

Lec-1 Special Relativity & Electrodynamics

: The Lorentz Transformation
— Shoaib Akhtar

(PS1)

* What is Relativity mean?

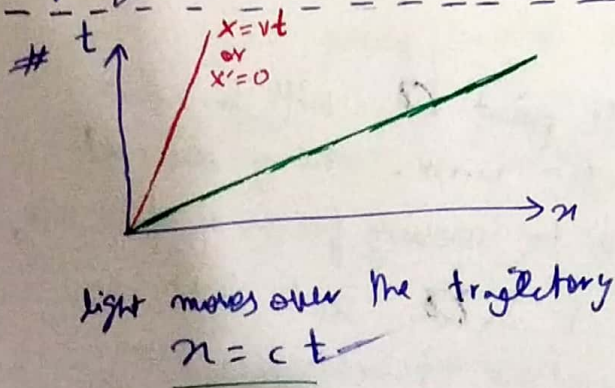
Relativity means that for all coordinate systems which are related to each other by uniform velocities; the laws of Physics are the same.

* Principle of Relativity: Laws of Physics are the same in every reference frame.
(not invented actually by Einstein)

* Einstein added ... law... Speed of light is c (3×10^8 m/s) ^{speed}
→ Combining these we can conclude light moves with same ~~velocity~~ ^{speed} "c" in every reference frame.

(Einstein said; its Law that "light moves with certain velocity")

→ and Principle of Relativity says that Laws of Physics should be unchanged in every reference frame (inertial)... so speed of light is same in all reference frame.



~~non-relativistic~~

$$\begin{cases} x' = x - vt \\ t' = t \end{cases} \text{ (Newton or Galileo assumption)}$$

→ This would be the transformation of coordinates. ~~to~~

* other frame is moving with velocity v .
(x', t') .. actually $x=vt$ equivalent to $x'=0$

∴ here; light moves in the moving frame;

$$\begin{aligned} x' &= x - vt \\ x &= ct \\ \Rightarrow x' &= ct - vt = (c-v)t \Rightarrow x' = (c-v)t \text{ or } \boxed{x' = (c-v)t} \end{aligned}$$

but $x' = (c-v)t'$ is violating Einstein rule.

* also; for light going in other direction $x = -ct$

$$\Rightarrow x' = -ct - vt \Rightarrow x' = -(c+v)t$$

$$\Rightarrow x' = -(c+v)t'$$

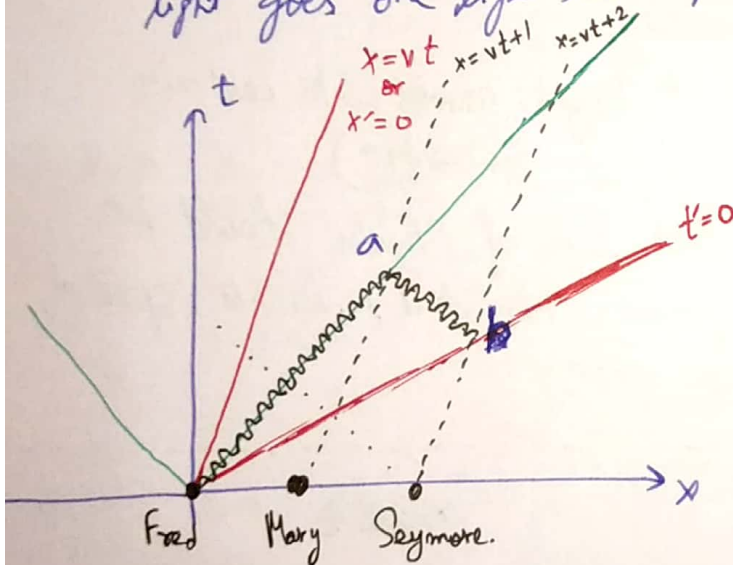
* One of the assumption he made

here is wrong: i.e. $t' = t$... i.e. simultaneity in every reference frame is same.

NOTE: for sake of simplicity & understanding ... we work in different units ...

new units: for distance $\Rightarrow$ light second.
for time $\Rightarrow$ second.

light goes one light second per second.



1 unit, 1 unit

steps
* F sends light signal to M
& at some point S also goint
to send light signal to M.
 $\hookrightarrow$ they send it in such a
way that they arrive at M
at same instant.

The point **a** will be called
 $t'=0$ w.r.t. moving observer
(because in moving frame of reference,
the points F & **a** sent light rays
to the central observer M here
that arrive at same instant of
time.

* b must be a point
for which $t'=0$

finding point a :

$$x = t$$

(speed of light made 2)

(p93)

$$x = t$$

$$x = vt + 1 \Rightarrow t = vt + 1 \Rightarrow t(1-v) = 1 \Rightarrow t = \frac{1}{(1-v)}$$

$$\therefore t_a = \frac{1}{1-v} ; x_a = \frac{1}{1-v}$$

$\therefore$ now; line ab. ; $x + t = \text{constant}$
so; $x + t = x_a + t_a$

$$\Rightarrow x + t = \frac{2}{1-v}$$

$\therefore$ for point b; $x + t = \frac{2}{1-v}$ and $x - vt = 2$

$$\Rightarrow t(1+v) = 2 \left(\frac{1}{1-v} - 1 \right) \Rightarrow t(1+v) = 2 \left(\frac{1-1+v}{1-v} \right)$$

$$\Rightarrow t = \frac{2v}{(1-v^2)}$$

$$\therefore t_b = \frac{2v}{1-v^2} \equiv t'_b = 0$$

$$\therefore x_b = \frac{2}{1-v} - \frac{2v}{1-v^2} = \frac{2}{(1-v^2)} \cdot \{ (1+v) - v \} = \frac{2}{(1-v^2)}$$

$$\Rightarrow x_b = \frac{2}{1-v^2} \Rightarrow \text{so;}$$

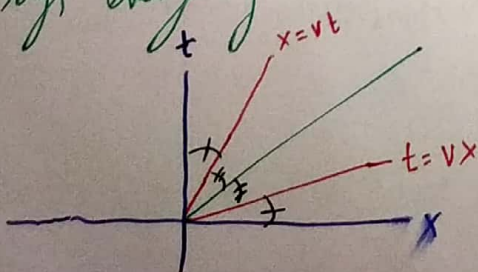
$$x_b = \frac{2}{1-v^2} ; t_b = \frac{2v}{1-v^2}$$

$\nexists$ now; line passing through origin & b is $t'_b = 0$.

$\hookrightarrow$ has slope $\frac{t_b}{x_b} = v$ slope of the line ; so; the line is ; $t = v x$

$\rightarrow$ All along this line, the moving observer

says everything is simultaneous.



* so; if Einstein is right; what is synchronous in one frame is not synchronous in other frame.

$x' = (x - vt) f(v)$; we know; $x = 0$; when $x = vt$

so; the equation of x' has factor of $(x - vt)^n$

∴ we assume $n = 1$

→ The function here is just the magnitude of the velocity;
 $v = |v|$
... assumption.

Similarly

$$t' = (t - vx) g(v)$$

∴ using speed of light is same

... i.e.; whenever; $x = t$ then $x' = t'$

so; $x' = x(1-v) f(v)$
 $t' = x(1-v) g(v)$ $f(v) = g(v)$

so;
$$\begin{cases} x' = (x - vt) f(v) \\ t' = (t - vx) f(v) \end{cases}$$

Now; who is to say whose frame is moving relative to ~~whom~~ whom with velocity v ; or ~~that for the~~ ~~later~~ latter moving w.r.t. former with velocity $-v$.

↳ so; whatever the relationship between two frames of reference are ; they should be symmetrical.

↳ i.e.; ~~we~~ instead of starting with x & t ; and drawing in x' & t' ; we can do exactly the opposite.

→ The only difference would be as far as I am concerned you are moving with velocity $-v$; as far as you are concerned ~~you~~ I am moving with $+v$.

so;

$$X = (x' + vt') f(v)$$

$$t = (t' + vx') f(v)$$

(p 5)

so;

is this compatible with : $x' = (x - vt) f(v)$
 $t' = (t - vx) f(v)$

actually from this we can't solve for x & t ...

★

$$X = ((x - vt) f(v) + v(t - vx) f(v)) f(v)$$

$$\Rightarrow X = \{ (x - vt) + v(t - vx) \} f^2(v)$$

$$\Rightarrow X = \{ (x - v^2 x) \} f^2(v)$$

only way this will make sense if $x = X$.

$$2) X = X (1 - v^2) f^2(v) \Rightarrow (1 - v^2) f^2(v) = 1$$

$$\Rightarrow f(v) = \frac{\pm 1}{\sqrt{1 - v^2}}$$

we could easily reject the -ve solution... by some logic

The -ve root corresponds to the possibility of reflecting the coordinates.
 (it should not depend on velocity)

so; for our purpose now; $f(v) = \frac{1}{\sqrt{1 - v^2}}$

There are Lorentz Transformation... They have been worked up to make sure that in every frame speed of light is c .

so;

$$X' = \frac{(x - vt)}{\sqrt{1 - v^2}} ; t' = \frac{(t - vx)}{\sqrt{1 - v^2}}$$

there should not be any point in velocity which you suddenly jump from one root to the other... that would be a very discontinuous thing to for the function of velocity f .

Note:

when $v = 0$; we want both frame should be same.

i.e. $x' = x$ & $t' = t$ (achieved by the root)

but negative root will result $\Rightarrow x' = -x$ & $t' = -t$.

★ If speed of light chosen c (in other units)

we must expect dimensional consistency

can also explicitly solve for them.

$$\boxed{\begin{aligned} x' &= \frac{(x - vt)}{\sqrt{1 - (v/c)^2}} \\ t' &= \frac{t - (vx/c^2)}{\sqrt{1 - (v/c)^2}} \end{aligned}} \quad \text{Lorentz Transformations.}$$

$$\boxed{\begin{aligned} y' &= y \\ z' &= z \end{aligned}}$$

$$x = ct$$

$$ct = z$$

$$\Rightarrow x = z$$

$$\text{so: } t = z/c$$

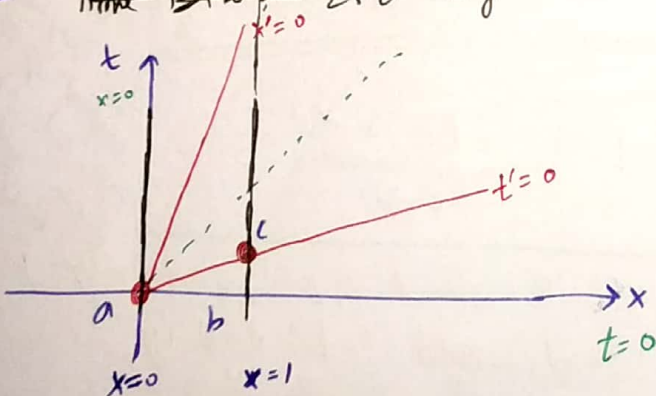
$$\text{so: } t \rightarrow t/c := t$$

if $v \ll c \Rightarrow x' = x - vt$ & $t' = t$ (Galilean Transformation as a special case of Lorentz Transformation)

★ by symmetry of the problem; the rest of $\perp$ direction.

... Lorentz Transformation would involve : $y' = y$
 $z' = z$

~~Time Dilation~~ Length Contraction.



∴ ~~non moving~~

* rest frame

say a, b ~~is length~~ are end of meter stick.

* moving frame

a, c are end of meter stick.

basically ; we want to know; $x'_c = ?$

$$\text{so: } t' = 0 \Rightarrow \underline{t = xv} ; \text{ but: } x = 1 \Rightarrow \underline{t = v}$$

$$\text{so: } x' = \frac{(x - v(vx))}{\sqrt{1 - v^2}} = \frac{x(1 - v^2)}{\sqrt{1 - v^2}} = x\sqrt{1 - v^2}$$

In moving frame the rod seems to be shorter.

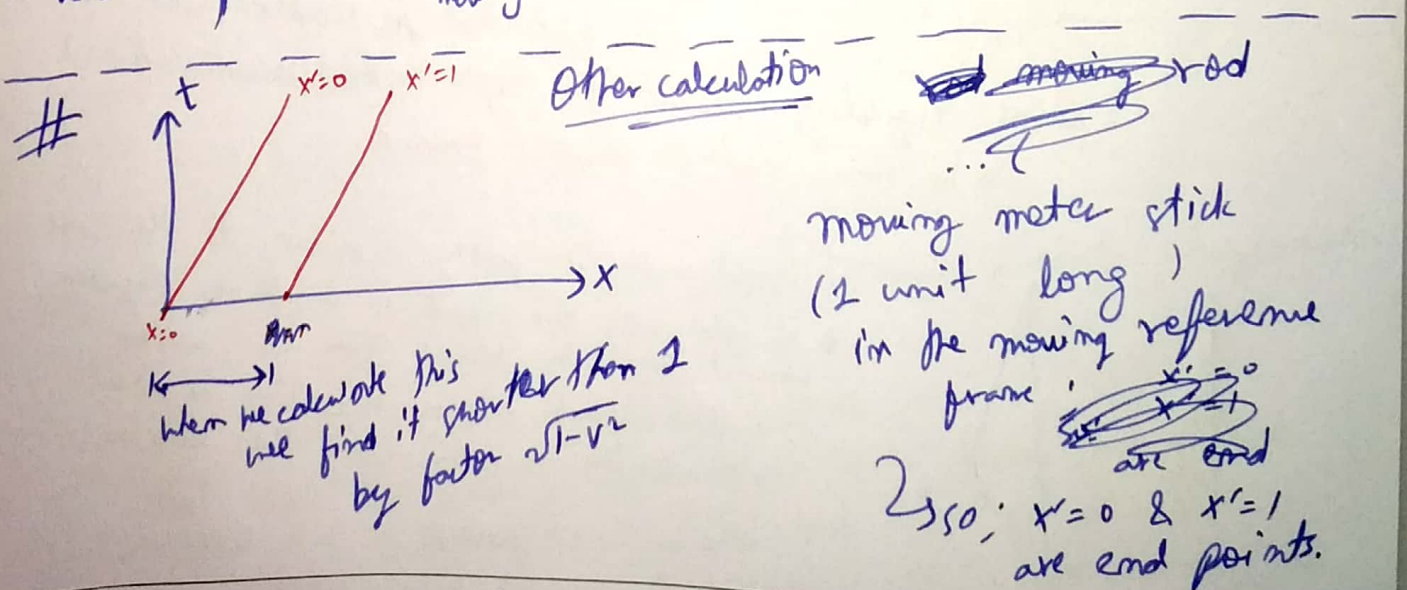
* Note ; we are actually talking about two different things.

(i) The meter stick at rest ; we are talking about distance between a and b , as measured by stationary frame ; ie ; to that metric : $d_{\text{stationary}}(a, b)$

(ii) In the moving frame : we are talking about distance between a and c , as measured by moving observer or moving measuring rods : ie ; by some other metric : $d_{\text{moving}}(a, c)$ * not actually a legitimate metric.

They are not really the same point of space & time here (b, c) .. so there is no real contradiction. That one looks shorter than the other.

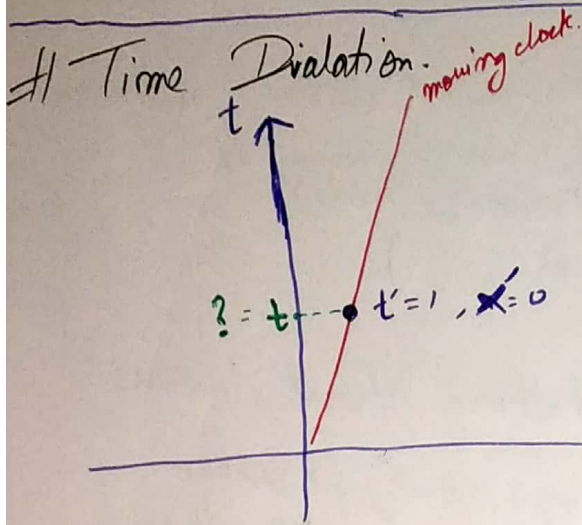
... and we found $d_{\text{moving}}(a, c) < d_{\text{stationary}}(a, b)$



- The moving meter stick looks short to stationary reference frame.
- The stationary " " " " moving " "

↳ NO contradiction... They are just talking about different things.

* They are actually talking about different notions what they mean by length.



$$x = \frac{x' + vt'}{\sqrt{1-v^2}}$$

$$x = \frac{x' + vt'}{\sqrt{1-v^2}}$$

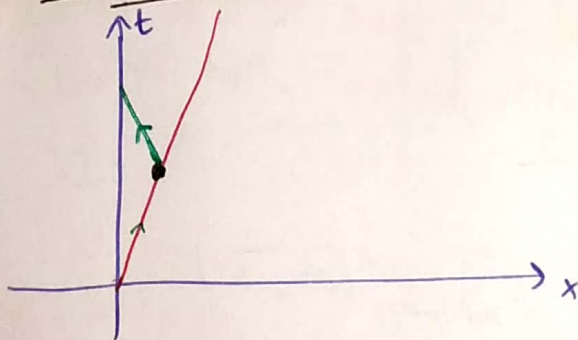
$$t = \frac{t' + vx'}{\sqrt{1-v^2}}$$

$$t' = 1, x' = 0$$

$$\Rightarrow t = \frac{1}{\sqrt{1-v^2}}$$

$$\Rightarrow t = \frac{1}{\sqrt{1-v^2}} > t' = 1$$

★ Twin Paradox



Remember these two frames are not symmetrical w.r.t. each other.

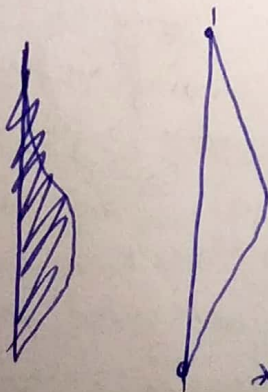
One of them under went acceleration.

- The twin who goes & come back experiences acceleration.

(he knew that he underwent a

curve trajectory... because he experienced acceleration)

→ he was not in inertial frame.



The mathematics is pretty similar to the case

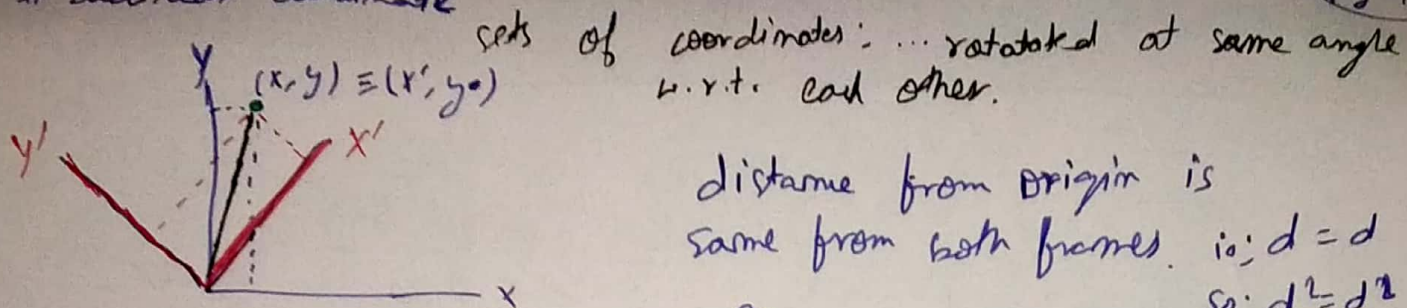
- if you take two points in a plane & you connect them by straight line or curve → then length along the line is not same as the length along the curve. ($d_{\text{line}} < d_{\text{curve}}$)

* And in Relativity case.

→ Time along the curve is little bit less than the time along the line. (There is something going on... different from length... but similar phenomenon... is $d_{\text{line}} \neq d_{\text{curve}}$)

Euclidean coordinate

(pg 9)



distance from origin is same from both frames. i.e. $d = d$
so: $d^2 = d^2$

$$\Rightarrow x^2 + y^2 = x'^2 + y'^2$$

so, the quantity $x^2 + y^2$ is invariant
it does not depend on which coordinate system you work it out in.

Q) Is there analogous quantity associated with Lorentz Transformation which is invariant.

Solution $(t', x') = \gamma(t, x)$

$\therefore$ let's try $t^2 + x^2 = t'^2 + x'^2$ (is it true)

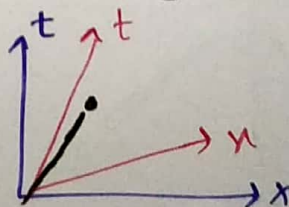
$$\times \frac{t^2 + v^2 x^2 - 2vtx}{1 - v^2} + \frac{x^2 + v^2 t^2 - 2vtx}{1 - v^2} \neq t^2 + x^2$$

$\therefore$ but: $t^2 - x^2 = t'^2 - x'^2$ (✓) is true:

$$\begin{aligned} \gamma(t, x) &= \frac{t^2(1 - v^2)}{(1 - v^2)} + \frac{x^2(v^2 - 1)}{(1 - v^2)} \\ &= t^2 - x^2 \end{aligned}$$

so, $t^2 - x^2$ is invariant quantity under Lorentz Transformation.

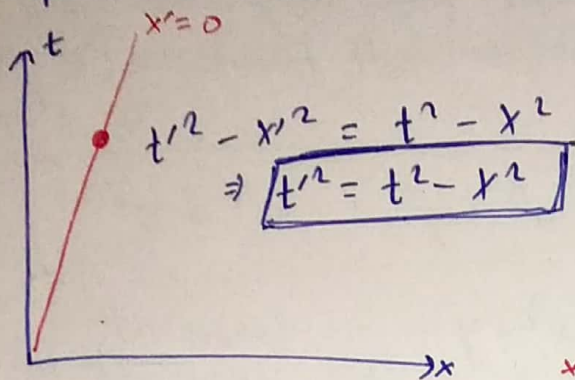
... this is almost like Pythagorean Theorem.
... that there is a kind of notion of distance from origin which is composed out of base of triangle (x) and height of triangle (t) stays same if you do Lorentz transformation, (in other words if you use other coordinates).



Let's define it

Spacetime distance $:= t^2 - x^2$

Special case.



t' is proper time...

$\hookrightarrow$ called Proper time of that clock.

$\hookrightarrow$ measure of space-time distance is called Proper time.

* That's this minus sign " $t^2 - x^2$ " which gives the triangle peculiar property that Hypotenuse is shorter than the height of triangle itself.

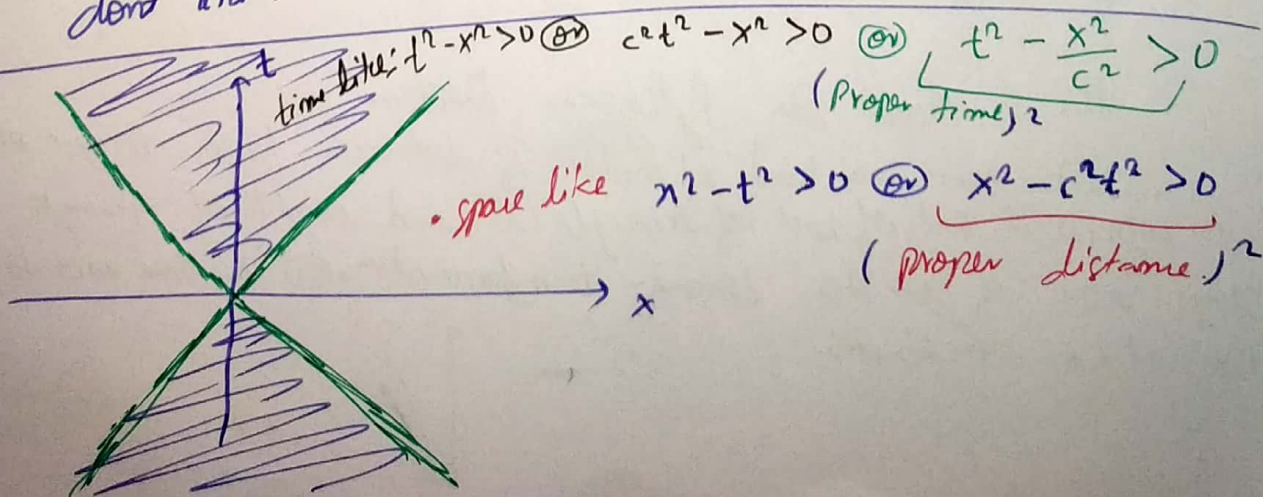
Let $x^2 - t^2 = 0 \Rightarrow \underline{x = \pm t}$ means a light ray (connected by a light trajectory)

$\hookrightarrow$ If the distance between $(t^2 - x^2)$ two point is zero, it does not mean the two points are the same; it means that a light ray can connect them.

... still it means proper time between two of them is zero.

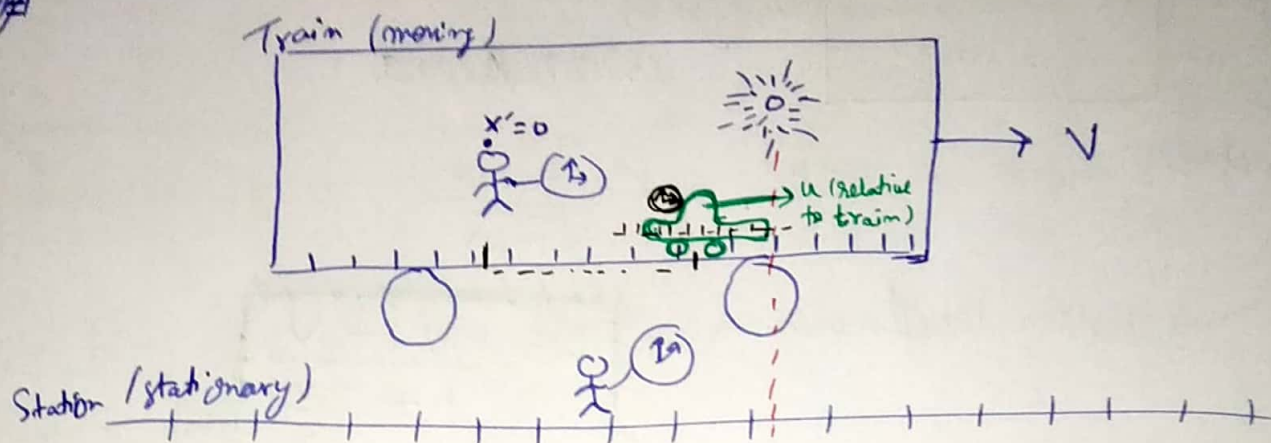
$\hookrightarrow$ In other words; if light beam would carry a clock with it; there would be zero ticks of ~~clock~~ clock between those points.

io: Clocks which move at the speed of light simply don't tick.



- If the two points are spacelike connected, you would call $x^2 - t^2$ proper distance between them.
 - If they are time-like separated; then you would call $t^2 - x^2$ proper time.
-

#



event
glowing of
a ~~light~~ bulb
say.

(x, t) : stationary observer records

(x', t') : moving observer records.

(x'', t'') : coordinates w.r.t. moving kitty car.

$$x' = \frac{x - vt}{\sqrt{1 - v^2}} ; y' = y$$

$$\text{also. } x = \frac{x' + vt'}{\sqrt{1 - v^2}}$$

$$t' = \frac{t - vx}{\sqrt{1 - v^2}} ; z' = z$$

$$t = \frac{t' + vx'}{\sqrt{1 - v^2}}$$

$$\therefore x=0 \Rightarrow \underline{x=vt}$$

let w be the velocity of car with ~~respect~~ respect to stationary observer; let's find $w(u, v)$

$$x'' = \frac{x' - ut'}{\sqrt{1 - u^2}} ; y'' = y'$$

$$t'' = \frac{t' - ux'}{\sqrt{1 - u^2}} ; z'' = z'$$

$$\begin{aligned} \therefore \text{so, } x'' &= \frac{\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - u \cdot \left(\frac{t-vx}{\sqrt{1-v^2}}\right)}{\sqrt{1-u^2}} = \frac{1}{\sqrt{(1-u^2)(1-v^2)}} \left\{ x - vt - ut + uvx \right\} \\ &= \frac{x(1+uv) - (u+v)t}{(\sqrt{1-u^2} \sqrt{1-v^2})} = \frac{x - \left(\frac{u+v}{1+uv}\right)t}{\sqrt{\frac{(1-u^2)(1-v^2)}{(1+uv)^2}}} \end{aligned}$$

so; $x''=0$; when ; $(1+uv)x=(u+v)t \Rightarrow \boxed{x = \frac{(u+v) \cdot t}{(1+uv)}}$

so; $\boxed{W = \frac{u+v}{1+uv}}$ speed of light 2

actually; $x'' = \frac{x - ut}{\sqrt{1-u^2}}$; $t'' = \frac{t - ux}{\sqrt{1-u^2}}$

Restoring the dimensions...

