2D CONFORMAL FIELD THEORY

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2D CONFORMAL FIELD THEORY

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Started on: 28-July-2020
Completed on: 30-July-2020

These notes are consequence of my self study; which I prepared while studying the subject. Some of the materials are motivated from lectures of Prof. G. Mussardo. For Fusion Rules and the Verlinde Formula, I explicitly used the lecture notes on the topic by Prof. Gaberdiel.

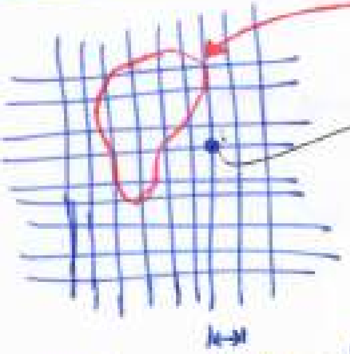
2d Conformal Field Theory

Shoaib Akhtar 28/7/2020 (Pg1)

Lec 1: Motivation for CFT via R.G., Perturbing CFT action by relevant operators ~~(also with irrelevant operators)~~

Classical Statistical Mechanics

We also have dynamical scale which emerge depending on the coupling constant the system is subjected to
↓
it is correlation length.



σ_i (fluctuating variable at each lattice site)

a ← microscopic scale,

Examples

1. **Ising Model** $\sigma_i \in \{\pm 1\}$

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - B \sum \sigma_i$$

This breakes $\mathbb{Z}_2$ invariance

Invariant under $\mathbb{Z}_2$; $\sigma_i \longmapsto -\sigma_i$

Preserves (& respect) $\mathbb{Z}_2$ symmetry.

2. **Potts Model**

$\sigma_i \in \{1, \dots, q\}$

variables assume q values (which we might colours)

Here the symmetry we want to study is S_q (permutation of q objects)

$$H = -J \sum_{\langle i,j \rangle} \delta_{\sigma_i, \sigma_j}$$

↳ nearest neighbours

3. **Spin Model** : here σ_i is vector $\vec{S}_i = (S_1, \dots, S_N)$

Symmetry group is $O(N)$ (continuous)

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

ξ correlation length

ξ depends on the coupling constant the system is subject to $\qquad \xi(g)$

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when $\xi \to \infty$; then system goes into phase transition

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When $\xi \to \infty$, since everything will depend on dimensionless quantities like $\left(\frac{\xi}{a}\right)$

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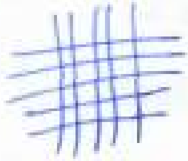
when $\left(\frac{\xi}{a}\right) \longrightarrow \infty$; for the model only to things will matter then.

(i) G - group of symmetry

(ii) d - dimensionality of space where the system is defined.

(All the rest we can essentially forget)

Once we have lattice  $\updownarrow a$

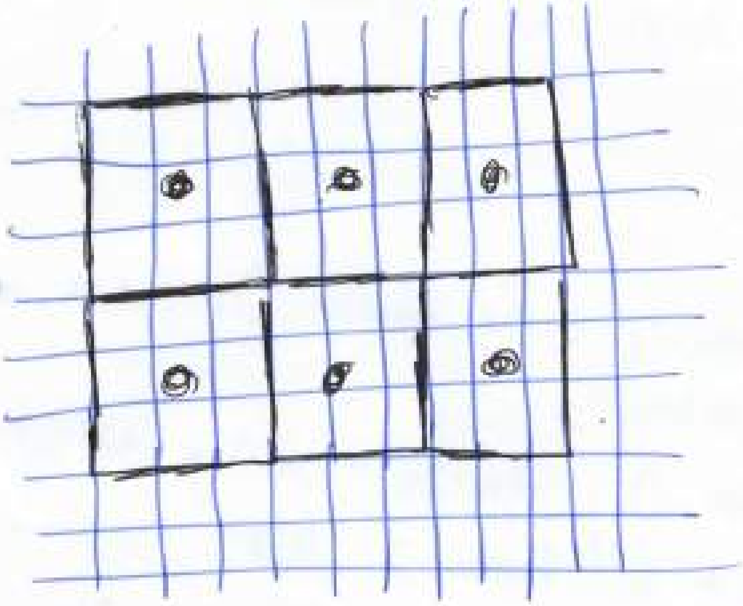
we have certain Hamiltonian on the lattice, which will be written in term of certain operators

 $\qquad H|_a = \sum g_i O_i$

$\hookrightarrow$ effective description of our system at lattice space a.

When $\xi \to \infty$;

The lattice microscopic details dont matter much ;
this suggests that we might look at blocks at different scales... making scaling transformations...
... & look system on a different scale.



So; the idea (push forward by ~~[illegible]~~ Cadanoff)
make the system 3×3 (say scale factor is 3)
So; on this new lattice ; we have to assign ~~eff~~ effective variable for the new block.
such that, physics will not change
~~[illegible]~~ constraint of Renormalization Group.
... i.e; partition function does not change.

$$\sigma_i \xrightarrow{f} \sigma_i'$$

what we will look at, is iterative maps of ~~[illegible]~~ do
this rule $f[f[f[f \ldots]]] \equiv f^n$

As far as we choose reasonable mapping f of this type ; The final result is largely ~~independly~~ independent of the form that we have taken for individual transformation. (We have much a Universal Behavior)

~~Now~~ Now, we want to change a to new ~~...~~ lattice site b(a) with certain rule

$$H\big|_{ba} = \quad \ldots$$

If we choose properly, the operator O_i (which are basis in space of operators).

If we choose O_i to be scaling operator;

The nice thing which will happen is that;

at different scale ba; The Hamiltonian will take the same functional form

$$H\big|_a = \sum g_i O_i$$

$$H\big|_{ba} = \sum g_i' O_i$$

The couplings constants g_i change multiplicatively.

$$g_i \longrightarrow g_i \, b_i^{d-\Delta_i}$$

Δ_i is scaling dimension of O_i.

Scaling Operator:

if $x \to \frac{x}{b}$, then $\varphi_i\left(\frac{x}{b}\right) = b^{\Delta_i} \cdot \varphi_i(x)$

then φ_i is scaling operator.

In technical terms ; (19.5)

Scaling operators are eigen vectors of Dilatation operator of field theory whose eigen value will be Δ_i.

Δ_i is called <u>Scaling dimension</u>.

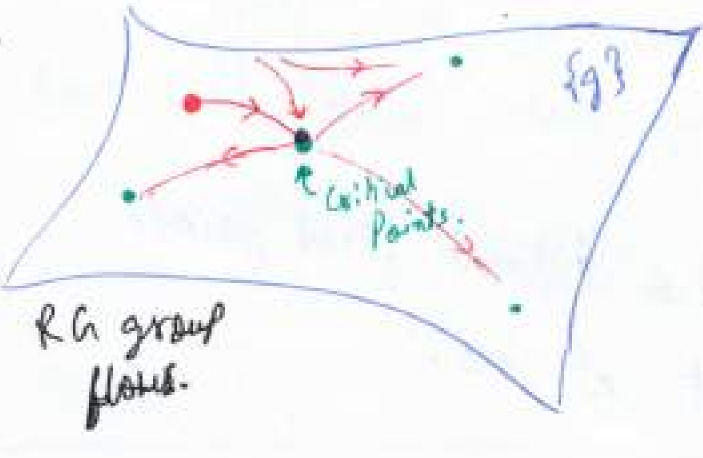
If we choose to write $H|_a$ in terms of Scalling operator (later they will form a basis ; so we can do it)

If we now do a rescalling: $H|_{ba}$

Then the Hamiltonian will look exactly as before ; ~~the~~ but with updated Coupling Constants.

$$H|_{ba} = \sum g_i' \mathcal{O}_i$$

Given a group of symmetry, we define ∞ dimensional coupling constant space associated to that symmetry



RG group flows.

Imagine that ; given a group of symmetry, we are able to identify all possible operators which have specific transformation around group of symmetry

↳ Essentially, they are irreducible representation of the group.

The coupling constant is Conjugate to scaling operator.

"by changing scale, we induce flow in the space of coupling constants".

Where this flows are going?

(Note: This is ∞ generalization of Dynamical system in mathematics)

Dynamical system ~~[illegible]~~ in mathematics are rules which assign transformation to variables ... & keep going.

↳ Sometime is done in descrete time or continuous time.

Ex) Logistic Map $\underline{X_{n+1} = r X_n (1 - X_n)}$

The trajectories in $\{g\}$ Space will flow without doing crazy things, and they will stop in some point which we will call critical points.

Critical Points are those; Such that $g_i' \equiv g_i$

(new g_i is equal to previous one .. will not flow)

There is simplest possible behavior of RG flows according to physics. They evolve untill they meet a fixed point

↳ In general they connect two different fixed points including the fixed point at ∞.

$$\frac{dg}{dt} \equiv \beta_i(\{g\})$$; Condition for fixed point is $\beta_i(\{g\}) = 0$

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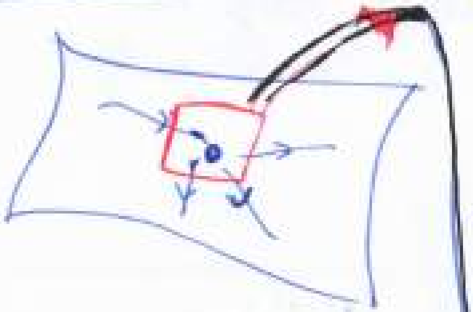
QFT of the fixed points
(QFT emerging out of fixed points) $\equiv$ Conformal Field theory (CFT)



- arrows comming in $\Rightarrow$ corresponds to irrelevant operators.
- ~~this~~ also has arrows pushing away ~~this~~

This meas if we write out Action ; it is then

$$A = A_{CFT} + \sum g_i \int \phi_i(x) \, d^d x$$

$\hookrightarrow$ CFT theory perturbed by operators.

If field ϕ_i, whose variable's conjugate to g_i is irrelevant, ie: $\Delta_i > d$
Then $g_i \to 0$ g_i goes to zero

If ϕ_i is relevant $\Delta_i < d$; then as you flow, then g_i grows & moves away from ~~this~~ initial point.

$\Delta_i > d$: Irrelevant
$\Delta_i < d$: Relevant.
$\Delta_i = d$: Marginal. } $\to$ going to higher order marginal might turn out to be marginal relevant or marginal irrelevant.

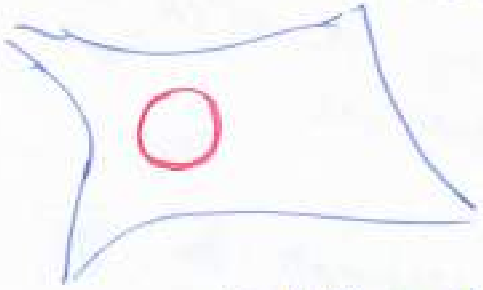
We can avoid crossing of trajectory at points which are different from fixed point.

This is physics.

Physics means, that once we are given description of a theory in thermal local variable; If we transform it, we should not have any ambiguity where the theory is going.

In general, we might have Limit Cycles

$t \to t + dt$ changing t, variable is circling around.



choose t to be Energy E.

We could ~~...~~ choose $E \frac{d}{dE}$, we can look ~~...~~ out the ~~...~~ theory respond to different change of energy.

If we have limiting cycle; it will produce very odd result in physics.

i.e: any observable $\Lambda(E)$ will be periodic with some period in energy $\Lambda(E) = \Lambda(E+T)$

☆ The only cases where operator become really marginal to all order when we have symmetry associated to it.

↳ for instance, if we have group of symmetry; then we have current.

Currents never Renormalize

because; current $J_i^{\mu}(x)$ and associated charges Q_i

ie; $J_i^{\mu}(x) \longrightarrow Q_i = \int dx\, J_i^{0}(x)$

If we have group, these charges Q_i satisfy
Commutation relation of our group of symmetry.

$[Q_i, Q_j] = i f_{ijk} Q_k$

And now we see; The fact that we have
quadratic relation under which we can express
each of them implies, this operator should always
contain the Engineering Dimension.

2d Conformal Field Theory

Shoaib Akhtar 28/7/2020 (1910)

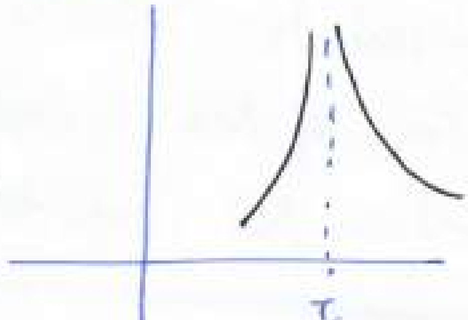
Lec 2: Critical Exponents, Functional Eqn for Free Energy, Wilson's idea.

$\frac{J}{T}, \frac{B}{T}$ or say T, B

We can ask how our correlation length diverge nearby the critical temperature.

$$\xi(T) \sim \begin{cases} \xi_0^+ \, (T - T_c)^{-\nu} \\ \xi_0^- \, (T_c - T)^{-\mu} \end{cases}$$

Correlation length has power law behavior around $T = T_c$



by thermodynamics, we can prove that $\mu = \nu$.

however ξ_0^+ & ξ_0^- are not same.

$\frac{\xi_0^+}{\xi_0^-}$ are universal numbers which is not necessarily 1. (These are fingre print)

~~exTri~~Critical Ising Model $\frac{\xi_0^+}{\xi_0^-} = 2\cos\frac{\pi}{18} = 1.28$

We also have behavior of Order parameters)

$$\langle \sigma(r)\, \sigma(0) \rangle \sim \frac{1}{r^{2\Delta}} \qquad [\sigma] = L^{-\Delta}$$

Specific heat $C = \frac{dU}{dT}$ U is Internal energy.

$$C \simeq \begin{cases} A_+ \, (T - T_c)^{-\alpha} \\ A_- \, (T_c - T)^{-\alpha} \end{cases}$$

$\frac{A_+}{A_-}$ is universal.

$$C = \frac{\partial u}{\partial T} \simeq \begin{cases} A_+ \, (T-T_c)^{-\alpha} \\ A_- \, (T_c - T)^{-\alpha} \end{cases} = \int dx \; \underbrace{\langle \varepsilon(x)\, \varepsilon(0) \rangle}$$

Two point function of the conjugate variable to the temperature which is Energy density.

~~so: $\varepsilon(x)$ and $\langle \varepsilon(x)\varepsilon(0)\rangle$ will also diverg~~

$$\chi = \frac{\partial \langle M \rangle}{\partial B} = \begin{cases} \chi_+ \, (T-T_c)^{-\gamma} \\ \chi_- \, (T_c - T)^{-\gamma} \end{cases} \sim \int dx \, \langle \sigma(x)\, \sigma(0) \rangle$$

Susceptibility

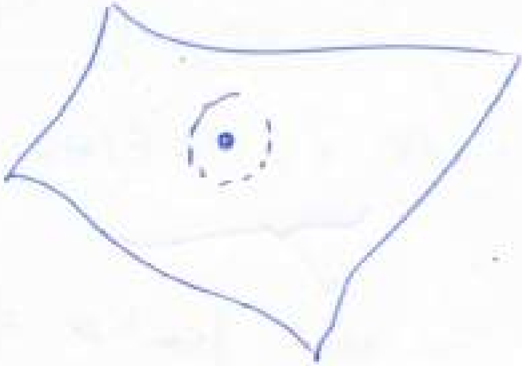
$$\langle \sigma \rangle \Big|_{B=0} = S \, (T_c - T)^{\beta}$$

$$B \Big|_{T=T_c} = \mathcal{D} \, [\langle \sigma \rangle]^{\delta}$$

Paper G.M. Fioravanti Sinnott PRE Tricritical Ising Model.

Exponents	Ising	Tricritical Ising	~~Potts~~ - Potts
α	0	28/9	11/3
β	1/8	1/24	1/9
γ	7/4	37/36	13/9
δ	15	77/3	14
ν	1	5/9	5/6
Δ	1/16	3/80	1/15

$d = 2$



Around a fixed point. We would like to ~~des~~ describe all the theories which are nearby there.

$$A = A_{CFT} + \sum_i g_i \int d^d x_i \; \varphi_i(x)$$

$\varphi_i(x)$ are scalling operator

$$x \to \frac{x}{b} \; : \; \varphi_i\left(\frac{x}{b}\right) = b^{\Delta_i} \varphi(x) \quad ; \; g_i \to g_i b^{d-\Delta_i}$$

$$Z = \int \mathcal{D}\varphi \cdot e^{-A} \quad \text{(partition function)}$$

$$Z = \int \mathcal{D}\varphi \cdot e^{-A} \equiv e^{-N \cdot f(\{g_i\})}$$

no. of blocks

free energy per unit of blocks which will depend on our couplings at ~~at the site~~ the size of our lattice scale a.

Now; we make a rescaling.

So, if ~~we~~ we do RG ~~go~~ trajectory we should have $e^{-N \cdot f(\{g_i\})} = e^{-N' \cdot f(\{g_i'\})}$

We have functional equation for free energy f.

because $N' = N b^{-d}$.

So;

$$f(\{g_i\}) = b^{-d} \cdot f(\{b^{d-\Delta_i} \cdot g_i\})$$

The solution can be given in full generality

We can keep going; and will do many iterations

↪ If we keep going, all the irrelevant operators are going to die.

We have K relevant operators in general.
(In any class of universality, the number of relevant operator is finite. The number of irrelevant operator is infinity)

We can have many different ways of writing most general solution depending on which coupling we select out to be one that at the end will be non-zero.

Select g_i which we put to zero ~~the~~ at last.

Then; $$f(\{g_j\}) = g_i^{-\frac{d}{d-\Delta_i}} \mathcal{F}\left(\frac{g_j}{g_i^{\phi_{ji}}} \cdots\right)$$

We are not able to fix.
But we know functional dependent of it.

where $$\phi_{ji} \equiv \frac{d-\Delta_j}{d-\Delta_i}$$

similar thing going here

We imagine that we have to solve wave equations

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) f = 0$$

General solution $\Rightarrow f = g_+(x-t) + g_-(x+t)$

We can ask, what is $\underbrace{\langle \sigma_j \rangle_i}$

order parameter j, when we keep coupling constant i different from 0.

$$\langle \sigma_j \rangle_i = \left.\frac{\partial f}{\partial g_j}\right|_{\substack{g_k = 0 \\ g_i \neq 0}} \propto g_i^{\frac{\Delta_i}{d-\Delta_i}}$$

$$\boxed{\langle \sigma_j \rangle_i = \left.\frac{\partial f}{\partial g_j}\right|_{\substack{g_k=0 \\ g_i \neq 0}} \propto g^{\frac{\Delta_i}{d-\Delta_i}}}$$

so; we see that $\frac{\Delta_i}{d-\Delta_i}$, which were previously our critical exponents β; we see that there is an algebraic equation for it.

~~before~~ previously we were doing
$\langle \sigma_j \rangle_i = |T_c - T|^{\beta}$

Here; g_i is playing role of $|T_c - T|$

β can be expressed in terms of Anamalous dismension.

From RG point of view; critical exponents are derived quantity from anamalous dimension.

$$\langle \sigma_j \rangle_i = \left.\frac{\partial f}{\partial g_j}\right|_{\substack{g_k=0 \\ g_i \neq 0}} = A \cdot g^{\frac{\Delta_i}{d-\Delta_i}}$$

A is some non universal number

we can identify the length dimension of A,
ie; how it depends on scaling.
And so we can construct ratio of them which are pure numbers. Every one should agree on pure numbers.

Nearby fixed point, field theory will be described by Conformal Field Theory

In any field theory, once we have all the order parameters which we label generically by φ_i

The object we care about are correlation functions $\langle \varphi_i(x_1) \dots \varphi_j(x_i) \rangle$

$$= \int \mathcal{D}\varphi \cdot e^{-S} \cdot \varphi_i \dots \varphi_j$$

Wilson idea;
we can compute $\langle \varphi_i(x_1) \dots \varphi_j(x_i) \rangle$ which are key points of the story.

If we make a hypothesis; the fact that two fields (scaling fields) ~~$\varphi_i\ \varphi_j$~~ $\varphi_i(x_1)\ \varphi_j(x_2)$ has singularity when $x_1 \to x_2$ which are catched up by power ~~law~~ law, and some coefficients

~~when $\varphi_i(x_1)\ \varphi_j(x_2)$~~ ~~$x_1 \to x_2$~~

i.e; If we make hypothesis that our field satisfy mathematics as follows

$$\varphi_i(x_1)\,\varphi_j(x_2) \underset{x_1 \to x_2}{\sim} \sum \frac{C^k_{ij}}{(x_1 - x_2)^{\Delta_i + \Delta_j - \Delta_k}} \cdot \varphi_k(x_2)$$

i.e; This Algebra, i.e; when $\frac{|x_i - x_j|}{\xi} \to 0$

Then we can compute any of $\langle \varphi_i(x_1) \dots \varphi_j(x_N) \rangle$ by systematically reducing it to 2 point functions, and collecting bunch of structure constants alongside.

