LIE GROUPS & LIE ALGEBRAS

SHOAIB AKHTAR

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These notes are consequence of my self study; and are mostly inspired from <u>Prof. Tibra Ali</u> lectures on **Lie Groups and Lie Algebras**.

Lie Groups and Lie Algebras — Shoaib Akhtar — 1/5/2020

Lecture 1 : Motivation and fundamental concepts of group theory. Examples of Group.

Motivations: Symmetries of Physical system

1905 - Einstein's Special Relativity ; Symmetries of Maxwell equations were not same as symmetry of Newtonian physics.

Standard Model of particle physics: $SU(3) \times SU(2) \times U(1)$ — Internal (gauge) symmetry.

Classical Field Theory — 100 years ago Emmy Noether ; Symmetries $\Longrightarrow$ Conserved quantities.

ex Expanding Universe, energy is not conserved.

Wigner's Theorem : Hilbert space of quantum system furnishes an unitary & linear representation of the symmetry group of the system.

$\longrightarrow$ Weinberg Vol I.

Fundamental Concepts

Defination: Group is a ~~set~~ set G equipped with a binary operation $\circ$ $G \times G \longrightarrow G$ known as group multiplication ; under which the set remains closed, and satisfies the following.

(i) Closure if $a, b \in G$ then $a \circ b \in G$

(ii) Associative $(a \circ b) \circ c = a \circ (b \circ c)$

(iii) $\exists$ a unique element $e \to$ identity :

$$a \circ e = a \quad \forall a \in G.$$

(iv) Inverses, for each a in G, $\exists$ a unique inverse a^{-1} s.t. $a \circ a^{-1} = e$

Remarks // $a^{-1} \rightarrow$ right inverse. (as we defined)

we can show a^{-1} is also left inverse

i.e. $a^{-1} \cdot a = e$

$\therefore$ ~~$a \circ a^{-1} = e$~~ ;

Proof // $a^{-1} \circ a = a^{-1} a e$

$= a^{-1} a a^{-1} (a^{-1})^{-1}$

$= a^{-1} (a a^{-1}) (a^{-1})^{-1}$

$= a^{-1} e (a^{-1})^{-1}$

$= a^{-1} (a^{-1})^{-1} = e$

Jambhekar's Book volume 2 (Reference)

we can show, $ea = ae$

(right identy is left identity also)

$$ea = (a a^{-1}) a = a \cdot (a^{-1} a) = a \cdot e$$

\# $G \rightarrow$ finite or infinite (countably infinite or uncountably infinite)

Size of G, order of G.

\# $ab \neq ba$ in general.

If all elements of G commute, we call the group Abelian.

If $\exists$ at least 2 element $a, b \in G$, s.t. $ab \neq ba$, then G is called non-Abelian.

Examples

(1) $C_4 = \{ \alpha, \alpha^2, \alpha^3, \alpha^4 = e \}$; $\alpha^n = \underbrace{\alpha \cdot \alpha \cdots \alpha}_{n \text{ times}}$.

cyclic group

$e = \alpha^4$

$a = \alpha$

$b = \alpha^2$

$c = \alpha^3$

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

2 $\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$

	+	0	1	2	3
$e \to$	0	0	1	2	3
$a \to$	1	1	2	3	0
$b \to$	2	2	3	0	1
$c \to$	3	3	0	1	2

$\mathbb{Z}/4\mathbb{Z}$ and C_4 are actually Isomorphic.

Group Isomorphism

eg $C_4 \cong \mathbb{Z}/4\mathbb{Z}$

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eg II Permutation Group P_3: (Reference) George I. Book in Lie groups.

(a, b, c) : The set of operations on (a, b, c) form a group.

Elements: ( ) $\to$ identity element _ do nothing
$(a, b, c) \longrightarrow (a, b, c)$

(1 2) exchange objects in position 1 & 2 : $(a, b, c) \to (b\ a\ c)$
(2 3) " " " " 2 & 3 ; $(a, b, c) \to (a, c\ b)$
(1 3) " " " " 1 & 3 : $(a\ b\ c) \to (c\ b\ a)$
(1 2 3) $1 \to 2 \to 3 \to 1$ $(a\ b\ c) \to (c\ a\ b)$
(3 2 1) " $(a\ b\ c) \to (b\ c\ a)$

Why does these operation create a group?
After each operation the ~~syte~~ system has a well defined configuration.

$\hookrightarrow$ an example of Transformation Group.

$\hookrightarrow$ Non Abelian. eg $(2\ 3)(1\ 2) = ?$
(12): $(a\ b\ c) \to (b\ a\ c)$
(2 3) $(b\ a\ c) \to (b\ c\ a)$

$\Rightarrow (2\ 3)(1\ 2) = (3\ 2\ 1)$

$(12)(23) = (123)$

$(23): (abc) \to (acb)$

$(12): (acb) \to (cab)$

<u>Cayley's Theorem</u>: Every finite group is a proper subgroup of a permutation group.

example 4) <u>Rotation in 2D</u>

$\vec{r} = \hat{i}x + \hat{j}y$; Rotate $\vec{r}$ by some angle θ

$$\vec{r} \to R(\theta)\, r$$

$$\Rightarrow \sqrt{r.r} \to \text{unchanged.}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

form group under multiplication:

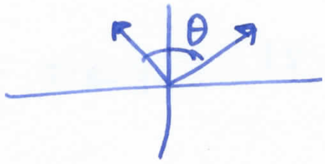
$$R(\theta_1) R(\theta_2) = R(\theta_1 + \theta_2)$$

$\det R = 1$; $R^T R = R R^T = \mathbb{I}$

This group is known as $SO(2)$

$$R \in SO(2) \quad ; \quad 0 < \theta \leq 2\pi.$$

All complex phases $e^{i\theta}$ form group



$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$$

$z = x + iy$; $e^{i\theta} z$; $\theta < 0 \leq 2\pi$.

The group of complex phases $e^{i\theta}$ forms the group $U(1)$

& $U(1) \cong SO(2)$

<u>Defination</u>

Subgroup: If a subset $H \subset G$ form a group by itself under the same multiplication law; then H is subgroup of G.

if $H \neq G$ & $H \neq \{e\}$ we say H is proper subgroup.

otherwise we say, H is an improper subgroup.

Coset

~~Right~~ Left Coset: let $H = \{h_1, h_2, \dots\}$ (it can be continuous) be a proper subgroup of G. let $g \in G$ which is not in H.

The set $\{g \cdot h : \forall h \in H\} \rightarrow$ Right coset gH

Right coset is not a group in general, it is a set; & depends on which element we choose.

$\boxed{\text{if } g \in H \text{; then } gH = H \text{ (haha..)}}$

gH in $G - H$

$G = H + g_1 H + g_2 H + \dots$

Coset Decomposition.

we can decompose the group as sum of sets which are non-intersecting

Coset Space $G \backslash H = \{H, g_1 H, g_2 H, \dots\dots\}$

Exercise! Consider $P_2 \subset P_3$; $P_2 = \{(\ ), (1\ 2)\}$

work out all the cosets. (left cosets) w.r.t. P_2.

