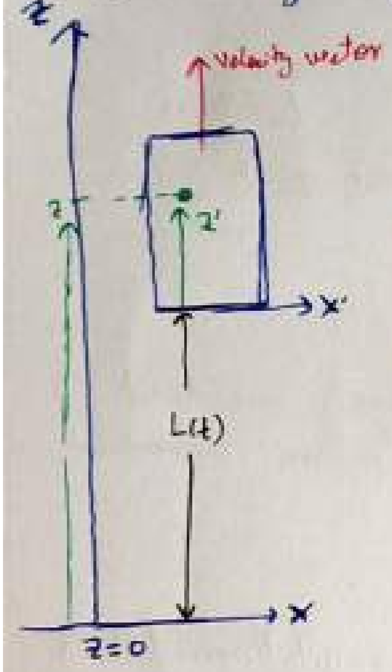
General Relativity : The Equivalence Principle & Tensor Analysis

- Shoaib Akhtar.

☆ Equivalence Principle "Gravity in some sense is same as acceleration".

Einstein's Thought experiment. (For simplicity ignore Special Relativity ... imagine speed of light to be infinitely fast) (assume slow velocity)



If we know laws of physics in frame z; surely ~~surely~~ we can figure out them in frame z'.

☆ Assume both frames are inertial reference. (means they are related by uniform velocity)

so; $L(t) = vt$; and $z' = z - L(t)$

we can assume $t' = t$. (assume low velocities)

& $x' = x$ (no velocity along x direction)

so; $z' = z - vt$; $t' = t$; $x' = x$; $y' = y$.

$F = m\ddot{z}$ (Law of motion in the z frame)

now; $\ddot{z}' = \ddot{z}$; and so; Law in primed frame: $F = m\ddot{z}'$ (and find Newton law in primed frame is exactly same as unprimed frame)

☆ Unprimed frame is accelerating.

take; $L(t) = \frac{1}{2} g \cdot t^2 \Rightarrow \ddot{L} = g$ and $\dot{L} = gt$

so, equation connecting coordinate transformation.

$z' = z - \frac{1}{2} g t^2$

$t' = t$

$x' = x$

now; $\ddot{z}' = \ddot{z} - g \Rightarrow \ddot{z} = g + \ddot{z}'$

and now we can write newtons eqn in primed frame of reference

so; $F = m\ddot{z}' + mg$

$$\Rightarrow \underbrace{F - mg}\ = m\ddot{z}'$$

↳ it looks exactly like force law of a particle at the surface of the earth. (or the surface of any kind of gravitating system)

↳ It looks like uniform gravitational field.

★ Gravitational forces are proportional to mass.

↳ implication; ∴ F = ma & if F ∝ m

↳ then mass cancels & the acceleration, or the total motion of the object does not depend on mass.

∴ $F - mg = m\ddot{z}$ → here is an example of a fictitious force mg (take g to be acceleration; then ma), mimicking the effect of gravity

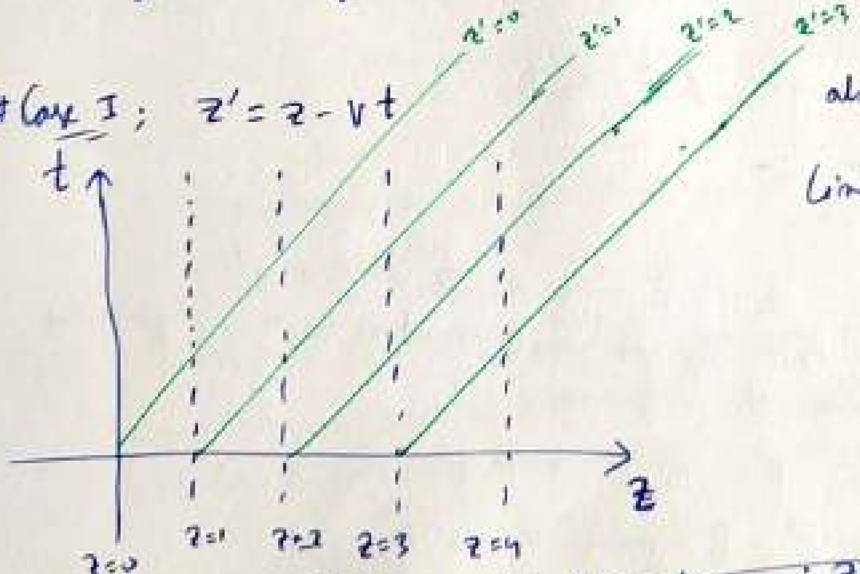
↳ ~~Most~~ Most people before Einstein considered this largely accident or coincidence... people just knew it.... that effect of acceleration mimicked the effect of gravity.

★ Einstein formulated that,
It is deep principle of nature, that gravitational forces cannot be distinguished from the effect of accelerated reference frame & indistinguishable.
↳ They are physically the same thing

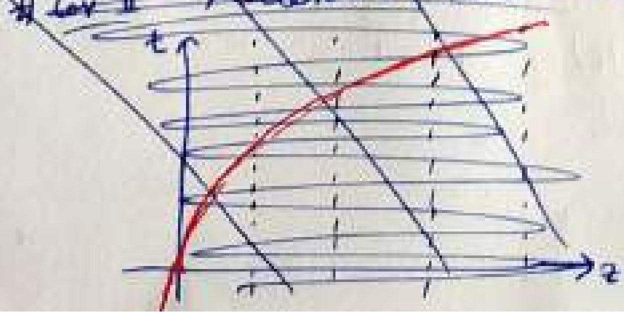
Case I; $z' = z - vt$

also; $t' = t$

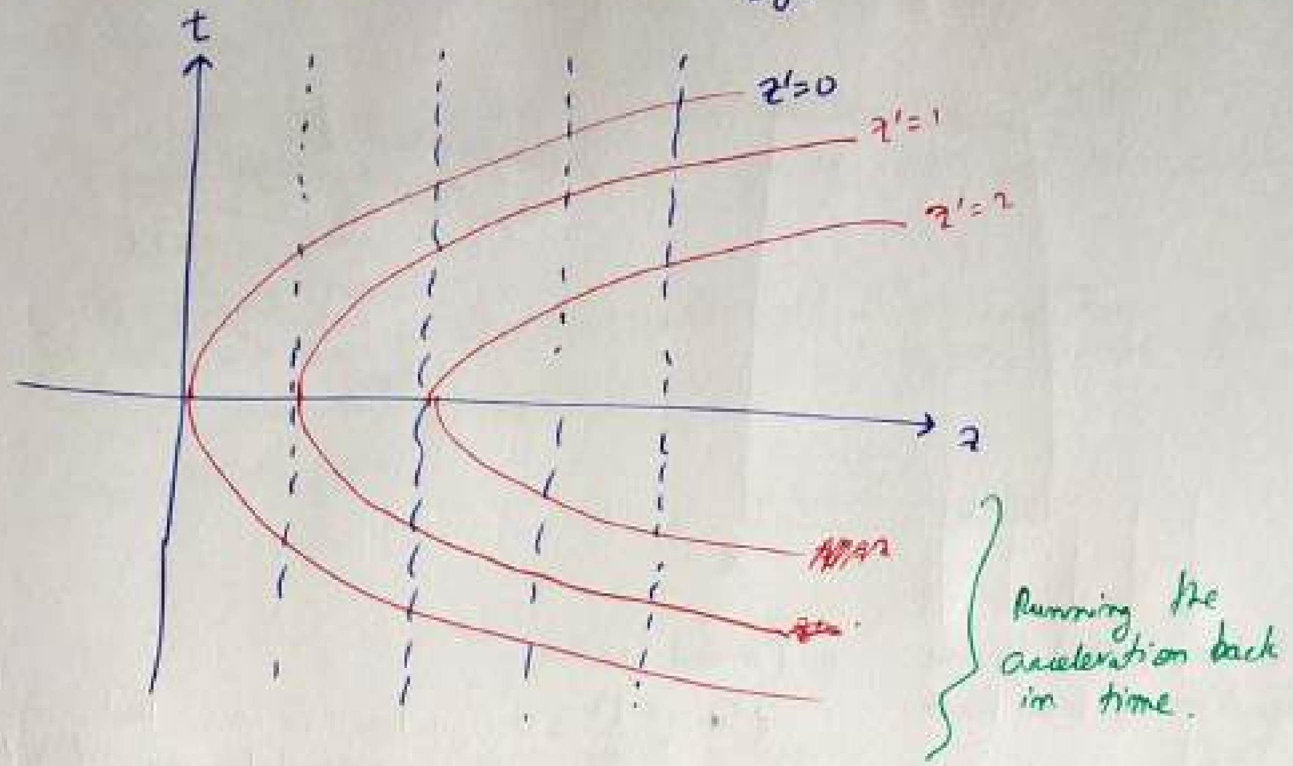
Linear transformation
line goes to line.



Case II ~~Accelerated coordinate system~~ ; $z' = z - \frac{1}{2}gt^2$



case II · Accelerated coordinate; $z' = z - \frac{1}{2}gt^2$



→ shifted parabolas.

→ This is curvillinear coordinate transformation.
• Line goes to curve.

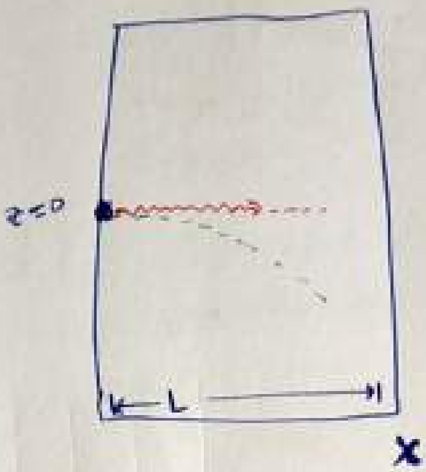
* This is something which Einstein understood very early, that there is connection between gravity and curvillinear coordinate transformation of space time.

⇒ That, special Relativity was all about Linear transformation (transformation which took uniform velocity to uniform velocity)

example // what is the influence of Gravity on light? (most physicist around 1905 or so thought that gravity has no effect on light... light just moves in straight line)

* But Einstein said; if it is true that there is principle, the equivalence between acceleration & gravity.
↳ Then Gravity will affect light.

Argument which Einstein gave.



~~Unprimed fr equation of light beam x=ct and also z=0~~

- Unprimed frame
  equation of light beam
  $x = ct$
  & $z = 0$

- Primed frame
  $z' + g\frac{t^2}{2} = 0$
  & $x' = ct$
  (because; $x' = x$)

⇒ here; we see that in primed frame of refference light is moving on a curved trajectory.

→ we can say that in primed frame; gravity is pulling light down. (and someone says that it is just that the acceleration that the elevator is accelerating upwards, and this makes it look like that light moves on curve trajectory)

→ Einstein said that they are the same thing. and this proved to him that the effect of a gravitational field on a light ray was to cause it to bend.

Is acceleration & gravity really indistinguishable?

↳ Lets talk about real gravitational field.

- There is no way globaly to introduce a coordinate transformation which is going to get rid of fact that there is gravitational field pointing inward towards the centre.

If you are in some small laboratory □ which is falling towards centre,

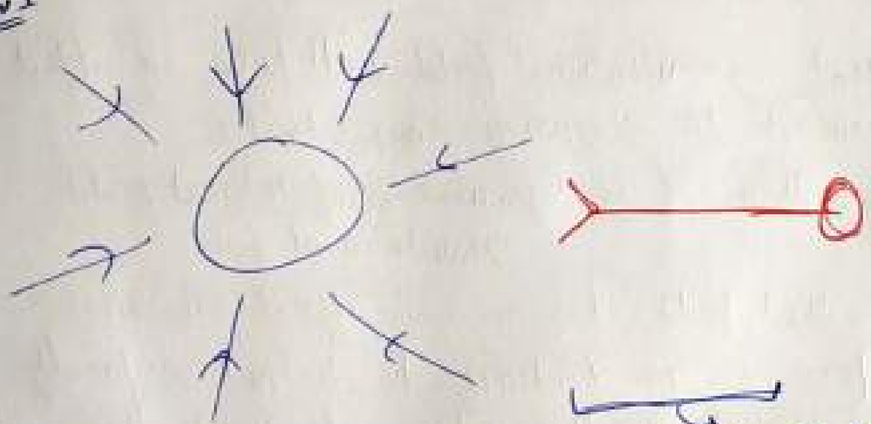
then if you make transformation like $z' = z - \frac{g}{2}t^2$ .. it might get rid of gravity there for the falling laboratory

→ But, the same transformation at other place (say diametrically opposite place) will just increase the gravitational field.

→ One way to understand why you can't get rid of it

because if you think of a large object... then....

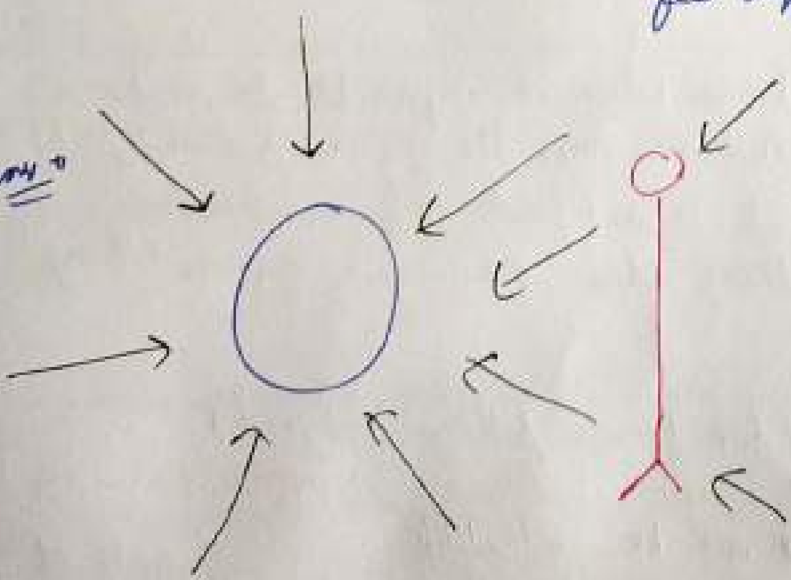
case I



will feel a sense of stretching

large so different part of him feel different gravitational fields.

case II



will feel a sense of compression.

* and these sense of stretching or compression can't be removed by any coordinate transformations.

---- These are Tidal forces.

So; it is not true that gravity is equivalent to going to an accelerated acceleration frame.

(Einstein knew this, .. but he was using it
↳ he actually said that .. small objects for small length of time cannot tell difference between gravitational field & accelerated frame of reference)
~~reference~~

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* If you have been given a complicated gravitational field
↳ and you have to check whether it is a genuine ~~field or~~ field or a faked one due to acceleration .. what you will do?

⇒ You calculate whether that gravitational field will have an effect on object which will cause it to sqeueeze and stretch.
... if it does, then it is genuine ~~gravitational~~ field gravitational field.

→ if you discovered that ∴ that field has no such effect on any object ∴.. no Tidal force ... no tendency to distort a freely falling system no ~~no matter th~~ matter where it is ∴ then its not a real gravitational field.

~~Question~~ Question • How do you know when it is possible to make a coordinate transformation that removes the gravitational field?

• How do you know when it is artifacts of coordinates and when it is real thing due to some gravitating mass. ?

How to check Tidal force. (for freely falling object)

↳ Through a chunk mass in the field ..... and if there are stresses & strains in the mass (say crystal here) ... it ~~will~~ will be detectable.

(The object should not be held by any sorts of rope or so.)

Tidal forces ~~[struck]~~ mean effectively distorting stresses on matter that squeeze it or pull it.

Carefully stating Equivalence Principle.

if you take a sufficiently small system (such that the gravitational field across it does not vary much) that is when Equivalence principle holds with accuracy.

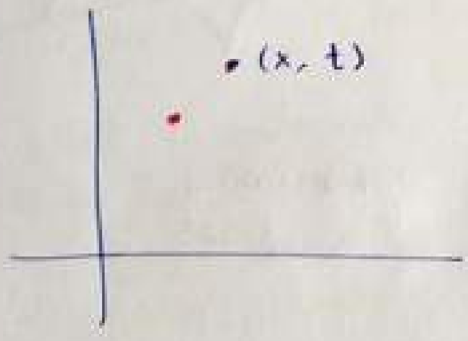
↳ Einstein knew this ... and so he asked himself a question that what kind of mathematics goes in to try to answer these kinds of question.

⇒ Other thing which came into it was Special Relativity (S.R.)

⇒ Einstein knew from the work of Minkowski that S.R. had a geometry associated with it.

• The geometry had a property that it had a kind of length associated with it ~~[struck]~~ ... proper time.

• S.R. has a length element



~~$d\tau^2 = dt^2 c^2 - dx^2$~~

$$d\tau^2 = dt^2 - \frac{dx^2}{c^2}$$

.. we can get ~~[struck]~~ rid of c by redifing x : $x \to x/c$

↳ and this is itself coordinate transformation; so: $d\tau^2 = dt^2 - dx^2$

sometime; people use; $ds^2 = dx^2 - dt^2$

☆ Einstein realized that, the question up there, ~~[struck]~~ i.e; are their coordinate transformations which can remove effect of tidal forces is very similar to a mathematics problem that was studied in great length by Reimann ... Namely the question of deciding whether a geometry was flat or not.

Einstein realised, that the two question whether

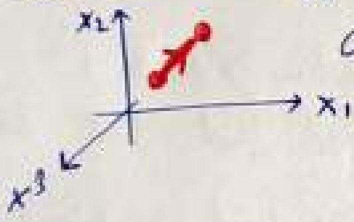
- a geometry is flat
- a spacetime has a real gravitational field in it

.... were very similar.

\# $ds^2 = dx^2 - dt^2$ → Reimann had not thought about the minus sign here

- Reimann was thinking about geometries similar to Euclidean Geometry.

☆ Euclidean Geometry



$dx^m \equiv (dx^1, dx^2, dx^3)$

here: $\underbrace{ds^2}_{\text{square of spacial distance}} = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

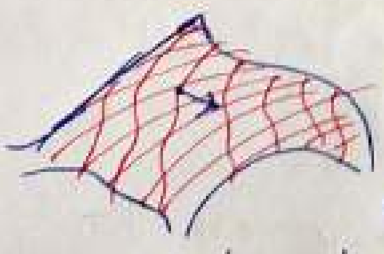
↳ formula for Euclidean distance in Euclidean Space.

- Gauss already understood that on curved surfaces, the formula for distance between two points was more complicated in general than Euclidean ~~one~~ one.

... and Reimann generalized this to arbitrary dimensions.

... from the ~~theo~~ theory of surfaces ....

↳ and what was known from the theory of surfaces ....

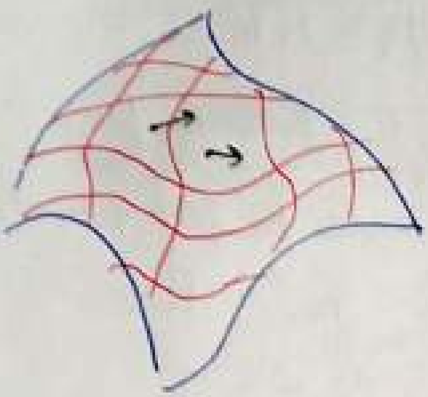


We ~~intro~~ introduce some coordinates on to it.

↳ we just lay out some coordinates ~~on it~~.

- We don't worry about whether the coordinate axes are straight lines or not. (because for all we know is that surfaces is real curved surface, there probably ~~wo~~ won't be things which we can call straight lines)

We lay coordinate and call them X^m s.

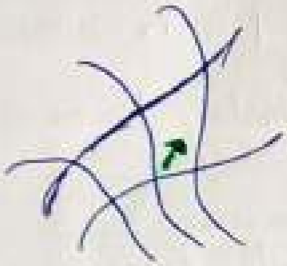


dX^m

and; $$ds^2 = \underbrace{\sum_{m,n} g_{mn}(x)\, dX^m dX^n}$$

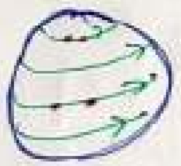
in general; square of distance between two points ~~xx~~ will be quadratic form in these little coordinate seperation.

→ Quadratic form means that you have a matrix ~~g~~ g_{mn}



... & in general g will depend ~~xx~~ on possition.

example



on sphere; $$ds^2 = R^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

* $g_{mn}(x)$ is called metric tensor.

What is mathematics of taking a metric tensor & asking is the space flat? or what it means to be flat.

⇒ What it means to be ~~xx~~ flat is that you can find a coordinate transformation ~~xx~~ in which the distance formula is just $(dy^1)^2 + (dy^2)^2 + (dy^3)^2 + (dy^4)^2 + \ldots + (dy^n)^2$

↪ The space is flat if: a coordinate transformation can make g look like identity matrix.

* you can diagonalize g at a given point.

↪ but you can't diagonilize it everywhere simultaneously

(so; you can't tell by looking at a point whether the surface is flat)

(looking at point always looks flat)

~~Tensor Analysis~~

\# So; given a metric tensor: g
↳ and we ask can it by coordinate transformation be reduced to Identity matrix everywhere.

$$g_{mn} \xrightarrow[\text{transformation}]{\text{coordinate}} \delta_{mn}$$

- if yes; the space is called flat.
- if no; " " " " curved.

$\delta_{mn} \, dx^m dx^n$
↳ formula for interval in cartesian coordinates for flat geometry

$\therefore$ $g_{mn} = \delta_{mn}$

i.e; $g = I$
(identity matrix)

Flat space in cartesian coordinates:
the metric tensor is just Identity matrix.

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* How things transform when you make coordinate transformations, and this is subject of Tensor Analysis.

Tensor Analysis

- let there be two sets of coordinates: x^m, and y^m
  i.e; $(y^1, y^2, y^3, \ldots, y^n)$ & $(x^1, x^2, \ldots, x^n)$
  ↳ and of course they are related.

so; $x^m(Y)$; $Y \equiv (\{y_i\}_i^m)$

& $y^m(X)$; $X \equiv (\{x_i\}_{i=1}^m)$

↳ Two coordinate system each one being function of the other.

$\therefore$ given; dx^m ... given a pair of points, neighbouring to each other ... and a little differential seperation between them

→ What is the corresponding differential seperation of the Y coordinates: dy^m.

$$dy^m = \sum_p \frac{\partial y^m}{\partial x^p} \cdot dx^p$$

* dx & dy represent an infinitisimal vector (same vector)

$$\underbrace{dy^m = \sum_p \frac{\partial y^m}{\partial x^p} dx^p}$$

→ This is transformation properties of components of that vector

... components of what is called Contravariant vector

Contravariant vector (simply means things which transform like above)

Defination: Contravariant vector.

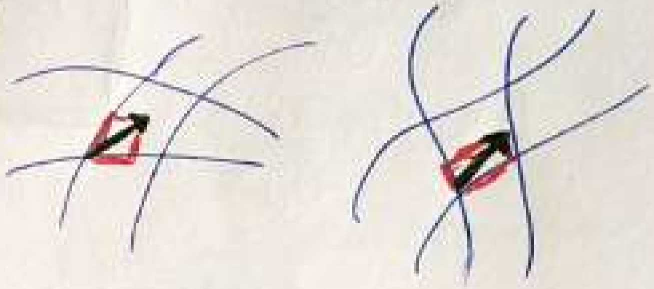
$$\underbrace{(V')^m = \sum_p \frac{\partial y^m}{\partial x^p} \cdot V^p}$$

Transformation property of Contravariant vector.

Contravariant vector transform in this way.

eg components of velocity also transform this way.

. primed frame is y frame
. unprimed " " x "



Unprimed frame

Primed frame

↳ The coordinates are different just because the axes the different.

Same vector... different coordinate.

Contravariant vectors are defined by their transformation properties.

$$(V')^m = \frac{\partial y^m}{\partial x^p} \cdot V^p$$

Covariant vectors.

Let have a scalar S ; then gradient is $\frac{\partial S(x)}{\partial x^m}$

(Scalar is the thing which does not transform) → It has well defined value at every point.

Note: Gradient does not transforms as contravariant vector.

∴ given we know: $\frac{\partial S}{\partial x^p}$ ... compute: $\frac{\partial S}{\partial y^m}$

so: $$\boxed{\frac{\partial S}{\partial y^m} = \frac{\partial S}{\partial x^p} \cdot \frac{\partial x^p}{\partial y^m}}$$

... use Einstein summation convention.

so; let: $W_m := \frac{\partial S}{\partial x^m}$ so: $$\boxed{(W_m)' = \frac{\partial x^p}{\partial y^m} \cdot W_p}$$

$(W_m)' := \frac{\partial S}{\partial y^m}$

(note contravariant had $(V')^m = \frac{\partial y^m}{\partial x^p} V^p$)

so; Things which transform as $(W_m)' = \frac{\partial x^p}{\partial y^m} \cdot W_p$

is called Covariant Vector.

Transformation Properties ...(actually defined by their transformation properties)

Contravariant Vector : $(V^m)' = \frac{\partial y^m}{\partial x^p} V^p$

Covariant Vector : $(W_m)' = \frac{\partial x^p}{\partial y^m} \cdot W_p$

→ They are just two different mathematical objects.
(later we will see connections)
→ then we could think of a vector having contravariant or covariant representation.

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let there be two different contravariant vectors.

V & U ∴ & define $T^{mn} = V^m U^n$

↳ here: T^{mn} is a special case of a tensor of Rank 2

↳ means Tensor with two indices.

so; $(T^{mn})' = (V^m)'(U^n)'$

$$= \left(\frac{\partial y^m}{\partial x^p} \cdot V^p\right)\left(\frac{\partial y^n}{\partial x^q} U^q\right)$$

$$\Rightarrow \boxed{(T^{mn})' = \frac{\partial y^m}{\partial x^p} \cdot \frac{\partial y^n}{\partial x^q} \cdot T^{pq}}$$

⇒ Any-thing which transforms like this is called Contravariant ~~Covariant~~ Tensor of Rank 2.

↳ for each index there is an operation.

↳ similarly:

$$\boxed{(T^{lmn})' = \frac{\partial y^l}{\partial x^p} \cdot \frac{\partial y^m}{\partial x^q} \cdot \frac{\partial y^n}{\partial x^r} \cdot T^{pqr}}$$

→ This will be transformation property of Rank 3 Contravariant Tensor.

covariant ⇒ downstair indices.
contravariant ⇒ upstair indices.

$W_m' Z_n' = \frac{\partial x^p}{\partial y^m} \cdot \frac{\partial x^q}{\partial y^n} \cdot W_m Z_n$

$$\Rightarrow \boxed{(T_{mn})' = \frac{\partial x^p}{\partial y^m} \cdot \frac{\partial x^q}{\partial y^n} \cdot T_{mn}}$$

⇒ This is transformation property of a thing with two covariant indices.

(or) This is the way how a covariant tensor of Rank 2 transforms.

Now the metric tensor transform.

~~$ds^2 = g_{mn} dx$~~

$$ds^2 = g_{mn}(X) \cdot dx^m dx^n$$; but:

$$dx^m = \frac{\partial x^m}{\partial y^p} \cdot dy^p$$

$$\Rightarrow ds^2 = g_{mn}(x) \cdot \left(\frac{\partial x^m}{\partial y^p} \cdot dy^p\right)\left(\frac{\partial x^n}{\partial y^q} \cdot dy^q\right)$$

$$\Rightarrow ds^2 = g_{mn}(x) \cdot \frac{\partial x^m}{\partial y^p} \cdot \frac{\partial x^n}{\partial y^q} \cdot dy^p dy^q .$$

... we will discover $g_{mn}(x)$ is tensor with two ~~covariant~~ covariant indices.

↳ and then comes the question, given this is transformation property of g; can we or can we not find a coordinate transformation which will ~~turn~~ turn g_{mn} in δ_{mn}.

(This is a mathematics question...)

"A good notation will carry you a long way"
↳ it will automatically tell you what to do next...

• Extrinsic curvature : has to do with the way a space is imbedded in a higher dimensional space.

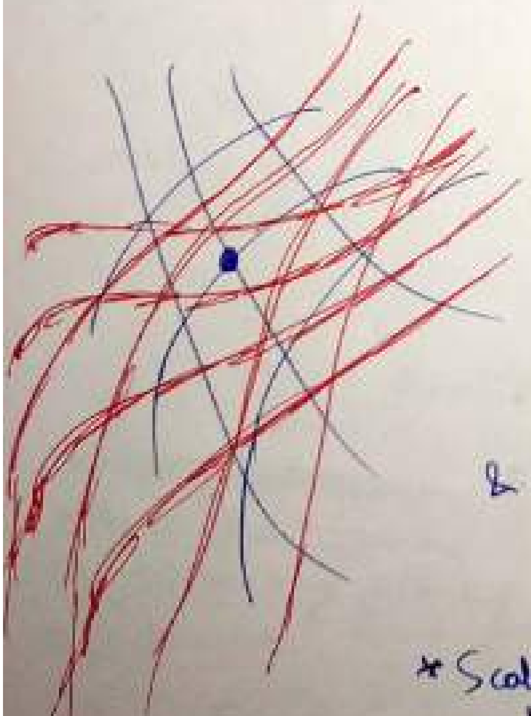
• Intrinsic geometry : can think of a tiny creature that moves along the surface ; cannot look out of the surface. ~~~~ → he does all kind of geometric studies along the surface (but never looks out of the surface) (and thus it never detects the facts that it might be embedded in a higher dimensional space in different ways)

• Intrinsic geometry is independent of the way surface is embedded in higher dimensional space.

General Relativity is about Intrinsic property of geometry.

☆ A field is a think which can vary from point to point. (we will now be ~~~~ interested in Tensor field) ★ simplest kind of field is scalar.

$S(x)$
↳ it is a quantity at each point of space where everybody in all coordinates ~~~~ agree about its value.



$y^m = y^m(x)$
⇓
we assume there is one-to-one correspondence between x & Y

& $x^n = x^n(Y)$

so; $X^m \longleftrightarrow Y^m$

Value of scalar field at actual point does not change

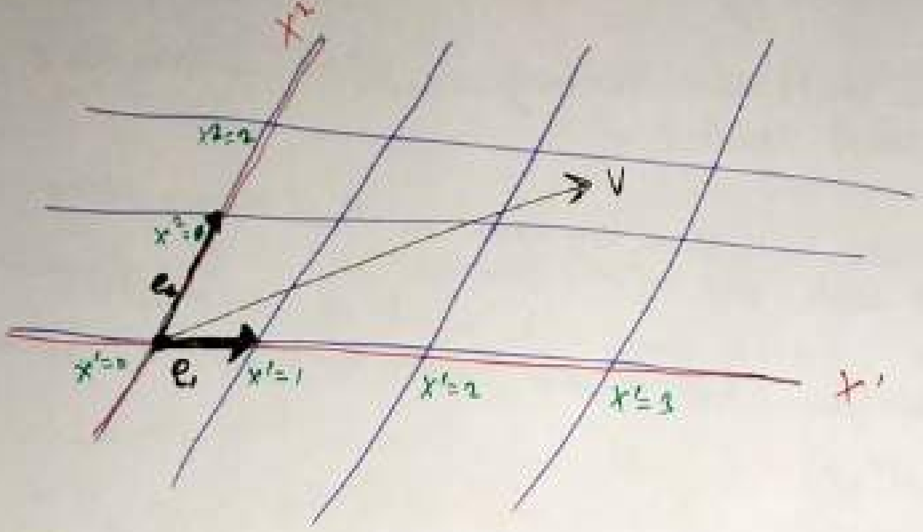
i.e.: $S'(y) = S(x)$
↳ prime denotes we are talking in coordinate system y.

★ Scalars transform's trivially.

Vectors

V^m :=) Contravariant vector

W_m :=) Covariant vector.



x^1 & x^2 could be perpendicular to each other

↳ but in general they can be at any angle

=) here assume they are not perpendicular.

~~here~~:

here; actual physical euclidian distance between $x^1=0$ & $x^1=1$ is not necessarily 1 unit.

↳ ~~They are~~

↳ They are neither perpendicular to each other, nor ~~do~~ these labels represents actual physical distance along the axis.

(They are just names of point)

e_1 & e_2 . ; we can lable them $\underline{\underline{e_i}}$. ; e_i are basis vectors.

take an arbitrary vector V

$$V = V^1 e_1 + V^2 e_2 + V^3 e_3$$

- e_i are vector in these formula.
- ~~are actually numbers~~
- V^i are actually numbers, but they are components.

=) $\boxed{V = V^m e_m}$ ~~convention~~

These components are called Contravariant components of vector V

(there is nothing required here to put indices in V^i upper, & ... it is just convention)

★ Contravariant components are the expansion coefficients. (pg 17)

Now, lets define projection of given vector onto axes.

$V \cdot e_i$ (if it was usual cartesian coefficients; then the coefficient V^i would be same as the dot products $V \cdot e_i$)

incidently $V \cdot e_i$ is called V_i with covariant index

i.e.; $V_n = V \cdot e_n$

so; ~~$V_n = V^m e_m$~~ $\Rightarrow V_n = (V^m e_m) \cdot (e_n)$

$\Rightarrow V_n = V^m (e_m \cdot e_n)$

$\therefore e_m \cdot e_n$ has two lower indices; and is infact the metric tensor.

lets calculate length of vector

$\therefore l^2 = V \cdot V = (V^m e_m) \cdot (V^n e_n)$

$l^2 = V^m V^n (e_m \cdot e_n)$; let; $g_{mn} := e_m \cdot e_n$

So; we have; $|V|^2 = V^m V^n g_{mn}$

↳ This is the character of metric tensor.
• Metric tensor tells you how to compute length of a vector.

example: the vector could be just dx^m & dx^n

then; length of little interval between neighbouring points would be $ds^2 = g_{mn} dx^m dx^n$.

Contravariant indices are the things which you use to construct a vector out of basis vectors.

Covariant indices are the dot products with the basis vector.

(They are different things ... but they would be same if we are talking of about ordinary cartesian coordinates)

If we change coordinates, keeping the vector fixed.

↳ we will change the coordinates.

↳ we will be clearly changing its components.

☆ so; how do contravariant components change when we change coordinates.

- primed means Y
- unprimed " X

$$\therefore \boxed{(V')^m = \frac{\partial y^m}{\partial x^n} \cdot V^n}$$

⇒ Transformation property of a contravariant object.

example ; $dy^m = \frac{\partial y^m}{\partial x^n} \cdot dx^n$

☆ example of covariant component of a vector ; let you have scalar S

$\frac{\partial S}{\partial x^m}$ } ⇒ this is unprimed ~~component~~ component of covariant vector.

$\frac{\partial S}{\partial y^m}$ } ⇒ this is primed component of covariant vector.

... of gradient vector.

$$\frac{\partial S}{\partial y^m} = \frac{\partial x^n}{\partial y^m} \cdot \frac{\partial S}{\partial x^n}$$

$$\boxed{(W')_m = \frac{\partial x^n}{\partial y^m} \cdot W_n}$$

⇒ Transformation property of a covariant object.

Tensor of higher rank (tensor with more indices)

- simplest example will be products of vectors (products of different components of vector)

$$\boxed{(W')^m{}_n = \frac{\partial y^m}{\partial x^p} \cdot \frac{\partial x^q}{\partial y^n} \cdot W^p{}_q}$$

↳ This is how a tensor of rank 2 with one contravariant & one covarient vector transforms.

↳ what makes this thing tensor is its transformation properties.

ex $(W')_{mn} = \frac{\partial x^p}{\partial y^m} \cdot \frac{\partial x^q}{\partial y^n} \cdot W_{pq}$

~~A tensor is something which~~

" A Tensor is an object which is characterized by its transformation properties"

- If scalar is zero in one frame; they are zero in every frame.
- " vector " " " " ; " " " " " " .
  (all components equal to zero) of the controvariant component V^m are zero.
  ↳ all of the components
-

★ If all of the components of a Tensor is zero in one frame; it is zero in every frame.
(Transformation property results this...)

↳ This means; once you have written down an equation in Tensor for; you can always transfer everything to LHS and set it equal to zero.
→ . and if the equation of that type is true in some frame; its true in every frame.

⇒ This is the basic value of Tensors;
they allow you to express equations of various kinds in a form where the same exact equation will be true in any coordinate system.

The statement : that two tensors are equal is an invariant statement.

ex $W^p{}_q = T^p{}_q$ (if true in one frame; true in all frame)

★ you could write; $W^p{}_q = T^{pq}$
(but here; RHS and LHS transform differently from each other) ⇒ then; even if this was true in one frame; it would not be necessarily true in other frame.

~~So kinds of equation which are ...~~

" So; The kinds of equation which are true in every frame ; are the once in which covariant & contravariant on LHS balances with RHS "

we can always write;

$$W^{pqrst\ldots}_{abcde\ldots} - T^{pqrst\ldots}_{abcde\ldots} = 0$$

Operations on Tensor

~~you can multiply a tensor by number~~

* you can multiply a Tensor by a number (this will multiply all the components of Tensors)

① Addition
② Multiplication
③ Contraction
④ Covariant Differentiation of Tensor.

These are different processes which you can do on tensors to make new tensors.

Adding Tensors (only add Tensors when their indices match & they are of same kind)

ex

$$T^{m\ldots\ldots}_{\ldots\ldots p} + S^{m\ldots\ldots}_{\ldots\ldots p} = (T+S)^{m\ldots\ldots}_{\ldots\ldots p}$$

Adding tensors define new tensor ... let call it $(T+S)^{m\ldots}_{\ldots p}$

you may call it W if you want i.e; $W^{m\ldots}_{\ldots p}$

but you know ... what are you doing... just adding term by term.

if $T^{m\ldots}_{\ldots p}$ & $S^{m\ldots}_{\ldots p}$ transform the same way (because indices match) ....so; you can factor out those $\frac{\partial x}{\partial y}$ ⇒ and you see that sum also transforms in the same way

hence; the sum is also a Tensor of the same kind ;i.e; of the same kind of indices.

Multiplication of Tensors.

(unlike addition; Multiplication of Tensors can be done with Tensors of any rank (no. of indices) ... ~~and independent of~~ ... whether the indices are upstairs or downstairs)

example: Multiplying vectors.

$V^m \otimes V_n$ → it is a Tensor with one upstair index m & one downstair index n.

$$V^m W_n = T^m{}_n$$

example $V^m \otimes W^n = T^{mn}$

★ a scalar is a Tensor; you can always multiply a Tensor by scalar.

"The only way you can make Tensor of same rank by multiplying; is for one of the factor to be scalar."

$$S V^m = T^m$$ } multiplying any Tensor by scalar; gives back Tensor of same kind.

★ (generally we get higher rank tensors with more indices when we multiply Tensors)

Note:

$V^m W^n = W^n V^m := T^{mn}$
$\neq V^n W^m$

→ when you write it as Tensor you must remember your convention that
. m corresponds to V
& n " " W.