$\boxed{W = \frac{u + v}{1 + \frac{uv}{c^2}}}$

ib $u \ll c$; $v \ll c$ $\Rightarrow W \approx u + v$; (Galileo law of velocity addition as a special case of Lorentz Transformations)

ex let; $c=1$; $u=0.01$; $v=0.01 \Rightarrow W = \frac{0.02}{1+0.0001} < W_{\text{Galileo}}$.
this is actually very small but the correction is very small

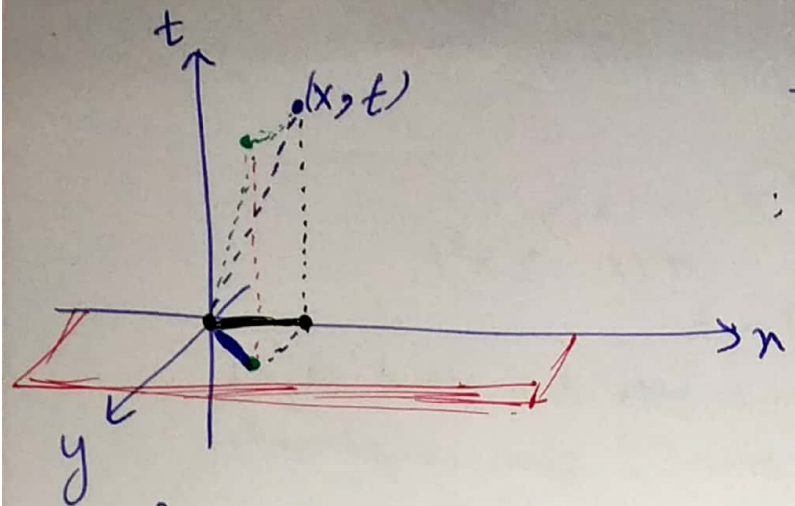
ex let; $c=1$; $u=0.9$; $v=0.9$; then; Newton or Galileo $\Rightarrow W=1.8$

instead; $W = \frac{1.8}{1+(0.9)^2} = \frac{1.8}{1.81} = \frac{1.8}{1.81} < 1$ (speed of light)

★ It is consistent to say all observers moves slower than the speed of light ... consistent with velocity addition formula & Lorentz transformation.
denominator will become imaginary if $v > c$

ex also; ; $u=c$; $v=c$; $W = \frac{c+c}{1+\frac{c^2}{c^2}} = c$ (Amazing)

ex $u=c$; v ; $W = \frac{v+c}{1+\frac{vc}{c^2}} = \frac{v+c}{1+v/c} = c \cdot \frac{(v+c)}{(c+v)} = c$ (Amazing)



$$\tau^2 = t^2 - x^2$$

$$= t'^2 - x'^2$$

$$= t''^2 - x''^2$$

$\therefore \tau$ is called proper interval

* All observer agree on this notion of proper time or interval over here.

* Proper Time is also the time read by clocks moving between these points.

* Let's consider a coordinate rotated physically w.r.t. ~~the~~ unprimed coordinate. (Both inertial) (not moving w.r.t. each other)

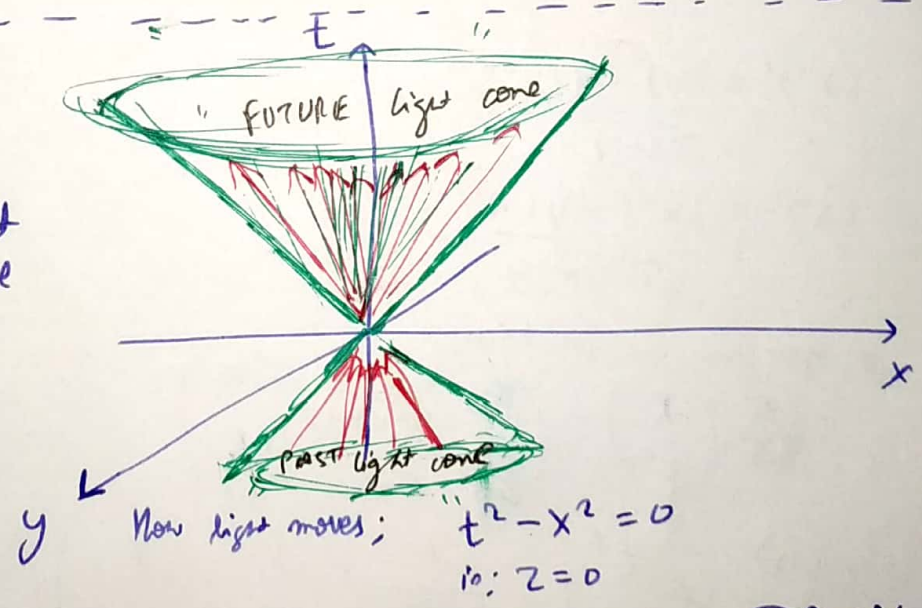
$$x^2 = (x_r)^2 + (y_r)^2 \quad \text{so;}$$

$$\tau^2 = t^2 - x^2 - y^2 - z^2$$

Invariant proper time. (in higher dimensions)

NOTATION'S

- Future light cone is all of the places that light can get to starting at the origin.
- Past light cone is all of the places that can send light to the origin.



note
 * In Newtonian mechanics there is an invariant quantity:
 t
 every body agrees on it.

in more dimensions ~~$t^2 - (x^2 + y^2 + z^2) = 0$~~
 $\therefore z^2 = 0$; but; $z^2 = t^2 - (x^2 + y^2 + z^2) = 0$
 Light cone is set of points that a light ray arrive at if its starts at ~~origin~~ origin.

FOUR VECTOR

& NOTATIONS

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$$\rightarrow (x, y, z) \quad \text{or} \quad x^i \rightarrow (x, y, z) \quad \text{or} \quad (x^1, x^2, x^3)$$

He had to add time t

→ so, vector becomes a four dimensional object with a time component & space components.

adding time component.

$$\rightarrow \cancel{(x, y, z, t)} \quad \text{or} \quad x^\mu \quad ; \mu = \{0, 1, 2, 3\}$$

$$(t, x, y, z)$$

$$; x^\mu \rightarrow (x^0, x^1, x^2, x^3)$$

$$\equiv (t, x, y, z)$$

~~$$z^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$$~~

$$z^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

* when you see i
⇒ index runs over only space

* when you see μ
⇒ index runs over all coordinates including time.

i ⇒ stands for space
 μ ⇒ " " space-time.

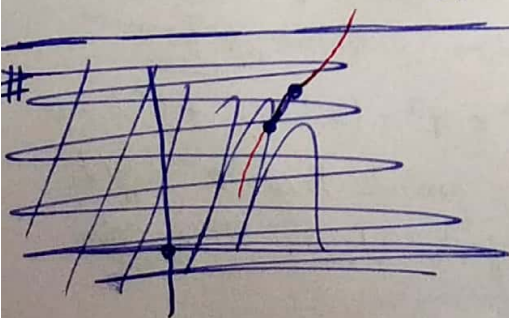
$$(x^1)' = \frac{(x^1) - v(x^0)}{\sqrt{1-v^2}}$$

$$(x^0)' = \frac{(x^0) - v(x^1)}{\sqrt{1-v^2}}$$

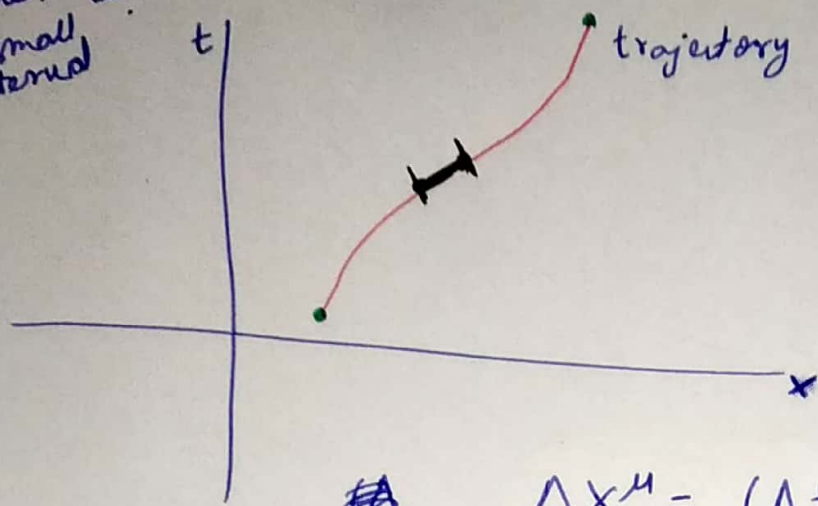
note: you can use matrices:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & -\frac{v}{\sqrt{1-v^2}} \\ -\frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\text{or} \quad \begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$



lets consider a small interval



consider a small interval along the trajectory.

~~Δ~~ $\Delta X^\mu = (\Delta t, \Delta x, \Delta y, \Delta z)$

→ change in the coordinate from ~~going to~~ tail of the vector to the head of the vector.

★ Four Velocity

(we divide by $\Delta \tau$... ie: w.r.t. invariance ... (not time))

$$u^\mu = \frac{\Delta X^\mu}{\Delta \tau}$$

$$(\Delta \tau)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

~~Square of invariance space time~~
Square of invariance space time distance between two points.

Lec-3 Special Relativity & Electrodynamics : Relativistic Laws of motion and $E = mc^2$. (Pg 19)
— Shoaib Akhtar.

∴ ~~1/11/11/11~~ Binomial theorem:

$$(1 + \epsilon)^p = \frac{1}{0!} + \frac{p \cdot \epsilon}{1!} + \frac{p(p-1)}{2!} \epsilon^2 + \frac{p(p-1)(p-2)}{3!} \epsilon^3 + \dots \infty$$

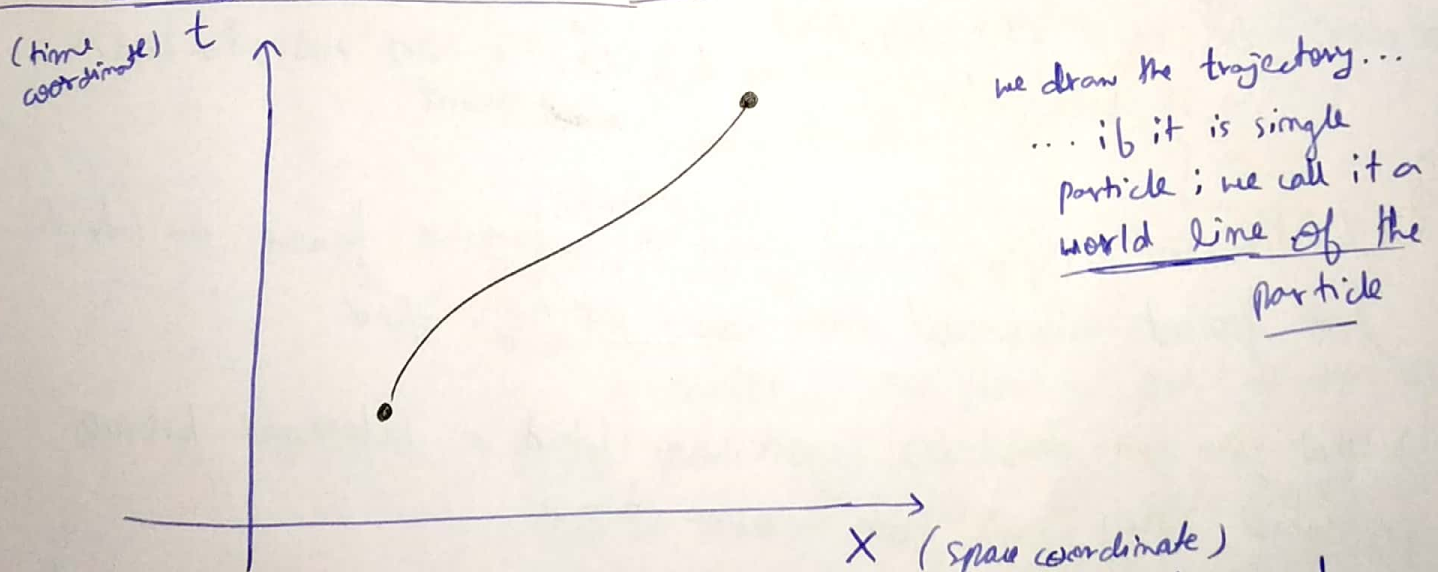
if $\epsilon \ll 1 \Rightarrow \boxed{(1 + \epsilon)^p \approx 1 + p \cdot \epsilon}$ (first order approximation)

$\sqrt{1 - v^2} = 1 + \frac{1}{2} \cdot (-v^2) = 1 - \frac{v^2}{2}$

if $v \ll c$ or $v \ll c$; here $\epsilon = -v^2$; & $p = 1/2$

$\frac{1}{\sqrt{1 - v^2}} = (1 - v^2)^{-1/2} = 1 + \frac{v^2}{2}$

Particle: A particle can be anything ~~approximate~~ legitimate.



* The trajectory of particle is time-like.

* Time-like:

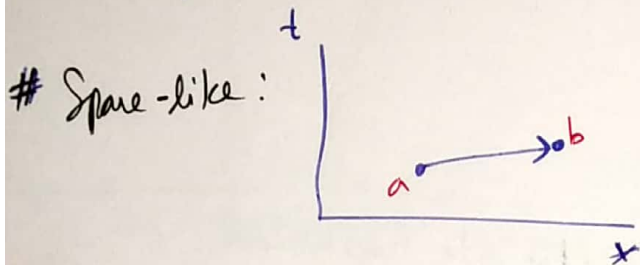
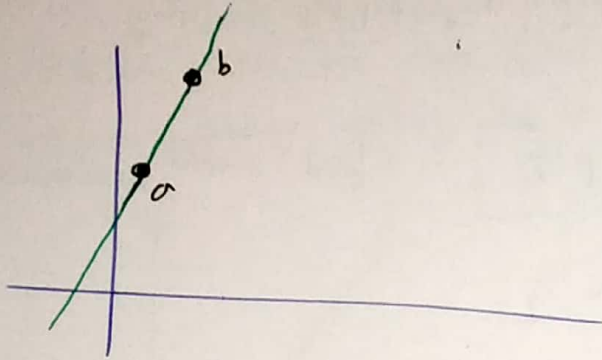
$(\Delta t, \Delta \vec{x}) \Rightarrow (\Delta s)^2 = (\Delta t)^2 - (\Delta \vec{x})^2$

* if spatial distance is less than time distance
 $(\Delta t)^2 > (\Delta \vec{x})^2 \Rightarrow (\Delta s)^2 > 0$

→ & the interval between these points is called time-like

→ Its got more time in it, than its got space in it. (Pg 20)

→ When interval is time-like, you can always find a Lorentz frame of reference, i.e. you can always transform to a frame of reference with the two points at the same position... it hasn't moved... → In fact, you simply go to the frame of reference where the ab line is at rest.



* separation is more space interval than time interval.

★ above: $\vec{\Delta x}$ meant $(\Delta x, \Delta y, \Delta z)$
→ its three vector

$$\therefore \Delta(\vec{\Delta x})^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

~~meant~~ meant

→ In this case; you cannot find a reference frame in which two points are at the same position space.

→ What you can do; is; you can find a reference frame in which they are at same time.

→ This means ⇒ there is not invariance significance to one end being later than the other ⇒ you can always find a frame of reference where two points are exactly at same time.

→ & even worse; there exist a reference frame in which point "b" is earlier than point "a" and also the frame where "b" is later than point "a".

* So, there is no invariance significance for points which are separated by spacelike interval.

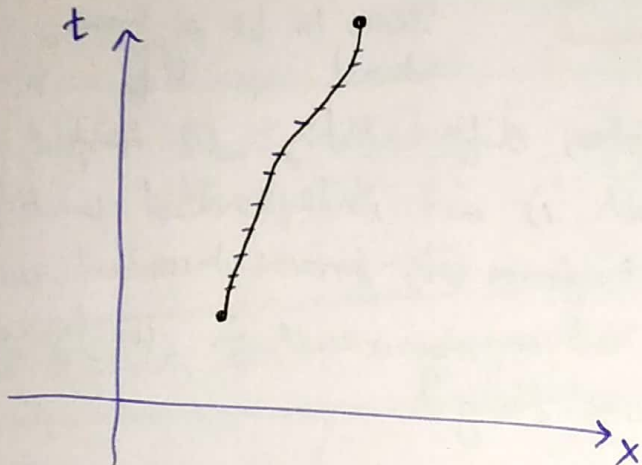
(pg 21)

↳ there is no invariance significance to which one occurred first, one being later to other or simultaneous.

↳ This is the relativity of Simultaneity.

* Particles move on time-like trajectories.

↳ everywhere along the motion of particle ...

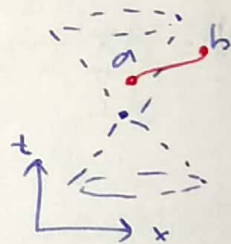


* every little piece of trajectory is timelike.

↳ Its just the statement that nothing can move faster than speed of light.

↳ if it would move

we get as confused faster ~~we would get~~ ~~confused~~ which is ~~starting~~ and; if we say particle goes from "a to b"

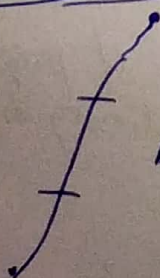


↳ we cant make sense of which one is earlier ... the starting point of particle & end point of particle.

↳ It would depend on reference frame (since the interval is spacelike)

↳ and Causality will not hold.

#



$$\Delta X^\mu = (\Delta t, \Delta \vec{x})$$

$$\Delta \vec{x} \text{ or } \Delta x^i$$

$$\Delta \tau = \sqrt{(\Delta t)^2 - \frac{|\Delta \vec{x}|^2}{c^2}}$$

★ ordinary velocity, ~~$\vec{v} = \frac{d\vec{x}}{dt}$~~
 $\vec{V} = \frac{d\vec{X}}{dt}$ or $v^i = \frac{dx^i}{dt}$

★ Four Velocity:

$\mu \in \{0, 1, 2, 3\}$

$$U^\mu = \frac{dX^\mu}{d\tau}$$

★ in spacetime;

Four dimensional velocity

① ~~velocity~~ Four velocity

seems to be a more natural quantity since

this involves differentiating with respect to τ ; which is an independent quantity

u.r.t. frame or frame invariant quantity

→ this will or may result in unifying various thoughts.

$v^i = \frac{dx^i}{dz} \cdot \frac{dz}{dt} = U^i \cdot \frac{dz}{dt}$

$$\Rightarrow v^i = U^i \cdot \frac{dz}{dt}$$

$\therefore U^0 = \frac{dt}{dz} \Rightarrow \frac{dz}{dt} = \frac{1}{U^0}$

$$v^i = \frac{U^i}{U^0}$$

$dz = \sqrt{dt^2 - d\vec{x}^2}$

$\Rightarrow dz = dt \sqrt{1 - \left(\frac{d\vec{x}}{dt}\right)^2}$

$\Rightarrow \frac{dz}{dt} = \sqrt{1 - |\vec{V}|^2}$

$$\Rightarrow \frac{dz}{dt} = \sqrt{1 - v^2}$$

→ This is what $\sqrt{1-v^2}$ is

$$\Rightarrow U^0 = \frac{1}{\sqrt{1 - v^2}}$$

→ i.e. Rate of change of proper time. relative to ordinary time.

* in non-relativistic limit:

$$\underline{U^0 \approx 1} \quad \text{ie; } \frac{dt}{dz} = 1$$

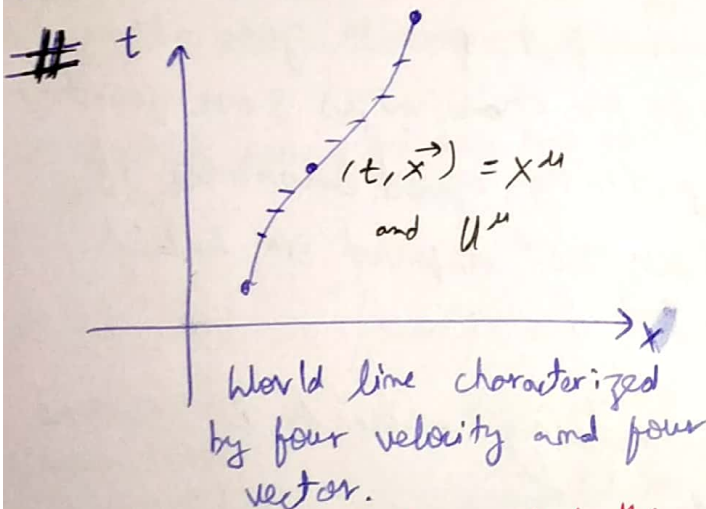
~~#~~ $V^i = \frac{U^i}{\sqrt{1-v^2}}$ #

$$V^i = U^i \cdot \sqrt{1-v^2}$$

U^0 = time component of Four Velocity; ; U^i = space component of Four velocity:

$$U^0 = \frac{dt}{dz} = \frac{1}{\sqrt{1-v^2}} ; U^i = \frac{V^i}{\sqrt{1-v^2}}$$

* Non relativistic limit: $v \ll 1$ or $v \ll c$: $U^i = V^i$; & $U^0 = 1$



(only one redundancy, that the four component of U are not completely independent)

Note
* Four vector velocity does not really have four independent components

→ It only has three independent components.

(reason: there is relation between three components of spatial velocity & U^i & U^0)

it is:

$$(U^1)^2 + (U^2)^2 + (U^3)^2 = \frac{v^2}{1-v^2}$$

∴ let $\vec{U} = U^i$

$$\Rightarrow (U^0)^2 - (U^i)^2 = \frac{-v^2}{1-v^2} + \frac{1}{1-v^2} = 1$$

$$\Rightarrow \boxed{(U^0)^2 - (U^i)^2 = 1}$$

$$\Delta z^2 = \Delta t^2 - \Delta \vec{x}^2$$

$$\Rightarrow 1 = \frac{(\Delta t)^2}{(\Delta z)^2} - \frac{(\Delta \vec{x})^2}{(\Delta z)^2} \Rightarrow \underline{\underline{1 = (u^0)^2 - (u^i)^2}}$$

We want to know what law governs the motion of particles.

↳ The law we are going to extract is from Principle of Least Action

★ We want our laws of physics to be same in every reference frame → This means, we should try to cast them in terms of quantities which are same in every reference frame.

* If we have a principle that says that particle goes along the trajectory which ~~minimizes~~ minimizes or maximizes some quantity called the Action ... we are doing good business if we make sure that the Action does not depend on which frame of reference it is evaluated in.

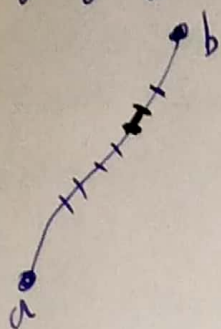
↳ We like Action to be something which is same in every reference frame.

- Action should be invariant.

* Other thing which we had learned from experience; is that Action should be built up incrementally as the sum of over the trajectory of little infinitesimal pieces for each segment of the trajectory. (This we can abstract from Classical Mechanics)

- Action being integral over the trajectory.
- The thing which is its integral should be quantity which is same in every reference frame.

* There is really one thing which is invariant from going from a position to a displaced position.
; and ; it is proper time along the trajectory.



So; a good guess is (and it is right guess)

$$\sum \Delta \tau = \text{Action}$$

→ Action is composed out of invariance; ⇒ so everybody will agree that a given path minimizes the action.
(or maximizes)

(Why we did not start with energy? ⇒ because it is invariant)
→ not even in Newtonian physics.

ex Shortest distance between two points in Euclidian space is an invariant concept.

* we had a freedom; we could multiply Action by a number. → it won't change the trajectory ~~Action = \sum \Delta \tau~~
(integer +ve or -ve) ... we would be maximising or minimising the Action.
ie; Stationary

→ & the equation is same; ie; Euler Lagrangian.

~~BD~~

$$\text{Action} = -m \sum \Delta \tau$$

→ multiplying by $-m$ changes ~~nothing~~ ... it is just convention ... complete convention → but the convention was not established by Einstein... ⇒ but long before him by the people who had no idea what Special Relativity was, but who had some prejudices about the way Classical mechanics works.
→ Its for the purpose of matching onto non-relativistic formulas.

$$\text{Action} = -m \int dz$$

along the trajectory

now we can write;

$$\text{Action} = -m \int \sqrt{dt^2 - dx^2}$$
$$= -m \int \sqrt{1 - \left(\frac{dx}{dt}\right)^2} dt$$

$$\text{Action} = -m \int \sqrt{1 - (\dot{x}^i)^2} dt$$

$\dot{x}^i = \frac{dx^i}{dt}$; $(\dot{x}^i)^2 = (\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2$

derivative w.r.t. ordinary time.

Lagrangian depends on position & velocities

→ when there are no forces ... it does not depend on position

→ only depends on velocities.

$$\mathcal{L} = -m \sqrt{1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}$$

→ This is the $\mathcal{L}$ that a relativistic version of Lagrange would have written down.

This is the $\mathcal{L}$ that Lagrange had written down.

Note $\mathcal{L}_L$ (Lagrangian of Lagrange)

→ The Lagrangian which Lagrange wrote down was

$$\mathcal{L}_L = \frac{m \dot{x}^2}{2}$$

... quite simple....

* What is the value of relativistic Lagrangian?

* The value is such that it encapsulates the motion of free particle in a way that is independent or same for every reference frame.

↳ It is guaranteed because the Action is independent of reference frame $\Rightarrow$ (making it stationary will get same answer in every reference frame)

m is mass.

$$\therefore \mathcal{L} = -m \sqrt{1 - v^2} \quad ; \quad \text{where}$$

* When: $v \ll c$ $\therefore \mathcal{L} = -m \cdot \left(1 + \frac{v^2}{2}\right) = -m + \frac{mv^2}{2}$

↳ adding a number to Lagrangian it does nothing

$$\mathcal{L}_{\text{non relativistic (important terms)}} = \frac{mv^2}{2} \quad (\text{matches with Lagrangian written down by Lagrange})$$

Momentum and Energy

(They are important because they are conserved... Net momentum & ~~energy~~ energy)

* What is meaning of momentum?

(It will be conserved because the Lagrangian is translationally invariant)... It only depends on velocities... not depends on position.

Its Translation invariant $\Rightarrow$ the momentum will be conserved.

$$P_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = -m \cdot \frac{1}{2} \cdot \frac{(-2 \cdot \dot{x})}{\sqrt{1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}} = \frac{m \dot{x}}{\sqrt{1 - v^2}}$$

$$P_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{m \dot{y}}{\sqrt{1 - v^2}} ; \quad P_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = \frac{m \dot{z}}{\sqrt{1 - v^2}}$$

$$\Rightarrow P_x = m \cdot \left(\frac{\dot{x}}{\sqrt{1 - v^2}} \right) = m \cdot \left(\frac{v^x}{\sqrt{1 - v^2}} \right) = m u^x$$

$$\Rightarrow \boxed{P_i = m u^i} \quad \text{ie:} \quad P_y = m u^y ; \quad P_z = m u^z$$

Suppose: you have a four vector: $\Delta t, \Delta x$; here: it just represents a little interval. (pg 29)

* Suppose: $\Delta x = 0$; can this invariant characterization of this interval.

↳ when you do Lorentz Transformation... it changes little bit
↳ it picks up Δt → so in that case it cannot be an invariant description of the vector.

* Suppose: $\Delta x = 0; \Delta t = 0$; is this is an invariant characterization: Yes.

so: saying a four vector is zero (all components of it)

↳ That an invariant statement

* to say some specific subset of component less than all of them is zero,
↳ is not an invariant statement

↳ Same is true for four velocity.

Momentum conservation.

* The components of momentum happen to be proportional to components of four velocity.

$$\vec{P}_{\text{initial}} = \vec{P}_{\text{final}}$$

$$\Rightarrow \vec{P}_{\text{initial}} - \vec{P}_{\text{final}} = 0$$

↳ This is actually three vector.

This is a vector equation; ~~usually~~
Is it Invariant vector equation?

(if it is true in one frame is ~~not~~ it necessarily true in other frame)

↳ not unless we can make an equivalent assertion about fourth component.

* so: if there is a fourth component that turns the momentum into a four vector ... then we say all four components are conserved ⇒ Then it becomes an invariant statement.

∴ The fourth component must be $\underline{m u^0}$

$$p_x = m u^x$$

$$p_y = m u^y$$

$$p_z = m u^z$$

$$p_0 = m u^0$$

→ Is there any significance to it.
and is it something that we already recognize
something ~~ought~~ which ought to be conserved.

→ yes; there is fourth thing conserved in
particle mechanics & that is energy.

★ so; is the fourth component of four momentum
nothing but energy?

so; look at Hamiltonian.

$$H = \sum_i \dot{x}_i p^i - \mathcal{L}$$

→ This is extremely formula for
any Hamiltonian or action
principle.

$$\therefore p^i = \frac{m \dot{x}^i}{\sqrt{1-v^2}}$$

so; ~~the~~ ~~the~~ ~~the~~

$$\therefore \mathcal{L} = -m \sqrt{1-v^2}$$

$$\text{so; } H = \sum_i \frac{m \dot{x}_i^2}{\sqrt{1-v^2}} - (-m \sqrt{1-v^2})$$

$$H = \frac{m v^2}{\sqrt{1-v^2}} + m \sqrt{1-v^2} \Rightarrow H = \frac{m[(v^2) + (1-v^2)]}{\sqrt{1-v^2}} = \frac{m}{\sqrt{1-v^2}}$$

$$\Rightarrow H = m \cdot u^0 = p^0$$

$$\Rightarrow H = p^0$$

so; conservation of momentum becomes the conservation
of Four momentum.

- conservation of x momentum.
- " " y "
- " " z "
- " " energy.

What is this new concept of energy has to do with old concept of energy? How different from the energy we had already established.

P231

$$\therefore mu^0 = \frac{m}{\sqrt{1-v^2}} \approx m \left(1 + \frac{v^2}{2}\right)$$

$$\Rightarrow mu^0 \approx m + \frac{mv^2}{2}$$

→ it is recognizable
(it is ordinary Newtonian kinetic energy)

∴ To make it dimensionally consistent,

$$\text{so: } E = mc^2 + \frac{mv^2}{2}$$

∴ so; here we see coming out a basic principle of mechanics that Conservation of quantity which is when evaluated for a system at rest is just mc^2 .

It applies to any object that can be identified as closed object... when the object is at rest.

→ we mean centre of mass.
(all the molecules may be vibrating)

Some people might say that "what they mean by rest mass is simply the sum of the rest mass of the particles"

→ this is not a very good invariant concept at all.

note They are conventions that if energy defined in one ways is conserved, it won't make any difference for its conservation in order to match with units if we multiply by speed of light or something.

"Conservation is Conservation"

$mc^2 + \frac{mv^2}{2}$
→ to match units of energy.

