Calculation of Invariants of Knots

using Prof Ramadevi formula

Shoaib Akhtar

1 Unknot 0_1

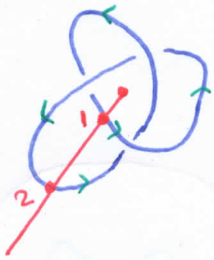
The invariant for the unknot is the q-dimensions of the representation living on it.

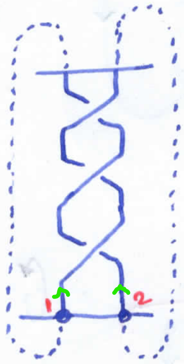
Let R_m be the Representation living on 0_1

$$\Rightarrow \boxed{V[0_1, R_m] = \dim_q R_m}$$

2 3_1 (Trefoil)

Write the briad which closes to form 3_1





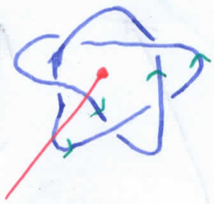
Ref Fig 4 of paper

$\Rightarrow$ This Braid is actually $\mathcal{L}_3(R,R)$

Hence $\boxed{V[3_1, R_m] = V[\mathcal{L}_3(R_m, R_m)]}$

use Theorem 5 & eqn 5.1 ; Fig 4

3 5_1

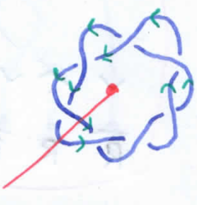


The Briad which closes to for 5_1 is $\mathcal{L}_5(R,R)$

$$\Rightarrow \boxed{V[5_1, R_m] = V[\mathcal{L}_5(R_m, R_m)]}$$

use Theorem 5 & eqn 5.1 ; Fig 4

4 7_1

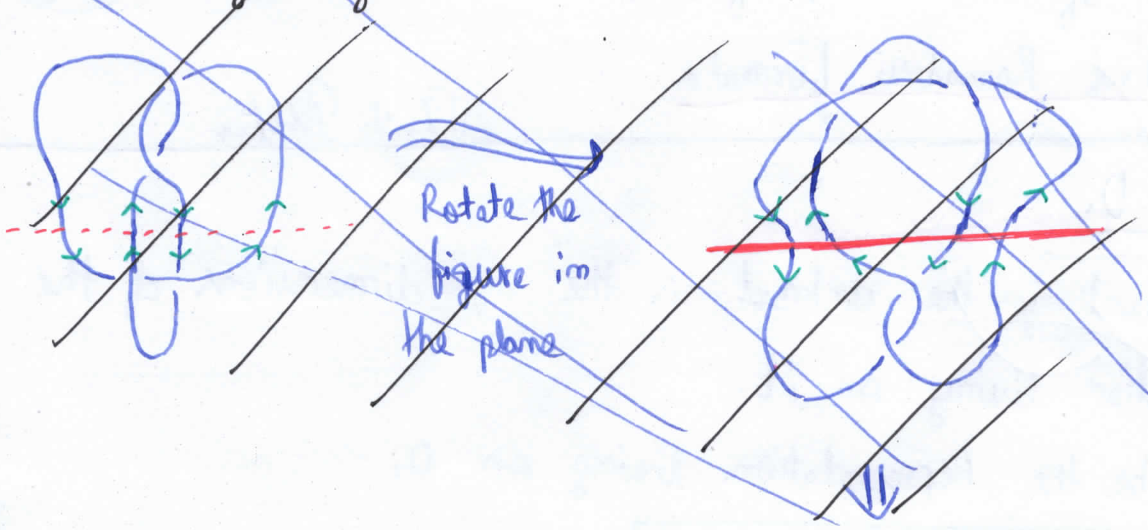


The Briad for this is $\mathcal{L}_7(R,R)$

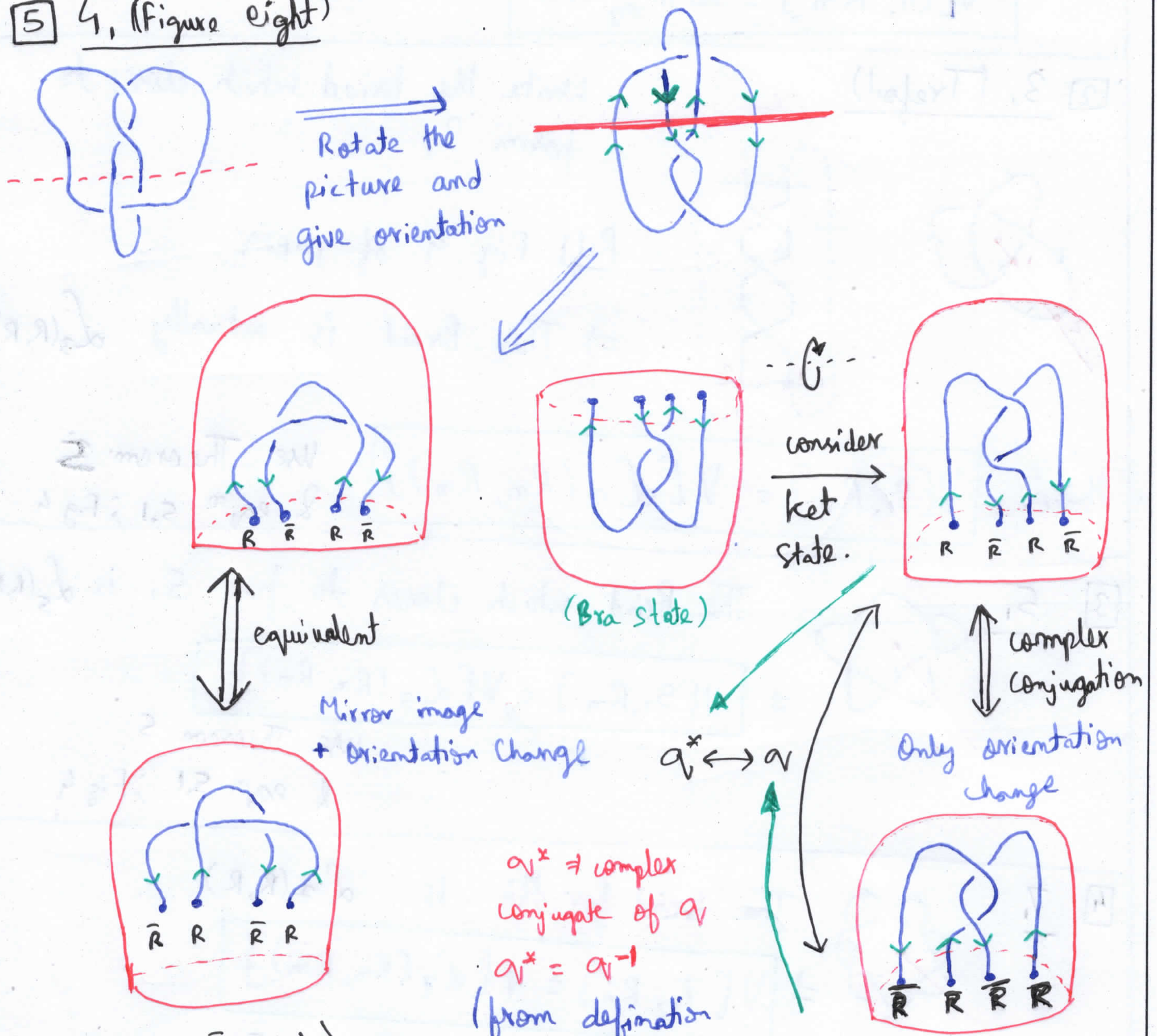
$$\Rightarrow \boxed{V[7_1, R_m] = V[\mathcal{L}_7(R_m, R_m)]}$$

