2} NJM^2$ (Hamiltonian with the Magnetic field)

Free energy: $f = \frac{JM^2}{2} - T \log \left(2 \cosh \left(\frac{MJ+B}{T} \right) \right)$

$$\frac{\partial f}{\partial M} = 0 \Rightarrow M = \tanh \left(\frac{MJ+B}{T} \right)$$



$\Rightarrow$



B small $\Rightarrow M$ small as well close to critical point.

use $\tanh(x) = x - \frac{x^3}{3} + \dots \infty$

$$M \approx \frac{MJ+B}{T} - \frac{1}{3} \left(\frac{MJ+B}{T} \right)^3$$

$$\chi = \frac{\partial M}{\partial B} = \chi \frac{J}{T} + \frac{1}{T} - \left(\frac{MJ+B}{T} \right)^2 \left(\chi \frac{J}{T} + \frac{1}{T} \right)$$

$B \rightarrow 0$ limit.

$$\chi = \left(\chi \frac{J}{T} + \frac{1}{T} \right) \left[1 - \left(\frac{M^2 J^2}{T^2} \right) \right] \quad J^2 = T_c^2$$

$$\Rightarrow \chi = \left(\chi \frac{J}{T} + \frac{1}{T} \right) \left[1 - \frac{M^2 T_c^2}{T} \right]$$

$$T > T_c ; M=0 ; \chi(1-\frac{T}{T_c}) = \frac{1}{T}$$

(pg 48)

$$\Rightarrow \chi = \frac{1}{T-T_c}$$

$$\chi \propto t^{-\gamma}$$

where $\gamma=1$

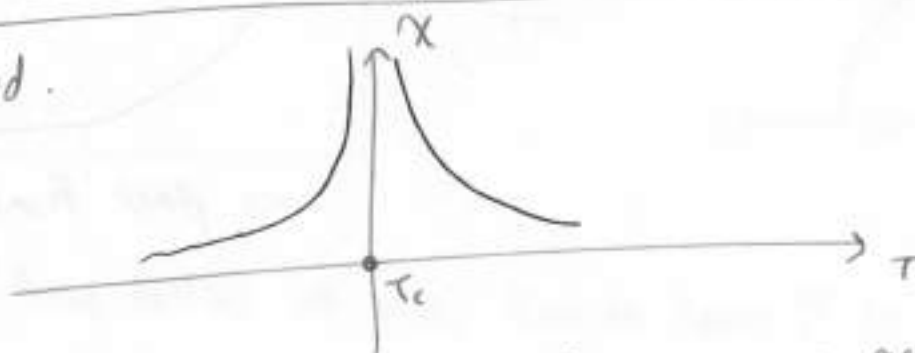
(another critical exponent.)

This is how susceptibility diverge as you go ~~critical~~ to critical temperature from above.

$$T < T_c ; M = \sqrt{-3t} ; \chi \approx \frac{1}{2} \cdot \frac{1}{T_c - T} \propto |t|^{-\gamma'}$$

$$\text{here ; } \gamma' = 1$$

So, we found.



The divergence goes like $\frac{1}{|T-T_c|}$ from both side.

★ Look at $T = T_c$; Now M behaves with B. (will give another critical exponent)

$$\Rightarrow M \approx M \frac{T_c}{T_c} + \frac{B}{T_c} - \frac{1}{3} \left(\frac{M T_c}{T_c} + \frac{B}{T_c} \right)^3$$

$$\Rightarrow M \propto B^{1/3} = B^{1/\delta}$$

~~higher order correction~~

$$\boxed{\delta = 3} \quad \delta \text{ critical exponent.}$$

Higher order term.

Specific Heat

pg 49

$$C_v = T \left(\frac{\partial S}{\partial T} \right)_v \quad ; \quad S = - \frac{\partial F}{\partial T}$$

$$\Rightarrow C_v = - \frac{T}{N} \left(\frac{\partial^2 F}{\partial T^2} \right)_v \quad \text{per lattice site}$$

$$\Rightarrow \boxed{C_v = - T \frac{\partial^2 f}{\partial T^2}}_v$$

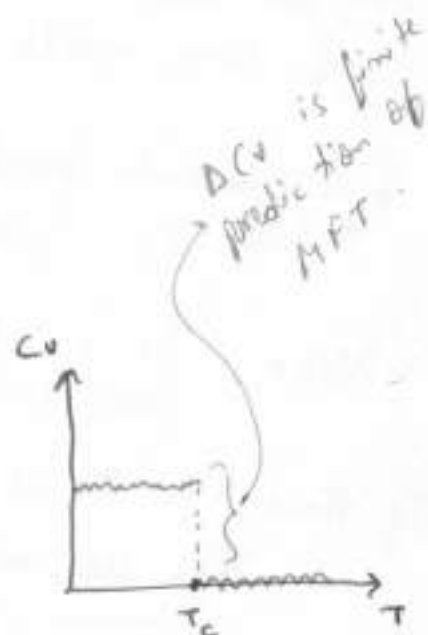
If $T < T_c$;

$$f \stackrel{M=(1-3t)^{1/2}}{=} - \frac{3}{4} \frac{1}{T_c} (T - T_c)^2$$

If $T > T_c$;

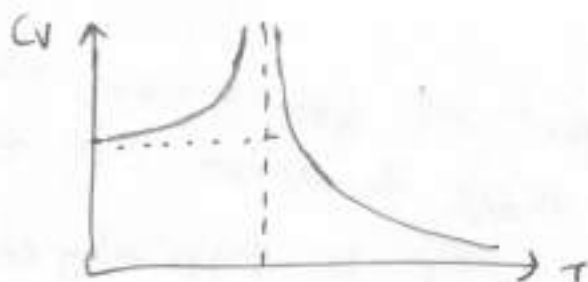
$$f \stackrel{M=0}{=} 0$$

$$\therefore C_v = \begin{cases} 0 & T > T_c \\ \frac{3}{2} \cdot \frac{T}{T_c} & T < T_c \end{cases}$$



MFT predicts jump in ~~specific~~ specific Heat.

Through observation we get.



Definition

$$C_v \propto |t|^{-\alpha'} \quad T < T_c$$

$$t^{-\alpha} \quad T > T_c$$

MFT prediction ; $\alpha = \alpha' = 0$; wrong prediction.

Critical exponents $\alpha, \beta, \gamma, \delta \rightarrow$ characterize our continuous phase transition.

(pg 50)

Landau's Theory

MFT results are very robust, and follow symmetry consideration.

Landau proposed,
All we need to know is the order parameter and the symmetries.

(in our case; $M = \langle \sigma_i \rangle$ (order parameter); & we know if of ising model had Z_2 symmetry $\sigma_i \rightarrow -\sigma_i$)

f is quadratic function of $M \rightarrow$ precisely ~~but~~ because original H has Z_2 symmetry. $\Rightarrow f(M) = f(-M)$

$\therefore$ Near T_c ; we must have non-analytic behavior of free energy.

If there is to be phase transition;

Then we must have $f = \eta M^2 + \mu M^4$

$\eta < 0$ in order to have phase transition.
~~this captures non-analytic behavior of f~~

we require $\mu > 0$.. because of "stability" property.

Prediction is Robust; many diverse systems may lead to the same "free energy".

$\rightarrow$ This is just the part of free energy which is important for phase transition ... a non analytic part. ... It is typically called.
LANDAU'S ENERGY.

$\Rightarrow$ Should have same critical exponents if they have same free energy. It is then said that they

belong to same universality class.

This is true, but critical exponents are actually predicted wrong They take values but not predicted by Landau's Theory.

* MFT Predicts universality classes (Landau Theory)

MFT is same as Landau Theory, ...

For our scalar order parameter; it predicts $\alpha=0, \beta=1/2, \gamma=1, \delta=?$

↳ The prediction is wrong

Experimentally; the value of critical exponent depend on dimensionality of space

A Nature of order parameter (whether scalar or vector, etc.)

* Symmetry of M.

MFT or Landau theory then obviously has to be wrong; because they are irrespective of dimensions; because d never appeared in these things.

Reason; why Landau theory or MFT breaks down is that we treated order parameter as constant. i.e; M is uniformly same throughout the lattice; that is we forbid the fluctuations & neglect the affect of fluctuations)

So, we want to treat order parameter as Thermodynamic parameter which can fluctuate.

(c) From ISING to Field Theory

(1952)

* HUBBARD - STRATONOVICH TRANSFORMATION

We want: $Z = \sum_{\{\sigma_i\}} e^{+\frac{1}{2T} \sum_{ij} \sigma_i \sigma_j J_{ij}}$

Calculate this using a trick of gaussian integration.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\phi e^{-\frac{1}{2}a\phi^2 + b\phi} = \frac{1}{\sqrt{a}} e^{\frac{b^2}{2a}} \quad \text{if } \text{Re}(a) > 0$$

This generalizes to higher dimensions

$$\begin{aligned} \frac{1}{(\sqrt{2\pi})^N} \int_{-\infty}^{+\infty} d\phi_1 \dots d\phi_N e^{-\frac{1}{2} \phi_i A_{ij} \phi_j + Y_i \phi_i} &= \frac{1}{\sqrt{\det A}} e^{\frac{1}{2} Y_i A^{-1}_{ij} Y_j} \\ &= \frac{1}{\sqrt{\det A}} e^{\frac{1}{2} Y_i A^{-1}_{ij} Y_j} \end{aligned}$$

Then we have

Then we have:

$$e^{\frac{1}{2} Y_i A_{ij} Y_j} = \frac{1}{\sqrt{\det A}} \cdot \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{+\infty} d\vec{\phi} e^{-\frac{1}{2} \phi_i A_{ij} \phi_j + Y_i \phi_i}$$

We recognize: $A_{ij} = \frac{J_{ij}}{T}$; & $Y_i = \sigma_i$

now:

$$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2T} \sum_{ij} J_{ij} \sigma_i \sigma_j} = \frac{T^{N/2}}{\sqrt{\det J}} \frac{1}{(2\pi)^{N/2}} \sum_{\{\sigma_i\}} \int d\vec{\phi} e^{-\frac{1}{2} \phi_i J_{ij} \phi_j + \sigma_i \phi_i}$$

$$\int d\vec{\phi} e^{-\frac{1}{2} \phi_i J_{ij} \phi_j + \sigma_i \phi_i}$$

$$Z = \frac{T^{N/2}}{\sqrt{\det J}} \cdot \frac{1}{(2\pi)^{N/2}} \sum_{\{\sigma_i\}} \int d\vec{\phi} \exp\left(-\frac{T}{2} \phi_i J_{ij}^{-1} \phi_j + \sigma_i \phi_i\right)$$

do the rescaling $\phi_i \rightarrow \frac{\phi_i}{T}$

$$\Rightarrow Z = \underbrace{\frac{1}{\sqrt{\det J}} \cdot \frac{1}{(2\pi T)^{N/2}}}_{\text{analytic factor}} \sum_{\{\sigma_i\}} \int d\vec{\phi} \exp\left[\frac{-1}{2T} \phi_i J_{ij}^{-1} \phi_j + \frac{1}{T} \phi_i \sigma_i\right]$$

This is an analytic factor
(if you want to study phase transition, this will not contribute much)

Approach 1

Since it's an analytic factor (it will give an additive analytic term in F) we set to 1... we don't care about it.

Approach 2

Let's change the integration measure... (effectively studying analytic factor to be 1)

$$Z \stackrel{\text{possible non-analytic part}}{=} \sum_{\{\sigma_i\}} \int d\vec{\phi} \cdot e^{-\frac{1}{2T} \phi_i J_{ij}^{-1} \phi_j + \frac{1}{T} \phi_i \sigma_i}$$

now we could do same easily because there is linear factor in exponential.

$$\begin{aligned} \text{Use } \sum_{\sigma_i = \pm 1} e^{\frac{1}{T} \phi_i \sigma_i} &= 2 \cosh(\phi_i / T) \\ &= e^{\log(2 \cosh(\phi_i / T))} \end{aligned}$$

$$Z = \int \mathcal{D}\phi \cdot e^{-S[\phi]}$$

$$S[\phi] = \frac{1}{2T} \phi_i T_{ij} \phi_j - \sum_i \log(2 \cosh(\frac{\phi_i}{T}))$$

$$\mathcal{D}\phi = \frac{1}{\sqrt{\det T}} \cdot \frac{1}{(2\pi T)^{N/2}} d\vec{\phi}$$

Advantage of
Hubbard - Stratonovich
transformation... we
transformed the problem in
path integral form.

Lecture 6: Spin $\Rightarrow$ Spin Correlation Function, Correlation Length, London - Ginzburg Functional.

At phase transition point: the theory is described by ϕ C.F.T.
 the theory is scale invariant ... you begin to see ~~fractals~~ fractals.
 "Fractals appear during phase transitions".

Spin-Spin Correlation Function

$$G_{ij} = (\sigma_i - \langle \sigma_i \rangle)(\sigma_j - \langle \sigma_j \rangle)$$

$$= \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$$

Spin-Spin correlation function

It measures how big blocks are in our system of all of spins.

To calculate;

$$H = -\frac{1}{2} \sum_{ij} J_{ij} - \sum_i B_i \sigma_i$$

Couple the Ising model to magnetic field; which can vary from site to site in lattice

Then we can show: $G_{ij} = -T \frac{\partial^2 F}{\partial B_i \partial B_j} \Big|_{B=0}$

(A trick to calculate G_{ij})

$$G_{ij} = -T \frac{\partial^2 F}{\partial B_i \partial B_j} \Big|_{B=0}$$

$$G_{ij} = \dots = \dots = \frac{a^3 T}{2\pi K} \frac{e^{-|\vec{r}_i - \vec{r}_j|/\xi}}{|\vec{r}_i - \vec{r}_j|^{d-2}}$$

coulomb like decay

exponential decay weighted by characteristic distance ξ

$$G_{ij} \stackrel{\text{M.F.T.}}{=} \frac{a^3 T}{2\pi K} \cdot \frac{e^{-|\vec{r}_i - \vec{r}_j|/\xi}}{|\vec{r}_i - \vec{r}_j|^{d-2}}$$

.....

(positive constant
(must very important))

(looks like
Yukawa potential)

$$\xi = \sqrt{\frac{K}{2(T-T_c)}}$$

Correlation Length. "measures how big the box are".

ξ diverges as $T \rightarrow T_c$ (means you will have blocks of the ξ size of the system).

More Generally;

$$G(r) \propto \frac{e^{-r/\xi}}{r^{d-2+\eta}}$$

where

$$\xi \propto |t|^{-\nu}$$

This introduces two more critical exponents: ν & η .

* $\nu \Rightarrow$ Critical Exponent for Correlation Length.

(prediction of M.F.T. $\nu = \frac{1}{2}$; predicts ξ diverges)

* $\eta \Rightarrow$ Anomalous Dimension.

$\Rightarrow$ says that ~~usual~~ usual dimension analysis is little bit wrong if you are near criticality;

& you can pick up extra dimension from the fractal structure which is coming up; which is described by the Anomalous Dimension..

At $T = T_c$; ξ diverges

We have observed the Fractal Structure.

system is described by CFT (Conformal Field Theory)

$T > T_c$



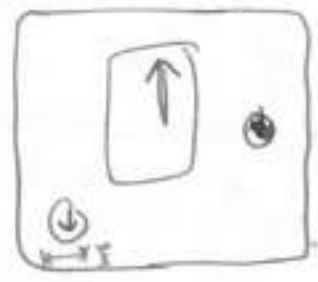
Completely random
(Random Configuration)
 $\langle \sigma \rangle = 0$

$T = T_c$



$\langle \sigma \rangle = 0$
FRACTAL
Structure
($\xi \rightarrow \infty$)

$0 < T < T_c$



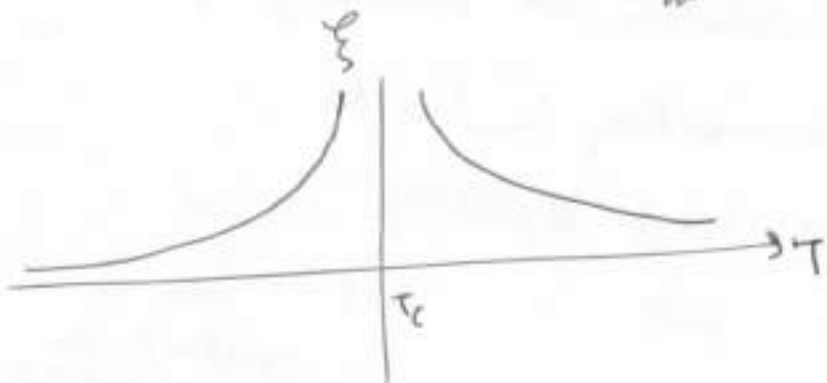
$\langle \sigma \rangle = 0.3$
thick blocks of $\uparrow$
 $\therefore$ but also have small
regions of $\downarrow$

$T = 0$



$\langle \sigma \rangle = 1$

If $T > T_c$
then ξ measure the size of spin up say
if then you go to $T < T_c$
 $\therefore$ it ξ measure the size of block
which has spin down.



That's why
it goes to
zero on both
side

M.F.T. $\eta = 0$

M.F.T.
prediction.
 $\alpha = 0$
 $\beta = 1/2$
 $\gamma = 1$
 $\delta = 3$
 $\eta = 0$
 $\nu = \frac{1}{2}$

~~3 are degenerate below~~

Only 2 are
independent.

(if you can calculate 2
correctly, then you can
calculate using the
algebraic relations; these
algebraic relation can also be wrong.)

We can show, based on very very generic arguments that for typical systems, only two out of these critical exponents are indeed independent. Ag 58

$$Z = \sum_{\{S_i\}} e^{\frac{1}{2T} \sum_{ij} J_{ij} S_i S_j}$$

using Gaussian Integral over some auxiliary fields $\phi_1, \dots, \phi_N$

we get.

$$Z = \frac{1}{\sqrt{\det J}} \cdot \frac{1}{(2\pi T)^{N/2}} \sum_{\{S_i\}} \int_{-\infty}^{+\infty} d\vec{\phi} e^{-\frac{1}{2T} \phi_i J_{ij}^{-1} \phi_j + \frac{1}{T} \sum_i S_i \phi_i}$$

$S_i S_j$ two spin interactions.

$\therefore$ but by Gaussian integral, we see the only place the spin appear is $e^{\dots \frac{1}{T} S_i \phi_i \dots}$ so reduce to

Interaction of spin with fields ϕ .

(Reminds of MFT, where we removed multiparticle interaction by having one point interaction with external field, like M)

$\rightarrow$ suggest ϕ can something be like M

$\therefore$ here; we have exact equality.

ϕ is a variable field

(M was a constant in MFT)

$\rightarrow$ This is why it gives wrong critical exponents)

Partition function being written as multidimensional integral over auxiliary fields ϕ_i .

$$Z = \int \mathcal{D}\phi \cdot e^{-S[\phi]}$$

$$S[\phi] = \frac{1}{2T} \phi_i J_{ij}^{-1} \phi_j - \sum_i \log \left(2 \cosh \left(\frac{\phi_i}{T} \right) \right)$$

EXACT (NO approximation)

Saddle point (WKB) Approximation

we want to minimize S .

we want $\frac{\partial S}{\partial \phi_i} \Big|_{\phi_i = \bar{\phi}_i} = 0$

lets call the solution to be $\bar{\phi}_i$
(ϕ average)

$$\Rightarrow \frac{1}{T} J_{ij}^{-1} \phi_j - \frac{1}{T} \tanh\left(\frac{\phi_i}{T}\right) = 0 \Big|_{\phi = \bar{\phi}}$$

$$\Rightarrow \frac{1}{T} J_{ij}^{-1} \bar{\phi}_j - \frac{1}{T} \tanh\left(\frac{\bar{\phi}_i}{T}\right) = 0$$

$$\Rightarrow \boxed{\bar{\phi}_i = \sum_j J_{ij} \tanh\left(\frac{\bar{\phi}_j}{T}\right)}$$

All lattice site are same; so $\bar{\phi}_1 = \bar{\phi}_2 = \dots = \bar{\phi}$

$$\Rightarrow \boxed{\bar{\phi} = J \cdot \tanh\left(\frac{\bar{\phi}}{T}\right)} \quad J = \sum_j J_{ij}$$

we have seen equation like $M = \tanh\left(\frac{MJ}{T}\right)$

if you make an identification of $\bar{\phi} = MJ$

then you return back to Mean Field Theory!

MFT is recovered by taking leading order of WKB approximation.

$$MFT = WKB \text{ Approximation to leading order.}$$

fields ϕ_i are confined to lattice points. (Pg 60)

So; we have something like field theory in discrete space time.

∴ we would like to take continuous limit.

Close to critical temperature, we can zoom out of the system which effectively makes the system continuous. So; effectively the lattice disappears.

→ Continuous limit (Effectively lattice disappears)

→ In the end; we will recover Classical Field Theory living in continuous space.

$$\star \log(2 \cosh(\phi/\tau)) = \log(2) + \frac{1}{2} \left(\frac{\phi_i}{\tau}\right)^2 - \frac{1}{12} \left(\frac{\phi_i}{\tau}\right)^4 + \dots$$

if ϕ is like m ; then
close to T_c ; ϕ is small

$$\log(2 \cosh(\phi_i/\tau)) = \log(2) + \frac{1}{2} \left(\frac{\phi_i}{\tau}\right)^2 - \frac{1}{12} \left(\frac{\phi_i}{\tau}\right)^4 + \dots$$

not interested in constant term

minus is important.

Lattice Fourier transform

$$\phi_i = \frac{1}{N} \sum_{\vec{a}} \tilde{\phi}(\vec{a}) e^{i \vec{a} \cdot \vec{r}_i}$$

→ ϕ is like
fourier
transform
of $\vec{a}$.

→ $\vec{a}$ momentum version of $\vec{r}$

convention: writing ϕ with argument $\vec{a}$
 $\phi(\vec{a})$ means we are taking about fourier
transform of $\phi(\vec{r})$

So:

$$J_{ij} = \frac{1}{N} \sum_{\vec{q}} J(\vec{q}) e^{i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)}$$

↪ because it only depends on $\vec{r}_i - \vec{r}_j$ difference; ... so only one summation over $\vec{q}$ works.

Orthogonality relations

$$\frac{1}{N} \sum_j e^{i(\vec{k} - \vec{k}') \cdot \vec{r}_j} = \delta_{\vec{k}, \vec{k}'}$$

$$\frac{1}{N} \sum_{\vec{q}} e^{-i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \delta_{ij}$$

Lets deal with the term

$$\sum_{i,j} \phi_i J_{ij}' \phi_j$$

... fourier transform everything.

$$= \sum_{i,j} \underbrace{\frac{1}{N} \sum_{\vec{q}} \phi(\vec{q}) e^{i\vec{q} \cdot \vec{r}_i}}_{\phi_i} \underbrace{\frac{1}{N} \sum_{\vec{q}'} J'(\vec{q}') e^{i\vec{q}' \cdot (\vec{r}_i - \vec{r}_j)}}_{J_{ij}'} \times \underbrace{\frac{1}{N} \sum_{\vec{q}''} \phi(\vec{q}'') e^{i\vec{q}'' \cdot \vec{r}_j}}_{\phi_j}$$

use then first orthogonality condition.

$$\sum_{i,j} \phi_i J_{ij}' \phi_j = \sum_{i,j} \frac{1}{N} \sum_{\vec{q}} \phi(\vec{q}) e^{i\vec{q} \cdot \vec{r}_i} \cdot \frac{1}{N} \sum_{\vec{q}'} J'(\vec{q}') e^{i\vec{q}' \cdot (\vec{r}_i - \vec{r}_j)} \cdot \frac{1}{N} \sum_{\vec{q}''} \phi(\vec{q}'') e^{i\vec{q}'' \cdot \vec{r}_j}$$

doing sum over i gives $\vec{q} & \vec{q}'$ become same.
 sum over j " $\vec{q}' & \vec{q}''$ become same.

$$= \frac{1}{N} \sum_{\vec{q}, \vec{q}', \vec{q}''} \phi(\vec{q}) J'(\vec{q}') \phi(\vec{q}'') \delta_{\vec{q}, \vec{q}'} \delta_{\vec{q}', \vec{q}''} \quad \text{Ⓢ}$$

$$= \frac{1}{N} \sum_{\vec{q}_j} \phi(-\vec{q}_j) J^{-1}(\vec{q}_j) \phi(\vec{q}_j)$$

1982

$$\sum_{i,j} \phi_i J_{ij}^{-1} \phi_j = \frac{1}{N} \sum_{\vec{q}} \phi(-\vec{q}) J^{-1}(\vec{q}) \phi(\vec{q})$$

$\Downarrow$
 Sum in real space (have two sum)

$\Downarrow$
 Sum in momentum space (have one sum)

$\Rightarrow$
 This is what we achieved...
 ... ~~the~~ ... in momentum space it is nicer; first of all we have just one sum

$$J^{-1}(\vec{q}) ?$$

let's do some approximation.

$$J^{-1}(\vec{q}) \text{ is } \frac{1}{J(\vec{q})} \text{ (This is not the approximation)}$$

$\therefore$ what is $J(\vec{q})$?

$$J(\vec{q}) = \frac{1}{N} \sum_{i,j} J_{ij} e^{-i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)}$$

$\hookrightarrow$ Transform of J_{ij} .

Now, we want to get the continuous limit.