* One can say ... "Think of the thing as a unit, and discuss its energy in a frame of reference where its total momentum is equal to zero."

(This we called centre of mass frame or rest frame of the combined system)

* $\rightarrow$ Since all ordinary systems are made up of particles that are moving here & there all over the place $\rightarrow$ a invariant concept is the energy of the system in a reference frame where its momentum is zero.

$\hookrightarrow$ If we call that the mass $\rightarrow$ Then the mass of the composite system would be greater than the sums of the two masses.

* If there are attractive forces, it might be less.

Massless particles. $\therefore \underline{v=1}$ or $\underline{v=c}$

$$\frac{m}{\sqrt{1-v^2}} = \frac{0}{0}$$

better than $1/0 \dots$

$\hookrightarrow$ It does not seem a good idea to try to think of energy of massless particles in terms of their velocity since all massless particles moves with exactly the same velocity.

$\hookrightarrow$ Can they have different energy?

Yes, the reason is $\frac{0}{0}$ is not determined.

* so; ~~thing~~ ^{think} of Energy as a function of momentum instead of the velocity.

* Classical Old mechanics:

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

$$p = mv$$

In relativity: $(u^0)^2 - (u^1)^2 - (u^2)^2 - (u^3)^2 = 1$

$$c=1 \Rightarrow m^2(u^0)^2 - m^2v^2 = m^2$$

$$\Rightarrow (p^0)^2 - p^2 = m^2$$

$$\therefore p^0 = E$$

$$\Rightarrow E^2 - p^2 = m^2$$

$$\Rightarrow E^2 = p^2 + m^2 \Rightarrow$$

$$E = \sqrt{p^2 + m^2}$$

putting speed of light;

$$E = \sqrt{p^2c^2 + m^2c^4}$$

Energy in terms of momentum & mass.

for massless particle; $m \rightarrow 0$;

$$E = pc$$

example; let a positronium; $m \approx 2$ (mass of two electron) (because it is bounded...)



electrically neutral particle with a mass of m .

Positronium $\rightarrow$ disappears & produces two photons moving away from each other.

calculate energy & momentum of these two photons.

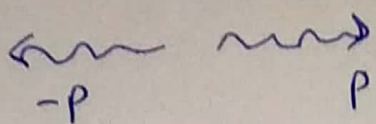
* assume positronium is at rest in your frame of reference.

$$\text{so: } \vec{P}_{\text{initial}} = 0$$

$\hookrightarrow$ so; photons must go in opposite directions (to conserve momentum)

↳ They must have equal and opposite momentum.

so:



$$E_{\text{initial}} = mc^2$$

$$E_{\text{final}} = 2 E_{\text{photon}} = 2c|p|$$

$$\Rightarrow |p| = \frac{mc}{2} \text{ this is momentum of each one of photons.}$$

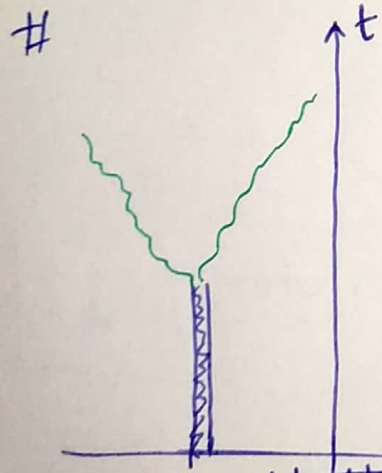
Relativistic decay

* Mass has always energy.

↳ but here..... so energy gets converted from ^{this} frozen form of rest mass into dynamical useful form of electromagnetic energy.

$E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ setting units: $E = \frac{m \cdot c^2}{\sqrt{1-(v/c)^2}}$

$[E] = [m] [v]^2$ units



If you trace back along the world line of photon, you will find out that they will intersect

- first of all ~~the~~ at the spacial location where the positronium was
- but they will intersect at exactly the same time. (or atleast at the same time if we ignore quantum uncertainties)

~~World line of positronium~~
World line of Positronium.

* Four velocity is constant, means not only in magnitude but in direction → says that the particles moves in a straight line in spacetime.

↳ that the motion of particle is straight line through space time ... and that says it moves along a line in space with constant velocity.

Lec-4 Special Relativity and Electrodynamics : Classical field theory. - Shoab Akhtar. (pg 35)

★ Spacetime has always one time coordinate & some no. of space coordinates.

(Mathematical Physicists ... can study spaces of any no. of time & any no. of space)

* Nobody knows how to make logical sense out of more than one time dimensions.

t, x^i

A field is an variable (a observable variable), a measurable quantity which depends on space & time $\rightarrow$ Which varies from place to place ... Varies across space, with time.

ex of fields: Temperature ; Wind velocity ;
(scalar field) (vector field)

Field is some function:

$$\phi(t, x^i)$$

* generic field is labelled as ϕ of t, x^i .

ϕ is greek F ... for function.

Non-relativistic motion of particle

think ϕ as position of particle (ordinary newtonian 1 space-dimensional world)

Action = $\int_a^b \left[\frac{m}{2} \dot{\phi}^2 - V(\phi) \right] dt$

= $\int_a^b \left(\frac{m}{2} \left(\frac{d\phi}{dt} \right)^2 - V(\phi) \right) dt$

$\therefore \mathcal{L} = \frac{m}{2} \dot{\phi}^2 - V(\phi)$

and equations of motion is given by

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi} \Rightarrow m \ddot{\phi} = - \frac{\partial V}{\partial \phi}$$

Newton's eqn.

This was a field theory in a world with no space & one time direction.

Imagine a world: time & no space : ~~one~~ zero dimensional space.

$\Rightarrow$ then we would be talking about theory ... of a function of time $\phi(t)$

$\rightarrow$ This is a theory of function that only depends on time ★
 $\hookrightarrow$ You can think of ordinary particle mechanics as a field theory in a world which

has no dimension of space, only dimension of time.

→ The same mathematics.. but different interpretation.

→ i.e. instead of thinking of as the position of particle, think of it as a field which depends only on a time.

→ so; we are going to think it as a field in a world of zero space dimension.

We are going to generalize field theory in a time dimension & a collection of space ~~dimension~~ coordinates.

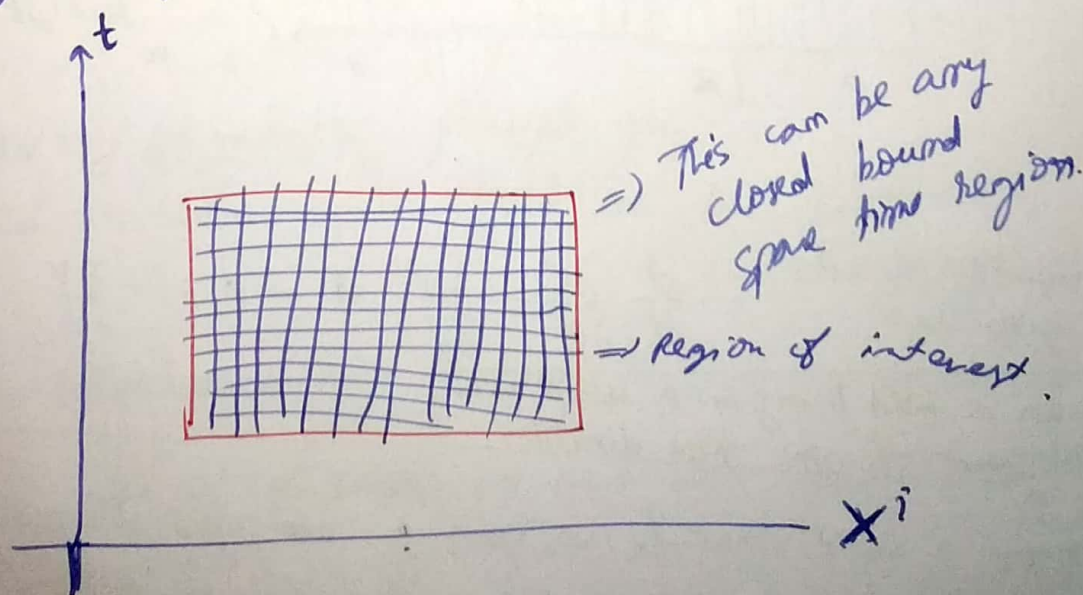
$$\phi(t, x^i)$$

* let's try to guess what principles of field theory are knowing what the principles are in a world of only one time & no space.

→ and best guess is that

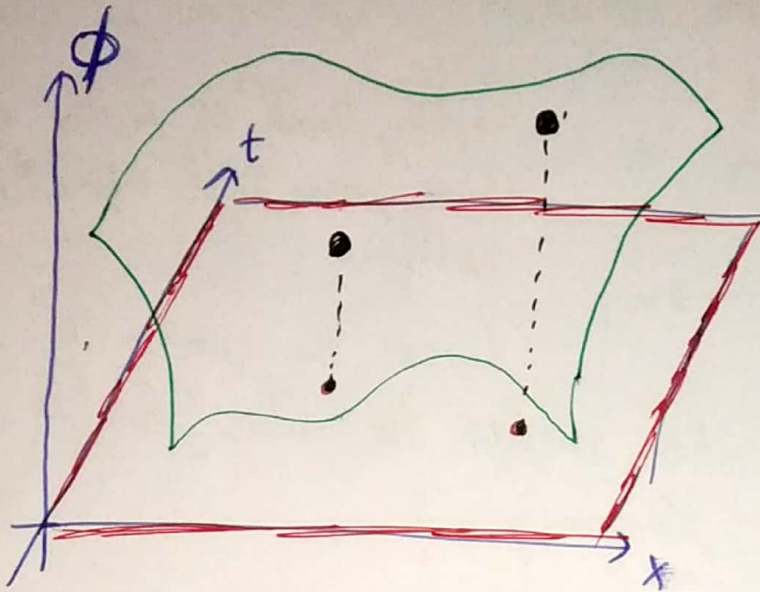
The world should be governed by principle of least action.

→ A very powerful principle which encodes & summarizes huge numbers of laws of physics



or can draw as ... where value of field whatever it is, on the vertical axis.

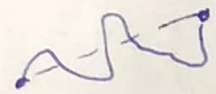
(p. 37)



~~draw field as a surface.~~ → History of field is drawn as a surface.

* bound space time by a rectangle (or in zero space ... we bounded the end
and talk about field within the rectangle. or closed points)
(just like we talked about field on the interval between "a" and "b" ^{curve or closed surface})

Action principle said ⇒ freeze the end points, you don't let them move and you say what's the minimum action subject to the condition that the end points are fixed.



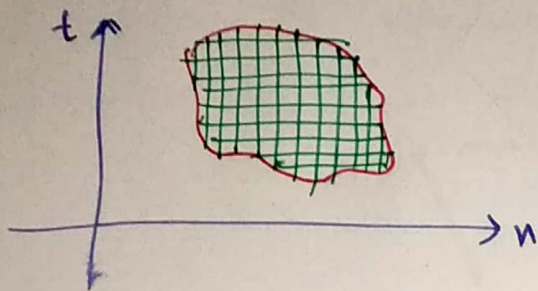
In the similar way ⇒ Action principle says,

"Hold fixed the value of the field on the boundary
(can be any space time region)

i.e. take a spacetime region ⇒ freeze the field on the boundary

⇒ let the value of ϕ wiggle up and down ~~inside~~ inside the region; eventually we find a configuration that minimizes the action. → And that will be history of field in space and time. (this will provide for us for sets of equations of motions)

In each small cell $\Rightarrow$ we specify action within that cell ... and add them all up.



$\hookrightarrow$ We think of Action as a sum over little infinitesimal cells.

$\hookrightarrow$ so, the question is what should we take for the Lagrangian?

~~math~~ $A = \int dx dy dz dt \mathcal{L}$

frequently to write integral over all four coordinates of space time we write d^4x in place of $dx dy dz dt$

$$d^m x = dt \cdot \prod_{i=1}^{m-1} dx^i ;$$

$\Rightarrow$ $A = \int d^4x \cdot \mathcal{L}$

* This Lagrangian is something which is not just going to be integrated over time ... but is to be integrated over space & time so we often call it Lagrange Density.

(it is just the analogue of Lagrangian)

On what it depends?

- Well its supposed to be the Lagrangian for field ϕ to tell you how ϕ behaves ... so, it will depend on ϕ .
- we saw earlier in usual case that $\mathcal{L}$ depends on ϕ .

→ If time and space are in the same footing in relativity, we might expect that the Lagrangian depends not only on the time derivative of ϕ but also depends on derivatives of ϕ with respect to spatial coordinates,

$$\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}.$$

$$A = \int d^4x \mathcal{L} \left(\phi, \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

$$\Rightarrow A = \int d^m x \cdot \mathcal{L} \left(\phi, \frac{\partial \phi}{\partial x^\mu} \right)$$

∴ Lagrangian might depend on x, y, z or t

~~time in the~~ t ~~to take in other case.~~ it depends on t .
 $\mathcal{L} = \frac{m}{2} \dot{\phi}^2 - V(\phi) = f(t)$
 (time $\phi(t)$)

→ but for closed system ordinarily, i.e., for the system which has energy conservation $\Rightarrow$ it means that Lagrangian does not depend explicitly on time.

→ & the same will be true here; it would depend not explicitly on time or space.

$\Rightarrow$ but it could.

Condensed notation;

$$\phi_t = \frac{\partial \phi}{\partial t} ; \phi_x = \frac{\partial \phi}{\partial x} ; \phi_y = \frac{\partial \phi}{\partial y} ; \phi_z = \frac{\partial \phi}{\partial z}$$

We will be writing down the equations which are equivalent of saying that the derivative of action with respect to ϕ at each grid point is equal to zero. $\rightarrow$ The corresponding Euler Lagrange equations.

originally: $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \frac{d\phi}{dt}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$] simple version

$\Downarrow$ changed one
 Instead of having a single term here with a time derivative... you have sum of terms, one for each direction of space-time.

$$\left[\sum_n \frac{\partial}{\partial x^n} \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \phi}{\partial x^n}} \right) \right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

example: $\mathcal{L} = \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 + \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right]$ (messing $\mathcal{L}$ for fluids).
 (we are not doing relativity now... just fluids)
 $\hookrightarrow$ If this would be Lagrangian; it would be completely the same for space and time $\rightarrow$ There would be no difference between space & time (it does not sound right... that the behaviour of waves can't tell the difference between time coordinate & space coordinate)

So; $\mathcal{L} = \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 - \left(\frac{\partial \phi}{\partial x} \right)^2 - \left(\frac{\partial \phi}{\partial y} \right)^2 - \left(\frac{\partial \phi}{\partial z} \right)^2 \right]$

(but exactly same kind of Lagrangian incidently would be if we were doing relativity...)
 $\hookrightarrow$ in that case; $\mathcal{L} = \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 - \left(\sum_i \left(\frac{\partial \phi}{\partial x^i} \right)^2 \right) \right]$

* But this might not be light,
it might be sound; elastic waves in solid.

↳ so; we place sound velocity, or elastic wave velocity
instead of c .

* some part of connection.

$$\mathcal{L} = \frac{m}{2} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 - \sum_i \left(\frac{\partial \phi}{\partial x_i} \right)^2 \right]$$

↳ you can think this
as part of potential
energy.

And potential energy has to
with everything that
does not depend on
time derivatives.

* officially definition of kinetic
energy has to do with time
dependence.

↳ so; we typically subtract potential energy; that's
is part of the ~~the~~ connection.

⇒ But, ultimately, the real reason is relativity.

* but we would add to $\mathcal{L}$; some potential...

... ⇒ Potential energy which is here just because
field has certain value.

$$\mathcal{L} = \frac{m}{2} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 \frac{1}{c^2} - \sum_i \left(\frac{\partial \phi}{\partial x_i} \right)^2 \right] - V(\phi)$$

This is the whole Lagrangian.

$$m \left[\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} \right] + \frac{\partial V}{\partial \phi} = 0$$

↳ some sort
of acceleration.

case; if $V=0$

→ Reduces to wave equation: $\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \sum_i \frac{\partial^2 \phi}{\partial x_i^2} = 0$

case; $V(\phi) = \frac{\mu^2}{2} \cdot \phi^2$

: [in case of ordinary classical mechanics μ^2 should have been spring constant (if ϕ stood for position of particle)]

so: $\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \sum_i \frac{\partial^2 \phi}{\partial x_i^2} + \mu^2 \phi = 0$

You might think it as higher dimensional version of harmonic oscillator.

if: $\sum_i \frac{\partial^2 \phi}{\partial x_i^2}$ would not have been there

→ it would read $\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \mu^2 \phi = 0$

→ it really would be just a harmonic oscillator.

The transformation laws with Lorentz Transformation applied to coordinates $\Rightarrow$ it also applies to differences of coordinates.
 $\Rightarrow$ & they apply to differentials also.

* $\Rightarrow$ That was the guiding principles, ... constructing things which are invariant.. (which are same in every coordinate system)

→ so; we have to start with collection of objects which are coordinate invariant

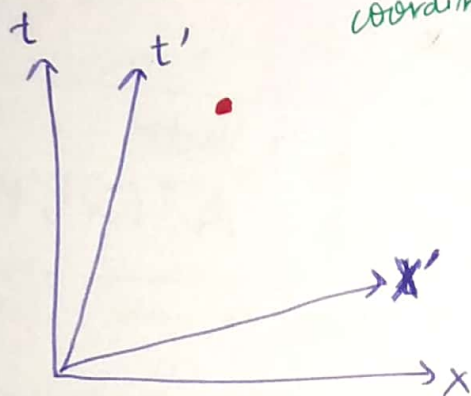
→ This brings us to the nature of the transformation properties of fields.

Scalar quantity is one, which no matter what frame of reference it looked out; what coordinate it looked from ... it is always the same.

* Scalar does not change when you rotate or Lorentz transform the coordinate.

$$\phi(x^\mu) = \phi'(x'^\mu)$$

transformed field ~~value~~ transformed coordinate
 value evaluated at transformed coordinate.



Vector field ; 4 Vector

x^μ ; dx^μ ; is 4 vector

→ if you multiply 4 vector by scalars or invariants ; you get back 4 vector
 (because invariants are completely passive when you transform)

so; $\frac{dx^\mu}{d\tau} = u^\mu$; $\frac{dx^\mu}{d\tau}$ is 4 vector.

→ what it means that it is a 4 vector ; it's a statement about how it transforms under Lorentz Transformation,

$$\left. \begin{aligned} t' &= \frac{t - vx}{\sqrt{1-v^2}} \\ x' &= \frac{x - vt}{\sqrt{1-v^2}} \\ y' &= y \\ z' &= z \end{aligned} \right\} \Rightarrow \text{Things which transform in this way are called four vectors.}$$

also true for $\underline{dx^\mu}$

$\hookrightarrow \quad \quad \quad \frac{dx^\mu}{dz}$

∴ transformation properties of these things are.

$$\begin{aligned} A^0 \\ A^x \\ A^y \\ A^z \end{aligned}$$

Four components of 4 vector.

$$(A^0)' = \frac{A^0 - vA^x}{\sqrt{1-v^2}}$$

$$(A^x)' = \frac{A^x - vA^0}{\sqrt{1-v^2}}$$

$$(A^y)' = A^y$$

$$(A^z)' = A^z$$

4 Vector
 $A^\mu(x^i, t)$

Imagine a fluid filling space time.

↳ so; every point in fluid: there was a four velocity.

$\nearrow U^\mu(x^i, t)$

* The four velocity of the fluid could be thought of as field.
↳ since it is 4 velocity... it is 4 vector.

Note you can make scalars out of four vectors -

- (i) If you take four components of the 4 vector.
- (ii) In fact you take any two 4 vectors: you can make a scalar out of them.

* Actually: every four vector can be used to make a scalar

(P355)

↳ example: 4 vector $A^\mu(x', t)$

∴ scalar: $(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$

→ It is scalar for exactly the same reason.

• once you know that these A 's transform the same way as x 's & t ⇒ Then by the same little bit of algebra, you can see that the difference of the square of the time component & the space component will not change under Lorentz Transformation.

... same as $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$.

Given a scalar ⇒ You can construct a 4 vector out of it.

example: given a scalar ϕ

∴ there are collection of components

$$\left[\frac{\partial \phi}{\partial x^\mu} \right]$$

→ These form a four vector.

* Now we have set of tools for ~~creating~~ creating Lagrangians.

What's the rule for making a Lagrangian or action invariant?

Ans) We require that Lagrangian itself is same in every coordinate frame.

→ The thing that we add up over all these little cells is the same in every coordinate system ... that means it is just the scalar.

→ That the Lagrangian should be a scalar.

So this is our rule, that Lagrangian should be a scalar under Lorentz Transformation.

You take your field : ϕ .

and you consider all the possible scalars that you can make out of it; and they are possible candidate to put into lagrangian.

- ϕ is itself a scalar.
- any function of ϕ is scalar.

so: $V(\phi)$ is scalar $\Rightarrow$ so its a candidate you can put in lagrangian.
any function of ϕ

* We certainly want to put in derivatives of the field
 $\Rightarrow$ (if we don't put them; field theory is going to be very very trivial)

$\hookrightarrow$ but we have to put derivatives in; in a way that it constitutes scalar.

$\therefore \frac{\partial \phi}{\partial x^\mu} \Rightarrow$ a constant the invariant
ie: $\left[\left(\frac{\partial \phi}{\partial t} \right)^2 - \left(\frac{\partial \phi}{\partial x} \right)^2 - \left(\frac{\partial \phi}{\partial y} \right)^2 - \left(\frac{\partial \phi}{\partial z} \right)^2 \right]$

* you can put in mass; but it is just a parameter;
(its not a dynamical thing)

$\hookrightarrow$ mass of what? ... you can put some numbers in.

generalization of this

Note: multiplying scalars give scalars.

so: $F(\phi) \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \sum_i \left(\frac{\partial \phi}{\partial x^i} \right)^2 \right]^P$

$\rightarrow$ you can put higher powers of them

let: $\frac{1}{2} \left[\frac{1}{c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \left(\sum_i \left(\frac{\partial \phi}{\partial x_i} \right)^2 \right) \right] F(\phi)$

usual classical $\mathcal{L}$: $\mathcal{L} = \frac{1}{2} m v^2$; you can use $\frac{m v^2}{2}$

↳ but its convention to use $\frac{1}{2}$

↳ so; ~~as~~ as force to make look good $F = m a$.

example $\mathcal{L} = \frac{1}{2} \left[\frac{1}{c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \left(\sum_i \left(\frac{\partial \phi}{\partial x_i} \right)^2 \right) \right] - \frac{\mu^2}{2} \phi^2$

↳ very simple field theory... & gives wave equation.

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \sum_i \frac{\partial^2 \phi}{\partial x_i^2} + \mu^2 \phi = 0$$

} This is simple wave equation.
(it is linear ~~diff~~ differential equation)

* General Lorentz Transformation is combination of rotation of space and transformation along the x axis.

ex you can have: $y' = \frac{y - vt}{\sqrt{1-v^2}}$; $t' = \frac{t - vx}{\sqrt{1-v^2}}$; $x' = x$; $z' = z$

(moving along y axis)

ex You can actually derive a general Lorentz Transformation.

* Any two frame which are related by angles in space & which are moving relative to each other in any direction

↳ This transformation can be built up as a combination of

- rotation of space first.
- and then transformation along next x-axis.
- " " another rotation of space. ~~(diff)~~

ie:

$$R(-\theta) L(\frac{v}{c}) R(+\theta)$$

pg 48

∴ be careful $L R$ may not commute.

you can't
 $RL = LR +$
 $\Rightarrow RL - RL + L$
 $\Rightarrow IL = L$

* as long as you make sure that your theory is invariant w.r.t. simple Lorentz Transformation: let's say X-Lorentz Transformation ~~and~~ and w.r.t. Rotation.

→ Then you will be invariant w.r.t. any Lorentz Transformation at all.

Is our Lagrangian $\mathcal{L}$ invariant w.r.t. rotations?

Yes. ; because $(\sum_{i=1}^3 (\frac{\partial \phi}{\partial x^i})^2)$ is sum of square of components of space vector. . . .

Conclusion Basic rules for classical field theory.

- Take your field.
- You figure out all the scalars that you can make out of your field & their derivatives.
- The field don't have to be just scalars (we can take example; where field are not just scalar; but also vectors, tensors)
→ and you build out of them all scalars that you can create.
- Once you have all the scalars that you can create; then you build the Lagrangian ~~and~~ in terms or functions of those scalars.
- Then you apply Euler Lagrange equations.
- These give you equation of motions of the field Equations describing ~~the~~ whatever that field theory is supposed to describe.

* All kind of Lagrangian you can think off

↳ They are all going to give classical mechanical system
 ↳ almost all of them don't survive the transition to Quantum Mechanics. ... That's where the fun happens.

↳ When you try to make transition to Quantum mechanics, you find that almost all of them are just inconsistent.

★ The field should be continuous

↳ if not; they have ∞ derivatives $\Rightarrow$ they have ∞ action, $\Rightarrow \infty$ energy.

But; There is lot ambiguity or possibility for choice of Lagrangian in classical physics.

Relation between Particle and Fields.

* Relation between ordinary particles & how they interact with fields.

• How might it interact with it & how might the presence of scalar field will affect the motion of the particle.

for this we need a Lagrangian.

(we can write down many many Lagrangians

↳ for the same reason that Classical mechanics simply doesn't have enough constraints in it to tell you in detail what the Lagrangian has to be)

• But we have some rules.

Let's think about Lagrangian of a particle moving in the presence of pre-established field.

(somebody had solved the equations of motion of field; and we know the field $\phi(t, x_i)$)

→ The particle is moving in that field

→ That particle might be coupled to the field.
(might be charged particle in Electromagnetic field)

Particle mechanics in presence of field.

$A = -m \int d\tau$ } Action of particle ..

↓
There was a parameter here which was called the mass (which turned out in fact to be the mass)
→ we discovered that it behaves like non-relativistic mass when particle was moving slowly -

$$A = -m \int \sqrt{dt^2 - d\vec{x}^2} = -m \int \sqrt{1 - \left(\frac{d\vec{x}}{dt}\right)^2} dt$$

→ This was our Lagrangian.

$$A = -m \int \sqrt{1 - v^2} dt$$

→ This is our Lagrangian

• we also expanded it out for low velocities, we found it out that it was good old Newtonian Lagrangian.

* now; what we can do to it ⇒ To allow the particle to couple to the field.

- allow the particle to interact with field.
- " " " to be affected by the field.

→ so; we need to put inside the Lagrangian field somewhere.

→ we have to put it in a way that it is Lorentz Invariant

x in other words; we want to put it in a way so it is same in every reference frame. (pg 51)

→ The only thing we have around involving fields which is same in every reference frame is scalar.

→ we can do lots of things with scalars.

but one very simple thing we could do, would be

$$-m dz + \phi \cdot dz$$

so; ~~a possible Lagrangian~~ could be

so; a possible Lagrangian for a particle moving in a pre established field. $\mathcal{L} = (-m + \phi(x^\mu)) \sqrt{1 - \left| \frac{d\vec{x}}{dt} \right|^2}$

∴ and hence the corresponding action is

$$A_{\text{action}} = (-m + \phi(x)) \int dt \sqrt{1 - \left| \frac{d\vec{x}}{dt} \right|^2}$$

so;

$$A_{\text{action}} = \int (-m + \phi(x^\mu)) \sqrt{1 - \left| \frac{d\vec{x}}{dt} \right|^2} \cdot dt$$

$$\text{i.e.; } A_{\text{action}} = \int (-m + \phi(x^\mu)) \sqrt{1 - v^2} \cdot dt$$

★ now; suppose; field like to migrate to some value for some reason

→ The field like to seek some stable value which was not zero.

→ we would like it to get frozen for some

reason away from zero ... i.e.; constant

→ i.e. ϕ then would be a constant (or at least approximately a constant)

→ The motion of particle then would be exactly identical to ~~the~~ particle whose mass was $(m - \phi)$

⇒ It would be exactly the same as if particle mass ~~was~~ was $(m = \phi)$. (p252)

↳ This is simplest example of how scalar field can give rise to a shift ~~of the mass~~ in the mass of the particle

(Higgs field is the thing which is most likely (a scalar field) in nature ... which does like this)

⇒ and you can see how it is possible that if the Higgs field gets shifted by some mechanics ... its value gets shifted, ~~and~~ in turn will shift the value of the mass of the particle.

★ In fact if mass of some particle naturally began as zero, but it has coupled to this field ϕ ... and this field ϕ naturally sort some particular value ...
↳ The particle would effectively have mass which would be related to this field ϕ .

Note This is not exactly the Higgs mechanism; it is closely connected with it ... ⇒ It is roughly what meant when people say that Higgs field gives the particle mass.
↳ Higgs field inter into the equation as if it were the part of the mass.