use Theorem 5 and eqn 5.1 ; Fig 4

5 4_1 (Figure eight)



5 4_1 (Figure eight)



q^* → complex conjugate of q

$q^* = q^{-1}$ (from defination of q)

also $\omega^* = \omega^{-1}$

$$\left|\chi\left(\hat{Q}^{H}_{2}\begin{pmatrix}\bar{R} & R\\ R & \bar{R}\end{pmatrix}\right)\right\rangle \qquad \left|\chi\left(\hat{Q}^{V}_{2}\begin{pmatrix}\bar{R} & R\\ R & \bar{R}\end{pmatrix}\right)\right\rangle$$

Hence we get

$$\boxed{V[4_1, R_n] = \left\langle \chi_{q^{-1}}\left(\hat{Q}^V_2\begin{pmatrix}\bar{R}_n & R_n\\ R_n & \bar{R}_n\end{pmatrix}\right) \middle| \chi_q\left(\hat{Q}^H_2\begin{pmatrix}\bar{R}_n & R_n\\ R_n & \bar{R}_n\end{pmatrix}\right)\right\rangle}$$

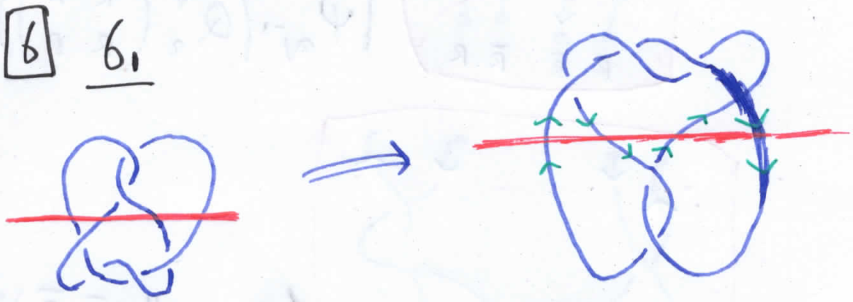
minus sign.

$$\Rightarrow V[4_1, R_n] = \sum_{l,j=0}^{n} \sqrt{\dim_q \hat{\rho}_j \cdot \dim_q \hat{\rho}_l} \cdot a_{\hat{\rho}_j \hat{\rho}_l} \cdot \omega^{j\cancel{/2} - l\cancel{/2}} \cdot q^{j^2\cancel{/2} - l^2\cancel{/2}}$$

no half factor; because we square eigenvalues

Equation (4.5), (4.6), (5.2) and figure 6 was used.