... if you zoom out, you ~~see~~ ~~the~~ have larger & larger distances in position space;

but in momentum space you have smaller & smaller chunks.

$\therefore \vec{q}$ is small because we want continuum limit ~~the~~

Now we can expand the exponential in Taylor series & can keep only few terms. (1968)

Then

$$J(\vec{q}) = \frac{1}{N} \sum_{ij} J_{ij} \left(1 - i \vec{q} \cdot (\vec{r}_i - \vec{r}_j) - \frac{1}{2} (\vec{q} \cdot (\vec{r}_i - \vec{r}_j))^2 + \dots \right)$$



Simple

gives J

* one sum gives J
= other sum gives N



$$A_{ij} = \vec{r}_i - \vec{r}_j$$

of course $A_{ij} = -A_{ji}$

but J_{ij} is symmetric

$$\text{So; } \sum_{ij} J_{ij} i \vec{q} \cdot (A_{ij}) = 0$$

(linear term does not contribute)

$$J(\vec{q}) = \frac{1}{N} \sum_{ij} J_{ij} - \frac{1}{2N} \sum_{ij} J_{ij} |\vec{q} \cdot (\vec{r}_i - \vec{r}_j)|^2 + \dots$$



$$J = \sum_{ij} J_{ij}$$

$$\Rightarrow J(\vec{q}) = J - \frac{1}{2N} \sum_{ij} J_{ij} |\vec{q} \cdot (\vec{r}_i - \vec{r}_j)|^2 + \dots$$

→ this depends on q^2

so;

$$\rightarrow -\frac{1}{2} K |\vec{q}|^2 \quad (\text{hahaha...})$$

$$J(\vec{q}) = J - \frac{1}{2} K |\vec{q}|^2 + \dots$$

* K should not contain N.
... hahaha...
easy to check.

$R \Rightarrow$ Range of Interaction

K is called spin stiffness.

same K which appears in correlation function.

$$K = \frac{R^2 \cdot J}{d}$$

$d \Rightarrow$ no. of dimensions.

Calculation of G_{ij} (correlation function) proceeds in similar way.

(pg 64)

$$S = S_0 + S_{\text{int}} \rightarrow S \text{ interacting.}$$

$$S_0[\phi] = \frac{1}{2T^2} \sum_{ij} T \phi_i J_{ij}^{-1} \phi_j - \frac{1}{2} \sum_i \frac{\phi_i^2}{T^2}$$

$$S_{\text{int}}[\phi] = + \frac{1}{12} \sum_i \frac{\phi_i^4}{T^4}$$

~~$$S_0[\phi] = \frac{1}{2T^2 N} \sum_{\vec{a}} \phi(\vec{a}) \left(\frac{T}{J(\vec{a})} - 1 \right) \phi(\vec{a})$$~~

$$S_0[\phi] = \frac{1}{2T^2 N} \sum_{\vec{a}} \phi(\vec{a}) \left(\frac{T}{J(\vec{a})} - 1 \right) \phi(\vec{a})$$

$$\frac{T}{J \left(1 - \frac{1}{2} |\vec{a}|^2 \frac{K}{J} \right)} - 1$$

→ small if $|\vec{a}|$ is small.

$$= \frac{T}{J} \left(1 + \frac{1}{2} |\vec{a}|^2 \frac{K}{J} \right) - 1$$

In M.F.T. we know $J = T_c$

$$= \left(\frac{T - T_c}{T_c} \right) + \frac{1}{2} |\vec{a}|^2 K \cdot \frac{T}{T_c^2}$$

$$= t + \frac{1}{2} |\vec{a}|^2 K \cdot \frac{T}{T_c^2} \rightarrow \text{positive quantity}$$

we it...

$$S_0[\phi] = \frac{1}{2T^2N} \sum_{\vec{q}} \phi(-\vec{q}) \cdot \left[t + \frac{1}{2} |\vec{q}|^2 K \frac{T}{T_c^2} \right] \phi(\vec{q}) \quad (1965)$$

we want to
redefine ϕ so that
we get rid of this

... we want to set
this to be 1.

... we absorb this in ϕ ...

$$\phi \rightarrow \frac{T_c \sqrt{2d}}{R a^{d/2}} \phi$$

introduce new variable $q = \frac{2dt}{R^2}$

$$S_0[\phi] = \frac{1}{2} \left(\frac{1}{Na^d} \cdot \sum_{\vec{q}} \phi(-\vec{q}) (1 + q^2) \phi(\vec{q}) \right)$$

can be turned to an integral over
 $\vec{q}$ in the limit.

$$S_0[\phi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \phi(-\vec{q}) (1 + q^2) \phi(\vec{q})$$

$$S_0[\phi] = \frac{1}{2} \int d^d x (|\nabla \phi|^2 + \lambda \phi^2)$$

$$S_{\text{int}}[\phi] = \int d^d x \cdot \frac{u}{4!} \phi^4$$

$$S[\phi] = \int d^d x \left[\frac{1}{2} |\nabla \phi|^2 + \frac{\lambda}{2} \phi^2 + \frac{u}{4!} \phi^4 \right]$$

Landau Ginsberg
function.

→ This is what we have in continuous limit.

* $\eta \propto t$

(changes sign as you cross T_c)

pg 66

* $u > 0$

Now, in Landau - Ginsburg function: what improved is that fields can spatially fluctuate because of the $|\nabla\phi|^2$.

... we get something like scalar field theory

In Euclidean Space (with imaginary time)

(no time direction in ~~fact~~ this field theory: everything is just purely spatial)

→ The field theory lives only on same

.... if you forget the fluctuations $|\nabla\phi|^2$, you sort of return back to Landau's Theory.

$|\nabla\phi|^2$ is the main thing which Ginsberg added.

Lecture 7: Wilsonian Renormalization 1: Basic idea:
Integration over fast modes, Flowing Couplings.

From Ising to Field Theory

$$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2\tau} \sum_{ij} G_{ij} \sigma_i \sigma_j} \approx \int D\phi e^{-S[\phi]}$$

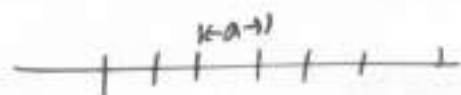
$$S[\phi] = \frac{1}{2\tau} \sum_{ij} \phi_i T_{ij}^{-1} \phi_j - \sum_i \log(2 \cosh(\phi_i/\tau))$$

Separation near
Critical Point
(effectively setting
lattice spacing a
goes to zero)

$$\int d^d x \left[\frac{1}{2} |\vec{\nabla} \phi|^2 + \frac{r}{2} \phi^2 + \frac{u}{4!} \phi^4 \right]$$

Landau - Ginzburg Action.

$d \Rightarrow$ no. of space dimensions.



we are not actually setting
 a to zero; but rather rather
concentrating on the scale of order
 ξ which goes to infinity

$\Rightarrow$ This is why
we can take
continuum limit
 $\therefore$ not because $a \rightarrow 0$
but because we are
looking at large
scales.

$\xi \rightarrow \infty$ large spatial scale

$\Rightarrow$ so $a \Rightarrow$ small

$$\gamma \propto t = \frac{T - T_c}{T_c} \quad (\text{change sign as we cross } T_c)$$

M.F.T. $u > 0$. neglected fluctuations: If there are fluctuations then
true T_c will be lower than predicted by T_c .

Respect symmetry of original Hamiltonian.

(Pg 68)

$$\phi_i \rightarrow -\phi_i$$

$\hookrightarrow$ which translates to $\phi \rightarrow -\phi$.

So; only have even power of ϕ .

$\therefore$ add $|\nabla\phi|^2$ to add fluctuations.

$\therefore$ Is there possibility for higher derivatives of ϕ ; but they will not be much relevant.

• Higher powers of ϕ or ∇ are irrelevant.
and so this is everything you can write down.

• Dimensional Analysis. "units of energy"

$$[k] = \dim(k) = +1 \quad ; k \neq \text{momentum}$$

$$[x] = -1$$

$\uparrow$
Spatial variable.

$$[S] = 0 = \underset{-d}{[dx^d]} + \underset{+2}{[\nabla^2]} + \underset{2 \times [\phi]}{[\phi^2]} \Rightarrow [\phi] = \frac{d-2}{2}$$

$$[v] = [\nabla^2] = +2 \quad [u] = \dots$$

$$\begin{aligned} 0 &= [dx^d] + [v] + 4[\phi] \Rightarrow 0 = -d + [v] + 4 \times \frac{(d-2)}{2} \\ 0 &= [v] + (2d-4) \\ \Rightarrow 4-d &= [v] \end{aligned}$$

define $\epsilon = 4-d$

So;

$$[u] = \epsilon$$

Definition Coupling constant g is Relevant is $[g] > 0$
(it really plays role for our problem)

RELEVANT $[g] > 0$
 MARGINAL $[g] = 0$
 IRRELEVANT $[g] < 0$

Classification of coupling constants $\Rightarrow$ depends on no. of dimensions

in $d=4$ dimensions; $[g] = 2$ & $[V] = 0$

So; if you added another coupling $V \propto \phi^6$

$\therefore [V] = 6 - 2d$ in 4 dimensions: $[V] = -2$

So; V is irrelevant by the classification

$V \rightarrow$ marginal in $d=3$

$\rightarrow$ relevant in $d \leq 2$

Irrelevant $d \geq 4$

"Lower dimensions you go; you should include more terms ~~dimension~~ in Landau - Ginzburg Functional"

So; Landau - Ginzburg as truncated here upto ϕ^4 is only good in $d=4$.

• How to calculate Z ?

- Perturbation theory in u .

(but here; it might not work; because u may not be small)

(it works in Q.E.D; there the coupling constant is $\alpha = \frac{1}{137}$... its small...)

This fails

- Perturbation theory: in fluctuation $\delta\phi$.
(This also fails)

The right way to proceed is crazy way

• Do Perturbation theory in small ϵ
..... ϵ -Expansion.

(similar idea is used in study of black holes; d = spacetime dimension. We see higher d black holes are simpler to study; so we study it & find results for $d=4$ by $1/d$ expansion.... (Have a look ... use it show!)).

here; we are solving for $d=4$ dimensions

(p 70)

its ~~unphysical~~ unphysical... best things are easily to calculate in $d \dots$ ~~to~~ to calculate for $d=2$ or $d=3$ we do expansion in d

... (expansion in ϵ is equivalent ~~to~~ in some sense to expansion of d ; because $\epsilon = 4-d$)

(d) WILSONIAN (Momentum Space) RENORMALIZATION

PART 1 | Main Idea

"It is a formal way of doing coarse graining."

Father of Renormalization is actually Gell-Mann; but he did it in real space which was not much helpful. It was Wilson who did it in momentum space which was more useful.

We want to calculate $Z = \int \mathcal{D}\phi e^{-S[\phi]}$

We want to calculate this in small steps.

When you calculate two type of divergences

as in QFT; large momenta: Ultraviolet Divergences.

small momenta (because ~~it is~~ in highly energy processes; we can create out of vacuum anything: The loops will cause some problems)

Here in our subject; we are basically treating condensed matter system. In Condensed matter system only one type of divergence exist; the one for small k .

2) This is which is responsible for our ~~phase~~ ~~to~~ renormalization of our phase transition; because we are really interested in whole system: so small k .

* For large k does not happen.

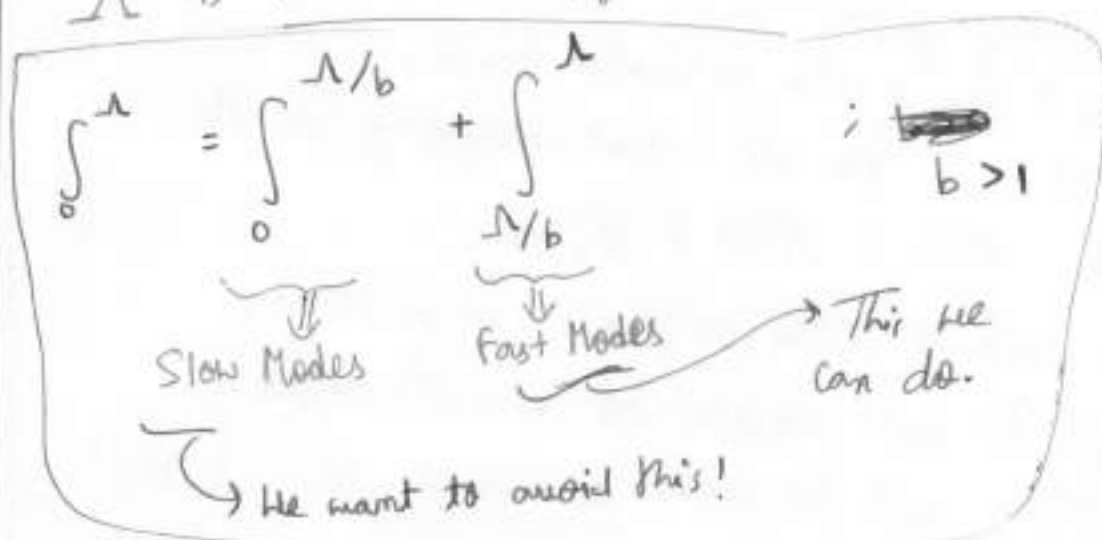
because we have lattice. so; There is a cut off on how big momentum can be. It does not make sense to probe ~~on~~ anything smaller than lattice spacing. That's why higher momenta are cut off.

Avoiding divergences in small k .

*
$$\phi(x) = \int \frac{d^d k}{(2\pi)^d} \phi(k) e^{i k \cdot x}$$

0 $\Rightarrow$ This will be the problem... small k .

Λ is Natural Cut off ; ~~$\Lambda \propto \frac{1}{a}$~~ $\Lambda \propto \frac{1}{a}$



~~$\phi = \phi_k$~~

In momentum space; split ϕ in $\phi_<$ (ϕ slow) & $\phi_>$ (ϕ fast)

$$\phi = \phi_< + \phi_>$$

$$\phi_<(k) = \begin{cases} \phi(k) & 0 < k < \Lambda/b \\ 0 & k > \Lambda/b \end{cases}$$

$$\phi_>(k) = \begin{cases} 0 & 0 < k < \Lambda/b \\ \phi(k) & k > \Lambda/b \end{cases}$$

Trik is to integrate fast only.

(1972)

* Split S as $S = S_0 + S_{int}$

$$S_0 = \frac{1}{2} \int d^d x [(\nabla \phi)^2 + r \phi^2]$$

$$= \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \phi(-k) (k^2 + r) \phi(k) \quad (\text{by fourier transform})$$

$$S_0 = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \phi(-k) (k^2 + r) \phi(k)$$

$$= \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} (k^2 + r) |\phi(k)|^2$$

?

$$\boxed{\phi(-k) \phi(k) = |\phi(k)|^2}$$

→ Non interacting part.

Now we will see a better justification for this.

$|\phi(k)|$; no we start plugging in the

$$\text{Splitting } \phi(k) = \phi_<(k) + \phi_>(k)$$

k can either be slow or fast at a time.

it cannot be both ~~simultaneously~~ ~~be~~ simultaneously ..

... There will be never be mixed $\phi_<$ & $\phi_>$

because here we have only one k .

$$\text{so; } S_0 = S_0[\phi_<] + S_0[\phi_>] \text{ because no}$$

k_{fast} and k_{slow} mixing.

→ So this is something which we mean by non interactive part.

$$(\phi(x) \text{ is real} \Rightarrow \phi(k) = \phi^*(k) \dots \text{check})$$

$$* S_{int} = \frac{u}{4!} \int d^d x \phi^4$$

plug fourier transform for each ϕ .

Integration over x gives delta function over sum of k s.

$$S_{INT} = \frac{u}{4!} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{d^d k_4}{(2\pi)^d} \phi(k_1) \dots \phi(k_4) (2\pi)^d \delta(k_1 + \dots + k_4) \quad (1978)$$

$$S_{INT} = \frac{u}{4!} \int \frac{d^d k_1}{(2\pi)^d} \dots \frac{d^d k_4}{(2\pi)^d} \phi(k_1) \dots \phi(k_4) \cdot (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4)$$

→ have four different momentum; out of which three are independent.

... so there can be mixing of slow & fast mode.

$$\text{So; } S_{INT} = S_{INT}[\phi_<, \phi_>]$$

but it can't be splitted as sum.

$$S_{INT} = S_{INT}[\phi_<, \phi_>]$$

Now, we want to integrate over fast modes.

$$Z = \int \mathcal{D}\phi_> \mathcal{D}\phi_< e^{-S[\phi_<, \phi_>]}$$

→ we do this integral.

$$\Rightarrow Z = \int \mathcal{D}\phi_< e^{-\tilde{S}[\phi_<]}$$

This is how it looks after integrating out fast modes
 ... we get some new action $\tilde{S}$ which now depends only on slow modes.

if we are in $d=4$

∴ Then irrelevant terms will drop out.

.... more generally;

$\tilde{S}[\phi_<]$... has to have the same form we started with.
 (can generate more terms; but these are irrelevant)

Pg 77

$$\tilde{S}[\phi_c] = \frac{1}{2} \int_0^{N_b} \frac{d^d k}{(2\pi)^d} |\phi_c(k)|^2 \cdot (\tilde{\gamma}_2 k^2 + \tilde{\gamma}_0) + \frac{\tilde{\gamma}_4}{4!} \int_0^{N_b} \frac{d^d k_1 \dots d^d k_4}{(2\pi)^{4d}} \phi_c(k_1) \dots \phi_c(k_4) (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4)$$

originally what stands in front of k^2 is 1 ... which is also a coupling constant. As you are integrated over fast mode; the coupling constant 1 may change to something else.

γ will also change.

; let's call $\tilde{\gamma}_2$ the coupling constant arising from 1

& $\tilde{\gamma}_0 \dots$ arising from γ .
& $\tilde{\gamma}_4 \dots$ arising from u .

$$\tilde{S}[\phi_c] = \frac{1}{2} \int_0^{N_b} \frac{d^d k}{(2\pi)^d} |\phi_c(k)|^2 \cdot (\tilde{\gamma}_2 k^2 + \tilde{\gamma}_0) + \frac{\tilde{\gamma}_4}{4!} \int \dots$$

$$\tilde{S}[\phi_c] = \frac{1}{2} \int_0^{N_b} \frac{d^d k}{(2\pi)^d} |\phi_c(k)|^2 \cdot (\tilde{\gamma}_2 k^2 + \tilde{\gamma}_0) + \frac{\tilde{\gamma}_4}{4!} \int_0^{N_b} \frac{d^d k_1 \dots d^d k_4}{(2\pi)^{4d}} \phi_c(k_1) \dots \phi_c(k_4) (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4)$$

we scale the momenta & fields to get the integral from 0 to Λ back.

To bring this to exactly the same form we started with we scale.

we have to introduce new momentum $\tilde{k} = kb$

" " " " " b ; $\tilde{\phi}(\tilde{k}) = \frac{\phi_c(k)}{2^{d/2}}$

$\tilde{k} = kb, \quad \tilde{\phi}(\tilde{k}) = \frac{\phi_c(k)}{2^{d/2}}$

This is not partition function

The fact that you have to change fields as well, so for ~~historical~~ historical reason it is called

$$\tilde{k} = kb, \quad \tilde{\phi}(k) = \frac{\phi(k)}{Z} \quad \text{Wave function Renormalization.} \quad (p. 75)$$

The fact that you have to change fields as well: so for historical reasons it is called Wavefunction Renormalization. This is not partition function. ... just a constant which will be determined later.

Of course we don't have any wavefunction here ... but anyway we are normalizing some field.

$$\begin{aligned} S[\tilde{\phi}] &= \frac{1}{2} \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \cdot b^{-d} \cdot (\tilde{\kappa}_2 b^{-2} \tilde{k}^2 + \tilde{\kappa}_0) |\tilde{\phi}(k)|^2 Z^2 + \dots \\ &= \frac{1}{2} \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \cdot [b^{-d} Z^2 \tilde{\kappa}_2 b^{-2} \tilde{k}^2 + Z^2 \tilde{\kappa}_0 b^{-d}] |\tilde{\phi}(k)|^2 \\ &\quad + \frac{1}{4!} \tilde{\kappa}_4 \cdot b^{-4d} Z^4 \cdot b^d \int_0^\Lambda \frac{d^d k_1}{(2\pi)^d} \dots \frac{d^d k_4}{(2\pi)^d} \tilde{\phi}(k_1) \dots \tilde{\phi}(k_4) (2\pi)^d \times \\ &\quad \times \delta(k_1 + \dots + k_4) \end{aligned}$$

Coming from δ function.

We are also changing k in $\delta(\quad)$ function.

$$\text{we: } \delta(f(x)) = \sum_{\{x_i\}} \frac{\delta(x - x_i)}{|f'(x_i)|}$$

s.t. $f(x_i) = 0$.

We have new coupling constants.

Now we have lots of freedom to set one of the coupling constant as 1 because we have also rescaled the field

we set the constant which stands in front of k^2 to be 1

(976)

set

$$b^{-d} z^2 \tilde{\chi}_2 b^{-2} k^2 = 1$$

$$z^2 \tilde{\chi}_0 b^{-d} = \tilde{\chi}$$

$$\tilde{\chi}_4 b^{-d} z^4 b^d = \tilde{u}$$

Now, drop $\sim$ from ϕ & $k \dots$; it does not matter because they are auxiliary variable which are getting integrated over

$$\int d\tilde{x} = \int dx$$

So, we require;

$$1 = b^{-d-2} z^2 \tilde{\chi}_2$$

Whatever $\tilde{\chi}_2$; we can write it as some function of a parameter b

$$\therefore \tilde{\chi}_2 = b^\eta ; b > 1$$

η is called Anomalous Dimension.

Choose this $b > 1$ if you want it

~~η has anomalous dim~~

(This is same as the one we see in definition of Spin-Spin correlation function)

So;

$$1 = b^{-d-2+\eta} z^2$$

$$z = b^{\frac{d+2-\eta}{2}}$$

$$z = b^{\frac{d+2-\eta}{2}}$$

to be same as our previous anomalous dimension

$\therefore$ we will see later.

Then

$$\phi(k) = b^{-\frac{d-2+\eta}{2}} \phi_c(k)$$

(pg 72)

check: prof work: $\phi(k) = b^{-\frac{d+2-\eta}{2}} \phi_c(k)$
check.

→ This is how wave function renormalizes in order to maintain that coupling constant to be equal to 1.

$$\tilde{\eta} = Z^2 \tilde{\eta}_0 b^{-d} = b^{d+2-\eta-d} \tilde{\eta}_0$$

$$\Rightarrow \boxed{\tilde{\eta} = b^{2-\eta} \tilde{\eta}_0}$$

$$\boxed{\tilde{u} = b^{4-2\eta-d} \tilde{u}_0}$$

We now have

$$S[\phi] = \int d^d x \left[\frac{1}{2} (\nabla \phi)^2 + \frac{\tilde{\eta}}{2} \phi^2 + \frac{\tilde{u}}{4!} \phi^4 \right]$$

→ $S[\phi]$ after integrating ~~over~~ over fast modes (changes coupling constants)

Effectively As we integrate over fast modes, coupling constant changes in each step.

$$(\eta, u) \longrightarrow (\tilde{\eta}, \tilde{u})$$

∴ Now, we do this repeatedly many times
... and at each step ~~the~~ η, u will change
 $(\eta, u) \longrightarrow (\tilde{\eta}, \tilde{u}) \longrightarrow (\tilde{\tilde{\eta}}, \tilde{\tilde{u}}) \rightarrow \dots$

The coupling changes

⇒ so we are flowing in the space of coupling constants.

A miracle can happen & ~~we may find that~~ it may happen that after ~~few steps~~ some steps the coupling constants don't change any more. This is called the Fixed Point.

We may find a Fixed Point (g^*, u^*) .

ie: after that; if we again do one more step
 $(g^*, u^*) \rightarrow (g^*, u^*) \dots$ ~~don't change~~
 $\dots$ don't change.

~~we are~~
In each step we are introducing new momenta $\vec{k} = k/b$; $b > 1$;
which means that if you think about in position space
 $\rightarrow$ Then in each step your new $\tilde{x} = x/b$
 $\therefore$ and so this must be true for any spatial
thing; in particular for correlation length.

$\therefore$ In each step, the correlation length
 ~~ξ~~ $\xi = \xi(g, u) \rightarrow \xi(\tilde{g}, \tilde{u}) = \frac{\xi(g, u)}{b}$

$\rightarrow$ Just the consequence of
how we changed the momenta

but At fixed point we found $\xi(g^*, u^*) = \frac{\xi(g^*, u^*)}{b}$

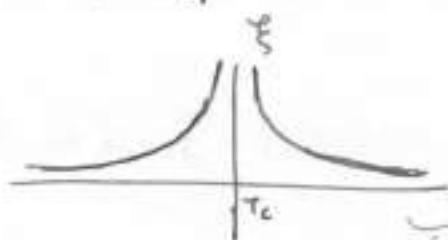
but $b > 1$

$\Rightarrow$ So we have two possibilities.

$$\xi(r^*, u^*) = \begin{cases} 0 & \dots \text{Phase } (T \neq T_c) \\ \infty & \dots \text{Critical Phase } (T = T_c) \end{cases}$$

∞ is interesting from the view of phase transition.

$\xi = \infty$ corresponds to critical point; so we are at $T = T_c$.