± equations of motion. (doing for the case particle move in only one dimension)

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{[m - \phi(x, t)] \dot{x}}{\sqrt{1 - \dot{x}^2}} ;$$

now: $\frac{d}{dt} \left((m - \phi) \frac{\dot{x}}{\sqrt{1 - \dot{x}^2}} \right) = \frac{\partial \phi}{\partial x} \cdot \sqrt{1 - (\dot{x})^2}$

$$\Rightarrow \left[\frac{d}{dt} \left(\frac{\dot{x}}{\sqrt{1 - \dot{x}^2}} \right) \right] (m - \phi) + \left(\frac{\dot{x}}{\sqrt{1 - \dot{x}^2}} \right) \cdot \left(-\frac{\partial \phi}{\partial t} \right) = \frac{\partial \phi}{\partial x} \sqrt{1 - (\dot{x})^2}$$

$$\Rightarrow (m - \phi) \cdot \frac{d}{dt} \left(\frac{\dot{x}}{\sqrt{1 - \dot{x}^2}} \right) = \frac{\partial \phi}{\partial x} \sqrt{1 - (\dot{x})^2} + \frac{\partial \phi}{\partial t} \cdot \frac{\dot{x}}{\sqrt{1 - \dot{x}^2}}$$

$\therefore$ if ϕ only depends on x ; $\dot{\phi}$ is not zero.
(it depends on $x \dots$ & x is not constant)

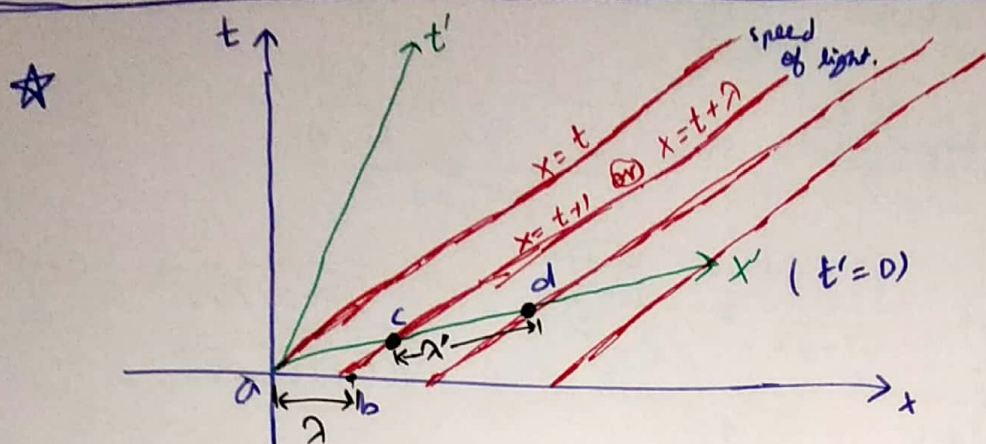
$$\therefore \dot{\phi} = \frac{\partial \phi}{\partial x} \cdot \dot{x} \quad \text{where } \phi(x)$$

$$\Rightarrow \text{so; } (m - \phi) \cdot \frac{d}{dt} \left(\frac{\dot{x}}{\sqrt{1 - \dot{x}^2}} \right) - \frac{d\phi}{dx} \cdot \frac{\dot{x}^2}{\sqrt{1 - \dot{x}^2}} = \frac{\partial \phi}{\partial x} \cdot \sqrt{1 - \dot{x}^2}$$

$$\Rightarrow (m - \phi) \cdot \frac{d}{dt} \left(\frac{\dot{x}}{\sqrt{1 - \dot{x}^2}} \right) = \frac{d\phi}{dx} \cdot \frac{1}{\sqrt{1 - \dot{x}^2}}$$

$\rightarrow$ put speed of light in... & take the limit when velocity is much slower than speed of light... it must come then little familiar.
 $\hookrightarrow$ You will find a pattern... the ~~pattern~~ pattern will be a familiar thing.

PS 55



Doppler shift
from geometry of
Lorentz Transformation.

Imagine there is a wave



/// $\Rightarrow$ light moving wave... (we are following some peaks)

- * λ is the distance between two maxima what unprimed coordinate calls at an instant of time.
- * and the wavelength to primed frame, is distance between two maxima what unprimed ~~frame~~ coordinate or frame calls at an instant of time.

so; $\Rightarrow$ if you work out Lorentz Lorentz transformation

Go & look for distance "ac" in ~~unprimed~~ primed frame

and compare with distance "ab" in unprimed frame.

$\hookrightarrow$ You will find them different $\rightarrow$ This will give Doppler shift.

- * let $\lambda = 1$; ~~locate~~ ~~points~~ ~~c, d~~; they are at ~~$t=0$~~ $t' = 0$

$$\text{so; } t' = 0 \Rightarrow \frac{t - vx}{\sqrt{1 - v^2}} = 0 \Rightarrow \underline{t = vx}$$

so; intersection; i.e. point c is ; $x = vx + 1 \Rightarrow x = \frac{1}{1 - v}$

$$\text{then; } t = \frac{v}{1 - v}$$

$$\text{so; } x_c = \frac{1}{1-v} \quad ; \quad t_c = \frac{v}{1-v}$$

→ now: we will calculate: x_c' .

$$\begin{aligned} \text{so; } x_c' &= \frac{x_c - v t_c}{\sqrt{1-v^2}} = \frac{\left(\frac{1}{1-v}\right) - \frac{v^2}{1-v}}{\sqrt{1-v^2}} \\ &= \frac{1}{(1-v)} \cdot \frac{(1-v^2)}{\sqrt{1-v^2}} = \frac{\sqrt{1-v^2}}{1-v} = \frac{\sqrt{(1-v)(1+v)}}{\sqrt{(1-v)(1+v)}} \end{aligned}$$

$$\Rightarrow x_c' = \sqrt{\frac{1+v}{1-v}} > 1$$

$$\boxed{\lambda' = \lambda \sqrt{\frac{1+v}{1-v}}} \text{ this is Relativistic dopler shift formula.}$$

$$\lambda' = \lambda \sqrt{\frac{1+v/c}{1-v/c}} \Rightarrow \boxed{\lambda' = \lambda \sqrt{\frac{c+v}{c-v}}} > \lambda$$

→ if going* in other direction:

$$\lambda' = \lambda \sqrt{\frac{c-v}{c+v}} < \lambda$$

so; if you shooting against the direction of light ray: you see shorter wavelength.

• if you are moving along the light ray you see a longer wavelength.

→ What happens when your velocity gets up near the speed of light

→ moving along light ⇒ it would get bigger & bigger... become ∞

• moving against ⇒ it would become smaller & smaller ... become 0.

In Quantum mechanics: fields & particles are the same thing ... (but as us here, fields are fields & particles, particles)

B57

... we are not doing Quantum Mechanics here)

• A field affects the particle in a certain way (example by creating forces on it)

must it not also be the case that particle affects the field.

↳ why it has to be the case ... why if a affects b does b necessarily have to affect a.

→ and this is built into idea of Action ... Action Principle (it simple basic idea of Action & Reaction)

★ let; you have two coordinates; x, y
 $\mathcal{L}$ depends on $x, y, \dot{x}, \dot{y}$; $\mathcal{L}(x, y, \dot{x}, \dot{y})$

Possibility: $\mathcal{L} = \mathcal{L}(x, \dot{x}) + \mathcal{L}(y, \dot{y})$

→ If this is the case and you look at the equation of motion for x ; ~~that~~ that would not involve y ; vice versa.
 ↳ so; in this case x does not affect y ; and y does not affect x .

* what can you say; if you know that y does affect x .
 ↳ this tells you that $\mathcal{L}$ must have something which somehow affect both x and y .

$\mathcal{L} = \frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} - V(x) - V'(y) + (xy)$
 ↳ if you put something in $\mathcal{L}$ which contains both x and y in a way that you can't ~~unravel~~ unravel it. → The EOM for x will involve y , & vice versa.
 ↳ only this term ... x & y not going to affect each other.

So: ⁶⁶ If you have both of them in Lagrangian in a way which is coupled together in this way;

1958

There is no ~~that~~ way it is possible that one can affect the other without the other affecting the one"

↳ This is why "If a affects b:
b affects a"

Particles and field.

- let's talk about how fields affects particle motion

- and then ask, how presence of particles, ~~also~~ affects the field.

* "This is very much an analogue of Electric & Magnetic field; affecting motions of charged particles; & charged particles creating and modifying the electromagnetic field".

* They are not two independent thing; $\Rightarrow$ They come from the same Lagrangian.

let; $\phi(x, t)$

$$\mathcal{L}_{\text{particle}} = -m\sqrt{1-v^2}$$

ordinary Lagrangian of particle.. but it does not have anything in it that describes field affect the particle..

$\Rightarrow$ let's put in something;

$$\mathcal{L}_{\text{particle}} = - (m + g\phi(x, t)) \sqrt{1 - \dot{x}^2}$$

$$\mathcal{L}_{\text{particle}} = - (mc^2 + g\phi(x, t)) \sqrt{1 - \frac{\dot{x}^2}{c^2}}$$

g is called coupling constant $\Rightarrow$ it measures the strength by which field affects motion of particle.

* let's go to non-relativistic limit : when $\ll v$
(for illustration purposes... to understand in easy case)

$$\text{so: } \mathcal{L}_{\text{particle}} \rightarrow -(mc^2 + g\phi) \left(1 - \frac{\dot{x}^2}{2c^2}\right)$$

$$= -mc^2 + \frac{m\dot{x}^2}{2} + g\phi - \frac{g\phi\dot{x}^2}{2c^2}$$

it is just a number... it is of no interest

↓
very small

$$\mathcal{L}_{\text{non relativistic particle}} = \frac{m\dot{x}^2}{2} + g\phi$$

identical to $T - V$; V is identified as $g\phi$.

* g is just a constant.

↳ for example, In electromagnetism,
but in

the strength of the coupling of a field on a particle is just electric charge.

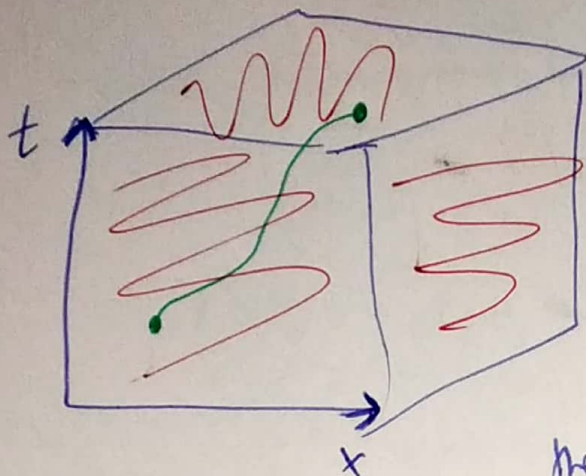
↳ so, it's natural that there be some constant multiplying something about the field.

→ Represents effect of field on particle.

Now, what about effect of particle on the field.

* Important thing to understand : is that there is only one Action → The total Action all together.

(The total action includes an action for the field and the Action for the particle)



we are trying to study the combined system of a field ; & a particle moving through the field.

→ & we are trying to minimize the action ; not just w.r.t. motion of the particle ; but w.r.t. motion of particle and motion of field.

* For that ; we simply explore around ; changing the field ; ~~let~~ letting it wiggle in different ways ; change the particles trajectory untill we ~~minimize~~ minimize the action → and that will give us the trajectory & the field that corresponds to the principle of Least Action.

* So ; lets write ; whole Action ⇒ including the field & the particle.

$$\int \mathcal{L}_{\text{Field}} \cdot d^4x = \int \frac{1}{2} \left[\frac{1}{c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \sum_i \left(\frac{\partial \phi}{\partial x^i} \right)^2 \right] d^4x$$

→ This is action for the field → now ; lets add to that the Action of the particle.

We will consider a very special simple case.

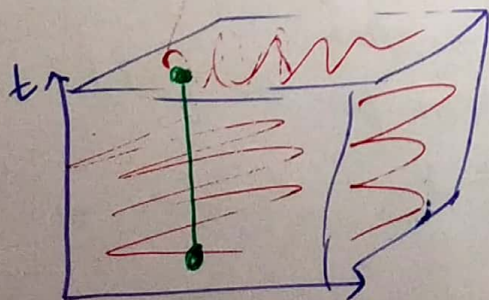
→ The particle is at rest.

* lets put the particle at rest.

→ lets assume that there is solution to the equation where the particle is just at rest.

* (After all there are situations where an electron might be at rest)

→ and ask what additional term in action can be.



$$\int \mathcal{L}_{\text{particle}} \cdot dt = \int - (mc^2 + g \cdot \phi) \sqrt{1 - v^2} dt$$

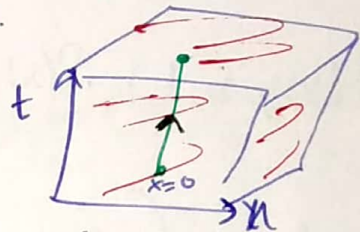
no contribution: ... just a constant

$$= -mc^2 \int_{t_{\text{initial}}}^{t_{\text{final}}} dt + \int g \phi dt$$

Contribution to ~~the~~ Total Lagrangian from this piece over here is just

$$- \int g \cdot \phi(x, t) dt$$

it's an integral along the world line of particle.



* Reason, the particle responded to the field because the Lagrangian for the particle involves ϕ .

* Reason, the field responds to the particle is because the same Lagrangian also has ϕ and positions of particle in it.

* Now we can add this term to field term.

* $\int \mathcal{L}_{\text{particle}} \cdot dt$ is integral over time $\Rightarrow$ integral over world line

* $\int \mathcal{L}_{\text{field}} \cdot d^4x$ is " over space time $\Rightarrow$ integral over all space-time.

$\hookrightarrow$ We want to add it ~~zero~~ in a way so it corresponds to integral over space-time.

for convenience and simplicity: let's put particle at $x=0$

$$\therefore \int \mathcal{L}_{\text{particle}} \cdot dt = - \int (g \cdot \phi(0, t)) dt$$

now

$$\phi(0, t) = \int \phi(x, t) \delta^3(\vec{x}) d^3x$$

→ dirac delta function.

$$\delta^3(\vec{x}) = \delta(x) \delta(y) \delta(z)$$

↳ it is non zero when all three δ functions are non zero.
it is at $(x, y, z) = (0, 0, 0)$

→ ~~This~~ It is function of three coordinates of space
↳ but has value at only one point of space.
(and at that point; it is enormously big.)

$$\iiint \phi(x, y, z, t) \cdot \delta(x) \delta(y) \delta(z) \cdot dx dy dz = \phi(0, 0, 0, t)$$

$$\text{so, } A = \int L_{\text{particle}} \cdot dt = \int \left(\iiint \phi(x_i, t) \delta^3(x_i) d^3x \right) dt$$

$$= \int \phi(x_i, t) \delta^3(x_i) d^4x$$

↳ we did this; because we wanted to add this with L_{field} .. ~~getting total action.~~

$$\text{Action} = \int \left\{ \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \sum_i \left(\frac{\partial \phi}{\partial x^i} \right)^2 \right] - g \cdot \delta^3(x_i) \phi(x^4) \right\} d^4x$$

↳ and this is the Lagrangian.

$$\text{Lagr } L = \frac{1}{2} \left[\frac{1}{c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \sum_i \left(\frac{\partial \phi}{\partial x^i} \right)^2 \right] - g \delta^3(x_i) \phi(x^4)$$

it has good old field action

↳ this represents effect of particle on field.

now; let's find equation of motion.

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = \left[\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \right] : \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = \left(- \frac{\partial^2 \phi}{\partial x^2} \right) m$$

↳

$$\text{so; } \left[\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} \right] = \frac{\partial \mathcal{L}}{\partial \phi} = -g \delta^3(x_i)$$

So; it is wave equation; but there is one point which is going different.

↳ What's going on is the particle there

↳ It's affecting the equation of motion; but only at one point.

★ let's suppose; we are looking for solution of eqⁿ (after all particle is standing still in this example) $\Rightarrow$ There ~~will be a solution~~ might be a chance, that there is solution of equation where the field itself does not change with time.

↳ just the effect standing still particle affecting the field is likely to make a field which also does not vary with time

so; we might look for solution when $\frac{\partial \phi}{\partial t} = 0$

$$\text{so; } \left[\sum_i \frac{\partial^2 \phi}{\partial x_i^2} \right] = g \delta^3(x_i)$$

↳ This is poisson equation.

source

$$\nabla^2 \phi = g \delta^3(x_i)$$

source density term.

- The same term in the Lagrangian which tells the particle to move as if there was potential energy $g \cdot \phi$ (in other words which exerts forces on the particle)
- Also the same term when used for the equation of motion for ϕ field; tells us that the ϕ field has a source.

→ These are not independent things $\Rightarrow$ The fact that field affects particle tell us particle affects field.

→ and with an specific simple example; it even tell us how;

$$\nabla^2 \phi = g \delta^3(x)$$

$$(\text{or}) \nabla^2 \phi = g \delta^3(\vec{x} - \vec{a})$$

where particle is at position $\vec{a}$ in space.

* the same parameter g ;

- The parameter g determines how strong the effect of particle on the field is.
- it also tell us how strong is the effect of field on the particle is.

→ So; fields & particle affect each other;

through a common term in Lagrangian.

example one dimensional space;

$$\nabla^2 \phi = g \delta^3(x)$$

$$\Rightarrow \frac{d^2 \phi}{dx^2} = g \delta(x)$$

now; suppose particle is moving;

→ all you would do;

$$\begin{aligned} \nabla^2 \phi &= g \delta(x - a) \\ &= g \delta(x - a(t)) \end{aligned}$$

$$\nabla^2 \phi = g \delta^3(\vec{x} - \vec{a}(t))$$

→ and what this tells is that

there is source of field; but the source of field is moving & at any given time it's at position $\vec{a}(t)$

→ so we can accommodate a moving particle.

→ If we did have the particle moving we would not expect field to be time ~~independent~~ independent.

→ so;

$$-\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = g \delta^3(\vec{x} - \vec{a}(t))$$

→ so; moving, or accelerating, or vibrating particle affect the field ... and give it a time dependence.

→ it will radiate waves.

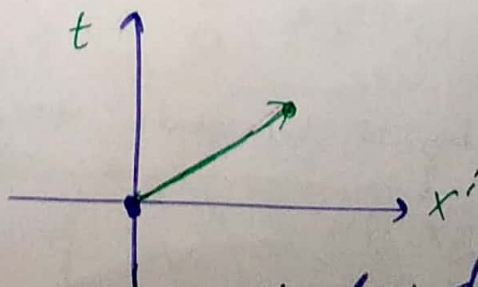
$$\square \phi = g \delta^3(\vec{x} - \vec{a}(t))$$

~~misinterpretation~~
Note! $g \cdot \phi(x^u)$ has to be energy.

Notation: (Partly invented by Minkowski, Einstein...)

$$x^u = (t, x, y, z) \\ = (x^0, x^1, x^2, x^3)$$

putting index up.



→ if this quantity thought of as displacement from origin is a vector. (A vector is a ~~very~~ statement about the way it transforms)

* All will agree zero vector is zero vector.

~~It is~~ • It is invariant.

(zero transforms to itself after transformation)

~~It is~~ dx^μ .

~~# We can~~

We can construct proper time out of displacement.
(a scalar)

Scalar : $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$.

→ If a scalar is equal to zero in one frame of reference, it's zero in every frame of reference.

→ because scalar is the thing which is same in every reference frame.

A^μ ; $(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$

→ A with upper index.

We invent a matrix to condense notation.

→ It is called metric (in special relativity it is denoted by η)

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

∴ we are going to represent 4 vectors by column.

$$A^\mu = \begin{pmatrix} A^t \\ A^x \\ A^y \\ A^z \end{pmatrix} = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

$$\sum_{\nu} \eta_{\mu\nu} A^{\nu} \quad \text{matrix multiplication} \quad = (\eta A)_{\mu}$$

matrix multiplication.

$$\therefore \text{we}; (\eta A)_{\mu} := A_{\mu}$$

$$\Rightarrow A_{\mu} = \sum_{\nu} \eta_{\mu\nu} A^{\nu}$$

$\rightarrow$ A with a lower index.

$\hookrightarrow$ It is not the original A. ~~It is~~ It is the transformed A as per matrix η (if it were unit matrix ($\eta = I$)... then you would get same thing)

$\hookrightarrow$ but; $\eta \neq I$.

$$A_{\mu} = \begin{pmatrix} -A^t \\ A^x \\ A^y \\ A^z \end{pmatrix} = \begin{pmatrix} -A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

$$A^{\mu} = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

$\hookrightarrow$ In other words; when we perform the operation, we simply change the sign of time component.

★ Einstein summation convention: whenever you see down stair index & upstairs index the same; it is automatically the rule that you sum over it.

so; $A_{\mu} = \eta_{\mu\nu} A^{\nu}$

~~sometimes~~
Sometimes people also use other convention;

$$\eta = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Vector with upper index (A^μ) is called Contravariant index.
- " " lower " (A_μ) " " Covariant vector index.

→ and you go between them by matrix η .

$$\begin{aligned}
 A^\mu A_\mu &= A^t A_t + A^x A_x + A^y A_y + A^z A_z \\
 &= \cancel{A^t A_t} \\
 &= A^t (-A^t) + (A^x)^2 + (A^y)^2 + (A^z)^2 \\
 &= -(A^t)^2 + (A^x)^2 + (A^y)^2 + (A^z)^2
 \end{aligned}$$

→ well; this is exactly the quantity which he think is Scalar ... its the difference of square of time component and space components.

* ~~we may actually~~

$$\# A^\nu A_\nu = A^\mu A_\mu$$

$$\begin{aligned}
 \mu &\in \{0, 1, 2, 3\} \\
 \nu &\in \{0, 1, 2, 3\}
 \end{aligned}$$

* greek index refer to four components.
* latin, i, j, refer to space components.

$$\# \boxed{A^\mu A_\mu = A^\mu \eta_{\mu\nu} A^\nu} \quad \text{Its scalar.}^*$$

→ It all stands for is A time squared minus A space squared.

let take two four vectors. $A^\mu B_\mu$ is it a scalar?

→ To prove it is scalar; we need to prove sum of two ~~Scalars~~ & difference of two scalars is also a scalar.

$$* (A+B)^\mu (A+B)_\mu - (A-B)^\mu (A-B)_\mu$$

→ Take a four vector & contract it with itself.

you could also show, $A^\mu B_\mu = A_\mu B^\mu$

so; if we take any two 4-vectors and contract them in this way ~~$(A+B)^\mu (A+B)_\mu = (A+B)^\mu (A+B)_\mu = (A+B)^\mu (A+B)_\mu$~~ ... we get another scalar ...

→ slight generalization of saying that just $A^\mu A_\mu$ is a scalar.

so, $A^\mu B_\mu$ is also a scalar.

$A^\mu B_\mu$ is sometimes said to be contracting A & B 4-vectors.

→ you can think of Minkowski version of dot product. 'Lingo'!

$\phi(x^\mu)$

* difference between ϕ at two neighbouring points; infinitesimally separated.

$$\frac{\partial \phi(\vec{x})}{\partial x^\mu} \cdot dx^\mu$$

→ This is small change in ϕ in going from one place to another.

$$\text{i.e. } \frac{\partial \phi}{\partial x^\mu} \cdot dx^\mu$$

known; ϕ is a scalar $\Rightarrow$ so we will agree that the value of difference of ϕ is also scalar;

Theorem: suppose you have a known 4-vector (to say that it is 4-vector means that it transforms in a certain way)

~~A~~

known 4-vector

~~to another quantity~~

A, B is another quantity.

such that $A_\mu B^\mu$ you get something which is scalar.

→ then it is possible to prove that if the product is independent of reference frame $\Rightarrow$ then B is also a 4-vector. (it also transforms as a 4-vector)

→ and dx^μ is clearly itself a 4-vector

hence: $\frac{\partial \phi}{\partial x^\mu}$ is also a 4-vector.

$$\frac{\partial \phi}{\partial x^\mu} = \left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

Now: In order to make sense as ~~product~~ sum of product: $\frac{\partial \phi}{\partial x^\mu} \cdot dx^\mu$

- $\frac{\partial \phi}{\partial x^\mu}$ had better correspond to a covariant vector
- if the ~~dx~~ dx^μ corresponds to contravariant one.

→ so; we found all that derivatives of ϕ w.r.t. x ; form a 4-vector ... but they form the covariant components of 4-vector.

so; $\frac{\partial \phi}{\partial x^\mu} := \partial_\mu \phi$

→ is a covariant vector.

→ represented this way
→ index it put lower so as to indicate that it is covariant thing.

Now we can use notational device to invent $\partial^\mu \phi$

$$\partial^\mu \phi \Rightarrow \left(-\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

$$\partial_\mu \phi \Rightarrow \left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

(*) Now we can compose a new scalar.

(pg 71)

$$\partial^\mu \phi \cdot \partial_\mu \phi = \left[- \left(\frac{\partial \phi}{\partial t} \right)^2 + \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right]$$

↳ (This type of term appeared in Lagrangian.)

↳ What this is telling us that Lagrangian is scalar

↳ The reason it is good that Lagrangian is scalar.
(it says action is same in every reference frame)

$$L_{\text{scalar field}} = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi$$

we could add other things to it.

↳ The rule is "Lagrangian should be a scalar"
so as to make sure that Action is same in every reference frame.

• i.e. If you have a scalar quantity which does not depend on frame of reference & you integrate it over all spacetime; then the result does not depend on which frame of reference it has been evaluated in.

↳ So; if your principle of Physics = i.e. Principle of Least Action.

... & the Action does not depend on which frame it is written in → Then the same Action Principle will give you equations of motion in each frame which has same form in every reference frame.

↳ It will give you equations which are invariant under Lorentz Transformation.

• If you leave out statement about light and take only the statement of Principle of Relativity; that lots of physics are same in every reference frame → Then...

Ordinary Newtonian Physics or Galilean physics.

... $F=ma$ type physics ... $\Rightarrow$ does have relativity associated with it.

$\hookrightarrow$ It is not the relativity of Lorentz Transformation.

$\hookrightarrow$ It is the relativity of the limit of Lorentz Transformations when speed of light goes to infinity.

$$\left. \begin{array}{l} x' = \frac{x - vt}{\sqrt{1 - (v/c)^2}} \\ t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} \\ y' = y ; z' = z \end{array} \right\} \begin{array}{l} \text{Taking limit} \\ \text{of this} \\ \text{transformation} \\ c \gg v \end{array} \Rightarrow \begin{array}{l} x' = x - vt \\ t' = t \\ y' = y \\ z' = z \end{array}$$

$\hookrightarrow$ So, there is an invariance of non-relativistic Newtonian mechanics ... it's called Galilean invariance & requires equations to be invariant under these replacement: $x' = x - vt$, $t' = t$.

$t' = t \Rightarrow$ says that all frames of reference share the same time $\rightarrow$ simultaneity is same for everybody.

$$\# (A')^t = \frac{A^t - v A^x}{\sqrt{1 - v^2}} ; (A')^x = \frac{A^x - v A^t}{\sqrt{1 - v^2}}$$

$$(A')^t = -(A')_t ; A^t = -A_t^*$$

$$\pm) -(A')_t = \frac{-A_t - v A_x}{\sqrt{1 - v^2}}$$

$$(A')_x^* = \frac{A_x + v A_t}{\sqrt{1 - v^2}}$$

$$\Rightarrow (A')_t = \frac{A_t + v A_x}{\sqrt{1 - v^2}} ; (A')_x = \frac{A_x + v A_t}{\sqrt{1 - v^2}}$$

Note: Be careful about scalars here.

- The energy density of gas will be different in frame of reference where box of gas is at rest from the frame moving fast relative to box.

∴ The moving one will record larger.

→ So; Energy density is something on which different frames will not agree on. → so it is not scalar

* (but don't confuse with usual scalar ... which has only magnitude)

→ Here we mean different thing.

let's solve wave equation.

... imagine you have wave field.

$$\mathcal{L} = \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 - \left(\frac{\partial \phi}{\partial x} \right)^2 - \left(\frac{\partial \phi}{\partial y} \right)^2 - \left(\frac{\partial \phi}{\partial z} \right)^2 \right] + \left(-\frac{\mu^2}{2} \phi^2 \right)$$

→ we added one more term
• it is also a scalar.

Now; this is field theory analogue of ~~the~~ harmonic oscillator.

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = -\mu^2 \phi$$

...
↓
analogue of
acceleration term

$$\Rightarrow \boxed{\frac{\partial^2 \phi}{\partial t^2} - \sum_i \frac{\partial^2 \phi}{\partial x_i^2} + \mu^2 \phi = 0}$$

KLEIN-GORDON Equation.

(They wrote this equation ... this was their attempt before Schrodinger ... to write down the equation for Quantum Mechanical particle.
↳ They tried to be relativistic ... if they would not have been tried to be relativistic; they would have written what now is known as Schrodinger equation.)

There are lot of solutions of this
 ↳ They are sums of plane waves.

* let's write plane waves in exponential form.

(it is useful when thinking about oscillating systems could pretend that the coordinate is complex $\rightarrow$ & at the end just take the real part of it)

★ we are going to look for solutions which oscillate in time & over space.

$$e^{-i\omega t} \cdot e^{i(k_x x + k_y y + k_z z)}$$

~~$$= e^{i k_\mu X^\mu}$$~~

$$= e^{i k_\mu X^\mu} = e^{i k_\mu X^\mu}$$

$\therefore (k_x, k_y, k_z)$ are just three numbers... they are called wave numbers or wave vector.

~~if we took k_μ as 4 vectors with~~

if we think of k as 4 vector with components

$$k_\mu \rightarrow (-\omega, k_x, k_y, k_z)$$

$$X^\mu \rightarrow (t, x, y, z)$$

$\Rightarrow$ from Klein Gordon equation. ; so; our guess for ϕ is $\phi = e^{i k_\mu X^\mu}$

$$-\omega^2 \phi + k_x^2 \phi + k_y^2 \phi + k_z^2 \phi + \mu^2 \phi = 0$$

$$\Rightarrow (-\omega^2 + k_x^2 + k_y^2 + k_z^2 + \mu^2) \phi = 0$$

$$\text{so; } \omega = \pm \sqrt{k_x^2 + k_y^2 + k_z^2 + \mu^2}$$

• for every k_x, k_y, k_z ; there are two solutions for ω .
 ↳ These are solutions of Klein-Gordon equation.

* Note : Klein-Gordon equation

↳ This is the classical field version of an equation which describes a quantum mechanical particle with mass m , energy ω , & momentum k .

$$E = \sqrt{p^2 + m^2} \quad \text{similar to} \quad ; \quad \omega = \sqrt{|k|^2 + m^2}$$

→ After you have constructed your solution in $e^{ik_\mu x^\mu}$ form; you can take real part of it.

Euler Lagrange equation for field.

$$\frac{\partial}{\partial x^\mu} \cdot \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi}{\partial x^\mu}} = \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

↓
has a form of containing lower index

↓
has a form of containing upper index
∴ $\frac{1}{v_p} = p \dots$ Notation.

