

LIE GROUPS & LIE ALGEBRAS

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These notes are consequence of my self study; and are mostly inspired from Prof. Tibra Ali lectures on **Lie Groups and Lie Algebras**.

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Lecture 1: Motivation and fundamental concepts of group theory.
Examples of Group.Motivations: Symmetries of Physical system

1905 - Einstein's Special Relativity; Symmetries of Maxwell equations were not same as symmetry of Newtonian physics.

Standard Model of particle physics: $su(3) \times su(2) \times u(1)$ — Internal (gauge) symmetry.

Classical Field Theory —

100 years ago Emmy Noether;

Symmetries $\Rightarrow$ Conserved quantities.Expanding
or Universe, energy is not conserved.

Wigner's Theorem: Hilbert space of quantum system furnish an unitary & linear representation of the symmetry group of the system.

 $\rightarrow$ Weinberg Vol I.# Fundamental ConceptsDefinition: Group is a ~~set~~ ^{set} G equipped with a binary operation $\circ$ $G \times G \rightarrow G$ known as group multiplication; under which the set remains closed, and satisfies the following.(i) Closure if $a, b \in G$ then $a \circ b \in G$ (ii) Associative $(a \circ b) \circ c = a \circ (b \circ c)$ (iii) $\exists$ a unique element $e \rightarrow$ identity;

$$a \circ e = a \quad \forall a \in G.$$

(iv) Inverses, for each a in G , $\exists$ a unique inverse

$$a^{-1} \quad \text{s.t.} \quad a \circ a^{-1} = e$$

Remark 11 $a^{-1} \rightarrow$ right inverse. (as we defined) Pg 2

we can show a^{-1} is also left inverse
i.e. $a^{-1} \cdot a = e$

$\therefore$ ~~$a = a^{-1} = e$~~ ; Proof 11 $a^{-1} \cdot a = a^{-1} a e$

Jambhakar's Book volume 2 (Reference)

we can show: $e a = a e$

(right identity is left identity also)

$$\begin{aligned} &= a^{-1} a a^{-1} (a^{-1})^{-1} \\ &= a^{-1} (a a^{-1}) (a^{-1})^{-1} \\ &= a^{-1} e (a^{-1})^{-1} \\ &= a^{-1} (a^{-1})^{-1} = e \end{aligned}$$

$$e a = (a a^{-1}) a = a \cdot (a^{-1} a) = a \cdot e$$

$G \rightarrow$ finite or infinite (countably infinite or uncountably infinite)

Size of G , order of G .

$ab \neq ba$ in general.

If all elements of G commute, we call the group Abelian.

If $\exists$ at least 2 element $a, b \in G$, s.t. $ab \neq ba$, then G is called non-Abelian.

Examples

① $C_4 = \{ \alpha, \alpha^2, \alpha^3, \alpha^4 = e \}$
cyclic group

$$\therefore \alpha^n = \underbrace{\alpha \cdot \alpha \cdots \alpha}_{n \text{ times}}$$

$$e = \alpha^4$$

$$a = \alpha$$

$$b = \alpha^2$$

$$c = \alpha^3$$

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

(2) $\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$

(133)

+	0	1	2	3
e $\rightarrow$ 0	0	1	2	3
a $\rightarrow$ 1	1	2	3	0
b $\rightarrow$ 2	2	3	0	1
c $\rightarrow$ 3	3	0	1	2

$\mathbb{Z}/4\mathbb{Z}$ and C_4 are actually Isomorphic.

Group Isomorphism

ex $C_4 \cong \mathbb{Z}/4\mathbb{Z}$

ex 11 Permutation Group P_3 : (Reference) George I. Book in Lie groups.

(a, b, c) : The set of operations on (a, b, c) form a group.

Elements: $() \rightarrow$ identity element, do nothing
 $(a, b, c) \rightarrow (a, b, c)$

$(1\ 2)$ exchange objects in position 1 & 2: $(a, b, c) \rightarrow (b, a, c)$

$(2\ 3)$ " " " " 2 & 3: $(a, b, c) \rightarrow (a, c, b)$

$(1\ 3)$ " " " " 1 & 3: $(a, b, c) \rightarrow (c, b, a)$

$(1\ 2\ 3)$ $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ $(a, b, c) \rightarrow (c, a, b)$

$(3\ 2\ 1)$ a $(a, b, c) \rightarrow (b, c, a)$

Why does these operation create a group?

After each operation the ~~sys~~ system has a well defined configuration.

$\hookrightarrow$ an example of Transformation Group.

$\hookrightarrow$ Non Abelian. $\because (1\ 2\ 3)(1\ 2) = ?$

$$(1\ 2): (a, b, c) \rightarrow (b, a, c)$$

$$(3\ 2\ 1): (b, a, c) \rightarrow (b, c, a)$$

$$\neq (1\ 2\ 3)(1\ 2) = (3\ 2\ 1)$$

$$(12)(23) = (123)$$

$$(123):(abc) \rightarrow (acb)$$

$$(12):(acb) \rightarrow (cab)$$

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Cayley's Theorem: Every finite group is a $\heartsuit$ subgroup proper

of a permutation group.

example 4 Rotation in 2D

$\vec{r} = \hat{i}x + \hat{j}y$; Rotate $\vec{r}$ by same angle θ

$$\vec{r} \rightarrow R(\theta) \vec{r}$$

$$\Rightarrow \sqrt{\vec{r} \cdot \vec{r}} \rightarrow \text{unchanged.}$$

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

form group under multiplication:

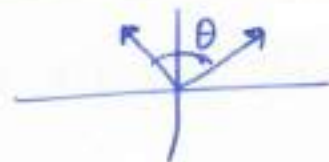
$$R(\theta_1) R(\theta_2) = R(\theta_1 + \theta_2)$$

$$\# \det R = 1 ; R^T R = R R^T = I$$

This group is known as $SO(2)$

$$R \in SO(2) ; 0 < \theta \leq 2\pi.$$

All complex phases $e^{i\theta}$ form group



$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$$

$$z = x + iy ; e^{i\theta} z ; 0 < \theta \leq 2\pi.$$

the group of complex phases $e^{i\theta}$ forms the group $U(1)$

$$2 \quad U(1) \cong SO(2)$$

Definition

Subgroup: If a subset $H \subset G$ form a group by itself under the same multiplication law; then H is subgroup

of G .
if $H \neq G$
& $H \neq \{e\}$ we say H is proper subgroup.

otherwise we say, H is an improper subgroup.

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Coset

Left Coset: let $H = \{h_1, h_2, \dots\}$ (it can be continuous)
be a proper subgroup of G . let $g \in G$ which is not in H .

The set $\{g \cdot h : \forall h \in H\} \rightarrow$ Right Coset gH

Right Coset is not a group in general, it is a set;
& depends on which element we choose.

if $g \in H$; then $gH = H$ (trivial case)

gH in $G - H$

$G = H + g_1H + g_2H + \dots$

Coset Decomposition.

We can decompose the group as sum of sets which are non-intersecting

Coset Space $G/H = \{H, g_1H, g_2H, \dots\}$

Example: Consider $P_2 \subset P_3$; $P_2 = \{(), (12)\}$

Work out all the cosets. (left cosets) w.r.t. P_2 .

Normal or Invariant Subgroups

If $\forall g \in G$ we have $gH = Hg$

then H is a normal or invariant subgroup.

Definition

Simple Group: If e and G are only normal subgroup of G , then we say G is simple group.

Centre: A subset Z of G is called the centre if
 $\forall z \in Z$ and $\forall g \in G$, we have $zg = gz$.

Lec 2: Fundamental Concepts of Representation Theory,
Groups and Quantum Mechanics, Lie groups
- Shoab Akhtar 1/5/2020.

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Lagrange's Theorem ~~For cyclic groups the order of proper~~
~~subgroup~~ For any finite group the order of proper
subgroup is a divisor of the order of the group.

$$H \subset G ; \quad \text{then} \quad n \cdot O(H) = O(G) ; n \in \mathbb{N}.$$

if $O(G) = \text{prime} \Rightarrow$ Then G cannot have proper subgroups
 $\Rightarrow$ so; G is simple group.

Basic definitions of Representation Theory.

Interested in mainly Representations of group in terms of
linear operators (differential operators, matrices, etc).

Defⁿ 1 The representation of G is a map D of the
elements of G onto a set of linear operators such that
$D(e) = \mathbb{1}$, where $\mathbb{1}$ is identity operator acting
on the vector space.

group multiplication is respected;

$$\text{if } a, b, c \in G \text{ s.t. } ab = c ; D(a)D(b) = D(c)$$

Remark, In QM (Quantum Mechanics) a phase is
possible; known as Projective Representations

Example: 1 $C_4 = \{ \alpha, \alpha^2, \alpha^3, \alpha^4 = e \}$

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; D(a) = \frac{1}{2} \begin{pmatrix} 1 + e^{i\pi/2} & -1 + e^{i\pi/2} \\ -1 + e^{i\pi/2} & 1 + e^{i\pi/2} \end{pmatrix}$$

$$D(b) = [D(a)]^2 ; D(c) = [D(a)]^3$$

Example 2/ $D(e) = 1 = D(a) = D(b) = D(c)$

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↪ Trivial Representation.

(In QFT we will see; the scalar field will be trivial representation in some sense of the Lorentz group; although ~~an~~ an infinite dimensional)

Faithful Rep. If D is a rep of G s.t. every element $g \in G$ is mapped to a distinct element $D(g)$; then D is a faithful representation.

otherwise $\Rightarrow$ Unfaithful rep.

Equivalent Reps Two representations D & D' are said to be equivalent if $\exists S$ (an operator S) s.t.

$$D'(g) = S D(g) S^{-1} \quad \forall g \in G.$$

(connected by ~~equiv~~ similarity transformation)

ex A representation of C_4

$$D'(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D'(a) = \begin{pmatrix} e^{i\pi/2} & 0 \\ 0 & 1 \end{pmatrix}$$
$$D'(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D'(c) = \begin{pmatrix} e^{3\pi i/2} & 0 \\ 0 & 1 \end{pmatrix}$$

is faithful rep.

we can verify; $D'(g) = S D(g) S^{-1}$

where; ~~$S = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$~~ $S = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

So; D' & D are equivalent reps.