Normal or Invariant Subgroups

If $\forall g \in G$ we have $gH = Hg$

then H is a normal or invariant subgroup.

Definition

Simple Group: If e and G are only normal subgroup of G, then we say G is simple group.

Centre : A subset Z of G is called the centre if $\forall z \in Z$ and $\forall g \in G$, we have $zg = gz$.

Lec 2: Fundamental Concepts of Representation Theory, Groups and Quantum Mechanics, Lie groups

— Shoaib Akhtar 1/5/2020.

Lagrange's Theorem ~~For cyclic groups the order of proper subgroup~~ For any finite group the order of proper subgroup is a divisor of the order of the group.

$$H \subset G \; ; \quad n \cdot O(H) = O(G) \quad ; n \in \mathbb{N}.$$

if $O(G) =$ prime $\Rightarrow$ Then G cannot have proper subgroups

$\Rightarrow$ so; G is simple group.

Basic definations of Representation Theory.

Interested in mainly Representations of group in terms of linear operators (differential operators, matrices, etc).

Defn 1| The representation of G is a map D of the elements of G onto a set a linear operators such that

$D(e) = \mathbb{1}$, where $\mathbb{1}$ is identity operator acting on the vector space.

group multiplication is respected;
if $a, b, c \in G$ s.t. $ab = c$; $D(a)\,D(b) = D(c)$

[Caveit, In QM (Quantum Mechanics) a phase is possible; known as Projective Representations]

Example: 1) $C_4 = \{\alpha, \alpha^2, \alpha^3, \alpha^4 = e\}$

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \quad D(a) = \frac{1}{2}\begin{pmatrix} 1 + e^{i\pi/2} & -1 + e^{i\pi/2} \\ -1 + e^{i\pi/2} & 1 + e^{i\pi/2} \end{pmatrix}$$

$$D(b) = [D(a)]^2 \quad ; \quad D(c) = [D(a)]^3$$

Example 2/ $D(e) = 1 = D(a) = D(b) = D(c)$

$\hookrightarrow$ Trivial Representation.

(In QFT we will see; the scalar field will be trivial representation in some sense of the Lorentz group; although ~~??~~ an infinite dimensional)

Faithful Rep. If D is a rep of G s.t. every element $g \in G$ is mapped to a distinct element $D(g)$; then D is a faithful representation.

otherwise $\Rightarrow$ Unfaithful rep.

Equivalent Reps Two representations D & D' are said to be equivalent if $\exists$ S (an operator S) s.t.

$$D'(g) = S D(g) S^{-1} \quad \forall g \in G.$$

(connected by ~~??~~ similarity tranformation)

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ex A representation of C_4

$$D'(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D'(a) = \begin{pmatrix} e^{i\pi/2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$D'(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D'(c) = \begin{pmatrix} e^{3\pi i/2} & 0 \\ 0 & 1 \end{pmatrix}$$

is faithful rep.

We can verify; $D'(g) = S D(g) S^{-1}$

where; $S = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

So; D' & D are equivalent reprs.

Irreducible Representation

A rep D is said to be reducible if it has an invariant subspace.

If the action of $D(g)$, $\forall g \in G$ on a subspace of the full space remains in the subspace.

~~In other words, P is projection operator onto subspace on V. Then if~~ In other words, P is projection onto subspace on V. Then if $PD(g)P = D(g)P \quad \forall g \in G$

P is some subspace of the vector space

In practicle terms, there exists some S s.t.

$$S D(g) S^{-1} = \begin{bmatrix} D_1(g) & 0 & \cdots \\ 0 & D_2(g) & \vdots \\ \vdots & & D_3(g) \\ \vdots & \cdots & & \ddots \end{bmatrix} \quad \forall g \in G$$

where $D_i(g)$ form reps of G.

Fully Reducible If $D(g)$ can be brought to the form

$$D(g) = \begin{pmatrix} A_1 & 0 & 0 & \cdots \\ 0 & A_2 & & \\ \vdots & & \ddots & \\ \vdots & & & A_n \end{pmatrix} \quad \forall g \in G$$

where A_is are not necessarily faithful representation of G, then D is fully reducible.

Groups and QM

Symmetries plays a rule in Q.M,

due to a theorem by Wigner 1931:

Theorem: The group of symmetries of quantum systems is represented by linear, unitary or antilinear, antiunitary operators acting on the Hilbert Space of the theory.

(Weinberg QTF volume 1)

Outline of proof: $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$. Under symmetry transformation

$$|\psi_1\rangle \longrightarrow U|\psi_1\rangle$$
$$|\psi_2\rangle \longrightarrow U|\psi_2\rangle$$

~~But~~ Because this is symmetry transformation, the probabilities should be invariant. $|\langle\psi_2|U^\dagger U|\psi_1\rangle|^2 = |\langle\psi_2|\psi_1\rangle|^2$

$\Rightarrow$ $U \to$ unitary : $\langle\psi_2|U^\dagger U|\psi_1\rangle = \langle\psi_2|\psi_1\rangle$

another solution

$U \to$ antiunitary : $\langle\psi_2|U^\dagger U|\psi_1\rangle = \langle\psi_2|\psi_1\rangle^*$

Antilinearity : $U(\alpha|\psi\rangle + \beta|\phi\rangle) = \alpha^* U|\psi\rangle + \beta^* U|\phi\rangle$

ex// If time Reversal T is symmetry ; then $[T, H] = 0$

then $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$

$T|\psi(t)\rangle = |\psi(-t)\rangle = T e^{-iHT} T^{-1} T|\psi(0)\rangle$

$\boxed{\because T^2 = \mathbb{1} \Rightarrow T^{-1} = T}$

we also know: $|\psi(-t)\rangle = e^{+iHt}|\psi(0)\rangle$

$\Rightarrow T|\psi(t)\rangle = Te^{-iHt}T(T|\psi(0)\rangle)$

$= Te^{-iHt}T|\psi(0)\rangle$

$T|\psi(0)\rangle = T\psi(-0)\rangle$
$= |\psi(0)\rangle$

In other word ; $|\psi(-t)\rangle = Te^{-iHT}T|\psi(0)\rangle$

$\Rightarrow \boxed{Te^{-iHt}T = e^{iHt}}$ $(*)$

if $[T, H] = 0$ is true ; and T is a unitary or linear operator ; then $(*)$ cannot be true.

$\therefore$ so; $T(i) = -iT$; then $(*)$ makes sense.