ξ goes to zero means you are in very well defined phase

$\xi = 0$ corresponds to phase; so $T \neq T_c$

→ i.e. $\xi \rightarrow 0$ means either in the phase below T_c or above T_c (either paramagnetic or ferromagnetic phase)

Now π changes as you flow in ~~space~~ space of couplings... that will be called RG flow & the corresponding equation will be called RG equation.

... of course we are interested in when couplings don't change... then we are at fixed point.

Part 2 Explicit Calculation.

$$Z = \int \mathcal{D}\phi, \mathcal{D}\phi_c e^{-S} = \int \mathcal{D}\phi_c e^{-\tilde{S}[\phi_c]}$$

$$e^{-\tilde{S}[\phi_c]}$$

Remember $S = S_0 + S_{int}$

$$= S_0[\phi_c] + S_0[\phi_s] + S_{int}[\phi_c, \phi_s]$$

$$S = S_0[\phi_c] + S_0[\phi_s] + S_{int}[\phi_c, \phi_s]$$

not touched
by fast integration.

$$Z = e^{-S_0[\phi_c]} \int \mathcal{D}\phi_s$$

$$Z = \int \mathcal{D}\phi_c e^{-S_0[\phi_c]} \int \mathcal{D}\phi_s e^{-S_0[\phi_s] - S_{int}[\phi_c, \phi_s]}$$

$$Z = \int \mathcal{D}\phi_c \cdot e^{-S_0[\phi_c]} \cdot \frac{\int \mathcal{D}\phi_s e^{-S_0[\phi_s] - S_{int}[\phi_c, \phi_s]}}{\int \mathcal{D}\phi_s e^{-S_0[\phi_s]}} \cdot \underbrace{\int \mathcal{D}\phi_s e^{-S_0[\phi_s]}}_{Z_0}$$

lets call this Z_0

$$Z_0 \equiv \int \mathcal{D}\phi_s e^{-S_0[\phi_s]}$$

} Partition function over
non interacting part
over fast mode.

This is just the definition of average of
 $e^{-S_{int}[\phi_c, \phi_s]}$ in field theory
over fast mode

$$\langle e^{-S_{int}} \rangle_{\text{FAST}} = \frac{\int \mathcal{D}\phi_s e^{-S_0[\phi_s] - S_{int}[\phi_c, \phi_s]}}{\int \mathcal{D}\phi_s e^{-S_0[\phi_s]}}$$

In general: $\langle \cdot \rangle = \frac{\int \mathcal{D}\phi \cdot e^{-S_x[\cdot]}}{\int \mathcal{D}\phi e^{-S}}$

Now, we get a beautiful formula for $\tilde{S}$.

(pg 81)

$$\tilde{S}[\phi_c] = S_0[\phi_c] - \log \left(\langle e^{-S_{\text{int}}} \rangle_{\text{FAST}} \right) - \log Z_0$$

Z_0 does not depend on slow modes ϕ_c
→ it is just a constant (so not much interesting)

Cumulant Expansion

given Ω be any random variable

then $\langle e^{-\Omega} \rangle = e^{\langle \Omega \rangle + \frac{1}{2} [\langle \Omega^2 \rangle - \langle \Omega \rangle^2] + \dots}$

Cumulant Expansion

given Ω be any random variable

then $\langle e^{-\Omega} \rangle = e^{\langle \Omega \rangle + \frac{1}{2} [\langle \Omega^2 \rangle - \langle \Omega \rangle^2] + \dots}$

Using Cumulant Expansion.

$$\tilde{S}[\phi_c] = S_0[\phi_c] + \langle S_{\text{int}} \rangle_{\text{FAST}} - \frac{1}{2} [\langle S_{\text{int}}^2 \rangle - \langle S_{\text{int}} \rangle^2] + \dots$$

once we know $\tilde{S}$ after calculating,

then we can find $\tilde{n}$ & $\tilde{u}$

Then we know recursion relation between new & old couplings.

Lecture 8: Wilson Renormalization 2: Explicit Calculation: Cumulant expansion, RG equations

$$\tilde{S}[\phi_c] = S_0[\phi_c] + \underbrace{\langle S_{int} \rangle_{FAST}}_{1^{st} \text{ order}} - \underbrace{\frac{1}{12} [\langle S_{int}^2 \rangle_{FAST} - \langle S_{int} \rangle_{FAST}^2]}_{2^{nd} \text{ order}}$$

0th order approximation for $\tilde{S}$

Zeroth Order $S_0[\phi_c] = \frac{1}{2} \int_k^{\Lambda/b} (k^2 + m^2) |\phi_c(k)|^2$

Integral over k

$$\int_k \equiv \int \frac{d^d k}{(2\pi)^d}$$

First Order $S_{int} = \frac{\mu}{4!} \int_{k_1 \dots k_4} [\phi_c(1) + \phi_s(2)] \dots [\phi_c(4) + \phi_s(4)] \gamma_m^d \delta(\sum k_i)$

$$= \frac{\mu}{4!} \int_{k_1 \dots k_4} \gamma_m^d \delta(k_1 + \dots + k_4) \left[\underbrace{\phi_c(1) \phi_c(2) \phi_c(3) \phi_c(4)}_{\substack{\text{terms where} \\ \text{every term is} \\ \text{slow.} \quad A}} + \underbrace{\phi_s(1) \phi_s(2) \phi_s(3) \phi_s(4)}_{\substack{\text{terms where every} \\ \text{term is fast} \quad B}} \right]$$

$$+ 4 \underbrace{\phi_c(1) \phi_s(2) \phi_s(3) \phi_s(4)}_{\substack{\downarrow \\ \text{mixed with one slow} \quad C}} + 4 \times \underbrace{\phi_s(1) \phi_c(2) \phi_c(3) \phi_s(4)}_{\substack{\text{mixed with one fast} \quad D}}$$

$$+ 6 \times \underbrace{\phi_c(1) \phi_c(2) \phi_s(3) \phi_s(4)}_{\substack{\text{mixed with 2 fast \& 2 slow} \quad E}}$$

now we integrate over fast mode.

B will just give a number; no slow mode left (not interesting)

A $\nrightarrow$ don't need to anything $\nrightarrow$ This term survives as it is.
When we do average $\langle \dots \rangle$ go away: only E contributes.

Remember that doing the average means

$$\langle \cdot \rangle_{\text{FAST}} = \frac{1}{Z_0} \int \mathcal{D}\phi \, e^{-S_0[\phi]} \cdot (\cdot)$$

$$\text{where } Z_0 = \int \mathcal{D}\phi \, e^{-S_0[\phi]}$$

Example: $\langle \phi(x) \phi(y) \rangle = \frac{1}{Z_0} \int \mathcal{D}\phi \cdot \phi(x) \phi(y) e^{-\int (\cdot)}$

$$\neq \langle \phi(x) \phi(y) \rangle = \frac{1}{Z_0} \int \mathcal{D}\phi \, \phi(x) \phi(y) \exp\left(-\int \frac{d^d k}{(2\pi)^d} \cdot (k^2 + \lambda) |\phi(k)|^2\right)$$

→ This is just path integral formula of two point function.

→ can think of it as $\phi_k \cdot A_{kk} \phi_k$

here matrix A is an operator $k^2 + \lambda$

after

we get

→ Propagator $(\text{or } G_0(k))$

$$G_0(k) = \frac{1}{k^2 + \lambda}$$

$$\neq \langle \phi(x) \phi(y) \rangle = G_0(k) (2\pi)^d \delta(k_3 + k_4)$$

Wicks Theorem (multi point interactions $\Rightarrow$ product of 2 point interaction.)

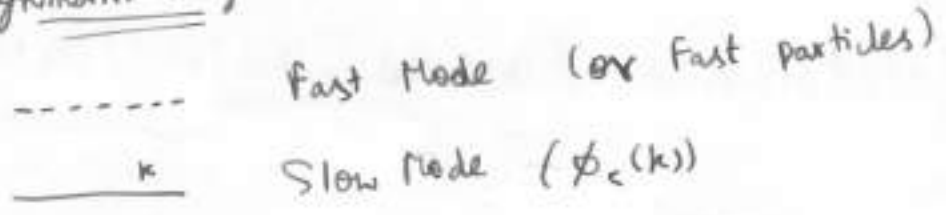
$\langle \phi_1 \dots \phi_k \rangle = 0$ for k odd. (because we cannot decompose it into 2 point interactions)

(C and D goes zero ~~to~~ because of this.)

Consequently terms C & D don't contribute.
 & only E is interesting.

Lets simplify our life by using Diagrammatic Notation.

Feynman Diagrams



$$= \frac{u}{4!} (2\pi)^d \delta(\sum k_i)$$

Processes of Averaging means connecting two fast modes

ex $\langle \text{diagram} \rangle_{\text{fast}} = \text{diagram with loop}$

where loop over fast mode loop^k is $= G_0(k) (2\pi)^d \delta(k+k_1)$
 $\text{loop}^{k_2} = G_0(k) (2\pi)^d \delta(k+k_2)$

$S_{int} =$

$\langle S_{int} \rangle =$

gives number (not interesting)

can connect two; but one left becomes zero

$$6 \times \frac{k_1 \dots k_3}{k_1 \dots k_3} = 6 \times \frac{u}{4!} \int_{k_1 \dots k_4} (2\pi)^d \delta(k_1 + \dots + k_4) \phi_c(k_1) \phi_c(k_2) \times \underbrace{G_0(k_3)}_{\delta(k_3 + k_4)} (2\pi)^d \delta(k_3 + k_4)$$

$$6 \times \frac{k_1 \dots k_3}{k_1 \dots k_3} = 6 \times \frac{u}{4!} \int_{k_1 \dots k_4} (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4) \cdot \underbrace{\phi_c(k_1)}_{-} \underbrace{\phi_c(k_2)}_{-} \cdot \underbrace{G_0(k_3)}_{\delta(k_3 + k_4)} (2\pi)^d \delta(k_3 + k_4)$$

$$= \frac{u}{4} \int_{k, q} |\phi_c(k)|^2 G_0(q)$$

$$k_1 = -k_2 = k$$

$$k_3 = -k_4 = q$$

↪ The integration factorizes

$$= \frac{u}{4} \int_q G_0(q) \cdot \int_k |\phi_c(k)|^2$$

denote this by
I

$$\therefore I := \int_q G_0(q)$$

$$I = \int_{N_b} \frac{d^d q}{(2\pi)^d} \times \underbrace{G_0(q)}_{\frac{1}{q^2 + \gamma}} \rightarrow \frac{1}{q^2 + \gamma}$$

$$= \int_{N_b} \frac{d^d q}{(2\pi)^d} \times \frac{1}{q^2 + \gamma}$$

(go to spherical coordinates
because integrand just depends
on q^2)

$$I = \int_{N_b} \frac{S_d}{(2\pi)^d} \cdot \frac{q^{d-1} \cdot dq}{q^2 + \gamma}$$

" S_d = area of
 S^{d-1} of unit radius.
(a unit sphere embedded
in d dimensional space)

$0^{\text{th}} + 1^{\text{st}}$ order

(1988)

$$\tilde{S}[\phi_c] = \frac{1}{2} \int \left[|k|^2 + \left(\lambda + \frac{u}{2} \mathbb{I} \right) \right] |\phi_c(k)|^2 + \frac{u}{4!} \int (2\pi)^d \delta(\sum k_i) \phi_c(k_1) \dots \phi_c(k_4)$$

Read off $\tilde{\lambda}_2 = 1 = b^\eta \Rightarrow \boxed{\eta = 0}$ ~~because $b \rightarrow 1$~~

$\tilde{\lambda}_0 = \lambda + \frac{u}{2} \mathbb{I} \Rightarrow$ The ~~anomalous~~ anomalous dimension comes out to be zero from 0^{th} & 1^{st} order perturbation theory.

$\tilde{\eta}_4 = u$

~~$\tilde{\lambda} = b^{2-\eta} \tilde{\lambda}_0 = b^{2-0} \tilde{\lambda}_0 = b^2 \tilde{\lambda}_0$~~

$\tilde{\lambda} = b^{2-\eta} \cdot \tilde{\lambda}_0 = b^2 \left(\lambda + \frac{u}{2} \mathbb{I} \right)$

$\tilde{u} = b^{4-d-\eta} \cdot \tilde{\eta}_4 = b^{4-d}$

Renormalization Group (R.G.) Recursion Relation (to First Order)

$\tilde{\lambda} = b^2 \left(\lambda + \frac{u}{2} \mathbb{I} \right)$

$\tilde{u} = b^{4-d} \cdot u$

Algebraic relation... generally algebraic relations are difficult to deal with ... so let's turn them into differential equation.

$\tilde{\lambda} = e^{2\Delta\ell} \left(\lambda + \frac{u}{2} \mathbb{I} \right)$

Using the ~~param~~ parametrization of b as $b = e^{\Delta\ell}$

$\Rightarrow \tilde{\lambda} \approx (1 + 2\Delta\ell) \left(\lambda + \frac{u}{2} \mathbb{I} \right)$

$\Rightarrow \tilde{\lambda} = \lambda + 2\Delta\ell \cdot \lambda + \frac{u}{2} \mathbb{I} + (\Delta\ell \cdot u \cdot \mathbb{I})$... so we will

because $\mathbb{I} \propto \Delta\ell \leftarrow$ not keep this term

This is actually higher order

$$\frac{d\lambda}{d\ell} = \frac{\tilde{\lambda} - \lambda}{\Delta\ell} = 2\lambda + \frac{u}{2} \frac{1}{\Delta\ell}$$

$$\Rightarrow \boxed{\frac{d\lambda}{d\ell} = 2\lambda + \frac{u}{2} \cdot \frac{S_d}{(2\pi)^d} \cdot \frac{\Lambda^d}{\Lambda^2 + \lambda}}$$

Similarly.

$$\boxed{\frac{du}{d\ell} = (4-d)u = \epsilon u}$$

Differential form of
R.G. relations.
(to first order)
 $\epsilon = 4-d$

More generally

for a coupling g ;
we get relations like

$$\frac{dg}{d\ell} = \beta_g$$

→ we call it β_g
it is called Beta Function. for the
corresponding coupling constant g .

$$\beta_\lambda = 2\lambda + \frac{u}{2} \cdot \frac{S_d}{(2\pi)^d} \cdot \frac{\Lambda^d}{\Lambda^2 + \lambda}$$

$$\beta_u = (4-d)u$$

Fixed point corresponds to $\beta_g = 0$.

We like to think of ℓ as some fictitious time or
something which parametrizes the curve in the space of
couplings. ... Now the coupling changes as you change
the scale.

~~is~~ l is fictitious time which corresponds to evolution of g in the space of couplings as we change the scale. (pg 90)

$$[x] = 2 ; [u] = 4 - d = \epsilon \quad \text{recall.}$$

In general, we can show that in the leading order we find $\frac{dg}{dl} = \beta_g = [\dim g] g + \dots$
 $\hookrightarrow$ higher order terms.

if $[\dim g] > 0 \Rightarrow g$ is increasing ; Relevant

$[\dim g] < 0 \Rightarrow g$ is decreasing ; Irrelevant

Dimension of g tells whether g increases or decreases (and this is related to relevant and irrelevant of the coupling constants)

To get interesting physics we have to go to 2nd order

To get Nobel Prize we need 2nd order

$$-\frac{1}{2} [\langle S_{int}^2 \rangle - \langle S_{int} \rangle^2]$$

This can have connected & disconnected diagrams

This is very convenient & actually removes all the disconnected diagrams

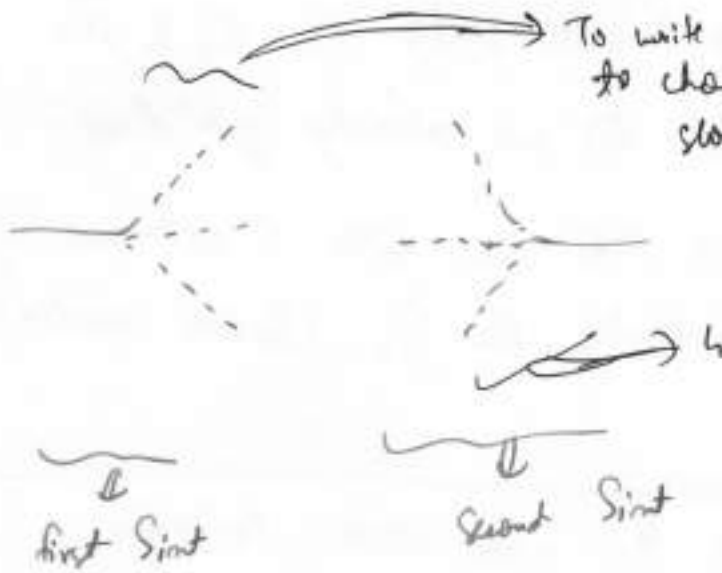
The result in the end is just connected diagrams

.. and now we can classify

them depending on how many external lines they have

i.e. with 0, 2, 4, 6 lines.

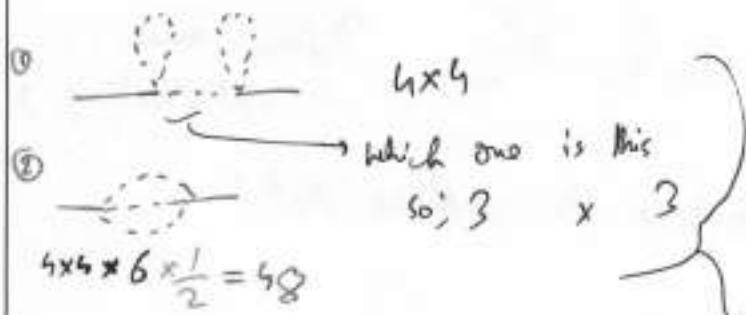
2 legs



To write this we have to choose which leg is slow leg; so 4 possibilities

(pg 91)

we get



with each diagrams, comes along the combinatorial factor related to the diagram

so; $4 \times 3 \times 3 \times 3 \times \frac{1}{2} = 72$

This is the pre-factor coming from $-\frac{1}{2} [(\sin^2) - (\sin)^2]$

So ~~72 diagrams which have 1 to 2 slow legs in end~~ ... in



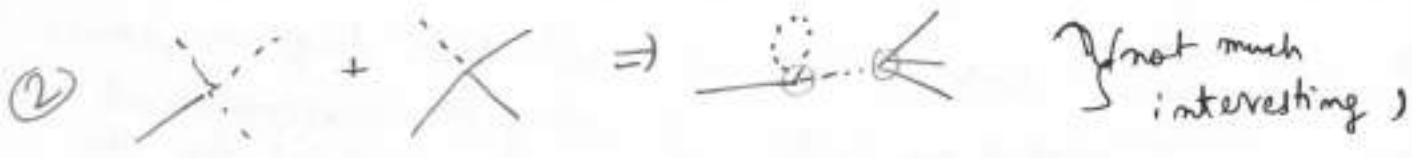
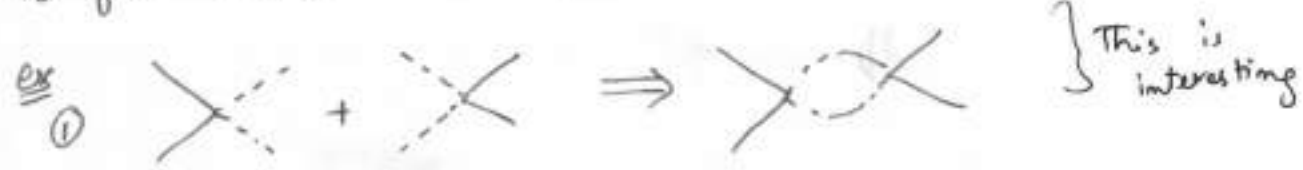
In the end we always end up with 2 as a slow legs

2 external leg can contribute only to $\tilde{\pi}_0$ & $\tilde{\pi}_2$

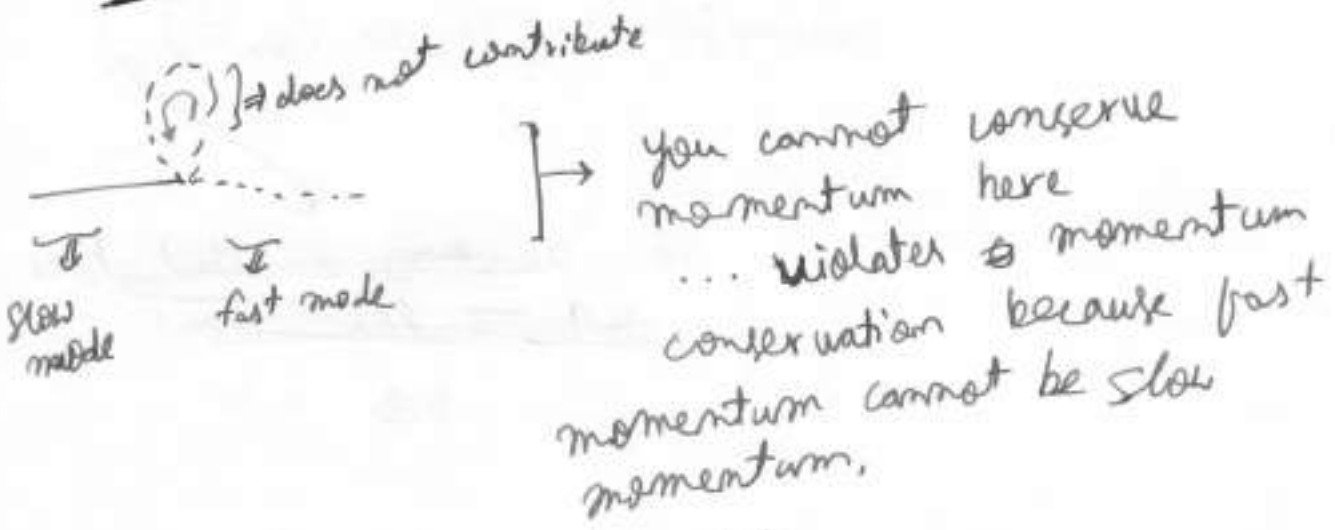
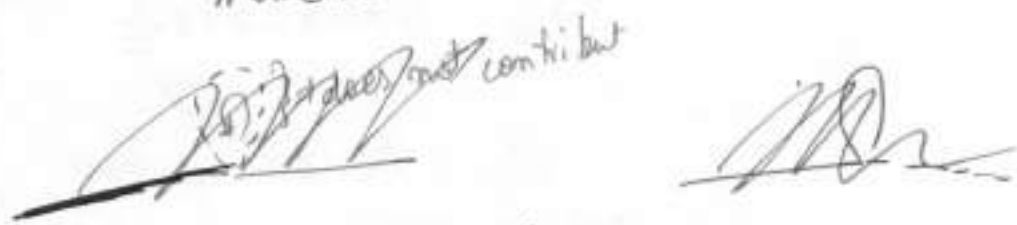
we don't care about this, we already got these.

$\therefore$ what we care about the one with 4 external legs;
which will contribute to $\tilde{\pi}_4$; hence contribute to
a coupling.

To find corrections to β_u , we need 4 legs.

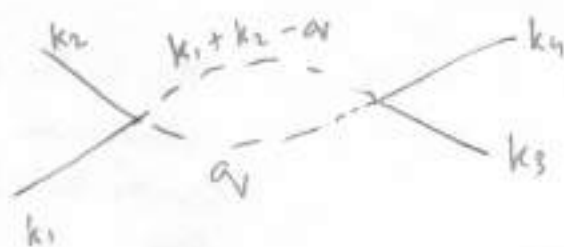


At each vertex you have to conserve total momentum



② is unphysical $\rightarrow$ zero ~~contribution~~ contribution; $\delta(\dots)$ will make it zero.

But we have to compute (1)



$$u_L = 6$$

for $\dots$

$$u_L = 6$$

for $\dots$

$\times$ Some factor for connecting

$$6 \times 6 \times 2 \times \frac{1}{2} = 36$$



$$\begin{aligned} \text{Diagram} &= \left(\frac{u}{4!}\right)^2 \times 36 \int (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4 - q) \phi_c(k_1) \phi_c(k_2) \phi_c(k_3) \phi_c(k_4) \times \\ &\quad \times G_0(q) \times G_0(k_3 + k_4 - q) \end{aligned}$$

$$\left(\frac{u}{4!}\right)^2 36 \int (2\pi)^d \delta(\sum k_i) \phi_c(k_1) \dots \phi_c(k_4) G_0(q) G_0(k_3 + k_4 - q)$$

$k_1, \dots, k_4, q$

k_3 & k_4 are slow modes
 $\rightarrow$ so to get feeling
 q is fast mode.

$$G_0(k_3 + k_4 - q) \simeq G_0(-q)$$

$$\text{but } G_0(-q) = G_0(q)$$

2.4

$$= \left(\frac{u}{4!}\right)^2 36 \int_{k_1, k_2} (2\pi)^d \delta(k_1) \phi_c(k_1) \cdot \phi_c(k_2) \times I_2$$

This is the only diagram which contributes for our u .

where; $I_2 = \int_{\mathcal{Q}} G_0^2(q)$

$$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{S_d}{(2\pi)^d} \cdot \frac{q^{d-1}}{(\eta + q^2)^2} dq$$

are length of interval to be small.

$$= \Delta l \frac{S_d}{(2\pi)^d} \times \frac{\Lambda^d}{(\eta + \Lambda^2)^2}$$

Correction of $\tilde{\pi}_4$:

$$\frac{u}{4!} - \left(\frac{u}{4!}\right)^2 \times 36 \times I_2 = \frac{\tilde{\pi}_4}{4!}$$

$$\tilde{u} = \tilde{\pi}_4 b^{4-d} = \left(u - \frac{3}{2} u^2 I_2\right) b^{4-d}$$

Now final R.G. equation with first order correction for both π and u

$$\frac{d\pi}{d\ell} = \beta_\pi = 2\pi + \frac{u}{2} \cdot \frac{K_d}{\pi + \Lambda^2}$$

$$\frac{du}{d\ell} = \beta_u = \varepsilon u - \frac{3}{2} u^2 \cdot \frac{K_d}{(\pi + \Lambda^2)^2}$$

where $K_d = \frac{S_d}{(2\pi)^d} \Lambda^d$

corrected R.G. equation to first order.