Irreducible Representation

A rep D is said to be reducible if it has an invariant subspace.

If the action of $D(g)$, $\forall g \in G$ on a subspace of the full space remains in the subspace.

~~In other words, P is projection operator onto subspace on V . Then if~~ In other words, P is projection onto subspace on V . Then if $P D(g) P = D(g) P \quad \forall g \in G$ (138)

P is some subspace of the vector space

In practice terms, there exists some S s.t.

$$S D(g) S^{-1} = \begin{bmatrix} D_1(g) & 0 & \dots \\ 0 & D_2(g) & \\ \vdots & & D_3(g) & \ddots \\ \vdots & \dots & & \ddots \end{bmatrix} \quad \forall g \in G$$

where $D_i(g)$ form repr of G .

Fully Reducible If $D(g)$ can be brought to the form

$$D(g) = \begin{pmatrix} A_1 & 0 & 0 & \dots \\ 0 & A_2 & & \\ \vdots & & \ddots & \\ 0 & & & A_m \end{pmatrix} \quad \forall g \in G$$

where A_i s are not necessarily faithful representation of G , then D is fully reducible.

Groups and QM

Symmetries plays a role in QM,

due to a theorem by Wigner 1931:

Theorem: The group of symmetries of quantum systems is represented by linear, unitary or anti-linear, antiunitary operators acting on the Hilbert Space of the theory.
(Weinberg QFT volume 2)

Outline of proof $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$. Under symmetry transformation

$$|\psi_1\rangle \rightarrow U |\psi_1\rangle$$

$$|\psi_2\rangle \rightarrow U |\psi_2\rangle$$

Because this is symmetry transformation, the probabilities

should be invariant. $|\langle \psi_2 | U^\dagger U | \psi_1 \rangle|^2 = |\langle \psi_2 | \psi_1 \rangle|^2$ (139)

$$\Rightarrow U \rightarrow \text{unitary} : \langle \psi_2 | U^\dagger U | \psi_1 \rangle = \langle \psi_2 | \psi_1 \rangle$$

another solution

$$U \rightarrow \text{antiunitary} : \langle \psi_2 | U^\dagger U | \psi_1 \rangle = \langle \psi_2 | \psi_1 \rangle^*$$

$$\text{Antilinearity} : U(\alpha|\psi\rangle + \beta|\phi\rangle) = \alpha^* U|\psi\rangle + \beta^* U|\phi\rangle$$

Ex 1 If time reversal T is symmetry ; then $[T, H] = 0$

$$\text{then } |\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$$

$$T|\psi(t)\rangle = |\psi(-t)\rangle = T e^{-iHt} T^{-1} T|\psi(0)\rangle$$

$$\therefore T^2 = \mathbb{1} \Rightarrow T^{-1} = T$$

$$\text{we also know: } |\psi(-t)\rangle = e^{+iHt} |\psi(0)\rangle$$

$$\begin{aligned} T|\psi(t)\rangle &= T e^{-iHt} T(T|\psi(0)\rangle) & T|\psi(0)\rangle &= T|\psi(-0)\rangle \\ &= T e^{-iHt} T|\psi(0)\rangle & &= |\psi(0)\rangle \end{aligned}$$

$$\text{In other words; } |\psi(-t)\rangle = T e^{-iHt} T |\psi(0)\rangle$$

$$\Rightarrow T e^{-iHt} T = e^{+iHt} \quad (*)$$

if $[T, H] = 0$ is true ; and T is a unitary or linear operator ; then $(*)$ cannot be true.

$\therefore$ so; $T(i) = -iT$; then $(*)$ makes sense.

Degeneracy and Groups

$G \rightarrow$ group of symmetries of Quantal system
(actually adjective for quantum)

D is Rep ; Then $[D(G), H] = 0$

$\&$ $|\psi\rangle$ be an eigenstate of H ; $H|\psi\rangle = E|\psi\rangle$

Let the subspace of $\mathcal{H}$ spanned by applying $D(G)$'s on eigenstate $|\psi\rangle$: $D(g)|\psi\rangle \quad \forall g \in G$.

be given by $\{|\psi_1\rangle, \dots, |\psi_n\rangle\} \rightarrow$ linearly independent set

Since $[D(g), H] = 0$: $H|\psi_i\rangle = E|\psi_i\rangle$ has same eigen value.

$$D(g)|\psi_i\rangle = \sum_j |\psi_j\rangle \langle \psi_j | D(g) | \psi_i \rangle$$

it is enough to decomposition of identity in this subspace $\{|\psi_1\rangle, \dots, |\psi_2\rangle\}$ because it is invariant.

$$\Rightarrow D(g)|\psi_i\rangle = \sum_j |\psi_j\rangle D(g)_{ji}$$

$D(g)_{ji} \rightarrow n \times n$ irrep of G .

Lie Groups and Lie Algebras

Introduced by Sophus Lee ; while trying to understand the geometry of solutions space of differential equations.

Continuous groups $\Rightarrow$ Elements are parametrized by set of continuous parameters.

Example 1 $U(1) \rightarrow e^{i\theta} : 0 \leq \theta \leq 2\pi$

we can identify parameter space with circle

so; we can write $U(1) \cong S^1$.

Because of continuous nature of parameters; it is natural to think of Lie groups as manifolds.

Def. of Lie group

A Lie group G is a group which is also a manifold.

It is equipped with multiplication: $G \times G \rightarrow G$ with the usual properties of group.

G includes a identity element & for each element there is a unique inverse.

It will turn out which one is identified as the identity element will depend on how we parametrize the group

Suppose; that the group elements are parametrized by $\Phi = (\phi_1, \dots, \phi_n)$
 we write the generic element $g(\Phi)$ and
 $e = g(\Phi = 0)$

No. of independent parameters is dimensions of the group.

We concentrate on Matrix group $\Rightarrow$ defined by some matrix equation.

Relationship between Lie Groups & Lie Algebra.

Consider a spin $\frac{1}{2}$ particle.

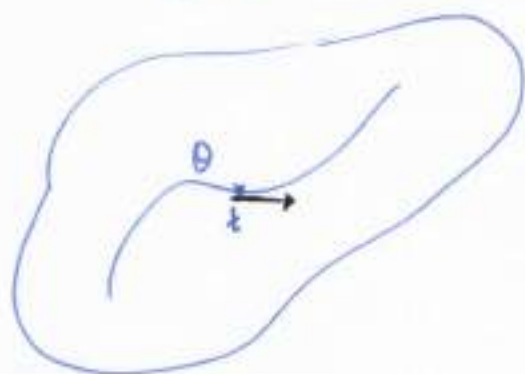
Its wavefunction is given by a vector $|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$
 Rotation by θ around some direction $\hat{n}$.
 Rotation is implemented by $e^{\frac{i}{2} \theta \hat{n} \cdot \sigma} |\psi\rangle$

$$\sigma = \{ \sigma_1, \sigma_2, \sigma_3 \}$$

Tangent Vector

Suppose we have a curve on the manifold.

$$\bar{\theta}(t) = \{ \theta_1, \theta_2, \theta_3 \}(t)$$



Tangent vector to $\bar{\theta}(t)$

$$\bar{V}(t) = \frac{d\bar{\theta}(t)}{dt}$$

Coordinate representation of tangent vector.

M

exercise $e^{i \frac{\theta_3}{2} \sigma_3} = \cos \frac{\theta_3}{2} + i \sigma_3 \sin \frac{\theta_3}{2}$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we $\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Pauli's don't commute.
 Groups formed by $e^{\frac{i}{2} \theta \hat{n} \cdot \sigma}$ is non-abelian
 group : $[e^{\frac{i\theta_1}{2} \sigma_3}, e^{\frac{i\theta_2}{2} \sigma_2}] \neq 0$

we can use $[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k$
 to prove.

Exponential Map Given a pauli matrix $\frac{i\sigma_a}{2}$

$$\exp : i \frac{\sigma_a}{2} \longrightarrow \exp \left[\frac{i\sigma_a \theta}{2} \right]$$

$\exp \left[\frac{i\sigma_a \theta}{2} \right] \sim$ form an Abelian subgroup.

$$e^{i\sigma_1 \theta_1/2} e^{i\sigma_1 \theta_2/2} = e^{i\sigma_1 (\theta_1 + \theta_2)/2}$$

$$e^{i\sigma_1/2} \cdot e^{-i\sigma_1/2} = \mathbb{1}$$

Lee 3: Lie Algebras from Lie groups. The adjoint Representation.

pg 13

-d'haud Akhtar 1/5/2020

Representation is reducible if it can be written in upper triangular form.

$$\text{ie: if } D(g) = \begin{pmatrix} D_1(g) & A(g) \\ 0 & D_2(g) \end{pmatrix}$$

$$V = \begin{pmatrix} W_1 \\ W \end{pmatrix}$$

there is an invariant subspace W which is preserved.

Fully Reducible if $D(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix}$

Given a manifold M .

However on a Lie Group of dim n , one can naturally define n vector fields.