$\boxed{6}$ $\underline{6_1}$

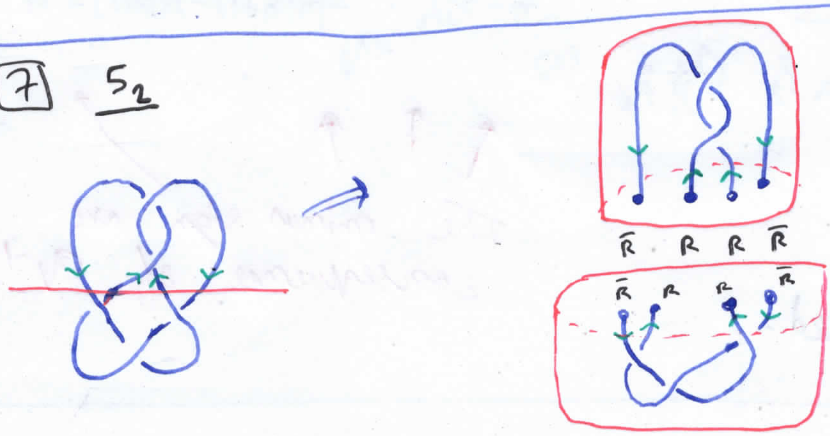


Hence following previous analysis.

$$\boxed{V[6_1, R_n] = \left\langle \chi_{q^{-1}}\left(\hat{Q}^V_2\begin{pmatrix}\bar{R}_n & R_n\\ R_n & \bar{R}_n\end{pmatrix}\right) \middle| \chi_q\left(\hat{Q}^H_4\begin{pmatrix}\bar{R}_n & R_n\\ R_n & \bar{R}_n\end{pmatrix}\right)\right\rangle}$$

$$\Rightarrow V[6_1, R_n] = \sum_{l,j=0}^{n} \sqrt{\dim_q \hat{\rho}_j \cdot \dim_q \hat{\rho}_l} \cdot a_{\hat{\rho}_j \hat{\rho}_l} \cdot \omega^{2j - l} \cdot q^{2j^2 - l^2}$$

$\boxed{7}$ $\underline{5_2}$



$$|\Psi_q(Q^V_2\begin{pmatrix}\bar{R} & R\\ R & R\end{pmatrix})\rangle$$

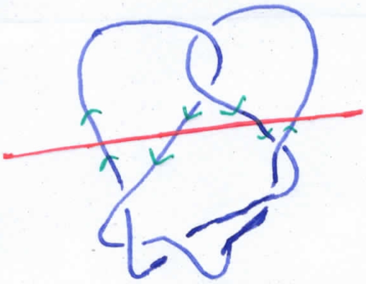
$$\langle \cancel{\chi}_{q^{*}}(\hat{Q}^H_3\begin{pmatrix}\bar{R} & \bar{R}\\ R & R\end{pmatrix})|$$

$$\exists\ \boxed{V[S_2, R_n] = \left\langle \Psi_q\left(Q^H_3\begin{pmatrix}\bar R_n & \bar R_n\\ R_n & R_n\end{pmatrix}\right)\middle|\ \Psi_q\left(Q^V_2\begin{pmatrix}\bar R_n & \bar R_n\\ R_n & R_n\end{pmatrix}\right)\right\rangle}$$

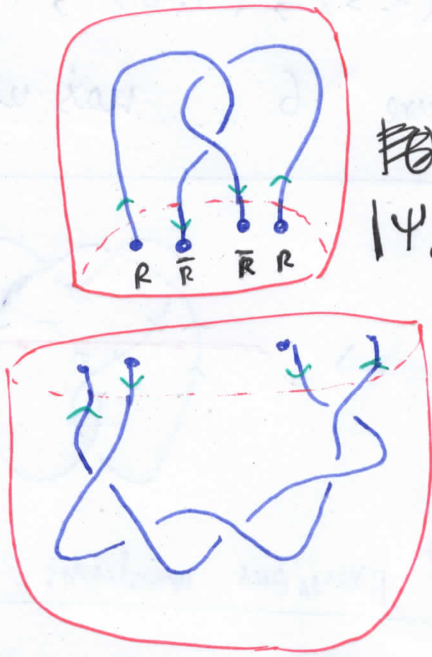
$$\exists\ V[S_2, R_n] = \sum_{l,j=0}^{n} \sqrt{\dim_q \hat\rho_j \dim_q \rho_l}\cdot a_{\hat\rho_j \hat\rho_l}\, \omega^{n+3j/2}\cdot q^{n(n+1)-l(l+1)+3j^2/2}$$

Figure 5, and Theorem 7 was used; along with eq^n (5.1) and (5.2).

$\boxed{8}$ $\underline{7_2}$



$\Rightarrow$



$$|\Psi_{q^{-1}}\left(Q^V_2\begin{pmatrix}\bar R & \bar R\\ R & R\end{pmatrix}\right)\rangle$$

$$\langle\Psi_{q^{-1}}\left(Q^H_5\begin{pmatrix}\bar R & \bar R\\ R & R\end{pmatrix}\right)|$$

$$\exists\ \boxed{V[7_2, R_n] = \left\langle \Psi_{q^{-1}}\left(Q^H_5\begin{pmatrix}\bar R & \bar R\\ R & R\end{pmatrix}\right)\middle|\ \Psi_{q^{-1}}\left(Q^V_2\begin{pmatrix}\bar R & \bar R\\ R & R\end{pmatrix}\right)\right\rangle}$$

$$\exists\ V[7_2, R_n] = \sum_{l,j=0}^{n} \sqrt{\dim_q \hat\rho_j \dim_q \rho_l}\cdot a_{\hat\rho_j \hat\rho_l}\cdot \omega^{-n-5j/2}\cdot q^{-[n(n+1)-l(l+1)]-5j^2/2}$$

The minus sign are consequence of q^{-1}.