~~And~~ At the end of the day; any of these $\langle \varphi_i(x_1) \dots \varphi_j(x_N) \rangle$ will be reduced to expectation value of one point $\langle \varphi_i \rangle$ which we can prove to be all zero but the identities.

$$\langle \varphi_i \rangle = \delta_{i,0}$$

<u>Lec</u> 3: Perturbing CFT Action by relevant (& irrelevant operators), Correlation function, Polyakov Theorem, Conformal Transformations, Conformal Killing equation, Mobius map, General Tensors, Scaling Operators, Quasi primary & primary operators

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G , d

symmetry group — dimension d

G

$\downarrow$ we can identify irreducible representation of this group

$\{g_1, g_2, \dots, g_k\}$

with associated fields $\{\varphi_1, \dots, \varphi_k\}$

And in terms of these, ~~...~~ we write down expressions which either satisfy group symmetry or explicitly break symmetry.

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ex $\mathbb{Z}_2$, $\varphi(x)$

The most general interaction which respect $\mathbb{Z}_2$ symmetry is $V(\varphi(x)) = V(-\varphi(x))$ in terms of even functions.

$$V(\varphi) = \sum C_{2m} \varphi^{2m}$$

These coefficients are our coupling constants which respect the symmetry

The one which breakes the symmetry is something like

$H(\varphi) = \varphi W(\varphi)$ where $W(\varphi) = W(-\varphi)$

$\Rightarrow H(-\varphi) = -H(-\varphi)$

ie: $H(\varphi) = \sum \beta_k \varphi^{2k+1}$

we have set of couplings which break the symmetry.

Fixed point is Characterized by all the β functions of the theory to be zero. $\beta_k(\{g\}) = \frac{d g_k}{dt} = 0$

$\Rightarrow \xi = \infty$ at this point.

$\xi = \infty \not\Rightarrow \beta_k(\{g\}) = 0$

☆ "Critical points are those for which $\beta_k(\{g\}) = 0 \;\; \forall k$"

example)

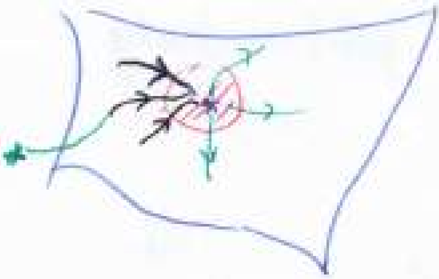
If we have massless flow between two non-trivial fixed points. Then along the trajectory the couplings are moving; ~~and~~ but $\xi = \infty$.

These critical points under very general assumption are associated to CFT

We localize our attention to one fixed point; and at that fixed point we ask what is operator content of the theory.

Say we describe our field theory in terms of Action, which we parametrize with some action of CFT, and with some couplings associated to relevant operators.

$$A = A_{CF} + \sum_{i}^{K} g_i \int \varphi_i(x)\, d^d x$$



This is description of class of universality associated to the fixed point spanned by finite no. of relevant operators.

Geometrically, this relevant operators are the unstable directions of the fixed point.
(relevant operators tell how we can get out of fixed point)
Associated to a fixed point, there is an infinite number of irrelevant operators that bring ~~us~~ us in.

↳ But for what ~~concerns~~ concerns in thermodynamics & things like that ; (Epistemological point of view) that ; The ~~...~~ irrelevant fields are comming from other fixed point where we are unstable.

↳ So, the typical approach (which is proved to be succesful) ~~for~~ is ~~to~~ to characterize field theory around one fixed point in terms of deformation of the fixed point action by a finite number of relevant operators.

We can try to characterize our action in terms of perturbing by irrelevant operators ; namely reversing the flow. !!!

In principle it can be done.
(But in absence of any further constraints which to might be something like integrability, supersymmetry, etc ∃ we are unable to get trajectory perturbatively stable)

ie; If we start doing perturbation theory w.r.t. A_{CFT} (that is within this theory; in principle we know all correlation functions)

~~interaction~~ $$A = A_{CFT} + \sum_{i=1}^{n} g_i \int \varphi_{i\,\text{relevant}}(x)\, d^d x$$

action like this is stable in perturbation theory meaning that at higher order, we might change the value of g_i to $\hat{g}_i$, but form of action remains exactly the same.

If we reverse the point of view, by perturbing the theory A_{CFT} by irrelevant operator even by one irrelevant operator

$$A = A_{CFT} + \Lambda \int \varphi_{irr}\, dx^d$$

as we start doing perturbation theory with this, The 2 point function of φ_{irr} w.r.t. A_{CFT} is divergent.

So; we have to add a new irrelevant operator at one loop term : $\hat{\varphi}_{irr}$

But these are also diverging itself; so we have to add another one. (We are not talking about higher order)

↳ This means that;
once if we perturb by irrelevant operators, then the action with which we even start with is unstable. To make it stable perturbatively; we need to add infinite number of counter terms, and fixing the normalization condition.

$$A = A_{CFT} + \Lambda\int\varphi_{irr} + \delta\int\varphi_{irr} + \dots$$

↳ This means; in absence of prescription of how we can fix these infinite number's of terms; The theory will not predict anything. (Because we need infinite set of condition to fix the parameters)

⟹ The only point, where this point of view might be useful if we know the trajectory we want to go through is integrable trajectory (means, its stable)

and hence there is only one way to keep the theory integrable perturbatively ⟹ & therefore this speaks uniquely the parameters.

$$\boxed{\langle \varphi_\Delta(x)\,\varphi_\Delta(0)\rangle = \frac{1}{r^{2\Delta}}\, f(r/\xi)}$$

↳ This is the most general from of 2pt function of a scaling operator of dimension Δ.

$$\langle \varphi_\Delta(x)\, \varphi_\Delta(0) \rangle = \frac{1}{r^{2\Delta}} f(r/\xi)$$

$r = |x|$

This gives pure scaling behavior

This is constant during scaling.

we can either think of (two ways of interpreting)

- $r << \xi$ (where ξ finite)
- or $\xi = \infty$, and r whatever.

When we keep ξ finite, but large: here; we think of being nearby fixed point (not sitting on fixed point; but sitting slightly away). Our Conformal Field Theory is ruling UV behavior of the theory ; CFT fully control the UV behavior of the theory, and somewhat ambitious to tell us all about how the theory behaves at short distances

~~too~~ We can also think of sitting on the point, then $\xi = 0$, and therefore $f(r/\xi)$ is absent (actually a constant $f(r/\xi)|_{\xi\to\infty} = f(0)$)

constant.

$\cancel{\langle \varphi_\Delta(x_1) \varphi_\Delta(x_2) \rangle} =$

Imagine we have $\mathcal{L}$agrangian to ~~desist~~ describe criticality. $A = \int \mathcal{L}\, d^dx$

$\mathcal{L}$ should have term like $(\partial\Phi)^2$.

Once we have $(\partial\phi)^2$. We are stuck.

A is pure number in appropriate unit.

d^dx is volume (volume is never renormalized)

$(\partial\varphi)^2$ are derivatives. So φ has well defined ~~sta~~ scaling behavior.

So: $[\phi] = L$ (some L)

So; The field which ~~are~~ we are talking about has very unique things like L.

but if $[\phi] = L$ is scalling behavior of this.

Then $\langle \phi(x)\ \phi(0)\rangle = \frac{1}{r^{(d-2)}}$ cannot have any other relation than this.

Then how can we have $\langle \phi(x)\phi(0)\rangle = \frac{1}{r^{2\Delta}} f(r/\xi)$??

The only way out is that the theory has some hidden scale.

(microscopic lattice or high energy cut off)

such that we can put something which can absorb dimen $(d-2)$; and then add $\frac{1}{r^{2\Delta}}$

$$\langle \phi(x)\ \phi(0)\rangle = \frac{\Lambda}{2^{(d-2)}} \cdot \frac{1}{2^{2\Delta}}$$

we can make out the power.

We do not have anomalogous behavior, if we dont have divergences. There fore underlying cut off has to be there for very good reason; although to simplify, we put $a=1$

RG 24

i.e. we actually have

$$\langle \varphi_\Delta(x)\, \varphi_\Delta(0) \rangle = \left(\frac{a}{r}\right)^{2\Delta} \cdot f\left(\frac{r}{\xi}\right) \cdot a^{-(d-2)}$$

and set $a=1$.

$$\langle \varphi_\Delta(x_i)\, \varphi_\Delta(x_j) \rangle = \left(\frac{a}{|x_i - x_j|}\right)^{2\Delta} \cdot f\left(\frac{|x_i - x_j|}{\xi}\right) \cdot a^{-(d-2)}$$

set $a=1$

If we want to describe CFT, we are sitting on one of the fixed points ; so the extra functions go away. They are just constants, which are fixed by renormalization of the field φ_Δ ; and so from now on we gonna take it 1.

Solve the dynamics of the fixed point(s)

fixed point intrinsically ~~or~~ are strong coupled theory because the degrees of freedom of a theory dynamically ~~...~~ is ξ^d. ξ is ∞ at fixed point

so; This theory is ∞ coupled theory.

This theory acquires scale invariance $x \to \lambda x$ because, since everything depends on $\frac{(\text{distance})}{\xi}$ and $\xi = \infty$; so we can rescale our distances by any factor we want.

So, the fixed point is Dilatation invariant : $x \to \lambda x$

We can solve theory, under a result due to Polyakov;

" If we have a theory, which is translational invariant, rotational invariant, local and invariant under dilatation.

Then the theory is invariant under Conformal Transformations "

---

lets imagine we have correlation function of several fields $\langle \varphi_i \dots \varphi_j \rangle$ which is given as path integral. $\langle \varphi_i \dots \varphi_j \rangle = \int \mathcal{D}\Phi \, e^{-S[\Phi]} \varphi_i \cdots \varphi_j$

$\left.\begin{array}{l} \text{- local} \\ \text{- Translation} \\ \text{- } \cancel{\text{loca}} \text{ Rotation} \\ \text{- Dilatation} \end{array}\right\}$ hypothesis of Polyakov.

The fact that theory is local;

So; if we ~~to~~ make a change of our coordinate

$x^a \to x^a + \xi^a(x) \iff \delta S \equiv -\int T_{\mu\nu}(x) \partial^\mu \xi^\nu$

(The fact that theory is local, is much as saying that there is a field associated to change of these coordinates; which is stress energy tensor.)

local object

locality means ~~δS~~ $\delta S \equiv -\int T_{\mu\nu} \partial^\mu \xi^\nu$

From General Relativity; we know that the response of a theory to a change of coordinate is dictated by Stress energy tensor.

So, Stress Energy tensor is nothing else than the variation of action with respect to the change of coordinate.

Translation invariance $x^a \rightarrow x^a + \underset{\text{constant}}{\varepsilon^a}$

This implies $\partial_\mu T^{\mu\nu} = 0$.

$T^{\mu\nu}$ is conserved quantity.
(& conserved charges are Hamiltonian & Momentum of the fields)

Rotational invariance $x^a \longleftrightarrow x^a + \omega^{ab} x^b$

infinitesimal antisymmetric.

$X_a \longrightarrow X_a + \omega_a{}^b X_b$ $\omega_{ab} = -\omega_{ba}$

Then $\delta S = \int T^{\mu\nu}\omega_{\mu\nu}$ & so $\int T^{\mu\nu}\omega_{\mu\nu} = 0$

The way to realize this is that $T^{\mu\nu}$ is symmetric.

Dilatation $x^a \longrightarrow x^a + \lambda x^a \ \ni \ T^\mu{}_\mu = 0$

because $\delta S = -\int T_{\mu\nu}(x)\partial^\mu \xi^\nu$ → This now results in computing trace.

local : $x^a \to x^a + \xi^a(x) \iff \delta S = -\int T_{\mu\nu}(x)\, \partial^\mu \xi^\nu$

Translational: $x^a \to x^a + \epsilon^a$ then $\partial_\mu T^{\mu\nu} = 0$

Rotational: $x^a \to \omega^a{}_b x^b$ then $T^{\mu\nu} = T^{\nu\mu}$

Dilatation: $x^a \to x^a + \lambda a^a$ then $T^\mu{}_\mu = 0$.

---

locality : existence of $T_{\mu\nu}(x)$

Translation : $\partial_\mu T^{\mu\nu} = 0$

Rotation : $T^{\mu\nu} = T^{\nu\mu}$

Dilatation : $T^\mu{}_\mu = 0$.

If our $T_{\mu\nu}$ satisfies this constraint, then the theory is invariant under larger set of coordinate transformations which full fill

$$\boxed{\partial^\mu \xi^\nu + \partial^\nu \xi^\mu = \frac{2}{d} g^{\mu\nu} \cdot (\partial \cdot \xi)}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g_{\mu\nu} \left(\frac{dx^\mu}{dx^a}\right)\left(\frac{dx^\nu}{dx^b}\right) dx^a dx^b$$

ie; $\hat{g} = g\,(\;)(\;)$

new metric

Jacobian factors

old metric.

Suppose: we impose that the new metric is just rescaling of the old by a factor called Weyl Factor.

Impose $\hat{g} = g \cdot \rho(x)$

imposing $\hat{g} = g \cdot \rho(x)$

implies that ~~under~~ the transformation $x^a \to x^a + \xi^a(x)$

has to satisfy

ie:

$$\partial^\mu \xi^\nu + \partial^\nu \xi^\mu = \frac{2}{d} g^{\mu\nu} \cdot (\partial \cdot \xi)$$

ie: $\partial^\mu \xi^\nu + \partial^\nu \xi^\mu \propto g^{\mu\nu}$

& the proportionality constant is fixed by taking trace.

~~when the do for flat sps~~

go to flat space.

Conformal transformations are those which satisfy $\underbrace{\partial^\mu \xi^\nu + \partial^\nu \xi^\mu = \delta^{\mu\nu} \cdot \left[\frac{2}{d} (\vec{\partial} \cdot \vec{\xi})\right]}$

This is ~~a~~ differentially equation which characterized geometrically ~~the~~ conformal transformation.

If we have transformation of type $\partial^\mu \xi^\nu + \partial^\nu \xi^\mu = \delta^{\mu\nu} \cdot \frac{2}{d} (\partial \cdot \xi)$

Then Action is invariant

$$\delta S = \int T_{\mu\nu} \cdot \partial^\mu \xi^\nu = \frac{1}{2} \int T_{\mu\nu} (\partial^\mu \xi^\nu + \partial^\nu \xi^\mu) = \frac{1}{2} \int T_{\mu\nu} \cdot \frac{2}{d} g^{\mu\nu} \partial \cdot \xi$$

$$= \frac{1}{d} \int T^\mu{}_\mu \cdot (\partial \cdot \xi) = \int 0 \cdot \partial \cdot \xi = 0.$$

The most general solution of Conformal Killing equation is.

$$\left\{\begin{array}{ll} x'_i = \Lambda_i^{\ k} x_k + a_i & \to \text{~~translation~~ translation} \quad \to \text{Lorentz transformation} \\ x_i' = \lambda x_i & \to \text{Dilatation} \\ \dfrac{x_i'}{|x_i'|^2} = \dfrac{x_i}{|x_i|^2} + b_i & \to \text{Special Conformal transformation} \end{array}\right.$$

(~~take → shrink → add → invert~~)

SCT (Special Conformal Transformation)

take vector → Invert (shrink) → add → Invert

Translation : $\frac{d(d-1)}{2} + d$

Dilatation : 1

SCT : d → $\frac{(d+1)(d+2)}{2}$ = # of parameters associated to Conformal group.

$$\text{SCT} : \boxed{\frac{x_i'}{|x_i'|^2} = \frac{x_i}{|x_i|^2} + b_i}$$

So; we began with a_i, Λ_{ik}, λ,
and we got the other one b_i

So; Message in a nutshell : We have enlarged our symmetry by d.

In 2 dimensions

$$2\,\partial^1\xi^1 = 1\cdot(\partial^1\xi^1 + \partial^2\xi^2) \qquad \mu=1,\ \nu=1$$

$$\Rightarrow \boxed{\partial^1\xi^1 = \partial^2\xi^2}$$

if $\mu=1, \nu=2$; Then $\partial^1\xi^2 + \partial^2\xi^1 = 0$

$$\boxed{\partial^1\xi^2 = -\partial^2\xi^1}$$

If we define $\begin{cases} z = x^1 + i x^2 \\ \xi(z) = \xi_1(z) + i\,\xi_2(z) \end{cases}$

Then we realize that Conformal Transformations (CT) in 2 dimensions; collapse to Cauchy Riemann equation for analytic function.

C.T. in 2 dimensions are given by general analytic function $f(z)$.

We know that, space of analytic function is in one to one correspondence with Cauchy Laurent coefficients $f(z) = \sum_{-\infty}^{+\infty} a_m \cdot \frac{1}{z^m}$

(or $f(z) = \sum_{-\infty}^{\infty} a_m \cdot \frac{1}{(z-z_0)^m}$; taking general point about which we are writing Laurent Expansion)

$$\boxed{f(z) = \sum_{-\infty}^{+\infty} a_m \cdot \frac{1}{(z-z_0)^m}}$$

Now we are in trouble.
Because space of conformal symmetry was finite dimensional. $\frac{(d+1)(d+2)}{2}$

in $d=2$; conformal group has 6 parameters.

But here; we have ∞ no. of parameters a_m because any arbitrary analytic function can be expanded as Laurent series; so a_m is also arbitrary as well.

$\rightsquigarrow$ And then we are telling that theory has to be invariant under $f(z)$; which means under ∞ no. of parameters !!

BAD NEWS: We are cheating somewhere?

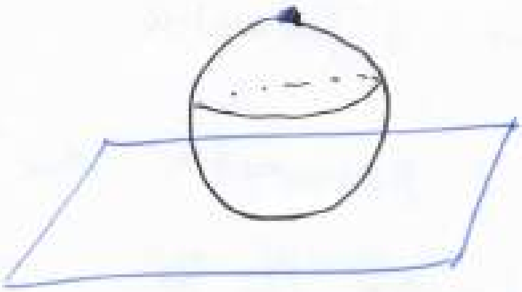
What is the only conformal transformation in 2d which is small everywhere. $\longrightarrow$ This has to be true everywhere (we did not told where it has to be true $\Rightarrow$ has to be true everywhere)

What is analytic function which is small everywhere? It is constant (Liouville's Theorem)

function f will explode ~~at ∞~~ somewhere.

The best we can do is:
We can say that our functions might explode at ∞.
(Somehow we put under carpet all bad things at ∞)

But these points is on same ~~footings~~ footings as other if we use Reiman ~~space~~ sphere (all points at ∞ get identified)



lets take the view that our complex plane is compactified to sphere..

The most we can do in order to have conformal transformation that is one to one define everywhere & localy is infinitisimal is

Mobius Transformation.

$f(z)$

we take only these

$$W(z) = \frac{az+b}{cz+d}$$ Mobius Transformation.

ex) we reject z^3 .... it is so nice, ... a polynomial.

Why? because when we invert, we have branch cut at origin; Therefore is not unique.

The new plane is related to old one by n sheets transformation. (not one to one)

The function which we are using in Polyakov argument has to be 1 - 1.

$S_R \Rightarrow$ Reiman sphere.

$$S_R \longrightarrow S_R \quad \text{1-1}$$

There is only one class of function which does so.

Mobius Maps (1) Pg 33

So: ~~[illegible]~~ "Global conformal transformation which rely on prove of Polyakov strings are Mobius maps; not the general holomorphic function"

If we use the general holomorphic function; action will no longer be invariant. It will change. And the way it change is controlled by "Conformal Ward Identity".

$$w(z) = \frac{az+b}{cz+d}$$ } How many parameters mobius has; it has 6 real parameters.

We can see; The set of mobius transformations forms a group.

$$\begin{tikzcd} z \arrow[r, "M_1"] \arrow[rr, bend right, "M_3"'] & v \arrow[r, "M_2"] & u \end{tikzcd}$$

Matrix associated to $w(z) = \frac{az+b}{cz+d}$ is $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

We use usual multiplication of matrices to get the tranformation (group).

Composition law are those 2x2 matrices.

We can choose $\det(M)$ to be 1; ie $\boxed{ad-bc=1}$

~~[illegible]~~. We see that, same mobius transformation is associated if we uptade each parameter by λ

$$\cancel{w(z) = \frac{\lambda a z + \lambda b}{\lambda c z + d}}$$

$$w(z) = \frac{\lambda a z + \lambda b}{\lambda c z + \lambda b} = \frac{az + b}{cz + d}.$$

↪ so; we can fix $\det(M) = 1$ without loss of generality.

Conformal Killing Equation gave holomorphic function as solution. But we showed only Mobius map is symmetry.

Under other Holomorphic function; δS will not be zero.

$$\delta S = -\int T_{\mu\nu}(x)\, \partial^{\mu} \xi^{\nu}\, d^d x$$

~~if~~ If ξ^{μ} fullfill Killing equation, then $\delta S = 0$ & it is symmetry.