Tangent space at the identity element:

$$\left. \frac{1}{i} \frac{d}{d\theta_a} e^{i\theta_a \frac{\sigma_a}{2}} \right|_{\theta_a=0} = \frac{\sigma_a}{2}$$

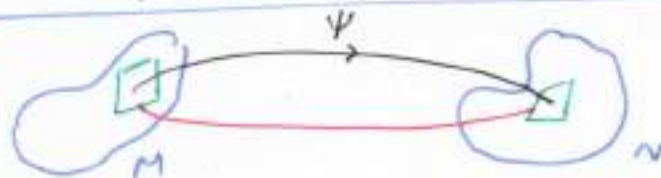
(no sum over a)

$\frac{\sigma_a}{2} \rightarrow$ tangent vectors at T_e .

$$g = g(\phi_1, \phi_2, \phi_3)$$

$$\frac{\sigma_a}{2} \rightarrow e^{i\frac{\vec{\phi} \cdot \vec{\sigma}}{2}} \frac{\sigma_a}{2} e^{-i\frac{\vec{\phi} \cdot \vec{\sigma}}{2}} = \frac{\sigma_a}{2}(\phi)$$

for this is what you get when you integrate the derivative



Includes push forward map.

$$\psi_* : T(M) \rightarrow T(N)$$

$g \in G$ is a map from $M \rightarrow M$.

(p15)

$\{\frac{\tilde{\epsilon}_a}{2}, 0\}$ forms vector space.

For $\frac{\tilde{\epsilon}_a}{2}$ for $a=1,2,3$: $g \frac{\tilde{\epsilon}_a}{2} g^{-1} \forall g \in G$.
gives vector field.

Lie Derivative

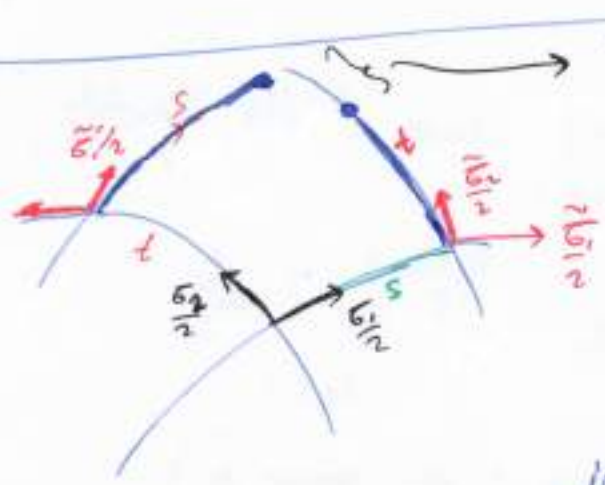
$$\mathcal{L}_X Y = [X, Y]$$

$XYf \neq YXf$ (in general) ; $[X, Y]f = Zf$ X, Y tangent vector
so: Z $\rightarrow$ tangent vector.

so: if we take $[\frac{\tilde{\epsilon}_a}{2}, \frac{\tilde{\epsilon}_b}{2}] = i f_{ab}^c \frac{\tilde{\epsilon}_c}{2}$ (sum over c)
 $\rightarrow$ some linear combination;

$\therefore f_{ab}^c = \epsilon_{abc}$ called Lie Algebra.

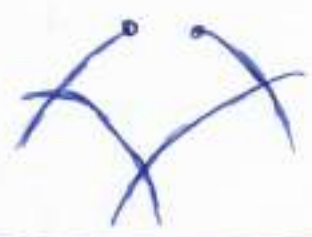
$f_{ab}^c \Rightarrow$ Structure Constant. : $f_{ab}^c = -f_{ba}^c$



These rectangles don't close in general.

$\hookrightarrow$ the Lie Derivative measures the failure of rectangles to close.

$\hookrightarrow$ we can think of it as a infinitesimal measure of non-commutativity.



$$[\frac{\tilde{\epsilon}_a}{2}(x), \frac{\tilde{\epsilon}_b}{2}(x)] = i f_{ab}^c \frac{\tilde{\epsilon}_c}{2}(x)$$

Tangent vectors at other points satisfy the same Lie Algebra. (nothing ~~special~~ about e)

Lie Bracket

(Pg 15)

On a typical Lie Algebra.

$$[T_a, T_b] = i f_{ab}^c T_c$$

$T_a \rightarrow$ generators (because they generate the group through exponential maps)

$f_{ab}^c \rightarrow$ structure constants.

Compact Lie groups

$f_{ab}^c = f_{abc} \rightarrow$ completely antisymmetric.

* $[X, Y] = -[Y, X]$

* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Satisfy Jacobi Identity.

$\hookrightarrow$ The relationship we get from this for structure constant is.

$$f_{ab}^d f_{dc}^e + f_{ca}^d f_{db}^e + f_{bc}^d f_{da}^e = 0$$

Notions of Reducibility of groups transfer over to Lie Algebra

$$e^{i \frac{\vec{\sigma} \cdot \vec{\theta}}{2}} |\bar{\Psi}\rangle \quad ; \quad \mathbb{D} \cdot \frac{\vec{\sigma}}{2} |\bar{\Psi}\rangle$$

$$\left(\mathbb{1} + \frac{i \vec{\sigma} \cdot \vec{\theta}}{2} \right)$$

$$e^{\frac{i \vec{\sigma} \cdot \vec{\theta}}{2}} \begin{pmatrix} |\Psi_1\rangle \\ |\Psi_2\rangle \end{pmatrix}$$

$\uparrow$ 2×2 matrix

$$\frac{\vec{\sigma}_a}{2} \begin{pmatrix} |\Psi_1\rangle \\ |\Psi_2\rangle \end{pmatrix}$$

$\uparrow$ 2×2 matrix

as matrix they have same dimensionality; so they can act on same vector space.

Given a Lie Group, $\longrightarrow$ always derive Lie Algebra.

Lie Algebra $\longrightarrow$ getting the group is not unique.

ex $\mathcal{L}(SU(2)) \begin{cases} \longrightarrow SU(2) \\ \longrightarrow SO(3) \end{cases}$

The Adjoint Action of g on $\mathcal{L}(G)$

$$\text{Adj}(g)(T_a) = g T_a g^{-1}$$

(Claim: The Adjoint action on $\mathcal{L}(G)$ by group G forms an irreducible representation of $\mathcal{L}(G)$ acting on itself.

$$\therefore g T_a g^{-1} \equiv \tilde{T}_a$$

we can check $\tilde{T}_a$ satisfy same Lie Algebra.

$$[\tilde{T}_a, \tilde{T}_b] = i f_{ab}{}^c \tilde{T}_c$$

so: $\tilde{T}_a$ should be able to expanded as linear combination of original Lie algebra. so: $\tilde{T}_a = C_{ab} T_b$

(because if T_a is in Lie Algebra, then also $\tilde{T}_a$ is in Lie Algebra ... that's what the claim is saying)

Lets take an infinitesimal element

$$g \approx 1 + i \theta_c T_c$$

since, as $\theta \rightarrow 0$, we must have $\tilde{T}_a = T_a$

$$\text{so: } C_{ab} \approx \delta_{ab} + i \theta_c F_{abc}$$

expanding both sides give

$$\cancel{T_a} + i \theta_c [T_c, T_a] = \delta_{ab} T_b + i \theta_c F_{abc} T_b$$

$$\Rightarrow [T_a, T_c] = -F_{abc} T_b$$

$$(F_c)_{ab} \equiv -F_{abc} \quad \text{antisymmetric}$$

define a matrix F_c

$F_c \rightarrow n$ dimensional representation of Lie Algebra.

$$\text{if we say } i f_{ab}{}^c = F_{abc} = (F_c)_{ab}$$

If we say $i f_{ac}^b \equiv -F_{abc} = (F_c)_{ab}$

$$(F_c)_{ab} = -i f_{ca}^b$$

now, we need to compute $[F_c, F_d]_{ab}$

$$[F_c, F_d]_{ab} = (F_c)_{ae} (F_d)_{eb} - (F_d)_{ae} (F_c)_{eb}$$

$$= -f_{ca}^e f_{de}^b + f_{da}^e f_{ce}^b$$

$$= -f_{ca}^e f_{de}^b - f_{ad}^e f_{ce}^b$$

$$= i^2 f_{cd}^e f_{ae}^b$$

using
Jacobi Identity

$$= i f_{cd}^e \cdot (-i f_{ea}^b)$$

$$= i f_{cd}^e (F_e)_{ab}$$

So, we have shown that: adjoint action gives rise to a representation which is the adjoint representation.

$$[F_c, F_d]_{ab} = i f_{cd}^e (F_e)_{ab}$$

Suppressing ^{matrix} indices: $[F_c, F_d] = i f_{cd}^e F_e$

$\{F_a\}$ forms an $n \times n$ dimensional irrep of $\mathcal{L}(\mathfrak{g})$

where $\dim \mathcal{L}(\mathfrak{g}) = n$.

Adjoint representation is very useful. It allows us to define an inner product on the space of Lie Algebras.

Inner Product: $\langle T_a | T_b \rangle \equiv \text{Tr}[F_a F_b]$

we can define inner product as the trace over adjoint representation of corresponding matrices.

~~$\{F_a, F_b\}$ is not an.~~

* $\text{Tr}[F_a F_b]$ is real and symmetric in a and b . (pg 18)

* This means: we can bring it to diagonal form.

$$\text{Tr}[F_a F_b] = K_a \delta_{ab} \quad (\text{no sum})$$

~~it means~~ out:

if G is compact,

Rescale the F_a so that $|K_a| = 1$

~~Compact Lie~~

so, for Compact Lie Group

we can choose $\lambda > 0$.

Then $\text{Tr}[F_a F_b] = \lambda \delta_{ab}$

later $\Rightarrow$ $\Downarrow$ Structure constants are completely anti-symmetric.

$$f_{abc} \equiv f_{ab}{}^c \quad (\text{just re-write for simplicity; we are not saying that in general } b \text{ \& } c \text{ are anti-symmetric})$$

$$\text{Tr}([F_a, F_b] F_c) = \text{Tr}(i f_{abd} F_d F_c)$$

$$\Rightarrow \text{Tr}[F_a F_b F_c - F_b F_a F_c] = i f_{abd} \cdot \lambda \cdot \delta_{dc}$$

$$\Rightarrow \text{Tr}[F_a F_b F_c - F_c F_b F_a] = i \lambda \cdot f_{abc}$$

-- (using cyclicity)

$$\Rightarrow \text{Tr}([F_b, F_c] F_a) = i \lambda f_{abc}$$

$$\Rightarrow i f_{bcd} \text{Tr}[F_d F_c] = i \lambda f_{abc}$$

$$\boxed{f_{bca} = f_{abc}}$$

$$\Rightarrow f_{abc} = -f_{bac} \Rightarrow \boxed{f_{bca} = -f_{bac}}$$

So; f_{abc} are completely anti-symmetric.