★ How do we prove that $V^m W^n$ is tensor
↳ we write down ~~the~~ transformation properties of components & prove that the resulting thing is a tensor of appropriate number of Indices.

Contraction. In order to prove that contraction leads to tensor, we need a theorem or say lema.

Lemma ~~(Argument of Theorem)~~

$$\underbrace{\frac{\partial x^b}{\partial y^m} \cdot \frac{\partial y^m}{\partial x^a}} = \frac{\partial x^b}{\partial x^a} = \delta^b{}_a$$

this implicitly means sum over m.

$$\boxed{\frac{\partial F}{\partial y^m} \cdot \frac{\partial y^m}{\partial x^a} = \frac{\partial F}{\partial x^a}}$$

now; if F = x^b so; $\frac{\partial x^b}{\partial y^m} \cdot \frac{\partial y^m}{\partial x^a} = \frac{\partial x^b}{\partial x^a} = \delta^b{}_a$

* example of Index contraction

let; $V^m W_n := T^m{}_n$

contraction means $V^m W_m$ (identify upper & lower indices; and sum over them)
lets see how it transforms

so;

$$(V^m W_n)' = \frac{\partial y^m}{\partial x^a} \cdot \frac{\partial x^b}{\partial y^n} \cdot (V^a W_b)$$

now; put : m = n

$$\text{so; } (V^m W_m)' = \frac{\partial y^m}{\partial x^a} \cdot \frac{\partial x^b}{\partial y^m} \cdot (V^a W_b)$$

$$= \delta^b{}_a \cdot V^a W_b$$

$\Rightarrow$ ~~$(V^m W_m)' = V^a W_a$~~

$$\Rightarrow (V^m W_m)' = V^a W_a$$
$$= V^m W_m$$

this says that $V^m W_m$ has same value in every frame ... its scalar.

★ actually:

$$T^{nmr\ldots}{}_{prs\ldots} = T^{nmr}{}_{prs}$$

↳ take any Tensor with bunch of indices ...

- take one index from upper indices & one from lower indices
  ↳ set them equal to each other

This is actually a 4 index object

• by contracting an upper with lower index of a Tensor of rank 6 ; we have lowered rank of Tensor by 2.

let: $T^{mn} := V^m W^n$

so; here; $(V^m W^n)' = \frac{\partial y^m}{\partial x^a} \cdot \frac{\partial y^n}{\partial x^b} \cdot (V^a W^b)$

now; if you put $m = n \Rightarrow (V^m W^m)' = \underbrace{\frac{\partial y^m}{\partial x^a} \cdot \frac{\partial y^m}{\partial x^b}}\cdot (V^a W^b)$

→ here; even if we sum over m; we will not get any sort of invariant quantity...
⇒ infact that is not transforming the way a Tensor should transform if we sum over m here (which is not a legitimate thing to do here)

Metric Tensor

definition: if we have a differential ~~...~~ length element dx^m.

dx^m are just expansion coefficients when you expand in terms of basis vectors $\{e_i\}$

dx^m is contravariant component of little displacement vector.

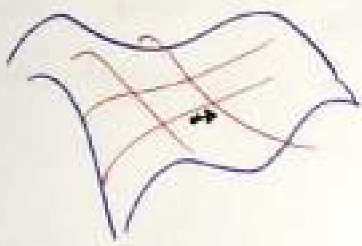
- What is the length of this vector.

(we haven't talked enough to tell what length of vector is.
→ This could be some arbitrarily shaped complicated space
& actually specifying what the space is, is in effect with specifying what all the lengths of all little elements are.)

∴ lets take square length. ds^2 ; it is quadratic in contravariant components.

$$ds^2 = g_{mn}(x)\, dx^m dx^n$$

⟹ This is basically the most general thing which can be written which is quadratic.



☆ we will stick with 4th dimensional space.

- How many independent components does g_{mn} has.

☆ $4 \times 4 = 16$ ... but ; $dx^1 dx^2 = dx^2 dx^1$

so; it is no point in having seperate g_{12} & g_{21}

↳ so; actually ; g_{mn} is symmetry.
(or say can be made symmetric)

so: $\begin{bmatrix} \circ & \circ & \circ & \circ \\ & \circ & \circ & \circ \\ & & \circ & \circ \\ & & & \circ \end{bmatrix}$; so; g_{mn} has 10 independent components.

if say: g_{12} & g_{21}

then: redefine; ~~$g_{12} = g_{21} = \frac{1}{2}(g_{12}+g_{21})$~~

$$g_{12} = g_{21} = \frac{1}{2}\left((g_{12})_{old} + (g_{21})_{old}\right)$$

because ; ~~the actual~~ the actual important thing is "$g_{12} + g_{21}$".

... lets now prove that g_{mn} is tensor.

* The basic guiding principle is that length of vector is scalar.

so; $ds^2 = g_{mn}(x) \cdot dx^m dx^n$; $dx^m = \frac{\partial x^m}{\partial y^p} \cdot dy^p$

~~also; $ds^2 = g_{pq}(y) dy^p dy^q$~~ ; also: $ds^2 = g'_{pq}(y) dy^p dy^q$

so; $ds^2 = g_{mn}(x) \cdot \left(\frac{\partial x^m}{\partial y^p} \cdot dy^p \right) \left(\frac{\partial x^n}{\partial y^q} dy^q \right)$

$$= g_{mn}(x) \frac{\partial x^m}{\partial y^p} \cdot \frac{\partial x^n}{\partial y^q} \cdot dy^p dy^q$$

but: $ds^2 = g'_{pq}(y) dy^p dy^q$

so: $$\underbrace{g'_{pq} = \frac{\partial x^m}{\partial y^p} \cdot \frac{\partial x^n}{\partial y^q} \cdot g_{mn}}$$

↳ but; this is just exactly the transformation property of a Tensor with two covariant indices.

so; we found g_{mn} is a Covariant Tensor. with two indices.

↳ called Metric Tensor

---

g_{mn} can be thought of as matrix ... symmetric matrix.

↳ when thought of as a matrix ... it has eigen values.

↳ Eigenvalues of g are never zero ; because: zero eigenvalue will correspond to a little vector of zero length.

* Symmetric matrices are invertible.

~~g~~ ★ Metric Tensor g when thought of as matrix is

- Hermitian
- Invertible (they have inverse)

none of the eigen values of g is 0 ... this is what allows us to invert the matrix

★ Metric Tensor has an inverse.

The inverse matrix to the Metric Tensor is also called g ... but you put indices upstairs; ie: g^{mn}

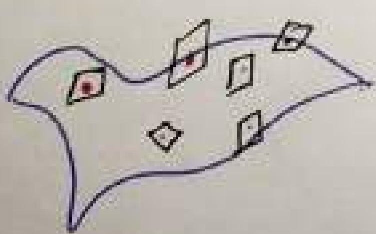
↳ it is a tensor & its defining property is that as a matrix its the inverse of g_{mn}.

★ multiplying matrix $\sum_n A_{mn} B_{nq} = (AB)_{mq}$

$$g_{mn}\, g^{np} = \underbrace{\delta^p_m}_{\text{when thought of as matrix it is Identity matrix}}$$

↳ This equation identifies g^{np} as inverse metric to g_{np}

* g^{mn} } ⇒ metric tensor with contravariant indices.
* g_{mn} } ⇒ " " " covariant " .



every point on surface has tangent space.

- Tangent space has exactly the same dimensionality as space itself.

∴ Tensors live in tangent space (mathematical way to describe the space they live in)

General Relativity : Flatness & Curvature - Shoaib Akhtar

Riemannian Geometry

we want to find a quantity R
(R should be a tensor ;... because if it is Tensor & is non zero ⇒ it is non-zero in every frame of reference)

- One dimensional space is either a line or closed curve. ⟶ if it is closed curve it is just characterised by its length.
  - ~~[illegible]~~
  - ↳ every closed curve of a given length "l" is identical to every other one in its intrinsic ~~property~~ properties.

☆ we want to find a thing called Riemann (R) such that ; if $R = 0 \Rightarrow$ Space is flat and there exist coordinates such that where $g_{mn} = \delta_{mn}$.

R is also called Curvature Tensor.

we know: $ds^2 = g_{mn}(x)\, dx^m dx^n$
and ds^2 is an invariant quantity.

NOTE (s standing for spacial distance)
(Remember now we are not doing relativity ... dont think like $-t^2 + x^2 + y^2 + z^2$
↳ its just Reimannian geometry with distances along ~~to~~ surfaces ... & all distances are positive)

g_{mn} is matrix & it has an inverse as a matrix g^{mn}

such that: $g^{mn} g_{nr} = \delta^m{}_r$

we can prove; if we take Kronecker Delta $\delta^m{}_n$ as a Tensor ; and we transform it pretending it a tensor ⇒ we just get back Kronecker Delta again.

- In Reimann geometry (as opposite to Einsteinian geometry) all lengths are positive ... more importantly square of all lengths are positive ⟶ and this says something about Metric Tensor

↳ It says that there are no direction along which if you evaluate this, it is not positive. : equivalent to the statement that all eigenvalues of Metric Tensor g in Reimannian Geometry is positive.
(it does say its positive definite)

#

$$\boxed{V^m \cdot g_{mn} = V_n}$$ → Metric Tensor g allows us to raise and lower indices.

(You can prove this)

This can also be thought of as defination of Covariant component in terms of contravariant ~~and~~ and Metric Tensor.

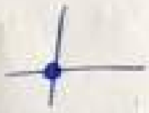
also $$\boxed{V^m = g^{mn} V_n}$$

we know:
$g_{mn} = e_m \cdot e_n$

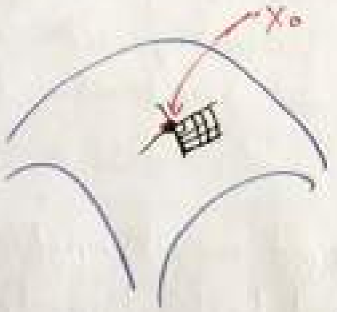
$$\begin{aligned} V_n &= V \cdot e_n \\ &= (V^m e_m) \cdot e_n \\ &= V^m (e_m \cdot e_n) \\ &= V^m \cdot g_{mn} \end{aligned}$$

To what extent given any space; can you find coordinates such that g_{mn} might be for example Kronekar Delta over some ~~f~~ limited region.

Theorem: There are coordinates which are good in immediate vicinity of point.
(They are not global coordinates... they dont cover the whole space)



coordinates called Gaussian Normal Coordinates.



The theorem ~~said~~ says:
- at every point you can choose coordinates (Gaussian Normal coordinate) such that at that point $g_{mn} = \delta_{mn}$ at $x = x_0$

(if you find ~~such~~ such coordinate... you can rotate it
→ so obviously there is more than one way to do it)

* you can choose your coordinate to a degree where

$$\frac{\partial g_{mn}}{\partial x^r} = 0 \quad ; r \in \{1, 2, \dots, D\}$$

at $x = x_0$; D is dimensionality of space.

... ie; derivative of metric w.r.t. any direction is zero.

→ if you try to extend that same coordinate away from that point, then in general (unless the space is flat) you will found out that ~~higher~~ derivates of g_{mn} are not zero.

... and also in general ... at $x = x_0$

$$\frac{\partial^2 g_{mn}}{\partial x^r \partial x^s} \neq 0 \quad \text{at } x = x_0$$

(ie; all second derivative can't be chosen equal to zero)

(cannot be all chosen equal to zero in general ... except the space is flat.)

↳ but never the less;

$$\frac{\partial g_{mn}}{\partial x^r} = 0 \quad \text{at } x = x_0$$

can be chosen.

$r \in \{1, 2, \ldots, D\}$

↳ so; at a point; there is no content really in saying that metric can chosen to be flat like, (ie: we mean $\delta_{mn} = \delta_{mn}$) even upto first derivatives $\frac{\partial g_{mn}}{\partial x^r} = 0$

(This can always be done)

* It is in the second derivatives that is somehow related ~~to~~ to whether ~~space~~ space is flat or curved.

Proof ; let our point of interest be origin: $x_0 = 0$ (without loss of generality)

now; suppose we have some general metric, and some coordinate y in which metric has some form.

assume $x = 0 \Leftrightarrow y = 0$ (ie; origin of both cooridinate is same)

lets look for some x (ie; new coordinate which will solve our purpose) ie; $x^m = y^m$ (will start out to be equal $0 = 0$

(but will not hold at far region) ... since we assumed origin for them to be same.

→ now; lets expand x as functions of y.

ie; $x^m = y^m + C^m_{nr}\, y^n y^r$ + ~~...~~ cubic term + ...

$\downarrow$ the linear term is just this since we made or assumed origin to be same for ~~simple~~ simplicity.

$\underbrace{C^m_{nr}\, y^n y^r}$ this is the most general quadratic term we can write off

→ There is no zeroth order term (or say constant) ∵ because origin of x & y frame ~~coincides~~ coincides.

let; choose $m = 4$; ~~there are~~ 10

- there are 10 components of metric
- how many combinations $y^n y^r$ are their ... ans 10 for a given m.
  ↳ so; C^m_{nr} has effectively 10 components. (can be made symmetrical matrix)

↳ for each n, r there is $m \in \{1, 2, 3, 4\}$ there is 40 independent
↳ so; this means; there is 40 independent coefficient here.

✱ The most general x you can build to quadratic order has ~~4~~ 40 independent coefficients.

✱ now; there are 10 independent components of g.
so; how many equations does $\frac{\partial g_{mn}}{\partial x^r} = 0$ at $x = 0$ corresponds to ... ans 10 equations for each r

and ~~since~~ since $r \in \{1,2,3,4\}$

↳ since "r" runs from 1 to 4

↳ so; $\frac{\partial g_{mn}}{\partial x^r} = 0$ gives 40 equations in total.
$\forall r \in \{1,2,3,4\}$

⇒ ~~So we have forty~~

⇒ ~~So we have 40 equations: $\frac{\partial g_{mn}}{\partial x^r} = 0$ and 40~~

⇒ So we have 40 equations "$\frac{\partial g_{mn}}{\partial x^r} = 0$"

and we have 40 unknowns "C^m_{nr}"

↳ and this allows us to be sure that we can solve these equations to Quadratic order.

→ but you fail at next order: $\frac{\partial g_{mn}}{\partial x^r \partial x^s} = 0$ or cubic order.

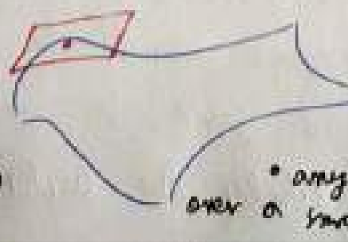
→ choosing a metric actually means choosing coordinate system (in this case choosing Riemann Normal coordinates) which in turn adjust the metric in flat coordinate system

☆☆ "So it is always possible chose a metric that has the property, that to quadratic order (in other words first derivative of metric is zero ($\frac{\partial g_{mn}}{\partial x^r} = 0$) & metric is equal to flat metric ($g_{mn} = \delta_{mn}$) at a point) i.e; Locally Flat."

any curve you have ... you have tangent at a point

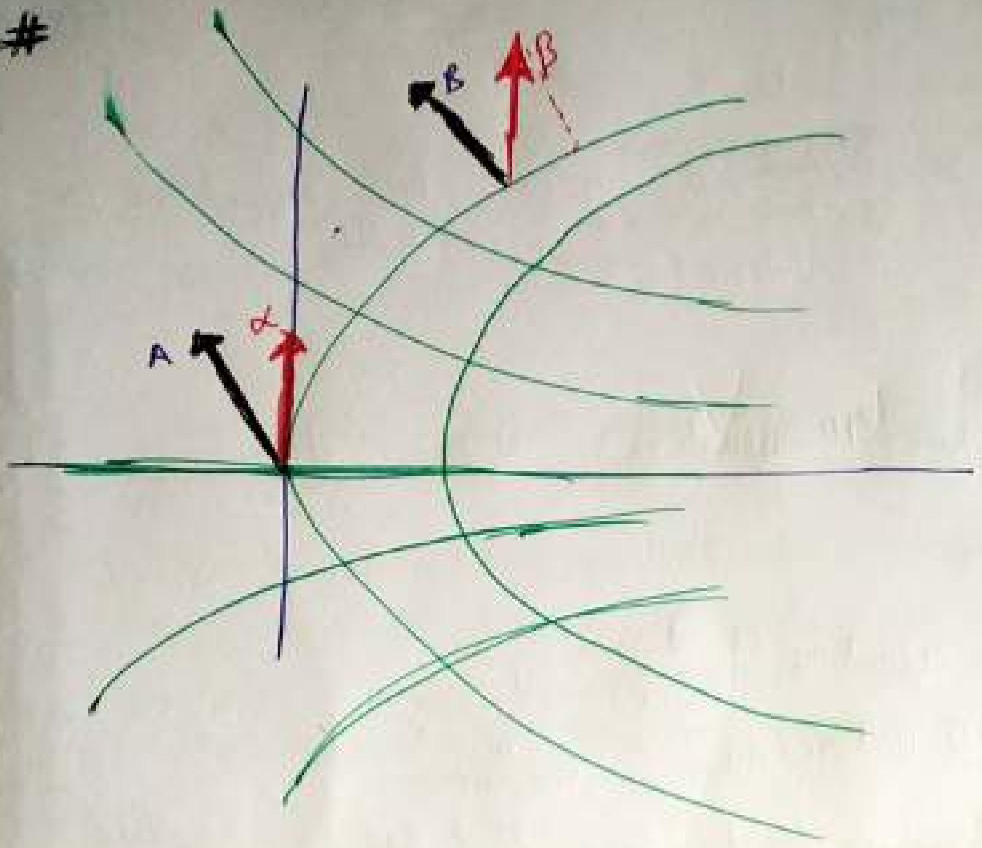
• small portion of curve can be visualized as a line segment (small line segment)

=



any surface you have
↳ you can have a tangent surface to it at a given point
• any smooth surface can be approximated over a small distance by a plane tangent to it.

#



What it means for vector field to be constant in space?
• (because the space is curved; it becomes hard to compare vector at one point & other point)

→ so; if coordinates can't be chosen to be everywhere flat; then what exactly do we mean by saying a vector over here is equal to vector at some other place.

↳ It does not really mean anything.
To compare vector at two different places; (unless you have nice flat coordinates) there is no unique way to do that.

In the above figure;
In blue coordinates; the components of A & B are same "+" (in our nice cartesian coordinate)

* But what about coordinates of A & B in green axis.
↳ easy example; α & β vectors (they are same in usual cartesian coordinates) only one non zero coordinate in green axis.
: α has only one non zero in green axis.
: β " both component non zero in green axis.

So in general you can't judge whether the vector is constant in whole space)

another way of saying:

that in some ~~coordinate~~ coordinate, ~~xr~~,

the derivative of m^{th} component of vector might be zero in one coordinate system i.e.; $\frac{\partial V_m}{\partial x^r} = 0$ (partial derivative w.r.t. a particular coordinate) w.r.t. x^r

; and not zero in another coordinate system.

→ there might be a case that all derivatives are zero in one coordinate system & not zero in other coordinate system (because coordinates are changing ...)
(not because vector may change from point to point; but because orientation of coordinate may change from point to point)

so; $\frac{\partial V_m}{\partial x^r} = 0$ is not an equation which is likely to be same in every ~~refere~~ reference frame

note (vector may also change in general)

→ so; it cant be a tensor equation.

⇒ so; derivatives of components of vector w.r.t. coordinates are not themselves components of Tensor.

☆ so; we need some better defination of vector (cant just directly differentiate components)
→ such that it satisfies properties of Tensor
(say zero in one frame then zero in every frame)

we make better defination; "Derivatives in Gaussian Normal Coordinates define the derivatives"

* When you differentiate a vector in general coordinate
→ you get two terms

- one term comes; because the vector may be changing.
- other " " ; " coordinates may be " .

* if you took ordinary derivative in Gaussian Normal Coordinate; & then re-express in general coordinate

↳ you will find that

$$\boxed{\underbrace{D_r V_m}_{} = \underbrace{\partial_r V_m}_{} - \underbrace{\Gamma^{\,t}_{r\,m} V_t}_{}}$$

derivative along the r axis of the components of vector V_m in arbitrary coordinates.

This is the usual term.
↳ arises because of the fact that V may be changing.

This term has to be proportional to V in some sense
↳ it is arising because coordinates may themselves be changing.

$$\partial_r V_m = \frac{\partial V_m}{\partial x^r}$$

→ It is called Covariant Derivative and it is a Tensor.

Γ^{t}_{rm} are either called Connection coefficients or Christoffel symbols.

☆ It's a theorem ; $\Gamma^{t}_{rm} = \Gamma^{t}_{mr}$

* In generalized Reimannian geometry ... geometry with torsion ... → In geometry with Torsion, the symmetry ($\Gamma^{t}_{rm} = \Gamma^{t}_{mr}$) is not true)

↳ But these geometry are not widely used in ordinary gravitational theory.

---

* Γ^{t}_{rm} is not tensor ; in Gaussian Normal coordinates they are zero
↳ but in general not.

Suppose we have a tensor T_{mn}

we want to calculate derivative of T_{mn} along s axis.

Γ^{t}_{rm} are not tensor

↳ In Gaussian Normal coordinate; they are all zero

↳ & otherwise with some curved coordinate; they are not zero

let; see $D_s T_{mn}$ ... there is a Christoffel ~~term~~ term for every index.

$$\boxed{D_s T_{mn} = \frac{\partial T_{mn}}{\partial x^s} - \Gamma^{t}_{sm} T_{tn} - \Gamma^{t}_{sn} T_{mt}}$$

☆ because metric tensor has zero derivative in Gaussian Normal coordinate.

↳ it mean that covariant derivative of metric tensor is zero.
(because Covariant derivatives are Tensor)

so; $D_s g_{mn} = \frac{\partial g_{mn}}{\partial x^s} - \Gamma^{t}_{sm} g_{tn} - \Gamma^{t}_{sn} g_{mt} = 0$

equation 1

(note; $D_s g_{mn}$ is zero; although $\frac{\partial g_{mn}}{\partial x^s} \neq 0$; because $\frac{\partial g_{mn}}{\partial x^s}$ is ordinary derivative in a given coordinate system ; not in Gaussian Normal Coordinates)

because ordinary derivative is zero in Gaussian Normal coordinates.

now; permute; s and m

so: $D_m g_{sn} = \frac{\partial g_{sn}}{\partial x^m} - \Gamma^{t}_{ms} g_{tn} - \Gamma^{t}_{mn} g_{st} = 0$

equation 2

also;

$D_n g_{sm} = \frac{\partial g_{sm}}{\partial x^n} - \Gamma^{t}_{ns} g_{tm} - \Gamma^{t}_{nm} g_{st} = 0$

equation 3

eq 3 + eq 2 − eq 1

$$\Rightarrow \partial_n g_{sm} + \partial_m g_{sn} - \partial_s g_{mn} = 2\,\Gamma_{mn}^{\ \ t}\, g_{st}$$

Note: ordinary derivative of metric tensor in G.N.C. at a point is zero at that point
(since G.N.C. is actually chosen in that way)
⇒ but ordinary derivative of metric tensor may not be zero in general coordinate at that point

how we get rid of g_{st}

Ans g_{st} has an inverse.

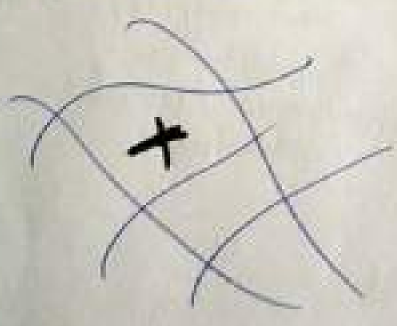
$$\Rightarrow \boxed{\frac{1}{2} g^{st}\left[\partial_n g_{sm} + \partial_m g_{sn} - \partial_s g_{mn}\right] = \Gamma_{mn}^{\ \ t}}$$

↳ This was true at a given point
↳ now that point is a general point
↳ so; it is true at every point.

i.e; The equation holds for a given $x = x_0$
↳ but x_0 belongs to a whole space and is any general point in whole space ∴ i.e; replace x_0 by x.
↳ so equation is actually true in whole space

★ Christoffel symbol can be non zero even if the space is flat.
(it ~~depen~~ also depends on what coordinates we are using)
example: polar coordinates.

★ How we are actually differentiating.



We want to know how to differentiate at a given point

(i) We go to Gaussian normal coordinates at the ~~the~~ given point
↓
(ii) I do my differentiating
↓
(iii) Then I relate it back to original coordinates, the general coordinates which had covered the whole space.

I want to know how to compare vectors at neighbouring points.

- so I go to ☆☆ Gaussian Normal Coordinates at that point, (and that tells me how to compare at that point)

☆☆ → The Gaussian Normal coordinat was just a trick to eliminate locally at each point the variation of metric tensor with position.

||note|| If Γ^{t}_{mn} is zero everywhere: it not only means that space is flat; it also means that we had chosen them cartesian coordinates.

☆ Covariant derivative of metric tensor is always zero.

→ this is not same as saying the metric is constant over space.

→ The statement that the metric tensor is constant would mean that ordinary derivative is zero; i.e; $\frac{\partial g_{mn}}{\partial x^{r}} = 0$.

"Covariant derivative of metric tensor is always zero; because in Gaussian Coordinates it is zero; and if Tensor is zero in one coordinate frame then it is zero in every coordinate frame"

- Covariant derivatives are Tensors.

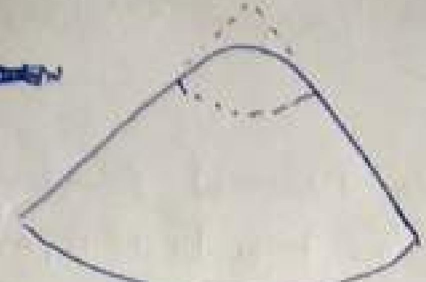
you can write your equation in Gaussian normal coordinates; (it may just involve ordinary derivatives)

→ but if you want same equations in general coordinates, then you replace ordinary derivatives by Covariant derivatives.

Curvature

example

a sort of rounded cone.



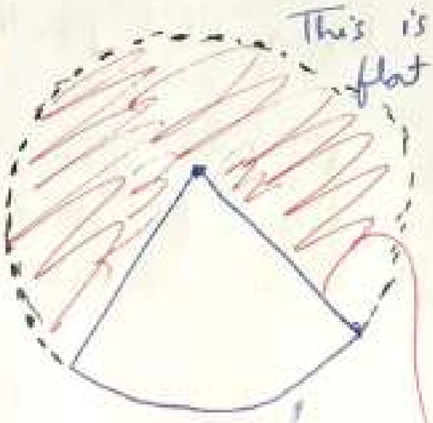
cut the cone & open it up.

• Such a cone is flat everywhere except near the top.

you cant flat down the cone on table without tearing it

↳ but if you cut it anywhere and open it up... you can lay down it flat.

This is clearly a flat space.



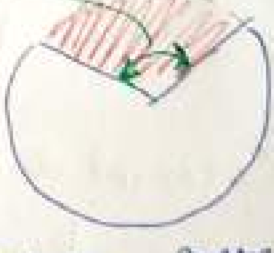
This is the region cutted out to make the cone.

★ The bigger the angle that you cut out

↳ the bigger is the deficite angle or Kroneker definite

⇒ The bigger the definite angle → The pointier the cone.

Deficite angle



actually in many cases that cutted region is small

• if Kroneker Deficite is small acute angle ..... it means cone is very flat on top.

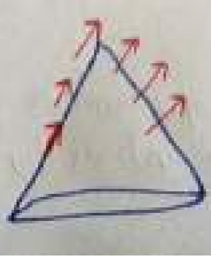
↳ large cut area ... more pointy cone

cone

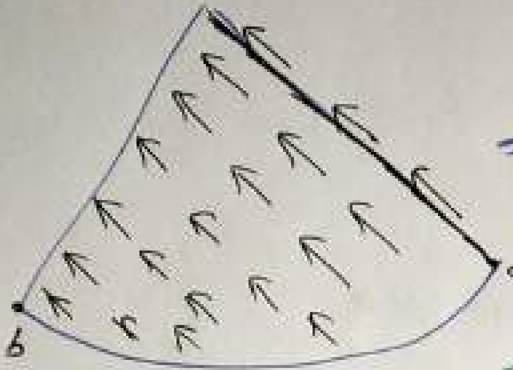
↳ small cut ...

cone

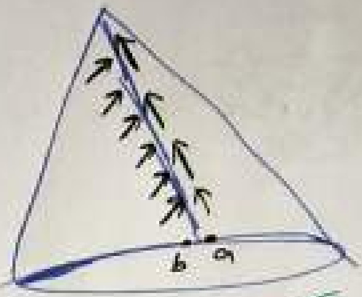
★ lets take a vector field on cone which is constant.



so: if you carry vector around ... when you get around other side ⇒ It has been rotated due to Kroneker deficite at the top.



fold to make cone



A region of curvation has the property:
that if you take a vector pointing in same direction as much as you can do ... keep it pointing in same direction & take it around the region of curvature the vector will rotate; even though you make sure that at every point you are moving it parallel to itself.

#

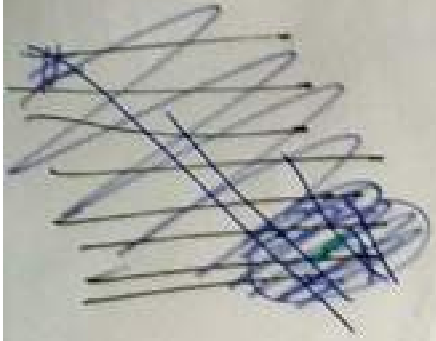
In other words; you tried to make sure that covariant derivative of vector was zero
↳ but nevertheless when you come around to other side, it is shifted.

We discover that there is funny jump in direction when we make cone (even though when you open it up, you see that they are all in same direction)

↳ exactly the same will be true in this geometry

↳ This is because the top of cone is curved.

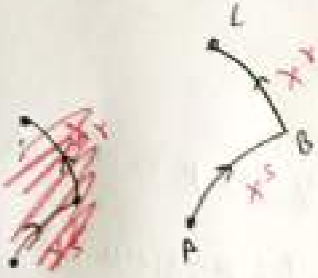
↳ Thats the effect of curvature.



Suppose you have some curvature in space.

- Take a vector field
  ↳ and then displace it by ~~infinitismal~~ infinetisimal distance along one axis

and then displace it by ~~some~~ infinetisimal distance along second axis.



⟹ The vector will change.

How the vector will change?

- It will change typically by differentiating the vector along these two axis in sequence.

⇒ first differentiate the vector along one axis;

then ; ~~... it along~~ differentiate it along second axis

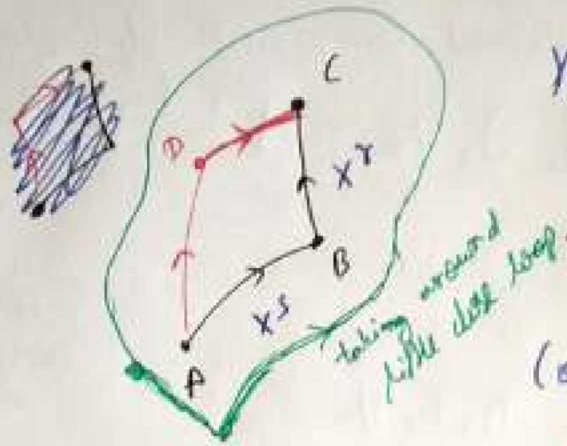
... and there is small change in vectors ~~be~~ because of these derivatives

So; if we take difference of vector ~~at~~ at "C" & "A"
↳ it consists of two changes
- change to go from A → B
- " " " " B → C

⟷ and total change in that vector is proportional to second derivative.

$D_r D_s V_m$

you could have first gone in r direction; then in s direction.



That will be opposite order of differentiation.

$D_s D_r V_m$

(ordinarily; in flat space $D_r D_s V_m = D_s D_r V_m$)

it is just the fact that ... derivatives.... it does not matter which order you take them.

but this is not true in curved space.

in curved space $(D_r D_s V_m - D_s D_r V_m)$ .. this operation can be thought of as taking the vector around the close loop.

taking it around little close loop we discovered that when there is curvature; you dont get back to where you started with.

* In curved space ⇒ Covariant Derivatives are not interchangeable.
 In flat " ⇒ " " " interchangeable

that the way we test whether space is flat or not.

* If $D_r D_s V_m - D_s D_r V_m = 0$ everywhere in space for any vector ⇒ Then space is flat.

* If we discover that there are places in space where order of differentiation gives different answers
↳ then we know space has some curvature there. analogous to the top of mountain.

---

let V_m be ~~cov~~ covariant components of vector.

$$\underbrace{D_s D_r V_m}_{T_{rm}} = D_s\left[\partial_r V_m - \Gamma_{rm}^{\ t} V_t\right]$$

this is a Tensor ... calculate

and take ~~[illegible]~~ covariant derivatives

▷ then compute ; $D_s D_r V_m - D_r D_s V_m$

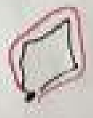
$$D_s D_r V_m - D_r D_s V_m = R_{sr\ m}^{\ \ t} V_t$$

where ;
$$R_{sr\ m}^{\ \ t} = \partial_r \Gamma_{sm}^{t} - \partial_s \Gamma_{rm}^{t} + \Gamma_{sm}^{p}\Gamma_{pr}^{t} - \Gamma_{rm}^{p}\Gamma_{ps}^{t}$$

i.e.;
$$\boxed{R_{sr\ m}^{\ \ t} = \underbrace{\partial_r \Gamma_{sm}^{t} - \partial_s \Gamma_{rm}^{t}}_{\text{The terms involving derivatives of Christoffel symbol}} + \underbrace{\Gamma_{sm}^{p}\Gamma_{pr}^{t} - \Gamma_{rm}^{p}\Gamma_{ps}^{t}}_{\text{The terms involving products of Christoffel symbol}}}$$

→ Reimann Curvature Tensor.

↳ Conceptually ; what is doing is just calculating the difference in a vector if you transport around here keeping parallel to itself unit it comes around.



↳ A little change in vector in going in a loop.

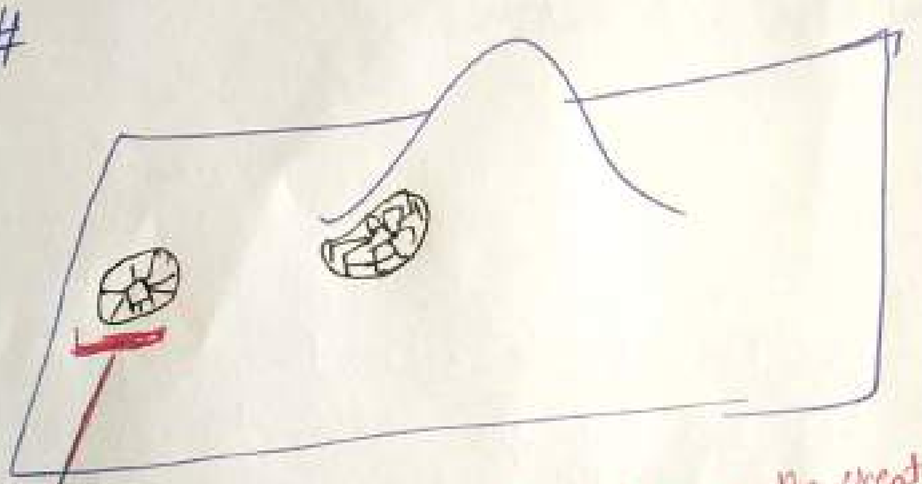
* If all the components of Curvature Tensor is Zero everywhere : The space is flat.

↳ ~~R tells you~~ $R_{sr}{}^{t}{}_{n}$ tells you whether the space is flat ; and if not how unflat it is.

↳ it is closely related to Tidal forces in gravitational physics

↳ Infact they really are Tidal forces (Tidal forces are represented by the Curvature Tensor)

#



flat far away and ~~smooth bud~~ somewhere in centre it has bulge.

Small creature moving in the space

- as long as the creature moves in flat space ... it is perfectly happy ... does not ~~get~~ stretched ... does not ~~get~~ ... does not ~~get to~~ get distorted ... not deform ... not squeeze.

- when you try to move it in curved ~~re~~ region : it cant follow the curvature without having to stretch its lengths. ... (it has to follow metric properties of curved space)

  ↳ it get stretched.

⟶ and the measure .. how much it get stretched locally is exactly the Curvature Tensor.

- Curvature Tensor is telling how the system gets deformed ~~as~~ as it moves up in the region of curvation.

  ↳ That's why it is connected with Tidal forces.

Tidal forces has much less effect on small object than large object.

~~... if small scale~~

... if system is very small ... then on the scale things vary will be very small

↳ & will not expricence great ~~differentiation~~ deformation.

☆ if large object ⇒ large deformation.

~~Summary of~~ # Summary of a Theorem (we had already seen in previous pages)

We can always chose a coordinate system; which in turn chooses or fixes the metric tensor (since metric tensor depends on coordinate system); such that the metric tensor is Same upto quadratic order to flat metric (δ_{mn}) at a given point x_0

i.e. $g_{mn} = \delta_{mn}$ at $x = x_0$

and $\dfrac{\partial g_{mn}}{\partial x^r} = 0$ at $x = x_0$

That chosed coordinate system (locally chosen at each point) is called Gaussian Normal Coordinate at a given point.

☆ Metic Tensor has zero derivative in Gaussian Normal Coordinate (G.N.C.)

- and since doing Covariant derivative; involves first doing ordinary derivative in G.N.C. and then expressing it in the given general coordinate.

⇒ but; in G.N.C.; the metric tensor can be chosen to be locally flat ⇒ so; ordinary ~~derivati~~ derivatives in G.N.C. are zero ⇒ and then if you express it in any coordinate frame; it comes out to be zero (property of Tensor... & remember Covariant Derivative is Tensor)

⇒ so; Covariant derivative of Metric Tensor is zero.

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Covariant Derivate (the point of covariant derivate is to write a Tensor such that in Gaussian Normal Coordinates at a given point it is just the ordinary derivative at that point.)

abbreviation
G.N.C. ⇒ Gaussian Normal Coordinate
or Most Cartesian Coordinates at a point. (MCC)

$D_m g_{rs} = 0$ (because in B.C the derivative of metric is zero)

$D_r V_s = \partial_r V_s - \Gamma_{r\ s}^{\ p} V_p$

GNC at a point is actually equivalent to MCC there → we mean same thing.

($\Gamma_{r\ s}^{\ p}$ would be zero in B.C. at a point.)

$$\Gamma_{r\ s}^{\ p} = \frac{g^{pm}}{2}\left[\partial_s g_{rm} + \partial_r g_{ms} - \partial_m g_{rs}\right]$$

or we may simply call it Best Coordinates at a point. (B.C.)