Now we say: ξ^{μ} locally satisfy Cauchy Reimann, (appear to be of Killing form) ; But cannot be a symmetry : because these are not infinitisimal (so this formula will not hold)

so; $\delta S \neq 0$

Can we find δS ?

And this will be argument of Conformal Ward Identity.

In field theory, each time we have Ward identities for a symmetry which are broken,

but it is reestablished by an equation that tells us how to control it.

We cant just throw away general Holomorphic functions.

Polyakov is satisfied by Mobius.

(Holomorphic) \ (Mobius) spoil Polyakov theorem;

so it is not invariance; but we can find δS.

And when we know what δS is; we are able to control it. ✓

Conformal Invariance

2 point & 3 point functions of (quasi)- primary operators are fixed !!!

↳ In this result; there is an assumption that, ① nearby a fixed point, there exist an ∞ no. of Scaling Operators around each point.

$$\boxed{\varphi_i(\lambda x) = \lambda^{-\Delta_i} \varphi_i(x)}$$

Scaling Operator.

Scaling operators are eigen-values of dilatation operator.

ie; There is a object D; (Dilatation operator)

Once we specify the critical point, it will give us the spectrum.

Spectrum of D is $\{\Delta_i\}$ at a specific point.

② Scaling operator forms a basis.

ie; any operator $\mathcal{O}(x)$ can be expanded in terms of scaling operators

$$\mathcal{O}(x) = \sum_{i=1}^{\infty} g_i \, \mathcal{O}_i(x)$$

Assumption

where $\mathcal{O}_i(x)$ are scalling operators.

We have to generalize the concept of tensor.

Mobius $w(z) = \dfrac{az+b}{cz+d}$

Generic holomorphic $f(z) = \sum_{-\infty}^{+\infty} a_n z^n$

Tensor in field theory is some quantity which has index

such that $G_{\mu\nu\rho\tau\dots} \, dx^\mu dx^\nu dx^\rho dx^\tau \dots$

is scalar.

ie; $\hat{G}_{\mu\nu\rho\tau\dots} (dx')^\mu (dx')^\nu (dx')^\rho \dots = G_{\mu\nu\rho\tau\dots} dx^\mu dx^\nu \dots$

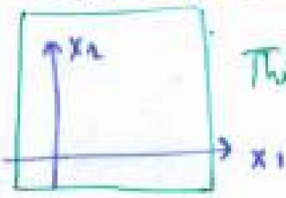
identity

which is just saying that

$$G_{\mu\nu\rho\tau\dots} = \hat{G}_{abc\dots} \left(\frac{dx'^a}{dx^\mu}\right)\left(\frac{dx'^b}{dx^\nu}\right)\dots$$

We generalize this Notion in following sense.

In 2 dimensions we have two real coordinates x^1, x^2



This is our physical space (say we want to study Ising model here ...)

Collect x^1, x^2 in complex variable

$x^1, x^2 \longrightarrow$ ~~$z = x^1 + ix^2$~~ $z = x^1 + i x^2$

Then define $\bar{z} = x^1 - i x^2$

Now; we pretend z, and $\bar{z}$ are independent variable.

ie; from now on; we pretend that;

we have a theory whose defining terms are two coordinates z and $\bar{z}$.

(A priori, $\bar{z}$ has nothing to do with z)

$(z, \bar{z})$ is $\mathbb{C}^2$ theory.

Our physical Theory is when we identify $\bar{z}$ with complex conjugation z^*

ie: $\bar{z} = z^*$ ie: $\mathbb{C}^2/_{(\bar{z} = z^*)}$

~~Now define (Quasi Primary field)~~

~~Quasi~~ We define <u>Quasi Primary fields</u> in tensor notations with weight Δ and $\bar{\Delta}$ which under mobius transformations transforms as follows:

($\bar{\Delta}$ is weight for dilatation of $\bar{D}$ part.

Δ is weight for dilatation of D part)

~~Φ~~

~~$\Phi_{\Delta,\bar{\Delta}}(z,\bar{z})(dz)^{\Delta}(d\bar{z})^{\bar{\Delta}} = \Phi_{\Delta,\bar{\Delta}}(z', \bar{z}')(dz')^{\Delta}($~~

$$\phi_{\Delta,\bar{\Delta}}(z,\bar{z})(dz)^{\Delta}(d\bar{z})^{\bar{\Delta}} = \hat{\phi}_{\Delta,\bar{\Delta}}(z', \bar{z}') \cdot (dz')^{\Delta}(d\bar{z}')^{\bar{\Delta}}$$

~~Scalar~~ Quasi Primary are associated to a ~~S~~ Scalar quantities which is the field labelled by $\Delta, \bar{\Delta}$ multiplied by infinitesimal volume dz, $d\bar{z}$ but raised to power $\Delta, \bar{\Delta}$ respectively.

A Quasi-Primary operator is that operator; that under only Mobius ~~of~~ ~~only~~ transforms as follows.

$$\boxed{\Phi_{\Delta,\bar{\Delta}}(z,\bar{z}) \longrightarrow \left(\frac{dw}{dz}\right)^{\Delta}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{\Delta}} \Phi_{\Delta,\bar{\Delta}}(w(z),\bar{w}(\bar{z}))}$$

Quasi Primary operators are Scalling operators.

Primary operator are those which transforms as tensor of order $\Delta, \bar{\Delta}$ under a generic analytic transformations

$$\Phi_{\Delta,\bar{\Delta}}(z,\bar{z}) = \left(\frac{df}{dz}\right)^{\Delta}\left(\frac{d\bar{f}}{d\bar{z}}\right)^{\bar{\Delta}} \phi_{\Delta,\Delta}(f,\bar{f})$$

$\hookrightarrow$ This is also a scalling operator.

Lec 4: Conformal Algebra, OPE Algebra, Virasoro Algebra, Ramanujam Partion Formula.

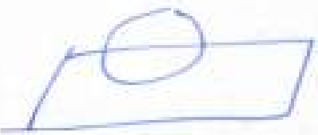
In 2d $\quad z \to f(z) = \sum_{-\infty}^{+\infty} a_n z^n$

$$\boxed{z \to w(z) = \frac{az+b}{cz+d}}$$ Riemann $\longleftrightarrow$ Riemann

$\to$ This functions diverge;

i.e. diverge at $z = -d/c$

(but this is sort of fake divergence because we can make a change or transformation by going around ∞, and redo the stuff again)

(roll the ball on plane )

no point is special.

Mobius map has some good properties

- Circle maps to circle
- line " " line.
- Symmetric point w.r.t. circle are preserved in the mapping. (If the coefficients are integer, this will constitute the so called modular group)

Observe that; with these transformations we can associate some ~~[illegible]~~ algebras (actually ∞ dimensional algebra)

Its useful to introduce two set of coordinates

$z = x + iy \quad , \quad \bar{z} = x - iy = z^*$

$(z, \bar{z}) \in \mathbb{C}^2$

Physical $\mathbb{C}^2/_{(z,\bar{z})}$

~~comm~~ "In conformal field theory, z and $\bar{z}$ are decoupled. At least algebraicly" $\Rightarrow$ will be justified when discussing Ward Identities.

Physically they are not decoupled.

For all algebraic manupulations, its like we are dealing with two copies of the same system in a certain way.

---

let us have z, $\bar{z}$ variable.

And we want to into holomorphic transformations into an algebra.

"Namely if we are going to apply it on set of functions: how this transformation is going to it".

The way of doing this is;

do some infinetisimal transformations

$$z \to z' + \varepsilon_m(z)$$

$$\varepsilon_m(z) \equiv -z^{m+1}$$ (This picks up one of the direction in which we can move our stuff)

Therefore, we will impose an operator $l_m = -z^{m+1}\partial_z$

as well as $\bar{l}_m = -\bar{z}^{m+1}\partial_{\bar{z}}$

l_m, $\bar{l}_m$ are meant to act on functions.

$$[l_m, l_n] = (m - n)\, l_{m+n}$$
$$[\bar{l}_m, \bar{l}_n] = (m - n)\, \bar{l}_{m+n}$$
$$[l_n, \bar{l}_m] = 0$$

(* *) Commutator relation

(We are dealing with ∞^{dim} algebra; because the no. of generatorms $l_m, \bar{l}_m$ are infinite.

CONFORMAL ALGEBRA

CONFORMAL

proof) $g(z)$ some function

then do the transformation $z \to z'$

Then $g(z) = g(z' + \varepsilon_m(z)) \simeq g(z) + \varepsilon_m \partial_z g(z)$

so; $g(z) \simeq g(z) + \varepsilon_m \cdot \partial_z g(z)$ } This is how the function infinitisimally transforms.

So: The action on set of function of infinitisimal analytic transformations are associated to these ε_m ; ε_m is any of the power law

ie: $\varepsilon_m(z) = -z^{m+1}$

Therefore, we see that if we do

$$-z^{n+1}\partial_z(-z^{m+1}\partial_z)\, g$$
$$-\left(-z^{m+1}\partial_z(-z^{n+1}\partial_z)\right) g$$
$$= (n+m)\, l_{n+m}\, g$$

$\varphi(x,y)$: Order parameter which originally & physically will depend on two coordinates x and y.

$\varphi(x,y) \Rightarrow \varphi(z,\bar{z})$ → Think in this form.

$\varphi(z,\bar{z})$ have certain property if we transform z part only or $\bar{z}$ part only.

If we combine the two algebra $(l_n + \bar{l}_n)$ in a symmetric way. and also $(l_0 + \bar{l}_0)$

- ★ Eigenvalues of $(l_0 + \bar{l}_0)$ corresponds to Anomalous dimension.
- ★ " " $(l_0 - \bar{l}_0)$ " " spin of the field.

Among the infinite no. of operators l_m, $\bar{l}_n$; can we identify those which corresponds to Mobius.

We can associate vector fields ; $V(z) = -\sum_{-\infty}^{+\infty} a_m \cdot z^{m+1} \partial_z$

Question) For which coefficients $\{a_m\}$ which makes the vector field $V(z)$ regular at the origin.

In order to be regular at the origin, we have to remove all a_m for $m < -1$

Field regular at origin $\Rightarrow m \geq -1$

we can do; $V(z) = -\sum_{-\infty}^{+\infty} a_m (z-z_0)^{m+1} \partial_z$

and can ask for regularity at generic z_0.

What about ∞; we change chart ...

... we can go around infinity by $1/z$ map.

$$z \longmapsto -\frac{1}{w}$$

So: $V(w) = \sum a_m \cdot \left(\frac{-1}{w}\right)^{m-1} \cdot \partial_w$

vector field around ∞

note ∂_z also transforms.

If we want V to regular at ∞;

Then we have to impose $w \leq 1$

So; The only field which is regular everywhere is

l_0, l_+, l_{-1}.

l_0 associated to Dilatation.

Out of ∞ set of operators $l_m, \bar{l}_n$.

There are three of them $\{l_0, l_{+1}, l_{-1}\}$ which have following properties:

① They close a subalgebra.

$[l_n, l_m] = (n-m)\, l_{n+m}$

$m, n \in \{0, \pm 1\}$ It will form sub algebr.

$$\boxed{\begin{aligned} [l_1, l_{-1}] &= 2 l_0 \\ [l_0, l_{\pm 1}] &= \mp l_{\pm 1} \end{aligned}}$$

Claim : l_0 is generator associated to Dilatation

because action of l_0 is $l_0 \longrightarrow -z \partial_z$

ie; $z \longrightarrow (1+\lambda) z$

l_0 will implement the transformation on functions $f(z)$ under $z \to (1+\lambda) z$

$l_{-1} = -\partial_z$: Associated to Translation

$l_1 = -z^2 \partial_z$: Associated to Special Conformal Transformation.

we realize that, l_1 is the extra transformation which enlarged the original transformation to a conformal group.

Till now everything is Classical.

At quantum level we have to add something on RHS of the algebra as displayed on (page 41), ie; equation (**)

↳ This will become the anamaly.

Dynamics

Experimental fact : Nearby fixed points, we have power law divergences

(for critical phenomenon)

Our dynamics will contain Scalling fields

$$\varphi_i(\lambda x) = \lambda^{-\Delta_i}\varphi_i(x) \qquad \text{Refination.}$$

Any field $\Psi(x)$ near any point x in Space.

$$\Psi(x) = \sum a_m \varphi_m(x)$$

$\varphi_m(x)$ forms basis (Its a hypothesis)

This implies that

$$\langle \ldots \Psi(x) \ldots \rangle = \sum_m \langle \ldots \varphi_m \ldots \rangle a_m$$

first assumption.

This is actually weak identity.

$\Psi(x) = \sum a_m \varphi_m(x)$ This is operatorial identity

it can be false.

It is true only in correlation functions... Weak identity.

What happens if $A(x)B(y)$

$\cdot x$

$\cdot y$

At critical point, we can stretch any seperation $|x-y| = d$ to the length we like, but 0.

$d \longrightarrow \lambda d$ (symmetry of the theory)

So, even if x & y are very far apart; by symmetry of the theory we can bring them very near.

so; we can bring x & y very near.

$A(x)\, B(y)$

This means that B local object under any purpose should behave as a local object.

so; $A(x)\, B(y)$ is a local object.

Say $$\underbrace{A(x)}_{\text{local}}\ \underbrace{B(y)}_{\text{local}} = \underbrace{C\left(\frac{x+y}{2}\right)}_{\text{local}} \ *$$

Now; consider
$$\varphi_n(x)\, \varphi_m(x) = \sum_k C^k_{n\,m}(x,y)\, \varphi_k\left(\frac{x+y}{2}\right)$$

The story of CFT is in these $C^k_{nm}(x,y)$ relations.

$A(x)\, B(y) = C\left(\frac{x+y}{2}\right)$

we say that locality of B is a fake thing; because we can shrink the difference $|x-y|$ as small as we want.

so; $A(x)\, B(y)$ under any purpose is local object.

If we want to be fair to both x & y; we say that it is localized in the middle $\frac{x+y}{2}$.

At the end, we will place $\frac{x+y}{2}$

with y, because $\frac{x+y}{2} = y + \eta$
↖ some displacement

but to get the displacement η; we can go from y to $y+\eta$ by L_{-1}.

So; without loss of generality;

we can write $A(x)\ B(y) = C(y)$

Note] $A(x)\ B(y) = C(y)$
→ This is not an equation.
It is a concept!!!

From symmetry, and completeness of the basis; we got the relation.

$$\underbrace{\varphi_n(x)\,\varphi_m(y) = \sum_k C^k_{n,m}(x,y)\,\varphi_k(y)}$$

Think of this equation as correlation function equation.

$(\ , \ ) = (\ )$

hence we get Algebra.

→ This is called OPE (Operator Product Expansion) Algebra

---

① Translation Invariance $\Rightarrow$ $C^k_{nm}(x,y) = C^k_{n,m}(|x-y|)$

② Scalling Invariance $\Rightarrow$ $C^k_{n,m}(x,y) = \dfrac{C^k_{n,m}}{|x-y|^{\Delta_n+\Delta_m-\Delta_k}}$

C^k_{nm} are constant coefficients.

↳ These are pure numbers.

(C^k_{nm} , Δ_i) are dynamical data of the theory which we have to compute.

Summary |

- Basis of scalling fields $\varphi_m(x)$ with scalling dim. Δ_m
- $\varphi_n(x)\, \varphi_m(y) = \sum \frac{C^k_{nm}}{|x-y|^{\Delta_n+\Delta_m-\Delta_k}} \cdot \varphi_k(y)$

Any CFT is solved by finding $\{\{\Delta m\}, \{C^k_{nm}\}\}$

CFT is solved; if we know how to compute generetic correlation function $\langle A(x) \cdots \cdots M(y) \rangle$

If we have the datas $\{\{\Delta m\}, \{C^k_{nm}\}\}$ it can be done. ✓

Analogy with SU(2)	OPE Algebra
here $S\frac{1}{2}$, $3/2$, whatever it is	Here the role is played by Δm
Clepshgordon coefficients	C^k_{nm}

There is also difference; since in OPE we are dealing with ∞ dimensional Algebra. and dealing with something which is point wise dependent.

Analogy/ for any CFT, if we want to classify the phase transition:

~~The~~ All different manifestation in nature of phase transition is nothing else than different representation of OPE algebra (Any different class of universality will be different realization; but on the same Algebra)

---

for an algebra we typically have $[I_n, I_m] = i f_{nm}^{k} I_k$

ex1 $[S_i, S_j] = \dots$; We choose one operator which we declare that in any representation it will be diagonal.

$i = 1, 2, 3$

SU(2)

usually we do S_z.

i.e. $S_z \sim \begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix} \updownarrow$ finite matrix.

In our case; we choose L_0 and diagonalize it. Since L_0 is the dilatation operator; then Δ will be eigenvalues of L_0.

In SU(2) we make linear combination, and get S_+, S_- operators.

and we find $[S_z, S_\pm] = \pm S_\pm$.

In SU(2)

$S_- |hw\rangle = 0$

← heighest weight vector

Ex In SU(2) ;

we can either start from bottom; and generate all vectors

or " " " " , top : " " " " :

let me choose as a basic vector, the lowest one.

These are called Highest Weight Vectors (HW)

↳ ie; once we get this vector ; we can construct freely the irreducible representation by keeping acting with S_+.

↳ ~~and~~ and once we collect all the vector along the way ; This is the irreducible representation.

At Quantum level, the Algebra we are dealing with is called Virasoro Algebra.

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{C}{12} n (n^2-1) \delta_{n+m,0}$$

$\boxed{[L_0, L_m] = -m L_m}$ if we act on a vector which is eigenvector of L_0; with L_m it will shift eigenvalue of L_0 by $-m$.

ie; imagine $L_0 |\Delta\rangle = \Delta |\Delta\rangle$

Then $L_m L_0 = \Delta L_m |\Delta\rangle$

want to understand what is $L_m |\Delta\rangle$.

ie; $L_m |\Delta\rangle$ } ⇒ This is eigenvector of L_0, with eigenvalue $(\Delta - m)$.

Proof/ $L_0 L_m |\Delta\rangle = (L_m L_0 - m L_m)|\Delta\rangle$
$= (\Delta - m) L_m |\Delta\rangle$

SU(2)

$S_-|hw\rangle = 0$

~~case~~ Our case

$L_m |\Delta\rangle = 0 \quad , \quad m \geq 1$

L_m
$m \leq 0$
builts the representation

all the ~~negatives~~ positive annhilate.

ie: we dont have problem of getting arbitrarily down.

arbitrarily down means very drastic for physics !!

recall $\langle \varphi \varphi \rangle = \dfrac{1}{|x|^{2\Delta}}$

if $\Delta > 0$: physics is OK. because correlation has to decay.

if $\Delta < 0$; then field get more & more correlated when we seperate them !!! (Does not work)

This is why we built  this

and not 

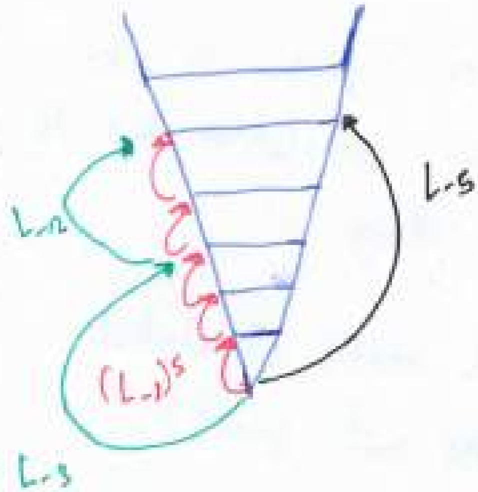
The no. of Δ can be finite; but no. of vector that we build up on any Δ will be infinite.

This will be all the theories about; on ~~...~~ this representation of ~~...~~ building up, how they interact, how they decompose, so forth.

Δ parametrizes the irreducible representation that will build up in terms of object; which are eigenvalues of L_0

i.e; $L_0|\Delta\rangle = \Delta|\Delta\rangle$ and $L_m|\Delta\rangle = 0$, $m \geq 1$

and rest of representation will be building up acting on Δ in all possible no. of ways $\rightarrow$ This no. of ways; is a combinatorial problem.



This combinatorial problem was solved by Ramanujan with Hardy

Ramanujan's Partition Formula

$$p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \cdot \frac{d}{dn}\left(\frac{1}{\sqrt{n-\frac{1}{24}}} \exp\left[\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right]\right)$$

where $A_k(n) = \sum_{0 \leq m < k,\ (m,k)=1} e^{\pi i\left(s(m,k) - 2nm/k\right)}$

Asymptotic expression for $p(n)$

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \rightarrow \infty$$

$P(n)$ will be degeneracy of n^{th} level

and it grows exponentially...

.... given by Ramanujan $P(n)$ formula.

Ex] lets reach the 3^{rd} floor!