Examples of Matrix Lie Group

* $GL(n, K) \rightarrow n \times n$ matrix with entries in field $K = \mathbb{R}$ or $\mathbb{C}$
(General Linear group) and $\det(M(n, K)) \neq 0$

where: $M(n, K) \in GL(n, K)$

* $SL(n, K) \rightarrow \det M = 1$

* $SO(n)$: group of rotation of a rigid body in n dimensions
" ; $O^T O = O O^T = 1$, $\det(O) = 1$

S = Special (it tells about determinant)

* $U(n)$ group; $n \times n$ unitary matrices entries are complex numbers;
 $U^\dagger U = U U^\dagger = 1$

* $SU(n)$ group, $\det(U) = 1$

$SO(n)$ is subgroup of $U(n)$

Special orthogonal group ($SO(n)$)



$$y' = O y ; \quad y'^T y' = y^T y$$

(length is unchanged under rotation)

$$(y'^T y' = y \cdot y)$$

$$\therefore y'^T y' = (O y)^T (O y) = y^T O^T O y = y^T y$$

$$\Rightarrow \boxed{O^T O = 1}$$

$$\boxed{O_{ij} O_{ik} = \delta_{jk}}$$

$\therefore$ We know that a real $n \times n$ matrix have n^2 parameters.

$\rightarrow$ but here due to constraints $O_{ij} O_{ik} = \delta_{jk}$ we have less independent parameters.

Let's take variation of it

$$\delta O_{ij} O_{ik} + O_{ij} \delta O_{ik} = 0$$

Symmetry in j and k ;

$$\frac{j+k}{2} = \frac{j-k}{2} \Rightarrow \frac{n+1}{2} \Rightarrow \boxed{n + \frac{n^2-n}{2}} \quad \text{not correct}$$

so; no. of independent parameters: $n^2 - \left(n + \frac{n^2-n}{2}\right) = \frac{n(n-1)}{2}$

$$\boxed{\dim SO(n) = \frac{n \cdot (n-1)}{2}}$$

$$\dim SO(2) = 1$$

$$\dim SO(3) = 3$$

$$O \simeq \mathbb{1} + \sum_a \xi_a T_a \quad ; \quad a = 1, \dots, \frac{n(n-1)}{2}$$

$$\xi_a \rightarrow \text{real}$$

$$O^T O = \mathbb{1}$$

To first order we have $\sum_a \xi_a (T_a + T_a^T) = 0$

$$\Rightarrow \boxed{T_a = -T_a^T}$$

generators are anti-symmetric

$$\text{i.e.} \quad \boxed{T^{ab} = -T^{ba}}$$

$$a, b = 1, \dots, \frac{n(n-1)}{2}$$

$$[T^{ab}]_{cd} = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c \quad \left. \vphantom{[T^{ab}]_{cd}} \right\} \text{generators of } SO \text{ matrices.}$$

$$O = \exp \left[\frac{1}{2} \xi_{ab} T^{ab} \right]$$

and the Lie Algebra is given by

$$[T^{ab}, T^{cd}] = \delta^{ac} T^{bd} - \delta^{bc} T^{ad} - \delta^{ad} T^{bc} + \delta^{bd} T^{ac}$$

Lie Algebra of $SO(n)$

$$O^T O = \mathbb{1} \quad ; \quad (\det O)^2 = 1 \Rightarrow$$

$$O = \mathbb{1} + \frac{1}{2} \xi \cdot T \quad ; \quad \text{choosing } \xi =$$

choose ξ such that $\xi \cdot T = \begin{pmatrix} 0 & \xi_0 & & 0 \\ -\xi_0 & 0 & & 0 \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}$

(1021)

∴ The whole matrix O looks like.

$$\begin{pmatrix} 1 & \xi/2 \\ -\xi/2 & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\Rightarrow \boxed{\det O = 1 + \frac{\xi^2}{4}}$$

$$P = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} : \det P = -1$$

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$



Lecture 4: Examples of Lie groups continued: $so(m, n)$, $SU(n)$ 1922

- Shouib Akhtar - 2/6/2020

Reference: # Grötdier & Schücker - chapter 8

* Weinberg Volume 1: Ch 3 for Wigner's Theorem.

Last time we finished with the statement ~~that~~ that;

Generators of $SO(n)$, $T^{ab} = -(T^{ba})^T$

$$T^{ab} = -T^{ba} \sim \frac{n(n-1)}{2}$$

Suppose; n of $SO(n)$ is $n=2l$; l integer.

Then we can define l linearly independent matrices

$$H^1 = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

$$H^2 = \begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 & \ddots \end{pmatrix}$$

$$H^l = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

$$[H^i, H^j] = 0 \quad \forall i, j$$

Definition Cartan Subalgebra (CSA) The maximal set of mutually commuting elements of a Lie Algebra is ~~the~~ known as the CSA.

$\{H^i\} \rightarrow$ generate rotations in planes which do not share a direction.

The ~~number of elements~~ ^{dimension} l of CSA is known as the rank of

The Lie group or the Lie Algebra.

(1923)

The vectors of a representation that the Lie Algebra acts will be labelled by their eigenvalues of $\{H_i\}$ in CSA.

Consider $n = 2l + 1$ (some odd number), the rank is still l (because the extra dimension that we have, can't be really paired with any other direction to give us new plane of rotation which does not share any axis with these rotations)

"For every ~~compact~~ Lie Algebra (arising from a compact Lie group) we can define the quadratic Casimir operator which commutes with all the elements of Lie algebra."

$$T^2 = T_a T_a \quad \text{(general Casimir operator for compact Lie group)}$$

Exercise $[T^2, T_a] = 0$

$SO(3) : T^2 = T_a T_a$

Schur's Lemma | Any matrix which commutes with all the matrices of an irreducible representation must be proportional to identity.

So if, $T_a^{(r)} \rightarrow$ generators in the irrep (r) , then

$$T_a^{(r)} T_a^{(r)} = C_2(r) \mathbb{1}$$

$\nwarrow$ some number which depends on r .

$C_2(r) \rightarrow$ characterizes the whole rep.

ex: for $SO(3)$ or $SU(2)$

$$C_2 = j(j+1)$$

$SO(m, m)$ group

(Pg 25)

$SO(m)$ → preserving the Euclidean metric

$$\text{ie: } O^T O = \mathbb{1}$$

Analogously for $\mathbb{R}^{n, m}$ which have metric ~~diag~~ $\text{diag}(-1, \dots)$

$$\eta_{\mu\nu} = \text{diag} \left(\underbrace{- \dots -}_{n \text{ minus sign}}, \underbrace{+ \dots +}_{m \text{ plus sign}} \right)$$

$$O(m, m) = \{ O \mid O^T \eta O = \eta \}$$

If we impose further that $\det O = 1$, we get a subgroup ~~$SO(m, m)$~~
 $SO(m, m)$ - $SO(m, m)$ is continuously connected to the identity

★ $SO(m, m) \rightarrow$ Non compact group

(because they are non-compact as group manifold)

★ There are no finite dimensional unitary, representation of non-compact Lie groups.
irreducible, non trivial

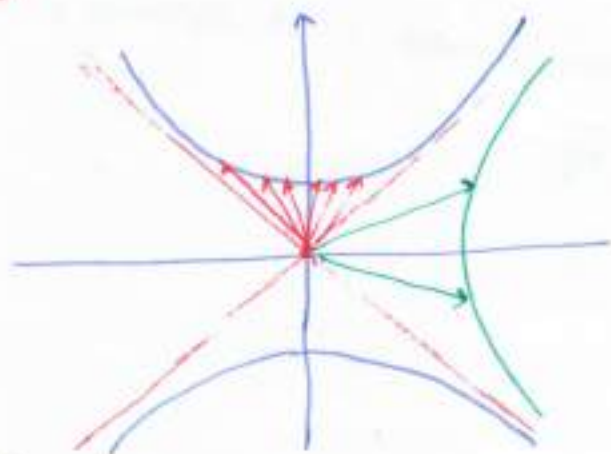
★ Lorentz group $O(1, 3)$.

$SO(1, 3)$ is a subgroup of $O(1, 3)$

~~$SO(1, 3)$~~ $SO(1, 3) \Rightarrow$ Proper Orthochronous Lorentz transformation

Proper : $\det O = 1$ (ie; continuously connected to $\mathbb{1}$)

Orthochronous : $O_{00} > 0$ (ie; you cannot change the direction of time)



(19/25)

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \sim \text{Parity}$$

$$T = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \sim \text{Time inversion}$$

$$\in O(1,3)$$

$$\notin SO(1,3)$$

Isometry group of AdS_n

AdS_n is defined by a hyperboloid in $\mathbb{R}^{2,n-1}$

$$-X_0^2 - X_1^2 + X_2^2 + \dots + X_n^2 = -R^2$$

radius of AdS .

Invariant under $SO(2, n-1)$ (also under $O(2, n-1)$)

* Conformal field theory in $(n-1)$ dimensional flat spacetime have symmetry ~~same as the~~ $SO(2, n-1)$ (same as the symmetry of AdS_n)

CFT in 2d space time $\Rightarrow$ Symmetry becomes infinite dimensional
(The Lie Algebra is Virasoro Algebra)

$$SL(2, \mathbb{R}) \rightarrow \text{Isometry of } AdS_3$$

The true time dimension is not really two time.

$\hookrightarrow$ The hyperboloid that we are defining eliminates one of the time like directions

"The intrinsic metric on this hyperboloid has only one time direction"

* Application to Holonomy Groups.



$$\Lambda \in SO(n)$$



Consider a general manifold M .