Covariant derivatives of Vector with Contravariant Components:

$$\boxed{D_m V^n = \partial_m V^n + \Gamma_{m\ r}^{\ n} V^r}$$

prove; $V^n = g^{np} \cdot V_p$

$\Rightarrow D_m V^n = (D_m g^{np}) V_p + g^{np} (D_m V_p)$

~~remember: Metric in B.C. has no deri~~

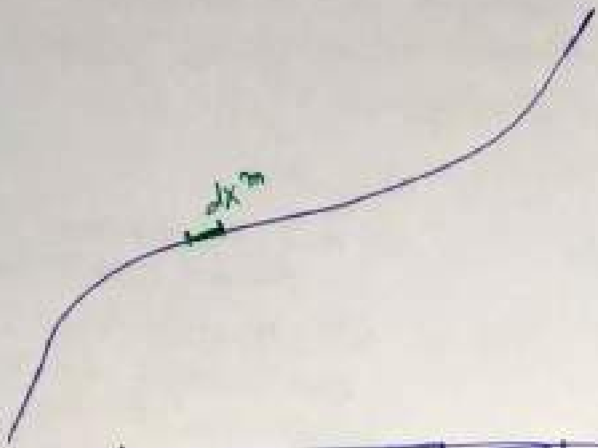
remember; Metric in B.C. has zero derivatives.
↳ so; B.C.; metric itself with lower indices is constant at least to first order (or say first derivatives is 0)

↳ so; this means that inverse matrix is also constant to first order ⇒ so one can also argue that

$D_m g^{np} = 0$

so; $D_m V^n = g^{np} (D_m V_p)$

Parallel Transport.



let have a curve on a given curved surface.

~~let we have curved surface~~

& we have vector at each point ~~of curve~~ along ~~the~~ curve and we want to know whether the field stays parallel to it self ; parallel to itself means that the covariant ~~de~~ derivative is zero along that direction.

$$D_m V^n = \partial_m V^n + \Gamma_{mr}^{n} V^r$$

* now consider derivative along the curve or trajectory... ... how the vector changes from point to point.

↳ This corresponds to taking covariant derivative & multiplying it between these points  by dx^m

$$\underbrace{D_m V^n dx^m}$$

} ⇒ This can just be called small change in the vector ~~...~~ going from point to point ~~keeping to~~ ... ~~...~~ except keeping track of the fact that the coordinates themselves may bend as you go from point to point. (and that why you have to ~~...~~ use Covariant Derivative)

This is change in vector V in going from one point to its neighbour ...let just call it ~~..~~ DV^n

$$DV^n = D_m V^n dx^m$$

} ⇒ more ~~...~~ precisely ; this is small change in the vector in Best coordinate at that point.

PJ 57

$$DV^n = \frac{\partial V^n}{\partial x^m} \cdot dx^m + \Gamma^n_{mr} V^r dx^m$$

This is covariant change in a vector in going from one point to another.

$\frac{\partial V^n}{\partial x^m} dx^m$ is just differential change in V^n ; i.e; dV^n

i.e; $dV^n = \frac{\partial V^n}{\partial x^m} dx^m$

$$\Rightarrow \quad DV^m = dV^n + \Gamma^n_{mr} V^r dx^m$$

covariant differential change in V

ordinary change in V

Christoffel symbol term times dx^m ; to keep account for the fact that the coordinates themselves be changing.

#

Defining vector parallel to itself when we move along the curve

→ means does not changes as you move along the curve.

(i) at each point ; you erect some Best coordinate.

⇓

(ii) In those coordinates you test whether the vector is changing.

(we are comparing vectors at neighbouring points... so we are doing that in the Best coordinate)

(in that B.C. ⇒ compare two vectors at point very close in the curve)

→ if not changing ; the vector is constant along little segment there.

⇓

(iii) Go to next segment... erect best coordinate at that point & do the same thing.

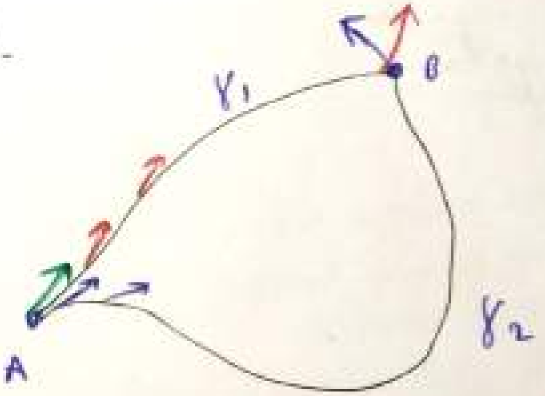
* If at every point the change in the vector is zero
  ↳ the vector is said to be parallel to itself along that curve. ~~match the vector at each point~~

↳ As you move along that curve; the vector maintains a relationship of being parallel to ~~itself~~ itself.

$\boxed{\text{note}}$ the Best coordinates are not unique.
  ↳ Just like an ordinary flat space the cartesian ~~coord~~ coordinates are not unique because you can always rotate them
  ⇒ you can also rotate the best coordinates at a point.
(This will not change whether the vector is parallel ~~to~~ to itself or not)

* Moving a vector from one point to another such that the covariant differential change (DV^n) is zero is called Parallel Transport of the vector along a certain curve.

• In general in curved spaces.
  ; if you take two points and • start a vector at same point (say A)
  ↳ and ~~parallelly~~ parallely transport the vector along different curves reach the same point B



↳ you will get different vectors at point B while transporting parallely ~~a along transpo~~ from point A to B through different curves (or say path) in ~~gen~~ general.
happens because of curvature in between.

★ So; the notion of parallel transport is relative to a particular curve.

note: $dV^m + \Gamma^m_{mr} V^r = 0$ parallel transport preserve the length of the vector.

~~★ In general; ★~~

---

* In general; the Tangent vector to a curve is not Covariantly constant

---

Geodesic is a curve

- shortest curve (in terms of length) between two points

or

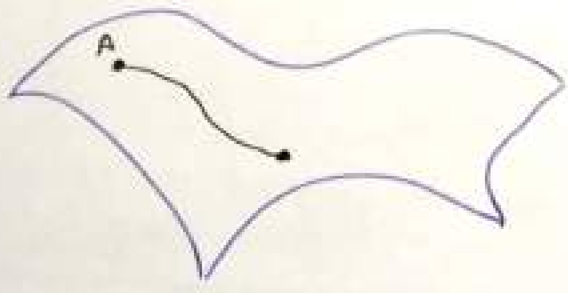
- A curve whose distance is stationary. (it could be longest or shortest .....)

~~★ Look along~~

★ Locally along the curve, and require that the curve is as straight as possible.

↳ if at each point along the curve; the derivative of tangent vector is zero ⇒ then that curve is as straight as possible

example



take a car at A (with size of wheels small enough as compared to curvature)

↳ take the steering wheel and point it absolutely ahead .... straight ...

... start driving ... dont turn the wheel.

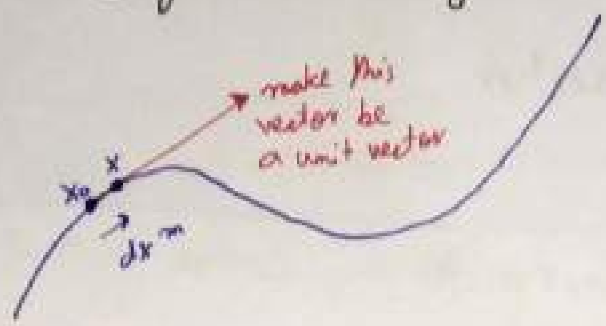
⟶ dont deviate;

just go straight ahead.

The curve that you will execute in the space will be geodesic.

Another way of saying the same thing,
is that tangent vector along the curve is constant.

☆ lets define a tangent vector.



given a curve

- go to a point x_0

⇓

- and take a neighbouring point seperated by a little interval called dx^m

⇓

- draw a vector through those two points.

⇓

- then take the limit in which x approaches x_0

⇓

Thats called the Tangent vector (the vector you get then)

$$\underbrace{t^m = \frac{dx^m}{ds}}$$

} → dividing by distance between two points.

... where; $ds^2 = g_{mn} \cdot dx^m dx^n$

↳ This defines unit vector (you can prove it is unit vector) along the curve.

... This is the notion of Tangent vector.

☆ what about the notion that the Tangent vector is constant.

ie; $Dt^m = 0$; ie; $\boxed{dt^n + \Gamma_m{}^n{}_r \cdot t^r \cdot dx^m = 0}$

ie;

go along the curve:

and if everywhere along the curve

$$dt^m + \Gamma_m{}^m{}_r \cdot t^r \cdot dx^m = 0 \Rightarrow \text{if it is:}$$

The curve is <u>Geodesic</u>

↳ dividing both side of equation ds

$$\Rightarrow \frac{dt^m}{ds} + \Gamma_m{}^m{}_r \cdot t^r \frac{dx^m}{ds} = 0 \Rightarrow \boxed{\frac{dt^m}{ds} + \Gamma_m{}^m{}_r \cdot t^r t^m = 0}$$

↳ Equation of motion of Geodesic.

we could also write it as

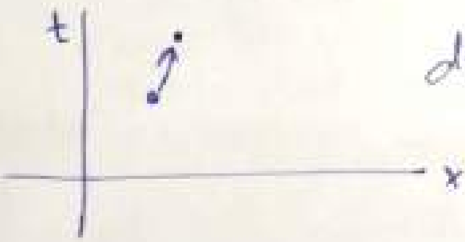
$$\frac{d^2x^m}{ds^2} + \Gamma_m{}^m{}_r \cdot t^r \cdot t^m = 0$$

$$\Rightarrow \boxed{\frac{d^2x^m}{ds^2} = -\Gamma_m{}^m{}_r \cdot t^r t^m}$$

* Particles in gravitation field, moves in straightest possible trajectory.
  ↳ it actually moves through straightest possible trajectory through space-time (not just through space)

Space Time

lets space-time distance between points as proper time.



$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$$

it is conventional to rewrite this as $-ds^2$

ie; $d\tau^2 = -ds^2$

according to convention; $\boxed{d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu}$

$g_{\mu\nu}$ is metric of space-time ; & $x^\mu = (t, x, y, z) \equiv (x^0, x^1, x^2, x^3)$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Metric Tensor
in Minkowski coordinates (which play the roles in special Relativity of Cartesian Coordinates)

~~In General~~

In General Relativity; Metric Tensor $g_{\mu\nu}$ becomes function of space-time;

ie; $g_{\mu\nu}(X)$; $X \equiv (x^0, x^1, x^2, x^3)$

so; $$\boxed{d\tau^2 = -g_{\mu\nu}(x)\, dx^\mu dx^\nu}$$

* Metric of space-time always has one negative eigenvalue and three positive eigenvalues.
(it means one dimension of time, and three dimension of space)

* If you want to know the space-time is flat mean?
flat here does not mean that there is a coordinate system in which the metric tensor is Kroneker Delta.
↳ it means; there is a coordinate system in which metric looks like "$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$" this.

↳ This is the notion of flat space-time.

~~* If metric cannot be brought to this form~~

~~###~~ ☆ If metric cannot be bought to this form

i.e; diag $(-1, +1, +1, +1)$

; then it is curved.

---

actually; ; $$\eta_{\mu\nu} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

---

example; if we were dealing with flat minkowski coordinates

we will write $d\tau^2 = -\eta_{\mu\nu}\, dx^\mu dx^\nu$

here ; special case : i.e; $g_{\mu\nu} = \eta_{\mu\nu}$

∴ however ; the general equation in any coordinate system is $d\tau^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$

\# We are going to deal with flat space

☆ we want to define the notion ~~of~~ of uniformly accelerated frame.

In special Relativity

→ There is a problem with notion of uniformly accelerated frame.

•——•——•——•——•——•

If you have bunch of points ; which are at a fixed separation.
↳ and you start accelerating them and you keep them at same fixed acceleration.

→ If they are going to accelerate with uniform acceleration; then they will maintain distance between them in the frame in ~~(all points are moving with same speed and acceleration~~ ~~so from frame of reference of any point ; all other points are at rest~~ ~~↳ and thus they maintain distance between them in the accelerating frame of the points)~~ which they are accelerating (because all points are moving in similar manner)

* but distances are supposed to transform when we go to frame of reference of points (that when we change frame of reference)

↳ and if we assume Lorentz Transformations to be true:
⇒ then we would discover in the rest frame of points the distance between them would get larger and larger.

$$l_{ground} = l_{point\ frame}\sqrt{1-(v/c)^2}$$

$$\Rightarrow \frac{l_{ground}}{\sqrt{1-(v/c)^2}} = l_{point\ frame} ;$$

~~now; l ground is constand~~

where; l_{ground} :⇒ distance between points in ground frame.

$l_{point\ frame}$:⇒ distance between points in the rest frame of points.

now; since; l_{ground} is constand

& v is uniformly increasing
(since we are assuming uniform acceleration)

↳ this implies: "$l_{point\ frame}$" is increasing as the velocity is increasing due to acceleration.

→ now;

if we imagine strings between the points

↳ so; as they start moving and so in an attempt to keep their distance same in ground frame

; they will stretch more and more in the rest frame of points ... and eventually break.

↳ This is obviously not which would be thought of as ordinarily accelerated frame

★ also; in Special Relativity

⇒ if you wait long enough; uniform acceleration will lead to moving greater than the speed of light. ~~you can't simply move points~~

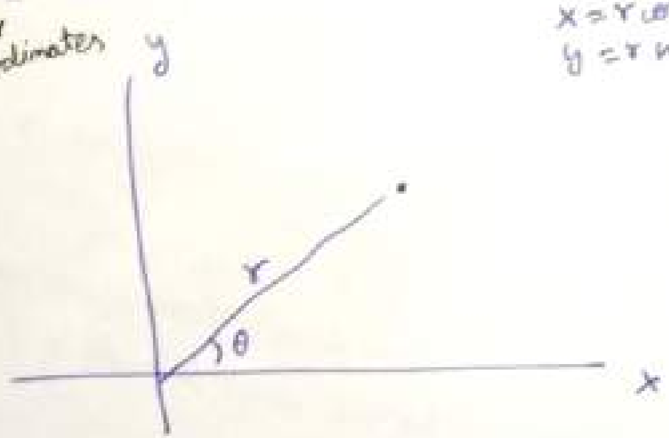
~~..... There has to be something else .....~~

There has to be something else;

so Uniform acceleration to the extent that it exist and make good physical sense is not as simple as moving these points all with same acceleration.

we will see later that uniformly accelerated coordinate system is ~~analogous to polar coordinates~~ analogue of polar coordinates.

Polar coordinates



$x = r\cos\theta$
$y = r\sin\theta$

$\Rightarrow \cos^2\theta + \sin^2\theta = 1$

$\Rightarrow x^2 + y^2 = r^2$

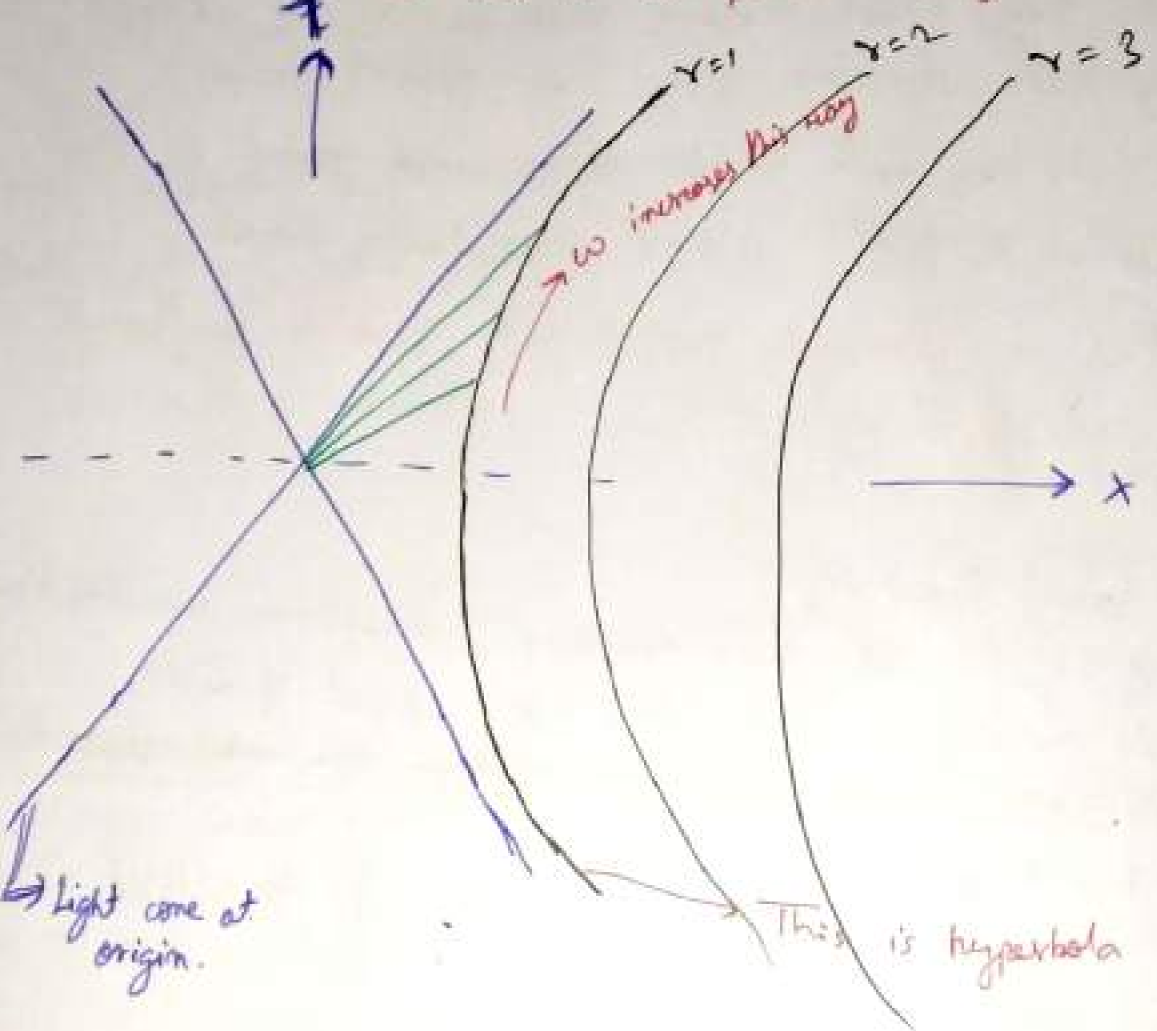
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Relativity

what is uniformly accelerated point in special relativity?

Ans: It is a point moving on a hyperbola.



★ What is analogous equation for hyperbolas. (analogous to polar coordinates ... or say circular motion)

$\cos\theta \longrightarrow \cosh\omega = \frac{e^{\omega}+e^{-\omega}}{2}$

$\sin\theta \longrightarrow \sinh\omega = \frac{e^{\omega}-e^{-\omega}}{2}$

$$\cosh^2\omega - \sinh^2\omega = 1$$

now; define: $\left.\begin{aligned} x &= r\cdot\cosh\omega \\ t &= r\cdot\sinh\omega \end{aligned}\right\} \Rightarrow$ This actually defines r & ω if you like

↳ it is actually coordinate transformation from minkowski coordinate x & t to r and ω.

* put $\omega = 0$; then increase r you go along x axis

↳ so; r is like space

* if you increase $\omega \Rightarrow$ you travel upward along the hyperbola
so; ω is like time

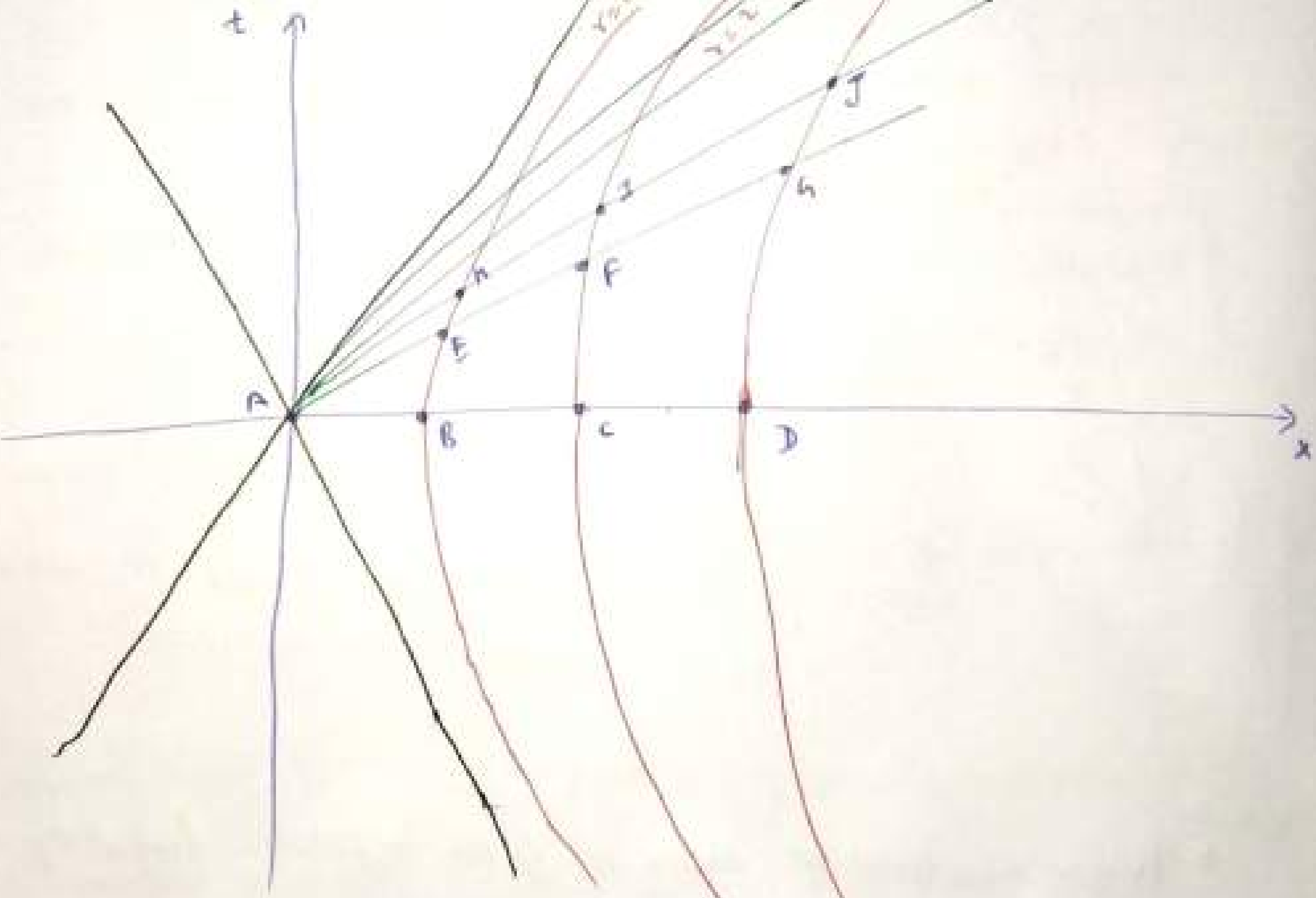
ω is time like coordinate.
r is space " " .

There is uniformity to circle that; at any point you can define r & it is constants on circles $\Longrightarrow$ you can define r in hyperbolic case which is constant on hyperbolas.

and finally; along any of the hyperbolas

$$X^2 - T^2 = r^2$$

this defines coordinates; r & ω
(those coordinates in special relativity are the closest thing that exist to uniformly accelerated coordinates



you can check ; $\underbrace{\text{distance}}_{\text{actually proper distance}}$ between : AB , BC , CD

AE , EF , FG ,

AH , HI ; IJ , etc

are same

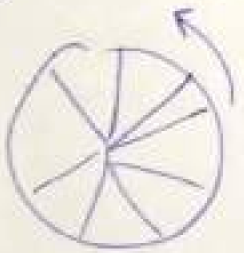
↳ Proper distance between these points are same.

→ This means ; that as the Lorentz Frame of Reference accelerates the distance between neighbouring points stay the same

↳ what is unusual here :
is that acceleration along the trajectories are different.

* particles on different hyperbolas (ie; for different r) will experience different acceleration ; but for a given hyperbolic trajectory (ie: given r) ; acceleration is same
(ie; acceleration on a given hyperbola is same.
~~actually proper~~

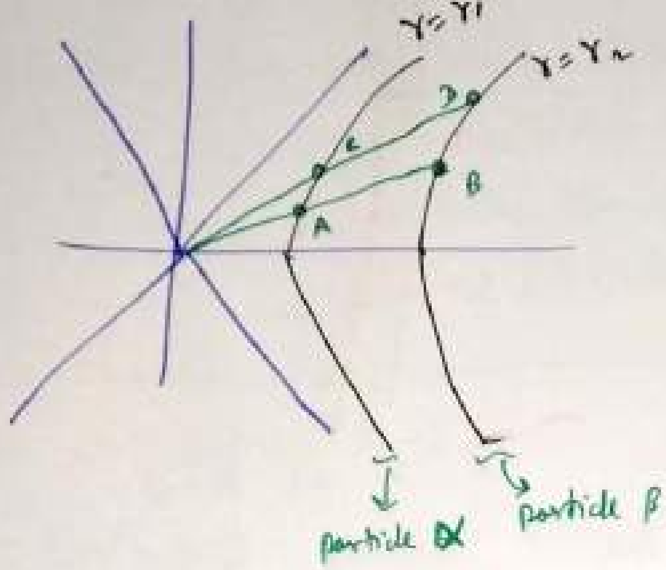
* analogous to particle executing uniform circular motion
↳ you have radially inward acceleration $\frac{v^2}{r}$ or $(\dot{\theta})^2 r$



⟹ And it varies from circle to circle (but is constant on a circle.

actually
* Proper acceleration along a given hyperbolic trajectory is uniform.

* i.e; particle moving on two different hyperbolic trajectory ; different r.



note;
Proper Acceleration
$$\frac{d^2 x^\mu}{d\tau^2}$$

The proper distance between points stay fixed
i.e; proper distance AB is equal to proper distance CD.

---

* proper acceleration A along such hyperbolic trajectories depends only on r ;

in units $c = 1$; $A = \dfrac{1}{R}$

in general units ; $A = \dfrac{c^2}{R}$

* Uniformly accelerated reference frame is defined relative to a particular origin.

Metric in ordinary polar coordinates;

we know: $ds^2 = r^2 d\theta^2 + dr^2$

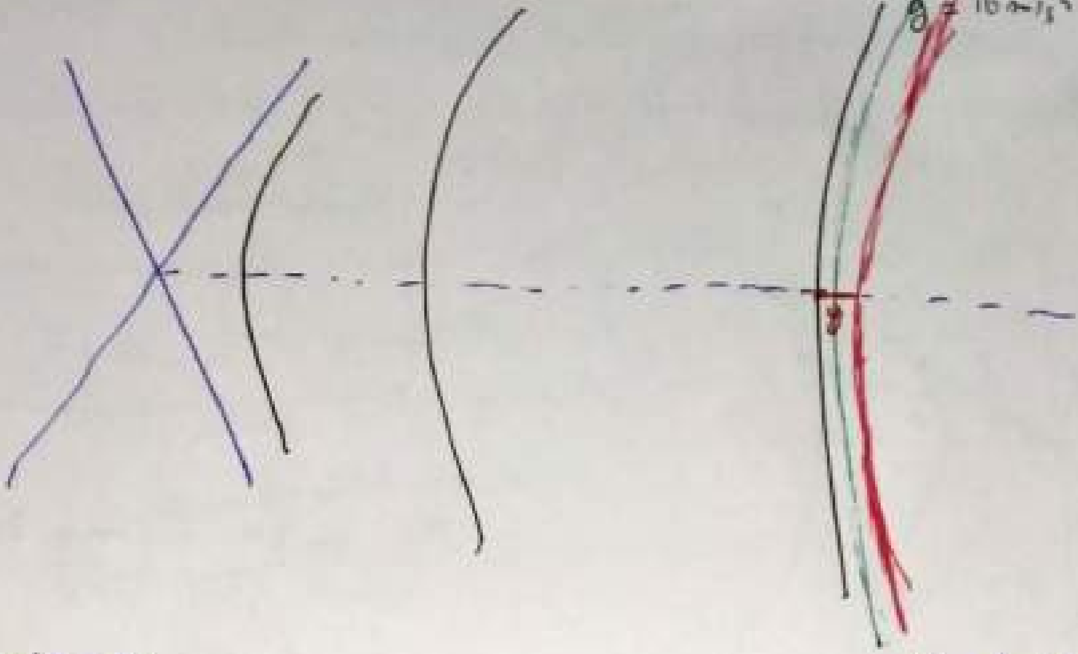
$\hookrightarrow$ This is metric of flat plane in polar coordinates.

Metric Tensor is $\begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix}$

$\hookrightarrow$ the ~~...~~ analogue here is; $d\tau^2 = r^2 d\omega^2 - dr^2$

$\hookrightarrow$ This is the metric of Space time

* for the purpose of problem here ... particle accelerating in a line ..... we effectively need one spacial dimention



~~go out to~~

* go out to a distance R where the acceleration is g
  ↳ & think think about a region close to R. ~~(like this)~~ (it is like being near surface of earth)
  (Space like distance between them be y)

$r = R + y$ (y is small compared to r)

∴ $A = g$

& $A = c^2/R$

$\Rightarrow \boxed{R = \frac{c^2}{g}}$

↳ to experience that acceleration we have to go very far away.

now; we re write the metric

$$d\tau^2 = (R^2 + 2Ry + y^2)\, d\omega^2 - dr^2$$

$$d\tau^2 = \left(1 + \frac{2y}{R} + \frac{y^2}{R^2}\right) R^2 d\omega^2 - dr^2$$

∴ let; $R\omega := t$ (we are actually going to talk about falling in uniform gravitational field)

so; $$d\tau^2 = \left(1 + \frac{2y}{R} + \cancel{\frac{y^2}{R^2}}\right) dt^2 - dr^2$$

neglect (since $y \ll R$)

$\Rightarrow$

$$\underbrace{d\tau^2 = \left(1 + \frac{2y}{R}\right) dt^2 - dr^2}$$

↳ This looks like space and time in ~~a~~ more or less ordinary way ↷ $d\tau^2 - dx^2$ ~~[struck]~~

but with a little correction term $\frac{2y}{R}$

↳ and this little correction is the thing which accounts for gravitation in this accelerated reference frame.

(Remember; we are really talking about flat space
↳ so; any gravitation we find here is in a sense same fake gravitation that we find in accelerated elevator)

$r = R + y$
$\Rightarrow dr = dy$

* in $c=1$ unit: $\frac{1}{R} = g$

so; metric becomes: $d\tau^2 = (1 + 2\underbrace{yg}) dt^2 - dy^2$

yg will account for potential energy per unit mass (in case of uniform gravitational field)

↳ we want to find out how the particle moves in this metric $\Rightarrow d\tau^2 = (1+2yg)dt^2 - dy^2$

↳ This is a metric tensor in the vicinity of a point ~~accelerating~~ accelerating with the acceleration $g = 10\,m/s^2$.

What are rules for Particle motion.

- Particles move on geodesic of space-time.

* $\frac{d^2x^n}{d\tau^2} \Rightarrow$ This is proper acceleration ; as long as the particle is moving slowly

↳ then proper & ordinary time are essentially the same

near here

& $\frac{d^2x^n}{d\tau^2}$ just becomes the ordinary acceleration.

so; here we need y component of motion

(time $\tau = R + y$.. & ~~spatial direction~~ thus

spacial direction in the metric is governed by y)

so; equation of geodesic gives;

$$\underbrace{\frac{d^2y}{d\tau^2}} = \underbrace{-\Gamma^y_{mr}\frac{dx^r}{d\tau}\cdot\frac{dx^m}{d\tau}}$$

this term is like acceleration... actually it is acceleration.

this has many terms.

but as long as all of the motion is slow

then only one of the combination is significant.

$$\boxed{\frac{dt}{d\tau} \simeq 1}$$

for slow motion.

& other components derivative is negligible.

space component divided by $d\tau$ is proportional to actual ordinary spacial velocity

(we are assuming spacial velocity is small compared to speed of light)... so neglect it

so; only important contribution comes here; when r & m are time components.

and then equation reduces to

$$\frac{d^2y}{d\tau^2} = -\Gamma^y_{tt}$$

$$\frac{d^2y}{d\tau^2} = -\Gamma_{t\,t}^{\,y}$$

; this says that $-\Gamma_{t\,t}^{\,y}$ must be the gravitational force.

i.e; Christoffel Symbol must be the gravitational force per unit mass
(it must be the derivative of gravitation potential energy per unit mass)

↳ lets check it

⇒ lets calculate $\Gamma_{t\,t}^{\,y}$ from its defination (or say formula)

$$g_{\mu\nu} = \begin{array}{c c} & \begin{array}{cc} t & y \end{array} \\ \begin{array}{c} t \\ y \end{array} & \begin{bmatrix} 1+2gy & 0 \\ 0 & -1 \end{bmatrix} \end{array}$$

⇒ you can calculate;

$$g^{\mu\nu} = \begin{array}{c c} & \begin{array}{cc} t & y \end{array} \\ \begin{array}{c} t \\ y \end{array} & \begin{bmatrix} -1 & 0 \\ 0 & +(1+2gy) \end{bmatrix} \end{array} \cdot \frac{1}{(-(1+2gy))}$$

⇝ we find ; only $g^{yy} = -1$

& $g^{yt} = 0$

so; $\Gamma_{t\,t}^{\,y} = \frac{g^{yy}}{2}\left(\cancel{\frac{\partial g_{yt}}{\partial t}}^{\,0} + \cancel{\frac{\partial g_{ty}}{\partial t}}^{\,0} - \frac{\partial g_{tt}}{\partial y} \right)$

these terms are exactly zero (there, is no g_{yt} term or say; $g_{yt} = 0$)

$$= \frac{-1}{2}\left(-\frac{\partial g_{tt}}{\partial y} \right) = \frac{1}{2} \cdot \left(\frac{\partial}{\partial y}(1+2gy) \right) = g$$

So; geodesic equation reduces to

$$\underbrace{\frac{d^2y}{d\tau^2} = -g}$$

~~[illegible]~~
we recognize this as equation of motion of a particle in uniform gravitational field.

note; here; $t \simeq \tau$ (since low velocity)

so; $\frac{d^2y}{d\tau^2} \simeq \frac{d^2y}{dt^2}$ $\Rightarrow$ so; $\underbrace{\frac{d^2y}{dt^2} = -g}$

~~at least as long as~~

The equation of motion of geodesic (at least as long as things are going slowly ..... and along with Newtonian approximation) is just Newton's equation in uniform gravitational field. ; i.e; uniform acceleration.

(we would have taken ~~[illegible]~~ acceleration to be a ...
↳ then we would have get Newtons equation

$$\frac{d^2y}{dt^2} = -a$$)

General Relativity : Metric ~~of~~ for a Gravitational field.

— Shoaib Akhtar.

$d\tau^2 = dt^2 - \dfrac{dx^2 - dx^2 - dz^2}{c^2}$

* if $d\tau^2 > 0$; then the vector is said to be time-like? and lie inside light cone
* if $d\tau^2 < 0$; then the vector is said to be space-like. } lie outside light cone.

(here we define : $ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$)

Light cone

* If $d\tau^2 = 0$; they are called Light Like. (lie in the cone)

$d\tau^2 = -g_{\mu\nu}\, dx^\mu dx^\nu$

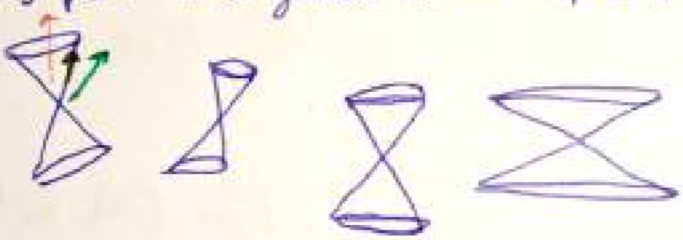
and : $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

* Proper time is time read of by moving clock in time like trajectory
* Proper distance along a space like surface is distance measured by a meter stick.

$\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$

so, $d\tau^2 = -\eta_{\mu\nu}\, dx^\mu dx^\nu$ (metric in minkowski coordinate in flat space time)

☆ every point in a general curved space time has a light cone associated with it.



The property of right number of positive and negative eigen value of metric tensor is called Signature of the metric

* Signature of the metric of ordinary flat space.
  black board would be +, +.

* Signature of Minkowski space in Special Relativity is −, +, +, +

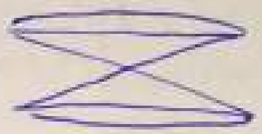
* we should make sure that metric should have signature −, +, +, + at every point.

we can check

* any time like vector is eigenvector of $\eta_{\mu\nu}$ with eigenvalue −1
* any space like vector is eigenvector of $\eta_{\mu\nu}$ with eigenvalue +1.
* any light like vector is eigenvector with eigen value 0.

by changing coordinates; we can squash the light cone, or say ~~tilt~~ tilt the light cone.

$\underline{ex}$ $d\tau^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$

* if c is large then light cone becomes 
* if c is small then " " " 

↳ it does not reflect a real change in the geometry.

~~ex~~ we could also have c depend on position
↳ then metric would have some funny position dependance.

; ~~so~~ and the metric as you move from one place to another we go from looking this way "" to looking this way "".

⟹ so, It is pure coordinate issue how the light cone looks.

⟿ but the Invariant statement is,
that there are space like vectors, time like vectors, and light like vectors ; and boundary between space like & time like vector is a cone.

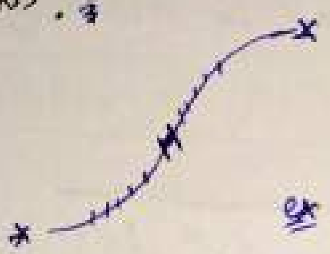
$\underline{\text{imp}!!!}$ $\eta_{\mu\nu}$ transforms under Lorentz Transformation to itself.

Geodesic $\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\tau}\cdot\frac{dx^\rho}{d\tau}$

[redacted] is

☆ Another way of saying the above geodesic statement is as following
we extremalize or (say minimize) the distance between two points.

we are interested in geodesic connecting two points.



Stationary distance between two points defines the geodesic.

ex Ordinary space

$$ds^2 = g_{mn}\, dx^m dx^n$$

$$\text{or } ds = \sqrt{g_{mn}\, dx^m dx^n}$$ } ⇒ distance along a little piece of path.