Degeneracy and Groups

$G \to$ group of symmetries of Quantal system
($\hookrightarrow$ actually adjective for quantum)

D is Rep ; Then $[D(g), H] = 0$

& $|\psi\rangle$ be an eigenstate of H : $H|\psi\rangle = E|\psi\rangle$

Let the subspace of $\mathcal{H}$ spanned by applying $D(g)$s on eigenstate $|\psi\rangle$: $D(g)|\psi\rangle$ $\forall g \in G$.

be given by $\{|\psi_1\rangle, \dots |\psi_n\rangle\}$ $\to$ linearly independent set

Since $[D(g), H]$; $H|\psi_i\rangle = E|\psi_i\rangle$ has same eigen value.

$$D(g)|\psi_i\rangle = \sum_j |\psi_j\rangle\langle\psi_j|D(g)|\psi_i\rangle$$

it is enough to decomposition of identity in this subspace $\{|\psi_i\rangle, \dots |\psi_2\rangle\}$ because it is invariant.

$$\Rightarrow D(g)|\psi_i\rangle = \sum_j |\psi_j\rangle D(g)_{ji}$$

$D(g)_{ji} \to n \times n$ irrep of G.

Lie Groups and Lie Algebras

Introduced by Sophus Lee ; while trying to understand the geometry of solutions space of differential equations.

Continuous groups $\Rightarrow$ Elements are parametrized by set of continuous parameters.

Example 1 $U(1) \to e^{i\theta}$; $0 < \theta \leq 2\pi$

we can identify parameter space with circle

so; we can write $U(1) \equiv S^1$.

Because of continuous nature of parameters; it is natural to think of Lie groups as manifolds.

Def. of Lie group

A lie group G is a group which is also a manifold.

It is equipped with multiplication : $G \times G \to G$ with the usual properties of group.

G includes a identity element & for each element there is a unique inverse.

It will turn out which one is identified as the identity element will depend on how we parametrize the group

Suppose, that the group elements are parametrized by $\Phi = (\phi_1, \dots, \phi_n)$

we write the generic element $g(\Phi)$ and

$e = g(\Phi = 0)$

No. of independent parameters is dimensions of the group.

We concentrate on Matrix group $\Rightarrow$ defined by some matrix equation.

Relationship between Lie Groups & Lie Algebra.

Consider a spin $\frac{1}{2}$ particle.

Its wavefunction is given by a vector $|\psi\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}$

Rotation by θ around some direction $\hat{n}$.

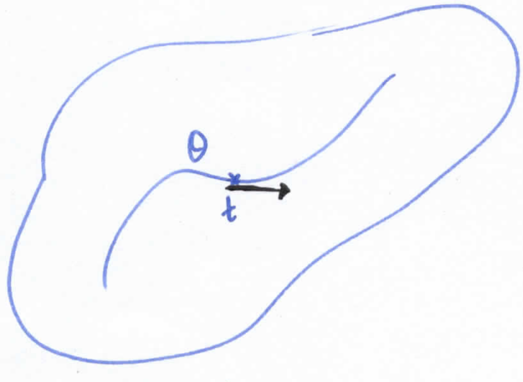
Rotation is implemented by $e^{\frac{1}{2}\theta \hat{n}\cdot\sigma} |\psi\rangle$

$\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$

Tangent Vector

Suppose we have a curve on the manifold.

$\theta(t) = \{\theta_1, \theta_2, \theta_3\}(t)$



Tangent vector to $\bar{\theta}(t)$

$\bar{v}(t) = \frac{d\bar{\theta}(t)}{dt}$

Coordinate representation of tangent vector.

M

exercise $e^{i\frac{\theta_3}{2}\sigma_3} = \cos\frac{\theta_3}{2} + i\sigma_3 \sin\frac{\theta_3}{2}$

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

use $\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Pauli's dont commute.

Groups formed by $e^{\frac{i}{2}\theta \hat{n}\cdot\sigma}$ is non-abelian group :

$$\left[e^{\frac{i\theta_3}{2}\sigma_3}, e^{\frac{i\theta_2}{2}\sigma_2}\right] \neq 0$$

we can use $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$.
to prove.

<u>Exponential Map</u> Given a pauli matrix $\frac{i\sigma_a}{2}$

$$\exp : i\frac{\sigma_a}{2} \longrightarrow \exp\left[\frac{i\sigma_a\theta}{2}\right]$$

$\exp\left[\frac{i\sigma_a\theta}{2}\right]$ ~ form an Abelian subgroup.

$$e^{i\sigma_1\theta_1/2}\, e^{i\sigma_1\theta_2/2} = e^{i\sigma_1(\theta_1+\theta_2)/2}$$

$$e^{i\theta\sigma_1/2} \cdot e^{-i\sigma_1\theta/2} = \mathbb{1}$$

Lec 3 : Lie Algebras from Lie groups. The adjoint Representation.

-Shoaib Akhtar 1/5/2020

Representation is reducible if it can be ~~?~~ written in upper triangular form.

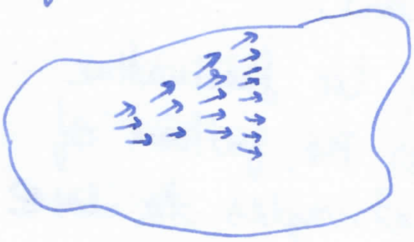
i.e; if $$D(g) = \begin{pmatrix} D_1(g) & A(g) \\ 0 & D_2(g) \end{pmatrix}$$

$$V = \begin{pmatrix} W_\perp \\ W \end{pmatrix}$$ there is an invariant subspace W which is preserved.

Fully Reducible if $$D(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix}$$

Given a manifold M.

However on a Lie Group of dim n, one can naturally define n vector fields.



Tangent space at the identity element:

$$\frac{1}{i}\frac{d}{d\theta_a} e^{i\theta_a \frac{\sigma_a}{2}} \Big|_{\theta_a = 0} = \frac{\sigma_a}{2}$$

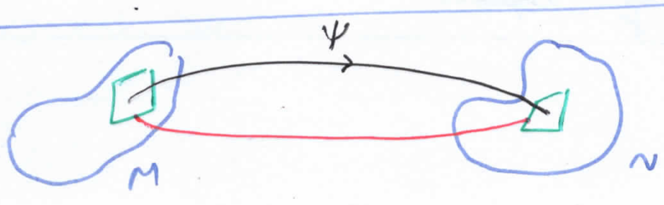
(no sum over a)

$\frac{\sigma_a}{2} \to$ tangent vectors at T_e.

$g = g(\phi_1, \phi_2, \phi_3)$

$$\frac{\sigma_a}{2} \longrightarrow e^{i\frac{\bar{\phi}\cdot\bar{\sigma}}{2}} \frac{\sigma_a}{2} e^{-i\frac{\bar{\phi}\cdot\bar{\sigma}}{2}} = \frac{\tilde{\sigma}_a(\phi)}{2}$$

$\Big\}$ ← This is what you get when you integrate Lie Derivative



Induces push forward map.

$$\psi_* : T(M) \to T(N)$$

$g e = g$ its a map from $M \to M$.

$\{ \frac{\sigma_a}{2}, 0 \}$ forms vector space.