& some generic closed curve.

& parallel ~~propag~~ propagate this vector on the close loop.

In general it will come back to original vector upto some rotation ; if we make the constrain that length of v does not change.

$$v' = \Lambda v \quad \text{if} \quad v'^T v = v'^T v$$

$$\Lambda \in SO(n)$$

We can actually give this a group structure.



I associate with every closed loop some element of $SO(n)$.



$\Rightarrow$ This loop rotates the vector upto Λ_1 .

The group multiplication $\Lambda_1 \Lambda_2$ consists of first parallel transporting around loop 2, then loop 1.

Group Inverse : if you ~~parallel~~ parally propagate the vector in one sense : its Λ

* if you do in other sense its Λ^{-1} .

Restricted Holonomy group.



We can show the rotation is independent of which vector v you choose. It only depends on loops. That's how we assign group element to the loops.

A manifold is simply connected if any closed loop can be deformed continuously to a point.

we will see; that the manifold for $SO(3)$ is not simply connected.

Special Unitary Groups $SU(n)$

Gauge Symmetry group of Standard Model is given by $SU(3) \times SU(2) \times U(1)$.

Accidental global $U(1)$ symmetry; Baryon # & Lepton # conservation.

$SU(n)$ consists of $n \times n$ complex matrices U which satisfy $U^\dagger = U^{-1}$ and $\det U = +1$.

If we did not impose $\det U = 1$, we get $U(n)$ group.

$U \rightarrow$ ~~matrix~~ as a complex matrix has $2n^2$ real parameters.

$$\text{so, } U^\dagger_{ji} U_{ik} = \delta_{jk}$$

$\therefore$ If set $i=k \Rightarrow n$ real constraint equations.

$\therefore$ If $i \neq k$, we only need to consider $i > j$; $2 \times \left(\frac{n(n-1)}{2}\right)$ real constraints.

$$\text{so, } 2n^2 - (n + n(n-1)) = n^2 \quad \dim(U(n)) = n^2$$

$\star \det U = 1$ eliminates one extra parameter

$$\text{so, } \dim(SU(n)) = n^2 - 1$$

~~det U = 1 eliminates~~

$\det U = 1$ eliminates $U = \begin{pmatrix} e^{i\theta} & & \\ & e^{i\theta} & \\ & & \dots & e^{i\theta} \end{pmatrix} \sim U(1)$ subgroup. Pg 28

$$\det U = e^{i\theta n}$$

but if $\theta = \frac{2\pi k}{n}$, $k \in \mathbb{Z}$; then $\det(U) = 1$

$\det(U) = 1$ eliminates $U(1)$ subgroup;

but there is still cyclic group $\mathbb{C}^k$.

$\begin{pmatrix} e^{i\frac{2\pi}{n}k} & & \\ & \dots & \\ & & e^{i\frac{2\pi}{n}k} \end{pmatrix}$ will generate $\mathbb{C}^3$.

Generators of $SU(n)$

$U = \exp(i \xi_a T_a)$, $\xi_a \rightarrow$ real $n^2 - 1$ parameters.

then $U^\dagger = U^{-1} \Rightarrow T_a^\dagger = T_a$

$\det U = 1 \Rightarrow \text{Tr}[T_a] = 0$ (generators have to be traceless)

because: we can show $\det(U) = \exp[\text{Tr}(i \xi_a T_a)]$

$$\Rightarrow \text{Tr}[i \xi_a T_a] = 0 \text{ if } \det(U) = 1$$

Since ξ_a are arbitrary parameters

$$\Rightarrow \text{Tr}[T_a] = 0$$

$$\forall a = 1, 2, \dots, n^2 - 1$$

$$U = e^{i \xi_a T_a} = 1 + i \xi_a T_a + \frac{(i \xi_a T_a)^2}{2!} + \dots$$

$$U^\dagger = 1 - i \xi_a T_a^\dagger + \dots$$

$$= e^{-i \xi_a T_a^\dagger}$$

$$UU^\dagger = e^{i \xi_a T_a} \cdot e^{-i \xi_a T_a^\dagger} = e^{i \xi_a (T_a - T_a^\dagger)}$$

use B.C.H. formula to be rigorous.

generators given by $T_a = \frac{\sigma_a}{2}$

Hermitian and traceless.

$$H_1 = \text{diag}(1, 0, \dots, 0, -1)$$

$$H_2 = \text{diag}(0, 1, \dots, 0, -1)$$

$\vdots$

$$H_{n-1} = \text{diag}(0, 0, \dots, 1, -1)$$

So, $n(n-1)$ remaining.

we can choose:

$\Rightarrow$ half to be real & symmetric, and non-diagonal.

$\Rightarrow$ other half to be pure imaginary & antisymmetric.

~~Hermitian~~ $H_{n-1} =$

we have $(n-1)$ mutually commuting generators.

$$\text{so Rank}(SU(n)) = (n-1)$$

Definition Simple Lie Group.

A simple Lie group, is a connected non-Abelian Lie group which does not have a non-trivial connected normal subgroup.

Note The excluded normal subgroups are Lie Groups.

A simple Lie group may have discrete normal subgroups.

Simple Lie Algebra: $\tilde{\mathfrak{g}} = \mathcal{L}(\mathfrak{g})$, $\tilde{\mathfrak{k}} \subset \tilde{\mathfrak{g}}$, $\tilde{\mathfrak{k}} \rightarrow$ ^{Subalgebra} invariant^{if} $[\tilde{\mathfrak{k}}, \tilde{\mathfrak{g}}] \subseteq \tilde{\mathfrak{k}}$ or 0

We say that a Lie Algebra is simple if it has no ~~invariant~~ non-trivial invariant subalgebra.

A simple Lie group gives rise to a simple Lie Alg. & vice versa

Suppose $\mathcal{L}(h)$ is simple, but the adjoint representation $\{F^a\}$ is reducible. $F_a = \begin{pmatrix} \text{///} & 0 \\ 0 & \text{///} \end{pmatrix}$

$$F_a = \begin{pmatrix} \tilde{a} \tilde{b} & \tilde{a} \bar{b} \\ \bar{a} \tilde{b} & \bar{a} \bar{b} \end{pmatrix}$$

$$(F^a)_{bc} = \begin{cases} F^a_{\tilde{b}\tilde{c}} \neq 0 \\ F^a_{\tilde{b}\bar{c}} \neq 0 \\ F^a_{\bar{b}\tilde{c}} = 0 \\ F^a_{\bar{b}\bar{c}} = 0 \end{cases}$$

According to definition: $F^a_{\tilde{b}\bar{c}} = -i f_{a\tilde{b}\bar{c}}$
 $= +i f_{\bar{b}a\tilde{c}} = -(F^{\bar{b}})_{a\tilde{c}}$
 $\rightarrow 0 \text{ if } a \neq \bar{a}$

So, $(F^{\tilde{a}})_{\tilde{b}\tilde{c}} = 0$

$(F^{\bar{a}})_{\tilde{b}\tilde{c}} = 0$

$\therefore \{F^a\} = \{F^{\tilde{a}}, F^{\bar{a}}\}$

So, $F^{\bar{a}} = \begin{pmatrix} 0 & 0 \\ 0 & \text{///} \end{pmatrix}$

$F^{\tilde{a}} = \begin{pmatrix} \text{///} & 0 \\ 0 & 0 \end{pmatrix}$

$\nRightarrow \mathcal{L}(h)$ is ^{not} simple.

$\therefore$ contradiction.

Lec: 5 Killing metric, Coset Manifold

(pg 31)

— Shouk Alshar 15/6/2020.

$$X \rightarrow X' = U X U^\dagger \quad ; \quad U \in U(2) \quad \text{but } SU(2)$$

$U(1)$ factor of $U(2)$ has trivial action on X

The part $SU(2)$ of $U(2)$ has non-trivial action on X .

$\therefore SU(2)$ is relevant homomorphism.

- # Abelian invariant subalgebra: Subalgebra that ~~commutes~~ commutes with every element in the Lie Algebra.
- # Semi simple Lie algebra: $\mathcal{L}(\mathfrak{g})$ which does not have an Abelian invariant subalgebra.

$$\text{ex} \quad \mathcal{L}(so(4)) = \mathcal{L}(su(2)) \oplus \mathcal{L}(su(2))$$

Some useful geometric concept

Killing metric

$g^{-1} dg \Rightarrow$ Left invariant 1 form

$g \rightarrow hg, h \in \mathfrak{g} \Rightarrow$ but constant
(ie: derivative operator does not act on it)

then $g^{-1} dg$ is invariant under transformation.

$$g^{-1} dg \in T^*(M) \otimes \mathcal{L}(\mathfrak{g})$$

$dg \cdot g^{-1} \Rightarrow$ Right invariant 1 form (ie: invariant under $g \rightarrow gh$)

$$\text{Bi-invariant} \quad \text{Tr}(g^{-1} dg g^{-1} dg) = \text{Tr}(dg g^{-1} dg g^{-1})$$

$\Downarrow$
left invariant by definition

$\hookrightarrow$ use the cyclic property of trace; we see that it is right invariant.

Using the fact that this is a form as well as Lie Algebra valid object:

we can expand it as follows.

$$g^{-1}dg = E^a_\mu(\varphi) T^a d\varphi^\mu$$

$G \Rightarrow$ Compact Lie group (so we can normalize $\text{tr}(T_a T_b) = \lambda \delta_{ab}$)

$\varphi^\mu \Rightarrow$ coordinates on manifold M : The manifold of Lie group G .