Give the diagrams which contribute to k^2 term?

(pg 95)

$\rightarrow \leftarrow$ gives contribution to $V \phi^6$... but it is ~~irrelevant~~
is irrelevant.

Lecture 9: Wilsonian Renormalization 3: Triumph: Wilson-Fisher Fixed Point, Critical-Exponents.



Renormalization Group flow.

Part III Fixed points in small ϵ - expansion.

Fixed point is where $\beta_r = 0 = \beta_u$

($\epsilon = 4 - d$): Expansion.

we will treat ϵ as small & expanding everything to leading order in ϵ . (leading order up to $O(\epsilon)$)

One of the fixed point is simple $(r^*, u^*) = (0, 0)$
(because each term in β_r & β_u are ~~proportional to~~ proportional to r or u)

$(r^*, u^*) = (0, 0)$ called
Gaussian Fixed Point
or MFT " " "

We can also have another fixed point.

if ϵ small then $u \propto \epsilon$; then from $\beta_r \Rightarrow r \propto \epsilon$
in β_u

so; we know that $r \propto \epsilon$
 $u \propto \epsilon$

so; we can neglect g in the denominator in RG equation.

Then $\beta_u = 0 \Rightarrow \epsilon u - \frac{3}{2} u^2 \frac{K_d}{\Lambda^4} = 0$

$$u = \frac{2}{3} \epsilon \frac{\Lambda^4}{K_d} \Rightarrow r = - \frac{u}{4} \frac{K_d}{\Lambda^2} = - \frac{\epsilon}{6} \Lambda^2$$

So; we have another fixed point

$$(r^*, u^*) = \left(-\frac{\epsilon}{6} \Lambda^2, \frac{2}{3} \epsilon \frac{\Lambda^4}{K_d} \right)$$

↳ This is interesting one which was awarded Nobel prize, called Wilson Fisher Point.

How things behave near fixed point

We want to linearize the RG equations around the critical point.

⇒ so, we Taylor expand the β -functions

Introducing a parameter $\delta r = r - r^*$, $\delta u = u - u^*$

Now, we can write down the linearization in matrix form since we have two parameters.

~~$\frac{d}{dt} \delta r$~~ ~~$\frac{d}{dt} \delta u$~~

$$\begin{aligned} \frac{dr}{dt} &= \beta_r \\ \delta \left(\frac{dr}{dt} \right) &= \frac{\partial \beta_r}{\partial r} \delta r + \frac{\partial \beta_r}{\partial u} \delta u \\ \therefore \frac{d}{dt} (\delta r) &= \frac{\partial \beta_r}{\partial r} \delta r + \frac{\partial \beta_r}{\partial u} \delta u \end{aligned}$$

$$\frac{d}{dt} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix} = \begin{pmatrix} \frac{\partial \beta_r}{\partial r} & \frac{\partial \beta_r}{\partial u} \\ \frac{\partial \beta_u}{\partial r} & \frac{\partial \beta_u}{\partial u} \end{pmatrix} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix}$$

$r = r^*$
 $u = u^*$

Taylor expansion written as matrix

→ This matrix is called M ; which captures the behavior of RG flow close to the fixed point.

$$M = \begin{pmatrix} 2 - \frac{u}{2} \frac{K_d}{(\lambda + \Lambda^2)^2} & \frac{1}{2} \frac{K_d}{\lambda + \Lambda^2} \\ \frac{3u^2 K_d}{(\lambda + \Lambda^2)^3} & \varepsilon - 3u \frac{K_d}{(\lambda + \Lambda^2)^2} \end{pmatrix}$$

$$\approx \begin{pmatrix} 2 - \frac{u}{2} \frac{K_d}{\Lambda^4} & \frac{1}{2} \frac{K_d}{\Lambda^2} \\ 0 & \varepsilon - 3u \frac{K_d}{\Lambda^4} \end{pmatrix}$$

→ This is our linearization matrix.

Evaluate M at corresponding ^{initial} ~~fixed~~ point.

- i) one for Gaussian fixed point.
- ii) another for Wilson Fick fixed point.

Want to find eigenvalues and eigenvectors. ~~at the~~

Gaussian Fixed Point

$$(\lambda^*, u^*) = (0, 0)$$

Then our Matrix $M = \begin{pmatrix} 2 & \frac{1}{2} \frac{K_d}{\Lambda^2} \\ 0 & \varepsilon \end{pmatrix}$

Eigen values 2 & ε .

$$\left. \begin{aligned} \lambda_t &= 2 \\ \lambda_{\tilde{u}} &= \varepsilon \end{aligned} \right\} \text{eigenvalues.}$$

Eigenvectors $(M - \lambda I)V = 0$

Then $V_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $V_{\tilde{u}} = \begin{pmatrix} 1 \\ \frac{2(\varepsilon - 2)\Lambda^2}{K_d} \end{pmatrix}$

In terms of new variable t & $\tilde{u}$; the matrix is diagonal; and these are corresponding principle direction.

So; R.G. equations in terms of new variables t & $\tilde{u}$ (Pg 100) becomes

$$\frac{dt}{d\ell} = \lambda_t t \quad ; \quad \frac{d\tilde{u}}{d\ell} = \lambda_{\tilde{u}} \tilde{u}$$

$$\Downarrow$$

$$t = t_0 e^{\lambda_t \ell}$$

$$\Downarrow$$

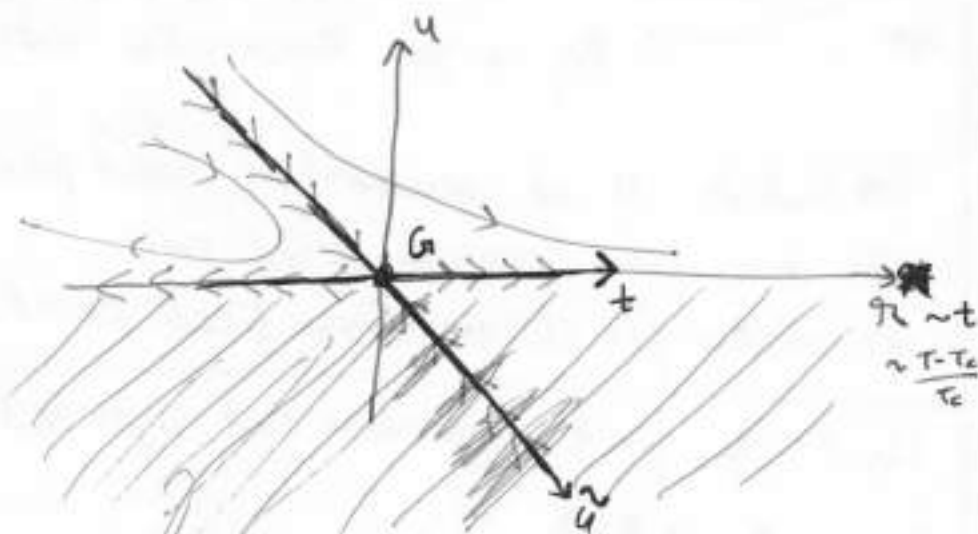
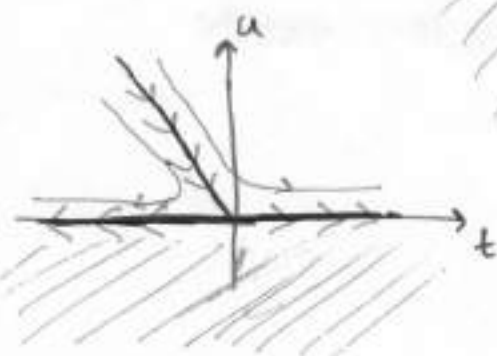
$$\tilde{u} = \tilde{u}_0 e^{\lambda_{\tilde{u}} \ell}$$

$$t = t_0 \cdot e^{2\ell}$$

$$\tilde{u} = \tilde{u}_0 \cdot e^{\varepsilon \ell}$$

Cases

(i) $\varepsilon < 0$ ($d > 4$)



negative values of u is unphysical
(for Landau's theory to be well defined)

We can never go to ~~into~~ fixed point, if we are not at $T = T_c$.

G is stable fixed point

$\because$ because $\tilde{u}$ direction is stable

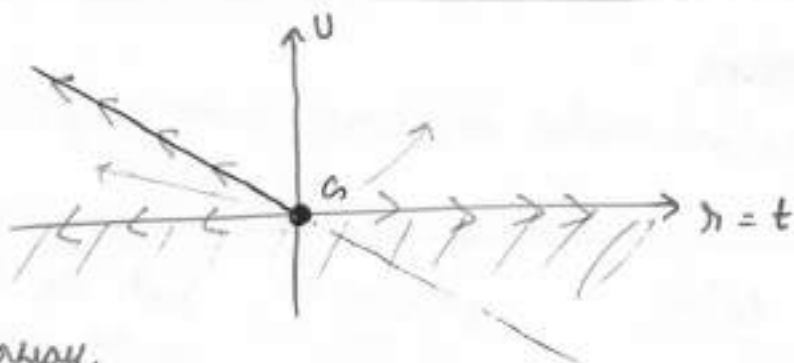
meaning it flows in.

(π direction is always unstable ... we are not at $T = T_c$)

G describes MFT critical point, and ~~to~~ it is because $T = T_c$ is where $\gamma = 0$; precisely happens at $T = T_c$ as predicted by MFT.

For $\epsilon > 0$ ($d < 4$)

Such G is unstable: if you are not at this G you flow away.



$\therefore$ if you are in vicinity of G ; you flow away.

$\hookrightarrow$ This is precisely the reason why MFT fails in $d < 4$ dimensions.

~~because MFT predicts~~

Reason as to why MFT fails in $d < 4$

because MFT predicts $T = T_c$ is physical ... but we found it is unstable so can't be physical.

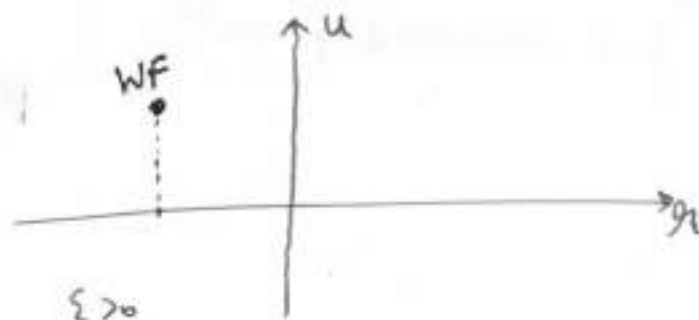
for $d = 4$; we ~~can't~~ can't comment right now; we will have to go to higher orders.

Wilson Fisher Fixed Point

$$(g^*, u^*) = \left(-\frac{\epsilon}{6} \Lambda^2, \frac{2}{3} \epsilon \frac{\Lambda^4}{K_d} \right)$$

cases

ii) $\epsilon > 0$ ($d < 4$)



True transition temperature occurs for $\epsilon < 0$ (pg 102)
 means $T < (T_c)_{\text{MF7}}$ as predicted by MFT.

~~Fluctuations~~
 Fluctuations make ordering more difficult $\therefore T_c < (T_c)_{\text{MF7}}$

$\epsilon < 0$ ($d > 4$) ... $u^* < 0$ $\rightarrow$ but we know $u < 0$ does not make any sense.

So; Wilson Fisher fixed point does not exist in $d > 4$.

That's why in $d > 4$, the only fixed point we have is gaussian fixed point & it is stable one; That's why MFT is correct.

In $d > 4$ gaussian fixed point $\rightarrow$ so MFT is correct.

for $\epsilon > 0$ ($d < 4$)

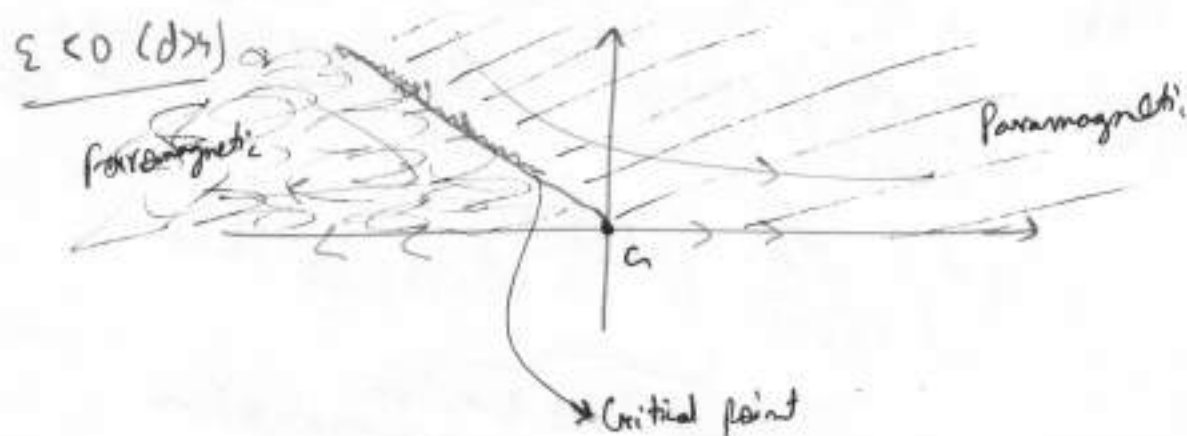
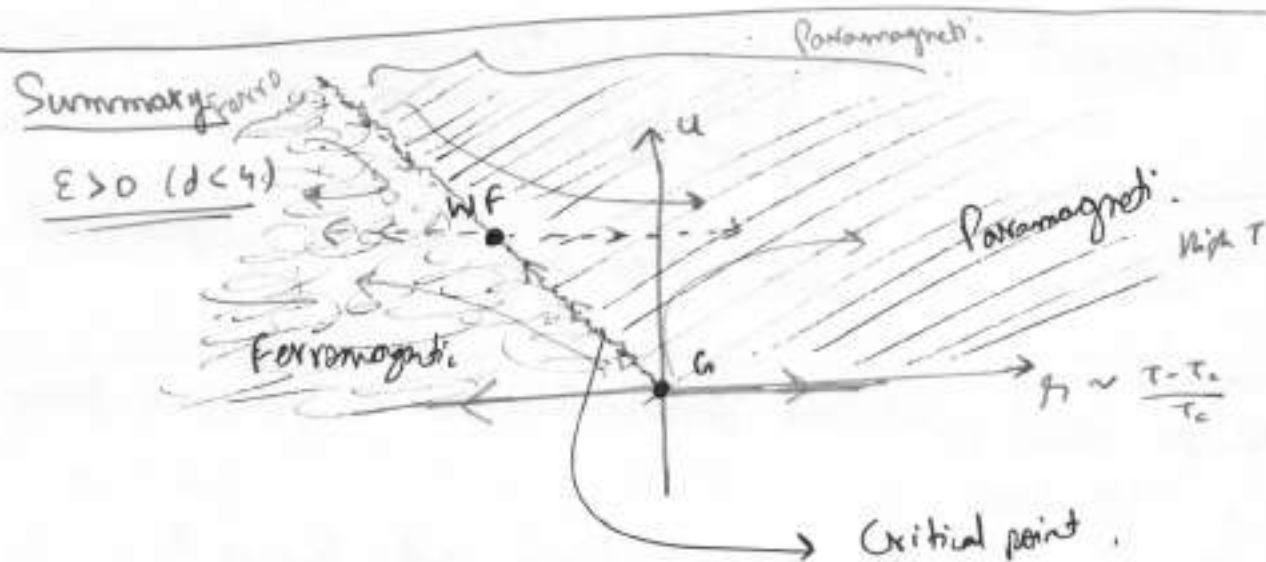
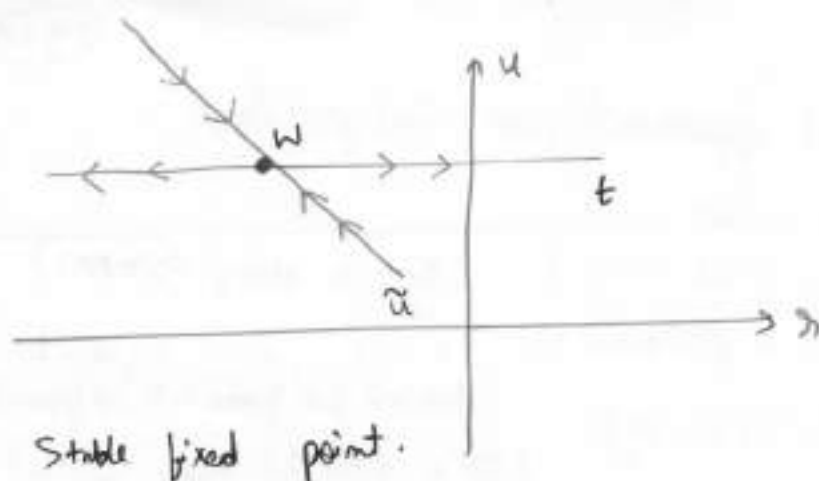
$$M = \begin{pmatrix} 2 - \frac{\epsilon}{3} & , & \frac{K_d}{2\Lambda^2} \\ 0 & , & -\epsilon \end{pmatrix}$$

$$\boxed{\lambda_t = 2 - \frac{\epsilon}{3}} \quad ; \quad \boxed{\lambda_{\tilde{u}} = -\epsilon}$$

Then we can find corresponding vectors

$$V_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$V_{\tilde{u}} = \begin{pmatrix} 1 \\ -\frac{4}{3} \frac{(3+\epsilon)\Lambda^2}{K_d} \end{pmatrix}$$



$d > 4$ MFT is correct.

PART IV Critical Exponents (Triumph)

Reminder

$$C \propto |t|^{-\alpha}, M \propto |t|^\beta, \chi \propto |t|^{-\gamma}, \beta \propto M^\delta$$

$$G(\eta) \propto \frac{\exp(-\eta/\xi)}{\eta^{d-2+\eta}}$$

$\eta \Rightarrow$ anomalous dimension

$$\xi \propto |t|^{-\nu}$$

Yesterday Tutorial

(pg 104)

Not all critical exponents are independent.

Scaling Laws:

Rushbrook: $\alpha + 2\beta + \gamma = 2$ (This is very generic)

Ozoriof: $\alpha + \beta(\delta + 1) = 2$ (" " " " " can be derived just from T.D. configurations)

Fisher: $\gamma = \gamma(2 - \eta)$ (It is little bit more subtle)

Josephson*: $\alpha = 2 - \nu d$ (dimension dependent; requires extra hypothesis, the hyperscaling hypothesis) (This may not be always valid)

⇒ If all are valid; then only 2 are independent.

Close to critical point and under R.G. flow in "fictitious time"

l. We have

(i) $\tilde{k} = kb$, $\tilde{x} = x/b$; we set $b = e^l$

(ii) $\tilde{\phi}(\tilde{x}) = \int \frac{d^d \tilde{k}}{(2\pi)^d} \tilde{\phi}(\tilde{k}) e^{i\tilde{k} \cdot \tilde{x}} = \int \frac{d^d k}{(2\pi)^d} b^{d+2-\eta} \phi(k) e^{ik \cdot x}$

$= b^d \int \frac{d^d k}{(2\pi)^d} \cdot b^{-\frac{d+2-\eta}{2}} \phi(k) e^{ik \cdot x}$

Wave function renormalization.

$\tilde{\phi}(\tilde{x}) = b^{\frac{d-2+\eta}{2}} \phi(x)$

→ This is how ϕ changes if you flow for finite l.

(iii) $\tilde{t} = t e^{\lambda_t l}$
 $\tilde{u} = u e^{\lambda_u l}$

(iv) $G(x) = \langle \phi(x) \phi(0) \rangle - \langle \phi(x) \rangle \langle \phi(0) \rangle$

Now we use that G is quadratic in ϕ .

$$G(\tilde{x}, \tilde{t}, \tilde{u}) = \cancel{b^{d-2+\eta}} G\left(\frac{x}{b}, te^{\lambda_t t}, ue^{\lambda_u t}\right)$$

because
quadratic in ϕ

$$G(\tilde{x}, \tilde{t}, \tilde{u}) = G\left(\frac{x}{b}, te^{\lambda_t t}, ue^{\lambda_u t}\right) \\ = b^{d-2+\eta} G(x, t, u)$$

Quadratic in ϕ

$$\Rightarrow \boxed{G(\tilde{x}, \tilde{t}, \tilde{u}) = b^{d-2+\eta} G(x, t, u)}$$

It is equivalent to saying

$$G(x, t, u) = b^{-(d-2+\eta)} G\left(\frac{x}{b}, te^{\lambda_t t}, ue^{\lambda_u t}\right)$$

→ This is the master formula from which you can find all critical exponents

let us start exactly at fixed point where $t=0=u$.

$$\text{Then we get: } G(x, 0, 0) = b^{-(d-2+\eta)} G\left(\frac{x}{b}, 0, 0\right)$$

so; exactly at critical point.

$$G(x, 0, 0) \propto \frac{1}{x^{d-2+\eta}}$$

η is the anomalous dimension which we get from how couplings transform

⇒ η is ~~not~~ anomalous dimension as before.

$\boxed{\eta=0}$ even to second order in R.G. flow.

lets move away from fixed point in t direction

$$\text{Now we consider: } G(x, t, 0) = b^{-(d-2+\eta)} G\left(\frac{x}{b}, te^{\lambda_t t}, 0\right)$$

true for any b .

So; the sneaky idea is to specifically choose $b = x$

Then we get

(Pg 106)

$$G(x, t, 0) = X^{-(d-2+\eta)} G(1, tx^{\lambda_t}, 0)$$

remember : $b = e^1$; $te^{\lambda_t 1} = t(e^1)^{\lambda_t}$
 $= tx^{\lambda_t}$

$$\Rightarrow \boxed{G(x, t, 0) = X^{-(d-2+\eta)} G(1, tx^{\lambda_t}, 0)}$$

so; we require $G(1, tx^{\lambda_t}, 0)$ is some function of $F(x/\xi)$.

so; tx^{λ_t} must be a function of $\frac{x}{\xi}$.

$$\text{so; } x^{\lambda_t} \cdot t = (x t^{\nu_{\lambda_t}})^{\lambda_t} = \left(\frac{x}{\xi}\right)^{\lambda_t}$$

$$\Rightarrow \boxed{\nu_{\xi} \propto |t|^{-\frac{1}{\lambda_t}}}$$

the critical exponent

$$\boxed{\nu = \frac{1}{\lambda_t}}$$

$$\nu = \frac{1}{\lambda_t} \stackrel{\text{u.f.}}{=} \frac{1}{2 - \frac{\varepsilon}{3}} \approx \frac{1}{2} + \frac{\varepsilon}{12} \quad (d < 4)$$

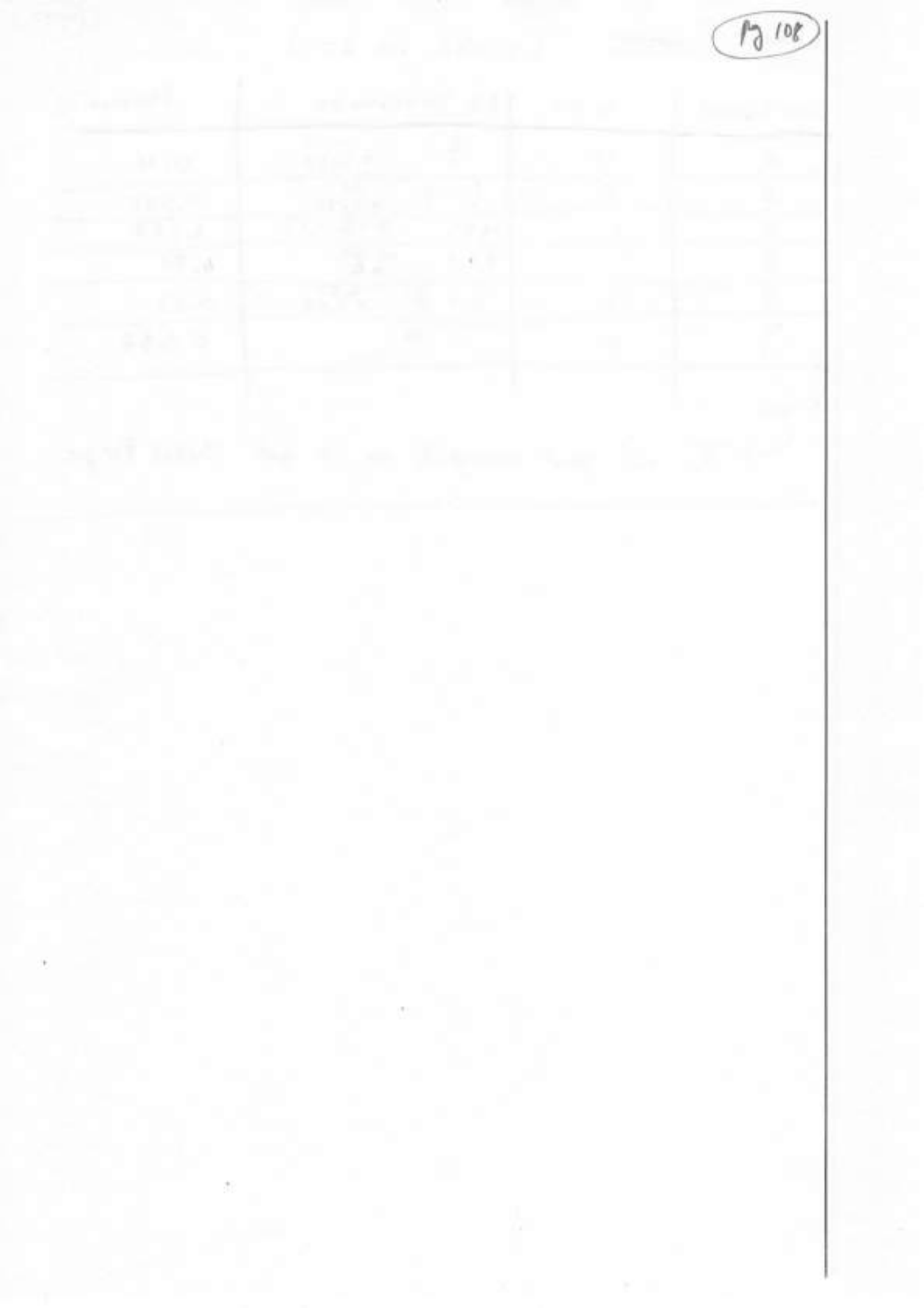
$$\stackrel{\text{u.f.}}{=} \frac{1}{2} \quad (d > 4)$$

Complete Summary(ISING in $d=3$)

pg 107

Critical Exponent	MFT	RG ϵ -Expansion	Numerics
α	0	$\frac{\epsilon}{2}$ $\stackrel{in 3D}{=} 0.17$	0.11
β	$\frac{1}{2}$	$\frac{1}{2} - \frac{\epsilon}{6}$ $\stackrel{in 3D}{=} 0.33$	0.326
γ	1	$1 + \frac{\epsilon}{6}$ $\stackrel{in 3D}{=} 1.17$	1.237
δ	3	$3 + \epsilon$ $\stackrel{in 3D}{=} 4$	4.79
ν	$\frac{1}{2}$	$\frac{1}{2} + \frac{\epsilon}{12}$ $\stackrel{in 3D}{=} 0.58$	0.63
η	0	0	0.036

→ This was good enough & so it got Nobel Prize.