For $\frac{\sigma_a}{2}$ ~~for~~ for $a = 1, 2, 3$; $\underbrace{g \frac{\sigma_a}{2} g^{-1}}_{\text{gives vector field.}}$ $\forall g \in G$.

Lie Derivative

$$\mathcal{L}_X Y = [X, Y]$$

$XYf \neq Zf$ (in general) ; $[X, Y] f = Z f$

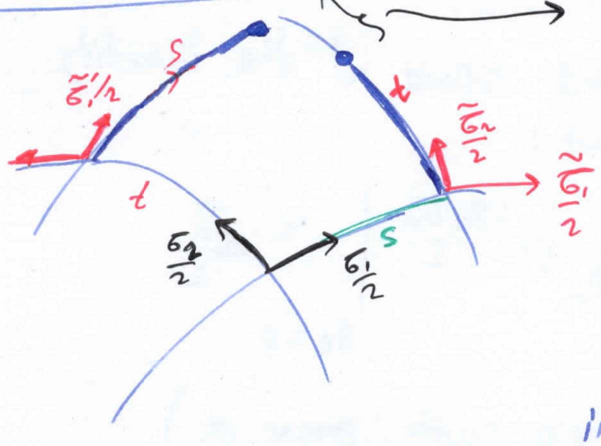
X, Y tangent vector
so: $Z \Rightarrow$ tangent vector.

$\therefore$ so; if we take $\underbrace{\left[\frac{\sigma_a}{2}, \frac{\sigma_b}{2} \right] = i f_{ab}{}^{c} \frac{\sigma_c}{2}}_{\text{called Lie Algebra.}}$ (sum over c)

$\hookrightarrow$ some linear combination;

$\therefore \quad f_{ab}{}^{c} = \varepsilon_{abc}$

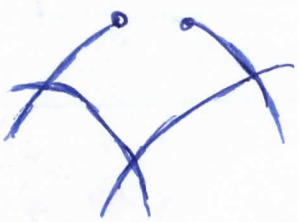
$f_{ab}{}^{c} \Rightarrow$ Structure Constant. : $\boxed{f_{ab}{}^{c} = - f_{ba}{}^{c}}$



These rectangles dont close. in general.

$\hookrightarrow$ The Lie Derivative measures the failure of ~~rect~~ rectangles to close.

$\hookrightarrow$ We can think of it as a infinitesimal measure of non-commutativity.



$$\left[\frac{\tilde{\sigma}_a(x)}{2}, \frac{\tilde{\sigma}_b(x)}{2} \right] = i f_{ab}{}^{c} \frac{\tilde{\sigma}_c(x)}{2}$$

Tangent vectors at other points satisfy the same Lie Algebra. (Nothing ~~special~~ special about e)

Lie Bracket

On a typical Lie Algebra.

$$[T_a, T_b] = i f_{ab}{}^{c} T_c$$

$T_a \rightarrow$ generators (because they generate the group through exponential map)

$f_{ab}{}^{c} \rightarrow$ structure constants.

Compact Lie groups $f_{ab}{}^{c} = f_{abc} \rightarrow$ completely antisymmetric.

* $[X, Y] = -[Y, X]$
* $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

satisfy Jacobi Identity.

$\hookrightarrow$ The relationship we get from this for structure constant is.

$$f_{ab}{}^{d} f_{dc}{}^{e} + f_{ca}{}^{d} f_{db}{}^{e} + f_{bc}{}^{d} f_{da}{}^{e} = 0$$

Notions of Reducibility of groups transfer over to Lie Algebra

$$e^{i\frac{\bar{\sigma}\cdot\theta}{2}} |\bar{\Psi}\rangle \quad ; \quad \theta \cdot \frac{\bar{\sigma}}{2} |\bar{\Psi}\rangle$$

$$\left(\mathbb{1} + \frac{i\bar{\sigma}\cdot\theta}{2}\right)$$

$$\underset{\uparrow \text{ 2x2 matrix}}{e^{\frac{i\bar{\theta}\cdot\bar{\sigma}}{2}}} \begin{pmatrix} |\Psi_1\rangle \\ |\Psi_2\rangle \end{pmatrix} \qquad \underset{\uparrow \text{ 2x2 matrix}}{\frac{\sigma_a}{2}} \begin{pmatrix} |\Psi_1\rangle \\ |\Psi_2\rangle \end{pmatrix}$$

as matrix they have same dimensionality; so they can act on same vector space.

Given a Lie Group, $\longrightarrow$ always derive Lie Algebra.

Lie Algebra $\longrightarrow$ getting the group is not unique.

ex

$$\mathcal{L}(SU(2)) \rightarrow SU(2)$$
$$\mathcal{L}(SU(2)) \rightarrow SO(3)$$

The Adjoint Action of g on $\mathcal{L}(G)$

$$Adj_{(g)}(T_a) = g T_a g^{-1} .$$

Claim: The Adjoint action on $\mathcal{L}(G)$ by group G form an irreducible representation of $\mathcal{L}(G)$ acting on itself.

$\therefore \quad g T_a g^{-1} \equiv \tilde{T}_a$

we can check $\tilde{T}_a$ satisfy same lie Algebra.

$$[\tilde{T}_a, \tilde{T}_b] = i f_{ab}{}^{c} \tilde{T}_c$$

so: $\tilde{T}_a$ should be able to expanded as linear combination of original lie algebra. so: $\underline{\tilde{T}_a = C_{ab} T_b}$

(because if T_a is in Lie Algebra, then also $\tilde{T}_a$ is in Lie Algebra... that's what the claim is saying)

Lets take an infinitesimal element

$$g \approx \mathbb{1} + i \theta_c T_c$$

since, as $\theta \to 0$, we must have $\tilde{T}_a = T_a$

so: $C_{ab} \simeq \delta_{ab} + i \theta_c F_{abc}$

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expanding both sides give

$$\cancel{T_a} + i \theta_c [T_c, T_a] = \cancel{\delta_{ab} T_b} + i \theta_c F_{abc} T_b$$

$$\Rightarrow \boxed{[T_a, T_c] = - F_{abc} T_b}$$

$$(F_c)_{ab} \equiv -F_{abc}$$

define a matrix F_c

$\underline{F_c \to n \text{ dimensional representation of Lie Algebra.}}$

~~If we say $if_{ab}{}^{c} = -F_{abc} = (F_c)_{ab}$~~

If we say $i f_{ac}{}^{b} \equiv - F_{abc} = (F_c)_{ab}$

$$\boxed{(F_c)_{ab} = - i f_{ca}{}^{b}}$$

now, we need to compute $[F_c, F_d]_{ab}$

$$[F_c, F_d]_{ab} = (F_c)_{ae}(F_d)_{eb} - (F_d)_{ae}(F_c)_{eb}$$

$$= - f_{ca}{}^{e} f_{de}{}^{b} + f_{da}{}^{e} f_{ce}{}^{b}$$

$$= - f_{ca}{}^{e} f_{de}{}^{b} - f_{ad}{}^{e} f_{ce}{}^{b}$$

$$= i^2 f_{cd}{}^{e} f_{ae}{}^{b}$$ using Jacobi Identity

$$= i f_{cd}{}^{e} \cdot (- i f_{ea}{}^{b})$$

$$= i f_{cd}{}^{e} (F_e)_{ab}$$

So, we have ~~to~~ shown that: adjoint action gives rise to a representation which is the adjoint Representation.

$$[F_c, F_d]_{ab} = i f_{cd}{}^{e} (F_e)_{ab}$$

supressing matrix indices: $[F_c, F_d] = i f_{cd}{}^{e} F_e$

$\{F_a\}$ forms an $n \times n$ dimensional irrep of $\mathcal{L}(G)$ where $\dim \mathcal{L}(G) = n$.