* $\int - (m + \phi) \sqrt{1 - v^2} \cdot dt = \text{Action of particle.}$

∴ (i) if ϕ is constant ⇒ The particle will effectively move with mass $(m + \phi)$.

(for all practical purposes) if ϕ is constant; it just shifts the mass of the particle)

(ii) It is also possible; you might have massless particle.

↳ so; the ~~the~~ intrinsic mass is due to ~~some~~ field.

* different coupling constant could lead to different masses.
⇒ so; for some reason, there is an environment in which ϕ is not zero, can not only shift the mass ⇒ it can create a mass for particle which did not even have one to begin with.

2) This is nature of Higgs Bosons.

• it creates mass roughly in this sense.

→ & because it create masses for particles (such as quarks, etc.)

the presence of those particle will react back on field

& create field.

* The strength with which a particle couples to the Higgs field will determine its mass.

=> Particles create field ... & field affects particles.

... Higgs Bosons is a particular case.

Special Point : Superconducting case is a situation where Electromagnetic field gets a mass ... and gets a mass from similar thing like above.

Lec-6 Special Relativity & Electrodynamics : The Lorentz Force Law. - Shear Hkatar. (Pg 77)

"Good notations can be extraordinarily powerful"

$A^\mu \rightarrow A^0, A^m$
 $dx^\mu \rightarrow dx^0, dx^m$

Contravariant Notation

A super μ : A^μ

$$A_\mu = \eta_{\mu\nu} A^\nu$$
 where $\eta = \text{diag}(-1, +1, +1, +1)$
 so: $A_0 = -A^0$
 $A_m = +A^m$
 covariant counterpart.
 (actually the same vector... but another way of describing it)

A sub μ : A_μ

$$\star d\phi = \frac{\partial \phi}{\partial x^\mu} dx^\mu = \partial_\mu \phi dx^\mu$$

$A_\mu B^\mu \Rightarrow \text{Scalar}$.
 (Quantity which does not change from frame to frame)

$$\frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi$$

let: $B^\mu(x)$
 a 4-vector which depends on space-time.

$$\partial_\mu B^\mu(x) = \frac{\partial B^0}{\partial x^0} + \frac{\partial B^1}{\partial x^1} + \frac{\partial B^2}{\partial x^2} + \frac{\partial B^3}{\partial x^3}$$

Scalar

(This is one of the example of forming scalars from vectors by what we learn to call Index Contraction)
 (is same as identifying upper & lower index; and then summing over)

(Scalars)' = (scalar)

$(A^\mu)' = L^\mu_\nu A^\nu$

it is a matrix element
 • has two entries, μ, ν
 → makes it a matrix

Example: Lorentz Transformation along x-axis

$$L = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} & 0 & 0 \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

→ standard Lorentz Transformation along x-axis.

if we write column vector.

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

~~Rotation of space is part of collection of Lorentz transformation.~~

* When we say rotation of space
 → we mean that time is not affected at all.

Note

• Rotation of space is part of collection of Lorentz transformation.

$$R(-\theta) L(\beta) R(+\theta)$$

Then: $\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = L \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$ } $\Rightarrow$ This was an example of Lorentz matrix along x-axis (pg 78)

~~an If we want to do~~

$$\# \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

We want to rotate by angle θ in y-z plane.
 • t is unaffected • x is unaffected.

Now you can take these matrices & start multiplying them, or combining them to make much more complicated transformations which for example are partly rotation along same axis, some are partly Lorentz Transformation along other axis.

$$\# \text{ given: } (A^\mu)' = L^\mu_\nu A^\nu$$

$$\text{then: } (A_\mu)' = M_\mu^\nu A_\nu$$

$$\text{where: } M = \eta L \eta$$

note here; L & M represents same physical transformations between coordinate frames S and S' so L & M must be connected.

$$\text{incidentally: } \eta^{-1} = \eta \quad (\eta \text{ is its own inverse})$$

$$\dots \text{ so; } \eta M \eta = L \quad \text{holds.}$$

• A 4-vector: (covariantly or contravariantly)

$\hookrightarrow$ What it stands for is an object which transforms from one frame to another in particular special way.
 $\hookrightarrow$ And the way is in parallel in the way coordinates transform.

Tensor ... they are simply a thing with more indices.

* Scalar is 0 rank Tensor.

* Vector " 1 rank Tensor.

* A Tensor of Rank 2 would be a thing with two indices.

Example let's take two vectors A & B .

$A_\mu B^\mu = \text{scalar}$ } we can make a product ... which is scalar.
(like this)

→ now we consider more general kind of product.

↳ more general kind of product has two indices. say μ, ν .

→ let's begin with Contravariant Version of it.

$$A^\mu B^\nu$$

This is an object ... it has $4 \times 4 = 16$ components.

This is a symbol which stands here for a complex of 16 different objects: $A^\mu B^\nu$
(different numbers)

↳ They are just a set of number you get by multiplying any component of A with any component of B .

... its got two indices & called ... & labelled in following way.

$$A^\mu B^\nu = T^{\mu\nu}$$

Now does this object $(T^{\mu\nu})$ now transforms.

* if we know how A & B transforms ... (which is same way here)
ie: $(A^\mu)' = L^\mu_\sigma A^\sigma$; $A'_\mu = M_\mu^\sigma A_\sigma$

$$(A')^\mu (B')^\nu = (L^\mu_\sigma A^\sigma) (L^\nu_\tau B^\tau) = (L^\mu_\sigma L^\nu_\tau) A^\sigma B^\tau$$

$$\Rightarrow L^\mu_\sigma L^\nu_\tau A^\sigma B^\tau$$

so; now we have an example: $(A')^\mu (B')^\nu = L^\mu_\sigma L^\nu_\tau A^\sigma B^\tau$

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→ so; here we have a new rule about a new kind of object with two indices which tells us how it transforms.

→ It transforms here with the action of Lorentz matrix on each one of the indices.

→ Now we can abstract from that

... and can say,

more generally; The way a Tensor transforms.

$$(T^{\mu\nu})' = L^\mu_\sigma L^\nu_\tau T^{\sigma\tau}$$

This is rule for Transformation properties of a simple tensor with two indices.

you can invent more complicated kinds of Tensor.... with three indices.

$$(T^{\mu\nu\lambda})' = L^\mu_\sigma L^\nu_\tau L^\lambda_\kappa T^{\sigma\tau\kappa}$$

There is a Transformer for each index.

we can also have;

$$A^\mu B^\nu + C^\mu D^\nu$$

• by assumption now: Adding Tensor gives us tensor.

} This is something which in general cannot be written as product of two vectors.

... but it's a Tensor

→ It's defined by its Transformation properties, not by the fact that it may or may not be associated with just pair of vectors.

example $T^{00} = (-1)^2 T_{00} = T_{00}$

(it is like relation between $A^0 B^0$ & $A_0 B_0$)

$\therefore A^0 B^0 = (-A_0)(-B_0) = (-1)^2 A_0 B_0 = A_0 B_0$

example $A^0 B^1 = -A_0 B_1$
 $= -A_0 B^1$

so: $T^{01} = -T_{01} = -T_{01} = T^{01}$

• everytime you lower or raise a time component ; you get a minus sign.

~~$T^{\mu\nu} = T^{\nu\mu}$~~
~~Symmetrical~~

in general: $A^\mu B^\nu$ is not same as $A^\nu B^\mu$
 $\Rightarrow A^0 B^1 \neq A^1 B^0$

... $T^{\mu\nu}$ is in general not same as $T^{\nu\mu}$ ~~changing order of indices might~~

In general, Tensors are not invariant under changing the order of the indices.

Special Cases.

Symmetric Tensors : $T^{\mu\nu} = T^{\nu\mu}$

example: $A^\mu B^\nu + A^\nu B^\mu$ } you can check that this is Symmetric Tensor.

example: $W^{\mu\nu} + W^{\nu\mu}$

Antisymmetric Tensors : $F^{\mu\nu} = -F^{\nu\mu}$

example: $A^\mu B^\nu - A^\nu B^\mu$

example: $W^{\mu\nu} - W^{\nu\mu}$

* Antisymmetric Tensors have fewer component than Symmetric Tensors $\Rightarrow$ reason is that their diagonal components vanish.

* If you think of it as an array $\rightarrow$ then an Antisymmetric Tensor is completely zero on the diagonal.

$\therefore F^{00} = -F^{00} \Rightarrow F^{00} = 0$
 $\hookrightarrow F^{55} = -F^{55} \Rightarrow F^{55} = 0$

~~more examples~~ $\rightarrow$ next page.

and is of the form.

x here; E & B labels
are down.
(later they will
make more sense)
↳ now it is just
for labelling purposes

$$\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ +E_1 & 0 & -B_3 & B_2 \\ +E_2 & +B_3 & 0 & -B_1 \\ +E_3 & -B_2 & +B_1 & 0 \end{pmatrix}$$

↳ This is notion of an Antisymmetric tensor.

• and it plays a key role in Electromagnetism where E stand
of course for Electric field & B for Magnetic field.

↳ we will see; The Electric & Magnetic field combine together
to form antisymmetric tensor in relativity.

→ The Electric & Magnetic field are not two independent things.

... in particular, what that means, is.

↳ that ~~when~~ when a Lorentz Transformation is
performed, The Electric field can become Magnetic field & vice versa.
just like x s can get mixed with t .

so; E can get mix with B .

⇒ so; what one can see as purely E field; ... one can see
it having some B component to it.

• The Dynamics of Electromagnetic field would be Equations of motions
that electromagnetic field satisfies... The Maxwell Equations

• Equations of a particle moving in an Electromagnetic field are
Lorentz Force Law.

Lorentz Force Law.

low velocity version: $m\vec{a} = e[\vec{E} + \vec{v} \times \vec{B}]$ } There is some speed of light in there... but set them equal to ∞ . 1.

Non-relativistic version of the Lorentz Force Law.

$$m\vec{a} = e[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}]$$

→ We are going to find relativistic version ... and we will find that the two terms " $\vec{E}$ " & " $\vec{v} \times \vec{B}$ " are really a part of a same thing written in a way where all reference frame will get same answer.

action of particle without field: $\int_1^2 -m dz = \int_1^2 -m \sqrt{1 - \sum_i \dot{x}_i^2} \cdot dt$

$$= \int_1^2 -m \sqrt{1 - v^2} \cdot dt$$

→ this identifies $-m\sqrt{1-v^2}$ as the Lagrangian of the particle.

★ What we add in an Electromagnetic field.

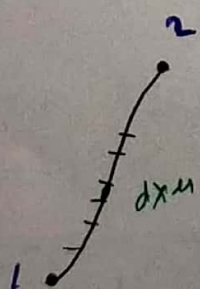
• a basic structure which enters in electromagnetic field is a 4-vector.

→ Electric & Magnetic fields are themselves derived quantities.

... The basic underlying quantity is $A_\mu(\vec{X}, t)$

→ This is the field describing electromagnetic waves (if you like) ... and it's a 4-vector.

→ This is called Vector Potential ... 4-component vector potential.
(and we will learn what it has to do with Electric & Magnetic fields...)



... we can make a scalar $A_\mu dx^\mu$ associated with a small segment of the path.

$$\int_1^2 dx^\mu A_\mu(\vec{x}, t)$$

... we will multiply it by a constant e
(it is the electric charge)

and because of notation that was first put in place by Benjamin & Franklin ... there is a minus sign.

$$-e \int_1^2 dx^\mu A_\mu$$

This is the other term in the action for a particle moving in an electromagnetic field.

$$\text{Action} = \underbrace{\int_1^2 -m \sqrt{1-v^2} dt}_{\text{Lagrangian for free particle}} - e \int_1^2 dx^\mu A_\mu(\vec{x}, t)$$

He wants to write Lagrange equations and show that they are just Lorentz Force law.

... but we will do it in a way which is relativistically invariant.

we write $-e \int_1^2 dx^\mu A_\mu$ in a form which will make it look like a Lagrangian: we just divide by dt & multiply by dt

$$-e \int_1^2 \frac{dx^\mu}{dt} A_\mu(x) dt \quad ; \quad \frac{dx^0}{dt} = 1$$

$$\begin{aligned} \text{so:} &= -e \int_1^2 [A_0(\vec{x}, t)] dt - e \int \dot{x} A_x dt - e \int \dot{y} A_y dt - e \int \dot{z} A_z dt \\ &= -e \int_1^2 A_0 dt - e \int \dot{\vec{x}} \cdot \vec{A} dt \end{aligned}$$

The reason that we wrote it this way with dt there, is to make it have a familiar look of Lagrangian action.

An Action written as integral over dt .

$$\text{so: } A = \int (-m \sqrt{1-v^2} - e A_0 - e \dot{x}^m A_m) dt$$

$$\text{so: } \boxed{\mathcal{L} = -m \sqrt{1-\dot{x}^2} - e A_0 - e \dot{x}^m A_m}$$

↳ A_m is field... we are going to imagine that $A_m(\vec{x}, t)$ is known function.

... and now we are just exploring motion of the particle in that known field.

$$\# \quad \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = \frac{m \dot{x}_i}{\sqrt{1-\dot{x}^2}} - e A_i$$

$$= m \frac{dx^i}{dz} - e A_i$$

note note
 $e \dot{x}^i A_i(\vec{x}, t)$
 ↓
 it is a mixed term
 ↳ depends on both position and velocity

$$\Rightarrow \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i} \right) = \frac{\partial \mathcal{L}}{\partial x^i} = -e \frac{\partial A_0}{\partial x^i}$$

$$\Rightarrow \boxed{\frac{d}{dt} \left[m \frac{dx^i}{dz} - e A_i \right] = -e \frac{\partial A_0}{\partial x^i} - e \dot{x}^m \frac{\partial A_m}{\partial x^i}}$$

Euler-Lagrange equation for each component of position.

looks little familiar

- LHS has something which sort of like m times acceleration.
- RHS " " " is minus the derivative of something.

... little similar to $m a = - \frac{\partial V}{\partial x}$

so we have;

$$\# \text{ i.e. } \mathcal{L} = -m \sqrt{1-\dot{x}^2} - e A_0 - e \dot{x}^m A_m$$

$$\Rightarrow \frac{d}{dt} \left[m \frac{dx^i}{dz} - e A_i \right] = -e \frac{\partial A_0}{\partial x^i} - e \dot{x}^m \frac{\partial A_m}{\partial x^i}$$

$$m \cdot \frac{d}{dt} \left(\frac{\dot{x}_i}{\sqrt{1-|\dot{x}|^2}} \right) - e \frac{\partial A_i}{\partial t} - e \frac{\partial A_i}{\partial x^m} \dot{x}_m$$

$$= -e \frac{\partial A_0}{\partial x^i} - e \dot{x}_m \frac{\partial A_m}{\partial x^i}$$

lets call $m \cdot \frac{d}{dt} \left(\frac{\dot{x}_i}{\sqrt{1-|\dot{x}|^2}} \right)$ as $m \cdot a$. ; take all others to RHS.
 in two kind of terms $\Rightarrow$ we are going to group them
 • one kind is just proportional to velocity.
 • other kind having velocity.

let $m \cdot \frac{d}{dt} \left(\frac{\dot{x}_i}{\sqrt{1-|\dot{x}|^2}} \right) := m \cdot "a";$

for simplicity; ie: $"a_i" := \frac{d}{dt} \left(\frac{\dot{x}_i}{\sqrt{1-|\dot{x}|^2}} \right)$
 just a notation for now

$\Downarrow$
 we are actually going towards
 $e \cdot E$ (which does not have velocity)

2 then $e \vec{v} \times \vec{B}$ (which does have velocity)

$\Downarrow$ and we are trying to identify these two terms.

so;

~~$$m "a_i" = e \left(\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x^i} + \dot{x}_m \left(\frac{\partial A_i}{\partial x^m} - \frac{\partial A_m}{\partial x^i} \right) \right)$$~~

$$so; m "a_i" = e \left(\frac{\partial A_i}{\partial x^0} - \frac{\partial A_0}{\partial x^i} \right) + e \left(\dot{x}_m \left\{ \frac{\partial A_i}{\partial x^m} - \frac{\partial A_m}{\partial x^i} \right\} \right)$$

This term wants to be electric field... has no velocity term.

This term wants to be magnetic field ... has velocity term.

* notice the pattern ... each of the groups has ~~terms~~ derivatives of A w.r.t. coordinates, antisymmetrized.
 $\rightarrow$ Notice the antisymmetry here ...

note: $\dot{t} = 1$; $\dot{t} = \frac{dt}{dt}$ @ $\dot{x}_0$

$$\text{so; } m^{\mu} a_{\mu} = e \left(\frac{\partial A_i}{\partial x_0} - \frac{\partial A_0}{\partial x_i} \right) \dot{x}_0 + e \left(\frac{\partial A_i}{\partial x^m} - \frac{\partial A_m}{\partial x^i} \right) \dot{x}_m$$

→ now; it looks symmetric w.r.t. space-time little bit

so; actually that can be condensed in the form.

$$m^{\mu} a_{\mu} = e \left(\frac{\partial A_i}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^i} \right) \dot{x}^{\mu}$$

where; $\dot{x}^{\mu} = \frac{dx^{\mu}}{dt}$
or $\frac{dx^{\mu}}{dx^0}$

and $a_{\mu} = \frac{d}{dt} \left(\frac{\dot{x}_i}{\sqrt{1-\dot{x}^2}} \right)$

note; write $\dot{x}^{\mu}$ here
not $\dot{x}_i$
~~visually $\dot{x}_i$ is not a 4-vector~~
actually; $\dot{x}_i \rightarrow \left(-\frac{dt}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$
 $\dot{x}^{\mu} \rightarrow \left(\frac{dt}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$

Note x_m or x^n means same thing ... (latin indices run over 1, 2, 3...)
 $x_m = +1 \cdot x^n$

∴ but $x_0 = -x^0$; so; be careful about x_{μ} and x^{μ} .
 μ : greek runs from 0, 1, 2, 3.

★ let try to write this formula in a way which is manifestly or obviously Lorentz Invariant

and the key are not now; because the RHS & LHS are not themselves 4-vectors.

→ but we can convert them to 4-vectors.

... write RHS & LHS being Tensor of same kind.

$$m \cdot \frac{d}{dt} \left(\frac{dx^i}{\sqrt{1-v^2}} \right) = m \frac{d}{dt} \left(\frac{dx^i}{dz} \right) = m \frac{d}{dt} (u^i)$$

$$\Rightarrow m \cdot \frac{d}{dt} \left(\frac{dx^i}{dz} \right) \frac{dt}{dz} = \frac{dt}{dz} \left(\right)$$

$$\Rightarrow m \cdot \frac{d}{dz} \left(\frac{dx^i}{dz} \right) = e \left(\frac{\partial A_i}{\partial x^0} - \frac{\partial A_0}{\partial x^i} \right) \frac{dx^0}{dz} + e \left(\frac{\partial A_i}{\partial x^n} - \frac{\partial A_n}{\partial x^i} \right) \frac{dx^n}{dz} \frac{dt}{dz}$$

↓
this is a part of 4-vector

→ This is a kind of acceleration.

~~This~~ ⇒ This gives us what standardly called proper acceleration

∴ $\frac{d^2 x^\mu}{dz^2}$ is proper acceleration.

* for slow moving particle it is ~~propto~~ close to ordinary ~~acc~~ acceleration.

So, we have;

$$m \cdot \frac{d^2 x^i}{dz^2} = e \left(\frac{\partial A_i}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^i} \right) \frac{dx^\mu}{dz}$$

→ This is three equation for space components of proper acceleration.

* Would it be nice if we had fourth equation which was just the time component some equation ... but for $m \frac{d^2 x^0}{dz^2}$.

→ If that fourth equation was correct ⇒ Then we could summarize the entire sets of equations by basically one equation.

$$m \cdot \frac{d^2 x^\mu}{dz^2} = e \left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \right) \frac{dx^\nu}{dz}$$

we have to check whether the equation for $\mu=0$ is true?

* The LHS is literally a 4-vector... so as the RHS.

" $\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}$ " is a Tensor with μ and ν index.

→ If you take a Tensor and combine it with a vector in the manner it is in RHS... you get a vector.

→ so; The equation is actually a Tensor equation in the sense because it transforms.

→ Because we took care from the beginning to make sure that the equations of motion were Lorentz Invariant (we did that by ~~making sure~~ making sure that action was scalar)

This is the logic: "Make sure that your Lagrangian respects the symmetries of the problem"

• If the symmetry of the problem is Lorentz symmetry, make sure your Lagrangian is Lorentz Invariant.

→ ~~Once~~ Once you do that, you don't have to worry again whether the theory is invariant u.r.t. Lorentz Transformation.

→ One you know; that the laws of physics are Lorentz Invariant and the first three components of a certain 4-vector are equal to first three " " some other 4-vector

→ Then you know automatically; that the fourth component will also match.

★ The only way systems of equation Lorentz invariant if the first three components are zero let's say,

→ the fourth component also has to be zero.

so; one can automatically conclude ~~that~~ from the fact that we built Lorentz Invariant theory; that the fourth equation must be true; if the other three equations are true.

→ It will form complex of equation which includes all four components: $m \frac{d^2 x^\mu}{d\tau^2} = T^{\mu\nu} \frac{dx^\nu}{d\tau}$

↳ In other words; if you took the equations of first three components & transform them; you will simply pick up the fourth equation.

* The first three equations are of the form $\frac{dp}{dt} = F$
tells how momentum changes with time $\leftarrow m a^\mu$ $\underbrace{\quad}_{T^{\mu\nu} \frac{dx^\nu}{d\tau}}$

* The fourth equation tells how kinetic energy change with time
 $\frac{dK}{dt} = F \cdot v$ } This equation tells that energy changes in a way that is consistent with the amount of work being done. ... and the work being done by the field.
not very surprising;

Remember that Momentum & Kinetic energy form a 4-vector.

→ so; it's not surprising that the 4-component equations are related: if not ~~that~~ to Energy & Momentum conservation; at least to the way in which energy and momentum change in response to the existence of force.

$$m \frac{d^2 x^\mu}{d\tau^2} = T^{\mu\nu} \frac{dx^\nu}{d\tau}$$

1 Rank Tensor

2 Rank Tensor

1 Rank Tensor

we contract one indices
→ and the contraction of indices produces a four vector.

(1 rank tensor)

... matching the four vector on LHS.

Three Fundamental Principles.

pg 92

① LOCALITY : "The way a thing changes, only depends on what's going on nearby to the thing"

- field over at some place does not change due to field having some value at other place located away.
(having some value) (not infinitesimally close)

• The motion of particle only depends on the value of the fields nearby $\Rightarrow$ It also does not depend on the value of the field at later time or an earlier time.

The equations of motions are equations between things that are nearby in small vicinity of time being related to other thing in the same small vicinity of time and space.

★ We implement Locality by making the Action: ~~that~~ that it should be integral over all space & time of some kind of Lagrange density which only depends on values of thing at a particular point neighbourhood x^μ & derivatives $(x_\mu, \dots)$

$\hookrightarrow$ on fields ϕ ; & their derivatives ϕ_μ .

$\hookrightarrow$ in other words it depends on data which has to do with local nearby concepts.

$$\text{Action} = \int d^4x \mathcal{L}(\phi, \phi_\mu)$$

* $\hookrightarrow$ and you add up the Action in point by point in space ... and what comes finally out after minimizing the action is differential equation.

* Differential equations are equations which relate how things change from one point to another to neighbouring things.

★ You could imagine Action has in it things which involves product of fields over here; and products of field far away.

$\hookrightarrow$ but then: When you work out the Euler Lagrange equations $\Rightarrow$ they will have the form that the way things change over here depends on what's going on at some other place.

~~No~~ No; that's not the way how Action principle works.

"The Action depends on fields ; and neighbouring fields" (Pg 93)

↳ neighbouring field now means derivatives $\phi_{,\mu}$;

$$\phi_{,\mu} := \frac{\partial \phi}{\partial x^\mu}$$

"So : the way the field changes at one point , is only determined by the value of the field at that point & neighbouring points"

★ The field change in a way which depends only on whats going on nearby in space-time.

• Action principle together with the idea that action is built up out of incremental pieces : each of which only depends on field & its neighbourhood (local ~~the~~ neighbours) ... ~~is more~~ ... is somewhat equivalent to Locality.

★ When we are talking about particle motion ; then we can substitute the statement :

"The Lagrangian along the world line of particle only depends on position of particles and neighbouring positions along the world line"

→ but it also depends on values of field at the position of particle.

↳ The acceleration of particle depends on value of field in neighbouring region where it is instantaneously ~~the~~ located.

That's the principle of Locality.

"The ~~the~~ response of the fields or particles depends only on whats going on infinitesimally close by"

② Lorentz Invariance.

build the lagrangian out of scalars.

(another way of saying; ~~that~~ that all will agree about the action within a certain volume of space-time.)

$$\mathcal{L} = \text{scalar}$$

③ Gauge Invariance..

The three above principle is extremely pervasive every theory that we know about



- General Relativity.
- Quantum Electrodynamics.
- Standard model of particle physics.
- Yang mill's theory.
- many more.

Lec-7 Special Relativity & Electrodynamics : The fundamental Principles of Physical Laws. (Pg 95)
 - Shoaib Akhtar.

• It seems to sort of be a Law of Nature; that,
 "Nature is expressed through mathematics."

- coordinates you label with upstairs index x^μ
- difference of coordinates " " " " dx^μ
- derivatives of scalars w.r.t. x ; $\frac{\partial S}{\partial x^\mu}$ we think of it as downstairs index.
- Things which transform under Lorentz Transformations as coordinates carry upstairs indices.
- Things which transform as derivatives; carry downstairs indices.

example; $\frac{\partial S}{\partial x^\mu} \cdot dx^\mu = \sum_{\mu=0}^3 \frac{\partial S}{\partial x^\mu} \cdot dx^\mu$

↳ The whole thing is change in "S" from going on point to neighbouring point i.e; $dS = \frac{\partial S}{\partial x^\mu} \cdot dx^\mu$

Einstein Summation convention: whenever you see a lower index next to an upper index with the same letter; it means sum.

↳ There is deeper geometric meaning to it ... we will see in General Relativity.

★ Every vector A ; has both contravariant like A^μ .
 & covariant like A_μ .

• If you follow the rules of upstairs and downstairs indices rigorously, you will find that it is very easy to write things which are invariant under Lorentz Transformations. ex $x^2 + y^2 + z^2 - T^2 = x^\mu x_\mu$.
 ↳ Invariant

★ Anytime you take partial derivatives $\Rightarrow$ it carries lower index.
 (most of the time)

(Invariant) $\frac{\partial}{\partial x^\mu} x^\mu = 1$

Covariant; $\partial_\mu := \frac{\partial}{\partial x^\mu} \rightarrow \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

Contravariant; $\partial^\mu \rightarrow \left(-\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

★ The usual rule to make a ~~new~~ theory (someday, the rule might break down; but so far they haven't)

↳ The usual rule for whatever reason they are

"The phenomenon should be described by an Action principle"

⇒ There is nothing that we know in nature that is not describable by Action Principle.

Principles

① Action: We look for an action such that equation of motion which follow from that Action describes the phenomenon that is discovered in the laboratory.
⇒ We look for an Action description; this guarantees that it will form the category of things with energy conservation and so forth.

② Locality: If you police a system at some point of time and space it has no immediate effect except on things in immediate vicinity of it.
↳ and this is guaranteed by requiring a principle of Locality in action; namely: that Action is an integral ...

Particle

$$Action = \int dt \mathcal{L}(x, \dot{x})$$

has units of energy



field

$$Action = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

has units of energy density.

... the principle of Locality states: that the thing we integrate (ie; Lagrangian) depends only on coordinates of particle and only the time derivatives. (and for field theory; it says that $\mathcal{L}$ depends on field ϕ ; and the various derivatives $\partial_\mu \phi$)

Notation: $\phi_{,\mu} := \frac{\partial \phi}{\partial x^\mu}$

★ The principle of Locality: comes down to statement in terms of equations of motion, that they are differential equations.

Entanglement: ... It is basically a quantum version of me having two coins Nickle & a Dyme ... closing my eyes ... shuffling them up ... and giving Alice one of them (I don't tell which) ... I give Bob the other one ... and then send them away to their homes ... There is no way that Alice has any direct effect on Bob by looking at her coin ... but instantly she knows what Bob has ~~by~~ is she looks at hers. ... "It is correlation; ~~it is not~~ it is not effect"

③ You require physics to be Lorentz Invariant.

(i.e. equations of motion ... the rules ... should be same in every reference frame)

→ This is easy to do ... Lagrangian is same in every reference frame ... means it is scalar. $\mathcal{L} = \text{scalar}$.

Lorentz Invariance $\Rightarrow$ (Incidentally this includes rotational invariance & so forth)
 $IR(+\theta) L(\varphi) IR(-\theta) \dots$

④ Gauge Invariance

Motion of particle in Electromagnetic field.

~~reintroduced~~ • He introduced 4-vector ... vector potential A_μ .
 (We will see where it comes from; but it is necessary in order to have an Action principle which leads to equations of motion of charged particle in Electromagnetic field)

$$W_{\text{Action}} = \int dt L_{\text{em}}$$

(its a particle ... not field theory ... field already established & is known ...
 ... so action is just integral over dt)

$$* \text{ ACTION} = \int [-m dz - e \cdot A_\mu dx^\mu]$$

for every little segment ... along the segment here Ξ
 $\hookrightarrow$ at the position of segment there is Vector potential.