↳ now we add them all up ... so we do the integral

$$S = \int ds = \int \sqrt{g_{mn}\, dx^m dx^n}$$

∴ now, the rule is to make S small as possible (i.e; to make it ~~minimal~~ extremal)

↳ we can think of S as some kind of action.

now; lets talk in space time

$$\tau = \int_1^2 d\tau = \int_1^2 \sqrt{-g_{\mu\nu}(x)\, dx^\mu dx^\nu}$$

↳ The action also depends on mass. (to get or define energy later)

$$\text{so; ACTION} = -m \int_1^2 \sqrt{-g_{\mu\nu}(x)\, dx^\mu dx^\nu}$$

putting this minus sign is just defination.

~~break~~ lets break up the interval into little time segments Δt (or say dt)

↪ and divide each of dx^μ in action by dt

so:
$$\boxed{\text{ACTION} = -m\int \sqrt{-g_{\mu\nu}(x)\cdot\frac{dx^\mu}{dt}\cdot\frac{dx^\nu}{dt}}\cdot dt}$$

but; Action by definition is : $A = \int \mathcal{L}(\dot{x}, x)\, dt$

so: $\mathcal{L} = -m\sqrt{-g_{\mu\nu}(x)\,\frac{dx^\mu}{dt}\cdot\frac{dx^\nu}{dt}}$

↪ the think inside the square root is proper time

↪ & it is always possible for timelike trajectories.

(particles always move on timelike trajectories)

note: $\frac{dx^\mu}{dt}$ is either one or velocity component.

now solve euler Langrange equation: $\frac{\partial \mathcal{L}}{\partial x^\mu}$

note: $\frac{dx^0}{dt}$ is actually $\frac{dt}{dt} = 1$

↪ so: $\mathcal{L}$ is just function of space component of velocity & x itself.

↪ so; we take $\mathcal{L}$ ... ~~and diffe~~

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^m}\right) = \frac{\partial \mathcal{L}}{\partial x^m}}$$

i.e. $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^m}\right) = \frac{\partial \mathcal{L}}{\partial x^m}$

note// if you work out Euler Langrange equation, you will discover that you exactly get the ~~conventional~~ usual geodesic equation. i.e. $\frac{d^2x^\mu}{d\tau^2} = +\Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau}$

Motion of particle in real gravitational field.

$$d\tau^2 = \underbrace{\left(1 + \frac{2U(x)}{c^2}\right)dt^2 - \frac{1}{c^2}(dx^2)}$$

A massive spherical symmetric object
↳ & we are now interested in metric of space outside the massive object.

→ we will check that this formula for metric really does makes sense; in the sense that it will give rise using the equations of motion to Newton's equation for particle moving in gravitational field (at least when you are far away from ~~the~~ the gravitating object where the gravitation is barely weak)

note The metric should reduce to usual flat space metric when you are far away.

then:

$$d\tau^2 = \left(1 + \frac{2 \cdot U(x)}{c^2}\right)dt^2 - \frac{1}{c^2}(dx^2)$$

* lets check: as long as particles are moving slowly: the Geodesic equation ~~[illegible]~~ just becomes Newtons Equation.

so; here: $\mathcal{L} = -mc^2\sqrt{\left(1+\frac{2U}{c^2}\right)dt^2 - \frac{1}{c^2}dx^2}$

$$A = -mc^2\int\sqrt{\left(1+\frac{2U}{c^2}\right) - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2}\cdot dt$$

$$\Rightarrow A = -mc^2\int\sqrt{\left(1+\frac{2U}{c^2}\right) - \frac{1}{c^2}\dot{x}^2}\cdot dt$$

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→ Now; Non-Relativistic Limit. ... i.e. large c

so; now lets approximate Action in this limit.

i.e $A = -mc^2 \int \cancel{\left(\sqrt{1 + \frac{1}{c^2}(2U - \dot{x}^2)}\right)} dt$

↳ now use binomial expansion.

$$\Rightarrow A = \int -mc^2 \left[1 + \frac{1}{2c^2} (2U - \dot{x}^2) \right] . dt$$

$$\Rightarrow A = \int \left\{ mc^2 - \frac{m}{2} (2U - \dot{x}^2) \right\} dt$$

$$\therefore \mathcal{L} = \underbrace{-mc^2}_{} - mU + \frac{m}{2} \dot{x}^2$$

↳ a constant in Langrangian makes no difference in equations of motion.

~~So; Action becomes~~

so; Action in non-relativistic limit becomes.

$$A = \int \left(-mU + \frac{m}{2} \dot{x}^2 \right) dt$$

$$\& \quad \mathcal{L} = \underbrace{\frac{m}{2} \dot{x}^2}_{\text{Kinetic energy term}} \underbrace{- mU}_{\text{potential energy term}}$$

→ Now when you solve Euler Langrange Equations; you will get Newtons equation.

$$m\ddot{x} = -m \frac{\partial U}{\partial x}$$

* we can say: $-g_{00} = \left(1 + \frac{2U}{c^2}\right)$

U is potential energy term

(or say approximately equal to) *

↳ if there was other terms like $\frac{1}{c^4}$ (but that would have not been much important in non-relativistic limit).

↳ so we can't say with complete confidence that $-g_{00}$ is $\left(1 + \frac{2U}{c^2}\right)$; but we can say that must be true for first order in small quantities.

now; $U(x) = -\frac{MG}{r}$

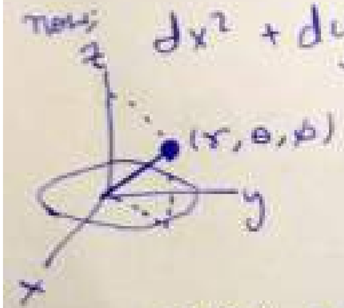
now; our first guess for the metric of spacetime surrounding a gravitational mass.

$$d\tau^2 = \left(1 - \frac{2MG}{rc^2}\right)dt^2 - \frac{1}{c^2}(dx^2)$$

i.e., $d\tau^2 = \left(1 - \frac{2MG}{rc^2}\right)dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) + \underbrace{\ldots}_{\text{small terms.}}$

now; $\underbrace{dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \cos^2\theta \, d\phi^2}$

↳ metric of ordinary three dimensional space in polar coordinates.



now; let; $d\Omega^2 := (d\theta^2 + \cos^2\theta \, d\phi^2)$

so; $dx^2 + dy^2 + dz^2$ becomes $dr^2 + r^2 d\Omega^2$

so; $$d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right)dt^2 - \frac{1}{c^2}dr^2 - \frac{1}{c^2}r^2 d\Omega^2.$$

$$= \left(1 - \frac{2GM}{rc^2}\right)dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right)$$

↳ so; if r is very small
$\left(1 - \frac{2GM}{rc^2}\right) < 0$; infact : $\left(1 - \frac{2GM}{rc^2}\right)dt^2$ changes sign at $r = \frac{2GM}{c^2}$

& thus; for small r
$d\tau^2 < 0$

↳ There is something wrong with this metric
- it is fine when you are far away.
- but when r becomes too small ; the metric becomes negative
  ↳ i.e; when you move too close to the centre : the metric tensor has four positive eigenvalues (instead of one negative & three positive)

↳ This implies that:
somehow pass into a region where there are four space direction and no time direction.
(This is bad... this is something we dont want)

↳ so ; we must change something.

* Einstein field equations suggests.
$$d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right)dt^2 - \left(\frac{1}{1 - \frac{2MG}{c^2 r}}\right)\frac{dr^2}{c^2} - \frac{1}{c^2}r^2 d\Omega^2$$

↳ so ; when we get to the point as you move in closer the Schwarzschild radius ($r = \frac{2GM}{c^2}$),
$\left(1 - \frac{2GM}{rc^2}\right)dt^2$ & $\left(\frac{1}{1 - \frac{2MG}{c^2 r}}\right)\frac{dr^2}{c^2}$ both changes sign.

↳ and because they both change sign; the signature of metric stays the same.

⟹ but something very odd ~~too~~ has happened

↳ the thing we were calling t becomes a space-like direction; and the thing we were calling r becomes time like direction for r less than $\frac{2GM}{c^2}$.

i.e; They flip.

↳ once you go inside that radius ($r = \frac{2GM}{c^2}$):

the coordinate r becomes time

& " " t " space direction.

☆

~~Something has hap As we passed~~

Something has happened as we passed in through a certain surface

... the surface being

$r = \frac{2MG}{c^2}$

This is mysterious now... but later ~~will~~ we will explore this.

↳ we will find out that the Geometry is ~~completely~~ completely smooth around $r = \frac{2MG}{c^2}$ ... nothing special happening.

☆ lets see how big is the correction

→ lets check in non relativistic limit. ... we want c to be very large

$$d\tau^2 = \left(1-\frac{2MG}{rc^2}\right)dt^2 - \left(\frac{1}{1-\frac{2MG}{rc^2}}\right)dr^2 - \frac{r^2}{c^2}d\Omega^2$$

note

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

so;
$$d\tau^2 = \left(1-\frac{2MG}{rc^2}\right)dt^2 - \Bigg(1 + \underbrace{\frac{2MG}{rc^2} + \left(\frac{2MG}{rc^2}\right)^2 + \dots}\Bigg)\frac{dr^2}{c^2} - \frac{r^2}{c^2}d\Omega^2$$

↳ if r is not too small

⟹ these correction terms can then be neglected

... they are too small to be important.

so; basically; the metric approximates to

$$d\tau^2 = \left(1-\frac{2MG}{rc^2}\right)dt^2 - \frac{dr^2}{c^2} - \frac{r^2}{c^2}d\Omega^2$$

---

lets see how particles moves in general in the metric

$$d\tau^2 = \left(1-\frac{2MG}{c^2 r}\right)dt^2 - \left(\frac{1}{1-\frac{2MG}{c^2 r}}\right)\frac{dr^2}{c^2} - \frac{r^2}{c^2}d\Omega^2$$

↳ we are not going to using any approximation;

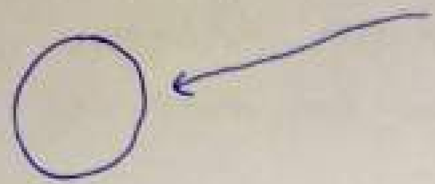
so lets work in the unit $c=1$

SCHWARTZSCHILD METRIC

$$\boxed{d\tau^2 = \left(1-\frac{2MG}{r}\right)dt^2 - \left(\frac{1}{1-\frac{2MG}{r}}\right)dr^2 - r^2 d\Omega^2}$$

↳ lets see how particles moves in this metric.

Radially falling body.



* somebody on the outside watching the falling body, uses the time "t" (uses his clock (conventional time...) ~~uses his clock~~.

* Somebody falling in uses proper time "τ" (uses his clock)

$$\text{Action} = -m\int\sqrt{d\tau^2}$$

$$;\ d\tau^2 = \left(1-\frac{2MG}{r}\right)dt^2 - \left(\frac{1}{1-\frac{2MG}{r}}\right)dr^2 - r^2 d\Omega^2$$

↳ since we are solving for radially falling body we can assume that the angle is not changing so; we can use; $d\Omega = 0$

so; $$\text{Action} = -m\int\sqrt{\left(1-\frac{2MG}{r}\right)dt^2 - \frac{1}{\left(1-\frac{2MG}{r}\right)}dr^2}$$

$$= -m\int\sqrt{\left(1-\frac{2MG}{r}\right) - \frac{1}{\left(1-\frac{2MG}{r}\right)}\left(\frac{dr}{dt}\right)^2}\ .dt$$

observe; $\frac{dr}{dt}$ is just the radial velocity $\dot{r}$

so; $$\text{Action} = -m\int\sqrt{\left(1-\frac{2MG}{r}\right) - \frac{1}{\left(1-\frac{2MG}{r}\right)}\cdot\dot{r}^2}\ dt$$

→ so; we have the Lagrangian.

$$\mathcal{L} = -m\sqrt{\left(1-\frac{2MG}{r}\right) - \frac{1}{\left(1-\frac{2MG}{r}\right)}\cdot\dot{r}^2}$$

$$P_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{m\dot{r}}{\sqrt{\left(1-\frac{2MG}{r}\right) - \dot{r}^2/\left(1-\frac{2MG}{r}\right)}}$$

so; we will now have a conserved quantity
h (the Hamiltonian) ... or say energy.

$$H = p_r \cdot \dot{r} - \mathcal{L}$$

$$\Rightarrow \quad H = \frac{m\left(1 - \frac{2MG}{r}\right)}{\sqrt{\left(1 - \frac{2MG}{r}\right) - \frac{\dot{r}^2}{\left(1 - \frac{2MG}{r}\right)}}}$$

↳ we know this is conserved

↳ so; we can conventionally write it as E
and we know it does not change with time.

$$E = \frac{m\left(1 - \frac{2MG}{r}\right)}{\sqrt{\left(1 - \frac{2MG}{r}\right) - \left(\frac{\dot{r}^2}{1 - \frac{2MG}{r}}\right)}}$$

→ so; we solve for $\dot{r}$ as function of energy & r

$$\dot{r} = \left(1 - \frac{2MG}{r}\right)\sqrt{1 - \frac{m^2}{E^2}\left(1 - \frac{2MG}{r}\right)}$$

when r is near to $2MG$ $\boxed{\dot{r} \simeq \sqrt{\frac{r - 2MG}{2MG}}}$

MG is radius of the horizon.

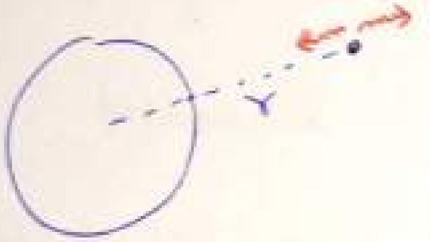
$r_s = 2MG$ (Schwarchild Radius)

This is telling us that as the particle is falling towards the horizon ; its velocity goes to zero
(it gets smaller and smaller and smaller ... instead of accelerating)
(which you might have expected) !

It slows down ... gets asymptotically slower & slower & never passes the horizon. ($r_s = 2GM$)

★ Metric : $d\tau^2 = \left(1 - \frac{2MG}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2MG}{r}\right)}$

now: think about a radially moving light ray.



now: light Ray is solution to the equation where the interval $d\tau^2$ is light like
; means ; $d\tau^2 = 0$

consider magnitude ⇒ $dr = \left(1 - \frac{2MG}{r}\right)dt$

(it can be falling in or moving out) ⇒ $\underbrace{\frac{dr}{dt} = \left(1 - \frac{2MG}{r}\right)}$

This is the light ray ... (thats the fastest thing which can ever be)

★ infalling light ray

$|\dot{r}| = \frac{r - 2MG}{r}$

(even light is slowing down as it is moving towards horizon)

but ; the speed of light has this property that in these coordinates the speed of light goes to zero as you get closer and closer

so: nothing (even including light ray) can pass the surface ever there ; $r = r_s$.

* when it is ~~far~~ far away : $\frac{dr}{dt} = 1$ ~~----~~ (usual speed of light)

---

note!! $\left(1 - \frac{2MG}{r}\right)$ is magnitude of velocity of light

↳ the magnitude of velocity goes to zero either for in-going or out-going radiation.

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2MG}{r}\right)^2$$ ⟹ There are two solutions when we take square root

$$\left(\frac{dr}{dt}\right)_{light} = +\left(1 - \frac{2MG}{r}\right)$$ ⟹ this is outgoing light

also; $$\left(\frac{dr}{dt}\right)_{light} = -\left(1 - \frac{2MG}{r}\right)$$ ⟹ this is light falling in.

* If you start a light ray right at the Schwarzschild radius, it wont get anywhere.
infact, a light ray will just sit at the Schwarzschild Radius.

- a light ray interior to Schwartzschild Radius, will fall in.
- " " " exterior " " " " : will go out

⟹ but the closer it starts to Schwartzschild Radius, the longer it will take to escape.

$$\boxed{R_s := \frac{2GM}{c^2}}$$ Schwartzschild Radius.

⟹ note: the solution we had found above is correct only outside the gravitating matter.

* for earth ; r_s = centimeter

& its radius $r_e \simeq 64000$ km

⟶ dont apply the whole solution in the interior of earth.

⟹ The solution & the peculiar r_s ; ~~only~~ only if the whole gravitating ~~object~~ object collapse down to a radius smaller than the Schwartzschild Radius.

(Then the solution found above is meaning & correct down to such small distances)

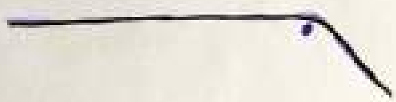
* Schwartzschild solution is idealized solution (idealized in the sense that Newtonian force law $F = \frac{-mMG}{r^2}$ .... its an idealization under the assumption that the mass creating the gravitational field is point at the centre...)

if mass spreads out... we dont believe that ~~that~~ ~~true~~ to be true in the interior of massive object.

(but of course it is true outside the mass... Newtons shell theorem)

so it is a sort of idealization for point mass)

* In Newtonian gravity.



→ if you have even a slight error from aiming at centre .... it will not hit the centre.

(it is infinitely unlikely for an in-falling point particle to actually hit the centre) even though it pulls so hard near centre.

(This is not so the case in General Theory of Relativity)

~~Basically~~
* if your aim is not perfect; the particle will have little bit of angular momentum;

~~Exa~~ because of this angular momentum; there will be effectively ~~be~~ centrifugal force which will keep it ~~going~~ from going through the centre.

• In General Theory Of Relativity ... gravity is so strong that it overwhelms the centrifugal barrier & pulls anything into the centre

Schwartzschild Metric

$$\boxed{d\tau^2 = \left(1-\frac{2MG}{rc^2}\right)dt^2 - \frac{1}{\left(1-\frac{2MG}{rc^2}\right)}\frac{dr^2}{c^2} - \frac{r^2}{c^2}d\Omega^2}$$

$$d\Omega^2 = d\theta^2 + \cos^2\theta\, d\phi^2$$

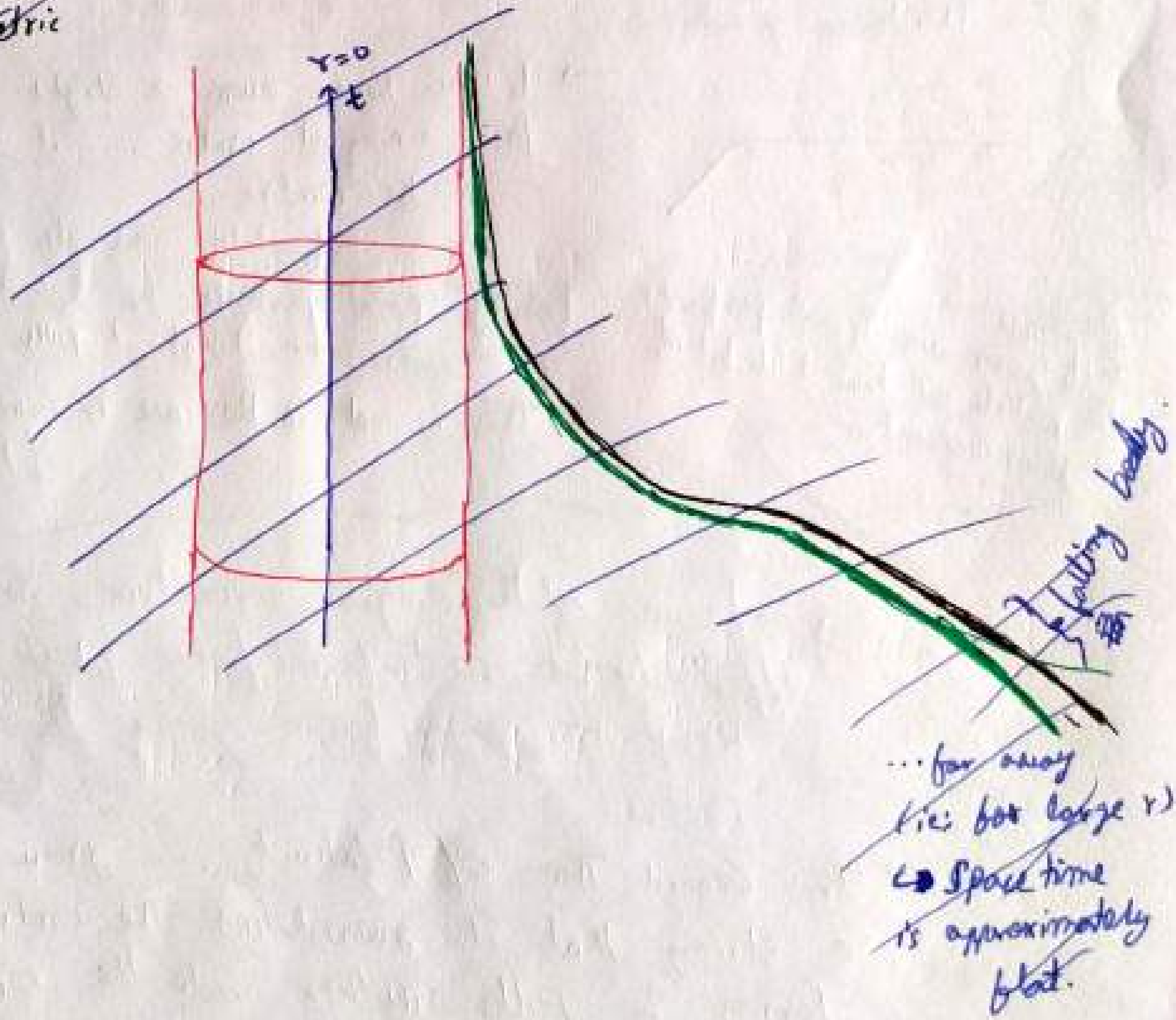
↳ it is just the metric of a unit sphere.

in units; $c = 1$

Schwartzchild Metric

$$\boxed{d\tau^2 = \left(1-\frac{2GM}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2MG}{r}\right)} - r^2 d\Omega^2}$$

☆ Picture of metric



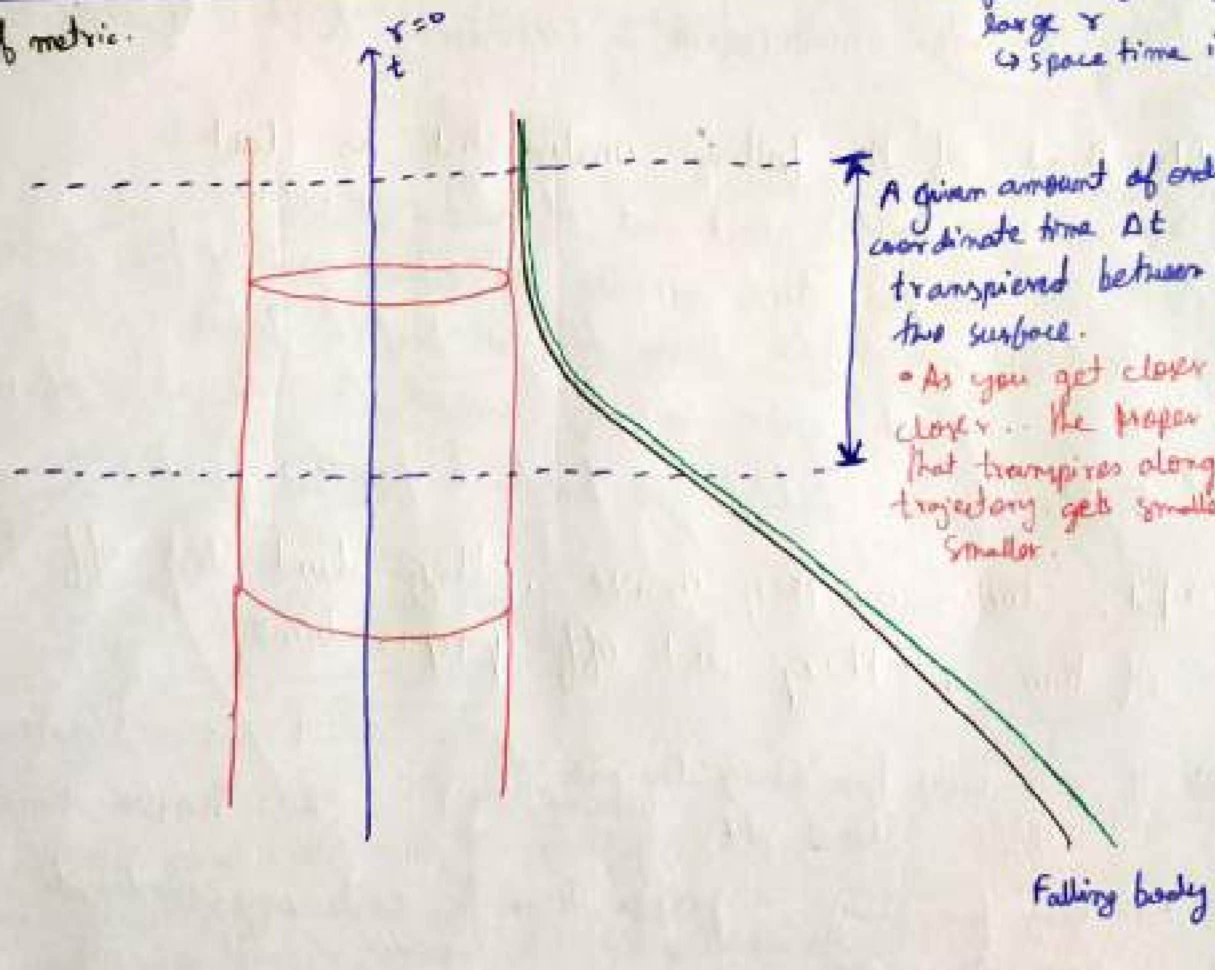
Note

* Nothing of any real physical significance is happening at $r = 2MG$.
  * nothing special, no large curvature is happening; no enormous deviation from flat space is ~~happening at~~ happening at $r = 2MG$.

* at $r = 0$; Something real is going on at $r = 0$;
  actually at $r = 0$ the curvature is becoming infinite.
  ↳ if we were to calculate Curvature Tensor to calculate how curved the geometry is ... we will find all of its components becoming infinite at $r = 0$.
  ↳ Curvature means tidal forces.
  ↳ anything falling in ... when it will hit $r = 0$ would experience infinitely strong tidal forces and would be torned to pieces.
  (it would be stretched radially & squeezed in the angular direction by infinite amount)

☆ Picture of metric.

* far away i.e. for large r
  ↳ space time is flat.

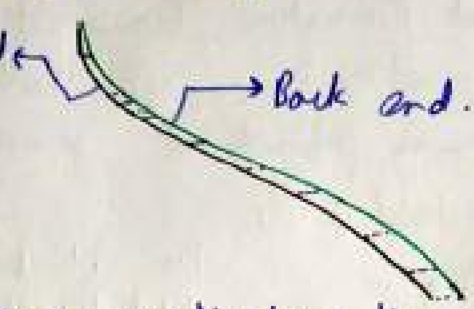


A given amount of ordinary coordinate time Δt transpired between these two surface.
* As you get closer & closer ... the proper time that transpires along this trajectory gets smaller & smaller.

* Somebody falling in with a clock; do they experience an infinite amount of time before they get to horizon or finite amount? (ordinary human being... your heart, your brain, etc is a clock)
↳ every time keeper you have will experience the ~~sta~~ same

Ans They experience the proper time. shift or relative size scale for beating of the clock $d\tau$. ~~[illegible]~~

* suppose ~~if~~ the object falling in has a length. (that's why in the picture we had drawn two curves... each curve for each end of the object)



at least in these coordinates; the gap in r between the front end of the meter stick & its back end shrinks to zero.
↳ This is actually a form of Lorentz contraction.

* "As something falls in, even though it appears to be slowing down; its momentum is increasing & it is Lorentz contracting"

* Also think of the falling meter stick as clock
(How can you build a clock out of meter stick?
Ans// You can put two mirrors on either end of the meter stick & allow a light beam to go back & forth;
and you simply count the number of pulses of light beam which go back & ~~forth~~ forth)
→ but; clocks as they move; they dont tick off " to kind of time; they tick off proper time.

t :: coordinate time
τ :: Proper time.

notice// if r is very far away; (we are not moving there) then $d\tau \simeq dt$
i.e.; very far away, proper time & ordinary coordinate time are the same.

But as we get closer and closer in ; for a given amount of dt ; dτ gets smaller and smaller.

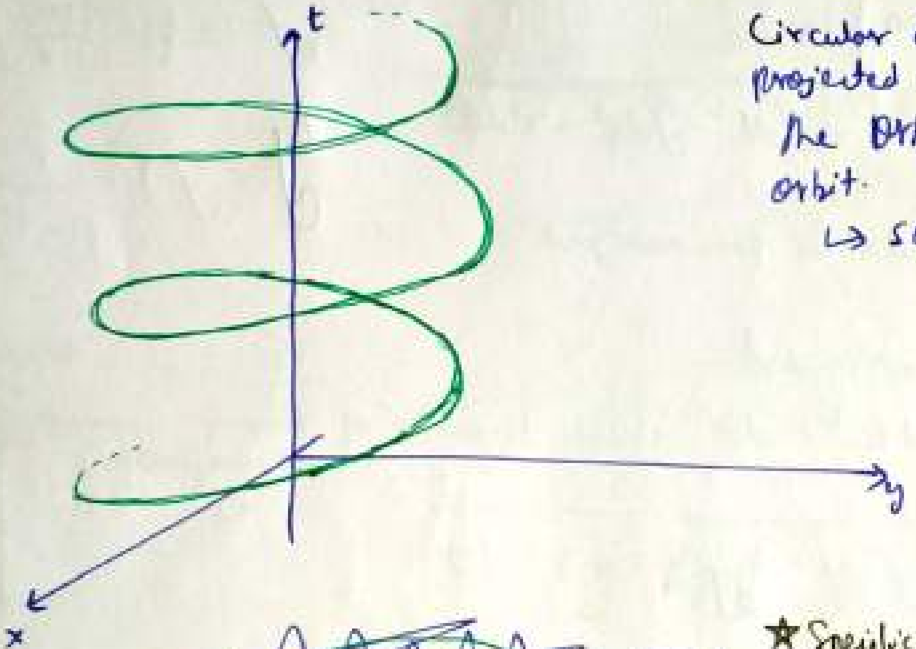
↳ so; while it appears that the amount of t ~~to~~ it takes to fall in into the black hole is infinite ; we could ask how much τ evolves by the time ~~the~~ in-falling object ~~arrives~~ eventually arrives at the horizon.... and this is finite.

Orbiting the black hole.

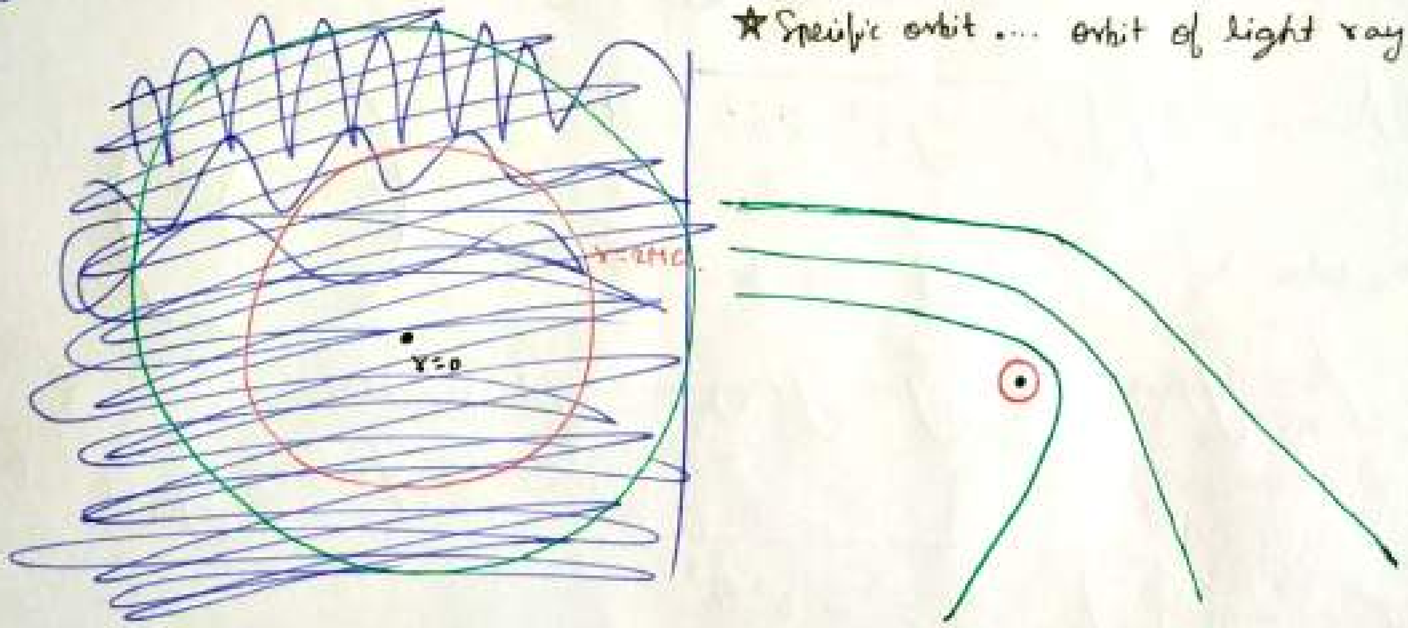
lets; take the case of a circular orbit.

Circular orbit means that if we projected the ~~orbit~~ orbit onto space the Orbit would just be a circular orbit.

↳ so ; if we are looking from the future down



★ Specific orbit .... orbit of light ray.



and at some point (it is not the horizon).... it is further out in the horizon. ; there exit trajectories which just go around.. circular trajectories. ∴ which just go right around the gravitating object.

Remember

* Unit of Lagrangian is energy.
* Unit of action is energy × time.

so for action: multiply by mc^2 or say m to $\int d\tau$.

Calculations:
$$\text{Action} = -m\int d\tau$$
$$= -m\int\sqrt{\mathcal{F}dt^2 - \mathcal{G}dr^2 - r^2 d\Omega^2}$$

; let: $\mathcal{F} := \left(1 - \frac{2MG}{r}\right)$

$$\mathcal{G} := (\mathcal{F})^{-1} = \frac{1}{\left(1 - \frac{2MG}{r}\right)}$$

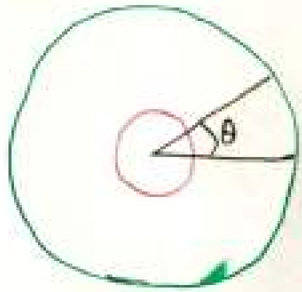
let; let the circular trajector be parametrized by single angle.

$$d\Omega^2 = d\theta^2 + \cos^2\theta\, d\phi^2$$

so; effectively $d\Omega^2 = d\theta^2$ (let trajectory of moving around the equator)

so; $$\text{Action} = -m\int\sqrt{\mathcal{F}dt^2 - \mathcal{G}dr^2 - r^2 d\theta^2}$$

$$\Rightarrow \text{Action} = -m\int\sqrt{\mathcal{F} - \mathcal{G}\dot{r}^2 - r^2\dot{\theta}^2}\; dt$$



remember that $\mathcal{F}$ & $\mathcal{G}$ are functions of r

$\mathcal{F} = \mathcal{F}(r)$ & $\mathcal{G} = \mathcal{G}(r)$; and remember:

$$\mathcal{F}(r) = \frac{1}{\mathcal{G}(r)}$$

$$\mathcal{L} = -m\sqrt{\mathcal{F} - \mathcal{G}\dot{r}^2 - r^2\dot{\theta}^2}$$

~~note~~ for circular orbit $\dot{r} = 0$.

There are two conservation laws here
- Conservation of energy.
- " " angular momentum.

* let L stand for angular momentum.

$$P_\theta = L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{m r^2 \dot{\theta}}{\sqrt{\mathcal{F} - \mathcal{G}\dot{r}^2 - r^2\dot{\theta}^2}} = m\Lambda$$

This is the quantity which is conserved.

where; $$\underbrace{\Lambda}_{\text{This is a conserved quantity since L is conserved.}} = \frac{r^2\dot{\theta}}{\sqrt{\mathcal{F} - \mathcal{G}\dot{r}^2 - r^2\dot{\theta}^2}} = \frac{L}{m}$$

* also; $$P_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{m\mathcal{G}\dot{r}}{\sqrt{\mathcal{F} - \mathcal{G}\dot{r}^2 - r^2\dot{\theta}^2}}$$

now; $$H = \sum P_q \cdot \dot{q} - \mathcal{L}$$

This is the defination of energy & it is always conserved as long as the Langrangian does not explicitly depends on time.

$$\Rightarrow H = P_\theta \cdot \dot{\theta} + P_r \cdot \dot{r} - \mathcal{L}$$

$$= \frac{m r^2 \dot{\theta}^2}{\sqrt{}} + \frac{m\mathcal{G}\dot{r}^2}{\sqrt{}} + m\sqrt{}$$

$$\Rightarrow H = \frac{m\mathcal{F}(r)}{\sqrt{\mathcal{F}(r) - \mathcal{G}(r)\dot{r}^2 - r^2\dot{\theta}^2}}$$

$\Rightarrow$ It is energy & it is conserved with time... let call the conserved quantity E.

So; $E = H$

so; conserved quantities are;

$$\Lambda = \frac{r^2\dot{\theta}}{\sqrt{\mathcal{F} - \mathcal{G}\dot{r}^2 - r^2\dot{\theta}^2}}$$

$$E = \frac{m\mathcal{F}}{\sqrt{\mathcal{F} - \mathcal{G}\dot{r}^2 - r^2\dot{\theta}^2}}$$

for circular orbit: $\dot{r} = 0$

so, $$E = \frac{m\mathcal{F}(r)}{\sqrt{\mathcal{F}(r) - r^2\dot{\theta}^2}}$$

⇒ This is an expression for energy that involves $\dot{\theta}^2$

* its partly kinetic energy due to the motion of θ.
* here is no kinetic energy due to motion of r (because r is not changing for circular orbits)

$$\Lambda = \frac{r^2\dot{\theta}}{\sqrt{\mathcal{F}(r) - r^2\dot{\theta}^2}}$$

→ Λ is usually called the reduced angular momentum $\left(\Lambda = \frac{L}{m}\right)$

$$\Rightarrow \Lambda^2\mathcal{F}(r) = r^4\dot{\theta}^2 + \Lambda^2 r^2\dot{\theta}^2$$

$$\Rightarrow \Lambda^2\mathcal{F}(r) = \dot{\theta}^2 r^2(r^2 + \Lambda^2)$$

$$\Rightarrow \dot{\theta} = \frac{\Lambda}{r}\sqrt{\frac{\mathcal{F}(r)}{r^2 + \Lambda^2}}$$

$$\Rightarrow E = \frac{m\mathcal{F}}{\sqrt{\mathcal{F} - \frac{\Lambda^2\mathcal{F}}{r^2+\Lambda^2}}} = \frac{m\mathcal{F}}{r}\sqrt{\frac{r^2+\Lambda^2}{\mathcal{F}}} \Rightarrow \boxed{E = \frac{m\sqrt{\mathcal{F}(r^2+\Lambda^2)}}{r}}$$

$$\Rightarrow E = \frac{m\sqrt{\mathcal{F}}\sqrt{r^2\Lambda^2 + r^4}}{r^2}$$

Rule: minimize energy as function of r for fixed angular momentum.

now: as $m \to 0$; $E \neq 0$; actually: $L = m\Lambda$ (think for photon for a given angular momentum)

so ; when $m \to 0$; Λ becomes big.