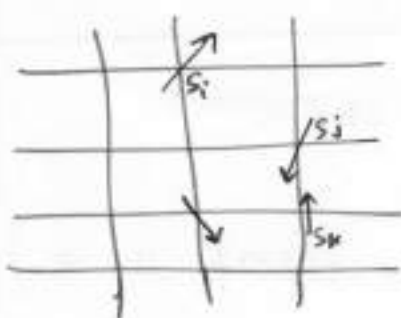
Lecture 10: Vector Model, Goldstone Modes, CMW Theorem and Lower Critical Dimensions

- Discrete $\mathbb{Z}_2$ Symmetry $\phi \rightarrow -\phi$
 - Order Parameter is scalar $\phi(x)$
- Then you will have same Landau-Ginzburg functional ... hence are under same ISING UNIVERSALITY CLASS

Now, we will go beyond Ising Universality Class: where we will have more than one order parameter & different kind of symmetry.

e) Systems with Continuous Symmetry

VECTOR MODEL - n



d-dimensional cubic lattice

we can have spins at each lattice site; and they will not only point up or down; but they can be along any directions: i.e. The spins will be arbitrary vectors

Vectors describing spins.

$$H = - \frac{1}{2} \sum_{i,j} \vec{S}_i \cdot \vec{S}_j J_{ij}$$

n -component vectors

(in general n can be different from d)

Then $\vec{S}_i = (S_i^1, S_i^2, \dots, S_i^n)$

now, we require $|\vec{S}_i| = 1$

- $n=3$: Heisenberg Model (interesting case)

Continuous $O(n)$ symmetry

(n -dimensional rotation)

(pg 110)

(arbitrary global rotation where all spins are rotated by same angle & along the same axis $\Rightarrow$ and the value of Hamiltonian remains unchanged)

Landau - Ginzburg

$$S = \int d^d x \left[\frac{1}{2} (\nabla \vec{\phi}) \cdot (\nabla \vec{\phi}) + \frac{g}{2} \vec{\phi} \cdot \vec{\phi} + \frac{u}{4} (\vec{\phi} \cdot \vec{\phi})^2 \right]$$

∇ in d -dimensions

$\vec{\phi}$ in n -dimensions

$\hookrightarrow$ (not 4!)

... for same reason
... will become clear later)

• In homework: we do small $\epsilon = (4-d)$ expansion

$\Rightarrow$ Qualitatively similar to Ising;

(except that: critical exponents now depend on n)

• However near $d=2$ physics is different (very different).

Let us write $\vec{\phi} = \rho(x) \vec{n}(x)$

$\hookrightarrow$ let's parametrise the field in this way.

where $\rho(x) = \sqrt{\vec{\phi} \cdot \vec{\phi}}$ (magnitude) ; $\vec{n} \cdot \vec{n} = 1$
 $\vec{n}$ = unit vector

$$\nabla \phi^a = n^a \nabla \rho + \rho \nabla n^a$$

$$(\nabla \vec{\phi}) \cdot (\nabla \vec{\phi}) = \nabla \phi^a \nabla \phi^a = \overset{\uparrow 1}{n^a n^a} (\nabla \rho)^2 + 2\rho \overset{\uparrow 0}{\nabla \rho} n^a \overset{\uparrow 0}{\nabla n^a} + \rho^2 (\nabla n^a)^2$$

$$(\nabla \vec{\phi}) \cdot (\nabla \vec{\phi}) = (\nabla \rho)^2 + \rho^2 \nabla n^a \nabla n^a$$

$$\vec{\phi} \cdot \vec{\phi} = \rho^2$$

(10/11)

So; Now; Landau - Ginzburg Functional becomes

$$S[\rho, \vec{n}] = \int d^d x \left[\frac{1}{2} (\nabla \rho)^2 + \frac{1}{2} \rho^2 \nabla n^a \nabla n^a + \frac{\pi}{2} \rho^2 + \frac{u}{4} \rho^4 \right]$$

Landau's Theory: $f(\rho) = \frac{\pi}{2} \rho^2 + \frac{u}{4} \rho^4$ (lets define free energy)
(equivalent to MFT)

ρ is now order parameter here.

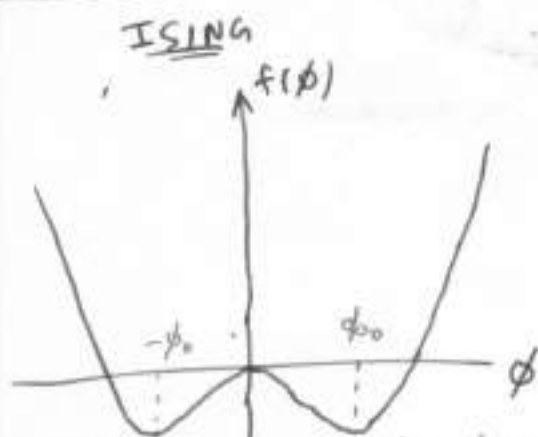
$$f'(\rho) = 0 \Rightarrow 2 \frac{\pi}{2} \rho + 4 \frac{u}{4} \rho^3 = 0 \Rightarrow \boxed{\rho = \rho_0 = \sqrt{-\frac{\pi}{u}}} \quad \pi < 0$$

$$= 0 \quad \pi > 0$$

Landau theory does not know anything about $\vec{n}$ because $\vec{n}$ only enters through the kinetic term; & we know Landau theory does not have the kinetic term.

So; in this approximation in Landau's Theory: $\vec{n}$ is arbitrary.

$\vec{n}$ is not controlled by Landau's approximation.

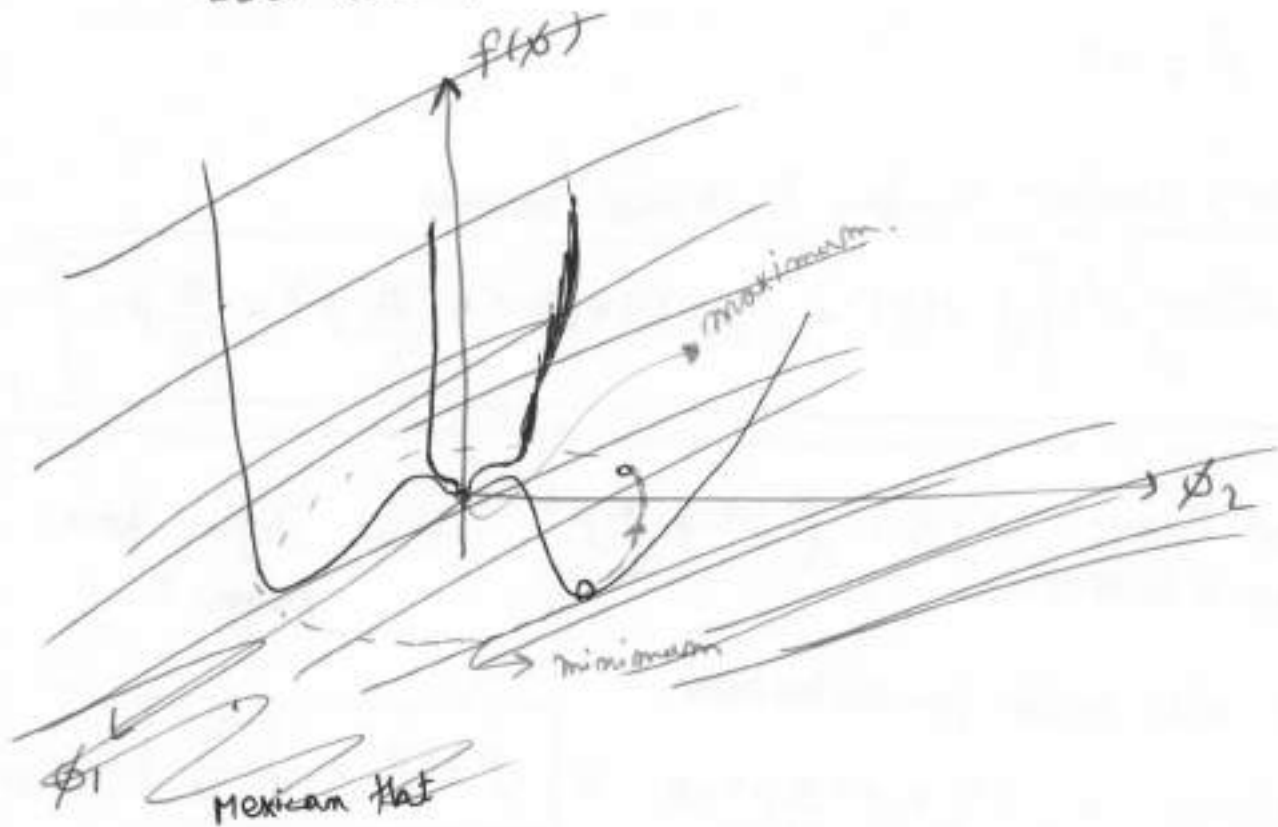


Spontaneous symmetry breaking. ; but to jump to other we have to overcome the barrier

This precisely (the barrier) corresponds to the fact that the symmetry we have broken is $\mathbb{Z}_2$ symmetry.

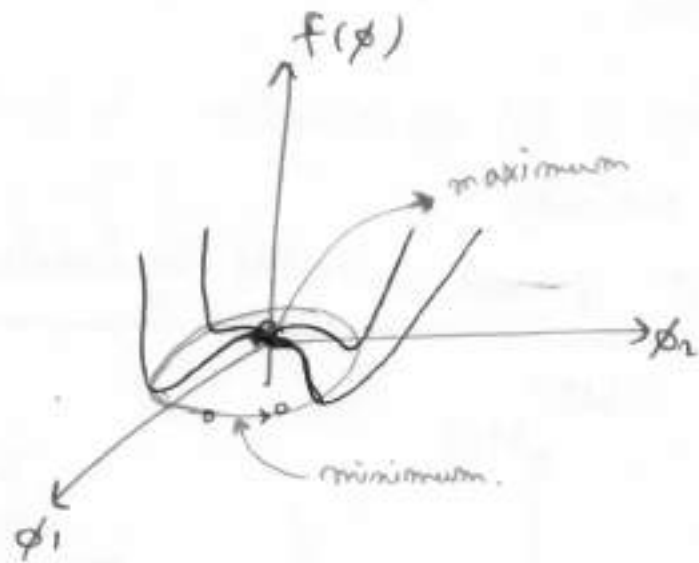
Vector ($n=2$)

(1/12)



Here we can travel from one minimum to other without costing much; i.e. minima connected by continuous symmetry : can go to to different minimas without crossing any potential barrier.

A better diagram (try)



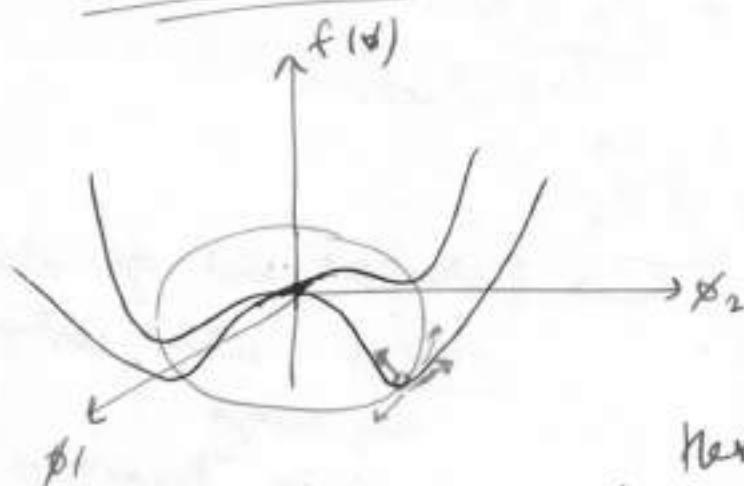
Fluctuations in Ising

(Pg 13)



once we take fluctuations into account : we can fluctuate around a minima, but the fluctuations will be such that ending up to other minima is very unlikely, because it will cost lot of energy.

Fluctuations in vector ($n=2$)



These will be called $\delta\phi_{||}$ (notation)

Here we can have two types of fluctuation.

& One of the fluctuations will not cost any energy & we can reach the other minima through thermal fluctuations without ~~costing~~ costing any ~~extra~~ energy.
i.e. fluctuations $\delta\phi_{\perp}$ are simple

So, we see here that physics is very different as compared to Ising.

So, Through $\delta\phi_{\perp}$ we can go around minimas without costing any energy.

Small Perturbations around MFT (Landau) Approximation Pg 115

Say that $\vec{n} = (1, 0, 0, \dots, 0)$ and write

$$\vec{\phi}(x) = \rho_0 \vec{n}$$

↳ MFT.

(ie; at a randomly chosen $\vec{n}$ as predicted by MFT;

so we chose $\vec{n}$ to be along ϕ_1 because it is completely arbitrary in MFT)

Now lets include fluctuations.

$$\vec{\phi}(x) = \underbrace{\rho_0 \vec{n}}_{\text{MFT}} + \underbrace{\rho_0 \delta \vec{\phi}}_{\text{Fluctuations.}}$$

where $\delta \vec{\phi} = \underbrace{\vec{n} \delta \phi_1}_{\text{fluctuations which go up & down ... cost energy}} + \underbrace{\delta \vec{\phi}_\perp}_{\text{fluctuations which go along without costing any energy ... its in transverse direction}}$



fluctuations which go along without costing any energy ... its in transverse direction

$$\vec{\phi} \cdot \vec{\phi} = (\rho_0 \vec{n} + \rho_0 \delta \vec{\phi}) \cdot (\rho_0 \vec{n} + \rho_0 \delta \vec{\phi})$$

$$= \rho_0^2 + 2\rho_0^2 \vec{n} \cdot \delta \vec{\phi} + \rho_0^2 \delta \vec{\phi} \cdot \delta \vec{\phi}$$

$$\boxed{\vec{n} \cdot \delta \vec{\phi}_\perp = 0} \text{ just by definition.}$$

$$\begin{aligned} \vec{\phi} \cdot \vec{\phi} &= \rho_0^2 + 2\rho_0^2 \delta \phi_1 + \rho_0^2 (\delta \phi)^2 \\ &= \rho_0^2 [1 + 2\delta \phi_1 + (\delta \phi_1)^2 + (\delta \phi_\perp)^2] \end{aligned}$$

(pg 15)

$$\vec{\phi} \cdot \vec{\phi} = \rho_0^2 (1 + 2\delta\phi_1 + (\delta\phi_1)^2 + (\delta\phi_\perp)^2)$$

$$(\vec{\phi} \cdot \vec{\phi})^2 = \rho_0^4 (1 + 4\delta\phi_1 + 6(\delta\phi_1)^2 + 2(\delta\phi_\perp)^2 + \dots)$$

neglect terms like $(\delta\phi_\perp)^4$ because treating small perturbations.

neglecting ~~high~~ higher order terms.

now,

$$S[\phi] = \int d^4x \left[\frac{\rho_0^2}{2} (\nabla \delta\phi_1)^2 + \frac{\rho_0^2}{2} (\nabla \delta\phi_\perp)^2 + \frac{\eta}{2} \rho_0^2 (1 + 2\delta\phi_1 + (\delta\phi_1)^2 + (\delta\phi_\perp)^2) + \frac{u}{4} \rho_0^4 (1 + 4\delta\phi_1 + 6(\delta\phi_1)^2 + 2(\delta\phi_\perp)^2) \right]$$

look at terms with $\delta\phi_1$:

$$\frac{\eta}{2} \rho_0^2 2\delta\phi_1 + u \rho_0^4 \delta\phi_1$$

now; we know $\rho_0^2 = -\frac{\eta}{u}$; plugging this here gives 0.

so; linear term in perturbation has to vanish.
 (we sort of knew this; we are expanding around minimum so the first order perturbation has to vanish)

looking at quadratic terms $(\delta\phi_1)^2$:

$$\frac{\eta}{2} \rho_0^2 (\delta\phi_1)^2 + \frac{3}{2} u \rho_0^4 (\delta\phi_1)^2 = \rho_0^2 |\eta| (\delta\phi_1)^2$$

something familiar

looking at $(\delta\phi_\perp)^2$:

$$\left(\frac{\eta}{2} \rho_0^2 + \frac{u}{2} \rho_0^4 \right) (\delta\phi_\perp)^2 = 0$$

New & happy.

The fact that perturbation ~~give~~ give non-trivial contribution here upto order $(\delta\phi_1)^2$ is familiar.

The fact that $\delta\phi_\perp$ vanishes upto second order is ~~new~~ new.

~~$$S[\phi] = \int d^4x \left[\frac{\rho_0^2}{2} (\partial \phi)^2 \right]$$~~

$$S[\phi] = \int d^4x \frac{\rho_0^2}{2} \left[(\nabla \delta \phi_1)^2 + (\nabla \delta \phi_2)^2 + 2\lambda (\delta \phi_1)^2 \right]$$

Something like
mass term

$\therefore$ The modes $\delta \phi_1$ has mass $\sqrt{2\lambda}$

No mass term for $\delta \vec{\phi}_\perp$. (These modes are massless)

$\hookrightarrow$ it is result of continuous $O(n)$ symmetry breaking that we have now massless mode which can go around the minimum of the Mexican hat.

These massless modes which originate in the spontaneous symmetry breaking of some continuous symmetry ~~are~~ are called Goldstone modes (Goldstone Bosons)

In general they arise when continuous symmetry is spontaneously broken.

$\hookrightarrow$ Since they are massless bosons; it does not take any energy to create them.

Write in k-space

$$\delta \vec{\phi}_\perp(\vec{x}) = \frac{i}{V} \sum_{\vec{k}} \delta \vec{\phi}_\perp(\vec{k}) e^{i\vec{k} \cdot \vec{x}}$$

$$\delta \vec{\phi}_1(\vec{x}) = \frac{1}{V} \sum_{\vec{k}} \delta \vec{\phi}_1(\vec{k}) e^{i\vec{k} \cdot \vec{x}}$$

(13/17)

Then ; we find that , the action in momentum space becomes

$$S = \frac{\rho_0^2}{2V} \sum_{\vec{k}} \left(k^2 |\delta \phi_1(k)|^2 + k^2 |\delta \phi_2(k)|^2 + 2|\lambda| |\delta \phi_1(k)|^2 \right)$$

if set $k \rightarrow 0$: then, goldstone modes don't cost ~~any~~ any energy.

we have potential term for $\delta \vec{\phi}_1$ example 1)
 don't have any potential term for $\delta \vec{\phi}_2$ $\delta \vec{\phi}_1$

crystallization of liquid.

→ continuous symmetry of liquid is broken & lattice is formed. ; we order our atoms in lattice.

so,

example 2) ACOUSTIC PHONONS

(breaking the continuous, translational symmetry as the ~~cryst~~ crystal crystallizes and forms a lattice)

We can show that these things have energy given by

$$\epsilon_k = \hbar c_s k$$

↖ some speed in the medium

as $k \rightarrow 0$; no energy $\Rightarrow$ So they are massless modes.

(photons in some sense are also ~~goldstone~~ goldstone modes)

now we have name for $\delta \vec{\phi}_2$; they are goldstone modes.

In lower dimensions; then there ~~are~~ massless modes
can destroy any order.

(pg 118)

↳ They are nasty things; but only in lower dimensions

Correlation Function

$$\langle \phi \rangle = \rho_0 \vec{n}$$

$$G_1(x-x') := \langle \delta \phi_1(x) \delta \phi_1(x') \rangle$$

$$= \frac{1}{V^2} \sum_k \langle \delta \phi_1(k) \delta \phi_1(-k) \rangle e^{i k \cdot (x-x')}$$

correlation function in
momentum space.

$$\frac{V}{\rho_0^2} \cdot \frac{1}{k^2 + 2\gamma}$$

picking up
some factor due to
normalization

$$\propto \frac{1}{k^2 + 2\gamma}$$

(reading it
from action)

$$m^2 = 2\gamma$$

$$G_1(x-x') = \frac{1}{|\vec{x} - \vec{x}'|^{d-2}} e^{-\frac{|\vec{x} - \vec{x}'|}{\xi}}$$

as expected, it
should be like
Yukawa

because ϕ_1 modes are massive
modes

$$\xi = \frac{1}{\sqrt{2|\kappa|}} \propto \frac{1}{t^\nu} \quad ; \quad \nu = \frac{1}{2} \quad (\text{same as MFT prediction})$$

For $\delta \phi_1$; there won't be any exponential; because the
correlation length ξ will be infinite.

On the other hand; for perpendicular modes we have

$$G_1(x-x') = \langle \delta \vec{\phi}_1(x) \cdot \delta \vec{\phi}_1(x') \rangle =$$

~~$$G_1(x-x') = \frac{1}{V} \sum_k \langle \delta \phi_1^a(k) \delta \phi_1^a(-k) \rangle e^{i k \cdot (x-x')}$$~~

$$G_1(x-x') = \frac{1}{V} \sum_k \langle \delta \phi_1^a(k) \delta \phi_1^a(-k) \rangle e^{i k \cdot (x-x')}$$

now reading from action we get

~~for each~~ $\frac{V}{\rho_0^2} \cdot \frac{1}{k^2}$ for each mode a .
(no mass term)

now we have $(n-1)$ modes $\phi_1 \dots$

$$\Rightarrow \frac{V}{\rho_0^2} \cdot \frac{(n-1)}{k^2}$$

massless $\Rightarrow \sum_1$ diverges.

lets look at

$$G_1(0) = \frac{n-1}{\rho_0^2} \cdot \frac{1}{V} \sum_k \frac{1}{k^2}$$

(turning to integral) $= \frac{n-1}{\rho_0^2} \int \frac{d^d k}{(2\pi)^d} \cdot \frac{1}{k^2}$

depends only on k^2 so we can go to spherical coordinates.

Λ some cutoff given by our lattice.

$$\Rightarrow G_1(0) = \frac{(n-1) S_d}{\rho_0^2 (2\pi)^d} \int_0^\Lambda dk \cdot k^{d-3}$$

The upper limit is not a problem
The integral can be divergent.

The lower limit can be problem in lower dimensions.

... So we put an upper cutoff.

lower limit 0 ;

The integral will be finite if $d > 2$
 " " " " infinite when $d \leq 2$

So; The lower limit 0 is the potential problem.

So, $d > 2$; then contribution of goldstone modes to fluctuations is finite; we can have ordered phase for some temperature $0 < T_c < (T_c)_{MFT}$

but if $d \leq 2$; then the goldstone mode completely change the picture & they have infinite contribution to fluctuations. So; we cannot order the system.
 The system will be disordered because of Goldstone modes.

Particle Physic ... called Coleman Theorem.

Statistical Physic ... " Mermin-Wagner Theorem.