Adjoint representation is very useful. It allows us to define an inner product on the space of Lie Algebras.

Inner Product: $\langle T_a | T_b \rangle = Tr[F_a F_b]$

we can define inner product as the trace over adjoint representation of corresponding matrices.

~~$Tr[F_a F_b]$ is [illegible]~~

* $\mathrm{Tr}[F_a F_b]$ is real and symmetric in a and b.
* This means; we can bring it to diagonal form.

$$\mathrm{Tr}[F_a F_b] = K_a \delta_{ab} \quad \text{(no sum)}$$

* ~~it turns~~ out;

~~if G is compact,~~

Rescale the F_as so that $|K_a| = 1$

~~Compact Lie~~

So, for Compact Lie Group

we can choose $\lambda > 0$.

Then $\mathrm{Tr}[F_a F_b] = \lambda \delta_{ab}$

~~Later~~ $\Rightarrow$ you $\Downarrow$

Structure constant are completely anti-symmetric.

$f_{abc} \equiv f_{ab}{}^{c}$ (just re-write for simplicity; we are not saying that in general b & c are anti-symmetric)

$$\mathrm{Tr}[[F_a, F_b] F_c] = \mathrm{Tr}[i f_{abd} F_d F_c]$$

$$\Rightarrow \mathrm{Tr}[F_a F_b F_c - F_b F_a F_c] = i f_{abd} \cdot \lambda \cdot \delta_{dc}$$

$$\Rightarrow \mathrm{Tr}[F_a F_b F_c - F_c F_b F_a] = i \lambda \cdot f_{abc}$$

-- using cyclicity.

$$\Rightarrow \mathrm{Tr}[[F_b, F_c] F_a] = i \lambda f_{abc}$$

$$\Rightarrow i f_{bcd} \mathrm{Tr}[F_d F_c] = i \lambda f_{abc}$$

$$\boxed{f_{bca} = f_{abc}}$$

$\cancel{\ldots} \Rightarrow \boxed{f_{bca} = -f_{bac}}$

so; f_{abc} are completely anti-symmetric.

Examples of Matrix Lie Group

* $GL(n,K) \to n\times n$ matrix with entries in field $K = \mathbb{R}$ or $\mathbb{C}$ (General Linear group)
  and $\det(M(n,k)) \neq 0$
  where; $M(n,k) \in GL(n,K)$

* $SL(n,K) \to \det M = 1$

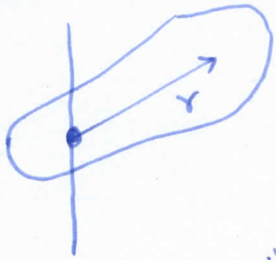
* $SO(n)$: group of rotation of a rigid body in n dimensions
  ie; $O^T O = O O^T = 1$, $\det(O) = 1$

  $S \equiv$ Special (it tells about determinant)

* $U(n)$ group; $n\times n$ unitary matrices entries are complex numbers; $U^\dagger U = U U^\dagger = \mathbb{1}$

* $SU(n)$ group, $\det(U) = \mathbb{1}$ — $SU(n)$ is subgroup of $U(n)$

Special orthogonal Group ($SO(n)$)



$r' = O r$; $r'^T r' = r^T r$
(length is unchanged under rotation)

$(r^T r = r \cdot r)$

$\therefore r'^T r' = (Or)^T (Or) = r^T O^T O r = r^T r$

$\Rightarrow \boxed{O^T O = 1}$

$\boxed{O_{ij} O_{ik} = \delta_{jk}}$

$\therefore$ We know that a real $n\times n$ matrix have n^2 parameters.

$\hookrightarrow$ but here due to constraints $O_{ij} O_{ik} = \delta_{jk}$ we have less independent parameters.

Lets take variation of it

$$\delta O_{ij}\, O_{ik} + O_{ij}\, \delta O_{ik} = 0$$

Symmetry in j and k ;

$$\underbrace{j \neq k}_{\frac{n^2-n}{2}} \; ; \; \underbrace{j=k}_{n} \qquad \frac{n^2-n}{2} + n \qquad \boxed{n + \frac{n^2-n}{2}}$$

so; no. of independent parameters. $n^2 - \left(n + \frac{n^2-n}{2}\right) = \frac{n(n-1)}{2}$

$$\boxed{\dim SO(n) = \frac{n \cdot (n-1)}{2}}$$

$\dim SO(2) = 1$
$\dim SO(3) = 3$

$$O \simeq \mathbb{1} + \sum_a \xi_a T_a \qquad ; \; a = 1, \dots \frac{n(n-1)}{2}$$

$\xi_a \to$ real

$$O^T O = \mathbb{1}$$

To first order we have $\sum_a \xi_a \left(T_a + T_a^T \right) = 0$

$$\Rightarrow \boxed{T_a = - T_a^T}$$

generators are anti-symmetric

i.e.: $\boxed{T^{ab} = - T^{ba}}$

$$a, b = 1, \dots, \frac{n(n-1)}{2}$$

$$\therefore \; [T^{ab}]_{cd} = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c$$

} generators of SO matrixes.

$$O = \exp\left[\frac{1}{2} \xi_{ab} T^{ab}\right]$$

and the Lie Algebra is given by

$$[T^{ab}, T^{cd}] = \delta^{ac} T^{bd} - \delta^{bc} T^{ad} - \delta^{ad} T^{bc} + \delta^{bd} T^{ac}$$

Lie Algebra of SO(n)

$O^T O = \mathbb{1}$; $(\det O)^2 = 1 \Rightarrow$

$O = \mathbb{1} + \frac{1}{2} \xi \cdot T$; ~~choose~~ =

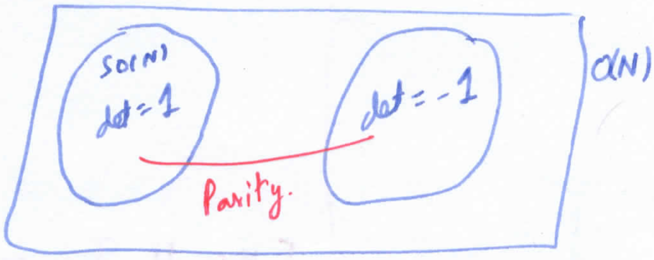
choose ξ such that $\xi \cdot T = \begin{pmatrix} 0 & \xi & & 0 \\ -\xi & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}$

So, The whole matrix O looks like.

$$\begin{pmatrix} 1 & \xi/2 & & & \\ -\xi/2 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

$$\Rightarrow \boxed{\det O = 1 + \frac{\xi^2}{4}}.$$

$P = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$; $\det P = -1$	$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$



Lecture 4: Examples of Lie groups continued : $SO(m,n)$, $SU(n)$

— Shoaib Akhtar - 2/6/2020

Reference: # Göckeler & Schücker — chapter 8

★ Weinberg Volume 1 : Ch 3 for Wigners Theorem.