Figure 5, Theorem 7, eq^m (5.1) & (5.2) was used.

Super-A-Polynomials for Twist Knots

2013 paper by prof Ramadevi

Shoaib Akhtar ; Sept, 2020.

An explicit calculation of some results from the paper.

[1] Rearranging Masbaum Cyclotomic expansion of the colored Jones polynomial for the twist knot K_p (which was obtained by Skein theory)

$$J_n = \sum_{k=0}^{\infty} C_{K_p}(k) \cdot \frac{\{n-k\}\{n-k+1\}\cdots\{n+k\}}{\{n\}}$$

where; $C_{K_p}(k) = (-1)^{k+1} q_I^{k(k+3)/2} \sum_{l=0}^{k} (-1)^l q^{l(l+1)p} \{2l+1\} \cdot \frac{\{k\}!}{\{k+l+1\}!\,\{k-l\}!}$

Notations: $\{n\} = q_I^n - q_I^{-n}$, $q_I^2 = q$, $\{n\}! = \{n\}\{n-1\}\cdots\{1\}$.

Now we use the q-Pochhammer symbols

$$(z;q)_k = \prod_{j=0}^{k-1} (1 - zq^j)$$

We can Rearrange $J_n(K_p;q)$ as follows

$$J_n(K_p;q) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} q^k \cdot (q^{1-n};q)_k \,(q^{n+1};q)_k \times$$

$$\times (-1)^l q^{l(l+1)p + l(l-1)/2} \cdot (1-q^{2l+1}) \cdot \frac{(q;q)_k}{(q;q)_{k+l+1}\,(q;q)_{k-l}}$$

proving the equality.

let $\alpha_{k,l}$ be the summand.

we have

$$\alpha_{k,l} = (-1)^{k+1} q_I^{k(k+3)/2} \cdot (-1)^l \cdot q^{l(l+1)p} \{2l+1\} \cdot \frac{\{k\}!}{\{k+l+1\}!\,\{k-l\}!} \times \frac{\{n-k\}\{n-k+1\}\cdots\{n+k\}}{\{n\}}$$

$$\beta_{k,\ell} = q^k \cdot (q^{1-m};q)_k \, (q^{m+1};q)_k \times (-1)^\ell \cdot q^{\ell(\ell+1)p + \ell(\ell-1)/2} (1-q^{2\ell+1}) \times$$

$$\times \frac{(q;q)_k}{(q;q)_{k+\ell+1} \, (q;q)_{k-\ell}}$$

$$\alpha_{k,\ell} = (-1)^{k+1} \cdot q_I^{k(k+3)/2} \cdot (-1)^\ell \cdot q^{\ell(\ell+1)p} \cdot \{2\ell+1\} \cdot \frac{\{k\}!}{\{k+\ell+1\}!\{k-\ell\}!} \times \frac{\{m-k\}\dots\{n+k\}}{\{n\}}$$

If we show the following, then we are done.

$\varphi_{k,\ell} =$

$$q^k \cdot (q^{1-m};q)_k \, (q^{m+1};q)_k \cdot q^{\ell(\ell-1)/2} \cdot (1-q^{2\ell+1}) \times \frac{(q;q)_k}{(q;q)_{k+\ell+1} \, (q;q)_{k-\ell}}$$

$$= (-1)^{k+1} \, q_I^{k(k+3)/2} \cdot \{2\ell+1\} \cdot \frac{\{k\}!}{\{k+\ell+1\}! \, \{k-\ell\}!} \times \frac{\{m-k\}\dots\{m+k\}}{\{n\}}$$

$$(q^{1-m};q)_k = \prod_{j=0}^{k-1} (1-q^{1-m+j})$$

$$= \prod_{j=0}^{k-1} q^{\frac{1-m+j}{2}} \cdot \left[q^{\frac{m-1-j}{2}} - q^{-\frac{(m-1-j)}{2}} \right]$$

$$= \left(\prod_{j=0}^{k-1} q^{\frac{(1-m+j)}{2}} \right) \left(\prod_{j=0}^{k-1} \left(q^{\frac{m-1-j}{2}} - q^{-\frac{(m-1-j)}{2}} \right) \right)$$

$$= q^{k/2} \cdot q^{-\frac{m}{2}k} \cdot q^{\frac{(k-1)(k)}{2\times 2}} \left[\prod_{j=0}^{k-1} \{m-1-j\} \right]$$

$$= q^{k/2} \cdot q^{\frac{k^2}{4} - k/4} \cdot q^{\frac{-mk}{2}} \left[\prod_{j=0}^{k-1} \{m-1-j\} \right]$$

$$= q^{k^2/4 + \frac{k}{4} - mk/2} \cdot \left[\prod_{j=0}^{k-1} \{m-1-j\} \right]$$

$$= q^{k^2/4 - mk/2} \cdot \{m-1\} \dots\dots \{m-k\} \cdot q^{k/4}$$

$$\boxed{(q^{1-n};q)_k = q^{k^2/4 - nk/2}\cdot\{n-1\}\cdots\{n-k\}} \times q^{k/4}$$