$T^a \Rightarrow$ Generators in the defining representation.

$$\begin{aligned} \text{Tr}(g^{-1}dg \cdot g^{-1}dg) &= (E^a_\mu E^b_\nu \cdot \lambda \delta_{ab}) d\varphi^\mu d\varphi^\nu \\ &= g_{\mu\nu}(\varphi) d\varphi^\mu d\varphi^\nu \end{aligned}$$

Identify $E^a_\mu E^b_\nu \lambda \delta_{ab}$ as metric $g_{\mu\nu}(\varphi)$

Then $\text{Tr}(g^{-1}dg \cdot g^{-1}dg)$ looks like line element on the Lie group.
 $\hookrightarrow$ called Killing Metric.

$g \in G$. if compact Lie group; then we can always write $g = e^{iA}$ this belongs to Lie algebra.
 let's write $g = e^{iP_a T}$

$$dg = g \cdot (iT \cdot d\rho)$$

$$g^{-1}dg = iTd\rho \Rightarrow g^{-1}dg = i T \cdot d\rho$$

This says it is in Lie Algebra

This says it's an element of cotangent bundle.

So; $g^{-1}dg \in T^*(M) \otimes \mathcal{L}(G)$

MB3
pg 33

~~$$g^{-1}dg = i T dg$$~~

$$g^{-1}dg = i \underset{\substack{\uparrow \\ \mathcal{L}(G)}}{T} \underset{\substack{\uparrow \\ T^*(M)}}{dg}$$



$$E^a_\mu(\varphi) d\varphi^\mu = \eta^a$$

These are one forms.

$$\text{Tr}(T^a g dg^{-1}) = \lambda E^a_\mu(\varphi) d\varphi^\mu = i d\rho^a$$

relation between φ & ρ

$g_{\mu\nu}$ can give rise to $\Gamma^{\rho}_{\mu\nu}$ = torsion free - connection
 Levi-Civita connection
 Christoffel symbols.

$$\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}$$

There are other connections on G . These connections are not torsion free. And the structure constants do contribute to them.

Group Actions: Let G be a Lie Group and M a manifold. The action of G on M is a map $\sigma : G \times M \rightarrow M$.

s.t. ① $\sigma(e, p) = p \quad \forall p \in M$

$\downarrow$
identity element.

$$\textcircled{2} \quad \sigma(g_2, \sigma(g_1, p)) = \sigma(g_2 g_1, p)$$

Transitive: If for any p_1 and $p_2 \in M$.
 $\exists$ an element $g \in G$ s.t. $p_2 = \sigma(g, p_1)$.
 Then the group action is called Transitive

Free: The group action is free; if for $g \neq e$ there is no fixed point of the action.

i.e. If $p \in M$, $\sigma(g, p) = p \Rightarrow g = e$
 Then we say that the action is free.

Counterexamples Examples

① The action of $SO(3)$ on $\mathbb{R}^3$ is not transitive
 i.e. If $|\bar{x}| \neq |\bar{x}'|$ (where lengths are not equal)
 where $\bar{x}, \bar{x}' \in \mathbb{R}^3$, then $\bar{x}$ and $\bar{x}'$ cannot be connected by ~~via~~ via a rotation.

② The action of $SO(3)$ on S^2 is transitive.
 (but it is not free)

Definition) Little group or Stabilizer Group

Let G be a Lie group that acts on M . The Little group of $p \in M$ is defined to be the subgroup which leaves p invariant.

ie: Little group of p $\rightarrow H(p) = \{ g \in G \mid \phi(g, p) = p \}$

In other words; p is fixed point of the action of $H(p)$.

Example $SO(2) \subset SO(3)$. $SO(2)$ that leaves the north pole invariant is the little group of north pole.

Coset Manifold

Suppose $H \subset G$.

Consider the set $G/H = \{ Gh \mid h \in H \}$

notation $Gh \rightarrow$ coset for the element h .

There are cosets

Coset Manifold

Suppose $H \subset G$, consider the set $G/H = \{ gH \mid g \in G \}$

Coset Space.

These are cosets

if $g_1 = g_2 h$ then $(g_1 \text{ and } g_2 \text{ are in same coset})$
we say $g_1 \sim g_2$. (Equivalence relation)

If $g_1 \not\sim g_2$ (ie: g_1 is near g_2), but $g_1 \not\sim g_2$
(ie: g_1 & g_2 are not in same coset)

Then ~~these~~ cosets corresponding to these group elements g_1 and g_2 are also nearby.

And therefore extend the structure of manifold on the coset space.

Example $SO(3)$ acting on S^2 ; $SO(3)/SO(2)$.
 $e_z \Rightarrow$ position vector pointing to the north pole.

$$\text{If } g_1 \sim g_2 \Rightarrow g_1 e_z = g_2 e_z$$

Starting from the north pole e_z we can reach any other point by applying a representative of a coset such that each coset corresponds to a unique point on S^2 .

So, we can think of S^2 as coset space
 $S^2 = SO(3)/SO(2)$

Finite dimensional representation of $SU(2)$

$$[J_a, J_b] = i \epsilon_{abc} J_c$$

we define ladder operators.

$$\cancel{J_{\pm}} \cdot J_{\pm} = J_1 \pm J_2 \sim \text{ladder operators}$$

These ladder operators are not Hermitian.

$$J_+^\dagger = J_- \quad ; \quad J_-^\dagger = J_+$$

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$[J_+, J_-] = 2J_3$$

$$\text{Rank}(\mathcal{L}(SU(2))) = 1.$$

$$\text{we choose } \text{CSA} = \{J_3\}$$

Label vectors $J_3 |m\rangle = m|m\rangle$

* $|m\rangle \Rightarrow$ a vector on which $\mathcal{L}(SU(2))$ acts.

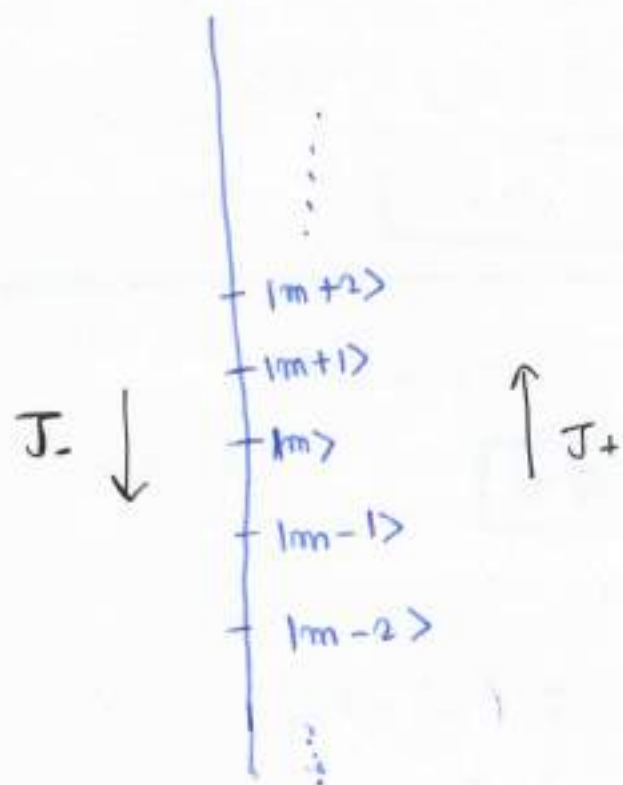
* $m \neq$ not assuming this is an integer.

$$J_3 |m\rangle = m|m\rangle$$

$$J_+ |m\rangle = ?$$

$$\begin{aligned} \text{so; } J_3 (J_+ |m\rangle) &= J_+ J_3 |m\rangle + J_+ |m\rangle \\ &= J_+ m|m\rangle + J_+ |m\rangle \\ &= (m+1) (J_+ |m\rangle) \end{aligned}$$

$$J_3 (J_- |m\rangle) = (m-1) (J_- |m\rangle)$$



We can assume that states are normalized $\langle m|m\rangle = 1$.

$$J_+ |m\rangle = C_{m+1} |m+1\rangle$$

$$J_- |m\rangle = D_{m-1} |m-1\rangle$$

~~C_{m+1}, D_{m-1}~~

C_{m+1}, D_{m-1} are constants to be determined.

$$\langle m+1 | J_+ | m \rangle = \langle m+1 | (J_+ | m+1 \rangle) = \langle m+1 |$$

$$J_- | m+1 \rangle = D_m | m \rangle$$

$$\langle m+1 | J_+ = D_m^* \langle m |$$

$$\langle m+1 | J_+ | m \rangle = D_m^* \langle m | m \rangle = D_m^*$$

$$\Rightarrow \boxed{C_{m+1} = D_m^*}$$

So, we can re write $J_- | m \rangle = C_m^* | m-1 \rangle$

Suppose $m=j$ is the highest possible eigenvalue of J_3

i.e. $J_+ | j \rangle = 0$

So, starting with $J_+ J_- = J_- J_+ + 2J_3$

$$\langle j | J_+ J_- | j \rangle = 0 + 2j \langle j | j \rangle$$

$$\Rightarrow C_j C_j^* \langle j-1 | j-1 \rangle = 2j \langle j | j \rangle$$

$$\Rightarrow \boxed{|C_j|^2 = 2j}$$

for generic m ;

$$\langle m | (J_+ J_- - J_- J_+) | m \rangle = 2 \langle m | J_3 | m \rangle$$

$$\boxed{|C_m|^2 - |C_{m+1}|^2 = 2m}$$

$$|C_{j-1}|^2 = |C_j|^2 + 2(j-1)$$

$$= 2j + 2j-1 = 2(2j-1)$$

$$|C_{j-2}|^2 = |C_{j-1}|^2 + 2(j-2)$$

$$= 2(3j-1-2)$$

$$|C_{j-s}|^2 = 2((s+1)j - 1 - 2 - \dots - s),$$

s is positive integer.

$$|C_{j-s}|^2 = 2(s+1)j - \frac{1}{2}s(s+1)$$

1939

~~we can choose~~

$$|C_{j-s}|^2 = (s+1)(2j-s)$$

$\Rightarrow$ we can choose C_{j-s} to be.

$$C_{j-s} = \sqrt{(s+1)(2j-s)}$$

Now; if we are looking at finite dimensional representation. Then apply J_- ~~ladder~~ ladder operator; at some point it should terminate;

which is possible only if one of the constant C_{j-s} is zero.

$$\Rightarrow \boxed{2j = s} \Rightarrow \boxed{j = \frac{s}{2}} \quad s \in \mathbb{N}.$$

$\Rightarrow m \rightarrow$ is also integer or half integer.

$$\begin{aligned} J_+ |m\rangle &= C_{m+1} |m+1\rangle \\ &= \sqrt{(j-m)(j+m+1)} \cdot |m+1\rangle \end{aligned}$$

$$J_- |m\rangle = \sqrt{(j+m)(j-m+1)} |m-1\rangle$$

When you apply J_+ enough number of times so that $m = j$

lowest you have $-j$.