L_{-3} , $L_{-1}L_{-2}$, $L_{-2}L_{-1}$

Is $L_{-1}L_{-2}$ and $L_{-2}L_{-1}$ really independent? No

$$L_{-1-2} = L_{-2}L_{-1} + \underbrace{[L_{-1}, L_{-2}]}_{\text{This is } L_{-3}}$$

So; we have to select out in all possible ways the ones which are independent.

The rule is $L_{-n_1} \cdots L_{-n_k}$ s.t. $\sum n_i = N$ where N is the level we want to reach.

and $n_1 \le n_2 \le \ldots$

(or opposite ordering.)

$$\sum_{N=0}^{\infty} P(N)\, q^N = \prod_{k=1}^{\infty} \frac{1}{1-q^k}$$

2d Conformal Field Theory

Shoaib Akhtar 29/7/2020 Pg 54

Lec 5: (Quasi-)Primary fields, 3pt func, 4pt func, Radial Quantization, Ward Identity, Central Charge, Deriving Quantum version of Conformal Algebra: The Virasoro Algebra

$O(x) = \sum \beta_m \varphi_m(x)$

$$\varphi_m(x)\varphi_n(x) = \sum \frac{C^k_{mn}}{|x-y|^{\Delta_m+\Delta_n-\Delta_k}} \cdot \varphi_k(y)$$

$\langle \varphi_k(x) \rangle = 0$ unless $\varphi_k(x) = \mathbb{1}$

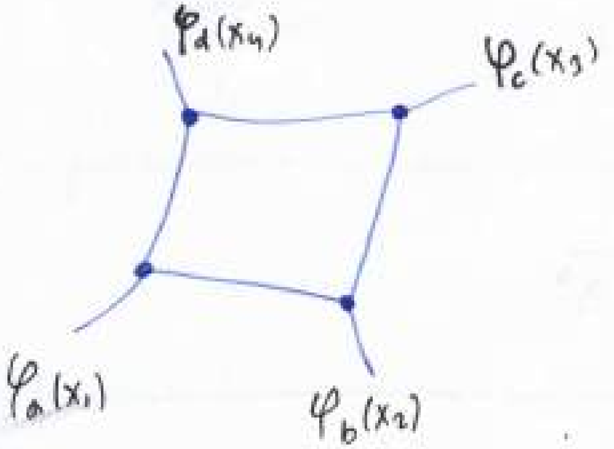
$\langle O_{n_1}(x_1) \dots O_{n_m}(x_m) \rangle \equiv \sum CCC\dots C$

Polyakov ~~by~~ tried to solve via Bootstrap approach.

- Any algebra in order to be consistent, has to be Associative

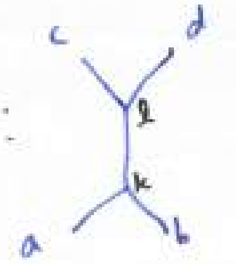
Associativity $\quad \varphi_a \; \varphi_b \; \varphi_c \;=\; \varphi_a \; \varphi_b \; \varphi_c$

Duality of 4-point functions



$\langle \varphi_a(x_1)\varphi_b(x_2)\,\varphi_c(x_3)\,\varphi_d(x_4) \rangle$

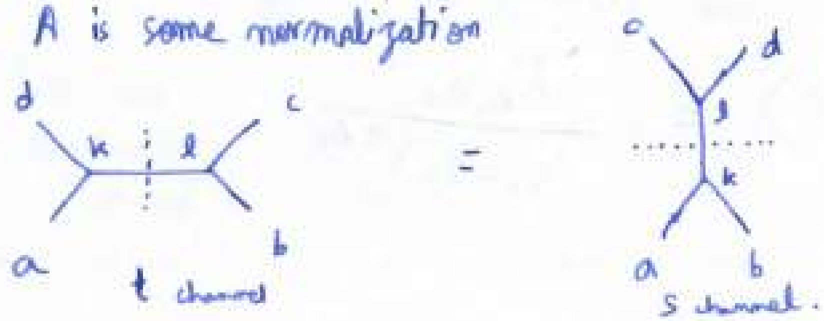
$$= \sum_k C^k_{db} \cdot C^l_{cd} \langle \varphi_k(x_2)\,\varphi^l(x_4) \rangle$$



$$= \sum_k C^k_{ab} C^l_{bd} \langle \varphi_k(x_3)\,\varphi_l(x_4) \rangle :$$

We will prove $\langle \varphi_k(x_1)\, \varphi_l(x_2) \rangle = \dfrac{\delta_{k,l} \cdot A}{|x_1 - x_2|^{\Delta_k + \Delta_l}}$

where A is some normalization



$$\sum_{k,l} C^k_{ab}\, C^l_{cd} \cdot \frac{A_{kl}}{|x_2 - x_4|^{\Delta_k + \Delta_l}} = \sum_{k,l} C^k_{ac}\, C^l_{bd}\, \frac{A_{kl}}{|x_3 - x_4|^{\Delta_k + \Delta_l}}$$

$\rightarrow$ Solving this for Δ_k, C^m_{nl} is very hard....

~~Representation theory for Conformal Algebra~~

☆ Simple consequences of Conformal invariance on (quasi)-primary fields.

$$\varphi(z,\bar z) = \left(\frac{dw}{dz}\right)^{\Delta} \left(\frac{d\bar w}{d\bar z}\right)^{\bar\Delta} \cdot \varphi(w(z), \bar w(\bar z)) \qquad (*)$$

Quasi-primary fields of Conformal Weight $(\Delta, \bar\Delta)$ are those fields which under Mobius transformation, transform as a generalized tensor of weight Δ & $\bar\Delta$ respectively w.r.t w & $\bar w$.

Primary fields under generic analytic transformation, transform as tensor of weight $(\Delta, \bar\Delta)$

$$\varphi(z,\bar z) = \left(\frac{df}{dz}\right)^{\Delta} \left(\frac{d\bar f}{d\bar z}\right)^{\bar\Delta} \cdot \varphi(f(z), \bar f(\bar z))$$

We want to show that;

as a result of Conformal Invariance

The 2 pt function of Quasi-Primary operator Φ ;

$$\langle \phi_{\Delta_1}(z_1) \cdot \phi_{\Delta_2}(z_2) \rangle = \frac{\delta_{\Delta_1 \Delta_2}}{(z_1 - z_2)^{2\Delta_1}}$$

we put the normalization to be 1

because; wherever these fields appear; there is a coupling constant associated to it $\lambda \int \phi$ .... so it just matters how we are sharing normalization between λ & φ.

proof) $w(z) = z + \varepsilon(z)$ Infinitisimal transformation.

for mobius $\varepsilon(z)$ can be a polynomial of at most order 2

$$\varepsilon(z) = \varepsilon_0 + \varepsilon_1 z + \varepsilon_2 z^2$$

$$\phi(z) = \left(\frac{dw}{dz}\right)^{\Delta} \Phi(w(z))$$

$$= \left(1 + \frac{d\varepsilon}{dz}\right)^{\Delta} \phi(1 + \varepsilon(z))$$

$$\Rightarrow \boxed{\delta\phi = \varepsilon \cdot \frac{d\phi}{dz} + \left(\Delta \cdot \frac{d\varepsilon}{dz}\right) \cdot \phi} \quad (*)$$

so: $\langle \phi_{\Delta_1}(z_1) \phi_{\Delta_2}(z_2) \rangle$

Since $w(z) = z + \varepsilon(z)$ is symmetry, the variation associated to it is zero

$$\delta(\langle \phi_{\Delta_1}(z_1) \phi_{\Delta_2}(z_2) \rangle) = 0$$

$$\langle \delta \phi_{\Delta_1}(z_1)\, \phi_{\Delta_2}(z_2) \rangle + \langle \phi_{\Delta_1}(z_1)\, \delta \phi_{\Delta_2}(z_2) \rangle = 0$$

$\Leftrightarrow$ Gives 3 differential equations

(i) $\varepsilon_0 = 0$, $\varepsilon_1 = 0$, $\varepsilon_2 \neq 0$
(ii) $\varepsilon_0 = 0$, $\varepsilon_1 \neq 0$, $\varepsilon_2 = 0$
(iii) $\varepsilon_0 \neq 0$, $\varepsilon_1 = 0$, $\varepsilon_2 = 0$

} i.e; vary $\varepsilon_0, \varepsilon_1, \varepsilon_2$ independently...

If $\varepsilon_0 \neq 0, \varepsilon_1 = \varepsilon_2 = 0$ $\quad (\partial_1 + \partial_2)\langle \quad \rangle = 0$

so; $\langle \quad \rangle$ can only depend on difference

i.e. $\langle \phi_{\Delta_1}(z_1)\, \phi_{\Delta_2}(z_2) \rangle = f((z_1 - z_2))$

If $\varepsilon_1 \neq 0, \varepsilon_0 = \varepsilon_2 = 0$, then $(\Delta_1 + \Delta_2 + z_1\partial_1 + z_2\partial_2)\langle \quad \rangle = 0$

This is euler equation for homogeneous function of weight Δ_1 & Δ_2.

so; $f = \dfrac{A}{(z_1 - z_2)^{\Delta_1 + \Delta_2}}$

If $\varepsilon_2 \neq 0$, $\varepsilon_0 = \varepsilon_1 = 0$; $\Rightarrow$ using this equation; it gives constraint on the constant + A

$\hookrightarrow$ and the result is ; "<u>Two scaling operators are orthogonal unless scaling operator share same Scaling dimension</u>"

Can do same with 3 pt function.

Impose $\delta . [\langle \phi_{\Delta_1}(z_1)\, \phi_{\Delta_2}(z_2)\, \phi_{\Delta_3}(z_3) \rangle] = 0$ because mobius transformation is an invariance.

~~In 3pt functions; we have 3 points:~~

~~& mobius~~

For 2pt function; (Mobius has 3 parameters)

using the three conditions we can constraint the behavior of function to scalling law, and then the ~~third one~~ third one fixes the normalization.

In 3pt functions; we have 3 points

$\hookrightarrow$ So; we shall be able to unique fix the behavior of the function in terms of the coordinate; but not its normalization.

$$\langle \varphi_{\Delta_1}(z_1)\varphi_{\Delta_2}(z_2)\varphi_{\Delta_3}(z_3)\rangle = C^{\Delta_3}_{\Delta_1,\Delta_2} \cdot z_{12}^{\Delta_3-\Delta_1-\Delta_2} \cdot z_{23}^{\Delta_1-\Delta_2-\Delta_3}\, z_{13}^{\Delta_2-\Delta_1-\Delta_3}$$

where $z_{ij} = z_i - z_j$

Claim: $C^{\Delta_3}_{\Delta_1,\Delta_2}$ is same structure constant which appear in OPE algebra.

2 point function is uniquely found in CFT.

Usually in QFT, 2pt function is very non-trivial.

It is infinite sum of infinite diagrams.

The fact that in CFT, we are able to uniquely pin down 2 pt function (note: this is interacting theory) is an amazing result.

---

4 pt function

Mobius has geometrical property of fixing three points

$$\delta \langle \phi_a(z_1)\phi_b(z_2)\phi_c(z_3)\phi_d(z_4)\rangle = 0$$

~~Th~~ Here we will not get much; because here we have 4 points

(using Mobius; we can put three points in the plane where ever we like)

↪ typically we put it at $(1, 0, \infty)$.

So; conformal invariance alone cannot fix uniquely the form of $\langle \phi_a(z_1)\phi_b(z_2)\phi_c(z_3)\phi_d(z_4)\rangle$

but can only give us constraints.

ie; It will be expressed as some unknown function but in terms of Harmonic Ratio.

$$\langle \phi_a(z_1)\phi_b(z_2)\phi_c(z_3)\phi_d(z_4)\rangle = F\left(\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_2-z_3)}\right)$$

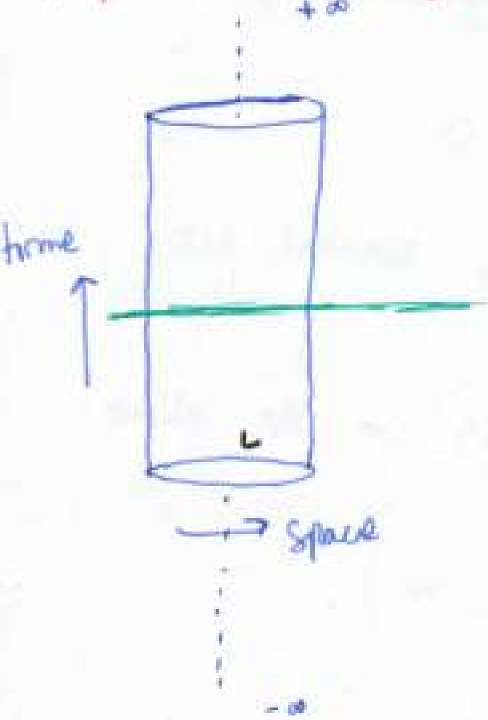
↪ Non trivial part : This depends on an invariant quantity of the symmetry group : Mobius

There can be trivial prefactors (which just depends on how we parametrize the function)

We want to interpret 2pt function in following way :

Suppose that we define our theory on cylinder.
(we can go from plane to cylinder through a conformal map (which is logarithmic))

$$w = (-\ln z) \cdot \frac{L}{2\pi}$$

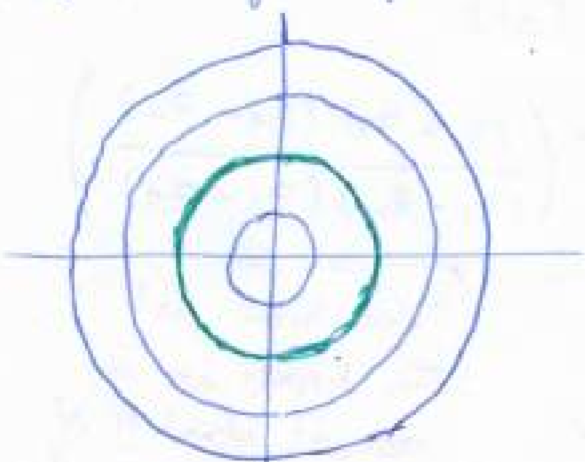


" Cylinder has a very natural way of interpretation of space coordinate & time coordinate "

↳ This is why we go to cylinder.

We can think of evolution of our theory in time starting from $-\infty$ & going to $+\infty$.

If we map time evolution (which is there in cylinder) to the original plane ; we get RADIAL QUANTIZATION



Because any radius r, corresponds to certain time t on cylinder.

$t \to -\infty$ corresponds to origin in the plane

$t \to +\infty$: the circle at ∞.

To define time evolution of State;
we multiply by a unitary operator e^{iHt}

i.e; $e^{iHt}\,|\psi_{in}\rangle = |\psi_{final}\rangle$

Final state.

Initial state

(A trick; multiply by e^{-iHt}; and take limit)
In order to identify states, we take trivial evolution of the things.

$$|\Delta\rangle_{initial} \equiv \lim_{z\to 0} \varphi_{\Delta}(z)\;|0\rangle$$

Vaccum

This vaccum is the only states which are invariant under Mobius group

i.e; $L_0|0\rangle = 0$
$L_1|0\rangle = 0$
$L_{-1}|0\rangle = 0$

Its now intuitive to define the bra (This has to do with things at ∞)

$$\langle\Delta| = \lim_{z\to\infty} z^{2\Delta}\,\langle 0|\,\varphi_{\Delta}(z)$$

There is this extra factor here; that takes out ~~the~~ evolution under the Hamiltonian of the theory.

With this defination; Any relation in CFT ~~is~~ can be interpreted as correlation functions of fields or as a scalar product of States.

$$\langle \varphi_{\Delta_1}(z_1)\varphi_{\Delta_2}(z_2)\rangle = \frac{\delta_{\Delta_1\Delta_2}}{(z_1-z_2)^{2\Delta_1}} \Longleftrightarrow \langle \Delta_1 | \Delta_2 \rangle = \delta_{\Delta_1\Delta_2}$$

While in ordinary field theories; there is a difference between states and fields.

(~~[illegible]~~ (ex Dirac; There are fields that are Majorana, etc. But the particle states are only fermions (electrons) & bosons)

In Conformal Field Theory, there is an isomorphism between all possible Conformal fields and states.

To any conformal states, we can associate a field, and vice versa.

$$|\Delta\rangle = \lim_{z\to 0} \varphi_\Delta(z)|0\rangle$$

$$\langle \Delta| = \lim_{z\to\infty} z^{2\Delta} \langle 0|\varphi_\Delta(z)$$

$H = \int T^{00}(x)\, dx$

Hamiltonian

This is really evolution according to Hamiltonian.

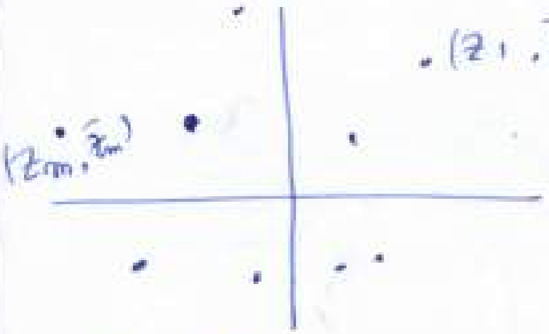
Proofs The Hamiltonian will play the role of L_0

The eigenvalues of H will be Δ ; i.e; $L_0|\Delta\rangle = \Delta|\Delta\rangle$

i.e; if we act Hamiltonian on primary states, we get the eigenvalues.

and we are putting $z^{2\Delta}$, because going from cylinder to the plane ∴ the role of time evolution is played by z ; The radial. (full proof later in notes)

$\langle \varphi_1(z_1, \bar{z}_1) \ldots\ldots \varphi_m(z_m, \bar{z}_m) \rangle$



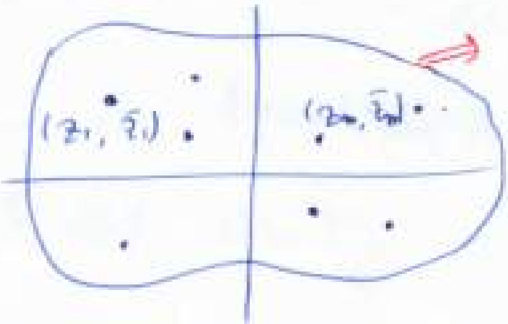
physical plane

$(z_1, \bar{z}_1)$ say correspond to $x = \frac{z + \bar{z}}{2}$

$y = \frac{z - \bar{z}}{2}$

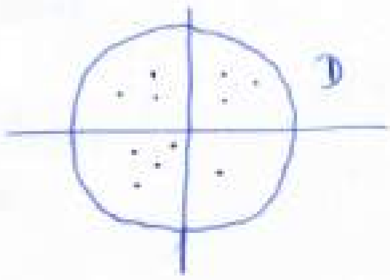
in physical plane.

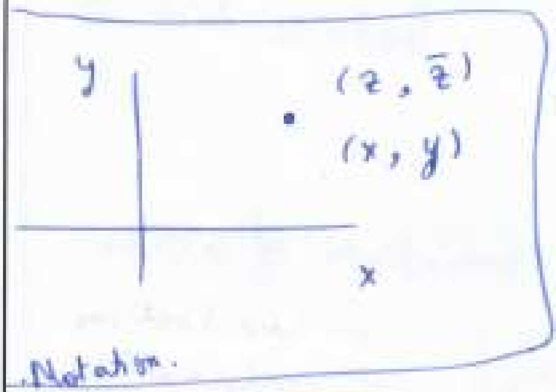
There will be some region which encloses all of them



We take a circle ;

Take a circle will contains all the point inside.





Notation.

Conformal Transformati's

$z \longrightarrow f(z)$

$\bar{z} \longrightarrow \bar{f}(\bar{z})$

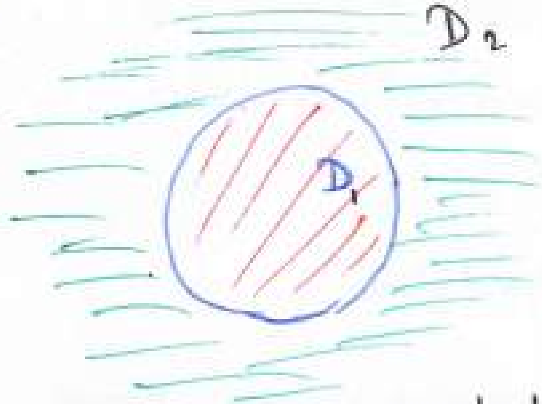
$x \longrightarrow x' + \varepsilon(x)$

where $\varepsilon(x)$ is analytic inside the disc ;
and outside small everywhere
(we connect the two the way we like it)

$x \to x' + \varepsilon(x)$

$\leftarrow$ we can take it to be full analytic function (with no singular points)

$\varepsilon(x)$ cannot be small everywhere



But; outside D, we take a function which is differentiable, etc; and goes to zero very rapidly.