... lets call

COLEMAN - MERMIN - WAGNER Theorem

For systems with continuous symmetry, there is no long range order at any finite temperature in $d \leq 2$ dimensions

~~We must~~ we motivated (but not derived) it by our previous analysis.

Two important dimensions

- Upper Critical Dimension: $d = 4$ ($d > 4$ MFT gives correct result)
- Lower Critical Dimension: $d = 2$ $\lim_{d \rightarrow 2^+} T_c(d) = 0$ for system with continuous symmetry.

Lecture 11: Near $d=2$, Vortices, Kosterlitz-Thouless Phase Transition

C-M-W Theorem: Systems with continuous symmetry there is no long range order for $d \leq 2$.

2 $\rightarrow$ it seems like we can't have phase transition for continuous ^{symmetry} system... but we can have certain types of phase transitions.

AIM: Avoid this theorem and show that we can have a phase transition in $d=2$ dimensions.

Avoid means to avoid.

$\Rightarrow$ So, we will have to break some of the assumptions of C-M-W Theorem to get phase transitions.

Near $d=2$ Fluctuations of ρ around ρ_0 are not important. But fluctuations of n^a are important (because $\delta\rho_\perp$ does not cost ~~energy~~ any energy)

now, set $\rho = \rho_0$.

$$S[\vec{m}] = \frac{\rho^2}{2} \int d^d x \nabla n^a \nabla n^a$$

(effective London Ginzburg functional...)

by changing units we can achieve

NON LINEAR ϕ -MODEL

$$S[\vec{m}] = \frac{1}{2T} \int d^d x \nabla n^a \nabla n^a$$

$\Downarrow$ Non-linear ϕ -Model

important for study around 2-dimensions.

looks like free theory, but it is not because there is a constraint $n^2 = 1$.
(This makes things difficult)

Set up a $d=2+\epsilon$ expansion

(pg 122)

We know $\lim_{d \rightarrow 2^+} T_c(d) = 0$ from CMW theorem

~~This does not~~ This does not work as good as expansion around $d=4$; ... but it works well.

We end up to following RG. equation.

$$\frac{d\tilde{T}}{d\ell} = -\epsilon \tilde{T} + (n-2)K_d \cdot \tilde{T}^2$$

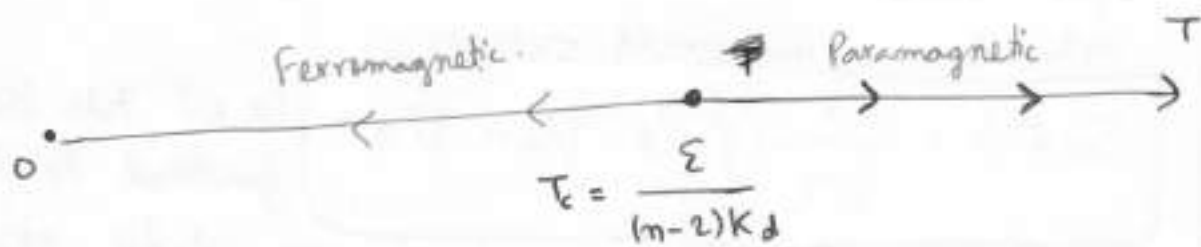
$$\epsilon = d-2$$

$$\tilde{T} = T \Lambda^\epsilon \quad (\text{rescaled } T)$$

$$K_d = \frac{S_d}{(2\pi)^d}$$

$\tilde{T}=0$ is obvious solution of $\beta_{\tilde{T}}=0$

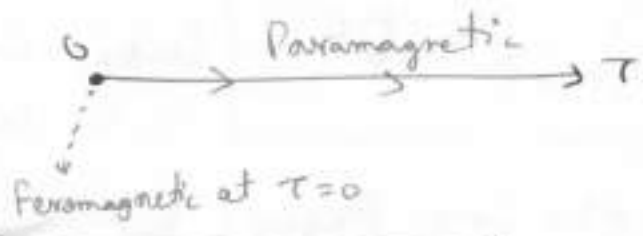
The other solution is $\tilde{T}^* = \frac{\epsilon}{(n-2)K_d}$



~~Note: $d=2$ exactly then $\tilde{T} = \frac{\epsilon}{(n-2)K_d} \tilde{T}^2$ equation at $d=2$~~

Note 1

$d=2$ exactly then



$$\frac{d\tilde{T}}{d\mu} = (n-2) k_n \tilde{T}^2$$

RG equation
at $d=2$

→ β function for Yang Mills in $d=4$

$AdS_5 \leftrightarrow CFT_{d=4} \leftrightarrow$ Spin Chains $d=2$
(by noting that β function is the same)

Very Special Case

$d=2=n$; $\frac{d\tilde{T}}{d\mu} = 0$... β function vanishes upto the leading order in ϵ (This is what we wrote here)

→ But it is actually true to any order in perturbation.

β function vanishes for any T ; so for any T we have like critical point (because fixed point)

(?) Line of critical points for every T (?)

Writing : $\vec{n} = \cos\theta \hat{x} + \sin\theta \hat{y}$

$$\nabla \vec{n} = -\sin\theta \cdot \nabla\theta \hat{x} + \cos\theta \cdot \nabla\theta \hat{y}$$

We want $\nabla n^a \nabla n^a$; so square $(\nabla \vec{n})^2$

$$\begin{aligned} \nabla \vec{n} \cdot \nabla \vec{n} &= \sin^2\theta (\nabla\theta)^2 + \cos^2\theta (\nabla\theta)^2 \\ &= (\nabla\theta)^2 \end{aligned}$$

$$S[\theta] = \frac{1}{2\pi} \int d^2x (\nabla\theta)^2$$

London - Ginzburg for $n=2$ in 2 dimensions.

looks like free theory; but not exactly because θ is angle & it is compactified from 0 to 2π
ie: $\theta \in (0, 2\pi)$... compact ... so not a free theory.

lets pretend θ is not compactified
so: we can calculate correlation functions.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

so: ~~we~~ we calculate

$$G(x) = \langle e^{i\theta(x)} e^{-i\theta(0)} \rangle$$

since the integral diverges.
... we introduce a
cutoff for lattice
spacing.

$$= \dots =$$

$$= \left(\frac{a}{x}\right)^{\frac{T}{2\pi}}$$

$$G(x) = \langle e^{i\theta(x)} e^{-i\theta(0)} \rangle = \left(\frac{a}{x}\right)^{\frac{T}{2\pi}}$$

Compare with $G(x) \propto \frac{e^{-x/\xi}}{x^{d-2+\eta}}$

we notice that we miss exponential part.

→ Comparing these we would conclude

$$\xi \rightarrow \infty$$

; ie: There is no more of these things.

$$\eta = \frac{T}{2\pi}$$

Critical exponents depend on T .

• Line of critical points with T-dependent critical exponent.

• $G(x \rightarrow \infty) \implies 0 \neq M^2$

→ so there is no long range order
because if there were ; then $G(x \rightarrow \infty)$ should go to M^2)

→ ~~Agree~~ Agrees with CMW - Theorem.

→ All these results are provided & not compact.

This is physically wrong.

because we are finding that for any temperature $\xi \rightarrow \infty$.
(and we cannot have ~~this~~ this)

We must have T_c such that the correlation length becomes finite above $T > T_c$.

Cannot have $\xi \rightarrow \infty$ for any T .

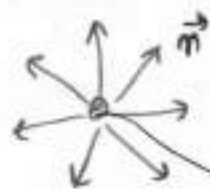
We must have T_c such that

$$G(x) \propto \frac{1}{x^{\eta(T)}} \longrightarrow G(x) \propto \frac{e^{-x/\xi}}{x^{\eta(T)}}$$

$$\xi = \begin{cases} \infty & T < T_c \\ \text{finite} & T > T_c \end{cases}$$

By assuming a non-compact we neglected a possibility of having vortices, i.e. VORTEX

19/26



all spins going out of a vortex

Vortex Core.

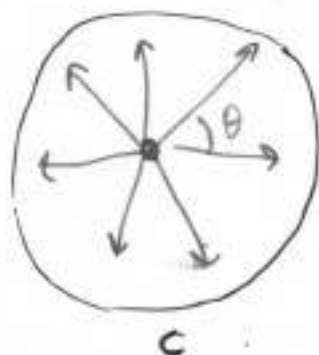
Vortex is something which is called Topological Defect; & we can detect it from very very far away...

... The presence of vortex is through Stokes theorem or Gauss theorem & with them we can detect vortices.

ex charge in a volume; we can calculate flux ~~to~~ very very far away & calculate charge.

↳ Same thing we do; but in less dimensions...

... use Stokes theorem instead of Gauss theorem.



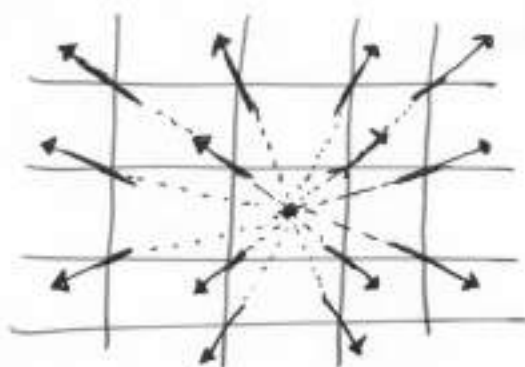
$$\oint_C \nabla \theta \cdot d\vec{l} = 2\pi m$$

m is winding number & characterizes strength of the vortex

(go around circle C)

$m = \dots -2, -1, 0, 1, 2, 3, \dots$

In Lattice



all the spins in the whole lattice has to twist due to presence of vortex in the middle.

(so; it is energetically very costly to create vortex because we to twist all spins)

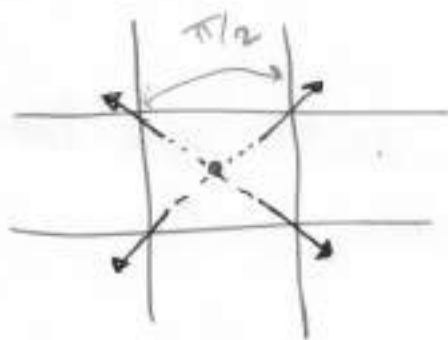
In continuous limit



$$\nabla \theta \propto \frac{1}{r}$$

; close to origin this is problematic.
($r=0$)

~~When integrating~~ When integrating energy we introduce cutoff a .



The biggest gradient you can have for spins if you are in lattice is something like

$$\frac{\pi/2}{a} = \frac{\pi}{2a}$$

Continuous limit breaks down at core.

But lattice ensures everything is well defined.

so you can't have infinite gradient
... max is $\sim \frac{\pi}{2a}$

Lets estimate the energy of the vortex.

19/12/8

here Action is actually the hamiltonian.

$$H = \frac{1}{2} \int d^2x (\nabla \theta)^2$$

$$\delta H = \frac{1}{2} \int d^2x (\nabla \delta \theta) \nabla \theta$$

$$= -\frac{1}{2} \int d^2x (\nabla^2 \theta) \delta \theta \quad (\text{doing by parts})$$

$$\text{Now } \delta H = 0 \Rightarrow -\int d^2x (\nabla^2 \theta) \delta \theta = 0$$

$\Rightarrow$ so we get that vortex is described by $\boxed{\nabla^2 \theta = 0}$.

now; still we have the constraint which defines the vortex

$$\oint_C \nabla \theta \cdot d\vec{l} = 2\pi m \quad (\text{constraint for non-trivial vortex})$$

so; we have to solve

$$\nabla^2 \theta = 0, \quad \oint_C \nabla \theta \cdot d\vec{l} = 2\pi m \quad \text{simultaneously.}$$

remember θ is the angle. now do the spins ~~the~~ actually twist around the core.

In polar coordinates; (r, ϕ)

we have $\boxed{\theta = m\phi}$ will solve both the equations.

indeed $\nabla\theta = \frac{m}{\hbar} \hat{\phi} \dots$ & satisfies.

Energy $E = \frac{1}{2} \int d^2x (\nabla\theta)^2$

$$= \frac{1}{2} \int_0^{2\pi} d\phi \int_a^L dr \frac{m^2}{\hbar^2} r$$

in polar coordinates

- * L size of the system
- * a lattice spacing.

$$E = m^2 \pi \ln\left(\frac{L}{a}\right)$$

→ This is how much energy it costs to create a vortex

costs lots of energy to ~~create~~ create a vortex;
(need to turn all spins in lattice)
however it can still be thermodynamically possible.

At finite T : to find if vortices are possible, consider the free energy $F = E - TS$

Entropy: $S = \log(\text{no. of states})$

→ ~ how many places we can place the vortices.

$$\approx \log(\text{no. of places we can place the vortex})$$

$$= \log\left(\frac{L}{a} \cdot \frac{L}{a}\right)$$

$\left(\frac{L}{a}\right)^2 = \text{no. of cells we have.}$

$$\Rightarrow \boxed{S = 2 \ln\left(\frac{L}{a}\right)}$$

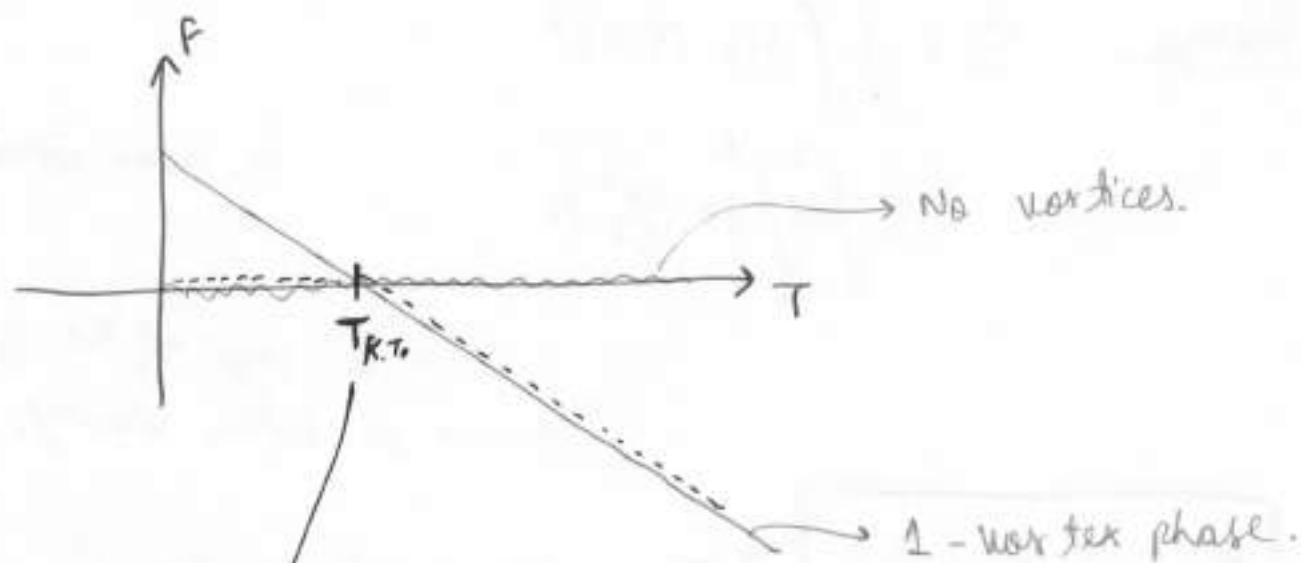
so; entropy is also huge.

$$\boxed{F = (\pi m^2 - 2T) \ln\left(\frac{L}{a}\right)}$$

Simplest possible vortex

19130

$$m=1 \quad ; \quad \Rightarrow \quad F = (\pi - 2T) \log(L/a)$$



So ; There exists a critical temperature $T_{K.T.}$

$$\therefore F=0 \Rightarrow \boxed{T_{K.T.} = \frac{\pi}{2}}$$

$T > T_{K.T.} = \frac{\pi}{2}$; we can have vortices.

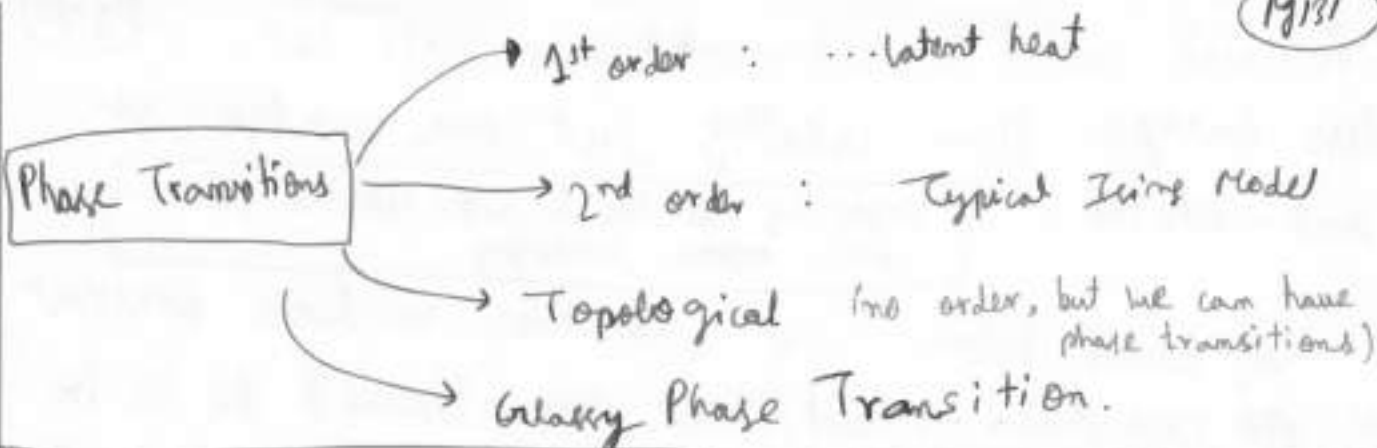
$T_{K.T.}$; K.T. for Kosterlitz & Thouless

$T_{K.T.}$: Kosterlitz & Thouless temperature.

awarded Nobel
Prize in 2016.

... also DUNCAN HALDANE.

It is not a standard phase transition; but it
is a Topological Phase Transition.



Vortex and Anti-vortex

There are like charge. ~~2~~ a

vortex is characterized by positive m .
 Anti-vortex " " " negative m .

Remark // Can have anti-vortices ($m < 0$)



$m = +1$

Vortex



$m = -1$

Anti-vortex.

We can sort of pop out of the vacuum, i.e. the lattice, both of vortex & anti-vortex simultaneously.

It's like, if you have vacuum fluctuations, i.e. you have vacuum, then you ~~are~~ ~~create~~ create a charge and anti-charge ... it does not cost much energy.

... we can't measure these effects if we are too far.

To create vortex & anti-vortex pair costs lot less energy than creating just one vortex or anti-vortex.

Separating vortex & anti-vortex pair costs ~~more~~ energy

So; even below T_{KT} , can have vortices present in ~~the~~ spin chain: but these are bound to be in the form of pair.

Can have vortices in pairs for $T < T_{KT}$ but not isolated ones.

$\therefore$ Isolated vortices can only happen for $T > T_{KT}$ where you have enough energy.

$$\xi = \begin{cases} \infty & T < T_{KT} \\ \text{FINITE} & T > T_{KT} \end{cases}$$

People sometime, give a physical interpretation for ξ as distance between two vortices.

- $T > T_{KT}$ we can have free vortices; ~~you~~ so you can have finite distance.
- for $T < T_{KT}$ you don't have any free vortices, so distance between them is ~~not~~ technically infinite.

~~Not~~ ★ Azimuth (search on internet)

Lecture 12: Monte Carlo Methods, Sampling, Markov Chain, Single-Spin-Flip Metropolis Algorithm.

Main Reference:

Newman & Barkema, "Monte Carlo Methods in Statistical Physics", chapters 1-3.

(Added reference in
lec 13: 1-4, 8.3.

Our goal is to develop numerical methods to study phenomena such as phase transition for many-body systems of N interacting particles.

Recall,

$$\langle Q \rangle = \text{Tr}(Q\rho)$$

In canonical ensemble

$$\rho = \frac{1}{Z} e^{-\beta H} \quad , \text{ means } \langle Q \rangle = \frac{1}{Z} \text{Tr}(Q e^{-\beta H})$$

remind; $\beta = \frac{1}{k_B T}$ (we will set $k_B = 1$)

$Z = \text{Tr}(e^{-\beta H})$ is the partition function.

For classical systems;

$$\langle Q \rangle = \sum_{\mu} Q_{\mu} p_{\mu} \quad ; \quad Z = \sum_{\mu} e^{-\beta E_{\mu}}$$

$$\text{where } p_{\mu} = \frac{1}{Z} e^{-\beta E_{\mu}}$$

μ represents a state of the system.

E_{μ} is the energy of state μ .

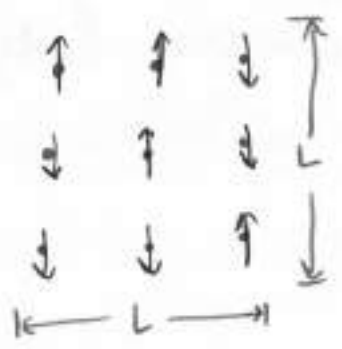
We would like to study expectation values and, ultimately, phase transitions in the thermodynamic limit.
($N \rightarrow \infty$)

For example,

19132

$$\langle E \rangle = \frac{\sum_n E_n e^{-\beta E_n}}{\sum_n e^{-\beta E_n}}$$

Lets consider; how to evaluate $\sum_n$ directly for a nearest-neighbour (nn) Ising model. on a d-dimensional cubic lattice with linear length L .



$$N = L^d$$

↑ no. of spins

On each site i , there is a binary variable $\sigma_i = \pm 1$.

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

↪ sum over nearest neighbours.

- $\sum_{\langle ij \rangle}$ denotes a sum over nn pairs.
- J is a coupling strength.
- h is an external field.

Exact solutions are known for

- $d=1$ (see Tutorial 2)
- $d=2$ (Onsagers, 1944)

The total # of ~~states~~ acceptable states is 2^N .
ie: $2^{(L^d)}$

For $N=25$, evaluating $\sum_n$ requires at least 1 minute.

N	time
25	$\sim 1 \text{ min}$
50	$\sim 60 \text{ years}$
80	$\sim 7 \times 10^{10} \text{ years}$ (larger than age of universe!)

We are limited to lattices ~~to~~ to 19138
 $L = 7 \text{ or } 8$ in $d = 2$.

Classical Monte Carlo (MC) Methods

Idea: Handle this exponential complexity of calculating

$$\langle Q \rangle = \frac{\sum_n Q_n e^{-\beta E_n}}{\sum_n e^{-\beta E_n}}$$

by considering only M states $\mu_1, \mu_2, \dots, \mu_M$ selected at random. We would like to get an accurate estimate for $M \ll 2^N$ for large N .

For example; if we choose each state μ_m ($1 \leq m \leq M$) ~~from~~ uniformly from the 2^N possible states, then we can estimate $\langle Q \rangle$ from

$$Q_M = \frac{\sum_{m=1}^M Q_{\mu_m} e^{-\beta E_{\mu_m}}}{\sum_{m=1}^M e^{-\beta E_{\mu_m}}}$$

"Estimator" $\nearrow$

Let's consider E_n for the Ising model in $d=1$.

$$\uparrow \downarrow \downarrow \dots \uparrow \quad H = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} \quad \left(\begin{array}{l} h=0, \\ \text{open boundary conditions} \end{array} \right)$$

From Tutorial 2, we know

$$Z = 2 [2 \cosh(\beta J)]^{N-1} \Rightarrow \langle E \rangle = - \frac{\partial \log Z}{\partial \beta}$$

$$= -(N-1) \cdot J \cdot \tanh(\beta J)$$

More generally, if the states are taken from a probability distribution w_n , then the estimator for $\langle Q \rangle$ is

$$Q_M = \frac{\sum_{m=1}^M Q_{\mu_m} \cdot W_{\mu_m}^{-1} \cdot e^{-\beta E_{\mu_m}}}{\sum_{m=1}^M W_{\mu_m}^{-1} \cdot e^{-\beta E_{\mu_m}}}$$

Important Sampling

We would like to sample the more "important" ~~states~~ (high probable) states more frequently.

~~Therefore~~ In particular, if $W_{\mu} = P_{\mu} = e^{-\beta E_{\mu}} / Z$, then

$$Q_M = \frac{1}{M} \sum_{m=1}^M Q_{\mu_m}$$

But how do we choose states from this distribution when we don't know Z ?

Markov - Chain Monte Carlo

Idea: Given state μ_m , move to the next state μ_{m+1} according to transition probability $T(\mu_m \rightarrow \mu_{m+1})$ (does not depend on history kind of thing)

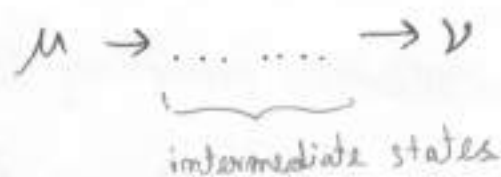
→ does not depend on $\mu_1, \mu_2, \dots, \mu_{m-1}$

$$\sum_{\nu} T(\mu \rightarrow \nu) = 1 \quad (\text{transition probabilities also have to be normalized})$$

The resulting set $\{\mu_1, \mu_2, \dots, \mu_m\}$ is called a Markov Chain of states.