Last time we finished with the statement ~~that~~ that;

Generators of $SO(n)$, $T^{ab} = -(T^{ab})^T$

$$T^{ab} = -T^{ba} \sim \frac{n(n-1)}{2}$$

Suppose ; n of $SO(n)$ is $n = 2l$; l integer.

Then we can define l linearly independent matrices

$$\left.\begin{aligned}
H^1 &= \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \\
H^2 &= \begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix} \\
&\vdots \\
H^l &= \begin{pmatrix} \ddots & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}
\end{aligned}\right\} \quad [H_i, H_j] = 0 \quad \forall\, i,j$$

Definition Cartan Subalgebra (CSA) The maximal set of mutually commuting elements of a Lie Algebra is ~~the~~ known as the CSA.

$\{H_i\} \rightarrow$ generate rotations in planes which do not share a direction. ~~an axis~~

* The ~~number of elements~~ dimension l of CSA is known as the rank of

the Lie group or the Lie Algebra.

The vectors of a representation that the Lie Algebra acts will be labelled by their eigenvalues of $\{H_i\}$ in CSA.

Consider $n = 2l+1$ (some odd number), the rank is still l (because the extra dimension that we have, cant be really paired with any other direction to give us new plane of rotation which does not share any axis with those rotation)

"For every ~~compact~~ Lie Algebra (arising from a compact Lie group) we can define the quadratic casimir operator which commutes with all the elements of Lie algebra." $T^2 = T_a T_a$ (general casimer operator ~~operator~~ for compact lie group)

Exercise: $[T^2, T_a] = 0$

$SO(3)$: $T^2 = J_a J_a$

Schur's Lemma: Any matrix which commutes with all the matrices of an irreducible representation must be proportional to identity.

So if, $T_a^{(r)} \longrightarrow$ generators in the irrep (r), then

$$T_a^{(r)} T_a^{(r)} = C_2(r) \mathbb{1}$$

some number which depends on r.

$C_2(r) \rightarrow$ characterizes the whole rep.

ex: for $SO(3)$ or $SU(2)$
$C_2 = j(j+1)$

$SO(n, m)$ group

$SO(n) \to$ preserving the Euclidean metric

ie; $O^T \mathbb{1} O = \mathbb{1}$

Analogously for $\mathbb{R}^{n,m}$ which have metric ~~diag(-1,~~

$$\eta_{\mu\nu} = \text{diag}(\underbrace{- - - \cdots -}_{n \text{ minus sign}} \; \underbrace{+ + \cdots +}_{m \text{ plus sign.}})$$

$$O(n,m) = \{ O \mid O^T \eta O = \eta \}$$

If we impose further that $\det O = 1$, we get a subgroup ~~S~~ $SO(n, m)$. $SO(n,m)$ is continuously connected to the identity

☆ $SO(n,m) \to$ Non compact group
(because they are non-compact as group manifold)

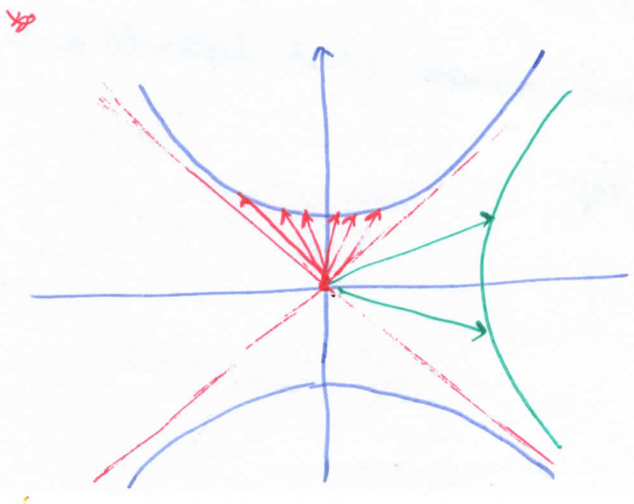
☆ There are no finite dimensional unitary, irreducible, non trivial representation of non-compact Lie groups.

☆ Lorentz group $O(1,3)$.

$SO(1,3)$ is a subgroup of $O(1,3)$ ~~SO(3)~~. $SO(1,3) \Rightarrow$ Proper Orthochronous Lorentz transformation

Proper : $\det O = 1$ (ie; continuously connected to $\mathbb{1}$)

Orthochronous : $O_{00} > 0$. (ie; you cannot change the direction of time)



$$P = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \sim \text{Parity}$$

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \sim \text{Time inversion}$$

$\left.\begin{matrix} P \\ T \end{matrix}\right\} \in O(1,3)$, $\notin SO(1,3)$

Isometry group of AdS_n

AdS_n is defined by a hyperboloid in $\mathbb{R}^{2,n-1}$

$$-X_0^2 - X_1^2 + X_2^2 + \dots + X_n^2 = -R^2$$

R ← radius of AdS.

Invariant under $SO(2, n-1)$ (also under $O(2, n-1)$)

☆ Conformal field theory in $(n-1)$ dimensional flat spacetime have symmetry ~~so we have for SO~~ $SO(2, n-1)$
(same as the symmetry of AdS_n)

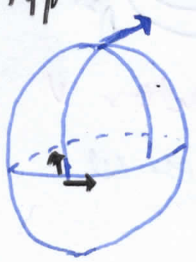
CFT in 2d space time ⇒ Symmetry becomes infinite dimensional
(The Lie Algebra is Virasoro Algebra)

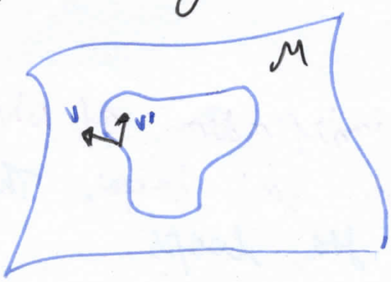
$SL(2, \mathbb{R}) \rightarrow$ Isometry of AdS_3.