Similarly (or just by replacing n with $-n$)

$$\boxed{(q^{1+n};q)_k = q^{k^2/4 + nk/2}\cdot\{-n-1\}\cdots\{-n-k\}} \times q^{k/4}$$

$$(q;q)_k = \prod_{j=0}^{k-1}(1-q\cdot q^j)$$

$$= \prod_{j=0}^{k-1} q^{\frac{1+j}{2}}\cdot\left[q^{-\frac{(1+j)}{2}} - q^{-\left(-\left(\frac{1+j}{2}\right)\right)}\right]$$

$$= q^{\frac{k}{2}}\cdot q^{\frac{k(k-1)}{2\times 2}}\prod_{j=0}^{k-1}\{-(1+j)\}$$

$$= q^{k/2 - k/4 + k^2/4}\cdot\prod_{j=0}^{k-1}\{(-1-j)\}$$

$$= q^{k/4 + k^2/4}\cdot\prod_{j=0}^{k-1}\{(-1-j)\}.$$

$$= q^{k/4 + k^2/4}\cdot\{-1\}\{-2\}\cdots\{-k\}$$

$$(-1)^k\cdot q^{\frac{k^2}{4}+\frac{k}{4}+\frac{nk}{2}}\cdot\{n+1\}\cdots\{n+k\}$$

$$(q^{1-n};q)_k = q^{k^2/4 + k/4 - nk/2}\cdot\{n-1\}\cdots\{n-k\}$$

$$(q^{1+n};q)_k = q^{\frac{k^2}{4}+\frac{k}{4}+\frac{nk}{2}}\cdot\{-n-1\}\cdots\{-n-k\}$$

$$(q;q)_k = q^{k^2/4 + k/4}\cdot\{-1\}\{-2\}\cdots\{-k\}$$

$$(q;q)_{k+l+1} = q^{\frac{(k+l+1)^2}{4}+\frac{(k+l+1)}{4}}\cdot\{-1\}\{-2\}\cdots\{-(k+l+1)\}$$

$$(q;q)_{k-l} = q^{\frac{(k-l)^2}{4}+\frac{(k-l)}{4}}\cdot\{-1\}\{-2\}\cdots\{-(k-l)\}.$$

We will use this.

$$\boxed{\{q;q\}_k = (-1)^k q^{k^2/4 + k/4}\cdot\{1\}\{2\}\cdots\{k\}} \quad \{q;q\}_k = (-1)^k\cdot q^{k^2/4+k/4}\{k\}!$$

$$\Psi_{k,l} = q^k \cdot (q^{1-n};q)_k \cdot (q^{n+1};q)_k \cdot q^{l(l+1)/2} \times (1-q^{2l+1}) \times$$

$$\times \frac{(q;q)_k}{(q;q)_{k+l+1}\,(q;q)_{k-l}}$$

$$\Psi_{k,l} = q^k \cdot q^{\frac{l(l-1)}{2}} \cdot (1-q^{2l+1}) \times \frac{q^{k^2/2+k/2}\cdot q^{k^2/4+k/4}}{q^{\frac{(k+l+1)^2}{4}+\frac{(k+l+1)}{4}} \cdot q^{\frac{(k-l)^2}{4}} \cdot q^{\frac{(k-l)}{4}}} \times$$

$$\times \frac{(\{-n-1\}\cdots\{-n-k\})\times(\{n-1\}\cdots\{n-k\})\times(\{-1\}\cdots\{-k\})}{\{-1\}\{-2\}\cdots\{k+l+1\}\cdot\{-1\}\{-2\}\cdots\{k-l\}}$$

$$= f(q)\cdot \Upsilon$$

The rest of the terms.

The factor of q

Note that; we need $\{2l+1\}$

Hence $$(1-q^{2l+1}) = (-1)(q^{2l+1}-1) = (-1)\, q^{\frac{2l+1}{2}}\left(q^{\frac{2l+1}{2}} - q^{-\frac{(2l+1)}{2}}\right)$$

$$= -\,q^{\frac{2l+1}{2}}\,\{2l+1\}$$

$$\Psi_{k,l} = \overbrace{(-1)\, q^{\frac{2l+1}{2}}\cdot \xi(q)}^{f(q)} \times \{2l+1\}\,\Upsilon \times$$

Rest of the q factors.

$$f(q) = -\frac{q^{\frac{2l+1}{2}}\cdot q^k \cdot q^{\frac{l(l-1)}{2}}\cdot q^{k^2/2\,\cancel{+k/2}}\cdot \cancel{q^{k^2/4}}\,{}^{+k/4}}{q^{k^2/4}\times q^{\frac{(l+1)^2}{4}}\times q^{\cancel{\frac{k(l+1)}{2}}}\times \cancel{q^{k^2/4}}\times q^{l^2/4}\cdot q^{\cancel{-kl/2}}\cdot q^{\cancel{k/2}+l/4}}$$