In total $(2j+1)$ steps in the ladder.

$$|j\rangle \xrightleftharpoons[J_-]{J_+} |j-1\rangle \xrightleftharpoons[J_{-1}]{J_+} |j-2\rangle \dots \xrightleftharpoons[J_{-j}]{J_+} |-j+1\rangle \xrightleftharpoons[J_-]{J_+} |-j\rangle \xrightarrow{J_-} 0$$

$\downarrow J_+$

$$J^2 = J_a J_a$$

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$$ii; J_a J_a |m\rangle = j(j+1) |m\rangle \quad \neq m$$

$J_a J_a \Rightarrow$ Casimir invariant (that labels the whole representation)

$$\langle m | J_a | m \rangle$$

Find the matrices for $j=2$ representation.
 $|2\rangle, |1\rangle, |0\rangle, |-1\rangle, |-2\rangle$

$SU(2)$ tensor and Young tableau

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

in group theory language

$$\text{we write as } 2 \otimes 2 = 3 \oplus 1$$

↓
 refers to
 the fact that
 spin $\frac{1}{2}$ is ~~2~~ 2
 dimensional

↓
 $\frac{1}{2}$ spin
 is 2 dimensional.

spin 1 is
 3 dimensional.

2 of $SU(2)$

$$\psi_a = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2$$

if you have Lie Algebra

$$[T_a, T_b] = i \epsilon_{abc} T_c$$

$$\tilde{T}_a \equiv (-T_a)^*$$

$$\Rightarrow \text{we can check } [\tilde{T}_a, \tilde{T}_b] = i \epsilon_{abc} \tilde{T}_c$$

$\tilde{T}_a$ satisfy the same Lie Algebra.

$\tilde{T}_a$ form conjugate representation. (Anti-fundamental)

However for $SU(2)$ $\tilde{T}_a = C T_a C^{-1}$
 where $C = i \sigma_2$

for $SU(2)$; the fundamental & anti-fundamental representation are related by similarity transformations.

The vector space ~~on~~ on which Anti-fundamental representation acts on:

their elements are ψ_a^* . 2^*

for $SU(2)$; $2 \cong 2^*$ (They are isomorphic)

$SU(m)$, $m \geq 3$; ~~on 2~~

$m \neq m^*$
 $\nearrow$ fundamental representation
 $\nwarrow$ Anti-fundamental representation.

for $SU(2)$ 2 & 2^* are same vector spaces.

① Fundamental Representation: 2

Under $SU(2)$ $\psi_a \rightarrow U_a^b \psi_b$

Invariant Tensors.

① ϵ_{ab} , ϵ^{ab}

Under $SU(2)$ $\epsilon_{ab} \rightarrow U_a^{a'} U_b^{b'} \epsilon_{a'b'}$

$$= \det(U) \cdot \epsilon_{ab}$$

$$= 1 \cdot \epsilon_{ab} = \epsilon_{ab}$$

Now, we pretend that 2^* is different representation ~~for~~ for the sake of good notations.

$$2^* : \psi^a = \psi_a^*$$

ZEE book.

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Under a $SU(2)$ transformation.

$$\psi_a \rightarrow (U^+)_{a'}^a \psi_{a'}$$

$$\begin{aligned} \delta_{ab} &\rightarrow (U^+)_{a'}^a (U^+)_{b'}^b \delta_{a'b'} \\ &= \delta_{ab} \end{aligned}$$

χ_a, ψ_b

we can contract to make singlet out of them; and get a representation which is trivial.

Contracting a downstairs index with upstairs index is done

by the tensor δ_b^a

which should transform as

$$\delta_b^a \rightarrow U_{b'}^{b'} (U^+)_{a'}^a \delta_{b'}^{a'}$$

$$= U_{b'}^{b'} (U^+)_{a'}^a \delta_{b'}^{a'}$$

$$= \delta_b^a$$

So δ_b^a is also invariant tensor of $SU(2)$

Correcting mistake of lecture 5

$$g = e^{i\zeta T}$$

Then $dg = e^{i\zeta T} d\zeta T$ (This is not true)

because $(\zeta T)(d\zeta T)$ don't commute.

$g^{-1}dg \in T^*(M) \otimes \mathcal{L}(G)$ This is true

ex in G.R. $T_\mu^\nu \in T^*(M) \otimes T(M)$

$SU(2)$

$2 = \psi_a ; a=1,2$

$2^* = \psi_a^* \equiv \psi^a$

2 and 2^* are related

i.e. $2 = 2^*$

$\Sigma^{ab}, \Sigma_{ab}, \delta^a_b$ are invariant tensor under $SU(2)$.

$\Sigma^{ab} \psi_a : \psi_a \rightarrow U_a^b \psi_b$

$$\begin{aligned} \Sigma^{ab} \psi_a &\rightarrow (U^\dagger)_{b'}^b (U^\dagger)_{a'}^a U_a^c \psi_c \cdot \Sigma^{b'a'} \\ &= (U^\dagger)_{b'}^b \delta_{a'}^c \psi_c \Sigma^{b'a'} \\ &\equiv \Sigma^{b'a'} (U^\dagger)_{b'}^b \delta_{a'}^c \psi_c \\ &= (U^\dagger)_{b'}^b \Sigma^{b'a'} \psi_c \end{aligned}$$

$\Rightarrow (\Sigma^{ab} \psi_a) \rightarrow (U^\dagger)_{b'}^b (\Sigma^{b'a'} \psi_c)$

define $\chi^b = \Sigma^{ba} \psi_a$

Then $\chi^b \rightarrow (U^\dagger)_{b'}^b \chi^{b'}$

if $\psi_a = 2$, then $\sum a_b \psi_b = 2^*$

(pg 44)

$SU(2)$ tensor with 2 indices ψ_{ab}

$$\text{then } \psi_{ab} = \psi_{[ab]} + \psi_{(ab)}$$

We can check that under $SU(2)$, $\psi_{[ab]}$ & $\psi_{(ab)}$ don't

mix
So, they actually form irreducible representation.

$\psi_{abcd} \Rightarrow$ not in general ~~is~~ irreducible.

$$\chi_a \psi_b \xrightarrow{2 \otimes 2^* \text{ contract}} \delta_a^b \chi_a \psi_b \quad \text{This transforms as a singlet.}$$

$$\square \quad \psi_{ab} = \chi_a \psi_b$$

$$\chi_a \psi_b = 2 \otimes 2$$

$$\chi_a \psi_b = \chi_{[a} \psi_{b]} + \chi_{(a} \psi_{b)}$$

↓
This has one component

↘ This has 3 components

So, $2 \otimes 2$ breaks up into $1 \oplus 3$

$$\text{ie: } \chi_a \psi_b = \chi_{[a} \psi_{b]} + \chi_{(a} \psi_{b)} = 1 \oplus 3$$

$$\Rightarrow 2 \otimes 2 = 1 \oplus 3$$

This we know from Quantum Mechanics; when you combine two spin $1/2$ particles, you get a vector representation & a scalar representation.

A singlet does not change under $SU(2)$ $\phi \rightarrow \phi$

(1995)

A fundamental under $SU(2)$: $\psi_a \rightarrow U_a^b \psi_b$

Young Tableau

a fundamental $\psi_a = 2 = \boxed{a}$

Rank 2 tensor with two indices which is anti-symmetric $\psi_{[ab]} = 1 = \left. \begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array} \right\} \text{antisymmetrised.}$

$$\psi_{(ab)} = 3 = \boxed{a} \boxed{b}$$

So; the statement $2 \otimes 2 = 1 \oplus 3$ in the box language can be written as

$$2 \otimes 2 = 1 \oplus 3$$

$$\square \times \square = \begin{array}{|c|} \hline \square \\ \hline \end{array} + \square \square$$

for $SU(2)$ $\begin{array}{|c|} \hline \square \\ \hline \end{array} \} \Rightarrow \text{something like this cannot exist.}$

ex) $2 \otimes 2 \otimes 2 = ?$

ie: $\square \times \square \times \square = ?$

$= (\square \times \square) \times \square$

~~ex)~~

So; the non-trivial part is.

$$\square \square \times \square = ?$$

we know

$$\square \times \square = \square \square + \begin{array}{|c|} \hline \square \\ \hline \end{array} \text{singlet}$$

$$\text{So; } \begin{array}{|c|} \hline \square \\ \hline \end{array} \times \square = \square$$

because $\begin{array}{|c|} \hline \square \\ \hline \end{array}$ is like 1
ie; like multiplying ~~scalar~~
scalar & fundamental

for $(\square \times \square)$ let's switch to tensor notation

then $3 \chi_{(a} \psi_{b)} \cdot \psi_c \in (\square \square \times \square)$

$$= (\chi_{(a} \psi_{b)} \psi_c + \chi_{(b} \psi_{c)} \psi_a + \chi_{(c} \psi_{a)} \psi_b) \\ + (\chi_{(a} \psi_{b)} \psi_c - \chi_{(b} \psi_{c)} \psi_a) \\ + (\chi_{(a} \psi_{b)} \psi_c - \chi_{(c} \psi_{a)} \psi_b)$$

expand last line

$$\frac{1}{2} [\chi_a \psi_b \psi_c + \chi_b \psi_a \psi_c + \chi_b \psi_c \psi_a + \chi_c \psi_b \psi_a + \chi_c \psi_a \psi_b \\ + \chi_a \psi_c \psi_b] \rightarrow \text{symmetric in } a, b, c$$

so; have 4 components in $SU(2)$.

$$\Psi(a, \dots, a_n) = n+1$$

expand second line

$$\frac{1}{2} [\chi_a \psi_b \psi_c + \chi_b \psi_a \psi_c - \chi_b \psi_c \psi_a - \chi_c \psi_b \psi_a]$$

expand third line ... (same as second line); just different label)

antisymmetric in c & a , and contract with ϵ^{ca} (we don't lose any information because it's already anti-symmetric in c and a)

$$\frac{1}{2} [\chi_a \psi_b \psi_c + \chi_b \psi_a \psi_c - \chi_b \psi_c \psi_a - \chi_c \psi_b \psi_a] \epsilon^{ca} = \eta_b$$

call it η_b ; because the only free index now left is b .

but $\eta_b = 2$

So; that second line object is a 2.