* whether a photon can orbit the geometry?
* for given energy & angular momentum of photon; We are interested in a photon of given angular momentum.

$$E = \frac{\sqrt{\mathcal{F}}\sqrt{r^2\Lambda^2 + r^4}}{r^2}\cdot m \quad \text{if } \Lambda \to \infty \Rightarrow E = \frac{m\sqrt{\mathcal{F}}}{r^2}\cdot r\cdot\Lambda$$

$$\Rightarrow E = \frac{1}{r}\sqrt{\mathcal{F}}\cdot(m\Lambda) = \frac{1}{r}\sqrt{\mathcal{F}}\,L \Rightarrow \boxed{E = \frac{L}{r}\sqrt{\mathcal{F}}}$$

where is the photon orbit?

The photon orbit will be where the function of r, ie; $E = \frac{1}{r}(L\sqrt{F})$ is stationary.

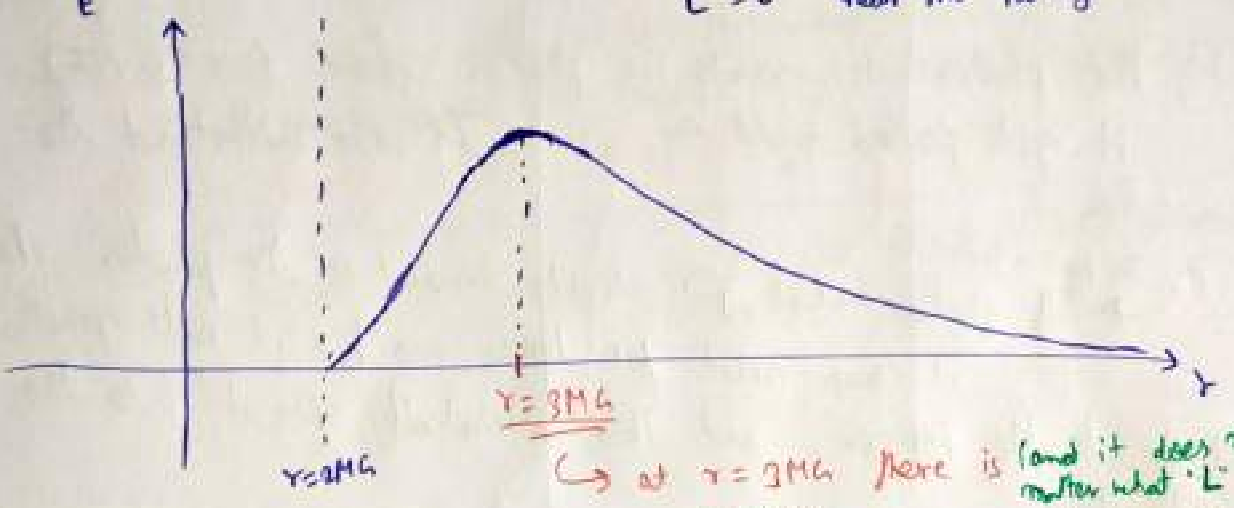
L is just a constant multiplicative factor here (we are exploring the situation for a given Angular Momentum)

so; we have to find at what values of r the combination

$$\frac{\sqrt{F}}{r}$$ becomes stationary.

so; $E = \frac{L\sqrt{1-\frac{2MG}{r}}}{r}$; $E \simeq \frac{L}{r}$ when you are very far away.

$E = 0$ near the horizon $r = 2MG$.



↳ at $r = 3MG$ there is maximum (and it does not matter what 'L' is)

(This corresponds to unstable equilibrium)

↳ so; we find that right at $r = 3MG$, a massless particle of photon can orbit the black hole.

↳ This radius is called photosphere or photon-sphere. (it's a sphere on which a photon can orbit) (The orbit is independent of what Angular momentum is)

* Since it is unstable equilibrium point ... so, if the photon starts even at slight displaced position, it will go back out.

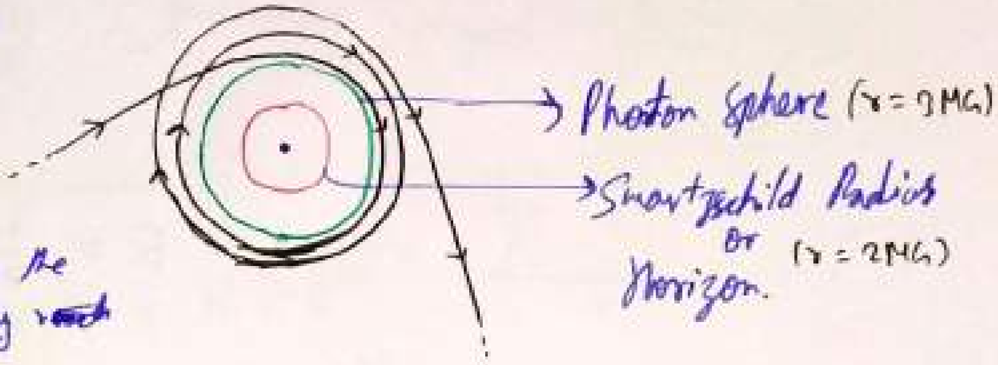
★ If your light ray is just outside the photon-sphere.. the photon will see whip around & then come out

↳ The closer you get to photon-sphere: the more angle the photon will sweep out before it eventually goes back out.

in particular:

* if you are very very close to photon sphere ($r > 3MG$): a photon comming in can go ~~many~~ around many many times but then go back out.

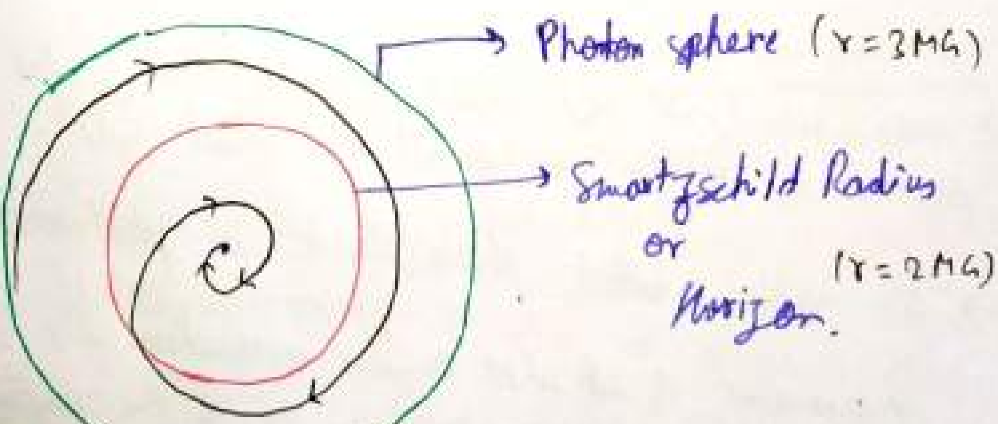
* if the light outside has suitable inward radial component of velocity, it will come inside ~~horizon~~ the photonsphere ↳ and then will cross the horizon & will eventually ~~reach~~ reach singularity.



☆ If the photon is inside the photon sphere ($r < 3MG$) ...it gets pulled right in ... → It cant withstand the Gravitation...

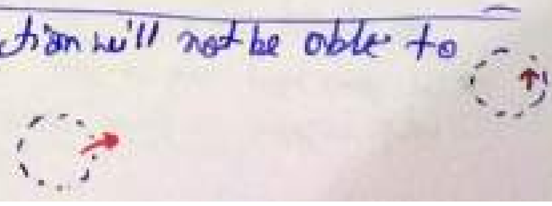
In particular:

→ If photon starts just slightly inward of the photon sphere ... it may orbit few times → but it will spiral right in to the horizon... and then eventually spiral into the singularity.



* A light ray which is moving in angular direction will not be able to escape from photon sphere.

* but light ray which is radially outward will escape.



* when can independently adjust angular momentum and radius for any general body... but it may or may ~~not~~ not orbit.

→ for circular orbits.. the angular momentum L & r are needed to be related.

→ but, for photon ... i.e. photon sphere $r = 3MG$ independent of the Angular Momentum.

* for photon the only circular orbit is at $r = 3MG$
(and it is unstable circular orbit)

* anything which is inside the photon sphere moving in angular direction (whether it got mass or does not have mass) will always wind up falling in.

→ there are no circular orbits for anything within photon sphere.

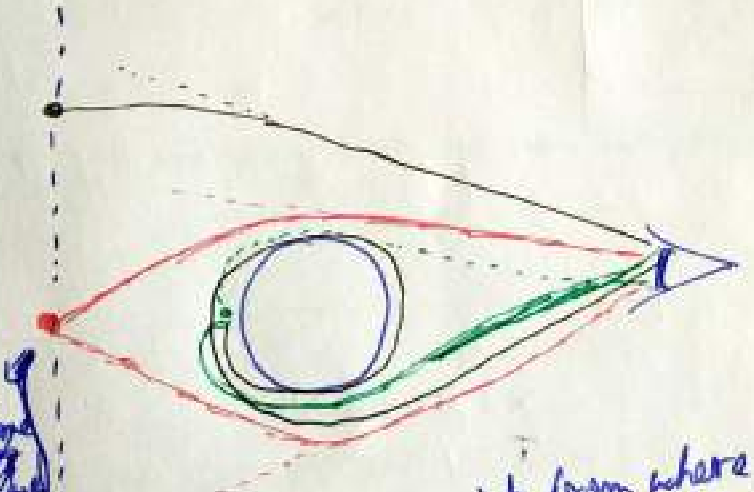
→ " " circular orbits for massive things outside the photon sphere

→ but for massless particles... only the photon sphere.

---

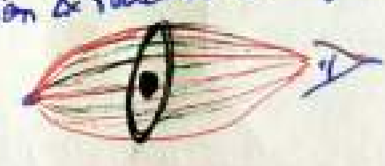
seeing by eye.

(we dont directly see Black Hole since it emits no light.



* we see very very complicated things when we view black whole... because light reaches our eye after following various curved trajectories.

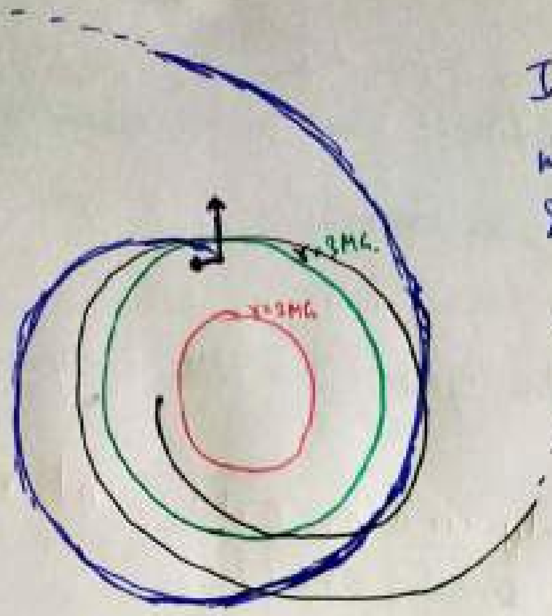
There are some points from where actually light goes in all 360° direction & reaches our eye.



↳ so actually what you see is ring around the Black Hole.

We see this background.

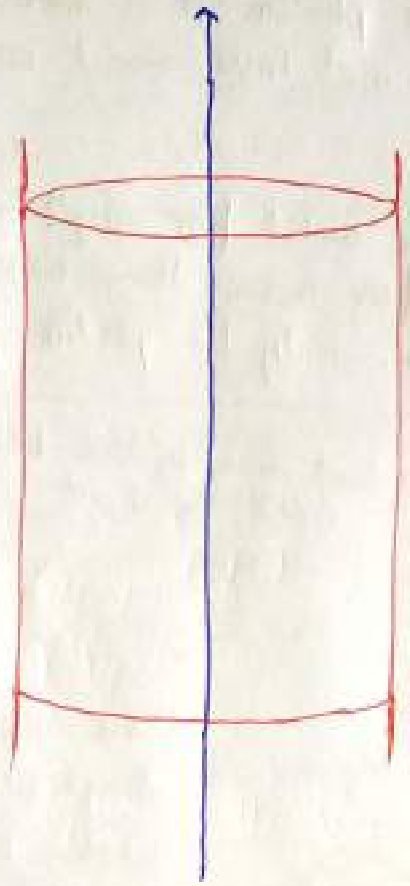
#



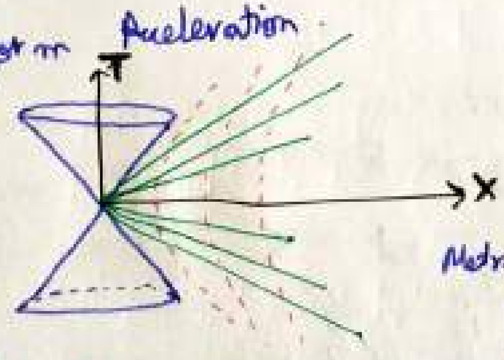
In particular a light ray which are sort of partly radial & partly angular (even inside photon sphere... but not inside Horizon ($r=2MG$))

→ will swing all ways around and come out and eventually escape.

#



★ Uniform Acceleration.



flat space

$$X = \rho \cdot \cosh \omega$$
$$T = \rho \cdot \sinh \omega$$
$$\Rightarrow X^2 - T^2 = \rho^2$$

Metric of ordinary flat space

$$d\tau^2 = \rho^2 d\omega^2 - d\rho^2$$

→ Analogue of metric of polar coordinates in ordinary geometry: $\rho^2 d\theta^2 + d\rho^2$

we are going to change coordinates little bit

$\rho^2 = 4\xi$ also $\omega = \frac{t}{2}$

now; rewrite the metric in these coordinates.

$2\rho\, d\rho = 4\, d\xi$
$\Rightarrow d\rho = \frac{2\, d\xi}{(\sqrt{4\xi})}$
$d\rho = \frac{d\xi}{\sqrt{\xi}}$

$$d\tau^2 = (4\xi)\left(\frac{dt^2}{4}\right) - \frac{d\xi^2}{\xi}$$

$$\Rightarrow \boxed{d\tau^2 = \xi\, dt^2 - \frac{d\xi^2}{\xi}}$$

here; we notice similarity with Schwartzchild metric. $\left(d\tau^2 = \mathcal{F} dt^2 - \frac{dr^2}{\mathcal{F}} - r^2 d\Omega^2\right)$

ξ changes sign when it becomes negative; i.e; around zero
(and negative ξ really does mean something....)

$\mathcal{F}$ changes sign around $r = 2GM$

so; this is a justified valid point.

* we want to focus on the place where $r = 2MG$.
i.e; we are going to study Swarzschild metric in the region where r is almost equal to $2MG$.
↳ just blow it up ... take a microscope ... zoom in .. ... & study the situation..

* now we are going to study near horizon, i.e; $r = 2MG$ .... so we set $r = 2MG$ in some (study in vicinity of $r = 2MG$)

so: $\frac{r - 2MG}{r}$

we approximate denominator to be approximately $2MG$.

we have to be little careful here; ... other-wise it will become zero
↳ but we know it not zero .. but just a very small number.

If we are interested in a particular black hole
↳ so; here for simplicity we can change units where Schwarzschild Radius is just equal to 1
(we cant do it for all black holes simultaneously ... but if we are interested in particular black hole we can always do that)

changing units where $2MG = 1$ ... makes the formula much simpler.

$$\boxed{d\tau^2 = (r-1)\,dt^2 - \frac{dr^2}{(r-1)}}$$

$r = 1$ Horizon

here we are not going to work out for radial terms ... let for simplicity there be no angular motion.

now; lets redefine $(r-1)$ & call it ξ.

$\therefore\ (r-1) := \xi$

$\Rightarrow$ we have changed coordinates so that it is measuring distance from the horizon.

$$\Rightarrow d\tau^2 = \xi\,dt^2 - \frac{dr^2}{\xi} \quad ; \quad r-1=\xi,\ dr = d\xi$$

$$\Rightarrow \boxed{d\tau^2 = \xi\,dt^2 - \frac{d\xi^2}{\xi}}$$

$\Rightarrow$ similar to flat space minkoski accelerated motion in hyperbolic coordinates

$\hookrightarrow$ This is telling us that in the vicinity of Horizon is in some mysterious way just flat space (or approximately flat space)

(because if by a coordinate transformation I can change the metric so that it looks like "$\xi dt^2 - \frac{d\xi^2}{\xi}$" $\Rightarrow$ it is saying that upto good approximation, the vicinity of Horizon is Locally flat)

$\hookrightarrow$ nothing very dramatic going on.
no large curvature
more or less flat space.

Why would be interesting in a coordinate system for gravitating object which is mimicking small acceleration?

~~And~~ I'm studying a gravitational or say gravitating object from say perspective of somebody standing still (being supported) $\therefore$ you effectively doing physics in uniformly accelerated reference frame. And what does physics feel like there, what it do there $\Rightarrow$ it does whatever the uniformly accelerated frame does.

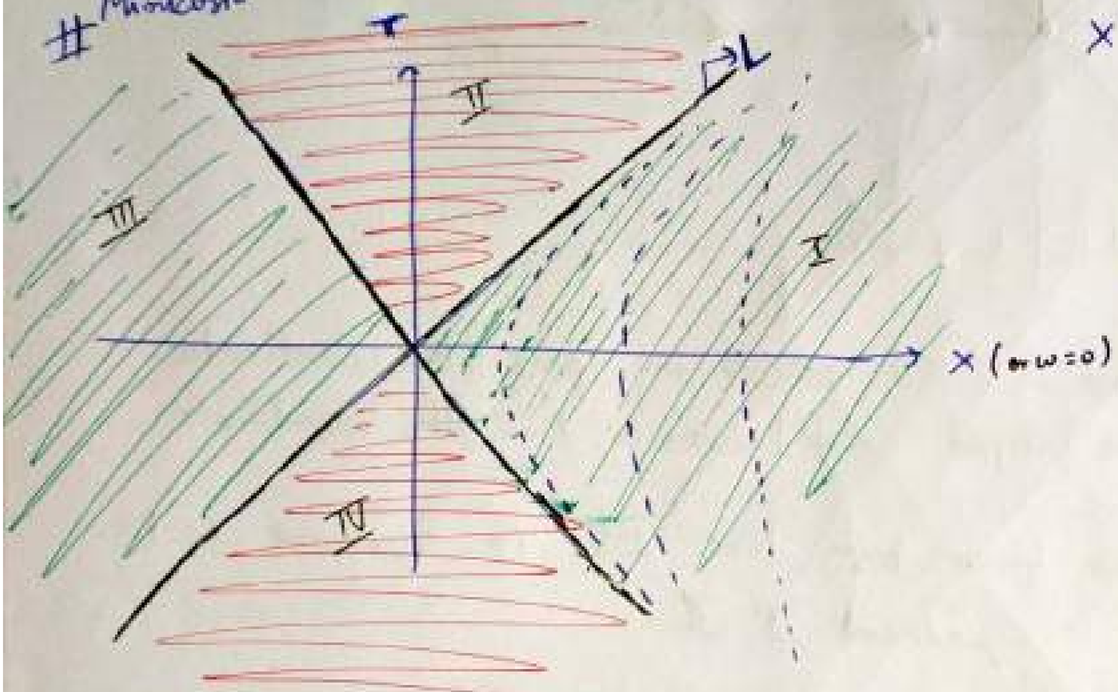
- We seeing that in the vicinity of Horizon of Black Hole is simply a uniformly accelerated coordinate system; but the space time itself is effectively flat.

Not much different from that.

$\xi = r - 1$ $\quad (\xi = r - 2MG)$

(ξ can change sign by going from outside to inside around the horizon)

Minkowski



$X^2 - T^2 = \rho^2$

$$\boxed{\frac{\rho^2}{4} = \xi}$$

↳ dont think that ξ cant be negative

→ think of $\rho^2 = X^2 - T^2$ & then it can be neative.

- Green shade ⇒ quadrant I & III
- Red shade ⇒ " II & IV

~~$\xi < 0$~~

* $\rho^2 > 0$ ⇒ hyperbola in I & III quadrant $(X^2 > T^2)$

* $\rho^2 < 0$ ⇒ hyperbola in II & IV quadrant. $(T^2 > X^2)$

∴ So: $\xi < 0$ ⇒ corresponds to point in Quadrant II & IV

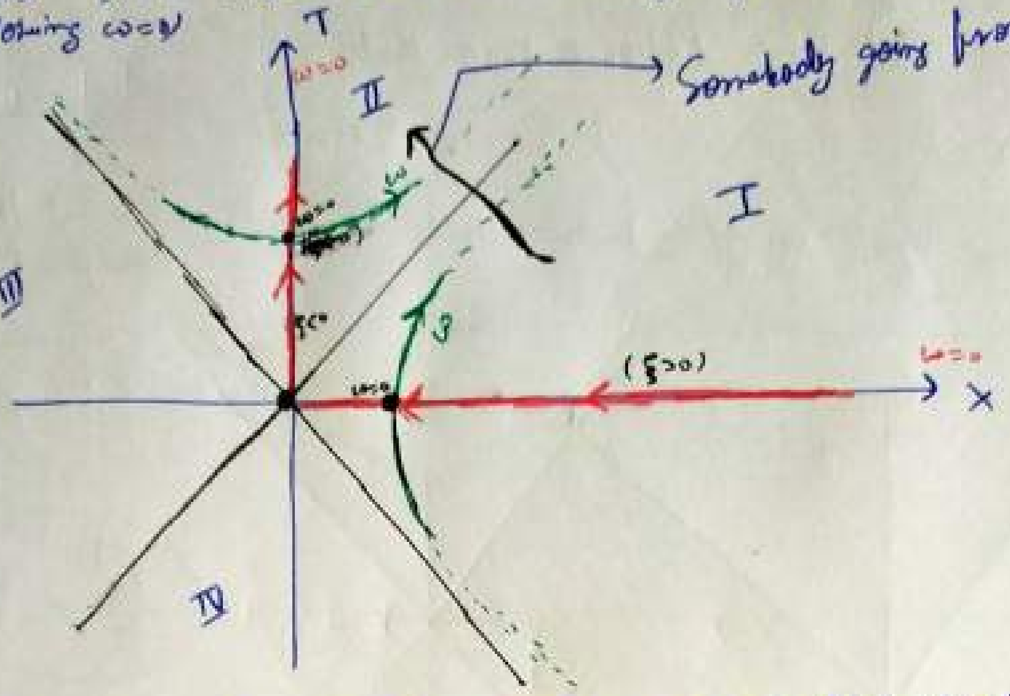
$\xi > 0$ ⇒ " " " " " I & III.

So in going from ξ positive to negative we are crossing the surface L (semi light cone at origin)

~~About~~ ~~Now~~, # Imagine now,

~~[illegible]~~ that we wanted follow all way in till we got to $\xi = 0$ ... and then where would go from there.
(following $\omega = 0$)

Somebody going from I to II.



what has happened is that ~~[illegible]~~ a space like coordinate namely ξ, has jumped and become a time like coordinate.

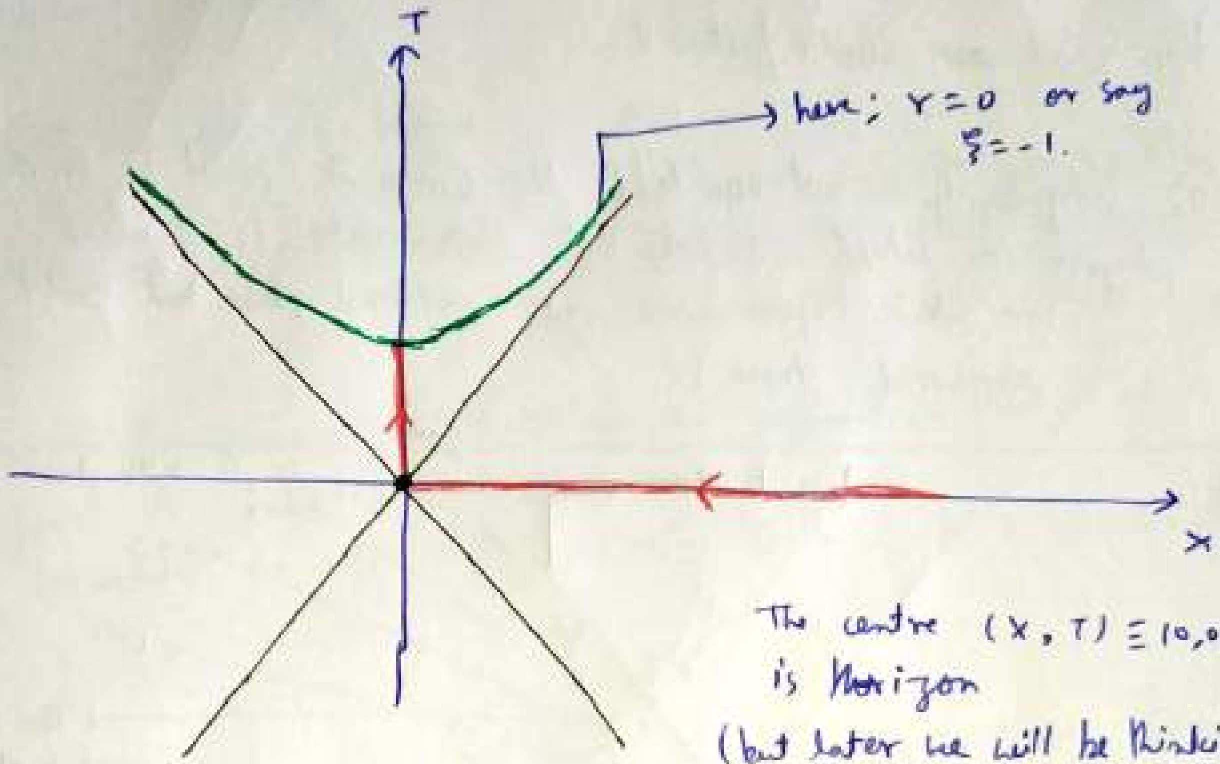
and ω goes from being time like in quadrant I to space like in quadrant ~~[illegible]~~ II.

Remember : ~~[illegible]~~ All this was just flat space ... there was nothing going on in space ... somebody who jumps from I to II sees nothing special ... there is nothing happening ... it is just peculiar coordinates that has some ~~[illegible]~~ funny behaviour as you go from one side to another.

$\Rightarrow$ The same is true for Schwarzschild Coordinates.

★ Same is true for Schwarzschild coordinates.

- when ξ changes sign you do exactly the same thing that you do in crossing the quadrant I to II in the minkowski flat space-time.
(i.e; crossing the horizon)



The centre (X, T) ≡ (0,0) is Horizon
(but later we will be thinking whole of the light cone at origin to be Horizon)

(i) • ξ is r - 1

(ii) • Varying r, until you get to r = 1, that is the horizon ~~to~~ (ξ = 0)

(iii) • And then continuing to vary r unit you move to r = 0
i.e; ξ = -1 (singularity)
eventually you get to r = 0

"Singularity at r = 0 is not a place ... it is time"

★ Singularity at r = 0 is not a single place ... many different places

... the surface r = 0 is not what you had usually thought it to be.

it was not a verticle ~~~~ line like this " | "

; but it was more horizontal, and more space-like than time-like.

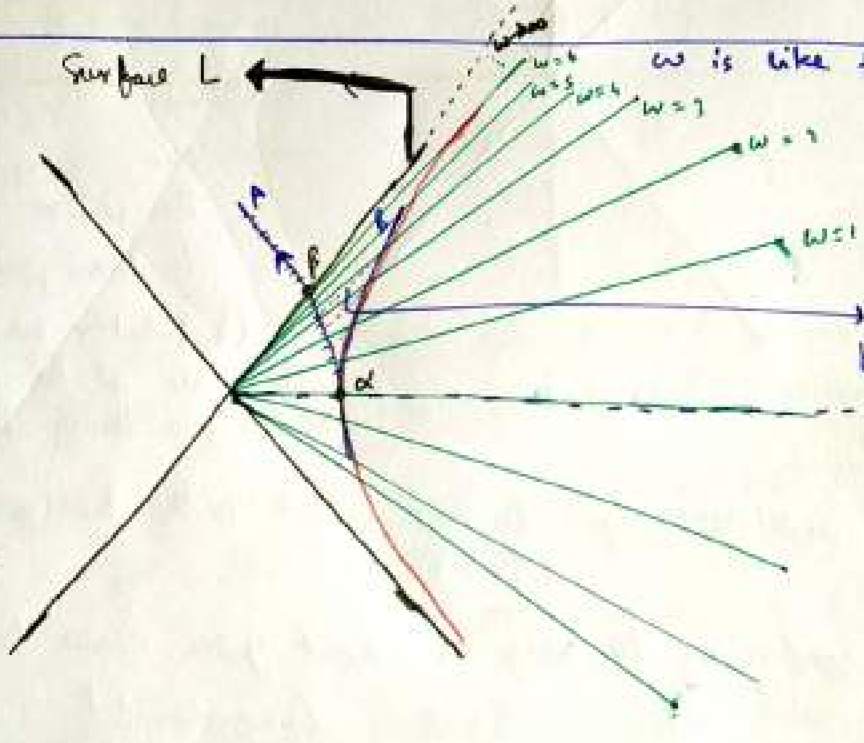
* ~~You cant avoid singularity when you get~~ in ~~behind the horizon~~
~~behind h~~

* You cant avoid hitting singularity when you get in behind the horizon because no matter what you do it is always ~~in~~ in your future ; and you will hit it.

* You cant ~~[redacted]~~ avoid future.

so; singularity is not avoidable the way it would be in newtonian physics in which centre of coordinate is a place.
(in which you can go around place but you can't go around time)

#



w is like time. ; $\omega = \frac{t}{2}$

→ This is B looking back at A ... B has to look back in time simply B because light takes some time to get to B.

let Alice go to (A) cross the Horizon
& let Bob be at (B) a constant distance
(so; B lies on hyperbola)

} represents time slices. time is simply the angular coordinate.

} position is constant along these hyperbolas.

Bob has coordinate which are just these polar type coordinates ω

Bob is staying outside the Black hole ... she is like I am now undergoing uniform acceleration ... she is standing on a floor above the horizon of Black hole.

- ω asymptotically goes to infinity when you get upto the light like direction there.

  (i.e; ⧖)

so; Bob says that infinitely late time corresponds to surface L there.

⇒ so; infact if Bob is looking at Alice... B looks back, & looks back... and it is not untill an infinite amount of her time has transpired that she actually sees Alice go through the horizon. ... actually B never sees A go through the horizon.

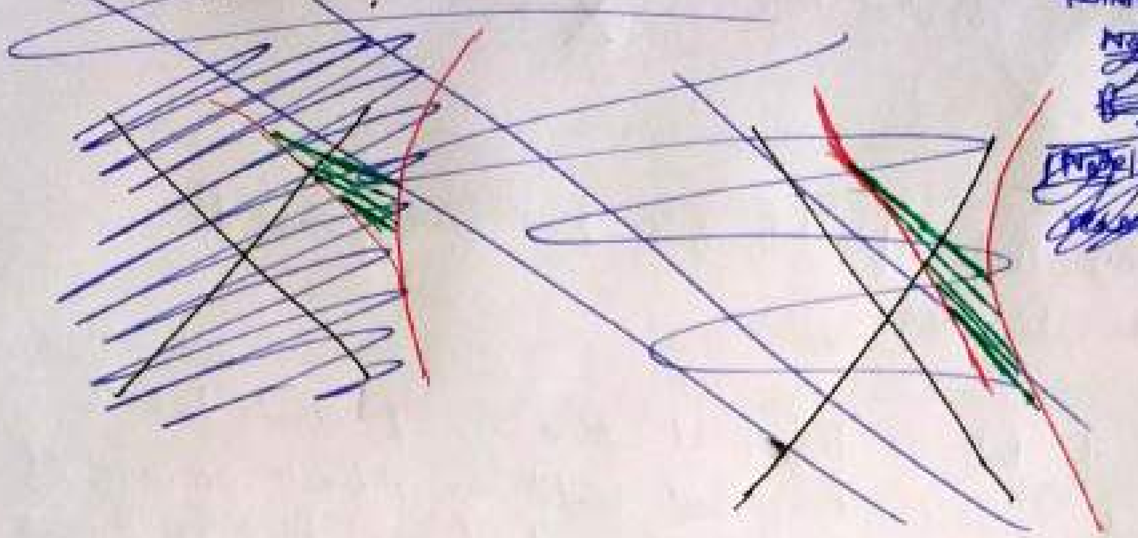
according to B reckoning, it takes an infinite amount of ω before A falls through the horizon.
(this translates to infinite amount of T in X, T coordinates.

- but, you can easily see that amount of proper time between α & β is finite. (just an ordinary trajectory)
  ↳ A is ticking off its time ... a finite amount of time in order to fall through the horizon. (i.e; to fall through the surface L ...)

note!! B obviously sees A slow down... because it takes infinite amount of time from Bobs perspective for Alice to go from α to β (i.e; to cross the horizon)
- each heart beat of Alice takes longer & longer from Bob's point of view. so; Bob see Alice slow down

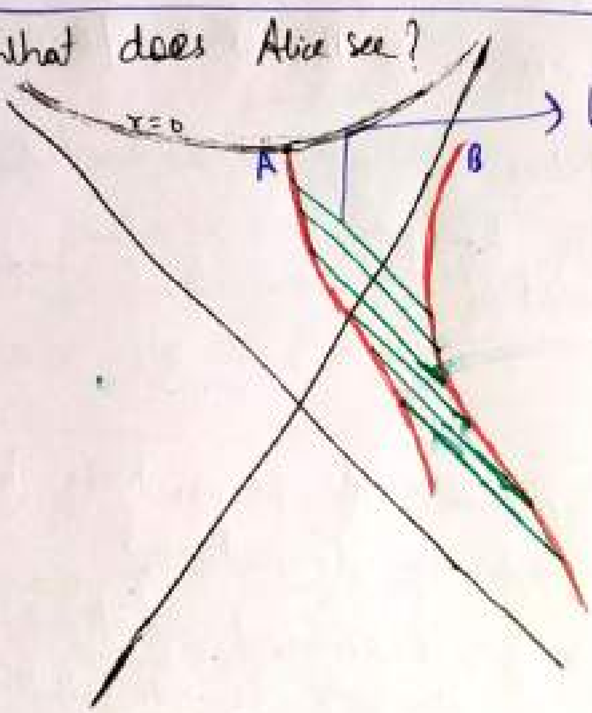
~~# what does Alice see?~~

~~What does Alice see ?~~



~~B's clock~~

☆ What does Alice see ?



→ light trajectories
... Through them Alices sees Bobs .

~~Alice sees slow~~

~~A also~~

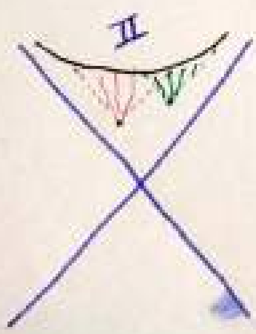
A nothing sees special .

⇒ Alice can also see Bob even after he passes through horizon (even though , B could never see A after horizon)

↳ but Alice has no trouble seeing Bob

☆ Alice just sees Bob accelerating away from her.

#



Anything that crosses behind L cannot escape .

- the coordinate we are using ... light moves at 45°

↝ b we can clearly see from any point inside light can't escape

↝ so anything slower than light will also not ~~stay~~ escape !

Note :
- Bobs clock record these hyperbolic angles as time.
- Alice " " proper time

↳ and it is because of this miss match which is governing all of peculiarities of motion near Black hole

---

☆ We are actually speaking about Classical Black holes
(we are not doing Quantum Mechanics now)

↳ Bobs see's (who is staying out) all of the matter that forms Black Hole not falling into the Horizon, but ~~sort of~~ sort of a kind of sedimentary motion with everything just asymptotically getting closer & closer to Horizon... ... flattening out & forming a kind of sedimentory layered structure which just asymptotically migrates closer & closer to the Horizon.

Note : light ray cannot come in & go out of photon sphere.

(although it is possible that light ray can ~~come out of~~ photon sphere & then again come back in... (and if it comes back in.. ... then sure it can't escape))

★ Write the metric in r' & t' (Same thing... just variable name changed)

so;
$$d\tau^2 = \left(1 - \frac{2MG}{r'}\right)dt'^2 - \frac{dr'^2}{\left(1 - \frac{2MG}{r'}\right)} - r'^2 d\Omega^2$$

let; $R_s := 2MG \Rightarrow d\tau^2 = \left(1 - \frac{R_s}{r'}\right)dt'^2 - \frac{dr'^2}{\left(1 - \frac{R_s}{r'}\right)} - r'^2 d\Omega^2$

now; let redefine r'; i.e; $r = \frac{r'}{R_s}$ (note that this r is dimentionless)

★ also redifine t'; [redacted]

we know in units $c=1$; the units of length & time are same

so; $t = \frac{t'}{R_s}$ (this t is also dimentionless)

" R_s is both a distance and a time "

↳ It is time in sense; that it is just the time it takes for a light ray to go the distance of one Swarzschild Radius.

now; lets rewrite the metric in these coordinates

$$d\tau^2 = \left(1 - \frac{1}{r}\right)dt^2 \cdot R_s^2 - \frac{dr^2 \cdot R_s^2}{(1 - 1/r)} - r^2 d\Omega^2 R_s^2$$

This containing all dimensions (dimensional analysis)

This is dimensionless

$$\Rightarrow d\tau^2 = R_s^2 \left[\left(1 - \frac{1}{r}\right)dt^2 - \frac{dr^2}{(1 - 1/r)} - r^2 d\Omega^2\right]$$

↳ notice; this is the metric which we would have been written down if $2MG = 1$

⇒ What we find out here is that basically the metric of all Schwarzschild Black Hole is the same except for an overall factor which is proportional to the square of R_s.

* for ~~[illegible]~~ most purposes in studying the geometry of the Black Hole we can set $R_s = 1$.

↳ Its like studying metric of sphere of different sizes. The metric of all spheres are same except the bigger & smaller sphere had to be multiplied by square of their radius. (other than that all spheres are same)

↳ in the same way, all Schwarzschild Black ~~[illegible]~~ Holes are same

---

$g_{\mu\nu}$ is dimensionless usually.

now: Γ contains derivation of g w.r.t. x & g inverse.

↳ so: unit of christofel symbol is inverse length.

$R \longrightarrow$ contains two kinds of terms

↳ curvature Tensor

$$\frac{\partial \Gamma}{\partial x} \quad \& \quad (\Gamma)^2$$

$\Rightarrow$ so: unit of R is inverse length squared.