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$$f(q) = -\frac{q^{l+\frac{1}{2}} \cdot q^{k} \cdot q^{\frac{l^2}{2}-\frac{1}{2}} \cdot q^{k^2/2} \cdot q^{k/4}}{q^{k^2/4} \cdot q^{\frac{(l+1)^2}{4}} \cdot q^{k/2} \cdot q^{l^2/4} \times q^{l/4}}$$

$$= -\left[q^{l+l/4} \times q^{k/2+k/4} \times q^{k^2/4} \times \frac{q^{\cancel{l^2/4}-l/2}}{\cancel{q^{\frac{l^2}{4}}}\, \cancel{q^{\frac{l}{4}}} \times q^{l/2}} \right]$$

$$= -q^{l-l} \times \cancel{q^{l^2/4}} \times q^{k^2/4} \times q^{k/2+k/4}$$

$$= -q^{\frac{k^2}{4}+\frac{3k}{4}} = -q^{\frac{k(k+3)}{4}}$$

$$f(q) = -q_I^{k(k+3)/2}$$ because $q_I^2 = q$

$$\boxed{f(q) = -q_I^{k(k+3)/2}}$$

$$\Upsilon = \frac{(\{-n-1\} \cdots \{-n-k\})(\{n+1\} \cdots \{n+k\}) \times (\{-1\} \cdots \{-k\})}{(\{-1\}\{2\} \cdots \{k+l+1\})(\{-1\}\{0\} \cdots \{k-l-2\})}$$

$$= \frac{(-1)^k \cdot (\{n+1\} \cdots \{n+k\} \cdot \{n-1\} \cdots \{n-k\}) \cancel{\{-1\}\{0\}\{1\} \cdots \{k-2\}}}{(\cancel{\{-1\}\{0\}\{1\} \cdots \{k-2\}}\{k-1\}\{k\} \cdots \{k+l-1\})(\{-1\}\{0\} \cdots \{k-l-2\})}$$

This is ~~ok~~ O.K., … we need this.

$$\Upsilon = (-1)^k \cdot \frac{\{n-k\} \cdots \{n-1\} \times 1 \times \{n+1\} \cdots \{n+k\}}{\{k-1\}\{k\} \cdots \{k+l-1\} \times (\{-1\}\{0\}\{1\} \cdots \{k-l-2\})}$$

$$\frac{\{k+l+1\}!}{\{k+l\}\{k+l+1\}} \times$$

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$$\Upsilon = \frac{(-1)^k}{\{k+l+1\}!} \cdot \frac{\{k+l\}\{k+l+1\}\left(\{n-k\}\cdots\{n-1\}\times 1\times\{n+1\}\cdots\{n+k\}\right)}{\{-1\}\{0\}\cdot \cancel{\{}}$$

Multiply & divide by ~~(k-l)!~~ $n!$

$$\Upsilon = \frac{(-1)^k}{\{k+l+1\}!}\cdot\left(\frac{\{k+l\}\{k+l+1\}}{\{-1\}\{0\}}\right)\times\left[\frac{\{n-k\}\{n-k+1\}\cdots\{n+k\}}{\{n\}}\right]$$

This factor!

$$\Upsilon = \frac{(\{-n-1\}\cdots\{-n-k\})(\{n+1\}\cdots\{n+k\})(\{-1\}\{-2\}\cdots\{-k\})}{(\{-1\}\{-2\}\cdots\{-(k+l+1)\})(\{-1\}\{-2\}\cdots\{-k-l\})}$$

$$= \frac{(-1)^k(-1)^k\{k\}!}{(-1)^{k+l+1}\cdot(-1)^{(k+l)}}\frac{(\{n-1\}\cdots\{n-k\})\times 1\times(\{n+1\}\times\cdots\{n+k\})}{(\{1\}\{2\}\cdots\{k+l+1\})(\{1\}\{2\}\cdots\{k+l\})}$$

$$= \frac{\cancel{(-1)^{k+l}}(-1)\{k\}!}{\{k+l+1\}!\,\{k-l\}!}\left(\{n-k\}\cdots\{n-1\}\times 1\times\{n+1\}\cdots\{n+k\}\right)$$

Multiply & divide by $\{n\}$

$$\Upsilon = \frac{\cancel{(-1)^{k+l}}(-1)\{k\}!}{\{k+l+1\}!\{k-l\}!}\cdot\left(\frac{\{n-k\}\{n-k+1\}\cdots\{n+k\}}{\{n\}}\right)$$

Hence everything is restored.

$\boxed{\text{Can't restore the } (-1)^{k+l} \text{ factor}}$

Got the result, but upto minus sign.