Inside disk we denote by D_1
Outside " " " " D_2

This transformation is actually a patchwork.

$$\varepsilon(z) = \begin{cases} \text{Analytic Function} & \text{Inside Disk } D \\ \text{Smooth, and goes to zero very rapidly} & \text{Outside Disk } D \end{cases}$$

At this point; There is gonna be variation of action because (it is not symmetry here) we have breaking of conformal invariance outside D_1

$$\delta S = -\frac{1}{2\pi} \int_{D_2} T_{\mu\nu}(x)\, \partial^\mu \varepsilon^\nu \cdot d^2x$$

Conformal invariance is violated outside disk D_1;

$\delta S \sim \int_{D_1} + \int_{D_2}$; $\int_{D_1} = 0$ because ε is analytic there

Conformal ~~in~~ invariance is violated in D_2:
because we dont have something analytic satisfying
Cauchy - Riemann so; $\int_{D_2} \neq 0$

Now define $\langle \ldots\ldots \rangle \equiv \int \mathcal{D}\phi \, e^{-S} \ldots\ldots$

A general trick (which here helps in deriving Ward Identities)

The transformation $z \longrightarrow z' + \varepsilon(z)$ is as far as
one ~~to~~ one; we are just reparametrizing our .
space.

So; The variation of correlation function has to
be zero !!!

$$\delta[\langle \ldots\ldots \rangle] = \delta[\int \mathcal{D}\phi \, e^{-S} \ldots] = 0$$

because we are not changing anything.
~~Any~~ Any point that was before; is now a new one;
however the variation ~~is~~ is now made of two
terms.

1. The variation of fields inserted ....
2. The variation of measure e^{-S}

(previously when we said that when we have invariance; we
only focused on the ~~...~~ variation of fields ....
while doing $\delta\langle \quad \rangle = 0$)

Here we will still be doing $\delta\langle\ \ \rangle = 0$
but now; will also take into account the variation of e^{-S}, because S is not invariant
(because the transformation is not a symmetry).

In General Relativity; each time we change our metric (here the analogy is of using different $\varepsilon(z)$); the reaction is through the Stress Energy Tensor.
Stress Energy tensor is uniquely defined this way.

Now further;
we eventually want to express $T_{\mu\nu}$ in terms of some basic fields ; $T_{\mu\nu}$ exists independently.
It is the field which react to the variation of the metric

Each time we do the $\varepsilon(z)$ transformation;
There is change of the theory through the δS change;
$\therefore$ i.e. The insertion of $T_{\mu\nu}$

$\hookrightarrow$ $T_{\mu\nu}$ exist in the theory independent of other things
& is the basic field (will see this later)

When we write something like $\langle \ldots \rangle$ we have to also specify in which geometry we are computing in.
If we make generic change of transformation:

$$\langle \ldots \rangle \longrightarrow \langle\langle \ldots \rangle\rangle$$

It will be correlation in other geometry.

What we are essentially doing, is that

$\langle\langle \dots \rangle\rangle \simeq \langle \dots \rangle$ + Something

It is related to $T_{\mu\nu}$.

(Deep down; we are taking into account my geometry has changed. And from GR we know that, the field which react to geometry is Stress Energy Tensor)

So; The variation of Correlation is.

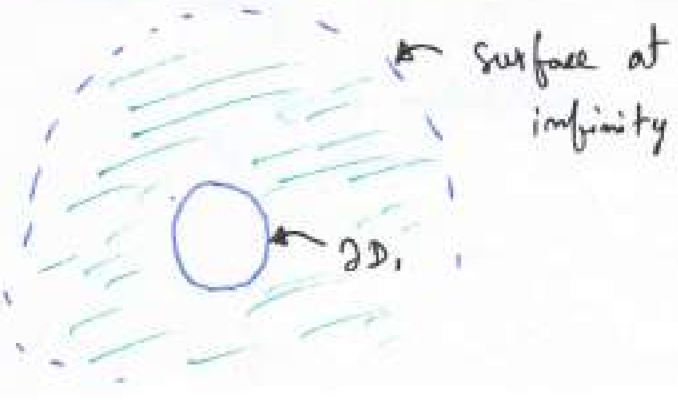
$$\sum_i \left[\left(\Delta_i \partial_i \varepsilon + \varepsilon \partial_i \right) + \left(\bar{\Delta}_i \bar{\partial}_i \bar{\varepsilon} + \bar{\varepsilon} \bar{\partial}_i \right) \right] \langle \phi_1, \dots, \phi_m \rangle$$
$$+ \frac{1}{2\pi} \int_{D_2} d^2 z \; \partial^\mu \varepsilon^\nu \langle T_{\mu\nu}(x) \phi_1 \dots \phi_n \rangle = 0$$

This is the extra piece we get because $\delta S \neq 0$.

for this piece; look at equation (*) on page 56

Since we have integral over plane $\int_{D_2} d^2 z$;

we can use divergence theorem. Integral on plane = Integral on surface.

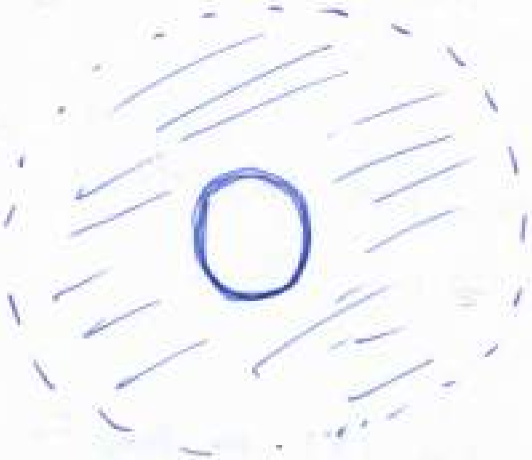


At infinity we want it to go to zero;
This is why we take ε smooth and goes to zero ~~at~~ very rapidly at ∞.
(ie; faster than $1/z^4$ if we want to be precise)

Then just remains integral over the boundary of D_1

$$\frac{1}{2\pi}\int_{D_2} d^2z\, \partial^\mu \varepsilon^\nu(x) \langle T_{\mu\nu}(x) \ldots \rangle = \frac{-1}{2\pi}\int_{D_2} \varepsilon^\nu(x) \langle \partial^\mu T_{\mu\nu}(x) \ldots \rangle$$

$$+ \frac{1}{2\pi}\int_{\partial D_2} d\Sigma\, n^\mu \varepsilon^\nu \langle T_{\mu\nu}(x) \ldots \rangle$$



note; $\partial^\mu T_{\mu\nu} = 0$ because $T_{\mu\nu}$ is conserved;

so; $$\frac{1}{2\pi}\int_{D_2} d^2z\, \partial^\mu \varepsilon^\nu(x) \langle T_{\mu\nu}(x) \ldots \rangle = \frac{1}{2\pi}\int_{\partial D_2} d\Sigma \cdot n^\mu \cdot \varepsilon^\nu \langle T_{\mu\nu}(x) \ldots \rangle \quad (*\,*)$$

$$\partial D_2 = \partial D_1 \sqcup \underbrace{(\text{Big circle at } \infty)}$$

$\longrightarrow$ This gives zero contribution because $\varepsilon \to 0$ rapidly at ∞.

In our theory ; which is conformal invariance.

we know : $T_{\mu\nu}$ is conserved , symmetric & traceless. → Two independent component out of 4 (in 2 dimensions)

So; we can make linear combination

$$T(z) = T_{11} - T_{22} + 2i\,T_{12}$$

$$\bar{T}(\bar{z}) = T_{11} - T_{22} - 2i\,T_{12}$$

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & -T_{11} \end{pmatrix}$$

by conservation; we can prove

$$\partial^\mu T_{\mu\nu} = 0 \iff \bar{\partial}_{\bar{z}} T(z) = 0$$

$$\partial_z \bar{T}(\bar{z}) = 0$$

ie; $T(z)$ does not depends on $\bar{z}$. $\bar{T}(\bar{z})$ does not depends on z. } Achieved using conservation law.

So; on the RHS of (**) ; we can change variable to parametrize the circle : we can write it in terms of Cauchy Integral.

$$\frac{1}{2\pi}\int_{\partial\Sigma}(d\Sigma)\, n^\mu \varepsilon^\nu \langle T_{\mu\nu}\cdots\rangle = \frac{1}{2\pi i}\oint dz\, \varepsilon(z)\langle T(z)\cdots\rangle - \frac{1}{2\pi i}\oint d\bar{z}\;\bar{\varepsilon}(\bar{z})\langle \bar{T}(\bar{z})\cdots\rangle$$

First ; we note that z and $\bar{z}$ part are completely decoupled.

~~So~~ So we have;

$$\sum_i (\Delta_i \partial_i \varepsilon + \varepsilon \partial_i) \langle \phi_1 \ldots \phi_n \rangle = \frac{1}{2\pi i} \oint dz\, \varepsilon(z) \langle T(z) \ldots \rangle \quad (*)$$

↳ obtained by identifying fully analytic & ~~...~~ fully ~~...~~ anti-analytic part ~~in equation~~ equation written on page 67..

We can write. computing derivative of analytic function; (use contour integrals...... with double pole..)

$$\sum_i (\Delta_i \partial_i \varepsilon + \varepsilon \partial_i) \langle \phi_1 \ldots \phi_n \rangle = \sum_i \oint \frac{dz}{2\pi i} \varepsilon(z) \left[\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i} \partial_i \right] \langle \phi_1 \ldots \rangle$$

$$\sum_i (\Delta_i \partial_i \varepsilon + \varepsilon \partial_i) \langle \phi_1 \ldots \phi_n \rangle = \sum_j \oint \frac{dz}{2\pi i} \varepsilon(z) \cdot \left[\frac{\Delta_j}{(z-z_j)} + \frac{1}{z-z_j} \partial_j \right] \langle \phi_1 \ldots \phi_n \rangle$$

→ converting differential operator in terms of pole

We want to convert $(*)$ into a local equation; ~~...~~ moreover of the type of OPE expansion.

And then; comparing RHS & LHS in the integral equation we get the following equality (note: $\varepsilon(z)$ is arbitrary ......)

WARD IDENTITY.

$$T(z_1)\, \phi_\Delta(z_2) = \frac{\Delta}{(z_1 - z_2)} \phi_\Delta(z_2) + \frac{1}{z_1 - z_2} \cdot \partial \phi_\Delta + h(z_1)$$

This is a holomorphic function; by Cauchys theorem
The contour integral over this is zero i.e. $\oint \varepsilon h$ because

E holomorphic in D.

So; There can be a residual harmonic function to ~~which~~ which the equation (*) is blind.

↳ here; we are not able to determine what is $h(z)$; but we just know that it is holomorphic with no poles. (later we might fix $h(z)$)

<u>Note)</u> The anamolous dimension of T is proper:
$T_{\mu\nu}$ is field in the theory whose anamolous dimension is 2 (in 2 dimensions)

↳ because if we integrate it; we get energy; ~~And energy is non anamolous~~. And energy is non-anamalous.

<u>Note)</u> Primary fields are those which are select out; that when we make its OPE with T; The power of singularity we get is mildest as possible, namely 2^{nd} order pole.

↳ If we take an arbitrary scaling field, and Then ~~make~~ make its OPE with T; Then we might have pole of higher order.

~~"Quasi primary has mildest OPE, compatible with everything (i.e, also with Möbius)"~~

$T(z)$ is a scalling field as any other.

We can ask what is OPE of T with itself.

$T(z_1)\,T(z_2)$

We can prove that,

T is a quasi-primary field of dimension 2.

Therefore: $T(z_1)\,T(z_2) = \frac{2}{(z_1-z_2)^2}\,T(z_2) + \frac{1}{(z_1-z_2)}\,\partial T + h(z_1)$

+ (can we have something higher order pole).

We can have 4^{th} order pole:

This shows that T is not primary

$$T(z_1)\,T(z_2) = \frac{c/2}{(z_1-z_2)^4} + \frac{2}{(z_1-z_2)^2}\,T(z_2) + \frac{1}{(z_1-z_2)}\,\partial T + h(z_1)$$

~~We are~~ when we are dealing with 2 point function.

$\langle T(z_1)\,T(z_2)\rangle$

Then we have to take something like.

$$\frac{\langle c/2\rangle}{(z_1-z_2)^4} + \underbrace{\frac{2}{(z_1-z_2)^2}\langle T(z_2)\rangle + \frac{1}{z_1-z_2}\langle \partial T\rangle}$$

These are zero; because expectation value of any scaling operator (but Identity) are zero.

So; $\langle T(z_1)\,T(z_2)\rangle = \frac{c/2}{(z_1-z_2)^4}$

~~[illegible] zero~~

We see an arbitrary parameter c , because

$\langle T(z_1)\ T(z_2) \rangle$

insert $\sum |\mu\rangle\langle \mu| = 1$

Then we understand that, The quantity is ~~intrinsically~~ intrinsically positive.

~~...~~

It is infinite sum between vacuum & bunch of other terms $\sum |\langle 0|\ T|\mu\rangle|^2 + \cdots$

So; These terms are not zero (by summing positive & negative terms) ; because it is intrinsically positive.

So; we need an extra term **c**

This **c** is called Central Charge

$$\langle T(z_1)\, T(z_2) \rangle = \frac{c/2}{(z_1 - z_2)^4}$$

$$T(z_1)\, T(z_2) = \frac{c}{2\,(z_1 - z_2)^4} + \frac{2\, T(z_1)}{(z_1 - z_2)^2} + \frac{1}{z_1 - z_2} \cdot \partial T$$

A nice thing about CFTs ; any OPE can be converted to ordinary commutator or anticommutator relation.

So; lets expand $T(z)$ around the origin in terms of mode
$$T(z) = \sum_{-\infty}^{+\infty} \frac{L_n}{z^{n+2}} \quad ; \quad L_n = \frac{1}{2\pi i} \oint_{C_0} dz \cdot z^{n+1} \cdot T(z)$$

Then

~~[illegible]~~

$$L_n L_m = \frac{1}{2\pi i} \oint_{c_1, c_2} d\xi_1\, \xi_1^{n+1}\, d\xi_2\, \xi_2^{m+1}\, T(\xi_1)\, T(\xi_2)$$

$- L_m L_n \Rightarrow$ want to compute this.

~~[illegible]~~

In order to compute $- L_m L_n$;

we have to swap the contours.

When two points of contour coincide, we have OPE;

The difference between two contours ~~[illegible]~~ will be equal to contour times the residue at the ~~[illegible]~~ point of incidence.

Pictorially

Out of this; we get the algebra.

$$\boxed{[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} n(n^2-1)\, \delta_{n+m,0}}$$

Lec 6 : Transformation of Stress Energy Tensor, Schwarzian derivative, Quantum field representation of Virasoro Algebra, also representation in terms of states;

$$\varphi_n(z_1)\,\varphi_m(z_2) = \sum \frac{C^k_{nm}}{(z_1-z_2)^{\Delta_n+\Delta_m-\Delta_k}}\cdot \varphi_k \qquad \text{OPE algebra}$$

$$\{C^k_{nm},\ \Delta_k\}$$

$$T(z_1)\,\varphi_\Delta(z_2) = \frac{\Delta}{(z_1-z_2)^2}\cdot\varphi_\Delta(z_2) + \frac{1}{z_1-z_2}\cdot\partial\varphi_\Delta(z_1) + h(z_1)$$

Holomorphic.

when $\varphi_\Delta(z) = \left(\frac{df}{dz}\right)^\Delta\cdot\varphi_\Delta(f(z))$: Primary Fields.

$$T(z_1)\,T(z_2) = \frac{c/2}{(z_1-z_2)^4} + \frac{2T}{(z_1-z_2)^2} + \frac{\partial T}{(z_1-z_2)} + g(z_1)$$

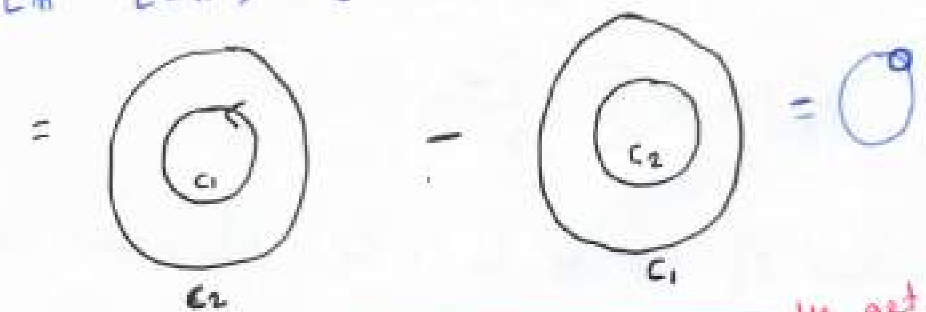
There is an isomorphism between OPE & Algebra.

Express $T(z)$ in modes around one point of Analyticity.

$$T(z) = \sum \frac{L_n}{z^{n+2}} \qquad L_n = \frac{1}{2\pi i}\oint dz\cdot z^{n+1}\,T(z)$$

As Cauchy Laurent

Then $L_nL_m - L_mL_n = [L_n, L_m]$



$$\boxed{[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\,n\,(n^2-1)\,\delta_{n+m,0}}$$

We get algebra parametrized by c.

$\Longleftrightarrow$ Quantum Version of Conformal Algebra

(The only difference is the presence of extra term. called the Central Extension of the Algebra".

Imagine OPE of currents.

Ex

$$J^a(z_1) J^b(z_2) = \frac{k\,\delta^{ab}}{(z_1-z_2)^2} + \frac{f^{ab}{}_c\, J^c}{z_1-z_2} + \dots$$

$J^a(z_1)$ are currents of dimension 1

Expanding it in modes $J^a(z) = \sum \frac{J^a_m}{z^{m+1}}$

and repeat the same contour difference argument; we get

$$[J^a_n, J^b_m] = i f^{ab}{}_c\, J^c_{n+m} + k \cdot n \cdot \delta^{ab} \cdot \delta_{n+m,0}$$

$\hookrightarrow$ ∞ dimensional Algebra.
Involves ∞ no. of modes; n, m taking ∞ no. of values
and a, b are values in Lie Algebra

If we have an operator $\varphi_\Delta(z)$ of dimension Δ;

Convention: when we are expanding mode,
we always put Δ here

$$\varphi_\Delta(z) = \sum \frac{\varphi_m}{z^{m+\Delta}}$$

$\Leftrightarrow$ If we do this; The defination which defines the mode is usual

---

Ex Take Majorana Fermions of

OPE $\psi(z_1)\psi(z_2) = \frac{1}{z_1-z_2} + \dots$

$$\psi(z) = \sum \frac{\psi_n}{z^{n+1/2}} \quad , \quad \psi_n = \frac{1}{2\pi i}\oint dz\, z^n \psi(z) .$$

→ Here we get Anti-Commutator

$$\{\psi_n , \psi_m\} = \delta_{n+m,0}$$

→ and doing the trick of swapping the contour outside ; we have to minus (because of fermions) So we get Anticommutator.

~~So OPE~~

In general, whether OPE corresponds to Commutator or an anti-commutator algebra is determined according to nature of fields.

Conclusion: We have converted the physical problem of classifying the critical behavior in the mathematical problem of getting irreducible representation of an object, (which we know how to control very well) ; which is an Algebra.

$$\delta S = -\frac{1}{8\pi} \int T_{\mu\nu}(x)\, \partial^\mu \epsilon^\nu (x)\, d^2 x$$

$T_{\mu\nu}(x)$ is like a response function

Variation of free energy $\delta F = \int h(x)\, \sigma(x)\, dx$

magnetization.

; Similarly $\sigma(x)$ is like a response.

$$\delta T = (2\partial\varepsilon + \varepsilon\partial)T + \frac{c}{12}\partial^3\varepsilon$$

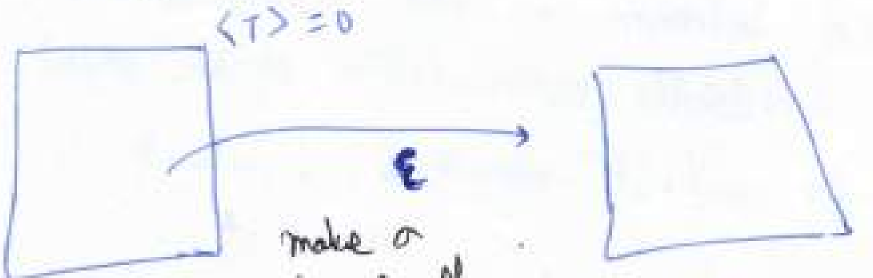
$$\langle \delta T \rangle = 0 + \frac{c}{12}\partial^3\varepsilon$$

because $\langle T \rangle = 0$ (one point)

$\Rightarrow$ $\boxed{\langle \delta T \rangle = \frac{c}{12}\partial^3\varepsilon}$ $\Rightarrow$ Now we understand that c is an <u>anamoly</u>.