$[g_{\mu\nu}] \Rightarrow$ dimensionless (no unit)

$$[\Gamma] = L^{-1}$$

$$[R] = L^{-2}$$

$\Rightarrow$ so: curvature at the ~~[illegible]~~ horizon : let say $R_{horizon}$ (at $r=1$)

so: $\underbrace{R_{horizon} \simeq \frac{1}{R_s^2}}$ (there will be a proportionality factor ...)

↳ we may solve it completely ... but we can avoid calculations by dimension analysis so; bigger the black hole ... bigger mass ⇒ bigger R_s

↳ implies smaller the curvature at the horizon.

so; At the Horizon,

The tidal forces associated with large black hole are less serior than that of small black hole.

\# set $R_s = 1$; $d\tau^2 = \left[\left(1-\frac{1}{r}\right)dt^2 - \frac{1}{(1-1/r)}dr^2 - r^2 d\Omega^2\right]$

Lets explore geometry of a unit black hole. (black hole for which you had set $2MG = 1$)

↳ proper distance between these points.

$$ds^2 = \frac{dr^2}{(1-1/r)}$$

↳ note; here we are actually displacing points without changing the time (not litterally moving in space time) ; just comparing two points at two different distances from black hole at the same time.

so; $dt = 0$

. furthermore ; we are not changing the angle ; just moving radially .. so $d\Omega = 0$

so: ds^2 reduces $+\frac{dr^2}{(1-1/r)}$ $\Rightarrow$ $ds = \frac{dr}{\sqrt{1-\frac{1}{r}}}$

$$ds = dr\sqrt{\frac{r}{r-1}}$$

let; call the proper distance from the horizon to be "ρ"

$$\rho = \int ds = \int_{r'=1}^{r'=r} dr' \sqrt{\frac{r'}{r'-1}}$$

$\Rightarrow \rho$ measures proper distance along radial direction.

$$\Rightarrow \underbrace{\rho = \int_1^r dr' \sqrt{\frac{r'}{r'-1}}}_{\rho(r)}$$

This is a function of r

This integral can be done ~~it is doable~~

let;

~~F(p)~~ ; we ~~let~~ will also redefine t

$$\omega := \frac{t}{2}$$

$$\frac{d\rho}{dr} = \sqrt{\frac{r}{r-1}} \Rightarrow \frac{d\rho^2}{dr^2} = \frac{r}{r-1}$$

$$\Rightarrow d\rho^2 = \frac{r}{r-1} d\rho^2$$

~~F(p) p² d~~ $d\omega^2$

~~r is known~~ ~~ρ is known a~~

* ρ is known as a function of r.

$\Rightarrow$ r can also be solved as a function of ρ

$\therefore$ let; express $\left(1-\frac{1}{r}\right)$ in the form $F(\rho)\cdot\rho^2$

ie; $F(\rho)\cdot\rho^2 := 4\left(1-\frac{1}{r}\right)$

some function of ρ times ρ squared.

So, metric becomes

$$d\tau^2 = F(\rho)\cdot\rho^2\cdot d\omega^2 - d\rho^2$$

(note; $\frac{dr^2}{(1-1/r)}$ $\Rightarrow$ is just $d\rho^2$ .. because ρ was defined to be proper distance (without varying time) so; $\boxed{d\tau^2 = F(\rho)\rho^2 d\omega^2 - d\rho^2 - (r(\rho))^2 d\Omega^2}$

Schwarzschild metric

$\Rightarrow$ in terms of ρ, horizon is $\rho = 0$.

you can find;

$$\lim_{\rho \to 0} \left(F(\rho)\rho^2 \right) = \left(4 \left(\frac{r-1}{r} \right) \right)_{\text{far away}} = 4$$

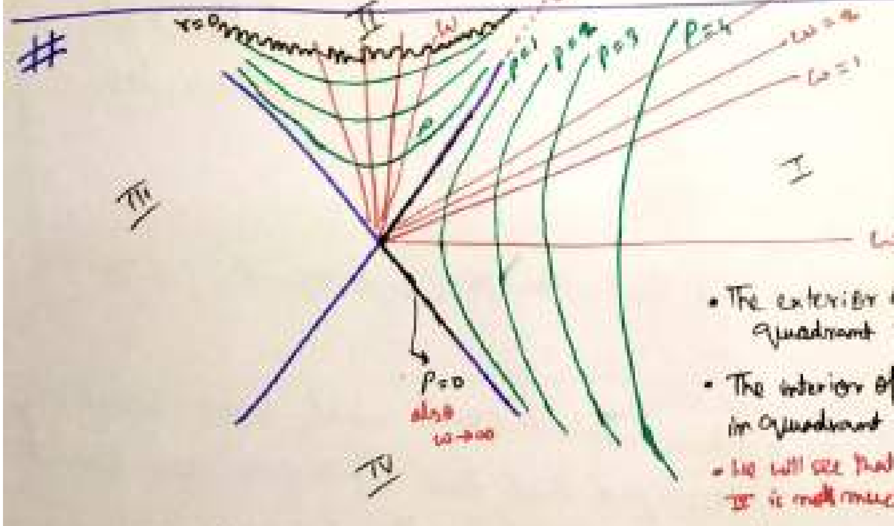
$$\lim_{\rho \to 0} F(\rho) = 1$$

$\Rightarrow$ so; close to horizon metric becomes
$$\underbrace{d\tau^2 = \rho^2 d\omega^2 - d\rho^2}$$
$\hookrightarrow$ and this is similar to exactly flat space in Hyperbolic Polar coordinates.

~~Approximation~~

$$\lim_{\rho \to 0} r(\rho) = 1 + \frac{\rho^2}{4} \approx 1$$

#



$\rho^2 d\omega^2 - \rho^2$

- The exterior of black hole is quadrant I. (w.r.t. Horizon)
- The interior of the Horizon is in quadrant II
- We will see that the quadrant III is not much of physical significance

so; near the horizon, the geometry is of the form flat space in ~~polar~~ hyperbolic polar coordinates.

Remember: $\omega = \frac{t}{2}$ $\Rightarrow$ So; we have discovered some coordinates where time is like Hyperbolic angle and $t \to \infty$ is just the light cone here (i.e; $\omega \to \infty$)

- in quadrant I; $r > 1$ & $\rho^2 > 0$
- In quadrant II; $r < 1$ & $\rho^2 < 0$

Note

as ρ gets smaller & smaller it tends to light cone

~~and eventually it becomes light cone at $\rho = 0$~~

$\rho = \varepsilon$ (small number)
and eventually it approaches 0
& the whole light cone becomes $\rho = 0$

horizon has $\rho = 0$

↪ so; Horizon is not just the point

↪ it is all along the light cone

$\rho = 0$

So; the point of no return is $\rho = 0$

↪ it is not actually a point .. but its the hole surface ie; The Horizon.

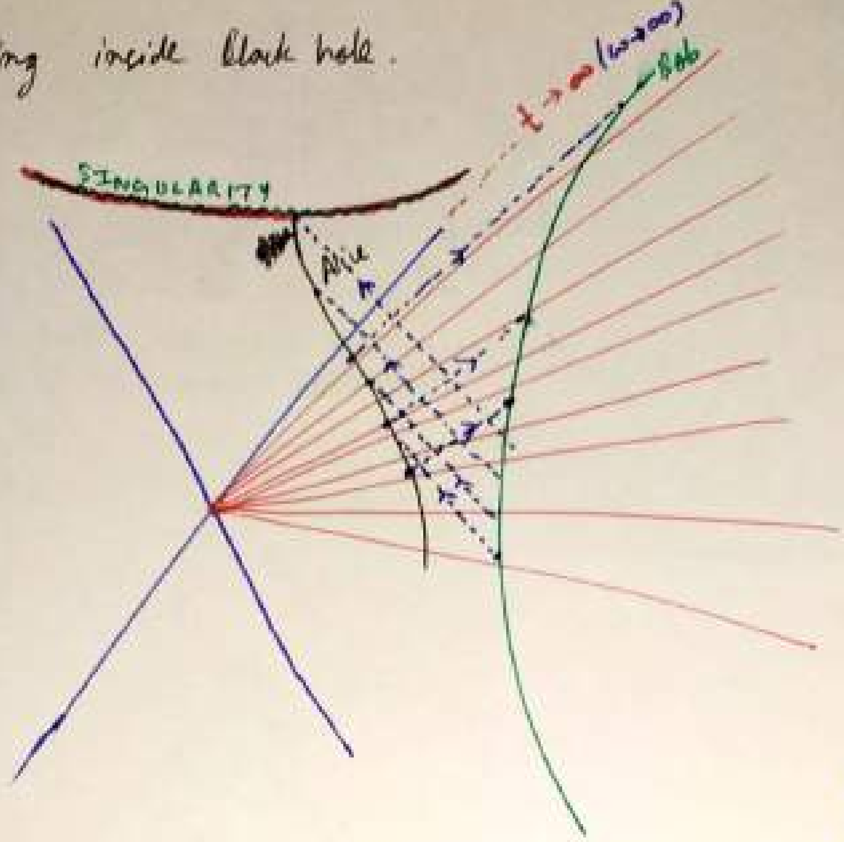
The coordinates we have constructed here are called Kruskal coordinates.

History

- at first people did not realized that Horizon was smooth, non-singular ordinary place.
- later in around 1950's Finklestein worked down what were essentially coordinates of above type ... and realized that the Horizon was point of no return (and not itself of significantly dangerous place)

* The term Black Hole was coined by Jhon Wheeler. (he coined the term when he wrote his first paper on Schwarzschild Metric) (earlier people called it a collapsed star)

falling inside black hole.



- Alice falling in

⇢ Bob watching alice

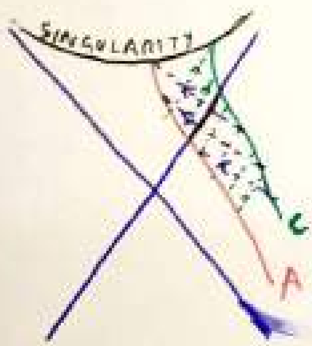
⇠ Alice watching Bob

↳ They see or watch when light from the thing which they are seeing reaches them.

They always see each other (visually) in past because speed of light is finite.

if Alice in her own frame of reference passes the horizon
↳ then she cant send signal to Bob to say that I have passed the horizon.

#

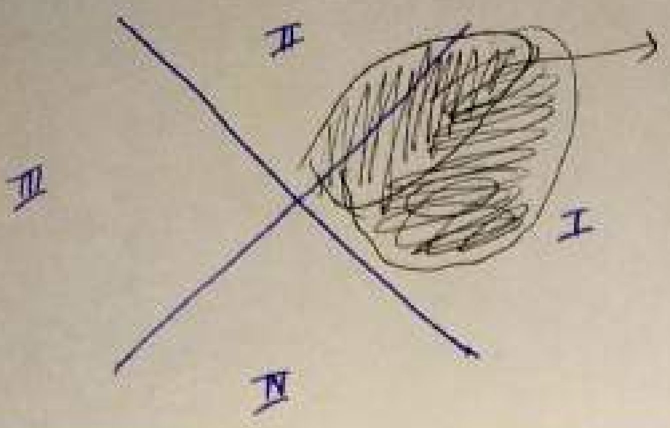


let charlie (C) follow Alice (A)

- When does C see A pass the horizon?
  Ans Exactly the same time C passes the horizon.

- But A will see C passing the horizon after it had herself pass the horizon (or might not see C passing horizon... if she reaches the singularity before then.

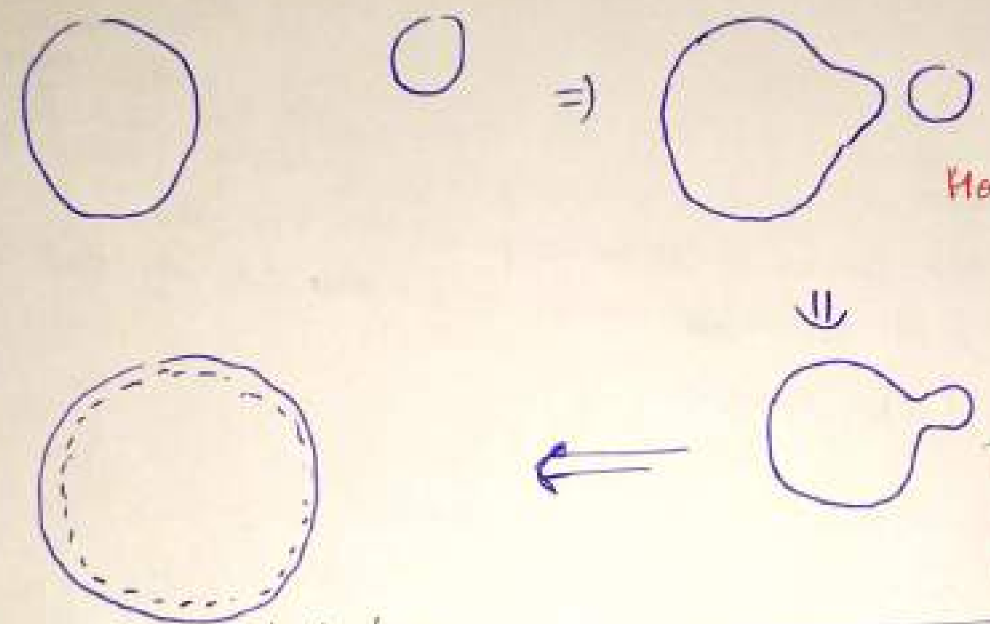
↳ remember when they are ~~seeing~~ seeing visually .. they see in past.



This region is pretty much flat space (i.e. being near horizon)

↳ but keep away from singularity. (there space is ~~..~~ not flat ... we have very high curvature ~~near~~ near singularity.)

Suppose you put some heavy mass



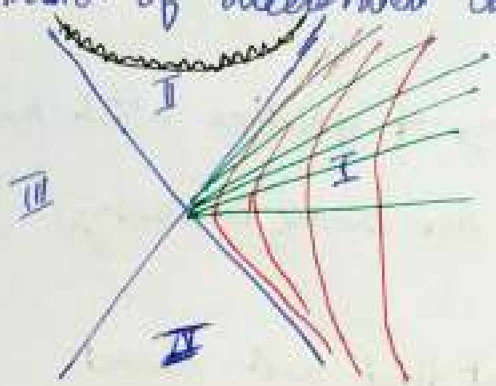
And because Black Holes have no hair, it will pull itself into sphere ... but the sphere will be slightly larger

(~~the~~ ~~the R_s will increases~~

↳ R_s will ~~....~~ increase.

$R_s = 2MG$

* The Horizon is a kind of Hyperbolic polar coordinate point similar to the behavior of accelerated coordinate frames.

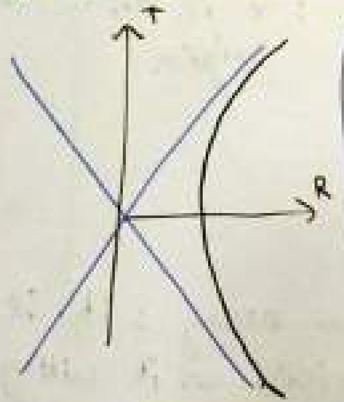


Quadrant III & IV dont have any ~~phy~~ physical significance for a real black hole.

* Outside a black hole somebody is experiencing a sense of acceleration ; just as the way we are experiencing sense of acceleration while standing on surface of earth (actually we have got used, so we dont ~~feel it~~ notice it, the fact that we are actually accelerating)

• but if somebody who have been outer space too long, in free fall ... and lost the sense ~~of~~ of what a gravitational field is like ... → and then you make ~~them~~ him stand on surface of the earth, he would be experiencing a sense of upward ~~or~~ acceleration.

⇒ In fact in the ~~diagram~~ diagram, somebody at rest does appear to be accelerating away from the horizon.



* The outside of black hole is quadrant I
* The inside " " " " " II.

outside - inside w.r.t. Horizon.

(i.e. $R_S = \frac{2MG}{c^2}$)

Note!! The above diagram representing the space-time in infinite
↳ to draw whole picture we need infinite page. (They are extending in ~~all~~ all directions)

⇒ We want some way to be able to draw the entire space-time on a finite piece of paper.

★ lets start with flat space-time

∴ lets redraw it or re-coordinatize it so that we do a coordinate transformation which pulls the whole thing into some finite region in paper.

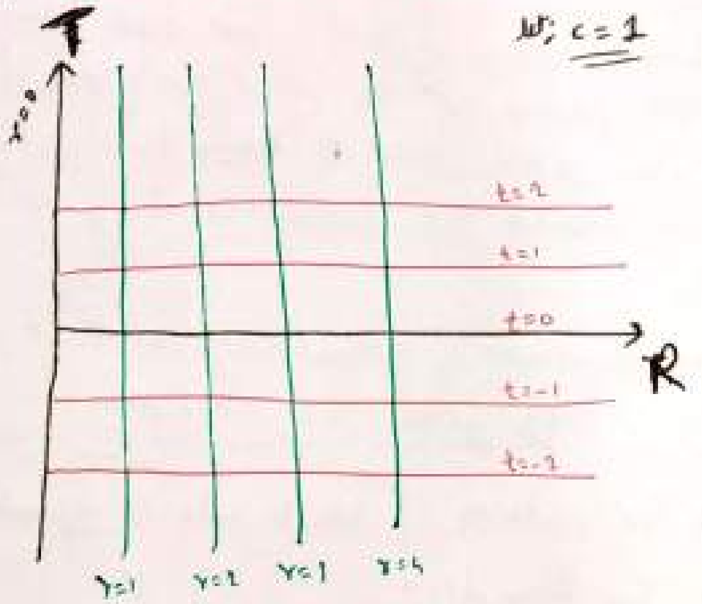
↳ This is useful incidently for geometry of spacetime which has Rotational symmetry.

Rotational Symmetries means that they are same in every direction.

∴ when system has Rotational symmetry; we dont think much about angular direction ... but we only think how things vary as we move away from center; i.e; the radial direction.

★ We think of ordinary flat space time; as having a time axis; and a radial direction (r is never negative)

let; $c = 1$

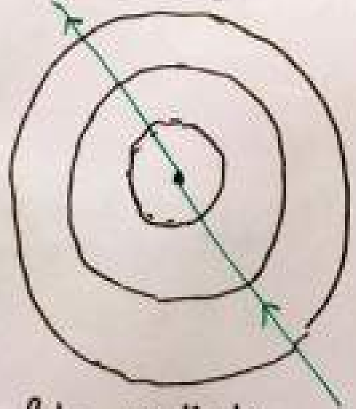


so; the entire space-time is to the right of the time axis (only because being to the left of time axis would mean negative r)

; r is distance from origin.

★ light Ray in polar coordinates.

(i)

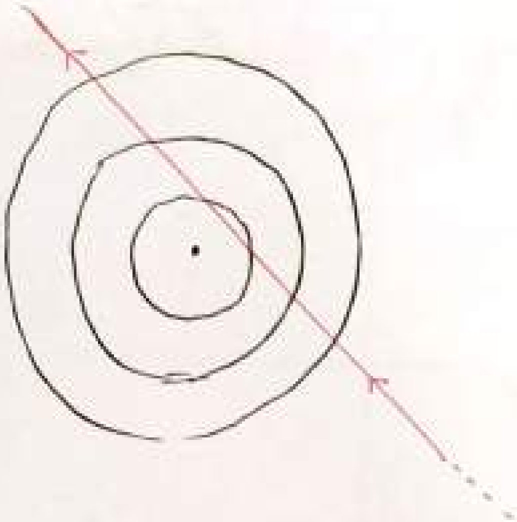


Polar coordinate

light ray comming from very far away; if it happens to be aimed right at the origin ⇒ it will go to the origin and then go back out.

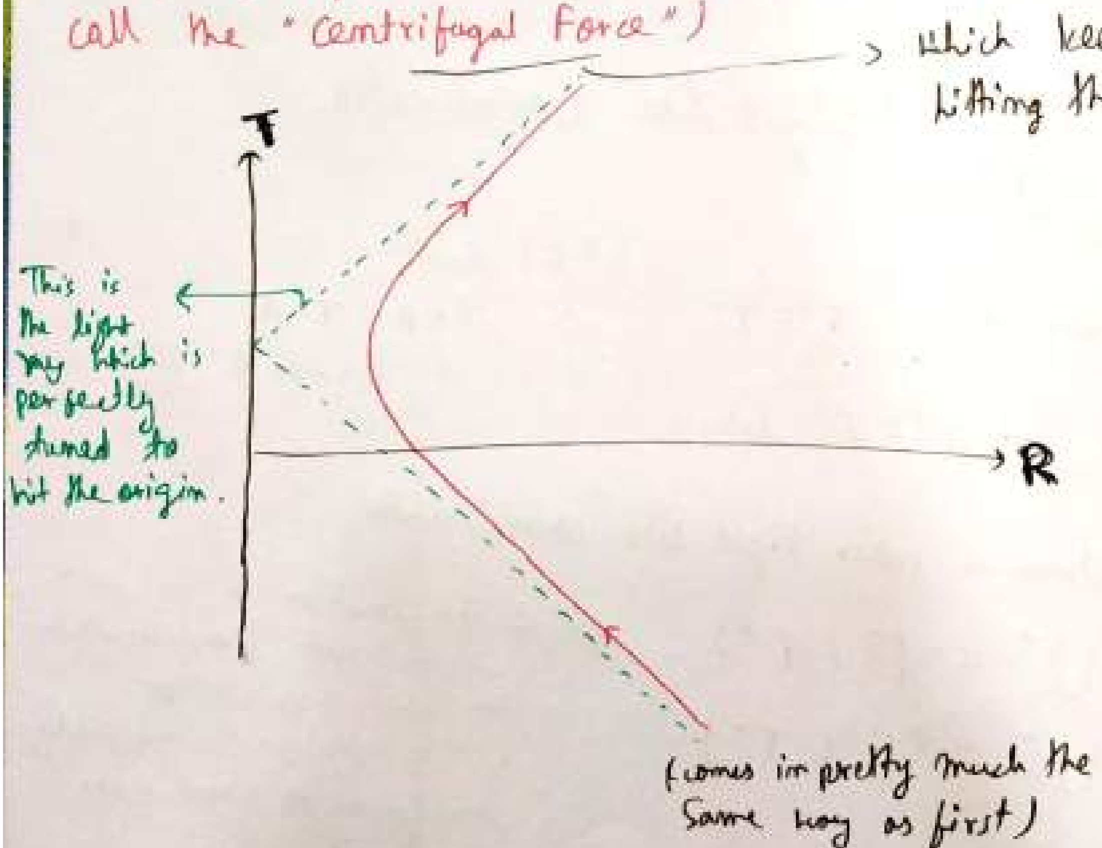
light come in at 45° from past ↖ & then bounce off at 45° ↗ in future in (t, r) coordinates.

(iii) Light ray which does not pass through the origin



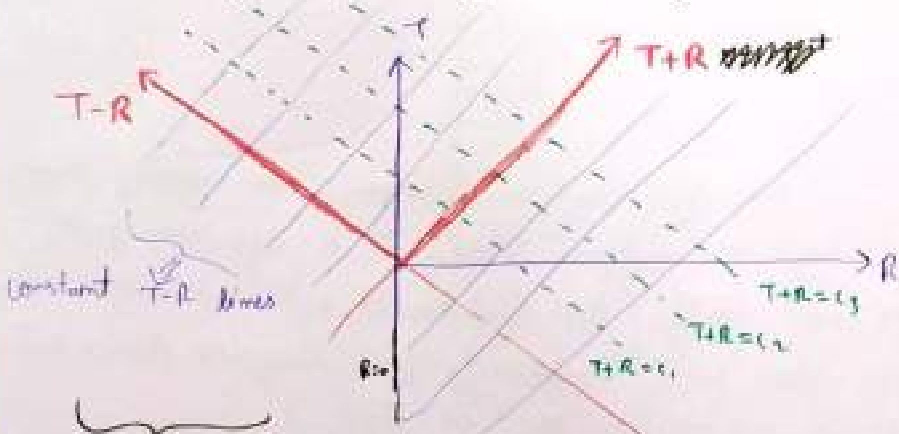
(so: light starts out comming in pretty much the same way as before (when it is very very far away you cant really tell whether it will hit origin or not unless you have extremely fine angular precision) : but it never gets to the origin)

(↳) In fact it almost appears to be repelled away from the origin and then goes back out.
(This repulsion is not a real repulsion; but it is what we call the "Centrifugal Force") → which keeps the light ray hitting the origin.



Now; we will try to deform the above diagram in various way so that the whole space can come in a finite region; but we will try to maintain that in that diagram light always moves at 45°. (because this is useful to us)

★ we introduce axes along 45° angles.



; $c_3 > c_2 > c_1$

by this; of course we dont mean to imply that there is anything ~~tef~~ behind left of T axis (ie; behind r=0)

notations
$$T+R := T^{+}$$
$$T-R := T^{-}$$
} Light like Coordinates.

note ; $R=0$; ie; T axis $\Rightarrow$ $T^{+}=T^{-}$; $T+R = T-R$

$\Rightarrow R = -R \Rightarrow \underline{\underline{R=0}}$

so; $R=0$ line is just $T^{+}=T^{-}$ line.

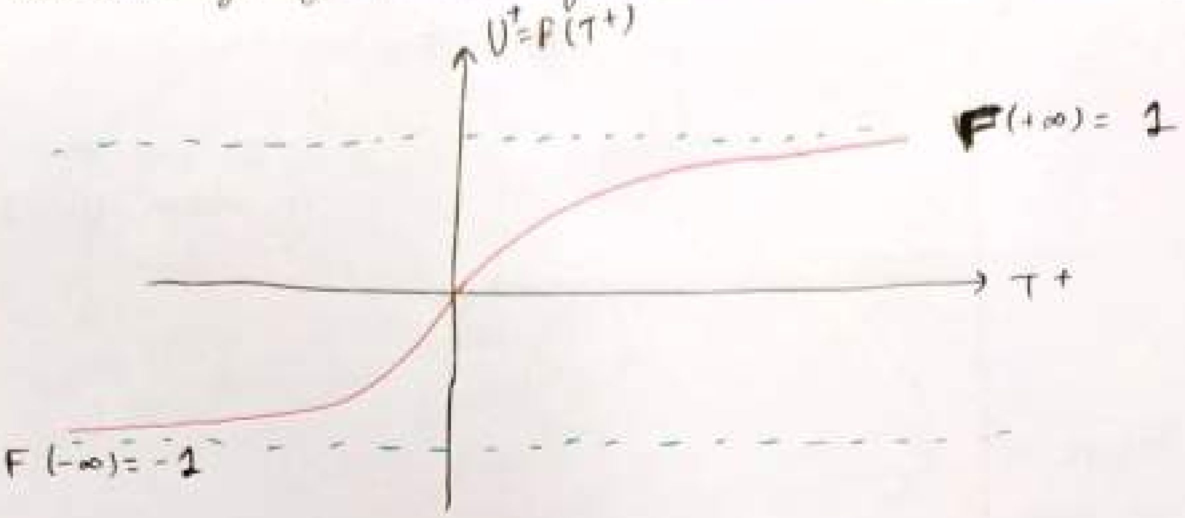
★ we will ~~now~~ introduce new light like coordinates:

$$U^{+} = F(T^{+})$$
$$U^{-} = F(T^{-})$$

; The function F is just a coordinate transformation in which ~~[illegible]~~ T^{+} & T^{-} just independently transform to some new coordinate

What kind of function we are going to use?

We are going to use function which looks as the following.



There are many functions which behave roughly the above way.

ex hyperbolic tangent (this is the one which is usually used)
(but there is nothing special about it)

So; we will use Tanh

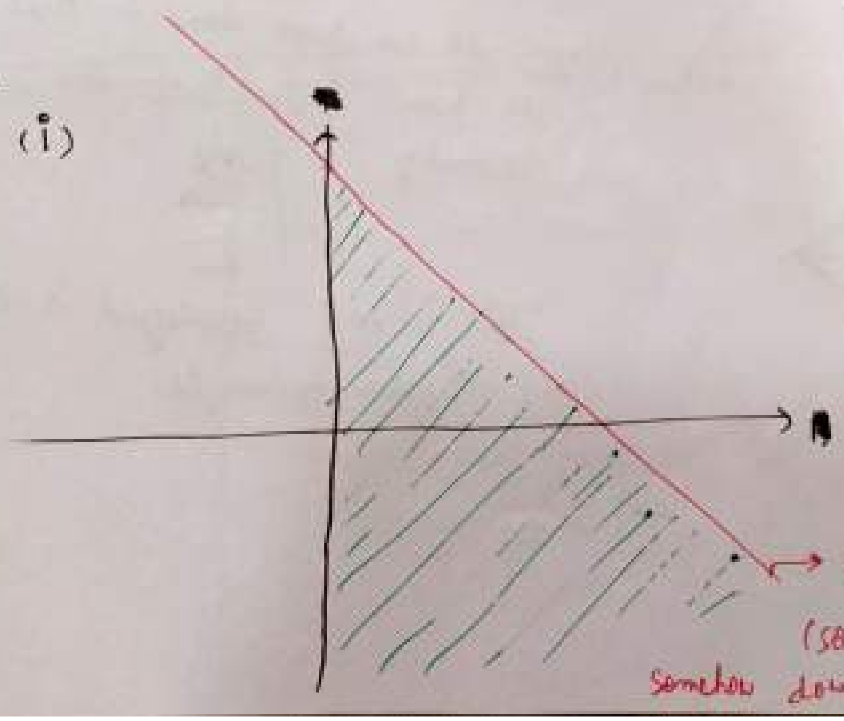
~~So $F(T^+) = \tanh$~~ So; we let: F to be

$$F(T^+) = \tanh(T^+)$$

So; $U^\pm = \tanh(T^\pm)$

; as T goes from $-\infty$ to $+\infty$
F(T) goes from -1 to $+1$
(F(T) is strictly increasing function)
; but it never gets below -1 and above $+1$.

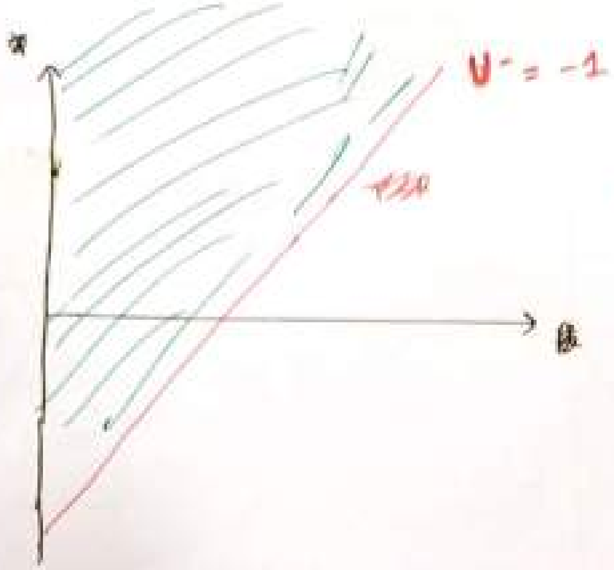
now
(i)



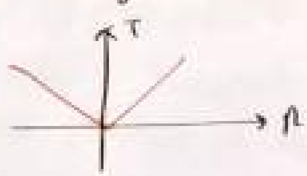
(so all of space time is mapped somehow down the line $U^+ = 1$)

(ii) U^- .. as we go down in Time ; ... U^- never gets smaller than -1 (ie; $T \Rightarrow$ ie; T^-)

so:



☆ what about the verticle line ; The original vertical line in

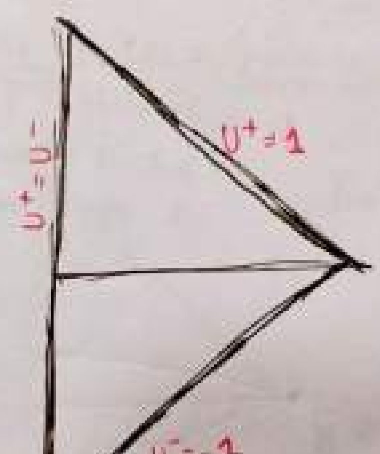


was $T^+ = T^-$

→ so ; here : $U^+ = U^-$

so: ↑ happens to be $U^+ = U^-$ & looks like verticle line.

so;



here ; all we have done is that ; we have squeezed the geometry " ┘→ " in this way ; and squeezed it onto a finite triangle.

now; what ~~it~~ is the trajectory of light ray.

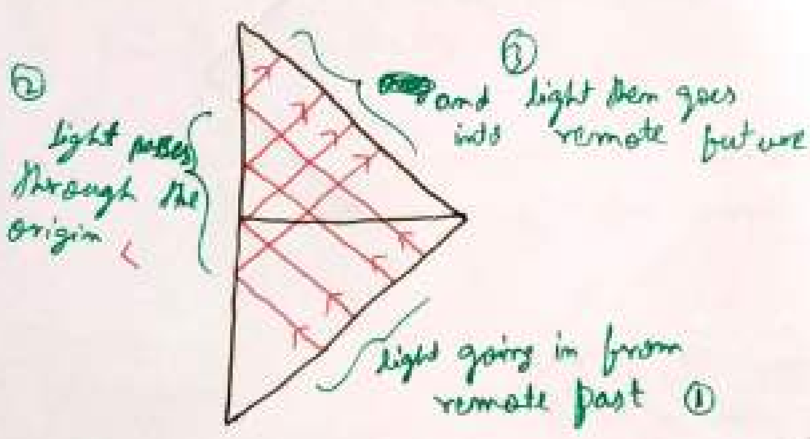
↳ in terms of T^+ & T^- ; it was that

- outgoing light ray is at constant value of T^-
- ingoing " " " " " " " T^+

implies U^- is fixed along this trajectory

implies U^+ is fixed along this trajectory.

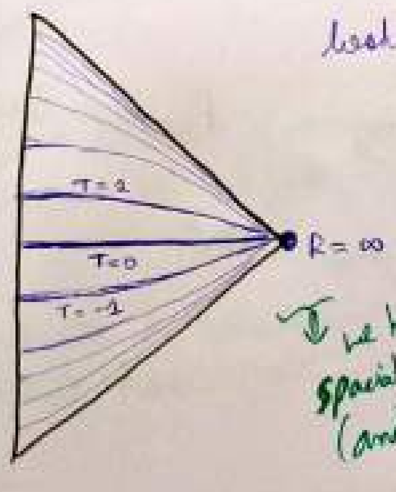
The upshot is that: if you make a coordinate transformation this way; it is still true that radial light ray moves on straight 45° angles.



So; we succeeded in taking all of spacetime and mapping it to a triangle that we can draw the whole thing on a finite piece of paper.

★ now; try to put in some more coordinates. ; plot various $T = c$ ... (constant T)

looking at fixed Time

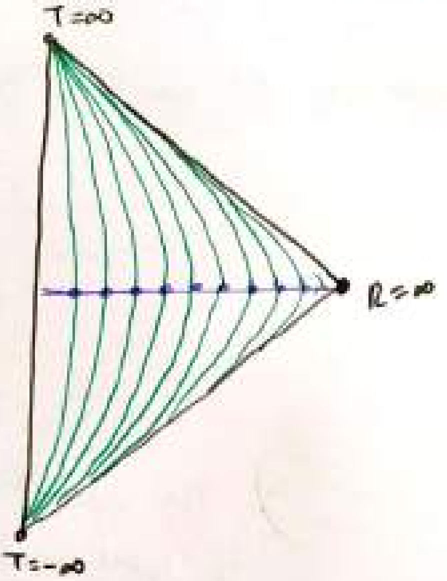


↳ we have squeezed all of spacial infinity into a point (and it is called Space like Infinity)

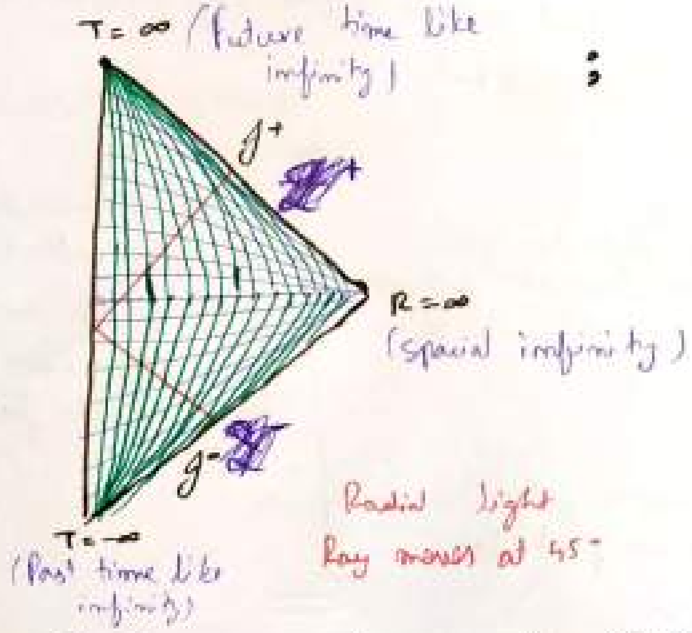
Scanned by CamScanner

* plot various $R = c$

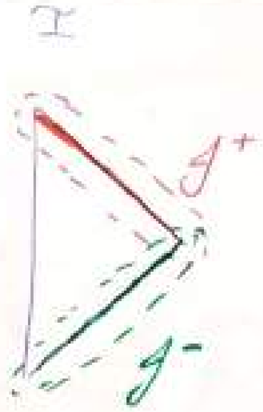
looking at fixed Spacial position.



so: we have



;



* "This process of squeezing everything onto a finite piece of paper without changing the speed of light is called "Compactification""

↳ it is also called making a Penrose Diagram.

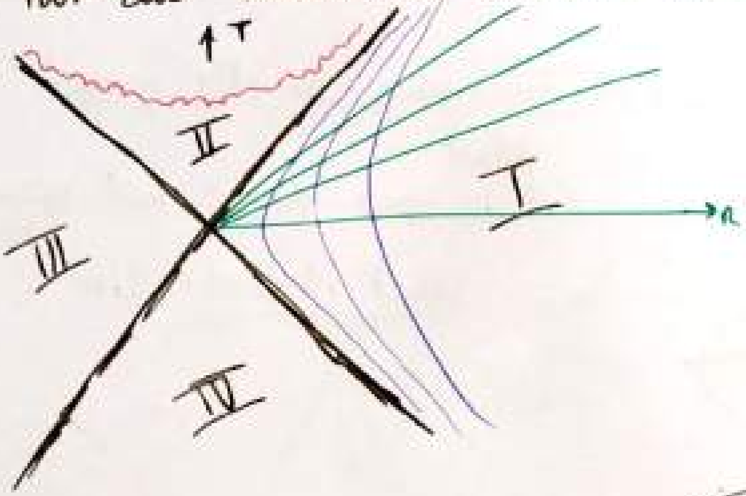
Penrose Diagram of flat space-time

it is also called Carter - Penrose diagram

script I − ; ie: $\mathcal{I}^-$ (or) scribe −
script I + ; ie: $\mathcal{I}^+$ (or) scribe +

: $\mathcal{I}^+$ is called future light like infinity : the place where light rays end
: $\mathcal{I}^-$ " " past " " " : " " " " begin

★ How does Schwarzschild Black Hole look like.