To reach states from a target probability distribution W_{μ} , an algorithm must satisfy the conditions

- ① Ergodicity: given μ and ν , it must be possible (non-zero probability) to reach ν from μ . (1913c)



- ② Detailed Balance (DB):

The rates of transitions into and out of any state μ must be equal. (if they are not equal, then our probability distribution would be changing with time)

(ie: otherwise W_μ would change with time)

$$\text{so: } \sum_{\nu} W_{\nu} T(\nu \rightarrow \mu) - \sum_{\nu} W_{\mu} T(\mu \rightarrow \nu) = 0$$

Rate of transition into μ from some state

Rate of transition out of μ to some other state ν .

One possible way to satisfy this condition:

$$W_{\mu} T(\mu \rightarrow \nu) = W_{\nu} T(\nu \rightarrow \mu) \quad \text{Detailed Balance (DB) condition.}$$

→ all our algorithm will satisfy this.

$$\text{So, where } W_{\mu} = e^{-\beta E_{\mu}} / Z$$

$$\frac{T(\mu \rightarrow \nu)}{T(\nu \rightarrow \mu)} = e^{-\beta(E_{\nu} - E_{\mu})}$$

$$\text{Notation: } T(\mu \rightarrow \nu) = \underbrace{g(\mu \rightarrow \nu)}_{\text{Selection Probability}} \underbrace{A(\mu \rightarrow \nu)}_{\text{Acceptance probability.}}$$

Selection Probability

Acceptance probability.

• $g(\mu \rightarrow \nu)$: Probability of proposing the move $\mu \rightarrow \nu$
 (probability that given state μ , our algorithm suggest moving it to state ν)

• $A(\mu \rightarrow \nu)$: Probability of ~~accepting~~ accepting the move
 (A is probabilistic; but in general we don't have to always ~~accp~~ accept the move)

We want $A(\mu \rightarrow \nu)$ to be as close to 1 as possible so that we sample as many different states as possible.

Single-Spin-Flip Algorithms for the Ising Model

Given a state μ_m of N spins at step m of the Markov chain; propose a new state μ_{m+1} that differs from μ_m by a single spin flip at some lattice site i .

ex

$\uparrow$	$\uparrow$	proposed move $\rightsquigarrow$	$\uparrow$	$\uparrow$
$\downarrow$	$\uparrow$		$\downarrow$	$\downarrow$
	$\uparrow$	$\sigma_i \rightarrow -\sigma_i$		

Lets say that the lattice site i is chosen with equal probability such that

$$g(\mu \rightarrow \nu) = \begin{cases} \frac{1}{N} & \text{if } \mu \text{ and } \nu \text{ differ by a single-spin flip (ssf)} \\ 0 & \text{if } \mu \text{ and } \nu \text{ differ by more than one ssf.} \end{cases}$$

By DB, for such a ssf

$$e^{-\beta(E_\nu - E_\mu)} = \frac{A(\mu \rightarrow \nu) \frac{1}{N}}{A(\nu \rightarrow \mu) \frac{1}{N}} = \frac{A(\mu \rightarrow \nu)}{A(\nu \rightarrow \mu)}$$

Let's consider two options for choosing the acceptance probabilities.

Option #1 $A(\mu \rightarrow \nu) = A_0 \cdot e^{-\frac{1}{2} \beta (E_\nu - E_\mu)}$

We are free to choose A_0 provided $A(\mu \rightarrow \nu) \leq 1$ for any ssf. Since the minimum energy difference for a ssf on a d -dimensional cubic lattice is $-4Jd$.



in 2d.

so: $A_0 = e^{-2\beta Jd}$

$$A(\mu \rightarrow \nu) = e^{-\frac{1}{2} \beta (E_\nu - E_\mu + 4Jd)}$$

Option #2 Metropolis Algorithm (1953)

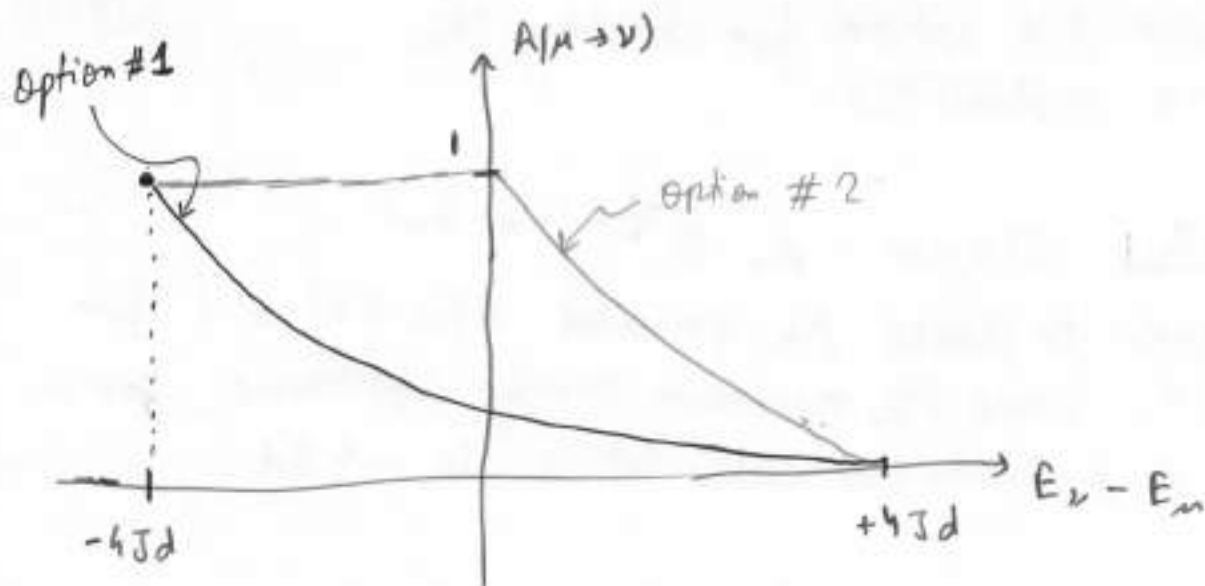
For a general proposed move $\mu \rightarrow \nu$ (not necessarily a ssf) and target distribution W_μ .

$$A(\mu \rightarrow \nu) = \min \left(1, \frac{g(\nu \rightarrow \mu)}{g(\mu \rightarrow \nu)} \cdot \frac{W_\nu}{W_\mu} \right)$$

For the ssf ($g(\mu \rightarrow \nu) = \frac{1}{N}$) and $W_\mu = e^{-\beta E_\mu} / Z$:

this tells
$$A(\mu \rightarrow \nu) = \begin{cases} e^{-\beta (E_\nu - E_\mu)} & \text{if } E_\nu - E_\mu > 0 \\ 1 & \text{otherwise.} \end{cases}$$

Exercise: show that this choice for $A(\mu \rightarrow \nu)$ satisfies DB.



He would like to use option #2 as compared to option #1 because acceptance ratio is higher.

Single-Spin-Flip Metropolis Algorithm for the d-dimensional Ising Model.

- ① Generate a random initial state μ .
 - ② Choose a site i of the lattice with equal probabilities.
 - ③ Calculate the energy difference ΔE associated with flipping $S_i \rightarrow -S_i$.
 - ④ Generate a random ~~number~~ number $r \in [0, 1)$ from a uniform distribution.
 - ⑤ If $\Delta E \leq 0$ or $r < e^{-\beta \Delta E}$, accept the flip and get $\mu_{m+1} \neq \mu_m$.
Otherwise, reject the flip ($\mu_{m+1} = \mu_m$).
- Repeat many times from ② with μ_{m+1} .

Lecture 13 Equilibration and Measurement, Monte Carlo for the 2D Ising Model, Finite-Size Scaling.

Specifically, we studied the single-spin flip (ssf).

Metropolis's algorithm, where we get μ_{m+1} from μ_m by ~~choosing~~ choosing site i at random and accepting $\sigma_i \rightarrow -\sigma_i$ with probability $A(\mu_m \rightarrow \mu_{m+1})$ given as

$$A(\mu_m \rightarrow \mu_{m+1}) = \min(1, e^{-\beta(E_{\mu_{m+1}} - E_{\mu_m})})$$

This is optimal.

From DB: $\frac{A(\mu \rightarrow \nu)}{A(\nu \rightarrow \mu)} = e^{-\beta(E_\nu - E_\mu)}$ we have this constraint.

For any μ, ν one of $A(\mu \rightarrow \nu)$ or $A(\nu \rightarrow \mu)$ is equal to 1. and we basically can't do better than that...

Outline:

- Data Analysis Practices
 - ↳ Equilibration
 - ↳ Measurement correlations
- Code: MC for the 2D Ising Model.
- Cluster Algorithms.
- Estimating T_c from specific heat and susceptibility.
- Finite-size scaling.
- May be: XY model.

Equilibration

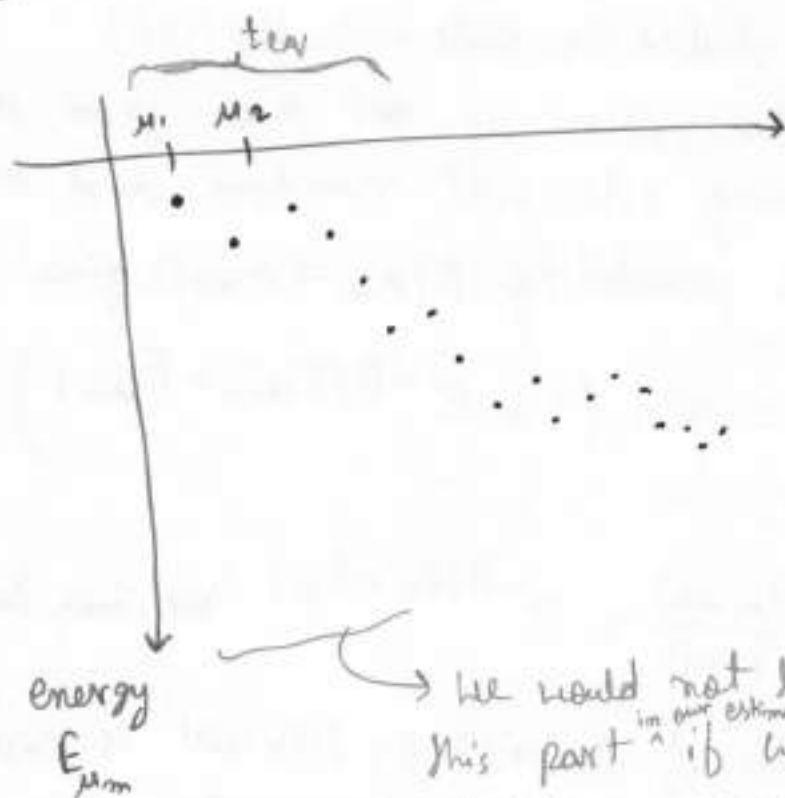
Recall we start our sampling from any μ_1 .

In many cases, μ_1 and the subsequent $\mu_2, \mu_3, \dots$

might have very low P_{μ} .

(pg 140)

For example, we might see the following for the energy of our samples v/s "Markov time".



we would not like to include this part ^{in our estimation} if we are trying to calculate an estimator for any expectation value

lets call it equilibration time t_{eq}

$$Q_M = \frac{1}{M - t_{eq}} \sum_{m=t_{eq}+1}^M Q_{\mu_m}$$

Measurement

When calculating Q_M , we should ideally use samples μ_m that are statistically independent.

(When we use ssf algorithm; our neighbouring states will be very correlated.. will be very similar to each other)

For ssf algorithms, we usually do a "sweep" of $O(N)$ proposed ~~updates~~ updates before performing a measurement. (19/4/1)

Then μ_m and μ_{m+1} are separated by many ssf's.

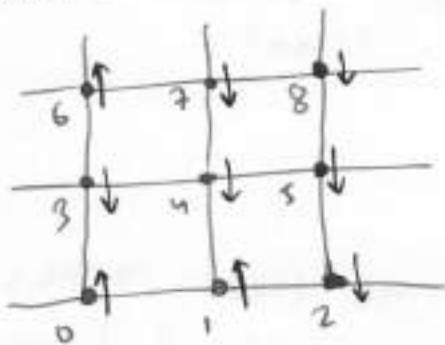
The number of updates required to get independent samples depends on T .

See section 3.31 of Newman & Barkema.

~~Code for today's tutorial using 2d~~

Code for today's tutorial ising 2d-mc.py

Ising model on an $L \times L$ lattice with periodic boundary conditions and $h=0$.



$$L=3$$

$$N=3 \times 3$$

$$H = -J \sum_{\langle ij \rangle} S_i S_j$$

we have a given state in our system;

Here we put periodic boundary condition
i.e. to say, 5 is also a nearest neighbour of 3.

We use an array called "spins" to store the current state:

$$\text{spins} = [+1, +1, -1, -1, -1, -1, +1, -1, -1]$$

+1 for up spin & -1 for down spin.

In order to calculate energies, we need to know the nn's of each site i . Store this in a two-dimensional "neighbours" array.

For example, when $L=3$;

(8/52)

$$\text{neighbours}[2] = \begin{bmatrix} 4 & 6 & 5 & 0 \end{bmatrix}$$

right, up, left, down

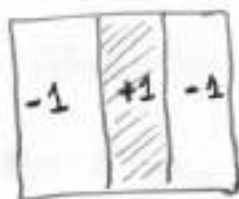
The code implements a ssf Metropolis algorithm to sample states μ and measure their energy and magnetization.

We do this for different temperatures.

The ssf algorithm has problems:

- ① It takes long time to generate independent samples. (especially near T_c)
- ② It can get stuck in local energy minima at low T , where it is very unlikely to accept moves that raise the energy. ($e^{-\beta \Delta E} \rightarrow 0$ when $T \rightarrow 0$ and $\Delta E > 0$) ($\beta \rightarrow \infty$)

For example, consider this state:



This is obviously not the ground state

here; there is no local move, no single spin flip that you can come up with that ~~would~~ would ever lower the energy to get us to the ground state.

→ To get from this state to ground state; we first have to lower the energy which is very unlikely from this algorithm.

There is no possible ssf that lowers the energy of this configuration (but it is not the ground state)

In the limit of low temperature; we will never Pg 153
 accept the move that raises energy
 because $e^{-\beta \Delta E} \rightarrow 0$ as $\Delta E > 0$ & $T \rightarrow 0$ i.e. $\beta \rightarrow \infty$
 So; means our ~~the~~ this algorithm can get stuck
 in this situation.

Cluster Algorithms

Idea Probabilistically build clusters of either all up or all down spins and propose flipping the entire cluster

↳ Swendsen and Wang (1987)

↳ Wolff (1989)

Estimating T_c from specific heat and susceptibility

Recall that near a continuous phase transition.

$$\left. \begin{aligned} C_v/N &\sim |t|^{-\alpha} \\ \chi/N &\sim |t|^{-\gamma} \\ \xi &\sim |t|^{-\nu} \end{aligned} \right\} \begin{aligned} &\text{lim the vicinity of phase transition)} \\ &\text{Specific heat per spin} \\ &\text{Susceptibility per spin} \\ &\text{Correlation length.} \end{aligned}$$

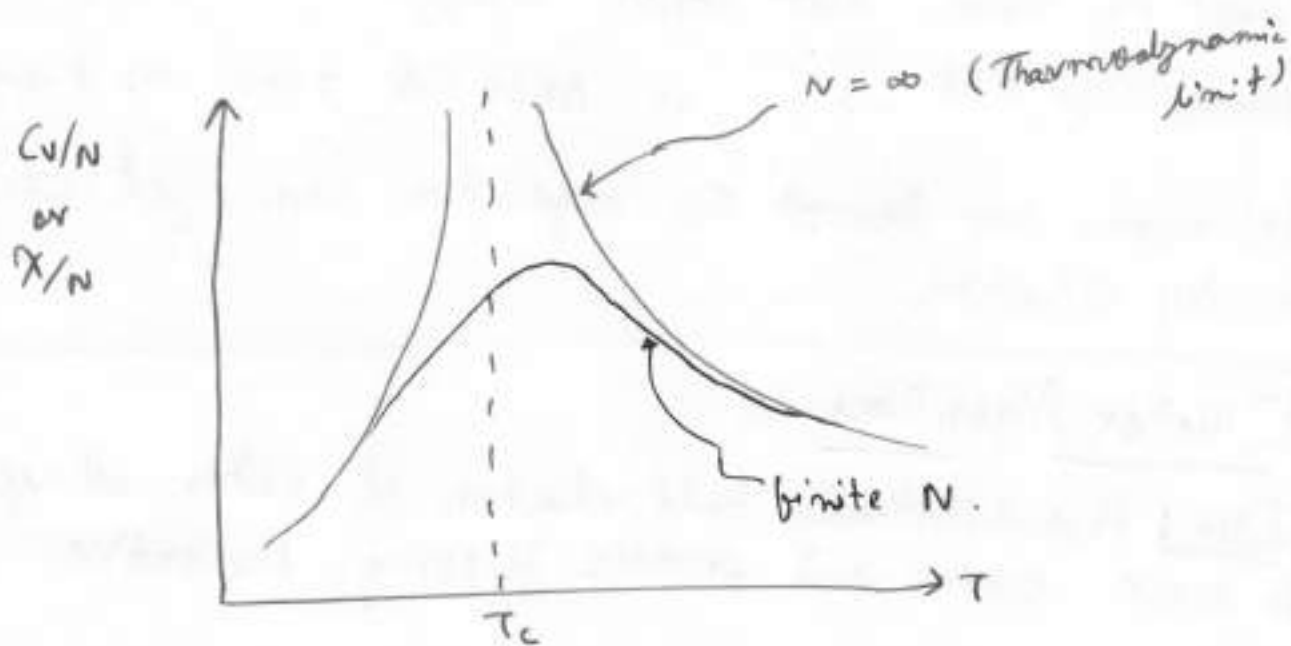
where, α, γ, ν are critical exponents.

• $t = \frac{T - T_c}{T_c}$ (a quantity which parametrizes how close we are to phase transition)

For the 2D Ising Model, C_v/N and χ/N diverge at T_c

However; on a finite ~~system~~ - size lattice, C_v/N &

χ/N diverge will peak near T_c but will not diverge. (19/14)



One can show, $C_v = \frac{\partial \langle E \rangle}{\partial T} = \frac{\langle E^2 \rangle - \langle E \rangle^2}{T^2}$

$$\chi = \left. \frac{\partial \langle M \rangle}{\partial h} \right|_{h=0} = \frac{\langle M^2 \rangle - \langle M \rangle^2}{T}$$

exercise ... prove it.

Finite - Size Scaling

↳ Technique for calculating T_c and critical exponents from calculations on finite lattices.

Note Since $\chi/N \sim |t|^{-\gamma}$ and $\xi \sim |t|^{-\nu}$:

$$\frac{\chi}{N} \sim \xi^{\gamma/\nu}$$

↳ this is for finite system.

ξ kind of measures distance on lattice over which spins are going to be correlated. But if we are on lattice (finite) ; then ξ is limited by size of order L (size of system)

On a finite system, correlations cannot exceed $O(L)$ and so

$$\frac{\chi}{N} \sim \begin{cases} \xi^{\gamma/\nu} & \text{when } \xi \ll L \\ L^{\gamma/\nu} & \text{when } \xi \gg L \end{cases}$$

Can write as: $\frac{\chi}{N} = \xi^{\gamma/\nu} f\left(\frac{L}{\xi}\right)$

$$\text{where } f\left(\frac{L}{\xi}\right) = \begin{cases} \text{constant} & \text{when } \xi \ll L \\ \left(\frac{L}{\xi}\right)^{\gamma/\nu} & \text{when } \xi \gg L \end{cases}$$

Lets eliminate ξ :

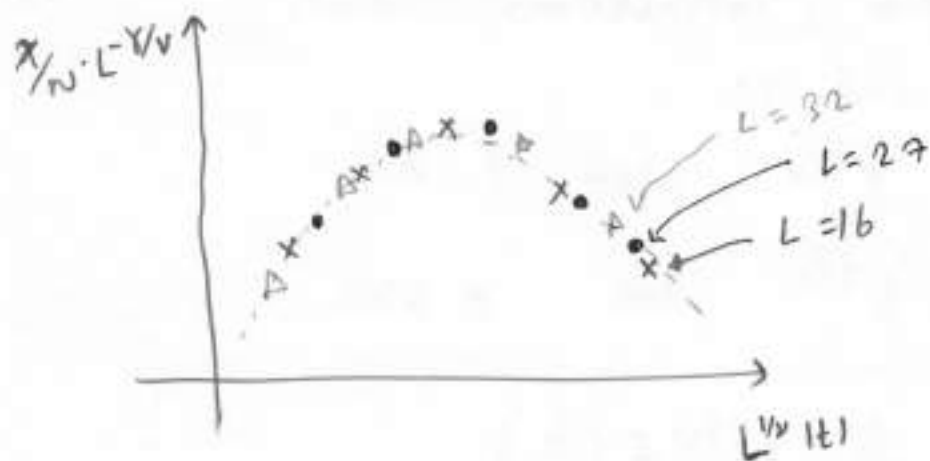
$$\begin{aligned} \frac{\chi}{N} &= L^{\gamma/\nu} \cdot \left(\frac{L}{\xi}\right)^{-\gamma/\nu} f\left(\frac{L}{\xi}\right) \\ &= L^{\gamma/\nu} (L^{-1/\nu})^{-\gamma/\nu} f(L^{1/\nu}) \\ &= L^{\gamma/\nu} \tilde{f}(L^{1/\nu}) \end{aligned}$$

$$\text{where } \tilde{f}(x) \equiv x^{-\gamma} f(x^\nu)$$

$$\Rightarrow \frac{\chi}{N} L^{-\gamma/\nu} = \tilde{f}(L^{1/\nu})$$

Plots of $\frac{\chi}{N} L^{-\gamma/\nu}$ v/s $L^{1/\nu}$ for various system sizes L should all collapse onto the same curve in the vicinity of T_c .

From this collapse, we can estimate γ , ν and T_c .

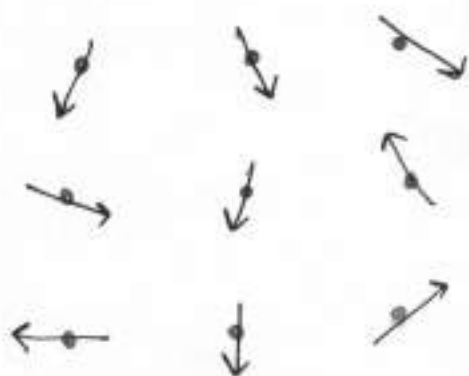


XY Model

$\vec{S}_i$ are allowed to rotate on circle. ($n=2$)

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

$$\theta_i \in [0, 2\pi)$$

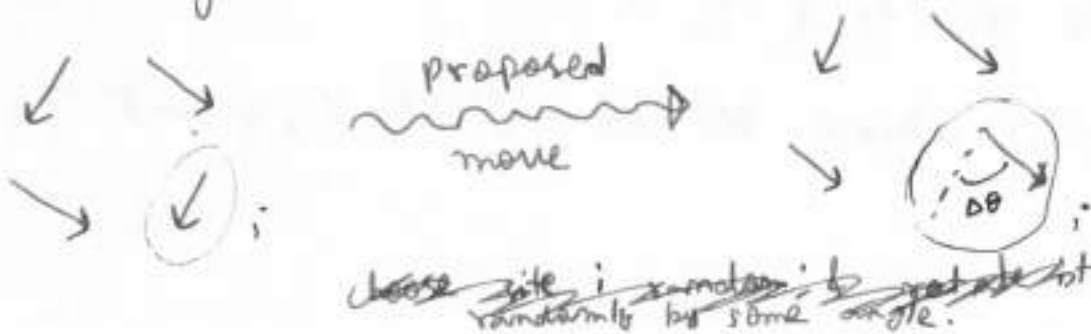


In 2D, there is no phase transition to a long-range ordered phase (M-W theorem).

Instead, there is a topological K.T. transition.

Single-Spin update

Similar to Ising, but include a random rotation.



choose site i at random; and rotate it by some random angle.

Propose the move $\theta_i \rightarrow \theta_i + \Delta\theta$

where $\Delta\theta \in [0, 2\pi)$ is selected from a uniform distribution.

At the KT transition, it is known that

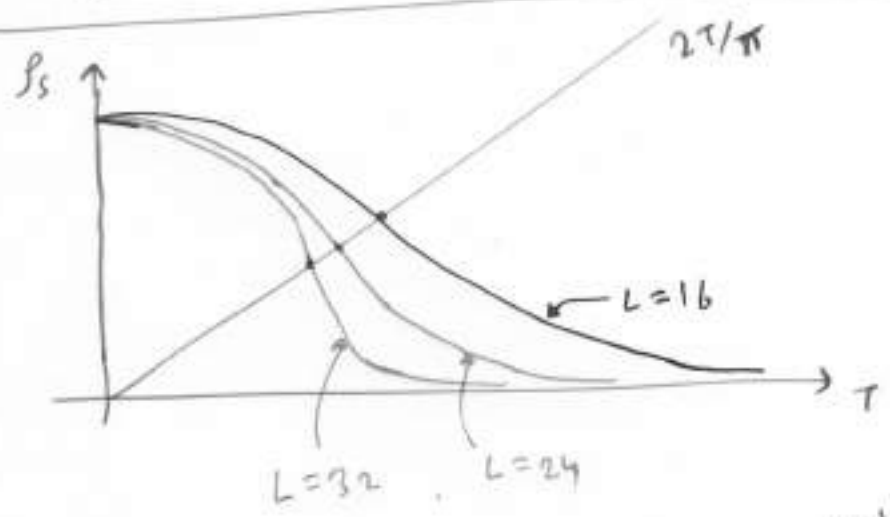
$$\beta_s(T_{KT}) = \frac{2T_{KT}}{\pi} \quad (\text{Nelson and Kosterlitz})$$

1977

↑ spin stiffness.