A Brief Summary of the inconsistency

Masbaum formula

$$J_n = \sum_{k=0}^{\infty}\sum_{l=0}^{k} (-1)^{k+1} \cdot q_I^{\frac{k(k+3)}{2}} \cdot (-1)^l \cdot q^{l(l+1)p} \cdot \{2l+1\} \times \frac{\{k\}!}{\{k+l+1\}!\,\{k-l\}!} \times \frac{\{n-k\}\{n-k+1\}\cdots\{n+k\}}{\{n\}}$$

call this summand $\alpha_{k,l}$

Rearranged Formula written in the paper

$$J_n(K_p;q) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} q^k (q^{1-n};q)_k (q^{n+1};q)_k \times (-1)^l \cdot q^{l(l+1)p} \times q^{\frac{l(l-1)}{2}} \cdot (1-q^{2l+1}) \times \frac{(q;q)_k}{(q;q)_{k+l+1}\,(q;q)_{k-l}}$$

call this summand $\beta_{k,l}$

To show that The formula is same we need to show $\alpha_{k,l} = \beta_{k,l}$

Note that: $q^{l(l+1)p} \cdot (-1)^l$ are common in $\alpha_{k,l}$ & $\beta_{k,l}$

Hence need to show;

$\cancel{(-1)^{k+1}\, q_I^{\frac{k(k+3)}{2}} \cdot (-1)^l \cdot q^{l(l+1)p}}$

$$(-1)^{k+1} \cdot q^{\frac{k(k+3)}{2}} \cdot \{2l+1\} \times \frac{\{k\}!}{\{k+l+1\}!\,\{k-l\}!} \times \frac{\{n-k\}\{n-k+1\}\cdots\{n+k\}}{\{n\}}$$

$$= q^k \cdot (q^{1-n};q)_k \cdot (q^{n+1};q)_k \times q^{\frac{l(l-1)}{2}} \cdot (1-q^{2l+1}) \times \frac{(q;q)_k}{(q;q)_{k+l+1}\,(q;q)_{k-l}}$$

now; I ~~sta~~ try to show LHS = RHS

But, before that, I need following results.

$$(q^{1-n};q)_k = q^{k^2/4 - nk/2 + \frac{k}{4}} \cdot \{n-1\} \ldots \{n-k\}$$

$$(q^{1+n};q)_k = (-1)^k \cdot q^{\frac{k^2}{4} + \frac{nk}{2} + \frac{k}{4}} \cdot \{n+1\} \ldots \{n+k\}$$

$$(q;q)_k = (-1)^k \cdot q^{k/4 + k^2/4} \cdot \{k\}!$$

↑ This doesnot give $(-1)^k$

$$LHS = q^k \cdot q^{\frac{l(l-1)}{2}} \cdot (1-q^{2l+1}) \times \frac{(q^{1-n};q)_k \, \overbrace{(q^{n+1};q)_k}^{(-1)^k} \, \overbrace{(q;q)_k}^{(-1)^k}}{\underbrace{(q;q)_{k+l+1}}_{(-1)^{k+l+1}} \, \underbrace{(q;q)_{k-l}}_{(-1)^{k-l}}}$$

Each of these gives $(-1)^k$

& so finally $\dfrac{(-1)^k (-1)^k}{(-1)^{k+l+1} (-1)^{k-l}}$

$= (-1)$

This minus sign is eaten away when $(1-q^{2l+1})$ is converted to $\{2l+1\}$

because;

$$(1-q^{2l+1}) = -q^{\frac{2l+1}{2}} \cdot \{2l+1\}$$

Hence these formula is not consistent; because of the factor of $(-1)^{k+l}$ hanging around.

Shoaib's Conclusion

Equivalence of double sum and multi sum formula of $J_n(K_p;q)$ (PGT)

We have the identity: $$(q;q)_{s_p}\sum_{s_{p-1}=0}^{s_p}\frac{\alpha^{(p)}_{s_{p-1}}}{(q;q)_{s_p-s_{p-1}}(xq;q)_{s_p+s_{p-1}}}=\sum_{s_p\geq\cdots\geq s_1\geq 0}\left(\prod_{i=1}^{p-1}x^{s_i}\cdot q^{s_i^2}\cdot\begin{bmatrix}s_{i+1}\\ s_i\end{bmatrix}_q\right)$$

where $$\alpha^{(l)}_n=(-1)^n\cdot x^{ln}\cdot q^{ln^2+\frac{n(n-1)}{2}}\cdot\frac{1-xq^{2n}}{1-x}\cdot\frac{(xq;q)_n}{(q;q)_n}$$

lets do for $K_{p>0}$

$$P_n(K_{p>0};a,q,t)=(-t)^{n+1}\sum_{s_p\geq\cdots s_1\geq 0}q^{s_p}\cdot\frac{(-atq^{-1};q)_{s_p}}{(q;q)_{s_p}}\cdot(q^{1-n};q)_{s_p}\cdot(-at^3q^{n-1};q)_{s_p}\cdot\times\prod_{i=1}^{p-1}(at^2)^{s_i}\cdot q^{s_i(s_i-1)}\begin{bmatrix}s_{i+1}\\ s_i\end{bmatrix}_q$$