The two time dimension is not really two time.
↳ The hyperboloid that we are defining eliminates one of the time like directions

" The ~~intr~~ intrensic metric on this hyperboloid has only one time direction "

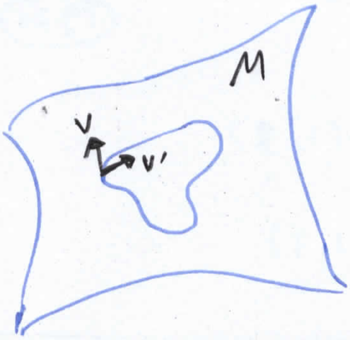
☆ Application to Holonomy Groups.





~~$V = \Lambda V$~~

$\Lambda \in \cancel{SO(3)}$



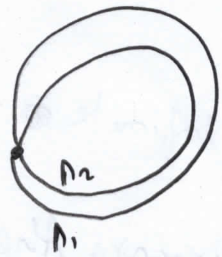
Consider a general manifold M.
& some generic closed curve.
& parallel ~~propa~~ propagate this vector on the close loop.

In general it will come back to original vector upto some rotation ; if we make the constrain that length of v does not change.

$v' = \Lambda v$ if $v'^T v = v' v$

$\Lambda \in SO(n)$

we can actually give this a group structure.



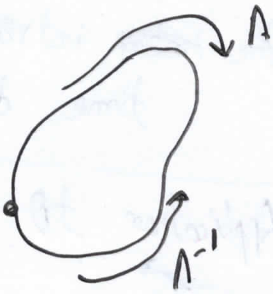
I associate with every closed loop some element of SO(n).

$\Rightarrow$ This loop rotates the vector upto Λ_1

The group multiplication $\Lambda_1 \cdot \Lambda_2$ consists of first parallel transporting around loop 2, then loop 1.

Group Inverse : if you ~~parallely~~ parally propagate the vector in one sense : its Λ

* if you do in other sense its Λ^{-1}.



Restricted Holonomy group

we can show the rotation is independent of which vector v you choose. It only depends on loops. Thats how we assign group element to the loops.

A manifold is simply connected if any closed loop can be deformed continuously to a point.

we will see; that the ~~manifold~~ manifold for $SO(3)$ is not simply connected.

Special Unitary Groups $SU(n)$

Gauge symmetry group of Standard Model is given by

$SU(3) \times SU(2) \times U(1)$.

Accidental global $U(1)$ symmetry; Baryon # & Lepton # conservation.

$SU(n)$ consists of $n \times n$ complex matrices U which satisfy

$U^{\dagger} = U^{-1}$ to $U^{\dagger}U = UU^{\dagger} = \mathbb{1}$ and $\det U = +1$

If we did not impose $\det U = 1$, we get $U(n)$ group.

$U \rightarrow$ ~~as~~ as a complex matrix has $2n^2$ real parameters.

so; $U^{*}_{ji} U_{jk} = \delta_{ik}$

$\therefore$ If set $i = k \Rightarrow n$ real constraint equation.

$\therefore$ If $i \neq k$, we only need to consider $i > j$; $2 \times \left(\frac{n(n-1)}{2}\right)$ real constraints.

so; $2n^2 - (n + n(n-1)) = n^2$ $\quad \dim(U(n)) = n^2$

$\det U = 1$ eliminates one extra parameter

So; $\dim(SU(n)) = n^2 - 1$.

~~det (U) = 1 eliminates~~

$\det U = 1$ eliminates $U = \begin{pmatrix} e^{i\theta} & & \\ & e^{i\theta} & \\ & & \ddots & \\ & & & e^{i\theta} \end{pmatrix} \sim U(1)$ subgroup.

$\det U = e^{i\theta n}$

but if $\theta = \frac{2\pi k}{n}$, $k \in \mathbb{Z}$; then $\det(U) = 1$

$\det(U) = 1$ eliminate $U(1)$ subgroup;

but their is still cyclic group C^k.

$\begin{pmatrix} e^{i\frac{2\pi}{3}k} & & \\ & \ddots & \\ & & e^{i\frac{2\pi}{3}k} \end{pmatrix}$ will generate C^3.

Generators of $SU(n)$

$U = \exp(i\xi_a T_a)$, $\xi_a \to$ real n^2-1 parameters.

Then $U^\dagger = U^{-1} \Rightarrow T_a^\dagger = T_a$

$\det U = 1 \Rightarrow Tr[T_a] = 0$ (generators have to be traceless)

because; we can show $\det(U) = \exp[Tr(i\xi_a T_a)]$

$\Rightarrow Tr[i\xi_a T_a] = 0$ if $\det(U) = 1$

since ξ_a are arbitrary parameters

$\Rightarrow Tr[T_a] = 0$

$\forall a = 1, 2, \dots, n^2-1$

$U = e^{i\xi_a T_a} = 1 + i\xi_a T_a + \frac{(i\xi_a T_a)^2}{2!} + \dots \infty$

$U^\dagger = 1 - i\xi_a T_a^\dagger + \dots$

$= e^{-i\xi_a T_a^\dagger}$

$UU^\dagger = e^{i\xi_a T_a} \cdot e^{-i\xi_a T_a^\dagger} = e^{i\xi_a(T_a - T_a^\dagger)}$ use B.C.H. formula to be rigorous.

SU(2)
generators given by $T_a = \frac{\sigma_a}{2}$

Hermitian and traceless.

$H_1 = diag(1, 0, \ldots, 0, -1)$

$H_2 = diag(0, 1, \ldots, 0, -1)$

$\vdots$

$H_{n-1} = diag(0, 0, \ldots, 1, -1)$

So; $n(n-1)$ remaining.

we can chose ;

$\Rightarrow$ half to be real & symmetric, and non-diagonal.

$\Rightarrow$ other half to be pure imaginary & anti symmetric.

~~We have~~ $\cancel{z, z\, H_{n-1}} =$.

we have $(n-1)$ mutually commuting generators.

So Rank $(SU(n)) = (n-1)$

Defination | Simple Lie Group.
A simple lie Group, is a connected non-Abelian Lie group which does not have a non-trivial connected normal subgroup.

Note | The excluded normal subgroups are Lie Groups.
A simple lie group may have discrete normal subgroup.

Simple Lie Algebra: $\tilde{g} = \mathcal{L}(G)$, $\tilde{k} < \tilde{g}$, $\tilde{k} \to$ invariant subalgebra if $[\tilde{k}, \tilde{g}] \subseteq \tilde{k}$ or 0

We say that a Lie Algebra is simple if it has no ~~invariant~~ non-trivial invariant subalgebra.