↳ because; usually when we have an infinite system we normalize the ground state energy to be zero at ∞.
(This is something which we can always do; expectation value is a constant (might be ∞); we subtract it to normalize $\langle T \rangle$ to be 0)

Imagine we have done this; and ~~that~~ so we get that: in the plane, we normalize our Stress Energy tensor to have expectation value 0.



Then we find $\langle T \rangle \neq 0$

ie; <u>Energy out of nothing</u> (just <u>from geometry</u>)

... Anamoly.

Recall that,
Mobius was at most quadratic. So if ε was associated to Mobius; Then $\langle \delta T \rangle = 0$

But if we do a generic analytic transformation that has non-zero third derivative

Then $\langle \delta T \rangle \neq 0$; i.e.; $\langle \delta T \rangle = \frac{c}{12} \partial^3 \varepsilon$

"So out of nothing, we get energy"

→ In physics, it is called Casimir Energy.

(A possibility that we might have density of energy just comming from geometry ; And it is something measurable)

Transformation of T under ~~under~~ general transformation (analytic)

$$T_{Plane}(z) = T_{New}(f) \cdot \left(\frac{df}{dz}\right)^2 + \frac{c}{12}\{f, z\}$$

where
$$\{f, z\} = \frac{\frac{d^3 f}{dz^3}}{\frac{df}{dz}} - \frac{3}{2}\left(\frac{\frac{d^2 f}{dz^2}}{\frac{df}{dz}}\right)^2$$

Schwarzian Derivative

T is not a tensor under general analytic map.

T is quasi-primary operator, as far as Mobius is concerned.

Imposing $\{f, z\} = 0$ gives f is Mobius map.

($\{f, z\} = 0$ Is differential equation satisfied by Mobius)

Properties of Schwarzian Derivative

$$\left\{ \frac{af+b}{cf+d} , z \right\} = \{ f , z \}$$

→ mobius transformation on new variable f

$$\left\{ f , \frac{az+b}{cz+d} \right\} = \{ f , z \} (cz + d)^4$$

→ Mobius transformation on the variable itself

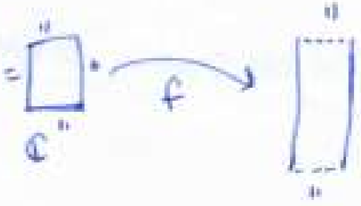
⇒ since everything involves derivatives we get Jacobian.

Suppose we do chain of transformations

$$z \longrightarrow w \longrightarrow u$$

$$\{ u , z \} = \{ u , w \} \left(\frac{dw}{dz} \right)^2 + \{ w , z \}$$

Suppose $f = \frac{L}{2\pi} \ln z$ will map original plane to cylingle.



$$T_{strip} = \left(\frac{2\pi}{L} \right)^2 \left[z^2 \, T_{plane}(z) - \frac{c}{24} \right]$$

now, recall we normalize T_{plane}

i.e. $\langle T_{plane} \rangle = 0$

so: $$\boxed{\langle T_{strip} \rangle = \frac{-\pi c}{6 L^2}}$$

If we integrate along strip (periodic) direction; in order to get ground state energy $E_0(L)$ we get $E_0(L) = \frac{-\pi c}{6L}$

There is unknown zero ground state energy which depend on geometry.

If we take $L \to \infty$; we recover $E = 0$.

If we take L as finite we will have force between the two sides of the strip

we have force on the sides of the strip.

How c can be different from 0 ?

Example 1] Free Bosonic Theory (massless) $\quad c = 1$

$$\mathcal{L} = \frac{g}{4\pi} (\partial\varphi)^2$$ with propagators

$$\langle \varphi(z,\bar{z})\,\varphi(0,0)\rangle = -\frac{1}{2g}\ln z \;-\frac{1}{2g}\ln \bar{z}$$

φ is not scaling field but we see $J = \partial\varphi$ has dimension 1 (and is scaling field) ... its like current.

using Noether Theorem

$$T(z) = -g :(\partial\varphi)^2:$$

we know $\quad \langle T(z_1)\, T(z_2)\rangle = \dfrac{c/2}{(z_1 - z_2)^4}$

lets find c & using Wick Theorem

$$\text{so; } \langle T(z_1)\, T(z_2)\rangle = g^2 \langle (\partial\varphi)^2 (\partial\varphi)^2\rangle$$
$$= \frac{1}{2} \cdot \frac{1}{(z_1 - z_2)^4}$$

Free Boson Theory has Central Charge $c = 1$

Example 2) Free massless fermionic Theory $c = 1/2$

(Majorana type)

$$\mathcal{L} = \psi \partial \psi + \bar{\psi} \bar{\partial} \bar{\psi}$$

Fermions in 2d has two components $\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$

ψ analytic

$\bar{\psi}$ purely anti-analytic.

so: $\bar{\partial}_{\bar{z}} \psi = \partial_z \bar{\psi} = 0$

Then $\langle \psi(z_1) \psi(z_2) \rangle = \dfrac{1}{(z_1 - z_2)}$

$$T = -\frac{1}{2} : \psi(z) \frac{\partial}{\partial z} \psi(z) :$$

$$\langle T\, T \rangle = \frac{1}{4} \langle \psi_1 \partial \psi_1 \;\; \psi_2 \partial_2 \psi_2 \rangle \overset{\text{use Wick's Theorem}}{=} \frac{1/4}{(z_1 - z_2)^4}$$

so: $\boxed{c = 1/2}$

Bosonization (Idea)

~~Bos~~

The possibility of representing Bosons in terms of fermions.

Complex fermion, is just 2 majorana fermion. Therefore has Central Charge 1. (because these are energy: c in proper unit is energy; & energy is additive)

i.e: $c = \frac{1}{2} + \frac{1}{2} = 1$ same as central charge as Bosons.

Representation Theory of Virasoro Algebra

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\, n(n^2-1)\, \delta_{n+m,0}$$

Approach 1// Giving Representation in terms of Quantum Fields

(Fields which depends on coordinates)

define Stress Energy tensor to be around some point $\quad T(z) = \sum_{-\infty}^{+\infty} \frac{L_m}{(z-z_1)^{m+2}}$

(We can shift the point about which we are expanding freely)

We want to define new field B, which we get by acting L_m on $A(z)$ $\quad$ ie; $B = (L_m A(z))$

~~ie B = (L_m A)~~

Defination: $\boxed{(L_m A(z_1)) \equiv \frac{1}{2\pi i} \oint dz \cdot (z-z_1)^{m+1}\, T(z)\, A(z_1)}$

Interpretation of the defination

$$(L_m A(z_1)) = \frac{1}{2\pi i} \underbrace{\oint dz \cdot (z-z_1)^{m+1}}_{\text{② multiplying}}\ \underbrace{T(z)\, A(z_1)}_{\text{① This has some OPE expansion}}$$

② multiplying by $(z-z_1)^{m+1}$ and integrating: we are filtering the field which are of the proper power.

What is $(L_0 \varphi_\Delta)$.

We find
$$\begin{cases} (L_0 \varphi_\Delta(z)) = \Delta \cdot \varphi_\Delta(z) \\ (L_{-1} \varphi_\Delta(z)) = \partial \varphi_\Delta \\ (L_n \varphi_\Delta(z)) = 0 \end{cases}$$

Primary fields are annihilated by all the positive modes.
L_0 is diagonal , L_{-1} is derivative

→ Consequences of these :

If in any representation if we have a field φ,
Then we shall have arbitrary derivatives of fields $\partial^m \varphi$ also in the representation.
(because; we can do say $(L_{-1})^3 \varphi$)

How we build up the Representation?

Representation means that we have vector space ; such that any operation we are going to do with generators we always get another vector of the space (nothing is left out; its an irreducible representation)

Span of vector : All possible commination with negative modes ordered in certain way

$$\{ L_{-m_1} L_{-m_2} \dots L_{-m_k} \varphi_\Delta \}$$

s.t. $0 \leq m_1 \leq m_2 \leq \dots \leq m_k$

$\cancel{\text{Span}\{L_{-m_1} \dots L_{-m_k} \mid k \in \mathbb{N} \cup \{0\} ; 0 \le m_1 \le m_2 \le \dots \le m_k\}}$

$$\text{Span}\{L_{-m_1} \cdots L_{-m_k} \mid k \in \mathbb{N} \cup \{0\} \; ; \; 0 \le m_1 \le m_2 \le \dots \le m_k\}$$

This is the vector space which forms an irreducible representation of our Algebra.

We use ordering to not overcount our states.

__Why we do ordering?__

$$\sum_{i=1}^{k} m_i = N \in \mathbb{N}$$

We can show

$$[L_0 , L_{-m}] = m L_{-m}$$

$$L_0 (L_{-m} \varphi_\Delta) = (\Delta + m) (L_{-m} \varphi_\Delta)$$

$L_{-m} \varphi_\Delta$ are eigenvalues of L_0 with eigenvalue $(\Delta + m)$.

__Examples__ $(L_{-1}^3 \varphi_\Delta)$, $(L_{-1} L_{-2} \varphi_\Delta)$, $(L_{-2} L_{-1} \varphi_\Delta)$, $(L_{-3} \varphi_\Delta)$

We can appriori think that the level 3 is 4 times degenerate. (So there are Three different fields which correspond to eigenvalue $\Delta + 3$)

False : because one of the two is linear combination of other.

__Check__ $L_{-1} L_{-2} = L_{-2} L_{-1} + [L_{-1}, L_{-2}] = L_{-2} L_{-1} + 3 L_{-3}$

so; $L_{-1} L_{-2} \varphi_\Delta$ is linear combination of $L_{-2} L_{-1} \varphi_\Delta$ & $L_{-3} \varphi_\Delta$.

Among the OPE's,
There is also identity field. (because its algebra; so it should be)

denoted by $\mathbb{1}(z)$ (it does not depends on z; its a constant)

its a field of dimension 0.

$$(L_{-1}\,\mathbb{1}) = 0$$

$$(L_{-2}\,\mathbb{1})(z) = \frac{1}{2\pi i}\oint \frac{1}{z-w}\,T(z)\,\mathbb{1}(w)$$

$$= T(z)$$

$$\boxed{(L_{-2}\,\mathbb{1})(z) = T(z)}$$

Stress Energy Tensor is not a primary; but is a descendent at the second level identity operator.

→ This is the claim; why its not the primary field: because it belongs to second floor of a family whose primary field is zero

This is the only family with this property. $\Longleftarrow$ $\Big\{$ No descendent at first order

So: Irreducible representation of Identity Famility is build up ~~to~~ all composite operator, by Stress Energy Tensor.

The way of counting degeneracy of level N is through combinatorics.

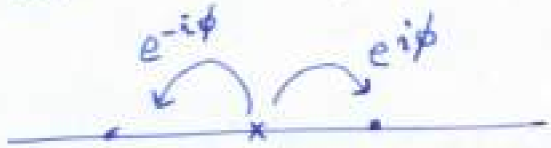
$$\underbrace{\prod_{n=1}^{\infty} \frac{1}{1-q^n}}_{\text{generating function}} = \sum_{N=0}^{\infty} P(N)\, q^N$$

$P(N)$ ← Degeneracy.

Asymptotically: $P(N) \propto \dfrac{\exp\left(\pi\sqrt{\frac{2N}{3}}\right)}{4\sqrt{3}\,N}$ HARDY & RAMANUJAN

Example: Constration of generating function in Brounian motion.

Construt dummy variables



we generate a function $(e^{i\phi}+e^{-i\phi})^N$

If we filter $e^{iN\phi}$ ∃ The coefficients count, how ~~tan~~ many times we have gone from origin to site N.

so: $(e^{i\phi}+e^{-i\phi})^N$ is generating function for Borwnian Motion.

Advantage of using Representation in terms of fields:

All correlation functions of descendent fields satisfy Linear Differentiel Equations.

If we have $\langle (L_{-m}\varphi)\, \varphi_1 \dots \varphi_n \rangle$

$$= \mathcal{D}^n \langle \varphi\, \varphi_1 \dots \varphi_n)$$

$\mathcal{D}^n$ is linear differential operator of order n.

$$\langle (L_{-m}\varphi)\, \varphi_1 \dots \varphi_n \rangle = \mathcal{D}^m \langle \varphi\, \varphi_1 \dots \varphi_n \rangle$$

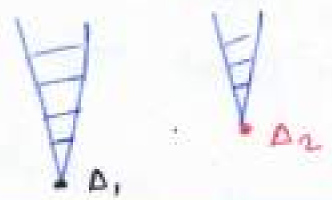
where;
$$\mathcal{D}^m \equiv -\sum_{j=1}^{n} \frac{(1-n)}{(z_j - z)^m} \cdot \Delta_j + \frac{1}{(z_j - z)^{n-1}} \cdot \frac{\partial}{\partial z_j}$$

$\Longleftrightarrow$ This implies; All correlation functions which involve descendent fields are not independent quantities.

They depen on primaries.

Consequence for two fields $\langle \Delta_1 | \Delta_2 \rangle = \delta_{\Delta_1 \Delta_2}$

This implied; all the fields which belong the two different representations are always orthogonal.



proof $\langle T(z)\, \phi_1(w_1) \dots \phi_m(w_m) \rangle$ Sandwhich the Ward Identity

$$= \sum_{i=1}^{n} \left(\frac{\Delta_i}{(z - z_i)^2} + \frac{1}{(z - z_i)} \partial_i \right) \langle \phi_1 \dots \phi_n \rangle$$

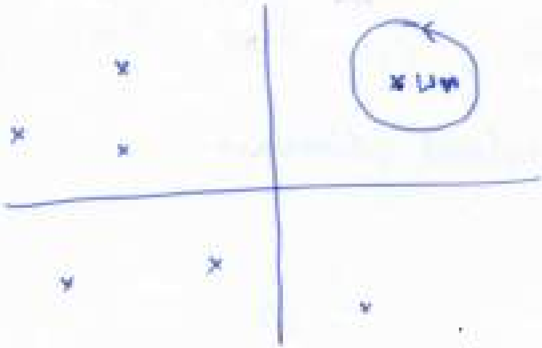
Now we do OPE of $T(z)$ with $\phi_n(w_m)$ in following way $\langle T(z)\, \phi_1(w_1) \dots \phi_m(w_m) \rangle$

$$T(z)\phi_m(w_m) = \sum_{k \geq 0}^{\infty} \left(L_{-k}\, \phi(w_m) \right) (z - w_m)^{k-2}$$

We do OPE around w_m involving desendents $L_{-k}\phi(w_m)$

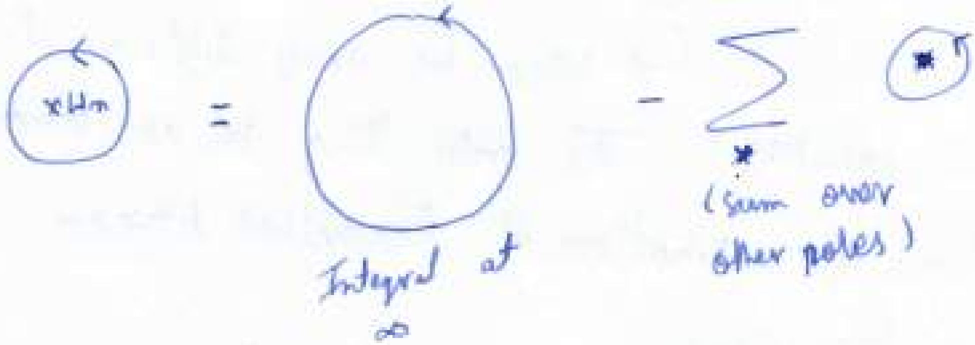
Definition of $L_{-k}\phi(w_m)$ is

$$\left(L_{-k}\phi(w_m)\right) = \frac{1}{2\pi i}\oint_C dz \cdot (z-w_m)^{-k+1} \cdot T \cdot \phi$$



<u>Theorem for Complex Analysis</u>

Sum of all residue's including residue at infinity is zero.



Residue at ∞ goes to zero; because T at ∞ goes like $\frac{1}{z^4}$

So:  $= 0$

When we do sum over all the other poles we get $\mathcal{D}^n$

" The Descendent field we need to build up ~~rep~~ representation; but for the dynamics concerned : they are just Algebra"

Once we have primary; we have them all.

↪ The structure constant $\{C^{k}_{nm}\}$:

we need only structure constants for primary.

~~Recall $\langle \varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_3)\rangle$ — C_{123} z_{12} z_{13} z_{23}~~

Recall $\langle \varphi_1(z_1)\, \varphi_2(z_2)\, \varphi_3(z_3)\rangle = C_{123}\; z_{12}^{\sim}\; z_{13}^{\sim} \cdot z_{23}^{\sim}$

$\left.\begin{matrix}\sim\\ \sim\\ \sim\end{matrix}\right\} \to$ just combinations of anamolous dimension.

When $\langle (L_{-1}\varphi_1)\, \varphi_2\, \varphi_3 \rangle = \mathcal{D}^2 [\quad \cdot \quad]$.

↪ When we apply differential operators; The only thing we are bringing down are bunch of combinations of Anamalous dimension.

So; Its Algebraic

Structure constant of desendent fields $\langle (L_{-2}\varphi_1)\, \varphi_2\, \varphi_3 \rangle$

$$C^{L_1}_{23} \propto C_{123}\; \underline{f(\Delta_1, \Delta_2, \Delta_3)}$$

↪ A polynomial of Anamalous dimension.

Hence; if $\langle \varphi_1(z_1)\, \varphi_2(z_2)\, \varphi_3(z_3)\rangle = 0$

Then $\langle (L_{-n}\, \varphi_1(z_1))\, \varphi_2(z_2)\, \varphi_3(z_3)\rangle = 0$

★ If primary donot talk with each other; then all the infinite desendents of them dont talk to each other.

Fusion

given two fields ; and fuse to third one

with a structure constant C_{abc} ; which rules the Fusion Rule.

If φ_a , φ_b & φ_c are primary

then $\langle \varphi_a \varphi_b \varphi_c \rangle = C_{abc} \cdots$

If $C_{abc} \neq 0$; Then a & b , can fuse in c.

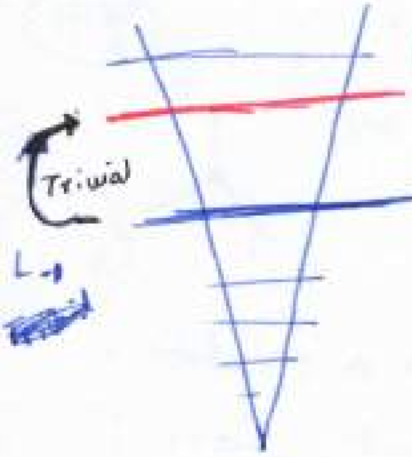
But if $C_{abc} \neq 0$; then we also have that :

a & b , can fuse to arbitrary desendent of c.

If $C_{abc} = 0$; Then we would not be able to couple any desendent fields.

* If they couple, They couple to all.
  (There are ∞ states which run as intermediate states)
* If they dont couple, They dont couple to anyone.

Any scalling field ; is Either a primary or a desendent of some primary.



What are sets which true at next level? (*)

Some sets of field here $A_1, A_2, \ldots, A_n$

(*) The trivial ones are all the derivatives of previous, ie: $\partial A_1, \partial A_2, \ldots, \partial A_n$.
But there might be more field than these.

So; Quasi Primary at the level N are the fields quotient with $L_{-1} Q_{n-1}$

$$Q_n \big/ (L_{-1} Q_{n-1})$$

Taking coset

ie; all the fields at that level which are not derivative of anyone before.

These are genuine new field appearing at next level.

Approach 2 // Representation theory in terms of States.

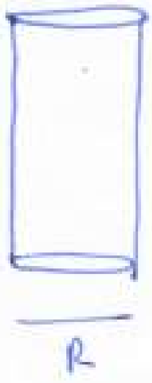
recall; States defined as $|\Delta\rangle = \lim_{z \to 0} \varphi_\Delta(z) |0\rangle$

s.t. $L_0|0\rangle = 0$
$L_1|0\rangle = 0$
$L_{-1}|0\rangle = 0$

$|0\rangle$ = vacuum state on which any observer agrees on.

$$\langle\Delta| = \lim_{z \to \infty} z^{2L_0} \langle 0| \varphi_\Delta(z)$$

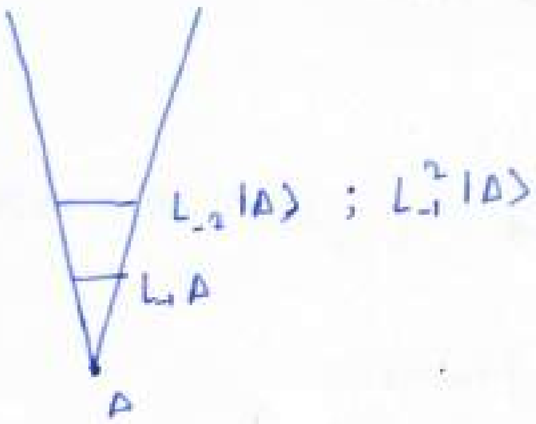
recall, L_0 played role of Hamiltonian on cylinder.