Vector potential is field ... depends on space & time; $A_\mu(x) \equiv A_\mu(t, x, y, z)$
 (x stands for (t, x, y, z))

$\hookrightarrow$ Note; Action is always of the form of
 integral over time dt .. (not dz)

$$\begin{aligned} \text{so: ACTION} &= \int -m \cdot \frac{dz}{dt} \cdot dt - e \cdot A_\mu(x) \cdot \frac{dx^\mu}{dt} \cdot dt \\ &= \int -m \sqrt{1 - \dot{x}^2} \cdot dt - e(A_0 + A_m \cdot \dot{x}^m) dt \end{aligned}$$

so; the Lagrangian for particle is

$$\mathcal{L} = -m \sqrt{1 - \dot{x}^2} - e(A_0(x) + A_m(x) \cdot \dot{x}^m)$$

$$\text{or } \mathcal{L} = -m \sqrt{1 - \dot{x}^2} - e(A_\mu(x) \cdot \dot{x}^\mu) \quad ; \quad \dot{x}^\mu = \frac{dx^\mu}{dx^0}$$

now; $\frac{\partial \mathcal{L}}{\partial \dot{x}^l} = \frac{m \dot{x}^l}{\sqrt{1 - \dot{x}^2}} - e A_l(x)$

$$= m \cdot \frac{dx^l}{dz} - e A_l(x) \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}^l} = m u^l - e A_l$$

now; $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^l} \right) = \frac{\partial \mathcal{L}}{\partial x^l}$ Euler Lagrange Equation

$$\Rightarrow \left[m \cdot \frac{d}{dt} u^l - e \frac{d}{dt} A_l(x) \right] = \frac{\partial \mathcal{L}}{\partial x^l}$$

$$\Rightarrow \left[m \cdot \frac{d}{dt} u^l - e \left(\frac{\partial A_l}{\partial x^0} + \frac{\partial A_l}{\partial x^m} \dot{x}^m \right) \right] = -e \left(\frac{\partial A_0}{\partial x^l} + \frac{\partial A_m}{\partial x^l} \dot{x}^m \right)$$

$$\textcircled{bv} \quad m \cdot \frac{d}{dt} u^l - e \left(\frac{\partial A_l}{\partial x^\mu} \cdot \dot{x}^\mu \right) = -e \left(\frac{\partial A_\mu}{\partial x^l} \cdot \dot{x}^\mu \right)$$

$$\Rightarrow m \cdot \frac{d}{dt} u^l = e \left\{ \left(\frac{\partial A_l}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^l} \right) \cdot \dot{x}^\mu \right\}$$

we let $E_l := \left(\frac{\partial A_l}{\partial x^0} - \frac{\partial A_0}{\partial x^l} \right)$ This called Electric field.

• This is the notation of an Electric field as described in terms of 4 vector: Vector Potential.

$$\cancel{m \cdot \frac{d}{dt}} m \cdot \frac{d}{dt} u^l = e \left(\frac{\partial A_l}{\partial x^0} - \frac{\partial A_0}{\partial x^l} \right) + e \left(\frac{\partial A_l}{\partial x^m} - \frac{\partial A_m}{\partial x^l} \right) \cdot \dot{x}^m$$

$\left(\frac{\partial A_l}{\partial x^m} - \frac{\partial A_m}{\partial x^l} \right) \cdot \dot{x}^m$ is identified as $(\vec{v} \times \vec{B})_l$

1th component of $\vec{v} \times \vec{B}$.

$\vec{v}$ is velocity
 $\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$

so; $m \cdot \frac{d}{dt} u^l = e (E_l + (\vec{v} \times \vec{B})_l)$

$$\Rightarrow m \cdot \frac{du^l}{dz} \cdot \frac{dz}{dt} = \left(\quad \right) + \left(\quad \right) \frac{dx^n}{dt}$$

$$\Rightarrow m \cdot \frac{d}{dt} u^l = \left(\quad \right) \frac{dz}{dt} + \left(\quad \right) \frac{dx^n}{dt} \cdot \frac{dz}{dt}$$

$$\Rightarrow m \frac{d^2 x^l}{d\tau^2} = e \left(\frac{\partial A_l}{\partial x^0} - \frac{\partial A_0}{\partial x^l} \right) \frac{dt}{d\tau} + e \left(\frac{\partial A_l}{\partial x^m} - \frac{\partial A_m}{\partial x^l} \right) u^m$$

$$\Rightarrow m \frac{d}{d\tau} u^l = e \left(\frac{\partial A_l}{\partial x^0} - \frac{\partial A_0}{\partial x^l} \right) \cdot u^0 + e \left(\frac{\partial A_l}{\partial x^m} - \frac{\partial A_m}{\partial x^l} \right) u^m$$

↳ this is a form of acceleration... but acceleration relative to τ .
 ... it's called Proper acceleration
 & it's a 4-vector. $a^\mu = \frac{d^2 x^\mu}{d\tau^2}$

$$\Rightarrow m \cdot \frac{d^2 x^l}{d\tau^2} = e \left(\frac{\partial A_l}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^l} \right) u^\mu$$

now let; $F_{lm} := \frac{\partial A_l}{\partial x^m} - \frac{\partial A_m}{\partial x^l}$

so; $m \cdot \frac{d^2 x^l}{d\tau^2} = e \cdot F_{lm} \cdot u^m$

; $F_{lm} \stackrel{\text{equivalent}}{=} F^l_m$

↳ restoring it as a 4-vector equation. ; although while writing or defining F_{lm}
 ↳ we have not been rigorous with symbols... but let now be;

$$m \cdot \frac{d^2 x^\mu}{d\tau^2} = e F^\mu_\nu \cdot u^\nu$$

where; $F^\mu_\nu := \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}$

↳ This is now properly constructed 4-vector equation.

note
 * If three components of a 4-vector equation are correct... it is automatically; that the fourth component equation is also correct... fourth component is also correct.

↳ That's the consequence of Lorentz Invariance.

* for space index; $m \cdot \frac{d^2 x^m}{d\tau^2} = e F^m_\nu \cdot u^\nu$

$$a^\mu := \frac{d^2 x^\mu}{d\tau^2}$$

ν can be any four indices... but it can't be m ... actually we mean to say F^m_m is 0 (because F is antisymmetric)

$$\Rightarrow m \cdot a^m = e (F^m_0 \cdot u^0 + F^m_l \cdot u^l)$$

4-vector equation:

$$m a^\mu = F^\mu{}_\nu u^\nu$$

$F^\mu{}_\nu$ is rank 2 tensor.

we will components of F in terms of electric and magnetic field.
(you can feed & check that it is actually Lorentz Force Law)

..... This is Maxwell tensor.

$$F^\mu{}_\nu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ +E_x & 0 & B_z & -B_y \\ +E_y & -B_z & 0 & B_x \\ +E_z & +B_y & -B_x & 0 \end{pmatrix}$$

These are space-time mixed components $\rightarrow$ They are components of Electric field. (everything in upper row, has one index which is time but other things are space components)

$$= F_{\mu\nu}$$

$\rightarrow$ These are space-space components $\rightarrow$ & are components of Magnetic field.

$\rightarrow$ This is Electromagnetic field tensor. ... and helps you to identify which components of the complex $F^\mu{}_\nu$ are equal to which component of electric and magnetic field.

$\downarrow$
when you raise index (for purpose of calculating something, ... which we will do)
 $\rightarrow$ if its time index you get minus sign.
 $\rightarrow$ if its space index, its not.

Gauge Invariance

* lets go back to the action for the particle along the trajectory.. Electromagnetic part of it
 $\rightarrow$ & discover the concept of Gauge Invariance.

Invariance : An invariance is a change in a system that does not affect anything... it does not affect the Action.

$\rightarrow$ strictly speaking ; its the change in system that does not affect the equations of motion.

example $F = ma$ is an equation of motion $\Rightarrow$ It has exactly the same form if you translate the centre of x axis from one point to another.
; that's Translational invariance ~~about origin~~ ~~relative to~~ ~~coordinate~~...

★ Let's take a field Lagrangian ... a field theory.

• simplest field theory we wrote down ;

↳ Theory of scalar field.

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0$$

This has Lagrangian given by : $\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi$

• It has Lorentz Invariance
but also has an additional invariance.

which is; supposing take field ϕ and add a constant to it.
 $\phi + C \Rightarrow$ This also satisfies equations of motion if ϕ does.

↳ Also action is unchanged by adding a constant.

• so; If the Action is such (or if you have a particular action) and you have field configuration which minimizes the action.

↳ ~~Then~~ Then; if you add a constant to the field; it would still minimize the Action.

↳ So; adding a constant to ~~field~~ a field is symmetry.

★ another example ; $\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \mu^2 \phi = 0$

where; $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\mu^2}{2} \phi^2$

now; what happens if we add constant to solution? ... is it solution anymore.

↳ Answer : It is not. (can be checked easily)

so; with an extra term; adding constant is not symmetry.

example $\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0$; $\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi$

what happens when you do the transformation $\phi \rightarrow \phi + \eta$

Is it ~~symmetry~~?

- The transformation does satisfy
- $\phi + n$ does satisfy equation of motion.

• Now; let's look at Lagrangian -

$$L = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\partial_x \phi)^2$$

$$= \left[\partial_t \phi + \cancel{\partial_t x}^0 \right]^2 - \left[\partial_x \phi + \cancel{\partial_x n}^1 \right]^2$$

$$= (\partial_t \phi)^2 - (\partial_x \phi + 1)^2$$

$$= \dot{\phi}^2 - (\partial_x \phi + 1)^2$$

$$= \dot{\phi}^2 - (\partial_x \phi)^2 - 1 - 2(\partial_x \phi)$$

$$\partial_t x = \frac{dx}{dt} = 0$$

because n & t are independent variables.

adding a constant to Lagrangian is completely insignificant.

adding a total derivative to Lagrangian does not change anything in equation of motion.

... we can prove.

→ we can prove; why adding a total derivative or derivative to Lagrangian does not change anything.

let go back to the integral $-e \int A_\mu dx^\mu$

now; consider the following operation to A

$$\underbrace{A_\mu}_{\text{covariant vector}} \rightarrow \underbrace{A_\mu}_{\text{covariant vector}} + \underbrace{\frac{\partial S}{\partial x^\mu}}_{\text{covariant vector or covariant vector}}$$



... does this change equations of motion?

" it make any change in the dynamics of the particle?

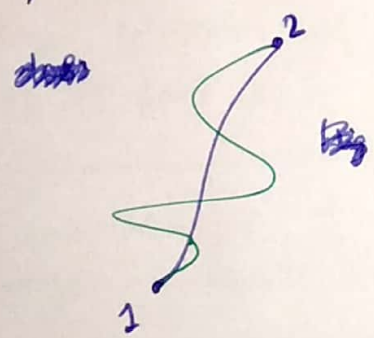
so; $-e \int_1^2 A_\mu dx^\mu \rightarrow -e \int_1^2 A_\mu dx^\mu + \frac{\partial S}{\partial x^\mu} \cdot dx^\mu$

$ds = \frac{\partial S}{\partial x^\mu} \cdot dx^\mu$

↳ represent incremental change of S along a small little piece of trajectory.

$$\begin{aligned} &= -e \int_1^2 A_\mu dx^\mu - e \int_1^2 \frac{\partial S}{\partial x^\mu} \cdot dx^\mu \\ &= -e \int_1^2 A_\mu dx^\mu - e [S(2) - S(1)] \end{aligned}$$

so; $\int_1^2 A_\mu dx^\mu \rightarrow \left(\int_1^2 A_\mu \cdot dx^\mu \right) + (S(2) - S(1))$



↳ This is actually an sort of constant if S is fixed
↳ & it depends on end points.
↳ so; S appears in action as an additive term which depends on end points.

hold the point 1 & 2 fixed;
and wiggle the trajectory in between.

↳ in wiggling the trajectory around holding end points fixed;
the term " $S(2) - S(1)$ " in the action will not change.

↳ so; if you find a minimum of the action, it will continue to be minimum of action when you put in this change of vector potential ($A_\mu \rightarrow A_\mu + \frac{\partial S}{\partial x^\mu}$) ... minimum w.r.t. trajectory.

↳ This has no effect on motion of particle
because it only affects action through end points.
and the rule is: vary the trajectory keeping the end points fixed.

★ So we have discovered a principle of invariance that motion of particle in an electromagnetic field does not change if $A_\mu \longrightarrow A_\mu + \frac{\partial S}{\partial x^\mu}$

This is called **Gauge Invariance**.

i.e. the vector potential itself can be changed in a certain way without any effect on the behavior of charged particle.

Any change that we make ⁱⁿ ~~the~~ vector potential; that does not change F^μ_ν will not affect the motion of particle.

Electromagnetic field tensor is Gauge Invariant
(it does not change when you add $\frac{\partial S}{\partial x^\mu}$ to the vector potential)

$$A_\mu \longrightarrow A_\mu + \frac{\partial S}{\partial x^\mu}$$

Direct Proof:

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$$

now: $A_\mu \longrightarrow A_\mu + \frac{\partial S}{\partial x^\mu}$

$$\begin{aligned} \Rightarrow (F_{\mu\nu})_{\text{new}} &= (F_{\mu\nu}) + \frac{\partial}{\partial x^\nu} \left(\frac{\partial S}{\partial x^\mu} \right) - \frac{\partial}{\partial x^\mu} \left(\frac{\partial S}{\partial x^\nu} \right) \\ &= (F_{\mu\nu}) + \frac{\partial^2}{\partial x^\nu \partial x^\mu} S - \frac{\partial^2}{\partial x^\mu \partial x^\nu} S \end{aligned}$$

but these cancel; because partial derivatives commute.

$$\Rightarrow (F_{\mu\nu})_{\text{new}} = (F_{\mu\nu})$$

so: $F_{\mu\nu}$ is unchanged.

(P2 105)

⑥ Why did we bother using Vector Potential, particularly if the vector potential is not unique (if you can change it and it does not affect the answers) ?

Ans There is no way to ~~ad~~ write action principle for motion of particle ~~that~~ that does not involve vector potential.

- Its only possible to write by auxiliary device of vector potential.
- Value of Vector potential is not a physically meaningful thing. (because if you change it by gradient ... the physics does not change)

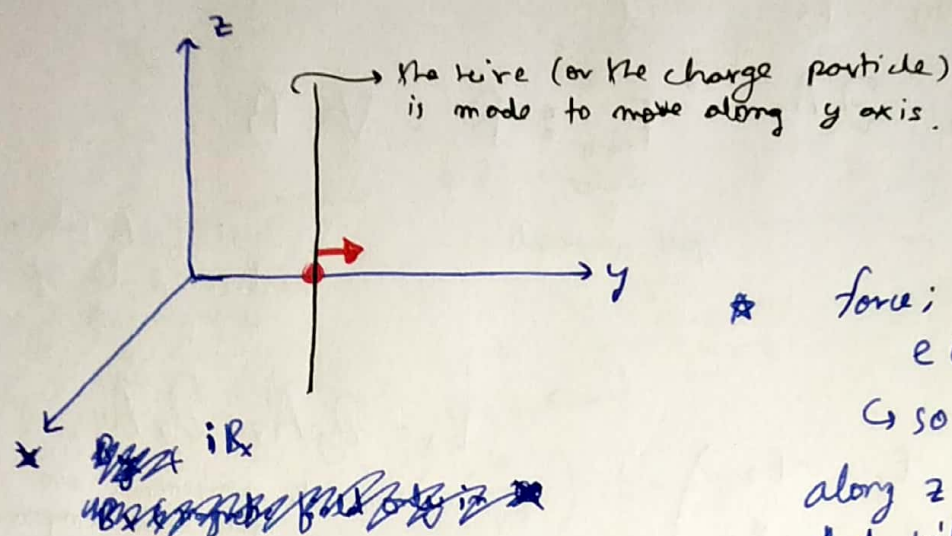
* Gauge Invariance is a feature of every known fundamental theory of ~~the~~ physics.

-
- You cannot express action of charged particle in a Lorentz Invariant way in terms of the Electric, Magnetic field, position & velocities of the particle ... Its a Theorem.

↳ and then you find out that this invariance that you can change the vector potential in some way without changing the physics.

* We ~~don't~~ know why nature always chooses these ~~G~~ Gauge Theories. ... its very persuasive.

Einstein first paper on Relativity: On electrodynamics of moving body.



★ force; ~~is~~
 $e(\vec{v} \times \vec{B})$

→ so; there is force along z axis... and that will cause a current to flow along the wire.

★ now; move the magnet; and keep the wire still.

→ here; a moving magnetic field... will create EMF... which effectively means an Electric field vertically which has the same effect
→ causes the charges to flow or exerts a force on stationary particle.
... it would have same effect.

⇒ Einstein said, look, the phenomenon is same; a vertical force... (we have two totally different description...)

- in one description; magnetic field exerts force on moving charge.
- in the other; the moving magnetic creates an Electric field; and the Electric field exerts a force on charge particle)

..... These are the same effect

⇒ & ~~the~~ the only explanation; is that what looks like a magnetic field (a pure magnetic field in one frame); in another frame of reference (here moving with particle); looks like a bit like electric field

→ so; he concluded; Electric & Magnetic fields are part of same complex of objects which transform into each other... (under Lorentz transformation) or motion.

Electromagnetic field Tensor.

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & +B_y & -B_x & 0 \end{pmatrix}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= \frac{\partial}{\partial x^\mu} := \partial_\mu$$

$$F_{0m} = E_m = \dot{A}_m - \partial_m A_0 \quad ; \quad F_{mn} = B = \nabla \times A$$

mixed space time components.

space components

curl of A when restricted to space-components.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & +B_x \\ E_z & +B_y & -B_x & 0 \end{pmatrix}$$

$$B_x = \partial_y A_z - \partial_z A_y$$

magnetic components are space-space components.

★ Now; lets see how; $F_{\mu\nu}$ transforms when we move from one frame to another.

Given an Electromagnetic field tensor in one frame ... how it transforms in other frame moving relative to the first frame.
What are new components of Electromagnetic field tensor, in new frame.

$$\# (X^\mu)' = L^\mu_\nu X^\nu \quad \text{or} \quad \begin{pmatrix} x^0' \\ x^1' \\ x^2' \\ x^3' \end{pmatrix} = \begin{pmatrix} L \\ \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad ; \quad \begin{matrix} t' = \frac{t - vx}{\sqrt{1-v^2}} \\ y' = y \\ z' = z \end{matrix} \quad ; \quad x' = \frac{x - vt}{\sqrt{1-v^2}}$$

Matrix L associated with Lorentz Transformation:

$$L = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} & 0 & 0 \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

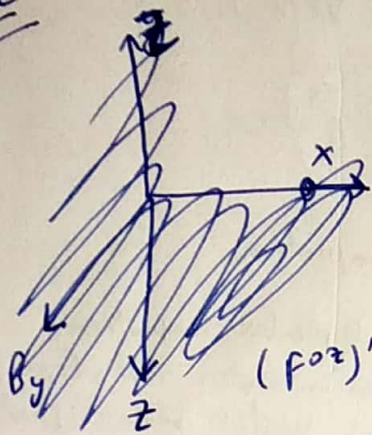
AKA stands for Also known as. abbreviation

→ This is matrix which represents Lorentz Transformation by velocity v along x axis.

$$(F^{\mu\nu})' = L^\mu_\sigma L^\nu_\tau F^{\sigma\tau}$$

Transformation of μ index
Transformation of ν index

example



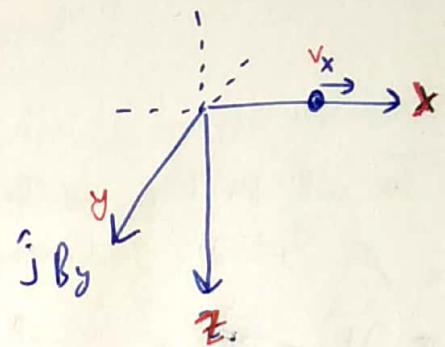
$$(E^z)' = (F^{0z})'$$

and we know; $F^{xz} = B_y$ → Only this component survives.

∴ & others are zero

$$(F^{0z})' = L^0_x L^z_z \cdot F^{xz}$$

$$= \left(\frac{-v}{\sqrt{1-v^2}} \right) \cdot 1 \cdot B_y$$



if $v \ll c$; $\Rightarrow (F^{0z})' = (E^z)' = -v B_y$

∴ $(F^{0z})' = -v B_y \cdot e$ along z axis

∴ but initial force was also $e(\vec{v} \times \vec{B}) = -v B_y e \hat{z}$...

→ so; The phenomenon is sort of same effect.

note; $\vec{A}$ stands for only three vector

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} S) = 0$$

Identity.

∴ now $\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

→ says that there are no magnetic charges.

also $\vec{E} = \left(\frac{\partial \vec{A}}{\partial t} \right) - \nabla A_0$

note

from three dimensional point of view; i.e. just space

→ The time component of A (4-potential) can be thought of as scalar.

i.e. $A_\mu \rightarrow (A_0, A_x, A_y, A_z)$: $\vec{A} \equiv (A_x, A_y, A_z)$ & A_0

$$(\nabla \times \vec{A})_x = \partial_y A_z - \partial_z A_y$$

$$\dots$$

$$(\vec{\nabla} \cdot \vec{A}) = \partial_x A_x + \partial_y A_y + \partial_z A_z$$

$$(\nabla S)_x = \partial_x S$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} A_0$$

$$\therefore \text{now; } \vec{\nabla} \times \vec{E} = \vec{\nabla} \times \left(\frac{\partial \vec{A}}{\partial t} \right) - \vec{\nabla} \times (\vec{\nabla} A_0)$$

$$= \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{A}) - 0$$

so; $\vec{\nabla} \cdot \vec{B} = 0$ $\Rightarrow$ Scalar Equation

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\Rightarrow$ Vector Equation.

$$\Rightarrow \vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$$

$\vec{\nabla} \cdot \vec{B} = 0$ is one equation.

$\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$ is four independent equations.
(but if we think of them in vector notation;
this is just one vector equation)

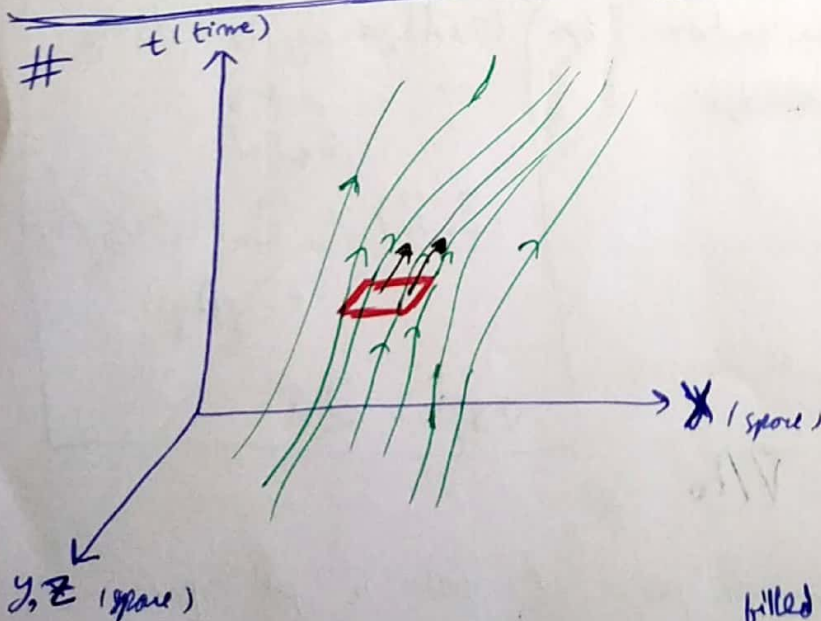
Now, we will write other two equations (they are dynamical equations; we will derive them from Action principle)

$\vec{\nabla} \cdot \vec{E} = \rho$ $\Rightarrow$ 3-dimensional scalar.

$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} = \vec{J}$ $\Rightarrow$ $\vec{J}$ is a 3-vector

• In 4-vector language we will discover that the three components of $\vec{J}$ and ρ together form a 4-vector.

ρ = charge density; $\vec{J}$ = Electric current density.



□ we take a little cell in space.

(we draw it as square ... but of course; it is a cube (it has 3 dimension))
 $\hookrightarrow$ & therefore it is small volume elements.

• We imagine little charge particles. They having world line.

$\hookrightarrow$ let's imagine that space & time are filled with a kind of a bundle of charge particles moving ... a fluid of them.

* now: think of the world line passing through the little cube. (PS III)

means:  at an instant; the particles which pass through cube are simply in the cube at that instant of time..

→ We take all electrons in there;... we count up all their charges

→ & we think of it as total charge passing through that little volume of space (passing through in the sense... that they were in the

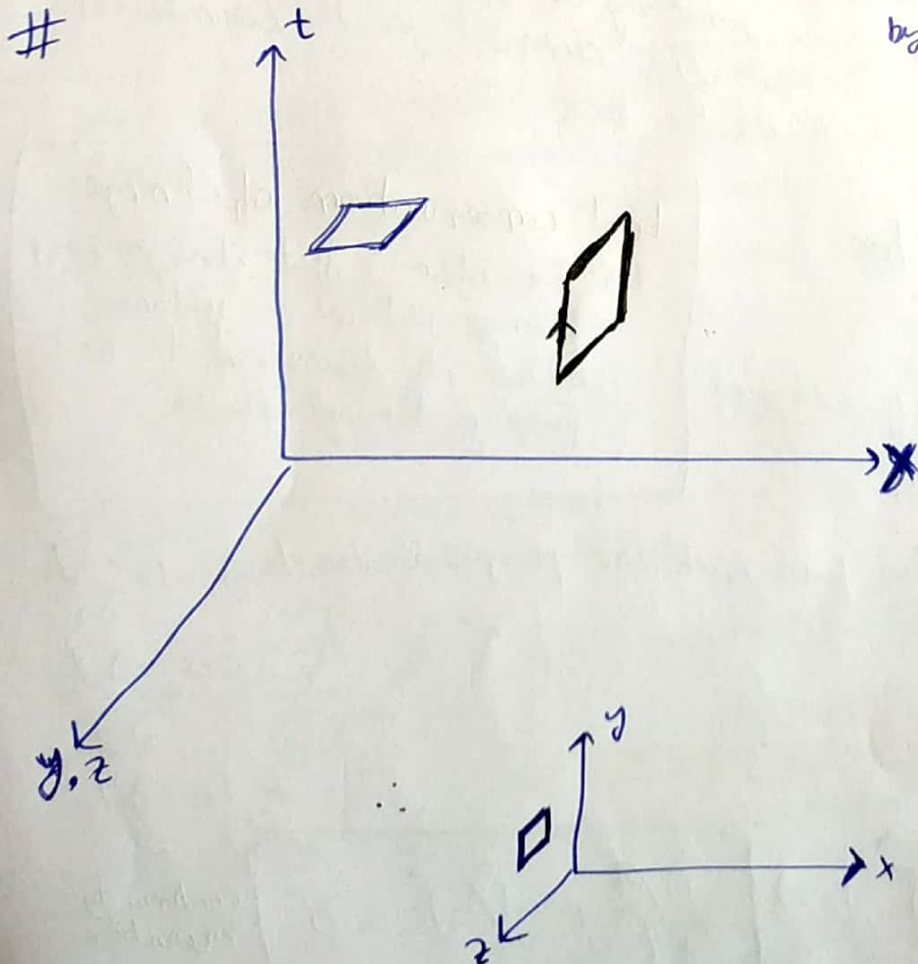
Past ; ... they were in the future  ..;

but at the instant we drew that little volume element

→ They were in the volume element)

- The total charge in that volume element divide by the volume element is the charge per unit volume (ρ).

$$\rho = \lim_{\Delta V \rightarrow 0} \frac{\Delta Q}{\Delta V}$$



- The current density can be understood by simply turning  on its side: ie - 

- Think of little cell in space time which is perpendicular to the x axis

(means; it is a little cube in space-time.

→ It has the edge along time axis t and it has two edges along the y & z axis.

 stands for two dimensional square... y & z square.

The amount of charge passing through little cell... per unit area ... per unit time. ~~Maths~~

→ This is called $\vec{J}$... in this case we get J_x
 ↳ The x^{th} component of $\vec{J}$.

$$J_x = \frac{\Delta Q}{\Delta A \Delta t}$$

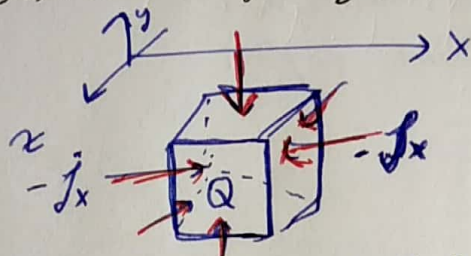
Area perpendicular to x axis (y-z area)

we will see;

$$J^\mu = (\rho, J_x, J_y, J_z)$$

↳ can be thought of as component of 4-vector ~~with~~ with upper index.

Conservation of Charge.



Charge in a region change if it passes through walls of that region.

↳ so: changes of charge in the interior of a box must be accompanied by something or rather having to do with the current on the boundaries of the box

$$(\nabla \cdot \rho) = \text{Change in the box.}$$

so: take; $\Delta V = 1$

∴ ~~the change~~ $\dot{\rho}$ = Rate of change of charge.

Local conservation of charge.

↳ The idea; that charge can't change within a volume unless it's discovered to be passing through walls.

so: contribution from two faces with are perpendicular to x is $-\frac{\partial \rho}{\partial x}$
 " " " " " " " y is $-\frac{\partial \rho}{\partial y}$
 " " " " " " " z is $-\frac{\partial \rho}{\partial z}$

$$\text{so: } \frac{\partial \rho}{\partial t} = -(\vec{\nabla} \cdot \vec{J}) \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

Continuity Equation.

or you like to call $\rho + \vec{\nabla} \cdot \vec{J} = 0$ as "Local Conservation" PS 113

↳ In other words; a conservation but a strong form of it.