Suppose $A_{ab} = -A_{ba}$ A anti-symmetry in 2 indices

$\begin{array}{c c} 1 & 2 \\ \hline 1 & 2 \end{array}$	$\begin{array}{c c} 2 & 1 \\ \hline 2 & 1 \end{array}$
--	--

only ~~2~~ 2 independent components.

for $SU(2)$; $\boxed{a_1} \boxed{a_2} \dots \boxed{a_n} = n+1$

$$\square \times \square \times \square = \square \square + \square + \square$$

so; $2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2$

→ note; the second & third line live in same vector space so they add up
 so; the second & third line just sum up to $\square$

for $SU(3)$ tensors :

$3 = \psi_a$: fundamental.

$3^* = \psi_a^* = \psi^a$: anti-fundamental. $\tilde{T}_a = -T_a^*$

$\nexists$ any C s.t. $\tilde{T}_a = C T_a C^{-1}$

Invariant tensors $\epsilon^{abc}, \epsilon_{abc}, \delta^a_b$.

Consider, $\psi_{[ab]}$ has 3 independent components in 3 dimensions.

$$\psi_{[ab]} = 3$$

then $3^* = \psi_a = \sum^{abc} \varphi_{[bc]}$

(pg 18)

this motivates $3 = \square = \psi_a$

but $3^* = \psi^a = \square$

motivation of $\square$
from $\psi_a = \sum^{abc} \varphi_{[bc]}$
because something is
being anti-symmetrized.

ex 11 $\sum^{abc}$

$\sum^{abc} \varphi_{[bc]} \rightarrow U$

note: $\square$ has 1 for $SU(2)$

but here: $1 = \square$ in $SU(3)$.

Examples

① $3 \otimes 3 = ?$

$\psi_a \chi_b = \psi_{[a} \chi_{b]} + \psi_{(a} \chi_{b)}$

$\downarrow$ $\downarrow$
 3 6

~~$3 \otimes 3 = 3^* \oplus 6$~~

$\square \times \square = \square + \square$

$3 \otimes 3 = 3^* \oplus 6$

② $3 \otimes 3^* = ?$

$\psi_a = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$; we are taking a fundamental of triplet of quarks; and 3^* is then anti-quarks.

we will find $\psi^a = \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix}$

$$\psi_a \psi_b = \underbrace{(\psi_a \psi_b - \frac{1}{3} \delta_a^b \psi_c \psi_c)}_8 + \underbrace{\frac{1}{3} \delta_a^b \psi_c \psi_c}_1 \quad (1939)$$

Now to represent 8

Answer



$$\cancel{\psi_a \psi_b} \quad \psi_b \psi_a = \underbrace{\epsilon^{bcd} \psi_{[cd]}}_{\text{8}} \underbrace{\psi_a}_{\text{1}}$$

i.e.:



here indices along column is antisymmetrized



but nothing along row.

This tell us that

$$3 \times 3 = \text{8} + \text{1}$$

$$3 \otimes 3^* = 8 \oplus 1$$

read Jones book for more details.

$$P_a^b = \psi_a \psi^b - \frac{1}{3} \delta_a^b \psi_c \psi^c$$

it turns out that ... we have wave functions of mesons.

$$P_a^b = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{-\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix}$$

$$3 \otimes 3 \otimes 3$$

(1950)

We can show $\square \times \square \times \square = \square\square\square + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$

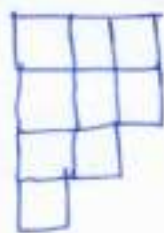
$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

Young's Tableau

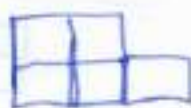
Refer to Jones Book.

~~① A legal Young's~~

① In a legal Young's Tableau, the length of any row cannot be larger than the row above.



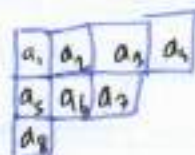
allowed



Not allowed

② The number of boxes in a column for ~~even~~ $SU(n)$ cannot be more than n due to antisymmetry.

③ Suppose we have a Young's Tableau ~~corresponding to~~ a rank n tensor with n boxes. We have a rank n tensor from which we have to construct the irrep. corresponding to the Tableau.



$$\psi_{a_1 \dots a_8}$$

Then

③ a) We first symmetrize the indices which lie on each row. $\psi_{(a_1 a_2 a_3 a_4) (a_5 a_6 a_7) a_8}$

③ b) Antisymmetrize the columns.

$$\psi_{(a_1 \dots a_4) (a_5 \dots a_7) a_8} \rightarrow \text{Antisymmetrize}$$

④ How to compute the dimension of a tableaux.

SU(N)



N	N+1	N+2
N-1	N	
N-2	N-1	
N-3		

6	4	1
4	2	
3	1	
1		

SU(3)

3	4
2	

3	1
1	

$$= \frac{3 \times 4 \times 2}{3 \times 1 \times 1} = 8$$

⑤ How to expand the product of 2 ~~tableaux~~ tableaux as a sum of tableaux.

T_1

$\times$

a	a
b	
c	

T_2

- i) Label the rows of T_2 by a for the first row,
b " " second " " ~~1~~,
c " " third

- ii) Add one box from T_2 to T_1 at a time starting from the top row making sure the following is satisfied. (at each step)

~~Each T_1 must be a legal!~~

(a) Each tableaux must be a legal Young Tableau at each stage.

(b) More than one box with the same label must not appear in the same column.

(c) For each box assign a number $n_a = \#$ of a's above and right to it.

Similarly assign a number $n_b = \#$ of b's above and right to it.

$$m_a \geq m_b \geq m_c \geq \dots$$

1252

(iii) If two tableaux are the same, then they are counted the same if their labels are the same. Otherwise they refer to distinct irreps.

(iv) ~~Can~~ Cancel columns with m boxes for $SU(m)$ because they are trivial.

ex for $SU(2)$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = \square$$

Example

$SU(3)$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline a & a \\ \hline b & \\ \hline \end{array}$$

$$= \left(\begin{array}{|c|c|c|} \hline & & a \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & a \\ \hline & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline a & \\ \hline \end{array} \right) \times \begin{array}{|c|c|} \hline a \\ \hline b \\ \hline \end{array}$$

$$= \left(\begin{array}{|c|c|c|c|} \hline & & a & a \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & a \\ \hline & a & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & a \\ \hline & & \\ \hline a & & \\ \hline \end{array} \right) \times \begin{array}{|c|} \hline b \\ \hline \end{array}$$

$$+ \left(\begin{array}{|c|c|c|c|} \hline & & & a \\ \hline & & a & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & a \\ \hline & a & \\ \hline a & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & a \\ \hline & a & \\ \hline & a & \\ \hline \end{array} \right) \times \begin{array}{|c|} \hline b \\ \hline \end{array}$$

$$+ \left(\begin{array}{|c|c|c|} \hline & & a \\ \hline & & \\ \hline a & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & a \\ \hline a & \\ \hline \end{array} \right) \times \begin{array}{|c|} \hline b \\ \hline \end{array}$$

$$= \begin{array}{|c|c|c|c|} \hline & & a & a \\ \hline & & & \\ \hline & b & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline & & & a & a & b \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & a & a \\ \hline & & & \\ \hline & b & & \\ \hline \end{array}$$

$m_a = 2$
 $m_b = 1$

have $m_a < m_b$
(so not allowed)

+

Intermediate steps

(1953)

$$= \left(\begin{array}{|c|c|c|c|} \hline & & a & a \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & a \\ \hline & a & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & a \\ \hline & & \\ \hline a & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & a \\ \hline & \\ \hline a & \\ \hline & \\ \hline \end{array} \right) \begin{array}{|c|} \hline b \\ \hline \end{array}$$

$$= \begin{array}{|c|c|c|c|} \hline & & a & a \\ \hline & b & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & a & a \\ \hline & & & \\ \hline & & & \\ \hline b & & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & a \\ \hline & a & b \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & a \\ \hline & & \\ \hline & a & \\ \hline b & & \\ \hline \end{array}$$

$$+ \begin{array}{|c|c|c|} \hline & & a \\ \hline & & \\ \hline & b & \\ \hline a & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline a & b & \\ \hline \end{array}$$

cancel three columns
because we are
doing SU(3)

$$= \begin{array}{|c|c|c|} \hline & & a|a \\ \hline & b & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & a & a \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & a \\ \hline & a & b \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & a \\ \hline a & \\ \hline & \\ \hline & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & a \\ \hline b & \\ \hline & \\ \hline & \\ \hline \end{array} + 1$$

$$= 27 \oplus 10 \oplus 10^* \oplus 8 \oplus 8 \oplus 1$$

[Faint, illegible handwriting throughout the page, possibly bleed-through from the reverse side.]

LIE GROUPS DON'T LIE.

THANK YOU

Shoaib Akhtar