Conformal Hamiltonian :

$$H = \frac{2\pi}{R}\left(L_0 + \bar{L}_0 - \frac{c}{24} \right)$$

This acts on all the states on cylinder.



When we have Hilbert space; The natural question to ask is Scalar Product.

Computing Scalar product of the theory (by using Algebra)

$\langle\Delta|\Delta\rangle = 1$ (Normalization)

Then $L_{-1}|\Delta\rangle \Rightarrow$ what is norm of this.

Defination: $L_m^{+} = L_{-m}$ (we can derive it from defination of L_m ...)

Then The norm is $\langle\Delta| L_1 L_{-1} |\Delta\rangle$

ie; $\langle\Delta| L_{-1}^{+} L_{-1} |\Delta\rangle = \langle\Delta| L_1 L_{-1} |\Delta\rangle$
$\qquad = \langle\Delta| L_{-1} L_{+1} |\Delta\rangle + 2\langle\Delta| L_0 |\Delta\rangle$

since $|\Delta\rangle$ is primary, so its annihilated by positive modes.

$$\langle\Delta| L_1 L_{-1} |\Delta\rangle = 2\langle\Delta| L_0 |\Delta\rangle$$
$$= 2\Delta\langle\Delta|\Delta\rangle = 2\Delta$$

$$\Rightarrow \boxed{\langle\Delta| L_{-1}^{+} L_{-1} |\Delta\rangle = 2\Delta}$$

Now we can build up ~~matr~~ the following matrices called Gram Matrices

i.e.; go to level 2

we have ~~....~~ $L_{-1}^2|\Delta\rangle$ and $L_{-2}|\Delta\rangle$ as a vector

so; we can define the Gram Matrix

$$\begin{bmatrix} \langle\Delta| L_{+2} L_{-2} |\Delta\rangle & \langle\Delta| L_1^2 L_{-2} |\Delta\rangle \\ \langle\Delta| L_2 L_{-1}^2 |\Delta\rangle & \langle\Delta| L_1^2 L_{-1}^2 |\Delta\rangle \end{bmatrix}$$

ie. computing all the scalar between them

We can program computers to give Gram Matrixes.

here;
$$\begin{bmatrix} 4\Delta + \frac{c}{2} & 6\Delta \\ 6\Delta & 4\Delta(1+\Delta) \end{bmatrix}$$

Can there be any relation between C & D : such that this linear space is not expand by 2, but just 1 vector

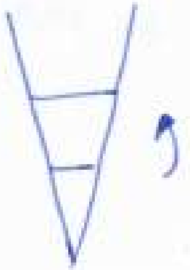
↪ ie; can there exist a null vector at that level.

So: Compute determinate & find roots.

here: $16\Delta^3 - 10\Delta^2 + 2\Delta\cdot c + \Delta\cdot c$

$= (\Delta - \Delta_{11})(\Delta - \Delta_{12})(\Delta - \Delta_{21})$ where $\Delta_{11} = 0$

ie; $\Delta = 0$ is a root. define $\Delta_{11} = 0$.



$\langle \Delta | L_1 L_{-1} | \Delta \rangle = 2\Delta$

The norm can be zero if

$\Delta = 0$

ie; at level 1 we can have null vector.

But if we have null vector at this level; we can move up by just applying L_{-1}.

There are extra roots also.

$$\Delta_{\substack{12 \\ 21}} = \frac{1}{16}(5-c) \pm \sqrt{(1-c)(25-c)}$$

plug $c = 1/2$ (Fermions, for example)

we get $\Delta_{12} = 1/2$, $\Delta_{21} = 1/16$

ie; If we ask; what are primary fields which has a null vector at second level: Answer: primary field with dimension $1/2$ & $1/16$.

These numbers are actually no. of ~~[illegible]~~;
~~[illegible]~~ easy modes.

$1/16$ is anamolous dimension of Magnetic field.

$1/2$ " " " Majorana field.

↳ Once we have null vector which is zero; we know exactly what is the expression which is null vector.

In this case;

The Null vector associated to these values are :

$$\left(L_{-2} - \frac{3}{2} \cdot \frac{1}{2\Delta+1} L_{-1}^{2} \right) \varphi_{\Delta} = 0$$

ie; if we insert this in any correlation function. for instance involving same field itself.

$$\left\langle \left(L_{-2} - \frac{3}{2} \cdot \frac{1}{2\Delta+1} \cdot L_{-1}^{2} \right) \varphi_{\Delta} \cdot \varphi_{\Delta} \varphi_{\Delta} \varphi_{\Delta} \right\rangle = 0$$

we know that any descendent is a differential operator

So; 4 point functions of fields, which are degenerate

ie; $\Delta = \Delta_{12}$ or Δ_{21} ; $\langle \varphi_{\Delta} \varphi_{\Delta} \varphi_{\Delta} \varphi_{\Delta} \rangle$

Satisfy linear Differential equation of order 2.

$$\mathcal{D}^{(2)} \langle \varphi_{\Delta} \varphi_{\Delta} \varphi_{\Delta} \varphi_{\Delta} \rangle = 0$$

will be hypergeometric equation

→ Hypergeometric equation, ... has coefficient which are gamma functions. When we go to limit, we get the value of structure constants.

→ So; Structure constants will be bunch of Gamma functions generically (Highly non-trivial numbers...)

We can get null vectors at arbitrary levels.

Lec : 7 : Verma module, Minimal modes, Kac Determinant, Diffⁿ equation for correlatⁿ function, Fusion Rules, Conformal Grid, Ising Model as an example,

lets take a primary field which has null vector at level 2 : $\varphi_{\Delta_{12}}$

$$\varphi_{\Delta_{12}}(x)\,\varphi_{\Delta} = \ldots\ ?$$

Answer// Whatever appears on RHS should be compatible with 2nd order linear differential equation which $\varphi_{\Delta_{12}}(z)$ satisfies.

$$\varphi_{\Delta_{12}}(x)\,\varphi_{\Delta} = \sum_{\Delta'} C^{\Delta'}_{(12)\,\Delta} \cdot \frac{\varphi_{\Delta'}}{(z_1 - z_2)^{\Delta_{12} + \Delta - \Delta'}} \qquad (**)$$

We want to constraint, what can be Δ'.

define $n = \Delta_{12} + \Delta - \Delta'$

we get an algebraic equation for n

$$\boxed{\frac{3n(n-1)}{2(2\Delta_{12}+1)} - \Delta + n = 0} \qquad (*)$$

we have function $f = \frac{1}{z^n}$

Then $\mathcal{D} f = \mathcal{D}\left(\frac{1}{z^n}\right) = 0$ at the leading order

(because ; do singularity analysis)

(*) has two solution.

So, There can be only two terms or say channels on RHS of (**)

ie; we are fusing two fields & asking how many primary fields run here

So; The only question we have to put is; how many primary run in that channel: because infinite no. is already ensured by the family.

What about if Δ is degenerate field itself?

Q. What is fusion rule of $\varphi_{12}\varphi_{mm}$

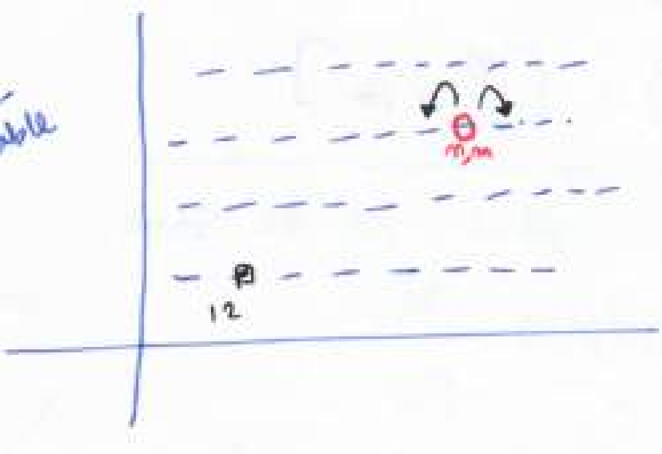
φ_{mm} belong to the table

Then it can have only two channels.

$$\varphi_{12}\,\varphi_{mm} = [\varphi_{m,m+1}] + [\varphi_{m,m-1}]$$

These are the channels which contribute.

Table



o ⇒ Take any field from the table. This is φ_{mm}

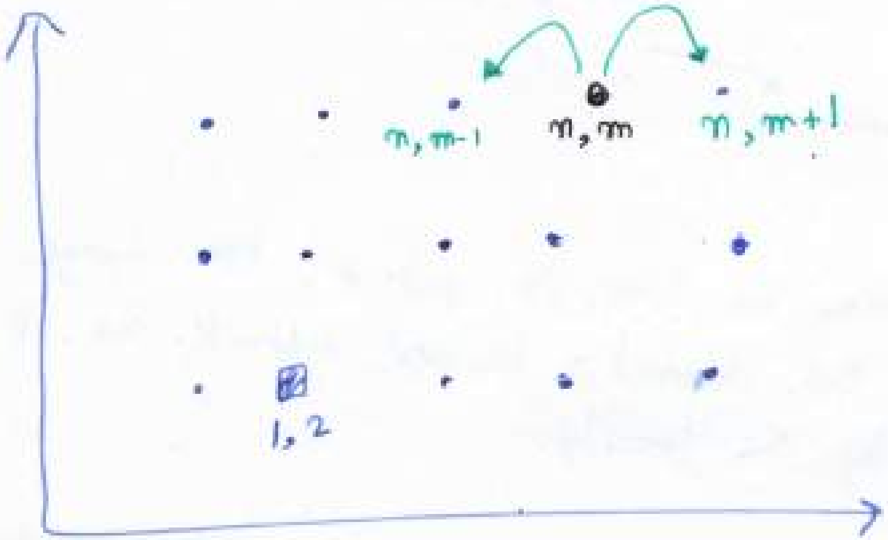
Take any field from the table; and ask what is its OPE with $\varphi_{\Delta_{12}}$

Simplicity write $\varphi_{\Delta_{12}} \equiv \varphi_{12}$

$\varphi_{\Delta_{n,m}} \equiv \varphi_{n,m}$

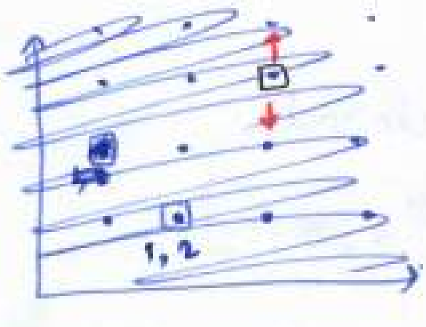
We get The channeles

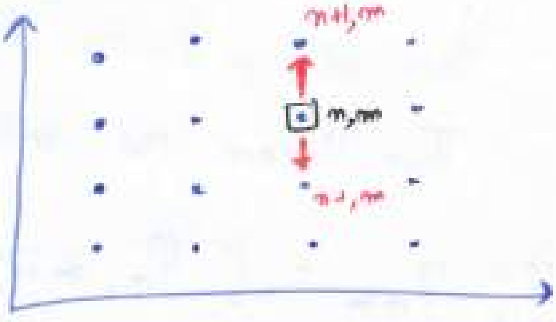
either $\varphi_{n,m-1}$ or $\varphi_{n,m+1}$



We can do the same with φ_{21}

Then $\varphi_{21}\varphi_{nm} = [\varphi_{n-1,m}] + [\varphi_{n+1,m}]$





$$\varphi_{12}\varphi_{n,m} = [\varphi_{n,m+1}] + [\varphi_{n,m-1}]$$

$$\varphi_{21}\varphi_{n,m} = [\varphi_{n-1,m}] + [\varphi_{n+1,m}]$$

φ_{12} & φ_{21} are like joystick of the game

- We can move vertically by applying φ_{21}
- " " " horizontally " " φ_{12}

$|\Delta\rangle$ is highest weight state ; i.e. $L_0|\Delta\rangle = \Delta|\Delta\rangle$
$L_n|\Delta\rangle = 0 \;, n>0$

The space of h.w. state & all its descendants is called Verma module : $V(c,h)$

$V(c,h)$ is mapped to itself by Virasoro Algebra.

<u>**Minimal Modes**</u> : (It's a subclass of CFT theories).
It has finitely many primary fields.

$|\chi\rangle \in V(c,\Delta)$ that fullfills $L_n|\chi\rangle = 0 \quad \forall\, n>0$.
and $|\chi\rangle \neq |\Delta\rangle$. Such $|\chi\rangle$ is called Singular Vector.
↳ It is also Null State.

Null States are orthogonal to whole Verma Mode.
i.e. $|\chi\rangle$ be singular ; And $L_{-k_1}L_{-k_2}\cdots L_{-k_m}^{\dagger}|\Delta\rangle$ a basis set state ~~state~~

Then the inner product is $\langle\chi|L_{-k_1}\cdots L_{-k_m}|\Delta\rangle$
$= \langle\Delta|L_{k_m}\cdots k_{k_1}|\chi\rangle^* = 0^* = 0.$

Also we find $\langle\chi|\chi\rangle = 0$ ✓.

Singular vectors are not the only null states ; Their descendants are orthogonal to whole Verma Module.

By quotienting out of the $V(c,\Delta)$ all the null submodules generated by the contained singular vectors ; The representation of the Virasoro Algebra is made irreducible.

i.e. Define an equivalence relation on $V(c,\Delta)$ by let $N(c,\Delta)$ denote null submodule

$|x\rangle, |y\rangle \in V(c,\Delta)$: $|x\rangle \sim |y\rangle$ if $|x\rangle - |y\rangle \in N(c,\Delta)$

Assume $V(c,h)$ is finite dimensional with basis vectors $\{|i\rangle\}$.

The Gram Matrix : $M_{ij} = \langle i | j \rangle$

if $|i\rangle$ & $|j\rangle$ are at different levels; Then $\langle i | j \rangle = 0$

~~Soos~~ This allows to write G_{ij} in Block Diagonal Form. Matri. with each block $M^{(N)}$ correspond to level N.

Kac Determinant

$$\cancel{\det M^{(l)} = \alpha_l \prod_{\substack{r,s \geq 1 \\ r,s \leq l}} \left[\Delta - \Delta_{r,s}(c)\right]^{P(l-rs)}}$$

where

$N=1$, $M^{(1)}(c,\Delta) = \langle \Delta | L_1 L_{-1} | \Delta \rangle = 2\Delta$

$\Rightarrow \det M^{(1)}(c,\Delta) = 2\Delta$

So; at level $N=1$, we have singular vector iff $\Delta = 0$

$N=2$ $\det(M^{(2)}(c,\Delta)) = 32(\Delta - \Delta_{11}(c))(\Delta - \Delta_{12}(c))(\Delta - \Delta_{21}(c))$

where the roots are

$$\Delta_{11} = 0$$

$$\Delta_{12} = \frac{1}{16}\left[5 - c - \sqrt{(1-c)(25-c)}\right]$$

$$\Delta_{21} = \frac{1}{16}\left[5 - c + \sqrt{(1-c)(25-c)}\right]$$

This means; There exist three states with zero norm at level $N=2$

Note: The root $\Delta_{11} = 0$ results from the descendant state of the singular vector at level 1.

Kac Determinant

$$\det M^{(l)} = \alpha_l \prod_{\substack{r,s \geq 1 \\ rs \leq l}} \left[\Delta - \Delta_{r,s}(c) \right]^{P(l - rs)}$$

where $$\Delta_{r,s}(c) = \Delta_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2$$

$$\alpha_{\pm} = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}$$

$$\Delta_0 = \frac{1}{24}(c-1)$$

and $P(l - rs)$ equals the no. of partitions of the integer $l - rs$.

Kac Determinant for Minimal Modes

If $\exists$ two coprime positive integers p and p', $p > p'$ such that $p\alpha_- + p'\alpha_+ = 0$.

Then we can write roots of the Kac Determinant & Central Charge c

$$\Delta_{r,s} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'} \qquad (**)$$

$$c = 1 - \frac{6(p-p')^2}{pp'} \qquad (***)$$

Properties:
$\Delta_{r,s} = \Delta_{r+p', s+p}$ (periodicity)

$\Delta_{r,s} = \Delta_{p'-r, p-s}$ (Symmetry)

$\Delta_{r,s} + rs = \Delta_{p'+r, p-s} = \Delta_{p'-r, p+s}$

$\Delta_{r,s} + (p'-r)(p-s) = \Delta_{r, 2p-s} = \Delta_{2p'-r, s}$

$$\Delta_{p'+r,\, p-s} + (p'+r)(p-s) = \Delta_{r,\, 2p-s} + (r)(2p-s)$$

Each singular vector results in a differential equation that constraints the Conformal weights & the central charge.

Due to ∞ no. of singular vectors; we have so many restrictions on $\Delta_{r,s}$ & such that after quotienting out the ~~[illegible]~~ null submodules, a finite set of Conformal family remains.

This finite set is closed under Fusion, and the no. of conformal families is limited by $1 \le r < p'$ and $1 \le s < p$ $(*)$

by using symmetry of ~~[illegible]~~ $\Delta_{r,s}$, we find that $\frac{(p-1)(p'-1)}{2}$ different conformal families are left.

$\to$ A model characterized by the coprime positive integers p & p' with a finite no. of primary field $\Phi_{r,s}$ restricted by $(*)$ with conformal weight & central charge as given by $(**)$ & $(***)$ is called a Minimal Model. $M(p, p')$

Differential Equations for the Correlation Functions.

To each descent $L_{-n}|\Delta\rangle$, There corresponds a descendant $\Phi^{(-n)}(w)$ of a primary field $\Phi(w)$ which is defined to be the field appearing in the operator product expansion of the primary with the energy momentum tensor

$$T(z)\Phi(z) = \sum_{n\geq 0} (z-w)^{n-2}\, \Phi^{(-n)}(w)$$

By performing an integration with deformed contours the descendant Φ^{-n} can be found to be

$$\Phi^{(-m)}(w) = \frac{1}{2\pi i}\oint_{w} dz \, \frac{1}{(z-w)^{m-1}} T(z)\Phi(z)$$

Given N primary fields $\{\Phi_i(w_i)\}_{i=1}^{N}$ with given conformal weight $(\Delta_i)_{i=1}^{N}$

Then $\langle \Phi^{(-m)}(w)\Phi_1(w_1)\ldots\Phi_N(w_N)\rangle = \mathcal{D}^{(m)} \langle \Phi(w)\Phi_1(w_1)\ldots\Phi_N(w_N)\rangle$ $(m\geq 1)$

where; the differential operator $\mathcal{D}^{(m)}$ is of the form

$$\mathcal{D}^{(m)}(w) = \sum_i \left(\frac{(m-1)\Delta_i}{(w_i-w)^m} - \frac{1}{(w_i-w)^{m-1}}\partial_{w_i} \right)$$

$\boxed{\text{proof}}$ $\langle \Phi^{(-m)}(w)\Phi_1(w_1)\ldots\Phi_N(w_N)\rangle$

$$= \frac{1}{2\pi i}\oint_{C(w)} dz \, \frac{1}{(z-w)^{m-1}} \langle (T(z)\Phi(w)\,\Phi_1(w_1)\ldots\Phi_N(w_N))\rangle$$

$$= \frac{-1}{2\pi i}\sum_{i=1}^{N}\oint_{C(w_i)} dz \cdot \frac{1}{(z-w)^{m-1}} \cdot \langle \Phi(w)\,\Phi_1(w_1)\ldots(T(z)\Phi_i(w_i))\ldots\Phi_N(w_N)\rangle$$

use that sum of residues (including residue at infinity is zero)

$$= \frac{-1}{2\pi i}\sum_{i=1}^{N}\oint_{C(w_i)} dz \cdot \frac{1}{(z-w)^{m-1}} \left[\frac{\Delta_i}{(z-w_i)^2} + \frac{1}{(z-w_i)}\partial_{w_i} \right] \langle \Phi(w)\Phi_1\ldots\Phi_N\rangle$$

$$= \mathcal{D}^{(m)} \langle \Phi(w)\,\Phi_1(w_1)\ldots\Phi_N(w_N)\rangle$$

Repeating this calculation shows,

That a correlator including a descendant of the form $\Phi^{(-k_1,\dots,-k_n)}(w)$ that corresponds to the state $L_{-k_1}\dots L_{-k_n}|\Delta\rangle$ in the Verma module can be replaced by a correlation functions of primaries acted on by a String of Diffun operators.

$$\langle \Phi^{(-k_1,\dots,-k_n)}(w)\,\Phi_1(w_1)\dots\Phi_N(w_N)\rangle = \mathcal{D}^{k_1}\dots\mathcal{D}^{k_n}\langle \Phi(w)\,\Phi_1(w_1)\dots\Phi_N(w_N)\rangle$$

Now;

we insert field corresponding to some singular vector of the reducible Verma module $V(c,\Delta_0)$ into a correlator.