↳ "which does not allow charge to disappear at one place & ~~re~~ reappear at another place; unless in between there is ~~some~~ a flow of charge (a current)"

Note Continuity equation can be thought off as consequence of maxwell equations:

$$\therefore \vec{\nabla} \cdot \vec{E} = \rho \Rightarrow \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}) = \frac{\partial \rho}{\partial t}$$

$$\& \vec{\nabla} \times \vec{B} + \frac{\partial \vec{E}}{\partial t} = \vec{J} \Rightarrow \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) + \vec{\nabla} \cdot \left(\frac{\partial \vec{E}}{\partial t} \right) = \vec{\nabla} \cdot \vec{J}$$

$$\Rightarrow 0 + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}) = \vec{\nabla} \cdot \vec{J}$$

So; $\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$

... so; in other words.. maxwells equations requires charges to be conserved.

Can we express Maxwells equations... all of these equations... in a relativistic notation which makes it clear that the collection of equation is invariant under Lorentz Transformation.
(Can we express them as Tensor Equations)

$$\star \frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0$$

↳ now; it becomes natural to think of it in notations which is manifestly Lorentz Invariant.

$$\frac{\partial \rho}{\partial t} + \frac{\partial J^m}{\partial x^m} = 0$$

lets; ~~define~~ ~~define~~ :

$$\vec{j}^m \Rightarrow (j_x, j_y, j_z) \Rightarrow (j^x, j^y, j^z)$$

lets; define ρ to be time component of j^m

ie; $j^0 = \rho$

Symbol

$$j^m \rightarrow (j_x, j_y, j_z) \equiv (j^x, j^y, j^z)$$

we can put index up
(because it is spatial component
... does not matter in this case)

put p & j^m together and call whole complex of four objects
 j^μ ... a 4-vector.

(They are $(p \text{ & } j^m)$ really the same thing
... remember how you defined them by orienting $\square$ in different direction in space time).

$$j^\mu \rightarrow (p, j^x, j^y, j^z)$$

Now in this notation:

so; Continuity equation has form of a 4-vector relativistically invariant equation;

$$\frac{\partial j^\mu}{\partial x^\mu} = 0$$

(or)

$$\partial_\mu j^\mu = 0$$

4-Vector
Continuity equation
(Lorentz Invariant form)

use summation convention.

so; its natural therefore to think of p together with j as the components of 4-vector ... in fact it can be proved that they really do transform as components of 4-vector.

Till now we know

- $\vec{E}$ & $\vec{B}$ form an anti-symmetric Tensor with the indices.
- p & j form a 4-vector (in other words Tensor with one index)

Let's begin with equation which were trivial ... some kind of definition.

$$\vec{B} \cdot \vec{B} = 0 \text{ and } \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

→ They are some kind of derivative acting on a component of F & the result is equal to 0.

$$\partial F = 0$$

• There are four such equations.

$$\partial_\sigma F_{\nu z} + \partial_\nu F_{z\sigma} + \partial_z F_{\sigma\nu} = 0$$

check
 ☆ let: $\begin{matrix} \sigma=x \\ \nu=y \\ z=z \end{matrix} : \partial_x B_x + \partial_y B_y + \partial_z B_z = 0$
 $\Rightarrow$ simply $\vec{\nabla} \cdot \vec{B} = 0 \dots$

we would get the same thing if we took: $\begin{matrix} \sigma=y \\ \nu=x \\ z=z \end{matrix}$

☆ let: $\begin{matrix} \sigma=y \\ \nu=x \\ z=t \end{matrix} : \left[\partial_y E_x + \partial_x (-E_y) \right] + \partial_t (B_z) = 0$

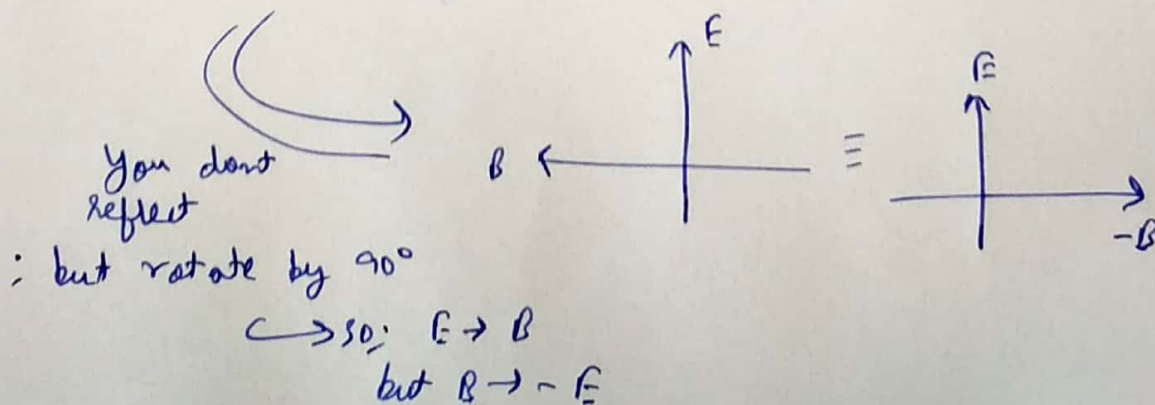
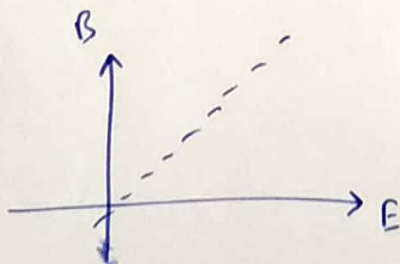
Relativistically Invariant Tensor notation
 (this is Tensor with three indices ... & the equation holds)

~~you could remember~~

$$F_{\nu z} = \frac{\partial A_\nu}{\partial x^z} - \frac{\partial A_z}{\partial A_\nu}$$

→ This complex eq equation has a name ... called Bianchi Identity.
 (its a special case of an identity called Bianchi Identity).
 • There is a similar identity in General Relativity.

Duality



Lec-9 Special Relativity & Electrodynamics : Lagrangian for Maxwell's Equation.
—Shoaib Akhtar.

(Pg 117)

$\vec{\nabla} \cdot \vec{B} = 0$

$-\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \vec{E}$

These are actually
Bianchi identities.

(These equations are
actually identities
... based on definitions
of $\vec{B}$ & $\vec{E}$)

* now; let's take the theory in the absence of
charge & current in empty space.

→ so; the other two equations become.

$\vec{\nabla} \cdot \vec{E} = 0$

$\frac{\partial \vec{E}}{\partial t} = +\vec{\nabla} \times \vec{B}$

These equations can be written in covariant form .. which is called Bianchi Identities.

$\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0$

These equation has form

$\partial_\mu F^{\mu\nu} = 0$

(them don't have
any charges)

equation in empty
space
(no charges & no
currents)

→ now we want to solve them ... we want to write down ~~generic~~
generic form of plane wave solution.

→ simply means a wave proceeding along some axis.

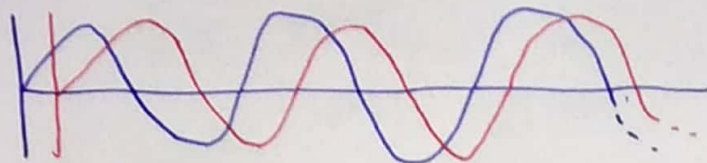
... we can always rotate the coordinate so that z-axis is along direction
of propagation of wave.

⇒ It has wavelength; $\lambda = \frac{2\pi}{k}$; k is called wavenumber.

• a typical ^{generic} plane wave of that form .. generally has functional form
 $\sin(kz - \omega t)$

$\therefore E_x(z, t) = E_x \cdot \sin(kz - \omega t)$
 $E_y(z, t) = E_y \cdot \sin(kz - \omega t)$
 $E_z(z, t) = E_z \cdot \sin(kz - \omega t)$

$B_x = \beta_x \cdot \sin(kz - \omega t)$
 $B_y = \beta_y \cdot \sin(kz - \omega t)$
 $B_z = \beta_z \cdot \sin(kz - \omega t)$



Phase Shift

$$\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \Rightarrow \boxed{E_z = 0}$$

so; E_z for wave moving along "z" axis must be 0
i.e. the "z" component of the field must be zero.

also
 $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow B_z = 0$

so; a plane electromagnetic wave; moving along an axis
↳ does not have any Electric or Magnetic field along that direction.

note; you can always rotate your x-y plane so that one of the E_x or E_y becomes zero.

↳ so; we had (without loss of generality) ... using our rotational freedom; we have now.

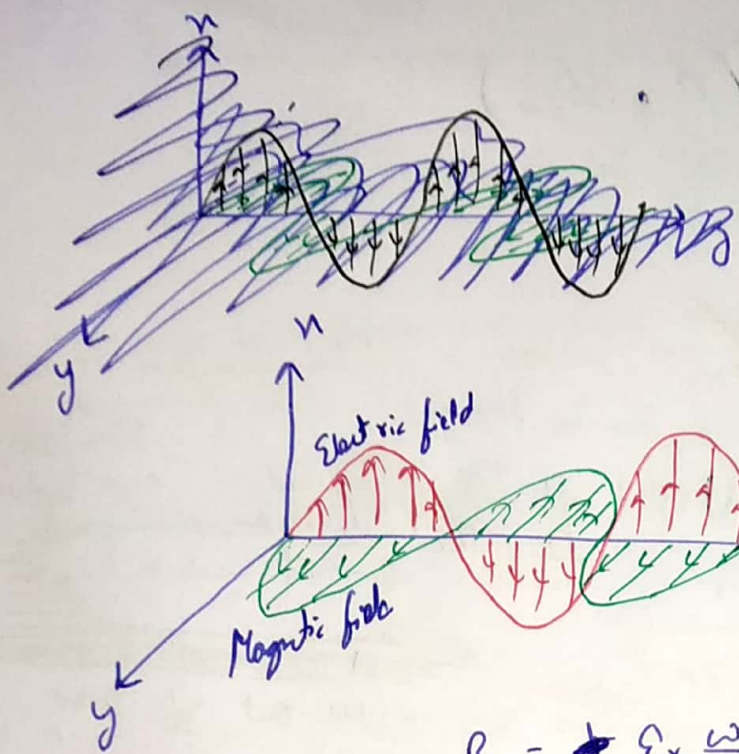
$$\begin{aligned} E_x &= E_x \cdot \sin(kz - \omega t) & B_x &= B_x \sin(kz - \omega t) \\ E_y &= 0 & B_y &= B_y \sin(kz - \omega t) \\ E_z &= 0 & B_z &= 0 \end{aligned}$$

let; now use; $\frac{\partial \vec{E}}{\partial t} - \vec{\nabla} \times \vec{B} = 0 \Rightarrow \left(\frac{\partial E_x}{\partial t} \right) = + (\vec{\nabla} \times \vec{B})_x = + (\partial_y B_z - \partial_z B_y)$
 $\Rightarrow \frac{\partial E_x}{\partial t} = \partial_z B_y$ i.e. $\partial_t E_x = -\partial_z B_y$

$\Rightarrow E_x \cdot (-\omega) \cdot \cos(kz - \omega t) = -B_y \cdot k \cos(kz - \omega t)$
 $\Rightarrow \boxed{B_y = + \frac{\omega}{k} \cdot E_x}$
($B_y \neq 0$)
(so: Electric and magnetic field cannot be parallel to each other)

now; $\partial_t E_y = -(\vec{\nabla} \times \vec{B})_y \Rightarrow 0 = + (\partial_x B_z - \partial_z B_x) \Rightarrow \partial_z B_x = 0 \Rightarrow B_x = 0$

so;
$$\begin{aligned} E_x &= E_x \cdot \sin(kz - \omega t) & B_x &= 0 \\ E_y &= 0 & B_y &= -\frac{\omega}{k} E_x \sin(kz - \omega t) \\ E_z &= 0 & B_z &= 0 \end{aligned}$$



→ direction of propagation.

$$B_y = \epsilon_0 \frac{\omega}{k} E_x$$

also we get. ~~also we get~~ $B_x = \epsilon_0 \frac{\omega}{k} E_y$ (from: $\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0$)

so; $\frac{\omega}{k} = 1 \Rightarrow \underline{\underline{\omega = k}}$ in the units where: $\underline{\underline{c=1}}$

so, $E_x = \epsilon_0 \sin(k(z - ct))$; $B_y = \frac{\epsilon_0 \omega}{k} \sin(k(z - \omega t))$

$\nabla \cdot \vec{E} = \rho = j^0$
 $-\frac{\partial \vec{E}}{\partial t} + \nabla \times \vec{B} = \vec{j}$

(or) $\partial_\mu F^{\mu\nu} = j^\nu$

written in Covariant Equation

→ Now you get this equations from Lagrangian.

use principles

Locality " whatever is going on at one instant of time is only related to neighbouring instant of time and position.

and it is always guaranteed if Action is integral over four dimensional space of a Lagrangian density.

where; Lagrangian density depends on all fields and derivatives of ϕ with respect to space time.

Notation: $\frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi$ (or) $\phi_{,\mu}$ (" , " comma means derivative & sub μ means derivative w.r.t. μ th component of x)

$$\text{Action} = \int d^4x \mathcal{L}(\phi, \phi_{,\mu})$$

~~Lorentz Invariance~~

Lorentz Invariance ; $\mathcal{L} = \text{scalar}$ (Lagrangian is scalar)

a simple Lagrangian which can be thought
~~some simple example~~ ; $\mathcal{L} = U(\phi)$

here U is some scalar.

(Lagrangian can't only contain $\partial_\mu \phi \dots$ this is itself not scalar)

note;

$$\partial_\mu \phi \partial^\mu \phi = \phi_{,\mu} \phi^{,\mu}$$

notations

~~it is a sort of convention~~
... we put $-\frac{1}{2}$ there.

↳ actually equal to $-(\partial_t \phi)^2 + (\partial_x \phi)^2 + (\partial_y \phi)^2 + (\partial_z \phi)^2$

~~Lagrangian~~ basically you can use $U(\phi)$
and $\partial_\mu \phi \partial^\mu \phi$ for Lagrangian
(where $U(\phi)$ is some scalar)

↳ you can combine them and make another Lagrangian.

• note ; it is convention to add $\partial_\mu \phi \partial^\mu \phi$ by multiplying it by a minus sign. (actually $-\frac{1}{2}$)

(if you want you can scale U & manage.)

if ; A is minimized or maximized ;
then so is $2A$)

$$-U - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi =$$

$$-\frac{U}{2} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi$$

↳ basically you can take $-\frac{1}{2}$ factor out of U
↳ & the whole theory remains ~~consistent~~ consistent.

so; a legitimate Lagrangian is

(p121)

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi)$$

(you can write down many many more things... but this is simple lagrangian)

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) = \frac{\partial \mathcal{L}}{\partial \phi} \Rightarrow \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = -\frac{\partial U}{\partial \phi}$$

★ and now; let's put Gauge Invariance; ~~let's put~~

~~let's now put A^μ~~
 (that Action should not change when you make Gauge Transformation)
 • the simplest way to insure that is to make lagrangian itself out of things which are Gauge ~~Invariant~~ Invariant.

• Things which are Gauge Invariant are $F_{\mu\nu}$

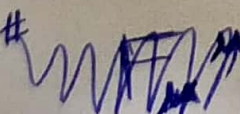
something which you can't add is $A_\mu A^\mu$ i.e. ~~$(A^0)^2$~~ $-(A^0)^2 + \sum_{x=1}^3 (A^x)^2$

↳ this is scalar but it is not Gauge Invariant.

★ now; if you have a Tensor $T_{\mu\nu}$

... a way which you can make ~~a~~ scalar out of it.
 (contract indices)

$$T_{\mu\nu} \xrightarrow{\text{raise an index}} T_\mu{}^\nu \xrightarrow{\text{contract it by setting indices equal.}} T_\mu{}^\mu$$

 # $\stackrel{\text{but}}{=} F_\mu{}^\mu = 0$

(what happens when you contract indices of F ; i.e.

$$F_{\mu\nu} \rightarrow F_\mu{}^\mu$$

because $F_{\mu\mu} = 0$ if diagonal elements are zero)

; it is 0)

* ~~Now the first & trivial things~~

* nothing linear in F works: $F_{\mu}^{\mu} = 0$

→ we can try things which are quadratic in F .

$F_{\mu\nu} F^{\mu\nu}$ → let's see what this has.

* first of all there will be mixed space time components.

$F_{0m} F^{0m}$... this is almost the square of the Electric field.

$$F_{0m} F^{0m} = -(E)^2$$

(minus sign comes because you raise time component ... and results in minus sign...)

→ but this term appears twice ... because you can have

$$F_{m0} F^{m0} = -(E)^2$$

so, $F_{\mu\nu} F^{\mu\nu}$ contains $-2(E)^2$

↳ but it also contains space-space components.

$$F_{12} F^{12} = B_z^2$$

$$F_{21} F^{21} = B_z^2$$

→ so, $F_{\mu\nu} F^{\mu\nu}$ contains $2(B)^2$

$$\dots F_{\mu\nu} F^{\mu\nu} = 2(-E)^2 + (B)^2$$

⇒ and we have a convention that Lagrangian begins with E^2 and $-B^2$: i.e. $E^2 - B^2$.

$$\cancel{F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)} - F_{\mu\nu} F^{\mu\nu} = 2(E^2 - B^2)$$

→ second part of convention is $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - B^2)$

$$\therefore -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - B^2)$$

$$\frac{\partial}{\partial x^\nu} \frac{\partial \mathcal{L}}{\partial \phi_{,\nu}} = \frac{\partial \mathcal{L}}{\partial \phi} \quad \begin{array}{l} \text{for each field} \\ \leftarrow \text{this is the Euler Lagrange equation.} \end{array}$$

• for each field you write separate equations... (Euler Lange eqⁿ)

What are the fields

↳ They are the vector Potentials,

A_0, A_1, A_2, A_3 four distinct fields.

↳ These are the independent fields.

★ let's introduce notation for derivative of field.

$$A_{\mu,\nu} := \frac{\partial A_\mu}{\partial x^\nu}$$

★ What about Field Tensor;

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \Rightarrow F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$$

NOTATION : $\phi_{[\mu,\nu]} := \phi_{\mu,\nu} - \phi_{\nu,\mu}$ ↳ Shorthand notation.

$$F_{\mu\nu} = A_{[\mu,\nu]}$$

★ In working out Euler Lagrange equations; we take four components of A to be independent separate fields.

↳ so; There is field equation for each one of them.

$$\mathcal{L} = -\frac{1}{4} (A_{\mu,\nu} - A_{\nu,\mu}) (A^{\mu,\nu} - A^{\nu,\mu})$$

↳ This is the Lagrangian... let's see if we can work out field equation. ↳ (without including $g^{\mu\nu}$ for the analysis now) ... later we will add and see.

• and let's pick up a particular component of A_μ .

* There will be equation for each component of A.

A_x or A_1

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial A_{x,\mu}} \right) = \frac{\partial \mathcal{L}}{\partial A_x}$$

so: $\frac{\partial \mathcal{L}}{\partial A_{x,y}} = ?$

$$\mathcal{L} = -\frac{1}{4} (A_{\mu,\nu} - A_{\nu,\mu}) (A^{\mu,\nu} - A^{\nu,\mu})$$

$$= -\frac{1}{2} \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right)^2 + \text{...}$$

→ This is the only term in the Lagrangian which contains $A_{x,y}$ (it also contains other things) ... but its the only term which contains ~~$A_{x,y}$~~ $A_{x,y}$.

$$\mathcal{L} = -\frac{1}{2} (A_{x,y}^2 - 2A_{x,y}A_{y,x} + A_{y,x}^2) + \text{...}$$

This term does not contain any of $A_{x,y}$.

→ now we will differentiate it w.r.t. to a particular component $A_{x,y}$.

so

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_{x,y}} &= -A_{x,y} + A_{y,x} \\ &= A_{y,x} - A_{x,y} = -(A_{x,y} - A_{y,x}) \end{aligned}$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial A_{x,y}} = -F_{x,y} = (-F^{x,y})$$

incidently; $F_{x,y} = F^{x,y}$ (space components can go ... up or down for free)

so; The General Formula is

$$\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} = -F_{\mu,\nu} \quad \text{or}$$

$$\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} = -F^{\mu,\nu}$$

DANGER
: don't confuse yourself
that $A_{x,y} = -A_{y,x}$
→ this does not hold.
: ($F_{\mu\nu} = -F_{\nu\mu}$ does)
→ You did the same mistake earlier.

Shwaits Observation

PS 125

$$\frac{\partial \phi^\alpha_\beta}{\partial x^\mu_\nu} = \begin{cases} \alpha \neq \nu \\ \beta \neq \mu \end{cases}$$

just where the indices goes:

$$\frac{\alpha_1}{1} = \alpha \quad (\text{upper})$$

$$\frac{1/\beta}{1} = \frac{1}{\beta} \quad (\text{lower})$$

$$\frac{1}{\mu_1} = \frac{1}{\mu} \quad (\text{lower})$$

$$\frac{1}{1/\nu} = \nu \quad (\text{upper})$$

Try to follow this ... you can have ~~but~~ beautiful pattern.

$$\therefore \frac{\partial \phi^\alpha_\beta}{\partial x^\mu_\nu} = \begin{cases} \alpha \neq \nu \\ \beta \neq \mu \end{cases}$$

↳ This is why I think

$\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}}$ is more often expressed in terms of

upper indices .. but not lower indices.

$$\# \text{ now; } \frac{\partial}{\partial x^\nu} \left(\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} \right) = - \frac{\partial F^{\mu\nu}}{\partial x^\nu} \quad \& \quad \frac{\partial \mathcal{L}}{\partial A_\mu} = 0$$

So; Euler Lagrange equation is

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = 0}$$

→ It is just the two of ^{Maxwell} Euler-Lagrange equations which happens to have $\vec{j}$ on right hand side

$$\text{ie; } \nabla \cdot \vec{E} = \rho = 0 \quad \& \quad \frac{\partial \vec{E}}{\partial t} + \nabla \times \vec{B} = \vec{j} = 0$$

(we havenot included the $\vec{j}$ for the moment)

→ The other two maxwell equations; ie: $\nabla \cdot \vec{B} = 0$ & $\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$

we dont have to do anything they are identities that follow from definition of A and its relationship to F .

There are no gauge Invariant things that we can make up just out of ~~the~~ A 's that don't involve $F_{\mu\nu}$. (P8 126)

→ We have only used one example
... and the fact is that it really does give rise to the real Maxwell Equations to take Lagrangian to be

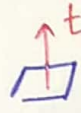
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

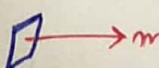
☆☆ The question is now; how you get RHS of the equation (that is the j^μ) ... where did that come from.

⇒ You have to add something to the Lagrangian.
↳ and that something ~~best~~ better involve the current vector (The four component of current vector)

... so; lets suppose; that there is some pre established electric current (charge density ... and a flow of charge : space & time component)

$$j^\mu = (\rho, j^x, j^y, j^z)$$

- Time component : Charge density (oriented along t axis )
- Space " : charge per unit area per unit time.

j^m is charge per unit area per unit time passing through little window oriented along m axis 

~~The real~~  = charge in there; $\rho \cdot dx dy dz$

∴ rate of change in there; $\dot{\rho} dx dy dz$

↳ if you use; local ~~charge~~ conservation; ... gives; $\dot{\rho} + \nabla \cdot \mathbf{j} = 0$

$$\text{i.e.} \quad \frac{\partial \rho}{\partial t} + \frac{\partial j^x}{\partial x} + \frac{\partial j^y}{\partial y} + \frac{\partial j^z}{\partial z} = 0$$

We can write it in relativistic notation;

$$\partial_\mu j^\mu = 0$$

It's called Continuity Equation.

→ ~~But~~ it represents charge conservation
(local charge conservation)

→ Represents the idea that no charge ~~app~~ disappears unless it is accounted for moving out of the region.

The action associated with current is

$$-\int dx^4 j^\mu A_\mu$$

→ This is a scalar
(it involves current & vector potential)

→ But it does not obviously look Gauge Invariant.

• We can check whether it is Gauge Invariant or not.

→ minus sign is convention... (it is due to Benjamin & Franklin)

Let's do Gauge Transformation;

$$-\int dx^4 j^\mu \left(A_\mu + \frac{\partial S}{\partial x^\mu} \right)$$

so; Change in Action under a Gauge Transformation := ΔA

$$\text{so; here; } \Delta A = -\int dx^4 j^\mu \frac{\partial S}{\partial x^\mu}$$

$$\int dx dy dz dt \left(j^0 \frac{\partial S}{\partial x^0} + j^1 \frac{\partial S}{\partial x^1} + j^2 \frac{\partial S}{\partial x^2} + j^3 \frac{\partial S}{\partial x^3} \right)$$

integrated over all space.

→ We are going to assume that if you go far enough away there is no current flowing. (no current at infinity)

$$\text{now; } \int j^1 \frac{\partial S}{\partial x} dx dy dz dt$$

→ use integration by parts

... since terms at end points of the integration region goes to zero (end points are at infinity)

so; there will be no additive term (that is zero)

→ you just flip derivatives & change sign.

so;
$$\int J' \frac{\partial S}{\partial x} dx dy dz dt = - \int \frac{\partial J'}{\partial x'} \cdot S dx dy dz dt$$

so; ~~we~~ actually summing all ~~these integrals~~ these integrals; we find,

$$\Delta A = - \int \frac{\partial J^\mu}{\partial x^\mu} \cdot S \cdot dx dy dz dt$$

But:
$$\frac{\partial J^\mu}{\partial x^\mu} = \partial_\mu J^\mu \quad \& \quad \frac{\partial J^\mu}{\partial x^\mu} = 0 \quad (\text{because of continuity equation})$$

so;
$$(\Delta A)_{\text{under Gauge Transformation}} = 0$$

~~minima:~~ so; we can add; ~~$J^\mu A_\mu$ to Lagrangian.~~

→ so; we can add $J^\mu A_\mu$ to Lagrangian.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu$$

($\mathcal{L}$ is gauge Invariant; iff ~~the~~ current is locally conserved)

so; now; the RHS of our Euler Lagrange equation becomes non zero.

~~$\frac{\partial \mathcal{L}}{\partial A_\nu} = \partial_\mu J^\mu$~~ ~~$\frac{\partial \mathcal{L}}{\partial A_\nu} = \partial_\mu J^\mu$~~

$$\frac{\partial \mathcal{L}}{\partial A_{\nu,\mu}} = \frac{\partial \mathcal{L}}{\partial A_\nu}$$

LHS came
$$-\frac{\partial F^{\mu\nu}}{\partial x^\nu}$$

and RHS came out to be
$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -j^\mu$$

so:
$$-\frac{\partial F^{\mu\nu}}{\partial x^\nu} = -j^\mu$$

so:
$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = j^\mu}$$

i.e: ~~$\partial_\nu F^{\mu\nu} = j^\mu$~~

i.e:
$$\boxed{\partial_\nu F^{\mu\nu} = j^\mu}$$

Conclusion • You can write Maxwell equation in Lagrangian form.

•
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

• $\mathcal{L}$ is Gauge Invariant if and only if the Current is Conserved.

~~Then~~ $\hookrightarrow$ If the current is not conserved: then the $\mathcal{L}$ is not Gauge Invariant & then you find out, that will lead to strange ~~results~~ results. (because it means equations of motion are not Gauge Invariant)

$\hookrightarrow$ It will be strange

because;

here: LHS ~~is~~ is certainly Gauge Invariant (only involves electromagnetic field tensors...)

∴ RHS is not... this means something very strange is going on... this just can't be the right equation.

★ We can derive continuity equation from Maxwell equations.

$$\rho = \nabla \cdot \vec{E} \quad ; \quad \dot{\rho} = \nabla \cdot \dot{\vec{E}}$$

$$\vec{J} = -\frac{\partial \vec{E}}{\partial t} + \nabla \times \vec{B} \quad ; \quad \nabla \cdot \vec{J} = -\nabla \cdot \dot{\vec{E}} + \nabla \cdot (\nabla \times \vec{B})$$

$$\Rightarrow \boxed{\dot{\rho} + \nabla \cdot \vec{J} = 0}$$

$$\Rightarrow \cancel{\nabla \cdot \dot{\vec{E}} + \nabla \cdot (\nabla \times \vec{B}) = 0}$$

• So; you can't have Maxwell equations & ~~it~~ have charge not conserved.

• Charge Conservation is connected to Gauge Invariance.

* Gauge is to do with measurements of ~~lengths~~ lengths.

* And actually Gauge Invariance has to do nothing with measurement of lengths.
(The word Gauge... came out of mis take)

→ It has to do with phases of Schrodinger wave function.
(it has to do with mixing of Quantum Mechanics...)

• Hermann Wile made a mistake (he thought it had to do with something, which it did not) ⇒ He very quickly corrected himself.
... but the term Gauge stuck.

• You can think of... Gauge here meant defining a zero of something
→ a sort of zero of this scalar field ϕ ... we ~~will~~ will understand the deeper meaning later.

You can invent a different vector potential with the property that equations of set A becomes equation of motion; & ~~equations~~ the ans of equation of set B become zero, ... becoming Bianchi identity.

$\nabla \cdot \vec{B} = 0$	$\nabla \cdot \vec{E} = \rho$
$\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E}$	$-\frac{\partial \vec{E}}{\partial t} + \nabla \times \vec{B} = \vec{J}$
Set A	Set B

→ Initially it was not a convention; it had to do with ~~absence~~ absence of Magnetic monopole.

(but now no one ~~any~~ anymore believe the absence of magnetic monopole)

→ In some sense it is convention in today's time.

→ It is this interchange of Electric and Magnetic, that it may be that real physics does have an interchange symmetry between Electric and Magnetic (Electric charges becoming magnetic charges)

→ Now it is not really a symmetry (because if it was symmetry, we would have discovered those magnetic charges already) ... they would have weighed ~~some~~ same as electron if there was real symmetry between them) ... it is not symmetry in the usual sense.

Special Relativity & Electrodynamics : Connection between Classical mechanics and field theory.