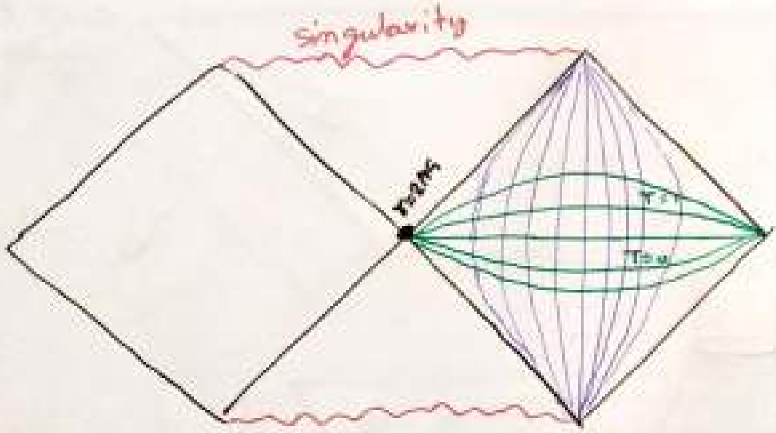
introduce again R, T

.... the Kruskal coordinates

↳ Introduce again light like coordinates T^+ & T^-

↳ do exactly the same operation & we get

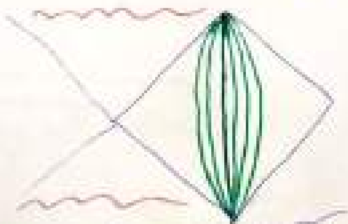
Drawing Penrose diagram: obtained by compactifying Kruskal coordinates in Black Hole geometry.



r=2M

I is exterior of black hole.

II is interior of blackhole.

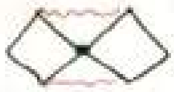


hovering close to horizon

hovering away from horizon

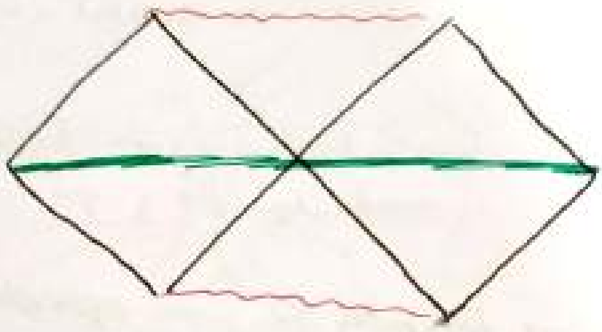
★ singularities are finite time away.
(it takes a finite amount of proper time to fall from horizon to singularity)

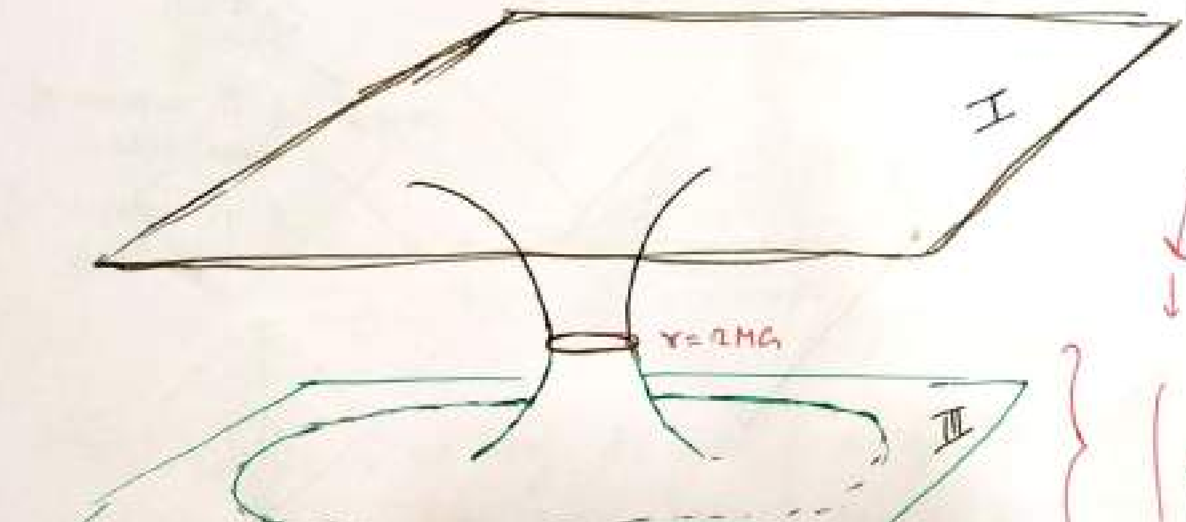
note: quadrant III & quadrant IV have no real meaning for a real physical black-hole.

What kind of geomentry is described by this full Extended Kruskal diagram ; i.e. "  " ?

☆ It seems to have two exterior regions ; I & III ; connected together at the horizon.

☆ We take a fixed slice ; The space on the green sheet
(it is space now ; not space-time)
What it looks like ?





very big region of space.

contracting

and starts expanding out again.

(represents quadrant III ...)

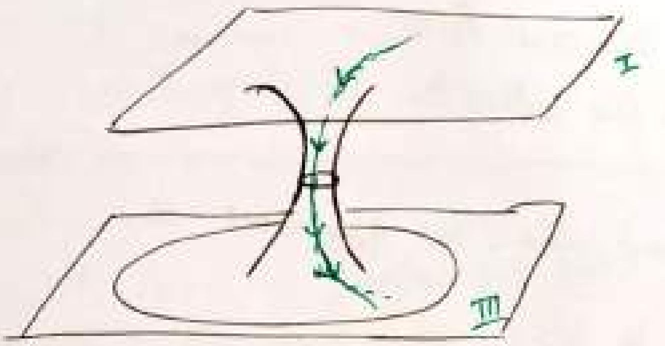
it looks like you can pass through from one side to the other by going through what people call a Wormhole ; also called

~~Einstein - Rosen bridge~~ Einstein - Rosen ~~Bridge~~ bridge

Einstein - Rosen bridge connects two ; what appeared to be external regions to the black whole which gets bigger and bigger as you move away from it.

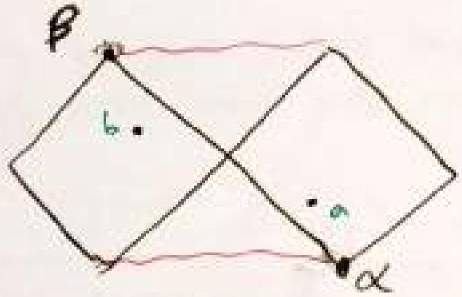
↳ and it looks like Black-Hole is connecting to two universes, i.e; two asymptotic regions.

∴ You might think you could pass through.



: but:
you can't pass.

: lets think somebody who starts on I ; and wants to pass to III.



: if they want to do that
i.e; to go from a to b
↳ its trajectory has to exceed speed of light

↳ so there is no way to get from quadrant I to quadrant III even if you start very very long in past (α) ; and you are willing to go way up in future (β)
↳ you will still have to exceed speed of light.

so; The diagram is little bit misleading : its everything at an instant of time ; but there is simply is not really time to go from one side to the other.

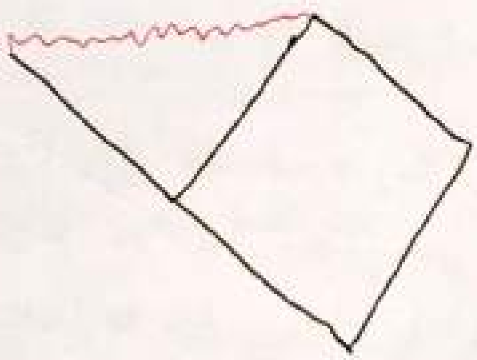
And so: Einstein Rosen - Bridge is not really a bridge.

; other way to think about it is that:
It opens up and closes again before anything can pass through it.

These kind of wormholes: which you cannot get through: are called non - traversable wormhole.

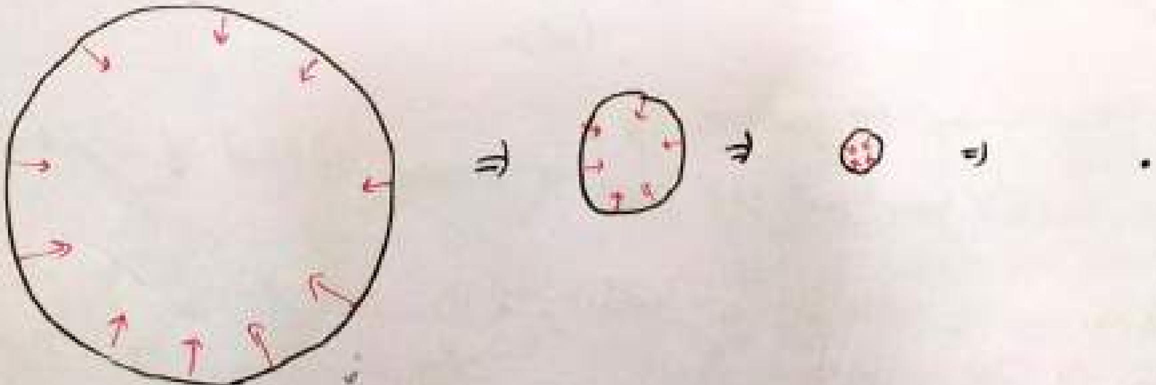
meaning to say you can't traverse them.
"You cant go from one side to the other"

∴ There is no real meaning for left hand side ; ie: quadrant III.
∴ there is no place to go.



Creating a black hole in infinite space by having some infalling matter..... (here we will do with special kind of matter...)

* starting with no black hole..



a thin shell of incoming radiation.

Radiation carries energy and momentum

↳ The shell is comming in at the speed of light

↳ At some point as we will see, the shell gets close enough together that there's so much energy in a small region that a Black Hole forms. (simplest version of Black Hole formation)

Newtons Theorem (Shell Theorem)

If you have mass distribution which forms a shell for which there is nothing in the interior; then there is no gravitational field in the interior. The interior is completely without gravitational field.

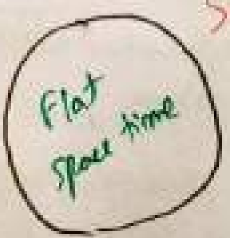
↳ There is gravitational field (Newtonian) outside which is identitical to gravitation field of a point mass (same amount) at the centre. But inside there is no gravitational field.

(it is even true if as the shell moves)

↳ as the shell moves in; more & more space will look like the field of point mass; and less & less will look like empty space.

⇒ This theorem is also true in General Relativity.

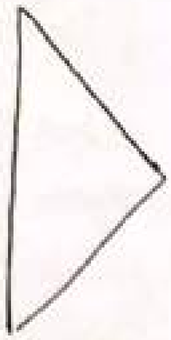
Statement: If you have shell of mass of any kind; then interior region is just flat space time (just as if there was no source). The exterior region looks like Schwarzschild solution.



Schwarzschild

so; If you have static non-moving shell; what you would do to construct the actual solution is you would sort of ~~paste to~~ paste together interior (which was flat) with an exterior which is a Schwarzschild.

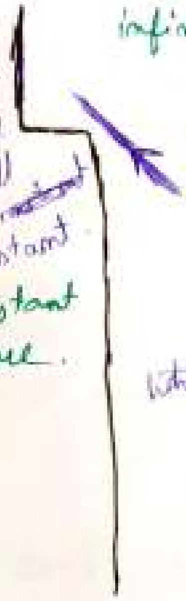
so;



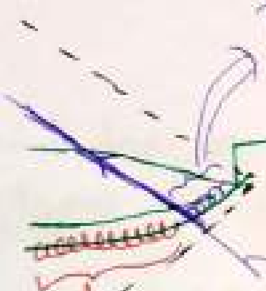
Flat space-time
with nothing in spacetime.

add the
idea of
incoming
shell of
radiation.
(as a pulse of
incomming photon)
A pulse ~~dot~~ distributed
on a nice-sphere of
incoming photons.

... and here they come from light like
infinity : $\mathcal{J}^-$.



(pulse of incoming photon)



This is the
part of space time
outside the shell
at a given instant

line of constant
time in space.

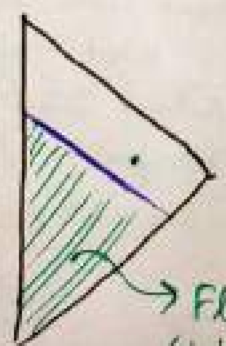
incomming
shell of
photons

This is the
part of space time
inside the
shell at a given
time.

~~where~~ where is the interior &
exterior of shell at any
given instant of time.

Shell theorem says; everything below incoming photon
trajectory is flat space time. (in other words; it is
correctly represented by the
space time we draw here "▷"
which was representation of
flat space time)

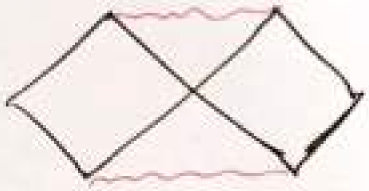
(flat penrose diagram is
correct representation of what is
going inside.

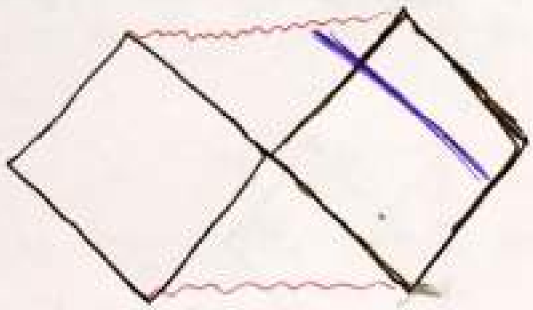


Flat space.
(interior region)

But outside region is wrong here... this is not correct representation of geometry of the infalling shell out beyond the shell.

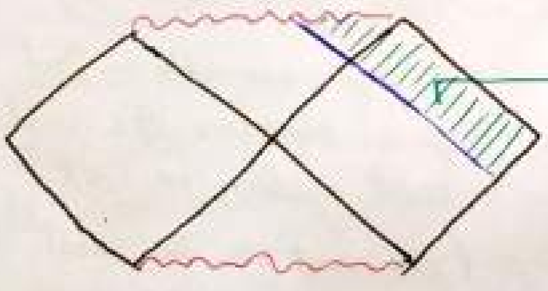
The correct representation is a Schwarschild Black Hole ; ie:





But which part correctly represents the exterior?
(we will only be taking the exterior piece from here
→ because we ~~...~~ know that interior is just flat space ... not a swarschild ~~...~~ solution)

... The exterior is the following shaded region.



→ Exterior region.

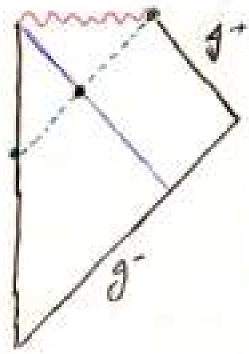
Now; how we combine this.
→ we through the wrong part in each piece
... and we are left with  and 
→ and paste the correct part and join along the infalling shell line.

Joining and redrawing we get ...

(the infalling shell (ie; the shell) is the thing they have in common)

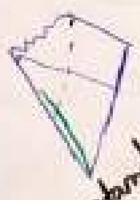
↪ so we paste them along the shell.

This is geometry of a black hole which is formed by an infalling shell.



Penrose diagram for formation of Black Hole

Not only you dont feel anything drastic... you dont feel anything ∴ nothing comming at you ∴ Nevertheless you are doomed. (you cannot get out)



is the horizon.

↪ even if you are inside the shell .. you are doomed

... you are still in flat space time → the shell has not gotten in yet (you cant see the shell if you look backward .. even on your light cone)

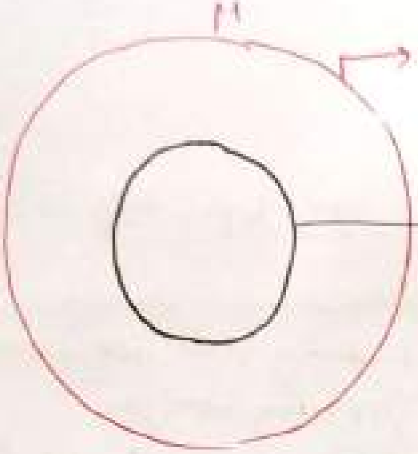
↪ The shell is coming in at you at the speed of light

Yet if you were standing over there; ie; you were somebody that just falling in there " ... just in free fall there

∴ you are already behind the horizon even though the shell has not even gotten in yet

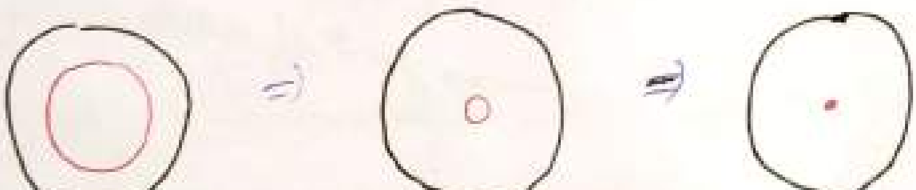
light has not gotten to you yet (you dont know the shell is comming)

This is the point where the shell crosses ~~the~~ its own Schwarschild Radius.



... as long as the mass is outside ... we can say mass is outside (hahaha ☺)

↪ but once the mass crosses the Schwarschild Radius: it is trapped it cannot get out ... it has formed Black hole.



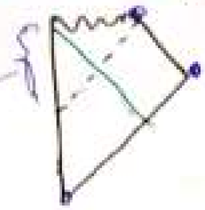
The defination of the horizon is the place which seperates those people who cant get out from those who can.
(it is defined even before shell crosses the ~~the~~ Schwarschild Radius)

... Horizon is something very general here.

→ here; the Horizon has materialized before the shell has actually gotten into ~~the~~ its own Schwarschild Radius.

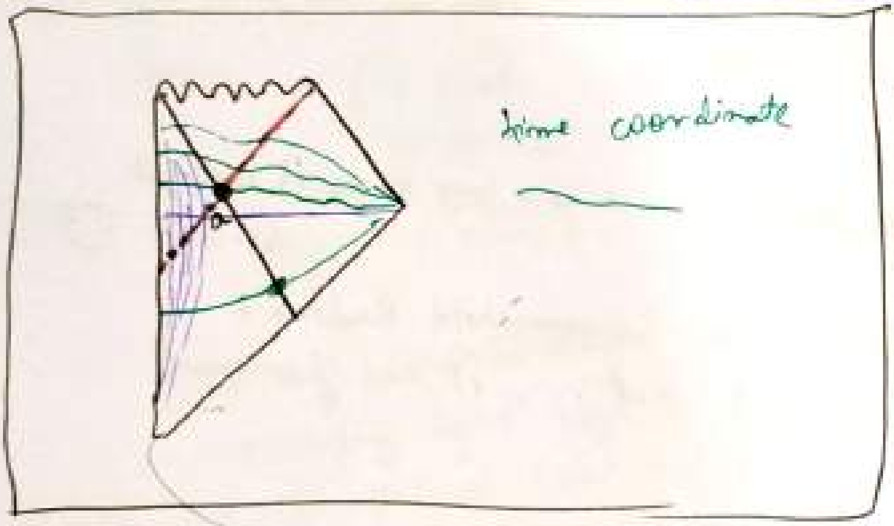
The defination of horizon is that it is simply the point of no return.

* somebody falling through the horizon has finite amount of time to survive ....



There is finite amount of proper time ~~between~~ here ...

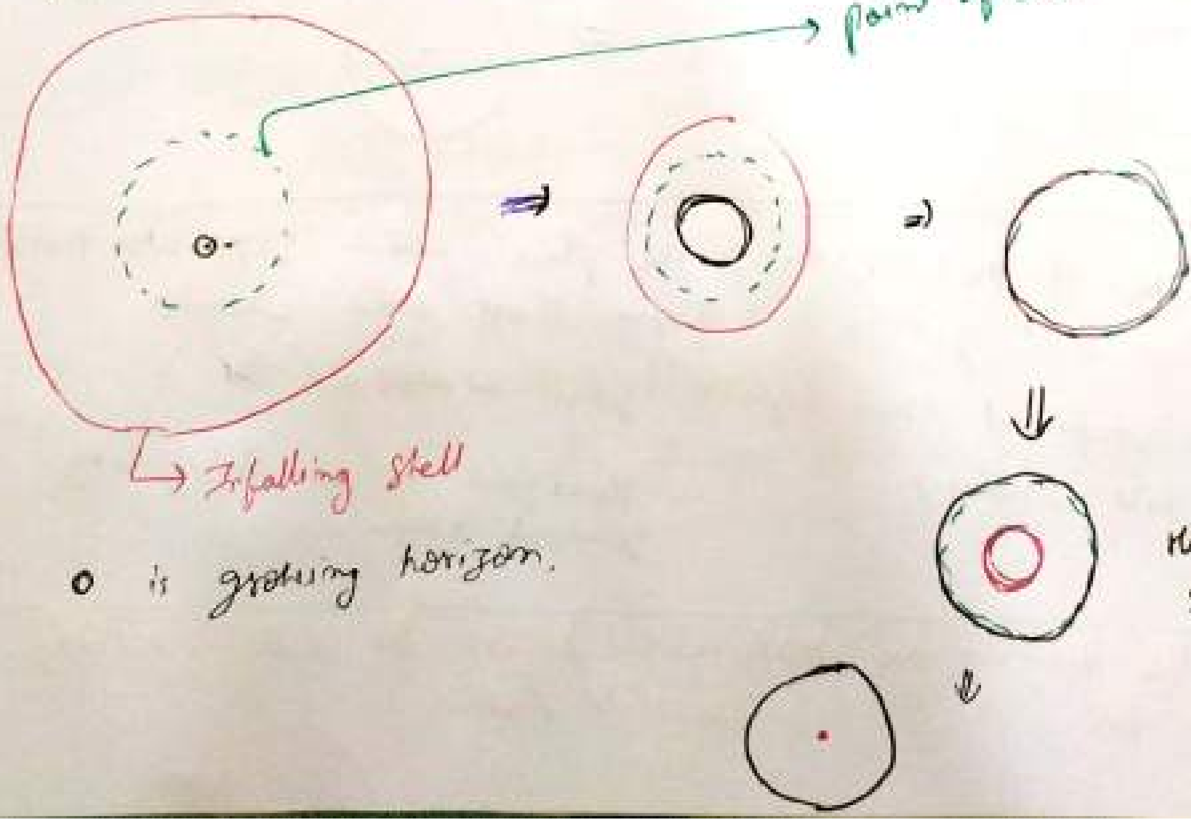
Only the corners are infinitely far away



The incoming shell crosses the growing horizon (at a) .... and pass that point → then shell is inside & horizon is outside.
((Horizon stops growing at the point shell crosses it)

initially the horizon is small

point of cross



Horizon now stops growing.

This is constant radius line. $\gamma(\gamma) = \gamma(\delta)$

This is not constant radius line
↳ radius increases as you go up along

$\alpha\beta$: $\gamma(\alpha) < \gamma(\beta)$

$$\gamma(\alpha) < \gamma(\beta) < \gamma(\gamma) = \gamma(\delta)$$

Note!! The diagrams here are not representing relative size of things (a small circle in corner is a huge space as compared to a circle inside)

↳ These diagrams are not correctly representing size scales.

⇝ They are representing motion of light rays.

⇝ in representing light rays... they are representing what things are in causal relationship to other things, what signals can propagate, who can send signal to whom.

⇝ ~~It represent~~ It reflects the Causal structure (the cause & affect)

i.e. what can influence what

→ and the rule is: that a thing can ~~influence~~ only influence the things in front of it where light can get too.

↳ it cannot influence the things outside the light cone.

Lec-9 General Relativity: Einstein field Equations — Shoaib Akhtar

"Principles of General Relativity are pretty simple; but it is computationally nasty."

Newton's Version of Einstein field equation

(Never the less; there are field equations equivalent to Newton)

(Newton did not actually think about field ... he did not have concept of field equation)

"Masses affect the Gravitational field; and Gravitational field affects the way masses move"

In the concept of newton

(i) field affects particle

statement: $F = -m \nabla \phi$

ϕ is gravitational potential;

(ii) $F = -m \nabla \phi(x)$ (This is one of the aspect .. where field tells particle how to move ... and in this case by telling them what their acceleration should be)

∴ but we know; $F = ma$ ⇒ $ma = -m \nabla \phi(x)$

$$\Rightarrow \boxed{a = -\nabla \phi(x)}$$

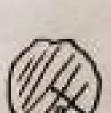
(iii) The equation which tells ~~how~~ Gravitational field what to be is the Poisson's equation.

$$\nabla^2 \phi = 4\pi G \rho$$

$\nabla^2 \phi$ is equal to something which has distribution of mass... which are actually the thing that are causing the field to be there.

∴ where ρ is density of mass (it is function of position & time) (mass per unit volume)

~~Exp~~ example



M

Rotationally symmetric mass distribution.

; solution

$$\phi = -\frac{MG}{r} \quad (\text{for } r > R)$$

out side solution.

now,

if we shrunk the mass to centre "." $\Rightarrow$ then the solution will be valid everywhere except at that point

i.e; $\phi = -\dfrac{MG}{r}$ (for $r \neq 0$)

$\underline{a = -\nabla\phi \,;\, \nabla^2\phi = 4\pi G\rho}$

$\hookrightarrow$ it is a kind of primitive field theoretic way to think about Newton's Gravitation.

$\hookrightarrow$ it is actually here Action at a distance ... because in Newtons theory if you move around the mass; ϕ instantly reacts to it and changes.

(it is here a way of writing ~~theory~~ Newtons theory, making it look like a field Theory)

★ Remember what we wrote down .. i.e; Schwarzschild metric (which is a solution of Einstein field equations)

set $\underline{c=1}$.

$$g_{00} = \left(1 - \frac{2MG}{r}\right) \Rightarrow g_{00} = 1 + 2\phi \;;\; \text{where: } \phi = \frac{-MG}{r}$$

time - time component of metric

$$\therefore \nabla^2 g_{00} = 2\nabla^2\phi$$

$$\Rightarrow \boxed{\nabla^2 g_{00} = 8\pi G\rho}$$

$\rightarrow$ this ~~sounds~~ sounds like matter (ρ) is affecting the geometry i.e; the metric

When we make this correspondance between Newtons ϕ and Schwarzschild Metric; i.e; $g_{00} = 1 + 2\phi$

$\hookrightarrow$ we see that "roughly it looks like matter telling geometry how to curve".

★ now; In General Relativity:

"$F = ma = -m\nabla\phi$" is replaced by the statement that ~~to~~

"once you know geometry, i.e; the metric ... the rule is that particles moves on space-time geodesic"

so; $ma = -m\nabla\phi$ is replaced by "Geodesic Rule"

$\nabla^2 g_{00} = 8\pi G \rho$

→ the RHS is density of mass

→ and from mass energy equivalence ; ie; mass really means energy

it is actually energy density

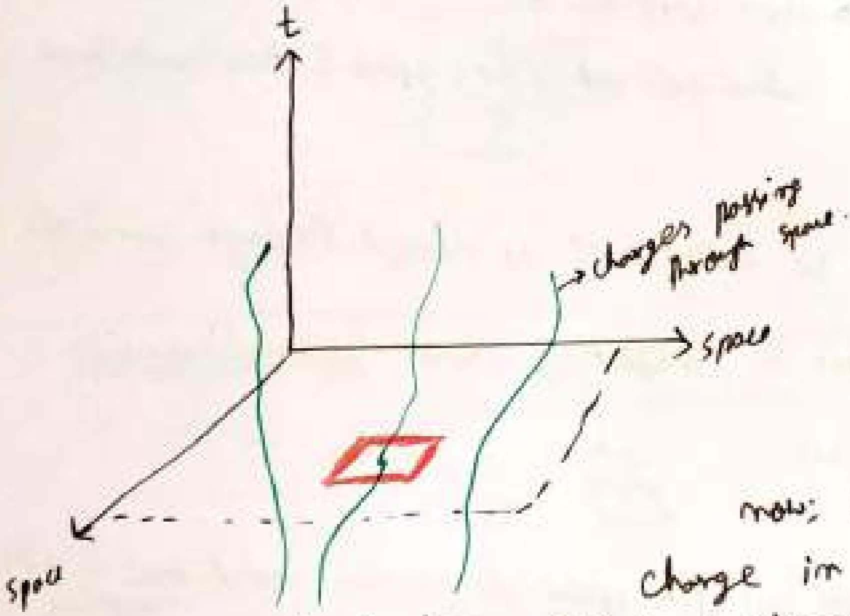
#. let Q be total electric charge of the system.

- let σ be charge density ; ⊛ charge per unit volume

☆

$$\sigma = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V}$$

$$= \lim_{V \to 0} \frac{Q}{V} \quad ; V \text{ is volume.}$$



; let ▱ be an element of space.

now; when we want to account charge in a volume of space at a given instant of time ⇒ that is almost like asking for the charges which pass through a little cubic area here (which is now being represented as a little square area " ▱ ")

☆ Flow of charge ... also called current.

~~take a lit~~ • take a little window in space " ▱ "

↳ the window is characterized by an area which we take to be infinitesimally small.

→ and orientation of window ⊛ a sense of direction through the window.

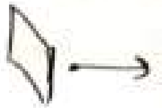
so; the window is characterized by a litle area, and an orientation and angle and a sense along it.

The current has to do with amount of charge passing through that window per unit time per unit area.

(windows have area .. they dont have volumes)

• and we also have to have a clock and allow the clock to proceed for small amount of time in order to ask how much charge flows through that window per unit time.

$$J^m = \frac{(Q \text{ through that window})}{(\text{Area})(\text{Time})}$$



note!! if $c=1$; J^m has ~~...~~ same unit as σ
(.. consequence of the fact that when we set $c=1$; space & time both have same unit)

J^m & σ ~~...~~ both can be thought of as charge through window

→ in case of σ ; the window is completely three-space dimension
... oriented along t axis

→ in case of J^m ; the window is two space dimension and one time dimension.

★ σ and J^m together form a four-vector in the sense of relativity)

$$J^\mu \longrightarrow (\sigma, J^m)$$ with J^0 = charge density
i.e. $J^0 = \sigma$

& J^1, J^2, J^3 be the three components of current.

Conservation of Charge (local idea)

Continuity

$-\dot{\sigma}$ ⇒ charge per unit time leaving the box ; and has to be equal to sum of currents ~~...~~ passing out through the box ; i.e. $\nabla \cdot \vec{J}$.

so; $\dot{\sigma} = -\nabla\cdot\vec{J}$; $\dot{\sigma} = \frac{\partial \sigma}{\partial t}$

$\Rightarrow \dot{\sigma} + \nabla\cdot\vec{J} = 0$

$\Rightarrow \frac{\partial \sigma}{\partial t} + \sum_m \frac{\partial}{\partial x^m} J^m = 0 \Rightarrow \frac{\partial J^0}{\partial x^0} + \frac{\partial J^m}{\partial x^m} = 0$

$\Rightarrow \boxed{\partial_\mu J^\mu = 0}$ i.e. $\boxed{\frac{\partial J^\mu}{\partial x^\mu} = 0}$

in curved coordinates; we replace derivatives by Covariant Derivatives ; i.e. $\boxed{\frac{D J^\mu}{D x^\mu} = 0}$ Continuity Equation

* when you go to curved coordinates: you should replace all derivatives by covariant derivatives... (otherwise equations will not be good Tensor Equation)

---

* ~~Each component~~ of energy and each component of momentum are like the charge ... They are conserved... They can flow (if the object is moving, the momentum & energy may be flowing)

Note

~~Difference~~

- Electric charge is an invariant (no matter how the charge is moving .. the charge of an electron is always the same)
- Density of charge and current of charge are not invariants.

* because of lorentz contraction; different observer assigns different volume to the charge ; and thus density & current change.
  - but it is ok... charge density is ~~an~~ a component of a 4-vector and it is not invariant.

- "charge density & charge current are not invariants ; they form together a four-vector".

* The total energy and momentum are not invariant.
(not just the density of energy & momentum)

↳ Energy and momentum unlike charge are not invariant.
(They together form the components of four-vector)

$$p^\mu = (E, p^m)$$

$\because m = 1, 2, 3$
m labels the directions of space.

$\therefore E = p^0$

$\Rightarrow p^\mu = (p^0, p^1, p^2, p^3)$

recall
in case of charge
... the density think
... density was time component of J
∴ because its density .. we assign a component "0" ... "0" for time.

so
p^0 is the energy.
lets think about time component of energy.

↳ i.e.; Time component of density (not the time component of total energy)

↳ In other words how much energy is in a small volume.

↳ we are going to call it T^{00}

the second not "0" indicates that we are talking about its density.

first not "0" indicates we are talking about energy

you can think of T^{00} as time component (meaning the density) of a think which is itself a time-component (namely the energy)

∴ T^{00} is a function of position & time $T^{00}(\vec{x}, t)$
↳ If you integrate over position it tells how much total energy we have.

but; Energy can also move ..... and like charge when energy disappears out of a region, it does so because it passes out through the walls of the region.

⇒ so; energy also has a flow.

.... it is exactly the same idea; the amount of energy per unit time, per unit area passing through a window " ◇ " is current of energy

↪ we dont call it current of energy

↪ we call it T^{0m} ; $m \in \{1,2,3\}$

T^{00} ⇒ Density of energy

↪ is density of energy

T^{01} ⇒ flow of energy along the direction X^1

T^{02} ⇒ " " " " " " X^2

T^{03} ⇒ " " " " " " X^3

↪ forms together flow of energy.

exactly the same way; the continuity equation of energy is derived.

$$\boxed{\frac{DT^{0\mu}}{DX^{\mu}} = 0}$$

analogous continuity equation for energy

But everything we said here about energy, we could now say for any one of the components of the momentum.

so; lets consider component p^m

↳ the component p^m also has a density.

component m of momentum, has density; and

that density is called T^{m0} ↳ density

talking about m^{th} component of 4-momentum.

∴ likewise; we also have; flow of m^{th} component of momentum ... flow along direction n

∴ i.e; T^{mn}.

"momentum is a conserved quantity; each of its components can flow along some direction"

∴ $T^{m\mu}(x)$; $x = (t, \vec{x})$

∴
$$\frac{DT^{m\mu}}{DX^{\mu}} = 0$$

---

so; we have ~~now~~ four continuity equations as follows

i.e;
$$\boxed{\frac{DT^{\nu\mu}}{DX^{\mu}} = 0}$$

• flow and densities of energy and momentum form a tensor with indices.

⟿ The first index tells who we are talking about (Energy or momentum)

⟿ The other index tells whether we are talking about density or flow.

$T^{\nu\mu}$ is called Energy - Momentum Tensor.

we can prove; it is general property of relativistic system which tells that $T^{\nu\mu}$ is symmetric.

corollary: $T^{m0} = T^{0m}$.

$$T^{\nu\mu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}$$

The flow and density of energy and momentum are combined into an Energy - Momentum Tensor.

---

so; there is some problem with $\nabla^2 g_{00} = 8\pi G \rho$

→ the notion of energy density here is incomplete

→ its a part of a Tensor.

→ so; LHS must also be a part of Tensor.

→ but whenever you have a Tensor equation; you cant have a tensor equation that says some particular component of Tensor is equal to ~~some other~~ some component of some other Tensor.

example A_{m} .... take A_3 : if there law of physics which

B_{m} says $A_3 = B_3$

→ it only makes sense as a law of physics if it is also true that ~~for~~ $A_1 = B_1$, $A_2 = B_2$.

we can rotate and ~~by~~ adjust it.

& so; if $A_3 = B_3$ in all frame ... by rotating we will get other components also equal. $A_m = B_m$

---

* When you go to relativity; the same thing is true even concluding the time component of equations.

* if $A^0 = B^0$ is true in every frame
  → the only way it can be true if all components are equal in every frame of reference; i.e; $A^\mu = B^\mu$.

" Good Laws of physics must be Tensor law of physics, in particular if they have to be true in every frame of reference "

Einstein generalization of Newton must read something as follows.

$$G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

~~RHS should be a~~ • RHS is a tensor with two components.

• LHS should be a tensor with two components;
  so; LHS should be tensor with same kind of Tensor structure, i.e; Symmetric here (because $T^{\mu\nu}$ is symmetric)

→ but LHS ~~should~~ should not be something which is made out of matter ... but it is something which is made out of metric & has to do with geometry .. and not with masses and sources.

* RHS is here something which is made out of matter.

... we will call LHS to be $G^{\mu\nu}$.

Things we know about $G^{\mu\nu}$
- it is made up out of metric
- it probably has two derivatives in it (to compare with $\nabla^2 g_{00} = 8\pi G\rho$ here)

⤳ we will find a good candidate for $G^{\mu\nu}$ and then can ask if we were in a situation where non-relativistic physics should be a good approximation ... does $G^{\mu\nu}$ reduce to just $\nabla^2 g_{00}$

↳ if it does not .. we leave it & try to find different $G^{\mu\nu}$.

---

what kind of tensor can ~~be~~ we make out of the metric and two derivatives
⤳ we have already talked about one such candidate
ie: The curvature Tensor.

$$\Gamma_{\nu}{}^{\sigma}{}_{\tau} = \frac{1}{2} g^{\sigma\delta}\left(\partial_\tau g_{\delta\nu} + \partial_\nu g_{\delta\tau} - \partial_\delta g_{\nu\tau}\right)$$

$$R_{\mu\nu\tau}{}^{\sigma} = \partial_\mu \Gamma_{\nu}{}^{\sigma}{}_{\tau} - \partial_\nu \Gamma_{\mu}{}^{\sigma}{}_{\nu} + \Gamma_{\nu}{}^{\sigma}{}_{\lambda}\Gamma_{\mu}{}^{\lambda}{}_{\tau} - \Gamma_{\mu}{}^{\sigma}{}_{\lambda}\Gamma_{\nu}{}^{\lambda}{}_{\tau}$$

↳ it has four components
... so we can make two components by contraction of ~~two~~ components.

but

$$R_{\mu\nu\sigma}{}^{\sigma} = 0 \;;\; R_{\mu\nu\tau}{}^{\mu} \neq 0 \quad ; \quad R_{\mu\tau} := R_{\mu\nu\tau}{}^{\mu}$$

$$R_{\mu\nu\tau}{}^{\nu} \neq 0$$

... note; $R_{\mu\nu\tau}{}^{\mu} = - R_{\nu\mu\tau}{}^{\mu}$ (same tensor effectively apart from sign)

⤳ its a Theorem ... there is only one tensor that you can build out of two derivatives acting on the metric which has only two indices. ... it is called Ricci Tensor.