$\boxed{x=at^2q^{-1}}$

$$=(-t)^{n+1}\sum_{s_p=0}^{\infty}q^{s_p}\frac{(-atq^{-1};q)_{s_p}}{(q;q)_{s_p}}\cdot(q^{1-n};q)_{s_p}\cdot\frac{(-at^3q^{n-1};q)_{s_p}}{((q;q)_{s_p})^{-1}}\sum_{s_{p-1}=0}^{s_p}\frac{\alpha^{(p)}_{s_{p-1}}}{(q;q)_{s_p-s_{p-1}}(at^2;q)_{s_p+s_{p-1}}}$$

now rename s_p by k ; & s_{p-1} by l

$$P_n(K_{p>0}:a,q,t)=(-t)^{n+1}\sum_{k=0}^{\infty}\sum_{l=0}^{k}q^k\frac{(-atq^{-1};q)_k}{(q;q)_k}\cdot(q^{1-n};q)_k\cdot\frac{(-at^3q^{n-1};q)_k}{((q;q)_k)^{-1}}\cdot\frac{\alpha^{(p)}_l}{(q;q)_{k-l}\cdot(at^2;q)_{k+l}}$$

$$=(-t)^{n+1}\sum_{k=0}^{\infty}\sum_{l=0}^{k} q^k\frac{(-atq^{-1};q)_k}{(q;q)_k}\cdot(q^{1-n};q)_k\cdot\frac{(-at^3q^{n-1};q)_k}{((q;q)_k)^{-1}}\cdot\frac{1}{(q;q)_{k-l}\cdot(at^2;q)_{k+l}}\cdot\Big\{(-1)^l\cdot(at^2)^{pl}\cdot q^{pl(l-1)+l(l-1)/2}\times\frac{1-at^2q^{2l-1}}{1-at^2q^{-1}}\times\frac{(at^2;q)_l}{(q;q)_l}\Big\}$$

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$$=(-t)^{n+1}\sum_{k=0}^{\infty}\sum_{l=0}^{k} q^k\cdot\frac{(-atq^{-1};q)_k}{(q;q)_k}\cdot(q^{1-n};q)_k\cdot(-at^3q^{n-1};q)_k\times(-1)^l\cdot(at^2)^{pl}\times q^{pl(l-1)+\frac{l(l-1)}{2}}\times\frac{1-at^2q^{2l-1}}{1-at^2q^{-1}}\times\frac{(q;q)_k}{\cancel{(q;q)_k}\,(q;q)_{k-l}\cdot(at^2;q)_{k+l}}\times\frac{(at^2;q)_l}{(q;q)_l}$$

$$=(-t)^{n+1}\sum_{k=0}^{\infty}\sum_{l=0}^{k} q^k\cdot\frac{(-atq^{-1};q)_k}{(q;q)_k}(q^{1-n};q)_k(-at^3q^{n-1};q)_k\times(-1)^l\cdot(at^2)^{pl}\cdot q^{pl(l-1)+\frac{l(l-1)}{2}}\cdot\frac{1-at^2q^{2l-1}}{1-at^2q^{-1}}\times\frac{(at^2;q)_l}{(q;q)_l}\times\frac{(q;q)_k}{(q;q)_{k-l}\cdot(q;q)_l}\times\frac{1}{(at^2;q)_{k+l}}$$

In paper; this is $(at^2q^{-1};q)_l$. $\boxed{\text{An error}}$!

My expression with minor change:

$$P_n(K_{p>0}; a, p, t) = (-t)^{n+1} \sum_{k=0}^{\infty} \sum_{l=0}^{k} q^k \cdot \frac{(-atq^{-1};q)_k}{(q;q)_k} \cdot (q^{1-n};q)_k \cdot (-at^3 q^{n-1};q)_k \cdot (-1)^l \cdot (at^2)^{pl} \cdot q^{(p+\frac{1}{2})l(l-1)} \times$$

$$\times \frac{1-at^2q^{2l-1}}{1-at^2 \cdot q^{-1}} \times \frac{1}{(at^2;q)_{l+k}} \cdot \frac{(at^2;q)_l}{\cancel{(q;q)_l}} \cdot \begin{bmatrix} k \\ l \end{bmatrix}_q$$

~~Note:~~ $\cancel{\dfrac{(at^2;q)_l}{(at^2;q)_{k+l}} = \dfrac{1}{(at^2;q)_k}}$

$$P_n(K_{p>0}; a, p, t) = (-t)^{n+1} \sum_{k=0}^{\infty} \sum_{l=0}^{k} q^k \cdot \frac{(-atq^{-1};q)_k}{(q;q)_k} \cdot (q^{1-n};q)_k \cdot (-at^3 q^{n-1};q)_k \cdot (-1)^l \cdot (at^2)^{pl} \cdot q^{(p+\frac{1}{2})l(l-1)} \times$$

$$\times (1-at^2q^{2l-1}) \cdot \begin{bmatrix} k \\ l \end{bmatrix}_q \times \left(\frac{(at^2;q)_l}{(1-at^2q^{-1})(at^2;q)_{k+l} \cancel{(q;q)_l}} \right)$$

Now, we need to manipulate;

$$\boxed{\frac{(at^2;q)_l}{(1-at^2q^{-1})(at^2;q)_{k+l} \cancel{(q;q)_l}}} \Rightarrow \text{Hope this is equal to } \boxed{\frac{1}{(at^2q^{l-1};q)_{k+1}}}$$

if the final result in double sum form in paper is correct!

$$\frac{(at^2;q)_l}{(1-at^2q^{-1})\,(at^2;q)_{k+l}} \overset{?}{=} \frac{1}{(at^2q^{l-1};q)_{k+1}}$$

I tried checking the equality for a particular value of q, t, a, k, l in mathematica; and they failed to be equal !