- Shoaib Akhtar

$$\# \vec{E}_M = -\frac{\partial \vec{A}}{\partial t} + \vec{\nabla} A_0$$

$$\# \vec{E}_L = +\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} A_0$$

Maxwell definition of Electric field.

- a sort of convention.
- (M for Maxwell)

~~Maxwell~~ Leonard definition of Electric field.

- Convention used by Leonard Susskind (L for Leonard) (will be used in the lecture)
- Convention used by Leonard Susskind (will be used in lecture)

To put it shortly;

$$\vec{E}_L = -\vec{E}_M$$

Convention of Maxwell and Leonard differ by a minus sign.

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

→ here; Maxwell and Susskind definitions or convention are same.

Maxwell equations which Maxwell wrote down.

$$\vec{\nabla} \times \vec{E}_M = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}_M}{\partial t} + \vec{J}_F$$

$$\vec{\nabla} \cdot \vec{E}_M = \rho_F$$

* Convention about $\vec{J}$ & ρ has not established by Maxwell... but by Benjamin & Franklin.

so; Franklin definition of ρ or $\vec{J}$ (current and charge density) ... denote by $\vec{J}_F$

$$\vec{J}_F, \rho_F$$

(current density, & charge density by Franklin definition)

⇓ changing the convention
Maxwell equations using Leonard's notations or conventions.

$$\vec{\nabla} \times \vec{E}_L = +\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = -\frac{\partial \vec{E}_L}{\partial t} - \vec{J}_L$$

$$\vec{\nabla} \cdot \vec{E}_L = \rho_L \quad (-\vec{\nabla} \cdot \vec{E}_L = -\rho_L \Rightarrow \vec{\nabla} \cdot \vec{E}_L = \rho_L)$$

* Leonard notation is opposite for Electric field... so naturally opposite for current & charge densities.

$$\vec{J}_L = -\vec{J}_F \quad \& \quad \rho_L = -\rho_F$$

The three concepts of Momentum.

~~three ways~~ (three ways of thinking about three different things)

① MECHANICAL MOMENTUM.

$$\mathbf{P}_m = m \dot{\mathbf{x}}_m \quad \text{or} \quad P^m = m \dot{x}^m$$

~~type~~ (space components... indices can be raised or down here)

② CANONICAL MOMENTUM

(~~comes when we start this~~ comes when we start thinking about Lagrangian formulation of mechanics)

let: L be Lagrangian;

$$L(q_i, \dot{q}_i)$$

$$P_i = \frac{\partial L}{\partial \dot{q}_i}$$

Note * Sometimes Mechanical & Canonical momentum are same. ; but not always.

example if $L = \frac{m}{2} \dot{x}_i^2 - V(x)$

then: $(P_i)_{\text{canonical}} = m \dot{x}_i$
 $= (P_i)_{\text{mechanical}}.$

③ $q_i' = q_i + \delta q_i$; so ; last concept of momentum ;

$$Q = \sum_i P_i \cdot \delta q_i$$

is a conserved quantity where you construct it by figuring out how all canonical coordinates in the system change when you do the operation, and then multiply small changes by conjugate momentum & then add them all up $\rightarrow$ This gives you conserved quantity... which is also called momentum.

$\rightarrow Q$ is conserved if the world is Translational invariant

Relationship between Mechanics & Field theory.

$$H = \sum_i P_i \cdot \dot{q}_i - L$$

$\downarrow$ Hamiltonian $\downarrow$ Lagrangian

* pick a frame... (don't try to keep the equation Lorentz invariant.)
 ... let's pick somebody's time.

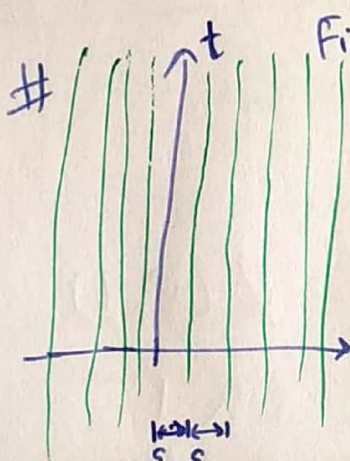
$\rightarrow$ we are doing this now $\rightarrow$ because this way it's easier to see connection with Classical Mechanics.

#

$$A = \int L(q, \dot{q}) dt$$

Classical mechanics have

- Time axis
- Set of coordinates which depend on time axis.
- & expression of action.
- and a principle of least action; ie; Action is stationary



Field theory ... ~~the have~~

- We have time axis.
- Coordinates of system functions of time
(they are not labelled by discrete index i)
- ↳ Coordinates of system are labelled by position.
- ↳ They are values of the fields at each position in space.

$$\phi(\vec{x})$$

(imagine discrete space for simplicity)

↳ so; at each point $\vec{x}$; there is separate &

independent field... ~~so~~ independent coordinate

↳ we use coordinate ~~to~~ in the same sense of q_i .
(Degree of freedom is a better word)

and $\vec{x}$ is simply a label in the same sense. as i is label in q_i

⇒ we also have $\dot{\phi}(\vec{x})$ (these are like $\dot{q}_i$)
time derivatives

now; Lagrangian also depends on derivatives w.r.t. space.

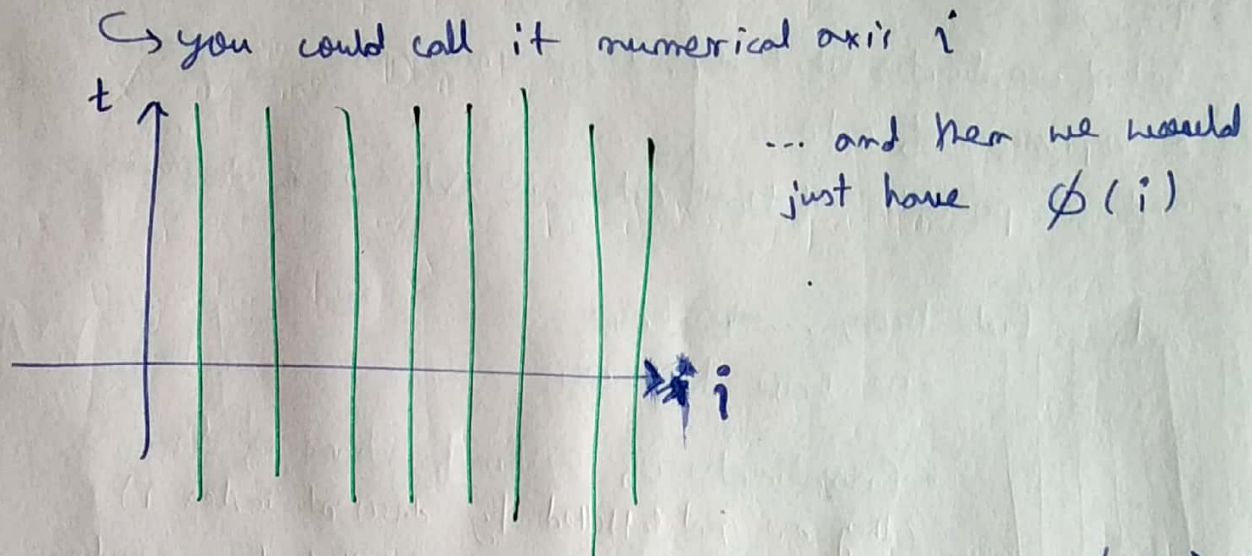
$$\frac{\partial \phi}{\partial x} = \frac{\phi(x+\epsilon) - \phi(x)}{\epsilon}$$

⇒ so; the derivatives are really just the function of ϕ .

↳ $\frac{\partial \phi}{\partial x}$ does not involve $\dot{\phi}$.

⇒ so; in places where Lagrangian might depend on spatial derivatives, that's really part of its dependence on ϕ .

~~* remember~~ * remember: x is just a label;



Wait; ϕ is in general function of $\vec{x}$ & t ; so $\phi(\vec{x}, t)$
 in the same sense q_i ~~is~~ depends on i & t
 (it depends on i just because which one you are talking about; ... and change with time) : $q_i(t)$

★ We treat each one of $\phi(\vec{x}, t)$ (easy to imagine in discrete form $\phi(i, t)$)
 as independent canonical degree of freedom.

★ Now we think about ... what we mean by Lagrangian or Action in field theory.

• Action in classical mechanics is always integral over time; $A = \int dt L$

~~↳ But in field theory~~

⇒ But in field theory... Lagrangian itself is going to be function of space (the integrating or summing elements depends on space)

→ ⁶⁶so; Action in field theory is also going to be sum over space or an integral over space

→ so; Action is integral over space & time.

$$A = \int dt \quad L = \int dt \left(\int d^n x \mathcal{L}(\phi, \dot{\phi}, \frac{\partial \phi}{\partial x^n}) \right)$$

"n is no. of dimensions of space"

the script "L" ... is the thing which in field theory is called ~~L~~ Lagrange Density.

$$A = \int dt \int d^3 x \mathcal{L}(\phi, \dot{\phi}, \frac{\partial \phi}{\partial x^n})$$

* You can think Field Theory as a special case of mechanics

$$A = \int dt \cdot d^3 x \mathcal{L}$$

• Field have energy ... and that energy is part of the universal conserved energy that is associated with time translation.

So how do we construct Energy ie; we want Hamiltonian.

$\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ } → This is canonical momentum conjugate to ϕ
→ ~~(Don't imagine this as usual momentum)~~

→ (Don't imagine this as usual momentum)
→ we don't want to call it P (we will call P for something else)

→ & we have standard notation for $\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$
... it is π or $\underline{\pi}_\phi$

$$\pi_\phi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}(\vec{x})$$

→ Canonical momentum conjugate to the coordinate ϕ

→ Both sides are function of position.

- There can be many fields in the problem.
($\phi_1, \phi_2, \phi_3 \dots$)

and there will be $\pi_{\phi_i} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}$ for each kind of fields.

Now; let's compute Hamiltonian.

- Hamiltonian is sum over all i . ~~Mechanics~~ } in Classical mechanics
i.e. $H = \sum_i p_i \dot{q}_i - L$

~~So~~

$\Rightarrow$ what is i stands for in field theory problem.

$\hookrightarrow$ they stand for $\vec{x}$

$\Rightarrow i$ and $\vec{x}$ are analogous thing;
they are labels that label the degrees of freedom.

So; sum over i is going to be integral over d^3x

so; ~~$H = \sum_i p_i \dot{q}_i - L$~~

$$H = \int d^3x \pi(\vec{x}) \dot{\phi}(\vec{x}) - L \quad \hookrightarrow \text{Total Lagrangian}$$

but: $L \equiv \int d^3x \mathcal{L}$

so: $\sum_i p_i \dot{q}_i \rightarrow \int d^3x \pi(\vec{x}) \dot{\phi}(\vec{x})$
 $L \rightarrow \int d^3x \mathcal{L}$

so:

$$H = \int d^3x [\pi(\vec{x}) \dot{\phi}(\vec{x}) - \mathcal{L}]$$

Now; look the form of Energy or ~~ham~~ Hamiltonian we had got.

↳ which is expressed as an integral over space.

→ so; we would call integrand Energy Density.

so; Energy Density = $\Pi(\vec{x}) \dot{\phi}(\vec{x}) - \mathcal{L}$

example ; for simplicity & to understand.. lets take one dimensional space.

$$\mathcal{L} = \underbrace{\frac{1}{2} \dot{\phi}^2}_{\text{Kinetic energy density term}} - \underbrace{\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2}_{\substack{\downarrow \\ \text{lets put in} \\ \text{this term}}} + \underbrace{V(\phi)}_{\substack{\Downarrow \\ \text{and finally just} \\ \text{add in this term} \\ \text{for generality.}}}$$

If we went back and thinking about Relativity; we will recognize this quantity as a scalar ...

& $V(\phi)$ is a scalar.

↳ and we will call it something like

$$-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi$$

now; $\Pi_\phi = \dot{\phi}$ and; $H = \int dx (\dot{\phi}^2 - \mathcal{L})$
 $= \int dx (\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + V(\phi))$

* in the Lagrangian density

↳ we have a kinetic energy term " $\frac{1}{2} \dot{\phi}^2$ " ⇒ related to time derivatives

↳ & we have other terms which does not have time derivatives in it

" $-\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 - V(\phi)$ "

obviously this whole thing is playing the role of potential energy.

(the piece of energy which does not have time derivatives)

* sometimes people call $V(\phi)$ as Field Potential energy.

* but it is really the whole combination " $\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + V(\phi)$ " ... (and) energy-thing which do

- but it is really the whole combination " $\frac{1}{2}(\frac{\partial \phi}{\partial x})^2 + V(\phi)$ " or everything which does not contain any time derivatives
 ↳ which is a kind of potential energy.

★ Energy is typically something which does not go negative.
 (needs more thought and proof... but can think here if $V(\phi)$ is some positive quantity)

→ $V(\phi)$ can also be negative too
 ↳ but you will get into trouble if it is unbounded from below.

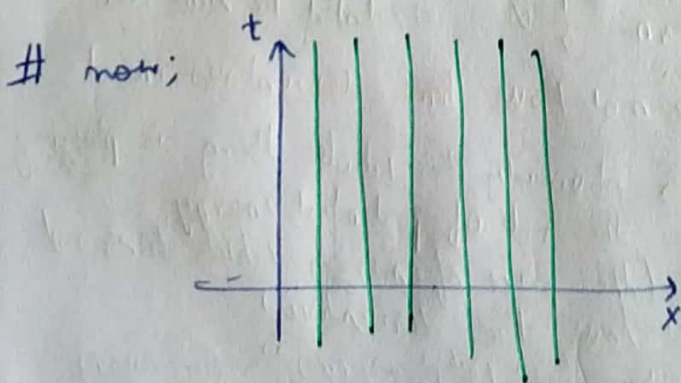
and ~~if~~; when it is bounded from below, then you can just add constant to it and make it positive... and it does not change anything
 (so you can arbitrarily assume $V(\phi)$ is positive)

- If $V(\phi)$ is not below, ~~the theory is not stable and not consistent.~~

He will give energy density a new name.

$$T^{00} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + V(\phi)$$

↳ Field energy density.



lets suppose that ϕ abruptly jumps
 (what is it that is requiring ϕ to vary smoothly)?

↳ why they cant be ~~so~~ wildly changing if they are really independent degrees of freedom?

⇒ That will result in bigger and ~~bigger~~ bigger spatial derivatives.

→ and this would mean that energy density will get bigger and bigger ... $T^{00} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + V(\phi)$

↳ so; if we say that we are interested in finite energies, or interested in configuration where energy is not going off to be something huge.

↳ it ~~may~~ requires the spatial derivatives to be finite → this will control ϕ and tells it to be smooth (and ~~contine~~ continuous).

now: what is the notation T^{00}

* In relativistic theories the Energy is a time component of a 4-vector. (in fact we usually call that 4-vector p^μ)

and so: $E = p^0$

* Whenever you have a conserved quantity in a relativistic theory ... a sensible theory ... the conservation is local.

↳ This means that the quantity does not just disappear over here and reappear at some place far away.

... so; when you have conserved quantities; they only change if you can detect a flux of them or a current of them passing through the walls of your laboratory.

↳ This gave rise to a concept of density (ρ) & a current (j)

↳ and they combine together to form another 4-vector j^μ

→ but density is like time component of 4-vector.

★ now we have density of energy.

- Energy itself is time component of something; so "o"
- Density is time component of something; namely current; so "o"

→ so; Density of energy is density of time component of a 4-vector.

→ This is why two o appear in T^{oo}
 (one of them stands for the fact that we are talking about energy; and the other one for the fact that we are talking about density)

$$T^{oo} \xrightarrow{\text{density}} \text{energy.}$$

$$T^{oo} = \Pi(\vec{x}) \dot{\phi}(\vec{x}) - \mathcal{L}$$

→ field Energy Density.

Now what about momentum density.

(here; we mean momentum which is conserved because of translation invariance) → the general concept due to Emmy Noether ... that every time there is symmetry in the world; there is a conserved quantity that goes with it.

we are going to call that quantity P .

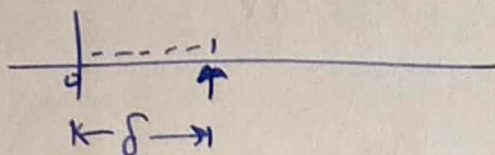
P ⇒ Momentum in the sense of Noether ... ~~having to~~ ... having to do with translational invariance.

~~DEFINITION~~

~~$P = \int d^3x \Pi \dot{\phi}$~~

$$P = \sum_i P_i S_{\text{ori}} \equiv \int \Pi_{\phi}(\vec{x}) \dot{\phi} d^3x$$

What is $\delta\phi$ when you make a shift of coordinates.



we are talking now about translation of coordinates
~~where~~ origin of coordinate is displaced
 (where from one point to neighbouring point)

what happens to ϕ

$$\phi(x) \rightarrow \phi(x - \delta)$$

after
shift of
coordinates.

so; change in ϕ : i.e. $\delta\phi = \phi(x) - \phi(x - \delta)$

$$\propto \frac{\partial\phi}{\partial x} \delta$$

$$\delta\phi = \frac{\partial\phi}{\partial x} \delta$$

$$P = \int \pi \cdot \frac{\partial\phi}{\partial x} \cdot \delta d^3x = \left(\int \pi \cdot \frac{\partial\phi}{\partial x} dx \right) \delta$$

so; if a thing is conserved... so; its multiple is also conserved

so; $\int \pi \cdot \frac{\partial\phi}{\partial x} dx^3$ is useful quantity)

→ & the official definition of ~~the~~ the momentum is defined in such way that small δ is divided out.

$$P = \int \pi_\phi(x) \frac{\partial\phi}{\partial x} d^3x \quad \Rightarrow \text{this is space component of } L\text{-vector.}$$

$$P^m = \int \pi_\phi(\vec{x}) \frac{\partial\phi}{\partial x^m} d^3x^m \quad (\text{here; we are shifting along } x^m \text{ axis})$$

* Now; each component of the momentum can be represented as integral of something w.r.t. that coordinate of space.

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$\Rightarrow$ so, it is pretty clear that the integrand " $\pi \frac{\partial \phi}{\partial x^m}$ " is momentum density.

its momentum density.... call it T

$\hookrightarrow m$ for the fact that it's the "m" component of the momentum that we are taking about.
(spatial component)

$\hookrightarrow 0$ for the fact that we are taking about density;

so: Momentum Density $\rightarrow$ $T^{0m} = \pi(\vec{x}) \frac{\partial \phi}{\partial x^m}$

$T^{00} \Rightarrow$ Density of energy
(density of time component)

$T^{0m} \Rightarrow$ density of space component of momentum

example: for our very simple field theory:

~~$T^{0m} = \pi \frac{\partial \phi}{\partial x^m}$~~ $T^{0m} = \frac{\partial \phi}{\partial t} \cdot \frac{\partial \phi}{\partial x^m}$

What is Momentum Density ... its physical significance

for example: ... for electromagnetic wave, @ wave of ϕ .

$\hookrightarrow$ well it is literally the momentum carried by that wave.
~~it's contained together~~ (it is combined together with the momentum carried by all the particles in the system)

• incidently, if you think about translation invariance. The system is only Translation invariance, if when you translate all the fields, you also translate all the particles.

→ If there are particles or charged particles, it is not okay to just translate the electromagnetic waves ~~and~~ and not translate the particles.

⇒ Invariance means you translate everything.

★ So; if we keep track, of not only the field, but all the particles that it might interact with; There will be contribution ~~from~~ to the momentum coming from these particles.

→ Now, we will be literally talking about Conservation of momentum of particles & fields.

ex Light wave has momentum.

★ The analog of Newton's laws would be field equations.

ie:
$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial V}{\partial \phi}$$
 for our simple example.

T^{0m} , T^{00} are some components of energy momentum tensor.

→ The other components of it are associated with flow of energy and momentum.

× Energy Momentum Tensor plays a huge role in General Relativity.
→ huge role of being sources of gravitational field.

Gauge Transformation: $A_\mu \rightarrow A_\mu + \frac{\partial S}{\partial A_\mu}$

→ we can use this freedom so that it can simplify our choice of A .

example

Given any A_0 ; it is possible to pick S such that the new transformed $(A_0)'$ is 0

i.e; $A_0 + \frac{\partial S}{\partial t} = 0 \Rightarrow \frac{\partial S}{\partial t} = -A_0$ & you can always find such S .

~~actually~~ ... it's called fixing the Gauge freedom.

(you have Gauge freedom ... that you could change vector potential without changing Physics ... & so you make a choice of A which is suitable)

$\Rightarrow$ So; we fix ; $A_0 = 0$, and ~~A~~ we are left with $A_m(x)$
(without loss of Generality in Physics due to Gauge Invariance)

So; $\vec{E}_m = -\frac{\partial \vec{A}}{\partial t} + \nabla A_0$; but here; $A_0 = 0$

So; $\vec{E}_m = -\frac{\partial \vec{A}}{\partial t}$ so; $(E)_m = -\frac{\partial A_m}{\partial t}$

& ; $\vec{B}_m = \nabla \times \vec{A}$

... the field ... the degrees of freedom are just the space component of vector potential here.

(since we choose a suitable gauge and made our time component of vector potential zero)

now; $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - B^2)$

$\Rightarrow \mathcal{L} = \frac{1}{2} \left[\left| \frac{\partial \vec{A}}{\partial t} \right|^2 \right] - \frac{1}{2} |\nabla \times \vec{A}|^2$

It involves only space derivatives ... looks quite similar to Potential.

$\hookrightarrow$ So; we identify this as Kinetic Energy term
(sums of squares of time derivatives of basic fields) ~~similar to~~ $\rightarrow$ similar to $\frac{1}{2} \dot{\phi}^2$

$\hookrightarrow$ similar to $-\frac{1}{2} (\nabla \times \phi)^2$

(similar to; $\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \times \phi)^2$)

$$\pi_{A_n} = \frac{\partial \mathcal{L}}{\partial A_{n,t}} = \frac{\partial A_n}{\partial t}$$

A_x, A_y, A_z

each of one of these can be thought of as field

There are three fields.
each one is of a degree of freedom of its own.

$$\textcircled{\text{or}} \pi_x = \frac{\partial A_x}{\partial t} = -E_x$$

$$\pi_y = \frac{\partial A_y}{\partial t} = -E_y$$

$$\pi_z = \frac{\partial A_z}{\partial t} = -E_z$$

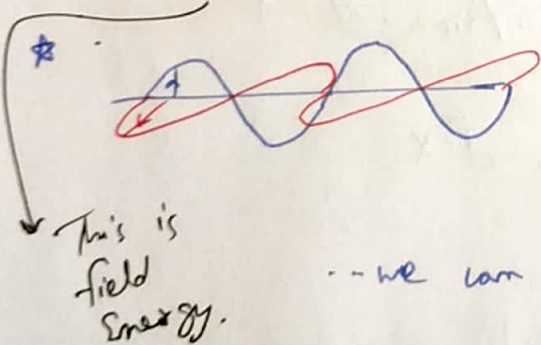
$$\text{note } \pi_m = -E_m \quad (m \in \{1, 2, 3\})$$

giving indices to π in this manner.

~~The canonical momentum~~
The physical meaning of the canonical momentum ~~increase~~ conjugate to the vector potential in this case is just the electric field.

$$\therefore H = \frac{1}{2} \left\| \frac{\partial \vec{A}}{\partial t} \right\|^2 + \frac{1}{2} \left\| \nabla \times \vec{A} \right\|^2$$

i.e. $H = \frac{1}{2} (E^2 + B^2)$ $\Rightarrow$ This tells us that electromagnetic wave moving down the axis has both Electric & Magnetic energy... & they happen to be same...



-- we can show; $\frac{1}{2} (E^2 + B^2) = B^2 = E^2$
(speed of light = 1 ... units...)

Momentum Density.

$$p^m = \int d^3x \pi \cdot \frac{\partial \phi}{\partial x^m} \quad \left. \begin{array}{l} \text{This is conserved quantity} \\ \text{by virtue of Noether's Theorem.} \end{array} \right\}$$

(if there were several fields in problem... we will add them up)

$\rightarrow$ and there are ~~several~~ several fields in this problem.

$$p^n \rightarrow \int d^3x \sum_m -E_m \cdot \frac{\partial A_m}{\partial x^n} \Rightarrow p^n = \int d^3x \left(\sum_m -E_m \cdot \frac{\partial A_m}{\partial x^n} \right)$$

$$P^n = \int -E^m \cdot \frac{\partial A_m}{\partial x^n} \cdot d^3x$$

$$E^m = E_m$$

spatial component

→ we do because we want to use summation convention.

↳ it is tempting to write

$$\int -E^m \left(\frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^m} \right) d^3x$$

because we ^{really} want to put magnetic field here (not just the damn trick with Vector Potential in it)

↳ does the extra term change P^n .

so; let's focus $\int E_m \cdot \frac{\partial A_m}{\partial x^n} \cdot d^3x$

... assuming that fields go to zero far away ~~that they are~~
(that there are no boundary terms)

$$\begin{aligned} \text{so: } \int E_m \cdot \frac{\partial A_m}{\partial x^n} d^3x &= - \int \frac{\partial E_m}{\partial x^m} \cdot A_m \cdot d^3x \\ &= - \int (\nabla \cdot \vec{E}) A_m \cdot d^3x \\ &= 0 \end{aligned}$$

but; if we don't have charge; ... $\nabla \cdot \vec{E} = 0$
(assuming no charge density)

so; $P^n = \int -E^m \left(\frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^m} \right) d^3x$

$$P^n = \int + (\vec{E} \times \vec{B})_n \cdot d^3x$$

$$\vec{P} = + \int (\vec{E} \times \vec{B}) d^3x$$

$\vec{E} \times \vec{B}$ is called Poynting Vector (Poynting was name)

~~MEMORISE~~ Poynting

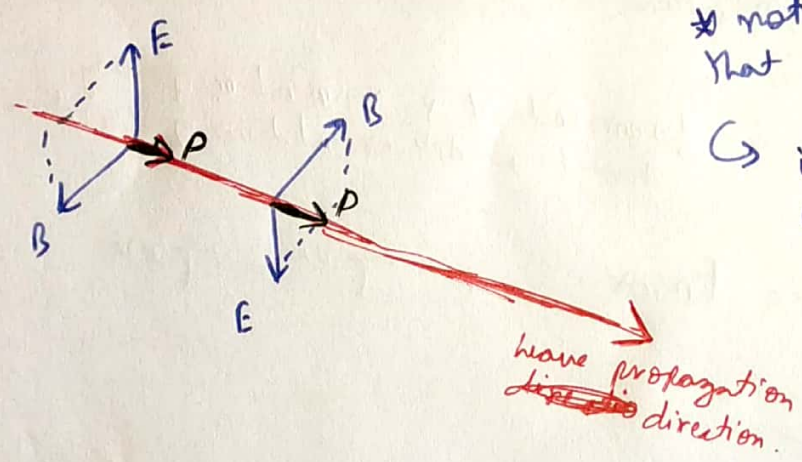
$\vec{E} \times \vec{B}$ is Poynting Vector; denoted by $\vec{S}$

$$\vec{S} := \vec{E} \times \vec{B}$$

$\therefore S^m$ is actually then Momentum Density

ie: $T^{0m} = S^m$

* If there is charge density; we have to account for the momentum of particles.



* notice how important ~~that~~ it is that E & B are $\perp$ to each other.

$\hookrightarrow$ if they were not perpendicular (they are not solutions of Maxwell equation) they would not carry any momentum.

* So the momentum @ the Poynting Vector is consequence of perpendicular nature of $\vec{E}$ & $\vec{B}$

... but it all traces back to Noether's theorem, about connection between conserved momentum and Translational invariance (spatial)

* T^{0m}
 $\downarrow$
 density of m^{th} component of momentum

- $\therefore T^{00} \Rightarrow$ density of energy

$T_{m0} \Rightarrow$ flux of energy in the m direction.

- first index ν says density (when $\nu=0$)
- second index μ says energy (when $\mu=0$)

T^{01} = Density of x component of momentum

$T'' \Rightarrow X$ -flux of the X momentum.

$T^2 \Rightarrow 4 - " " " \times "$

ТЗ' 22- " " " X "

$\Rightarrow$ amount of x momentum passing through window oriented in z direction

★ $T^{\nu\mu}$ forms a symmetric Tensor; .. i.e; $T^{\nu\mu} = T^{\mu\nu}$

→ now: what it means

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• example ; $T^0_0 = T^1_1$ i.e. ; $\times$ Flux of energy = Density of $\times$ momentum.

→ so; flow of energy along an axis is same as the density of momentum along the same axis.

Poynting vector has two meanings

- Poynting vector has two meanings
 • It is flow of energy (You can think of power per unit area flowing through a region)
 &
 • It is density of momentum.

Summary of Maxwell equations.

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Maxwell Equation in 4-vector notation..

- $\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0$ $\Rightarrow$ Bianchi Identity.
- $\partial_\nu F^{\mu\nu} = j^\mu$ $\Rightarrow$ Equation ~~got~~ from dynamics
... (by Lagrangian formalism)

$\hookrightarrow$ we can prove ~~that~~ that j^μ satisfies continuity equation from above two equations; i.e. $\partial_\mu j^\mu = 0$

$$\therefore \partial_\mu j^\mu = \partial_\mu \partial_\nu F^{\mu\nu} \quad \text{or} \quad j^\mu_{, \mu} = F^{\mu\nu}_{, \mu, \nu}$$

now:

$$\begin{aligned} F_{\mu\nu} &= \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \\ \Rightarrow F^{\mu\nu} &= \frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu} \end{aligned}$$

$$\Rightarrow \text{and } \partial_\mu \partial_\nu F^{\mu\nu} = 0$$

$$\text{so, } \underline{\underline{\partial_\mu j^\mu = 0}}$$