* Ricci Tensor is a contraction of Reimann Tensor
(it has less information... you can think in terms of no. of components)

Reimann Tensor has lot more components than Ricci Tensor.

$R_{\mu\nu} \longrightarrow R^{\mu}{}_{\nu}$

you can raise & lower indices

$R_{\mu}{}^{\nu}$

$R^{\mu\nu}$ ...

$R_{\mu\nu}$ (i.e; Ricci Tensor) is symmetric

$$\boxed{R_{\mu\nu} = R_{\nu\mu}}$$

R

---

so; (i) first possibility: $R^{\mu\nu} \stackrel{?}{=} 8\pi G\, T^{\mu\nu}$

∴ we can have; $R = R^{\mu}{}_{\mu}$; note: $R^{\mu}{}_{\mu} = R^{\mu\nu} g_{\mu\nu}$

(ii) Second possibility.

~~it is a scalar~~

it is a scalar

(This is called Curvature Scalar)
or Ricci Scalar.

∴ we can have possibility: $g^{\mu\nu} R \stackrel{?}{=} 8\pi G\, T^{\mu\nu}$

Which $G^{\mu\nu}$ we should chose.

If we belive; local conservation holds for energy & momentum

~ then we will conclude: $\underbrace{D_{\nu} T^{\mu\nu} = 0}_{\text{i.e; Continuity equation.}}$

so; from here; it will follow:

$$D_{\mu} G^{\mu\nu} = 0$$

lets calculate; $D_\mu(g^{\mu\nu}R) = (D_\mu g^{\mu\nu})R + g^{\mu\nu}(D_\mu R)$

$\because$ we know; covariant derivative of metric tensor is zero.

$\Rightarrow D_\mu(g^{\mu\nu}R) = g^{\mu\nu}(D_\mu R)$

$\because$ now; covariant derivative of scalar are just ordinary derivatives.

$\Rightarrow D_\mu(g^{\mu\nu}R) = g^{\mu\nu}\partial_\mu R$

$\because$ in general derivative of curvature scalar is not zero.

↳ if curvature scalar was constant everywhere (which is not true in general) .. ~~so~~ then only we would have $\partial_\mu R \equiv 0$.

so; we find: $D_\mu(g^{\mu\nu}R) = g^{\mu\nu}\partial_\mu R$

so; $G^{\mu\nu} = g^{\mu\nu}R$ will not work

$D_\mu(R^{\mu\nu}) = \frac{1}{2} g^{\mu\nu}\partial_\mu R$

so; $G^{\mu\nu} = R^{\mu\nu}$ will also not work

$\because$ ~~...~~ $\because$ but if we take linear combination; then it will work

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$$

this $G^{\mu\nu}$ will work ... and will satisfy $D_\mu(G^{\mu\nu}) = 0$

⇝ its a theorem that there is nothing else made up out of two derivatives acting on the ~~...~~ metric that has the property that its covariantly conserved (upto a factor)

i.e, $\underline{D_\mu(G^{\nu\mu}) = 0}$.

so; lets assume there is some factor, say k

so; $k\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = 8\pi G \cdot T^{\mu\nu}$

now we go to non-relativistic limit to calculate this k.

... in appropriate limit (non-relativistic)

$k(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 8\pi G T^{\mu\nu}$ should be same as $\nabla^2 g_{00} = 8\pi G \cdot \rho$

for $\mu = 0; \nu = 0$

And they are same with numerical coefficient

$k = 1$

$\Rightarrow$ $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ is called Einstein Tensor

$\because$ $R^{\mu\nu}$ is Ricci Tensor

& R is Curvature Scalar.

$\Rightarrow$ $$\boxed{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G\, T^{\mu\nu}}$$

now known as Einstein field Equations.

When we consider the case with either no sources; or when we are in a region of space where there are no sources (outside of all sources)

then; $T^{\mu\nu} = 0$ $\Rightarrow$ so we have $R^{\mu\nu} = \frac{1}{2} g^{\mu\nu} R$

lets get R by contracting indices.

$$R^{\mu}{}_{\nu} = \frac{1}{2} g^{\mu}{}_{\nu} R$$

$g^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$ it is Kronecker Delta.

$g^{\mu}{}_{\nu} = g^{\mu\delta} g_{\delta\nu} = \delta^{\mu}{}_{\nu}$

∴ and set: $\mu = \nu \Rightarrow$

$$R^{\mu}{}_{\mu} = \frac{1}{2} g^{\mu}{}_{\mu} R$$

$$\Rightarrow R = \frac{1}{2}(g^{\mu}{}_{\mu}) R$$

$$\Rightarrow R = \frac{1}{2} \delta^{\mu}{}_{\mu} R$$

$: \delta^{\mu}{}_{\mu} = \delta^{0}{}_{0} + \delta^{1}{}_{1} + \delta^{2}{}_{2} + \delta^{3}{}_{3} = 4$

$$\Rightarrow R = \frac{1}{2} \cdot 4 \cdot R \Rightarrow R = 2R$$

$\Rightarrow$ only solution is $\mathbf{R = 0}$

note $R = 0$ does not imply that Curvature Tensor is zero or Ricci Tensor is zero

~~so~~

but in the equation; $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}$

when $R = 0$ & $T^{\mu\nu} = 0$

$$\Rightarrow R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \times 0 = 0$$

$$\Rightarrow R^{\mu\nu} = 0 \Rightarrow \boxed{R^{\mu\nu} = 0}$$

Einstein field equations become little simple ... and this is usually called Vacuum Case.

i.e. in region of ~~sp~~ space where there are no sources.

The solutions are not trivial example: they contain Gravitational Wave with no sources (like electromagnetic waves)

(ii) Schwarzschild Metric

★ you can check ... and it is true

that ~~R~~ $R^{\mu\nu} = 0$ for Schwarschild metric.

↳ of course it will be ambiguous at singularity.
(but will be true for every where away from singularity)

" Every where away from singularity ; the Schwarschild metric is what is called Ricci flat ; i.e; $R^{\mu\nu} = 0$ "

saying $R^{\mu\nu} = 0$ is sometimes called Ricci flat.
(it is not the same thing as "flat")

$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G \, T^{\mu\nu}$

$\Rightarrow R^{\mu}{}_{\nu} - \frac{1}{2} g^{\mu}{}_{\nu} R = 8\pi G \, T^{\mu}{}_{\nu}$

$\Rightarrow R - 2R = 8\pi G (T^{\mu}{}_{\mu})$

$\Rightarrow -R = 8\pi G \cdot T^{\mu}{}_{\mu}$ ※ $\boxed{T^{\mu}{}_{\mu} = T_{\mu\nu} g^{\nu\mu}}$

$\Rightarrow 8\pi G \cdot T_{\mu\nu} g^{\nu\mu} = -R$

$\Rightarrow 8\pi G \, T = -R$

let; $T := T^{\mu}{}_{\mu}$
(the scalar that you get when you contract indices of $T^{\mu}{}_{\nu}$)

$\Rightarrow \underline{R = -8\pi G T}$

↳ we can put this back in Einstein equations

... ~~$R_{\mu\nu} + \frac{1}{2} g$~~

$R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \cdot 8\pi G \cdot T = 8\pi G \, T_{\mu\nu}$

$\Rightarrow \boxed{R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)}$ *

Note: Einstein equation can be represented as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$$

or

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

~~Einstein Equat~~ These equations are of similar form apart from a factor of $8\pi G$ ... set $8\pi G = 1$ (to see it clearly)

→ and they better be because T was representing matter, R " " geometry

↘ & they are related

$T = T^{\mu}{}_{\mu}$ is called Trace of Energy-Momentum tensor

---

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$$

or

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

---

eg. • for electromagnetic radiations T would be zero. (for massless particles)
• for massive particle; $T \neq 0$ in general.

---

* you can have geometries where $R^{\mu\nu} = 0$; and Reiman Tensor is not zero.

example Gravitational Waves.

★★ • In 4-dimensions; there is less information in Ricci Tensor than the Reimann Tensor.

• In 3-dimensions; the amount of information is same. (i.e. you can write one in terms of the other)

• In 2-dimension; all of the information is in the scalar R.

✿ Roughly speaking ; to any solutions you can add gravitational waves • → will satisfy Einstein field equation ....

∴ so ; gravitational waves must be something that contain more information than the Ricci Tensor.

Cosmological Constant can either be thought of as component to Energy Momentum Tensor called ; $T^{cos}_{\mu\nu} = \lambda g_{\mu\nu}$; i.e. $T_{\mu\nu} = T_{\mu\nu} + T^{cos}_{\mu\nu}$

so; if there was cosmological constant ... you might write.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\left((\lambda g_{\mu\nu}) + T_{\mu\nu}\right)$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G (\lambda g_{\mu\nu}) + 8\pi G\, T_{\mu\nu}$$

$$\therefore \underbrace{\lambda_{cosmological}}_{\text{cosmological constant}} = 8\pi G \lambda$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda_{cos}\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

$$\Rightarrow R_{\mu\nu} - g_{\mu\nu}\left(\frac{R}{2} + \lambda_{cos}\right) = 8\pi G\, T_{\mu\nu}$$

∴ $8\pi G \lambda g_{\mu\nu}$ i.e.; $\lambda_{cos}\, g_{\mu\nu}$ only depends on geometry .... you can shift it to other side of equation and think of it as a part of the geometric side of the equation.

∴ or you can leave it on the right side ; and think of it as part of the energy momentum.

Cosmological Constant can be thought in either way.

✿ The ordinary $1/R^2$ force law is the only power force law which allows orbit to be closed without precision of the orbit. (its a curious accidental mathematical fact)

* Weak Gravitational Waves
(amplitude of gravitational waves are small enough that you can make approximation such as the amplitude square of gravitational field is zero.)

Einsteins Equation : without matter (no energy momentum tensor on RHS) $\Rightarrow$ $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$ (or) $R_{\mu\nu} = 0$

* matter is another word for energy momentum tensor.

☆ Equilibrium Situation • means solution which has no time dependance.
• which also does not have any matter on RHS.

$\Rightarrow$ There is only one equilibrium situation
↳ the equilibrium situation is just empty space.
i.e; empty flat space.

↳ in that case the metric has a ~~simple~~ form in some coordinate system as follows; $g_{\mu\nu} = \eta_{\mu\nu}$

note! — The metric of flat space is not δ_{ij} ..... it depends on coordinate system we are using $\rightarrow$ but the thing is that for flat space; we can choose a coordinate where metric becomes δ_{ij}

$$\text{where: } [\eta_{\mu\nu}] = \begin{array}{c|cccc} & t & x & y & z \\ \hline t & -1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & 0 & 0 & 0 & 1 \end{array}$$

$\therefore g_{\mu\nu} = \eta_{\mu\nu}$ is an eq^m solution. It is a solution of Einstein field equation when there is no matter ; i.e; $R_{\mu\nu} = 0$.

☆ far from place (the place where something is going on... say a pulsar is rotating) $\rightarrow$ far from that $g_{\mu\nu}$ can be chosen as follows ; $g_{\mu\nu} = \eta_{\mu\nu}$ + (something small)

*small ~~xx~~ means that its components are much smaller than the components of $\eta_{\mu\nu}$

... let that small thing be called $h_{\mu\nu}$

so; we have $\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}}$

when we say position .. we mean position & time
$x = (t, \vec{x})$

$\Rightarrow \boxed{g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)}$

now; we will take the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; and calculate from it $R_{\mu\nu}$ and set it equal to 0.

→ This will give an equation for $h_{\mu\nu}$.

$$\underbrace{R}_{\text{structure of Reiman Tensor}} = \partial\Gamma - \partial\Gamma + \Gamma\cdot\Gamma - \Gamma\cdot\Gamma$$

(Ricci tensor is a combination of components of Curvature Tensor)

$$\underbrace{\Gamma}_{\text{Structure of Christoffel symbol}} = \frac{1}{2} g^{-1} \left(\partial g + \partial g - \partial g \right)$$

g^{-1} contains things like $\eta - h$

$\therefore \partial g^{\mu\nu} = \partial \eta + \partial h$
$= 0 + \partial h$
$\Rightarrow \partial g = \partial h$

so; $\Gamma \sim (\eta - h) \left(\partial h + \partial h - \partial h \right)$

$\therefore$ ∂h .i.e; derivatives of h is also assumed to be small.
→ h is small .. cant vary very sharply .... so small derivative.
(unless it has sharp point or something ... which we will assume it does not)

so; $\eta \cdot \partial h$ is once small

$h \cdot \partial h$ is twice small (it is quadratic in fluctuation)

so; we neglect; $h \cdot \partial h$

so; we have: $\Gamma \sim \eta \cdot \partial h$

or we absorb η ..

∴ we have $\Gamma \sim \partial h$

In the approximation of ~~some~~ weak gravitational radiation: the christoffel symbol is itself proportion to some collection of derivatives of h.

~~[illegible]~~ ☆ Ricci Tensor will contain derivatives of Γ

↳ and this is going to be second derivative of h ; ie; $\partial^2 h$

$\boxed{\text{incidently ; } h \text{ is called Gravitational field ... or field of the gravitational wave.}}$

∴ $R = \partial\Gamma + \Gamma \cdot \Gamma$

$R \sim \partial^2 h + \partial h \cdot \partial h$

↳ we neglect this

$\Rightarrow R \sim \partial^2 h$

(curvature Tensor or Reimann Tensor)

↳ we know Ricci Tensor is some combinations of Reimann Tensor

⟶ so Ricci Tensor will be here composed out of simple second derivatives of the metric tensor (or here say h)

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.... so our equation of motion is some kind of equation which looks like "$[\partial^2 h \ldots + \ldots]_{\mu\nu} = 0$"

→ there is equation for each μ & ν

→ Equations of this form, .. especially in relativity are usually wave equation.

Oridnary Wave .. Wave Equation : wave moving down z-axis
: let ϕ be wave field.

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial z^2}$$

Solution

$$\Rightarrow \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial z^2} = 0$$

... if we have more directions of space.
→ we would have

$$\frac{\partial^2 \phi}{\partial t^2} - \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) = 0$$

approximation we are using

we are dropping out quadratic terms (also higher order terms)
like h^2, $h \cdot \partial h$, $\partial h \cdot \partial h$ ...

→ But this is a well defined approximation ; technically it is Linearization of Einstein's equations (which simply means that you ~~through a~~ throw away everything of higher power)

... This is good approximation when Gravitational Wave is weak.

remember : It is not that metric of flat space-time is equal to $\eta_{\mu\nu}$; its that it can be chosen to be $\eta_{\mu\nu}$

* make a coordinate transformation on $\eta_{\mu\nu}$ and the metric changes .. but it is still exactly the same solution.

↳ This means that there must be solutions of these equations "$[\partial^2 h]_{\mu\nu} = 0$" which look like they are non trivial; but all that they really represent is flat space but in coordinates where the coordinates have little ripple in them.

---

in flat space ; i.e; δ_{mn}

* we can start with good cartisian coordinates X, Y

.. we can introduce new coordinates

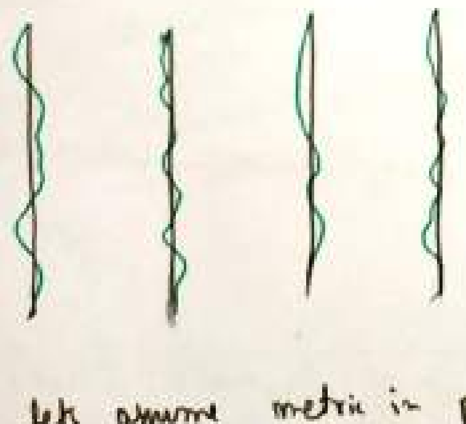
$$X' := X + f_x(x, y)$$
$$Y' := Y + f_y(x, y)$$

its a coordinate change.

small correction term.

(in this case for understanding our example)

* we have not changed space in any way .... all we have done is changed coordinates; and perhaps put some wiggle.



x', y'
x, y

; and now; if we re write the metric for the prime components

now; lets assume metric in prime coordinate is just $dx'^2 + dy'^2$ (we are supposing that primed coordinates are nice cartesion coordinates & ; x & y are the slightly curved one) .. we are working backwards

$$dx' = dx + \frac{\partial f_x}{\partial x^m} dx^m \qquad : \quad m = 1, 2$$

$$dy' = dy + \frac{\partial f_y}{\partial x^m} dx^m \qquad : \quad x^1 = x, \; x^2 = y$$

we will get; $dx'^2 + dy'^2 = dx^2 + dy^2 + \underbrace{h_{mn} \cdot dx^m dx^n}$

↳ This is small correction

assuming f & ∂f is small

↳ so; drop their higher powers

↳ i.e; do linearization.

... the small correction in the metric tensor does not mean that the geometry ~~has ch~~ has changed

... it just means that the coordinates we have used have wiggles in them.

you will get

$$\boxed{h_{mn} = \frac{\partial f_m}{\partial x^n} + \frac{\partial f_n}{\partial x^m}}$$

↳ these are some ~~pertu~~ perturbations (small change) on the original form of metric which are trivial.

- There are wide varieties of conditions that will allow us to eliminate the unphysical meaningless solutions of Einstein equations.

↳ once you have done that & imposed all of einstein equations

here "$[\partial^2 h]_{\mu\nu} = 0$"

↳ The equations become simple & they become wave equation for components of h.

$$\boxed{\frac{\partial^2 h_{\mu\nu}}{\partial t^2} - \nabla^2 h_{\mu\nu} = 0}$$

~~i.e~~ Each component of fluctuation $[h_{\mu\nu}]$ satisfy wave equation.

☆ now; there are more equations than one for each μ & ν.

↳ the reason is that; we have to divide out this spurious fake solutions which are there because of the coordinate ambiguity

↝ once you do that you will find out the physical solution

∴ i.e; the one that have really some physical meaning ...

.....

we will classify them as follows —

ex wave moving down z-axis

$\phi = \sin(k(t-z))$

solution of $\frac{\partial^2 h_{\mu\nu}}{\partial t^2} - \frac{\partial^2 h_{\mu\nu}}{\partial z^2} = 0$ are of same ~~type~~ type

i.e: $h_{\mu\nu}(t, z) = h^0_{\mu\nu} \cdot \sin(k(t-z))$

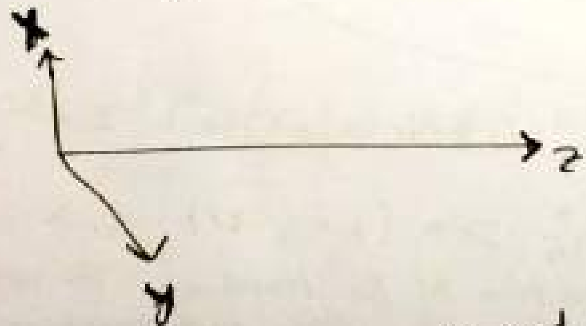
it is just numerical coefficient that multiplies the $\sin(k(t-z))$ i.e; the amplitude.

0 is not here.

This is the nature of gravitational wave ... each component behaving like A wave moving down the axis

... however .. there are ~~more~~ additional equations

and when you impose them all you find that Gravitational wave have ~~very~~ simple form.



only allowed components is

$h_{ij} = h^0_{ij} \cdot \sin(k(z-t))$

$\therefore i, j \in \{x, y\}$

.. and others components are zero.

Remind of Electromagnetism

here we do the same thing

- we solve maxwell field equations.
- but when we write down all the equations; we find some constraints
  - the constraints here are called Transversality of the field.

★ Transversality of Electromagnetic field means that Electromagnetic field alway point in direction perpendicular to the motion of the wave.

∴ not only solutions are wave; but they are Transverse wave.

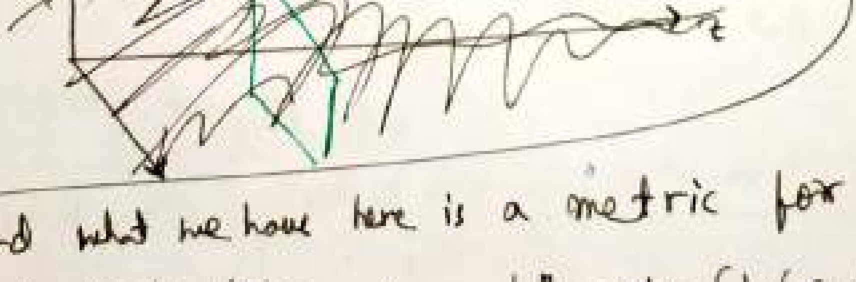
Very similar things happen in case of $h_{\mu\nu}$ waves

→ The waves have to be transverse

.... means that the time components and the z components of $[h_{\mu\nu}]$ are zero.

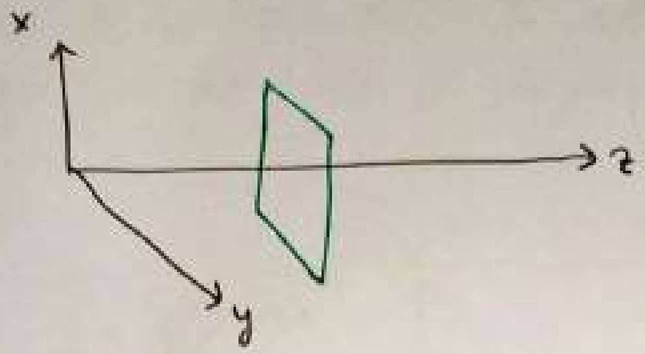
→ the only components of $[h_{\mu\nu}]$ which are allowed to be non zero are the components in the plane perpendicular to the direction of the wave.

☆ we slice the wave into two dimensional plane as we go down the z-axis at a fixed instant of time



→ and what we have here is a metric for each z (the slice at a given z)

which looks like : $h_{ij} \cdot \sin(k(z-t))$ with only components in the plane ⊥ to the motion of the wave ... The components are just numbers.



There is one more equation that comes from Einsteins field equations and it says that trace of h_{ij} is zero

i.e; $h_{xx} + h_{yy} = 0$

⇒ and these whole set of equations tell ;
that the metric of gravitational wave is as follows.

$h_{00} = 0$

$h_{z0} = 0$

$h_{zz} = 0$

$\left(h_{\text{anything with time component or } z\text{-component}} \right) = 0$

; The only components which are not zero is

h_{xx}

h_{yy}

h_{xy}

h_{yx}

; The metric tensor is symmetric

so; $h_{xy} = h_{yx}$

⇒ $h_{xx} = h_{xx}$

⇒ $h_{yy} = -h_{xx}$;

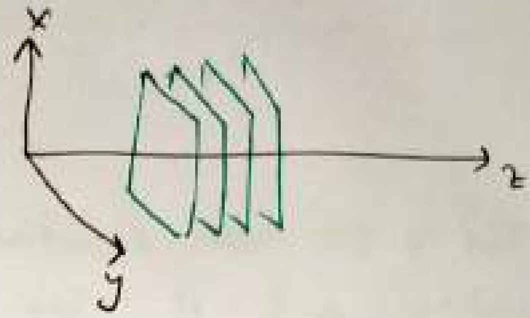
⇒ $h_{xy} = h_{yx}$

so; $\left. \begin{matrix} h_{xx} \\ -h_{xx} \\ h_{yx} \end{matrix} \right\}$ ⇒ all of these vary only with z & t (if wave goes along z axis) so; $h_{xx} = h^{\circ}_{xx} \cdot \sin(k(z-t))$

$h_{yy} = -h^{\circ}_{xx} \cdot \sin(k(z-t))$

$h_{xy} = h^{\circ}_{xy} \cdot \sin(k(z-t))$

$h_{yx} = h^{\circ}_{xy} \cdot \sin(k(z-t))$



* now think of a series of layers.
* lets look at fixed instant of time.

let; $t=0$ .. as we move down z axis; there is a perturbation on the metric: ~~the metric~~ the metric tensor is a little bit different from just flat space metric

↳ it oscillates as you go down the z-axis.

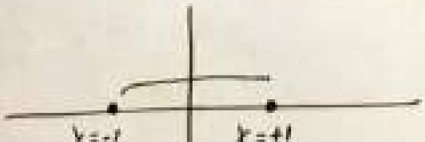
... and the components that oscillates are the components that have to do with the metric of the plane perpendicular to the wave.

* What does it mean to have a non-zero h_{xx}

→ this means g_{xx} is little bit different from 1

i.e; $g_{xx} = 1 + h^{\circ}_{xx} \cdot \sin(k(z-t))$

↳ means proper distances along x axis are little bit different, that actual distance between $x=-1$ & $x=+1$ is a little bit different then it would be if the perturbation was not there.



↳ it could be larger or smaller seperation depending on sign of h_{xx}.

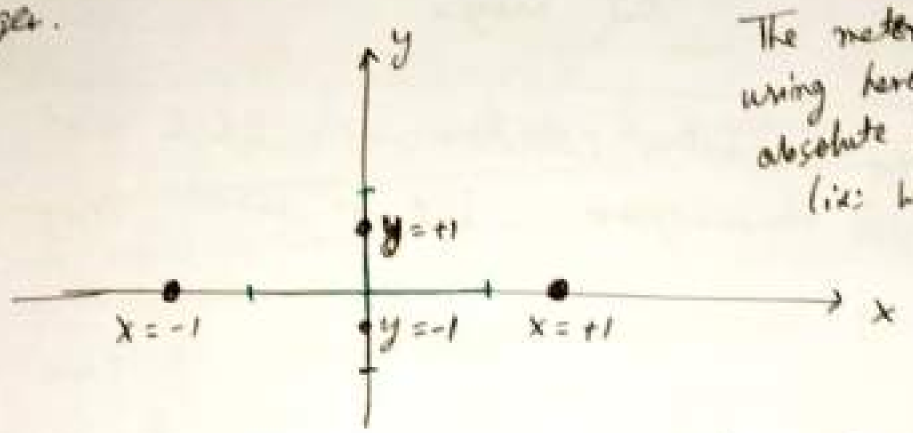
remember
Metric tensor tells you actual distances between points.

if h_{xx} is positive ; then the actual physical distance between $x=-1$ & $x=+1$ will be little bit larger then ~~one~~ from the one when we had no perturbations.

$g_{yy} = 1 - h_{xx}$

; when h_{xx} was positive ;

a meter stick oriented along y axis would look little bit longer.

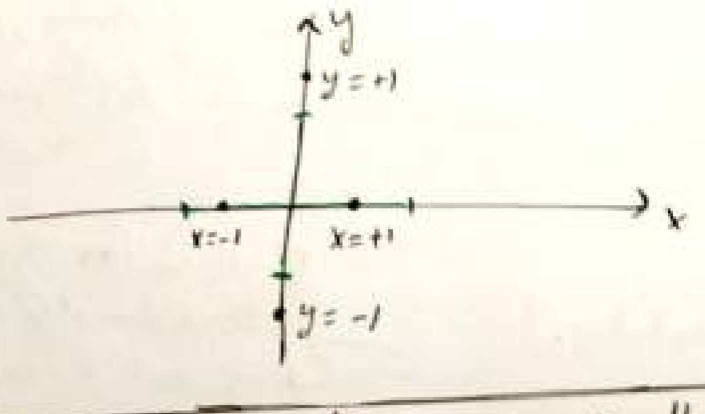


The meter stick we are using here is of its absolute ~~ls~~ length (i.e: when it was in flat spacetime) ... always one meter long.

This is nothing but ; a squeezing of coordinates.

and as the wave propagates... after a brief interval of time h_{xx} will change sign.

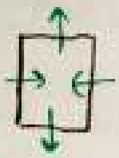
... the meter will look little short along y axis
& " " " long " x axis



The time dependance of this waves really does exert tidal forces.

\# if we took square piece of plywood □ ... it would deform as

and



... this way.

\# Interestingly there is another solution in which not h_{xx} & h_{yy} are non-zero ... but in which h_{xy} is non zero.

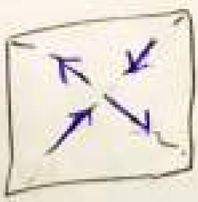
$$h_{xy} = h^{0}_{xy} \cdot \sin(k(z-t))$$

This would be a metric which will look like.

$$[\delta_{xy}] + \begin{pmatrix} 0 & h_{xy} \\ h_{yx} & 0 \end{pmatrix}$$

... this still has $h_{xx} + h_{yy} = 0$.

→ it is also exactly the same kind of thing, ~~but~~ except that the squeezing and stretching are along 45° axis.



... osillating.

... ~~think for $h_{xx} = h_{yy} = 0$ for simplicity~~

→ its not a new solution here, except that the original solution turned by an angle.

In fact; any solution that you write down which ... corresponds to some orientation of axis where along those axis the compression & expansion are along those tilted axes.

This wave; due to tidal forces will create real stresses in a body ... A strain gauge can be used to measure these compression & expansion.

• a steel one; and a wood one .. will stress by different amount

If you want to detect gravitational waves; then what we should do is create a system which has resonance at the frequency of gravitational wave.

... then you would have a driven system being driven at its own natural frequency .. and under those circumstances; the response will be particularly big.

• it is not time which is oscillating
↳ the wave is oscillating with time.

when we said orthogonal to z ... we meant space-directions orthogonal to z.

... there is time .. and the components of metric which have time component in them all are zero.

(it is very much like electromagnetic wave)

★ There is curvature in time direction.

→ the h in those direction (time) are non zero
→ but when you differentiate ... you can differentiate w.r.t. time and get the time into it

$\frac{\partial h_{xy}}{\partial t} \neq 0$ (not zero in general here)

---

* ~~[illegible]~~ * small oscillation about the equilibrium are linear.

→ small ~~[illegible]~~ oscillation ... so our Linearization of Einstein field equations worked well.

---

* whenever we do linearized approximation; we always check at the end that the corrections to it would be small.

$$\boxed{\text{"The wave will maintain its transverse character for fluctuations of any order" (small or high)...}}$$

The stiffness of material determines the velocity of propagation through the material → so you can ~~substitute~~ substitute ~~[illegible]~~ stiffness for velocity of propagation.

→ Electromagnetic & Gravitational wave are extremely still. They are stiffest things possible ~~(they travel at~~ (they are both governed by the speed of light)

→ we no longer think in terms of stiffness

... but if we would; we would then think that the space on which these waves moves are very stiff.

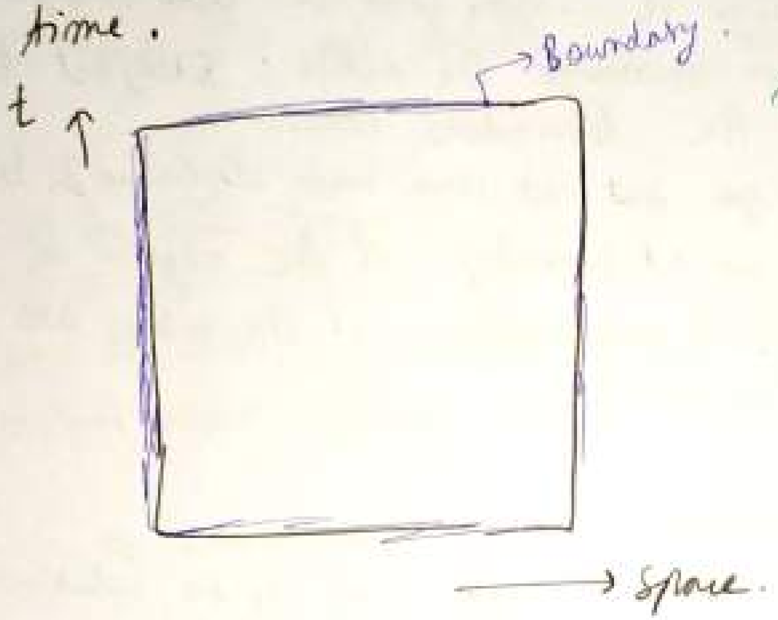
The principle of Least Action.

(all systems that we know about ; mechanical systems,
electromagnetic radiation,
Quantum field theory;
Standard model, particle physics!

↪ They are all governed by an action ~~pt~~ principle)

The action of a field is an integral over space and time.

- A field configuration is a value of field at every point of space and time.



⇒ A field history is labelled by a value of the field at every point here.

so; its some field ... lets call it generically ϕ.
... and its a function of $\vec{x}$ & t ; ie; $\phi(t, \vec{x})$

↪ and this is field configuration.

★ not every configuration is a solution of the equations of the theory.
↪ just like not every trajectory is a solution of the equations of motion.

(every trajectory is a thinkable trajectory ... but not every ~~traject~~ trajectory satisfies the equations of motion)

In the same way: every $\phi(x)$ is a thinkable possible value of field in space and time; but they all are not solutions.

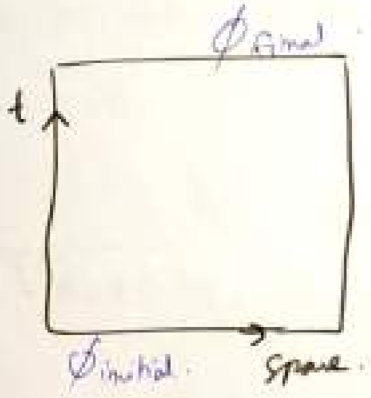
↳ and in general .. to the best of our knowledge

↳ every such field is governed by a principle of ~~least~~ least action.

$$\text{ACTION} = \int dx^{\mu} \cdot \mathcal{L}(\phi, \phi_{,\mu})$$

↳ now: you stationarize it ... you find a field configuration which minimize or maximize the action; subject to a constraint: i.e. the Boundary Conditions.

... you go out at some large distance & large time .. actually you go at boundary of the region of space time in concern .... and you say what the values are at the boundary.



we ask what is the solution which interpolates between ϕ_{final} & $\phi_{initial}$.

⟹ The answer will be the solution that make the action stationary.

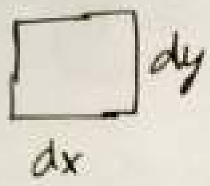
★ Einstein field equation can be derived by Action principle.

---

suppose we have ordinary space .. and we have some metric

$$ds^2 = g_{mn} \cdot dx^m dx^n$$

now; suppose we are interested in volume of a small region of space



; assume the metric has two components

g_{xx} & g_{yy}

(others zero)

and we now want to know area

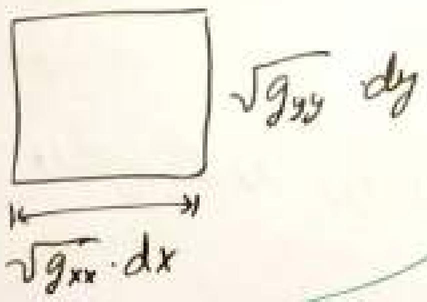
↳ we will call volume ... area ≡ volume in 2 dimensions (they are same concept)

Area of the little square?

The area of little square can be known; if we know the actual proper distance associated with dx .. & dy.

What is the actual proper distance associated with dx here?

Ans it is $\sqrt{g_{xx}} \cdot dx$



; so; Area of rectangle is

$$\sqrt{g_{xx} \cdot g_{yy}} \cdot dx\, dy$$

This is the special case where there is no off diagonal g_{xy} term.

note; here

$$g_{xx} \cdot g_{yy} = +\det(g_{\mu\nu})$$

If you work out in general case $\begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix}$

↳ The volume element of this little element will come out to be

$$\underbrace{\sqrt{Det(g)}\ dx\, dy}$$ } ⇒ it is usually called volume element.

This is the formula for the volume of small differential □ dx, dy element. □ in any curved space.

☆ If we have some large region : and if we want total volume of it ... we integrate.

$$\int \sqrt{Det(g)}\ \cdot dx dy \cdot dz$$

If you have four-dimensional (4-D) spacetime
↳ its useful to define similar concept which will just involve the fourth component.

... Space time volume element $\sqrt{det(g)}\ dx\, dy\, dz\, dt$

It is invariant.

Space time volume $= \int \sqrt{det(g)}\ \cdot dx dy dz dt$

☆ Now; if you take volume element: and integrate with any scalar (say $S(t,\vec{x})$)

$$\int \sqrt{\det(g)}\, S(t,\vec{x})\, dx\,dy\,dz\,dt$$

→ This is also invariant quantity which does not depend on coordinates.

Action in relativity theory is supposed to be same in every coordinate frame ... otherwise the laws of physics which will come out of it would depend on coordinate system.

So; it is natural to take the action to be as follows.

$$\text{ACTION} = \int d^4x \cdot \sqrt{\det(g)} \cdot (R + \Lambda)$$

~~What kind of scalar we make~~

☆ lets take Einstein theory without any energy momentum tensor on RHS ... just pure gravitational field ... What kind of scalar we can put there.

☆ How many scalars are there.. which we can make up only out of metric?

Ans.
- R can be made from metric which is scalar.
- any number.

$[R + \Lambda]$

→ we can add to this any numerical number ... and the number that you put here depends on the laws of physics (it is itself a law of physics)

→ it is called Cosmological constant .., Λ)

Λ will correspond to a term in action which is proportional to space-time volume itself.

... without cosmological constant ; the action is $A = \int d^4x \sqrt{\det(g)} \cdot R$

.. with " " ; " " $A = \int d^4x \sqrt{\det(g)} \cdot [R + \Lambda]$

☆ Einstein field equations equivalent to saying

$$\delta \int d^4x \sqrt{\det(g)} \cdot R = 0 \;..$$

---

note!! stress energy term would come in as additional term in action which would depend on material & other fields

ex there can be electromagnetic energy .. other kinds of energy.

"$\delta \int d^4x \sqrt{\det(g)}\, R = 0$ is vacuum theory

... later this solves & say $R = 0$ ... this is vacuum theory which governs things like Gravitational Waves; Schwarzschild Solutions"

even the $\delta \int d^4x \sqrt{\det(g)}\, [R + \Lambda] = 0$ is vacuum theory.

∴ .. there would be other things in action ... and those would be made out of different fields

... they always $A_{\text{tin}} = \int d^4x \sqrt{\det(g)}\, [R + \ldots]$ have $\sqrt{\det g}$ there.

...

example $\delta \int d^4x \sqrt{\det g}\, [R + F_{\mu\nu} F^{\mu\nu}] = 0$ would govern electromagnetic field.

★ $\int d^4x \sqrt{\det(g)} \cdot R$ is Hilbert-Einstein action.

⇒ note ~~what~~ what is used as field in space

∴ Answer the metric $g_{\mu\nu}(x)$

The Einstein-Hilbert action $\int d^4x \sqrt{\det(g)} \cdot R$ is made up of metric ...... you vary the metric a little bit ... and you look for places where you find minimum of action.

$$\boxed{\delta \int d^4x \sqrt{\det(g)} \cdot R = 0 \quad \text{is Einstein-Hilbert form of General Relativity.}}$$

History
Einstein ~~litera~~ literaly have to build all the idea of curvature & geometry ... (very less people at that time knew about Reimanian geometry) ... one of Einstein friend ~~Marcel Gr~~ Marcel Grossman helped him to do with some mathematics.