(this is something we can measure)

Plots of $\beta_s(T)$ v/s T should intersect the line $2T/\pi$ at $T = T_{KT}$ when $L \rightarrow \infty$.



Study the location of the crossing points as a function of size and extrapolate to $L \rightarrow \infty$.



Bonus Lecture 14: Black Hole Thermodynamics.(a) Motivation

Schwarzschild Metric: $ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2$; $f = (1 - \frac{2M}{r})$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

Asymptotic mass (Total Energy)

In G.R. it is very problematic to define energy. Gravity actually behaves holographically. It means that you cannot define energy locally. But you can say ^{only} what is energy inside some volume of space.

So we don't talk about local mass. Talk about Asymptotic mass. We include a big sphere; & talk about how much mass is inside that spacetime.

by Noether's Theorem, if there is symmetry $\leftrightarrow$ conserved quantity.

If this symmetry is something like time translation, then conserved quantity is something like mass of the spacetime.

Symmetries in G.R. $\leftrightarrow$ Killing Vectors



$$\mathcal{L}_\xi g_{\alpha\beta} = 0 \quad (\text{symmetry of metric})$$

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \quad (\text{Killing vector equations})$$

Schwarzschild is static, no time dependence.

pg 150

So: there is a killing vector $k = \frac{\partial}{\partial t}$

Komar Mass

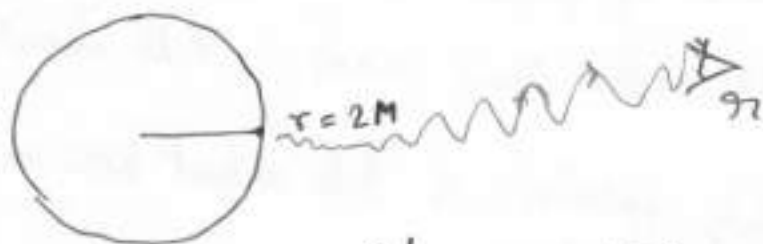
$$M = \frac{-1}{8\pi} \int_{S^2_\infty} * dk$$

$* dk$ is
2 form
... integrate
over 2-manifold
 S^2
at ∞ .

• Black Hole Horizon

$$r_+ = 2M$$

"Boundary of Black Hole"



(send radial photons)

$$\frac{v_{r_+}}{v_{r_+}} = \frac{\Delta \tau_{r_+}}{\Delta \tau_{r_+}} = \frac{\sqrt{f(r_+)}}{\sqrt{f(r)}} \cdot \frac{\Delta t}{\Delta t}$$

$$\Rightarrow \frac{v_{r_+}}{v_{r_+}} = \frac{\sqrt{f(r_+)}}{\sqrt{f(r)}}$$

$$\text{but } f(r_+) = 0$$

$$\Rightarrow \boxed{\frac{v_{r_+}}{v_{r_+}} = 0}$$

So; you can think of r_+ as surface of infinite redshift.

This is why it is called the BLACK HOLE.

(term coined by Wheeler)

(people ~~ear~~ earlier called it Frozen star)

Surface Area

$$A = 4\pi r_+^2$$

(Pg 151)

because of the chosen Area Change for r .

ie; surface of sphere is ~~such as~~
Such as it is in flat space.

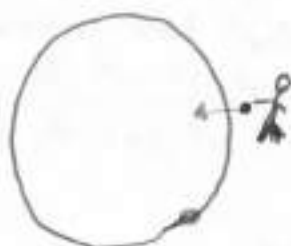
$$dt = 0 = dr$$

So; The induced metric ~~$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$~~
on the horizon is

$$dr^2 = r_+^2 d\Omega^2$$

$$A = \int \sqrt{\gamma} \cdot d\theta d\phi = \int_{\theta, \phi} r_+^2 \sin\theta d\theta d\phi = 4\pi r_+^2$$

Surface Gravity



What is ~~force~~ force to hold a particle at a fixed r on the horizon (just outside the horizon)

On the horizon, the force will be infinity (otherwise you could escape from it)



An observer at infinity can hold a particle on the horizon using a massless string; with finite force (due to red shift)

~~Because~~ Φ

Finite because, you are balancing whatever ∞ hold at horizon with ∞ redshift at observer at infinity ... & so force can be finite.

The better will be to calculate acceleration.

K : surface gravity. acceleration
(It is like gravitational acceleration of the particle on the horizon; however as perceived at infinity)
($m=1$)
(mass of test particle to be held)

$$K = \frac{f'(r_+)}{2}$$

(for any spherically symmetric spacetime)

$$\neq K = \frac{1}{2} \left(\frac{d}{dr} \left(1 - \frac{2M}{r} \right) \right) \Big|_{r=r_+} = \frac{1}{4M} = \frac{1}{2r_+}$$

$K = \frac{1}{4M} = \frac{1}{2r_+}$

 Think of K as; gravitational acceleration on the horizon of black hole normalized to infinity (otherwise it will be ∞ at just the horizon)

Observation use Newton's gravity.

so: ~~$g = \frac{M}{r^2}$~~ so; $g_{\text{Newton}} = \frac{M}{r^2} \Big|_{r=r_+}$

$$= \frac{M}{r_+^2}$$

remember $r_+ = 2M$

$$\Rightarrow g_{\text{Newton}} = \frac{1}{4M}$$

(The results coincides with the Newtonian theory; but of course the interpretation is slightly different)

Not generally true.

Pg 153

... but ~~sp~~ spherically symmetric spacetime are almost Newtonian.

Observation

$$A = 4\pi r_+^2$$

$$M = r_+/2$$

$$K = \frac{1}{2r_+}$$

now let's play with differentials

$$\boxed{dM = \frac{K}{2\pi} \frac{dA}{4}}$$

... interesting observation
... beginning of Black Hole
Thermodynamics.

(b) Black Hole Mechanics (1973)

Bardeen, Carter, Hawking considered most general Black Hole (it does not only have mass M , but it can also rotate with some angular momentum J , can also have charge Q)

... No hair theorem
(can't have scalar field...
... will not have
any scalar hair.)

Scalar field ~~would~~ would either actually go to infinity and radiate away or it would collapse into the B.H.; swallowed by the B.H.; and there would be no any trace of it.)

↳ so; we cannot have scalar field i.e. scalar hair; which would be another charge which will be characterising the B.H.

0th Law: $K = \text{constant on the horizon.}$

(Pg 155)

(all these laws are proved by purely using geometrical arguments by these gentlemen)

1st Law: $dM = \frac{K}{2\pi} \frac{dA}{4} + \underbrace{\Omega dJ + \phi dQ}_{\text{Work terms.}}$

Work terms.

Our black hole can have J ; & the thermodynamic conjugate quantity to that is like angular velocity Ω

$\Omega \Rightarrow$ angular velocity of horizon.

$Q \xleftarrow{\text{Conjugate Quantity}} \text{Electrostatic potential } \phi.$

$J \Rightarrow$ Angular Momentum

2nd Law: $\boxed{dA \geq 0}$ Hawking himself showed this.

3rd Law: It is impossible to reduce K to zero in finite no. of steps.

$\hookrightarrow$ These laws were proved on Geometrical Arguments.

These look like ~~the~~ laws of Thermodynamics.

If we identify $K \sim T$?

& $A \sim S$?

~~Then these laws look like laws of thermodynamics.~~
Then these laws will just be laws of Thermodynamics.
Can this be true?

if we say this; then a Black Hole would have finite temperature. This does not make any sense because in order to have finite temperature you have to radiate something.

But classically B.H. don't radiate.

Classically, this is impossible ~~becau~~ because Black Hole is an ultimate sponge.

so; T has to be equal to 0 : $T=0$ no other way.

(c) Black Hole Thermodynamics.

The key idea comes from Wheeler,
he asked his graduate students what happens if
we throw a cup of coffee into black hole.

it initially had entropy (the cup of coffee)
but B.H. has zero temperature. So where
does the entropy go.

Wheeler's Cup of Tea: where does the entropy go?

He asked to Bekenstein.

Bekenstein concluded $S \propto A$ so that B.N. can have entropy.

(pg 156)

→ People did not believe it at that point until Hawking ~~showed~~ one year later showed that indeed if you take quantum effects into consideration, then ~~you have~~ B.N.s have non-trivial temperature.

Hawking (1974) Hawking considered QFT in curved background. (The calculation is complicated and actually wrong)

and showed

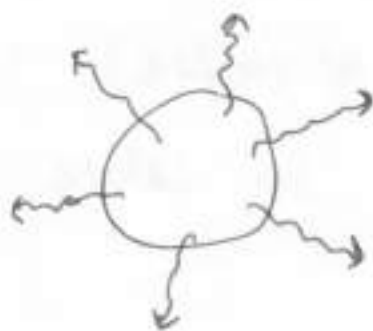
$$T = \frac{\hbar K}{2\pi K_B}$$

B.N. actually radiates if you take Quantum effects

to fix with Bekenstein; it also came out $S = \frac{A}{4\hbar G}$

$T = \frac{\hbar K}{2\pi K_B}$	$S = \frac{A}{4\hbar G}$
--------------------------------	--------------------------

⇒ Tells that really; Quantum Mechanically B.N. radiates particles.



its like Black Body.

as you go to infinity, the Black Body becomes Grey Body. ~~Grey~~ b

Corey body at infinity : because some of the modes are scatter back.

(19157)

... The particles which are created, they are to pass over a potential; so there some are scattered back.

At infinity you observe $T = \frac{\hbar K}{2\pi k_B}$; ~~here~~ and if you neglect scattering back at infinity; then you will see black body with temperature $T = \frac{\hbar K}{2\pi k_B}$

- Derivation criticized by many people (you cannot neglect back reaction near horizon)
- But, Euclidean Path Integral gives same result upto leading order.

LQ, String... all these ~~results give same approaches~~ approaches gives same result upto the leading order.

It can be shown that Hawking Radiation is a kinematical effect. (Does not require Einstein Equations.)

→ Analogue Systems.

- (Surface Waves on water)
→ measured ✓

(d) Euclidean Trick

★ Thermal Green functions have periodicity in imaginary time. (comes from huge similarity between path integral & partition function as we know in Statistical physics)

$$\tau = it$$

(pg 158)

$$\boxed{G(\tau) = G(\tau + \beta)}$$
 if this is true

Then we can show the periodicity
is $\beta = \frac{1}{T}$ (will be shown in
QFT 2)

~~blame for studying this~~ ~~we can study for~~

We can study fields in the vicinity of Euclideanized
Schwarzschild black hole. We find they have periodic green's
function $\Rightarrow$ So, ~~this~~ this means that they were in
thermal bath.

In particular; consider the Gravitational Field.

$\hookrightarrow$ Euclideanized Schwarzschild Metric.

$$ds_E^2 = f dz^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \quad (\text{Euclidean version of Schwarzschild Metric})$$

Near Horizon; we can write $f(r) = f(r_+) + (r - r_+) f'(r_+) + \dots$

$$f(r_+) = 0 \quad \Rightarrow \quad f(r) = \Delta r f'(r_+) + \dots$$

$$\Delta r = r - r_+ \quad \Rightarrow \quad \boxed{f(r) = 2\Delta r K}$$

So; close to horizon; metric looks like.

$$ds_E^2 = 2\Delta r K dz^2 + \frac{dr^2}{2\Delta r K} + r^2 d\Omega^2$$

$\frac{dr^2}{2\Delta r K}$ depends only on r ; so its a total differential.

So; let $dp = \frac{dr}{\sqrt{2K\Delta r}}$ (we can integrate it & get p)

by integrating we find

$$\Delta r = \frac{K}{2} \rho^2$$

pg 159

$$ds_E^2 = K^2 \rho^2 dz^2 + d\rho^2 + r_+^2 d\Omega^2$$

↳ not much interesting,
... just a sphere
of radius r_+ .

Almost

$$ds_E^2 = \rho^2 d\psi + d\rho^2 + r_+^2 d\Omega^2$$

↳ This is just 2D space written in
polar coordinates (ρ, ψ)

$\rho \Rightarrow$ radial ; $\psi \Rightarrow$ angle.

so: $\boxed{\psi = K\tau}$

The sneaky argument is ; $\rho^2 d\psi + d\rho^2$ will be non-singular only if ψ has periodicity 2π .

otherwise, we have conical singularity at origin $\rho = 0$.

↳ so; non-singular at $\rho = 0$ only if $\psi \sim \psi + 2\pi$.

$\rho = 0$ means $r = r_+$

(ie; we are ~~are~~ demanding that there will be no singularity at $r = r_+$)

↳ And it's ok ; there is no true singularity at horizon ; there is just coordinate singularity.

but ; $\psi = K\tau \Rightarrow \tau \sim \tau + \frac{2\pi}{K}$

if there is periodicity of β imaginary time

Pg 160

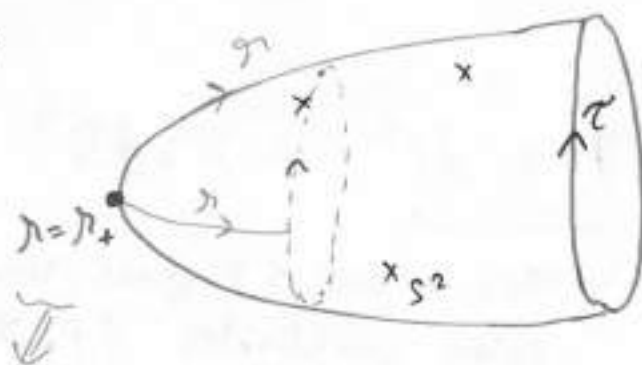
$\hookrightarrow$ so there must be temperature ; $\beta = \frac{2\pi}{k}$

$$T = \frac{k}{2\pi}$$

Black Hole has temperature.

How does Euclidean Geometry looks like (in τ and r)

$\approx$ Cigar



$$\tau \sim \tau + \beta$$

non-singular

at each point on surface ~~there is~~ of Cigar is a sphere S^2 sitting there.

Geometry in Euclidean space.

Partition Function

here field is metric.

$$Z = \int \mathcal{D}g \cdot e^{-S_E[g]}$$

path integral over metric

• Euclidean action for gravitational field.

WKB approximation. $\approx e^{-S_E[g_{\text{classical}}]}$

$$Z = \int \mathcal{D}g e^{-S_E[g]} \underset{\text{WKB}}{\approx} e^{-S_E[g_{\text{classical}}]}$$

This is the first approximation for Quantum Gravity that we have here.

$$S_E = \int_{\Omega} \frac{d^4x \sqrt{-g}}{16\pi G} \cdot R$$

Ernststein - Hilbert Action

~~Einstein - Hilbert Action~~

→ just writing this; variational principle is not well posed (say when Ω has boundary)

$$\text{so: } S_E = \int_{\Omega} \frac{d^4x \sqrt{-g}}{16\pi G} R \quad \left(+ \delta g = 0 \quad + \delta(\partial g) = 0|_{\partial\Omega} \right)$$

↳ But it kills many solutions, as we know from Classical Mechanics.

... so we put in boundary terms in action.

$$S_E = \int_{\Omega} \frac{d^4x \sqrt{-g}}{16\pi G_N} R + \int_{\partial\Omega} \frac{d^3x \sqrt{h} K}{8\pi G_N}$$

~~K = extrinsic~~
K = extrinsic curvature.

Gibbons - Hawking term
(makes variational principle well defined)

Now; to calculate actual value of partition function; we have to find actual value of action
... & they come out to be infinite even for flat space.

So; we add another boundary term to tune the value of action ; which is hiding in K_0 . pg 162

define K_0 such that S_E vanishes for flat space.

$$S_E = \int_{\Omega} \frac{d^4x \sqrt{g}}{16\pi G_N} R + \int_{\partial\Omega} \frac{d^3x \sqrt{h} K}{8\pi G_N} - \underbrace{\int_{\partial\Omega} \frac{d^3x \sqrt{h} K_0}{8\pi G_N}}_{\text{To tune value of } S_E}$$

(WKB or Geometric Optics Approximation ; are same thing)

After doing calculation for Saddlepointized Schwarzschild.
we get

$$S_E^{\text{Schwarzschild}} = \frac{\beta M}{2}$$

boundary integrals $\int_{\partial\Omega} d^3x$ has

& one of them is τ

So; $\int d\tau$ must give β .

So; now we calculate free energy.

$$Z = e^{-\beta M/2}$$

$$F = -T \log Z = -\frac{M}{2}$$

$$S = -\frac{\partial F}{\partial T} = \frac{1}{16\pi T^2} = \pi r_+^2 = \frac{A}{4}$$

remember $T = \frac{1}{8\pi M}$

Bekenstein.

Note Euclidean geometry has nothing inside horizon.

→ It looks like cigar...

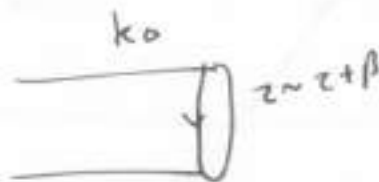
... we make the geometry nice and smooth at horizon & so whatever was inside horizon sort of disappeared.

So, in the integral, we are integrating what ~~happens~~ is happening outside horizon.

what effectively we do



True geometry



Flat thing with same periodicity

$$= \frac{M}{2} \cdot \beta$$

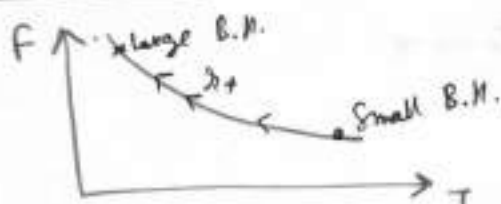
Black Holes can have interesting Thermodynamic Phase Transition.

$$F = M - TS$$

↑
mass

if this free energy is doing something bad; then we know that there is phase transition.

Schwarzschild B.H. $F = F(T, \dots)$ → other parameters.

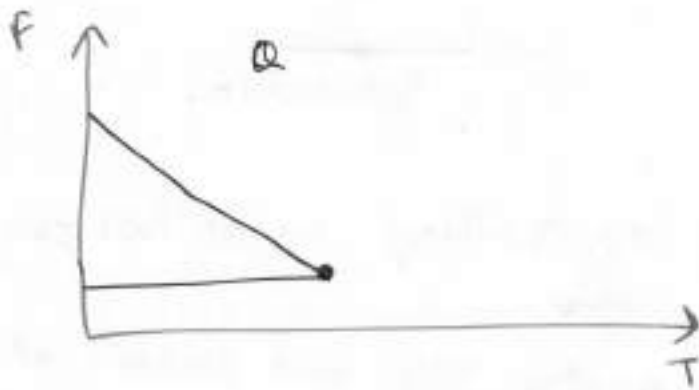


This is boring.

Schwarzschild + Charge

(Charged B.H.)

(pg 165)

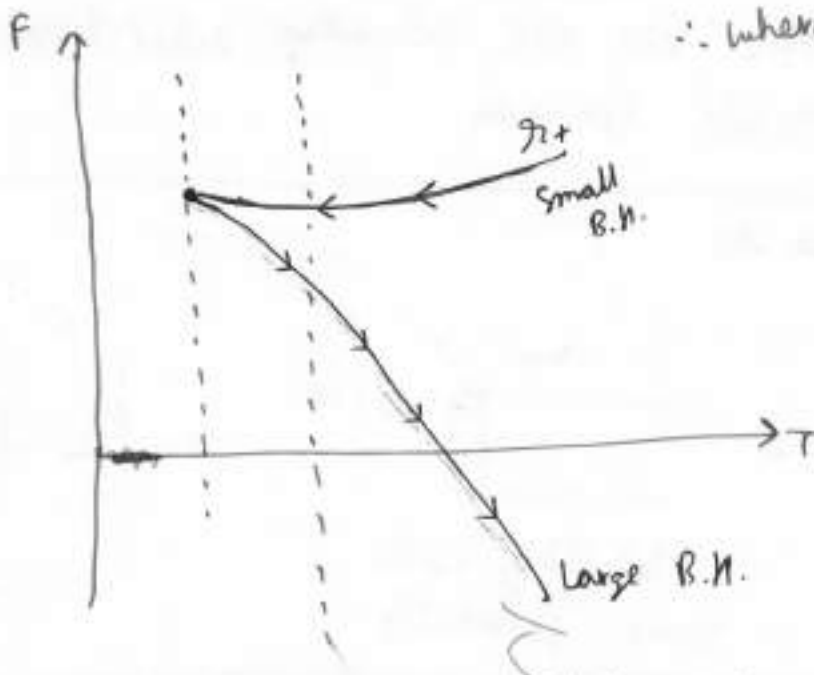


Not very interesting.

Becomes very interesting ~~go~~ as you go to AdS.

Schwarzschild + $\Lambda < 0$

$$f = 1 - \frac{2M}{r} + \frac{r^2}{l^2}$$



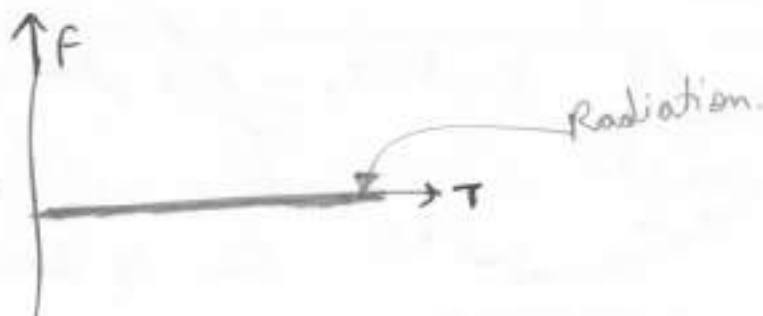
$\therefore$ where $\Lambda \sim -\frac{1}{l^2}$

(cosmological setting
... cosmological
constant characterized
by length scale l)

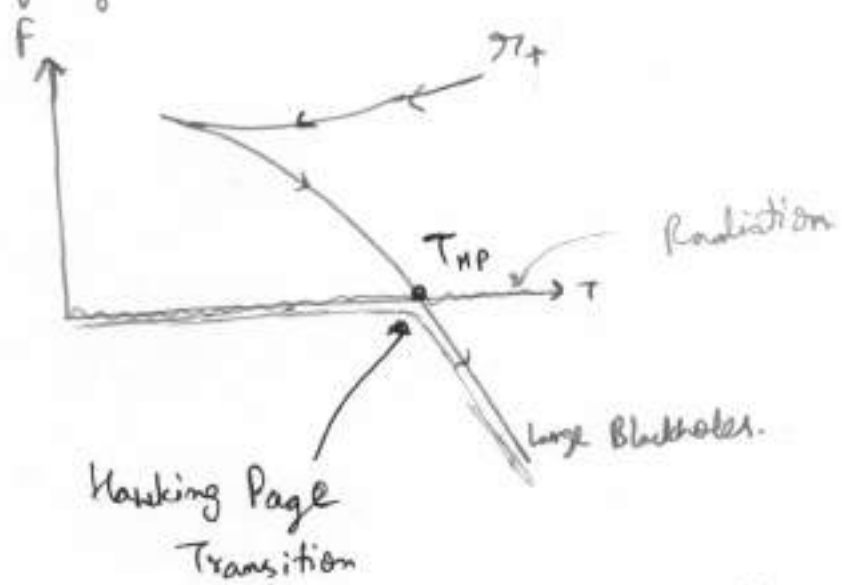
large black holes are
thermodynamically preferred.
(large B.H.s are preferred)

System = B.H. + Radiation

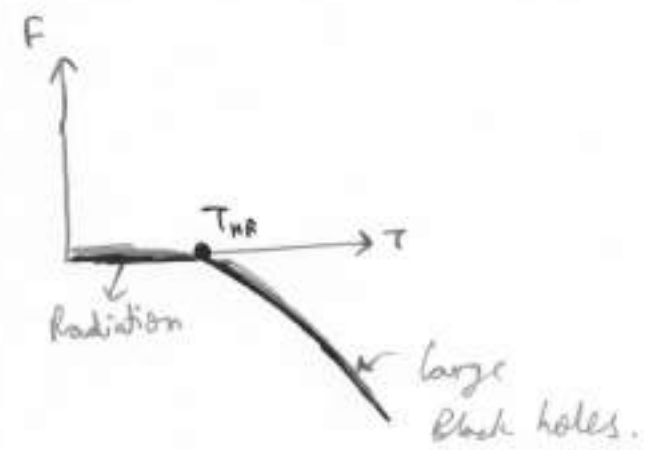
Radiation



So: F of system = BH + Radiation.



Radiation to large Black Hole phase transition.



ADS/CFT ... DECONFINEMENT of Quarks.



The background of the entire page is a light gray color with a subtle, repeating diamond-shaped pattern. Overlaid on this are several large, overlapping circles in various shades of gray, creating a layered, geometric effect. The circles are positioned in the top-left, top-right, and bottom-left corners, leaving the center area more open for text.

THANK YOU

Looking at the world from a
different viewpoint changes the
physics of it.

--Shoaib Akhtar