Suppose $\displaystyle |\Delta_0 + n_0\rangle = \sum_{Y,\,|Y|=n_0} \alpha_Y\, \mathcal{D}^{Y}_{*} |\Delta_0\rangle$ is a singular vector at level n_0

<u>Notations)</u> $Y = \{r_1,\dots,r_k\}$ $(1\le r_1 \le \dots \le r_k)$

$|Y| = r_1 + \dots + r_k$

$\mathcal{D}^{Y} = \mathcal{D}^{r_1}\dots\mathcal{D}^{r_k}$

$\Leftrightarrow$ Quotienting this singular vector out of the Verma module; means that we also set the corresponding field to zero.

let Φ_0 be the field corresponding to $|\Delta_0\rangle$

Of course; correlation function of this nulli fied field with a chain of primary fields must vanish

So, we get

$$0 = \langle \sum_{Y, |Y| = n_0} \alpha_Y \, \Phi_0^{(-r_1, \ldots -r_n)}(w_0) \, \Phi_1(w_1) \ldots \Phi_N(w_N) \rangle$$

$$= \sum_{Y, |Y| = n_0} \alpha_Y \, \mathcal{D}^Y \langle \Phi_0(w_0) \, \Phi_1(w_1) \ldots \Phi_N(w_N) \rangle$$

$$\boxed{\sum_{Y, |Y| = n_0} \alpha_Y \cdot \mathcal{D}^Y \cdot \langle \Phi_0(w_0) \, \Phi_1(w_1) \ldots \Phi_N(w_N) \rangle}$$

$\rightarrow$ This equation restricts the Conformal weights of the primaries appearing in the correlator.

Example | Given a Verma module $V(c, \Delta)$ and we want to find singular vector at level 2.

A level 2 state can be written as a linear combination of $L_{-2} |\Delta\rangle$ & $L_{-1}^2 |\Delta\rangle$

so; $|\chi\rangle = (L_{-2} + a L_{-1}^2) |\Delta\rangle$ for some a.

And for $|\chi\rangle$ to be a singular state; It must satisfy $L_n |\chi\rangle = 0 \quad \forall n > 0.$

note | Its enough to demand that this equation holds for $n = 1$, and $n = 2$.
(Since the Virasoro Algebra implies that $|\chi\rangle$ is also annihilated by the L_n-generator for $n \geq 3$.

$L_1 |\chi\rangle = ([L_1, L_{-2}] + a [L_1, L_{-1}^2]) |\Delta\rangle$

$$\Rightarrow \boxed{L_1 |\chi\rangle = (3 + 2a(2h+1)) \, L_{-1} |\Delta\rangle}$$

so; $L_1|\chi\rangle = 0 \Rightarrow a = -\frac{3}{2(2\Delta+1)}$ if $\Delta \neq 0$.

we don't have to impose any condition on a.

If $\Delta = 0$; we don't have to impose any condition on a.

To get Relation between Δ and c,
apply L_2 on $|\chi\rangle$

ie; $L_2|\chi\rangle = ([L_2, L_{-2}] + a[L_2, L_{-1}^2])|\Delta\rangle$

$\Rightarrow \quad L_2|\chi\rangle = (2\Delta(2+3a) + \frac{c}{2})|\Delta\rangle$

$$\boxed{L_2|\chi\rangle = [2\Delta(2+3a) + \tfrac{c}{2}]|\Delta\rangle}$$

$$\Rightarrow \quad \boxed{c = 2\Delta\cdot\left(\frac{5-8\Delta}{2\Delta+1}\right)}$$

$$\Rightarrow \Delta = \frac{1}{16}\left[5 - c \pm \sqrt{(c-1)(c-25)}\right]$$

Plugging the corresponding null field into a correlator with a product $X = \Phi_1(w_1)\cdots\Phi_N(w_N)$ of primary fields we get

$$\left[\mathcal{D}^{(2)} - \frac{3}{2(2\Delta+1)}\left(\mathcal{D}^{(1)}\right)^2\right]\langle\Phi(w)X\rangle = 0$$

which can be written as.

$$\left[\sum_{i=1}^{N}\left(\frac{1}{w-w_i}\partial_{w_i} + \frac{\Delta_i}{(w_i-w)^2}\right) - \frac{3}{2(2\Delta+1)}\partial_w^2\right]\langle\Phi(w)X\rangle = 0$$

here; $\Phi(w)$ is the field corresponding to $|\Delta\rangle$, plugging $X = \Phi(w_1)$ does not give anything new. (because diff^n is trivially satisfied)

Plug $X = \Phi(w_1)\,\Phi(w_2)$

and use the general form of 3 point function.

$$\langle \Phi(w)\,\Phi_1(w_1)\,\Phi_2(w_2)\rangle = \frac{C_{\Delta,\Delta_1,\Delta_2}}{(w-w_1)^{\Delta+\Delta_1-\Delta_2}\,(w_1-w_2)^{\Delta_1+\Delta_2-\Delta}\,(w-w_2)^{\Delta+\Delta_2-\Delta_1}}$$

$C(\Delta,\Delta_1,\Delta_2) = C_{\Delta,\Delta_1,\Delta_2}$ is constant depending on Conformal weights.

We get the following constraints on Conformal weights.

$$\Delta_2 = \frac{1}{6} + \frac{\Delta}{3} + \Delta_1 \pm \frac{2}{3}\sqrt{\Delta^2 + 3\Delta\Delta_1 - \frac{1}{2}\cdot\Delta + \frac{3}{2}\Delta_1 + \frac{1}{16}} \qquad (*)$$

If we choose for example $h = h_{2,1}(c)$ and
$h_1 = h_{r,s}(c)$

The the above formula $(*)$ gives us two possible solutions for Δ_2.

$\hookrightarrow$ Comparing it with the results of Kac determinant, we find that the solution is precisely

$[\Delta_{r-1,s},\ \Delta_{r+1,s}]$.

First Fusion Relation

The OPE of the fields $\Phi_{2,1}$ with an arbitrary primary field $\Phi_{r,s}$ in a minimal model may only contain the fields $\Phi_{r+1,s}$ & $\Phi_{r-1,s}$.

$$[\Phi_{2,1}] \times [\Phi_{r,s}] = [\Phi_{r-1,s}] + [\Phi_{r+1,s}]$$

Notational language to express the First Fusion Rule.

here $[\Phi_{(r,1)}]$ denotes the conformal family of $\Phi_{(r,s)}$ and its descendants.

R.H.S. says: "at most two conformal families appear in OPE, but their coefficients could also be zero).

By generalising the same method for higher level singular vectors, the closed algebra for all conformal families in a minimal model is

$$[\Phi_{r_1,s_1}] \times [\Phi_{r_2,s_2}] = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2 = 1 \bmod 2}}^{k=r_1+r_2-1} \; \sum_{\substack{l=1+|s_1-s_2| \\ l+s_1+s_2 = 1 \bmod 2}}^{l=s_1+s_2-1} [\Phi_{k,l}]$$

Example The Ising Model (2d Ising model)

The Conformally invariant action of the Ising Model at the Critical point of its second-order phase transition yields a ~~charge~~ central charge $c = \frac{1}{2}$.

In the holomorphic part of the theory; we have three conformal families arising from three different primary fields.

These fields are

Vacuum Field	$\mathbb{1}$	$\Delta_{\mathbb{1}} = 0$
Spin Field	σ	$\Delta_\sigma = \frac{1}{16}$
Energy field	ϵ	$\Delta_\epsilon = 1/2$

We can identify this model with the minimal model $M(4,3)$ characterized by $p=4$, $p'=3$.

→ plugging this in the expression for conformal weights $\Delta_{r,s}$, $(1 \le r < 3$, $1 \le s < 4)$ and the central charge c;

it leads to exactly the given values for Ising model.

We draw a Conformal Grid which shows the conformal weights in dependance on r and s;

(which is invariant by a rotation by π around the centre due to the symmetry $h_{r,s} = h_{p-r,\, p-s}$.

	$r=1$	$r=2$
$s=1$	0	$1/2$
$s=2$	$1/16$	$1/16$
$s=3$	$1/2$	0

We identify the following fields

$$\mathbb{1} \Longleftrightarrow \Phi_1 := \Phi_{1,1}$$

$$\epsilon \Longleftrightarrow \Phi_2 := \Phi_{2,1}$$

$$\sigma \Longleftrightarrow \Phi_3 := \Phi_{2,2}$$

Since the Kacs formula predicts the first singular vector of representation with highest weight $\Delta_{r,s}$ at level rs; we conclude that both the energy &

The spin primary fields have singular vectors at level 2 ($\Phi_3 := \Phi_{1,2}$ ✓ also works)

→ This will be easy.

The singular vectors at level $N=2$ are ~~exactly the~~ exactly the one we predicted. ✓

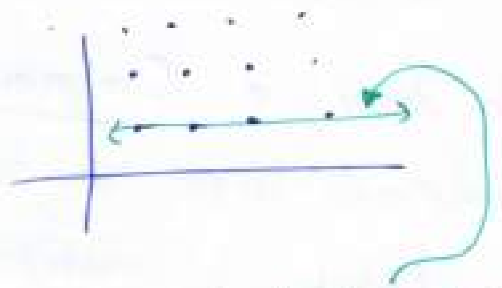
<u>Example</u> $\varphi_{12} \cdot \varphi_{12} = [\mathbb{1}] + [\varphi_{13}]$ Recall Ising model.

because $[\varphi_{11}] = [\mathbb{1}]$

Ising model. $\sigma \cdot \sigma = [\mathbb{1}] + [\varepsilon]$

$r=2$	$1/2$	$1/16$	0
$r=1$	0	$1/16$	$\frac{1}{2}$
	$s=1$	$s=2$	$s=3$

in field theory $\varepsilon = :\sigma^2:$

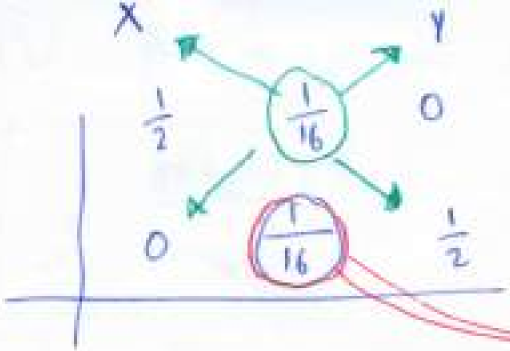


All the fields here are power laws of φ_{12} normal ordered.
ie; $:\varphi_{12}^{\,n}:$

$\Delta_{r,s}$ associated to $\mathcal{D}^{(r,s)}$

$\Delta_{(p-r),(q-s)}$ " " $\mathcal{D}^{(p-r)(q-s)}$

→ The same physical field algebraically sit in two different representations, And satisfy two differential equation of different order ⟹ The two has to be compatible



$[\frac{1}{16}] = [\Delta_{12}]$

What is

$\sigma \cdot \sigma = [\mathbb{1}] + [\varepsilon]$

This notation captures singularity of OPE.
(Not putting any power bcz because its fixed by dimension)

What about 2,2

$(2,2) = :\varphi_{12}\varphi_{21}:$

now writing OPE from this view point

$\sigma \cdot \sigma = [\mathbb{1}] + [\varepsilon] + [X] + [Y]$

(And at this level we cannot fix structure constant; because equations were linear. (it will come later)

but this has to be compatible with $\sigma \cdot \sigma = [\mathbb{1}] + [\varepsilon]$

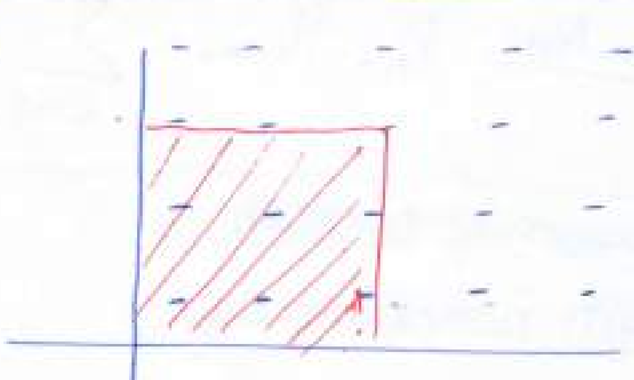
Therefore; Structure constant related to X & Y here is zero.

(Once we dont couple to primary; they dont couple to any of the Descendent)

Thinking about Conformal Table

run r & s for full plane

The only one which couple are those which full fill relation for C & $\Delta_{r,s}$.



→ So; if we are able to solve the dynamics of φ_{12} & φ_{21}; we are done.

Notation

$$\boxed{\varphi_{\Delta r,s} \equiv \varphi_{r,s}}$$

$$\varphi_{12} \cdot \varphi_{n\,m} = [\varphi_{n,m}] + [\varphi_{n,m-2}] + [\varphi_{n,m+2}]$$

Using $\Delta_{r,s} = \Delta_{p-r,\,q-s}$.

$\Downarrow$ corresponds to $\mathcal{D}^{(rs)}$

$\Downarrow$ this $\mathcal{D}^{(p-r)(q-s)}$ } Linear Differential Equation (LDE)

LDE span a linear space (A priory we know no. of solutions)

The fact that same field satisfies two different differential equation puts conditions which of the solution are the physical one. They have to be compatible

$$\boxed{\varphi_{12} \cdot \varphi_{12} = [\varphi_{11}] + [\varphi_{13}]}$$

~~Fusion theory~~

$$\varphi_{nm} = :\varphi_{12}^{\,n}\,\varphi_{21}^{\,m}:$$

Fusion Rules: $\varphi_{n_1 m_1} \cdot \varphi_{n_2 m_2} = \sum_{k=|n_1-n_2|+1}^{n_1+n_2-1} \sum_{l=|m_1-m_2|+1}^{m_1+m_2-1} [\varphi_{kl}]$

~~Complete Ising Model~~

~~Example~~ ~~Example of Nagat~~

Example/ Negative, Non unitary Minimal models.

Yang Lee Model $p=2, p=5$

Here we get $c = -22/5$

$$\begin{array}{|cccc} 0 & -1/5 & -1/5 & 0 \\ \hline \end{array}$$

even though we can $|\Delta\rangle = |-1/5\rangle$

and normalize it to 1

~~i.e~~ i.e; $\langle\Delta|\Delta\rangle = 1$

Then $\langle\Delta|L_1 L_{-1}|\Delta\rangle = 2\Delta\langle\Delta|\Delta\rangle = -\frac{2}{5}$

So; The theory is not positive definite.

---

c : Central Charge, ~~[illegible]~~ uniquely characterizes the Class of Universality

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Lec 8: Fusion Algebra, The Verlinde Formula (in brief)

The Fusion Algebra

OPE: $$\Phi_{\Delta_i}(z)\,\Phi_{\Delta_j}(w) \sim \sum_{\Delta_k} C^{\Delta_k}_{\Delta_i,\Delta_j}\,\Phi_{\Delta_k}(w)\,(z-w)^{\Delta_k-\Delta_i-\Delta_j}+\dots$$

Fusion Numbers: $$\mathcal{N}^{k}_{ij} = \begin{cases} 0 & \text{if } C^{\Delta_k}_{\Delta_i,\Delta_j}=0 \\ 1 & \text{otherwise.} \end{cases}$$

The fusion number counts the no. of independent possibilities to obtain a field Φ_{Δ_k} by fusing two fields Φ_{Δ_i} & Φ_{Δ_j}.

Generally: $\mathcal{N}^{k}_{ij}$ can take values larger than 1; but dont do so in Minimal Models

The Fusion Algebra: $$[\Phi_{\Delta_i}]\times[\Phi_{\Delta_j}] = \sum_k \mathcal{N}^{k}_{ij}\,[\Phi_k]$$

This indicates, which conformal families appear in the OPE of a field of the conformal family $[\Phi_i]$, and a member of the conformal family $[\Phi_j]$ without telling precise form of the OPE.

$\mathcal{N}^{k}_{ij} = \mathcal{N}^{k}_{ji}$ because of the way we are interpreting.

~~also~~ Due to associativity of Primary fields, Fusion algebra is also associative.

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Then $[\Phi_{\Delta_i}]\times([\Phi_{\Delta_j}]\times[\Phi_{\Delta_k}]) = [\Phi_{\Delta_i}]\times(\sum_l N_{jk}^{\,l}\, \Phi_{\Delta_l})$

$$= \sum_{l,m} N_{jk}^{\,l}\, N_{il}^{\,m}\, [\Phi_m]$$

and $([\Phi_i]\times[\Phi_j])\times[\Phi_k] = \sum_l N_{ij}^{\,l}[\Phi_{\Delta_l}]\times[\Phi_{\Delta k}]$

$$= \sum_{l,m} N_{ij}^{\,l}\, N_{lk}^{\,m}\, [\Phi_{\Delta m}]$$

$\Rightarrow$ $[\Phi_{\Delta_i}]\times([\Phi_{\Delta_j}]\times[\Phi_{\Delta_k}]) = ([\Phi_{\Delta_i}]\times[\Phi_{\Delta_j}])\times[\Phi_{\Delta k}]$

$$\Rightarrow \boxed{\sum_l N_{jk}^{\,l}\, N_{il}^{\,m} = \sum_l N_{ij}^{\,l}\, N_{lk}^{\,m}}$$

define matrix N_i with $(N_i)_{j,k} = N_{ij}^{\,k}$

Then $\boxed{N_i N_k = N_k N_i}$

The Verlinde Formula

Fusion matrices commute, and are normal.

so: $N_{ij}^{\,k} = S D S^{-1}$ $\qquad$ $S \Rightarrow$ Diagonalizing matrix

eigenvalues of N_i denoted by $\lambda_i^{(l)}$

$N_{ij}^{\,k} = (SDS^{-1})$

$$= \sum_{l,m} S_{jl}\, \lambda_i^{(l)}\, \delta_l^m\, (S^{(-1)})_{mk} = \sum_l S_{jl}\, \lambda_i^{(l)}\, (S^{-1})_{lk}$$

$$\boxed{N_{ij}^{\,k} = \sum_l S_{jl}\, \lambda_i^{(l)}\, (S^{-1})_{lk}}$$

Additionally; we can calculate $N_{i0}^{\,k} = \delta_{ik}$ trivial.

$\Phi_{\Delta_0} \mathbb{1} \sim$

$$S_{im} = \sum k N^k_{i0} S_{km} = S_{0m}\, \lambda^{(m)}_i$$

$$\Rightarrow \boxed{S_{im} = S_{0m} \cdot \lambda^{(m)}_l} \Rightarrow \boxed{\lambda^{(l)}_i = \frac{S_{il}}{S_{0l}}}$$

$$\left. N^k_{ij} = \sum_l \frac{S_{jl}\, S_{il}\, (S^{-1})_{lk}}{S_{0l}} \right\} \rightarrow \text{Verlinde formula.}$$

Erik Verlinde gave interpretation to this.

He stated Modular Transformation: $S : \tau \longrightarrow \frac{-1}{\tau}$ diagonalizes the fusion rule.

Character of a Verma Module $V(c,h)$ with c, and Δ

$$\chi_{c,\Delta}(\tau) = \mathrm{Tr}\; q^{L_0 - \frac{c}{24}} \qquad (q := e^{2\pi i \tau})$$

Since any state in the Verma module is an eigenstate of L_0 with an eigenvalue of the form $\Delta + N$,

we can write $\chi_{c,\Delta}(\tau) = q^{h - \frac{c}{24}} \cdot \sum_{N=0}^{\infty} P(N)\, q^N$

where $P(N)$ counts the number of states at level N
$\longrightarrow$ use Ramanujan Formula.

$$\chi_{r,s}\left(\frac{-1}{\tau}\right) = \sum_{(\rho,\sigma) \in E_{p,p'}} S_{rs,\rho\sigma}\, \chi_{\rho\sigma}(\tau)$$

$E_{p,p'} \Rightarrow$ set of all irreducible highest weight representations.
$\hookrightarrow$ consists of $\frac{(p-1)(p'-1)}{2}$ elements.

$$S_{rs;\rho\sigma} = 2\sqrt{\frac{2}{pp'}} \cdot (-1)^{1+s\rho+r\sigma} \cdot \sin\left(\pi \frac{p}{p'} r\rho\right) \sin\left(\pi \frac{p'}{p} s\rho\right)$$

Fusion no. for minimal modes;

$$\mathcal{N}_{rs,mn}^{kl} = \sum_{(i,j)\in E_{p,p'}} \frac{S_{rs,ij}\, S_{mn,ij}\, S_{ij,kl}}{S_{11,ij}}$$

Importance of Verlinde Formula

① It combines local as well as global properties in CFT.

② The fusion number $\mathcal{N}_{ij}^{k}$ contain information about the local OPE of the two fields

whereas; the modular transformation S is related to the global modular invariance of partition functions on the torus.

SYMMETRIES CAN GOVERN US.

THANK YOU

Shoaib Akhtar