A simple Lie group gives rise to a simple Lie Alg. & vice versa

Suppose $\mathcal{L}(G)$ is simple, but the adjoint representation $\{F_a\}$ is reducible. $F_a = \left(\begin{array}{c|c} & 0 \\ \hline 0 & \end{array}\right)$

$$F_a = \left(\begin{array}{c|c} \tilde{a}\tilde{b} & \tilde{a}\bar{b} \\ \hline \bar{a}\tilde{b} & \bar{a}\bar{b} \end{array}\right)$$

$$(F^a)_{bc} = \begin{cases} F^a{}_{\bar{b}\bar{c}} \neq 0 \\ F^a{}_{\tilde{b}\tilde{c}} \neq 0 \\ F^a{}_{\bar{b}\tilde{c}} = 0 \\ F^a{}_{\tilde{b}\bar{c}} = 0 \end{cases}$$

According to defination: $F^a{}_{\bar{b}\bar{c}} = -i f_{a\bar{b}\bar{c}}$

$$= +i f_{\bar{b}a\bar{c}} = -(F^{\bar{b}})_{a\bar{c}}$$

$$\to 0 \quad \text{if} \quad a \neq \bar{a}$$

so; $(F^{\tilde{a}})_{\bar{b}\bar{c}} = 0$

$(F^{\bar{a}})_{\tilde{b}\tilde{c}} = 0$

ie; $\{F^a\} = \{F^{\tilde{a}}, F^{\bar{a}}\}$

so; $F^{\bar{a}} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \end{array}\right)$

$F^{\tilde{a}} = \left(\begin{array}{c|c} & 0 \\ \hline 0 & 0 \end{array}\right)$ $\Rightarrow \mathcal{L}(G)$ is not simple.

$\therefore$ so contradiction.

Lec: 5 | Killing metric, Coset Manifold

— Shoaib Akhtar 15/6/2020.

$X \to X' = U X U^\dagger$; $U \in U(2)$ but $SU(2)$

$U(1)$ factor of $U(2)$ has trivial action on X

The part $SU(2)$ of $U(2)$ has non-trivial action on X.

$\therefore SU(2)$ is relevant homomorphism.

Abelian invariant subalgebra: Subalgebra that ~~comm~~ commutes with every element in the Lie Algebra.

Semi simple Lie algebra: $\mathcal{L}(G)$ which does not have an Abelian invariant subalgebra.

ex) $\mathcal{L}(SO(4)) = \mathcal{L}(SU(2)) \oplus \mathcal{L}(SU(2))$

Some useful geometric concept

Killing metric $\quad g^{-1}dg \Rightarrow$ Left invariant 1 form

$g \to hg$, $h \in G$ but constant (ie; derivative operator does not act on it)

then $g^{-1}dg$ is invariant under transformation.

$g^{-1}dg \in T^*(M) \otimes \mathcal{L}(G)$

$dg \cdot g^{-1} \Rightarrow$ Right invariant 1 form ; ie; invariant under $g \to gh$

Bi-invariant $\quad \underbrace{Tr(g^{-1}dg\, g^{-1}dg)}_{\text{left invariant by defination}} = Tr(dg\, g^{-1} dg\, g^{-1})$

Due the cyclic property of trace; we see that it is right invariant.

Pg 32

Using the fact that this is a 1form as well as lie Algebra valid object;

we can expand it as follows.

$$g^{-1}dg = E^a_\mu(\varphi)\, T^a\, d\varphi^\mu$$

$G \Rightarrow$ Compact Lie group (so we can normalize $tr(T_a T_b) = \lambda \delta_{ab}$)

$\varphi^\mu \Rightarrow$ coordinates on manifold $\mathcal{M}(G)$: The manifold of lie group G.

$T^a \Rightarrow$ Generators in the defining representation.

$$Tr(g^{-1}dg \cdot g^{-1}dg) = (E^a_\mu E^b_\nu \cdot \lambda \delta_{ab})\, d\varphi^\mu d\varphi^\nu$$

$$= g_{\mu\nu}(\varphi)\, d\varphi^\mu d\varphi^\nu$$

Identify $E^a_\mu E^b_\nu \lambda \cdot \delta_{ab}$ as metric $g_{\mu\nu}(\varphi)$

Then $Tr(g^{-1}dg \cdot g^{-1}dg)$ looks like line element on the lie group. $\rightarrow$ called Killing Metric.

$g \in G$. if compact lie group; then we can can always write $g = e^{iA}$ ← This belongs to lie Algebra.

lets write $g = e^{i\rho_a T}$

$$dg = g \cdot (iT \cdot d\rho)$$

$$g^{-1}dg = iTd\rho \Rightarrow g^{-1}dg = i\, T \cdot d\rho$$

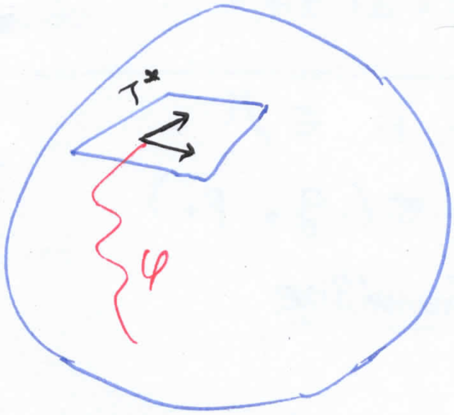
This says it is in Lie Algebra

This says its an element of Cotangent Bundle.

So; $g^{-1}dg \in T^*(M) \otimes \mathcal{L}(G)$

$\cancel{g^{-1}dg = i T d\rho}$

$$g^{-1}dg = i \underbrace{T}_{\mathcal{L}(G)} \underbrace{d\rho}_{T^*(M)}$$



$$E^a_\mu(\varphi)\, d\varphi^\mu = \eta^a$$

These are one forms.

$$Tr\,(T^a g\, dg^{-1}) = \underbrace{\lambda\, E^a_\mu(\rho)\, d\varphi^\mu = i\, d\rho^a}_{\text{relation between } \varphi \text{ \& } \rho.}$$

$$g_{\mu\nu} \xrightarrow[\text{to}]{\text{can give rise}} \Gamma^\rho_{\mu\nu}$$ = torsion free-connection

or

Levi-Civita connection

or

Christoffel symbols.

$$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$$

There are other connections on G. Those connections are not torsion free. And the structure constants do contribute to them.

Group Actions: Let G be a Lie Group and M a manifold. The action of G on M is a map $\sigma : G \times M \longrightarrow M$.

s.t. ① $\sigma(e, p) = p \quad \forall\, p \in M$

identity element.

② $\sigma(g_2, \sigma(g_1, p)) = \sigma(g_2 g_1, p)$

Transitive: If for any p_1 and $p_2 \in M$: $\exists$ an element $g \in G$ s.t. $p_2 = \sigma(g, p_1)$.

Then the group action is called <u>Transitive</u>

Free: The group action is free; if for $g \neq e$ there is no fixed point of the action.

ie; If $p \in M$, $\sigma(g, p) = p \Rightarrow g = e$

Then we say that the <u>action is free</u>.

~~Counterexamples~~ <u>Examples</u>:

(★1) The action of $SO(3)$ on $\mathbb{R}^3$ is <u>not</u> transitive

ie; If $|\bar{x}| \neq |\bar{x}'|$ (when length are not equal) where $\bar{x}, \bar{x}' \in \mathbb{R}^3$, then $\bar{x}$ and $\bar{x}'$ cannot be connected ~~by~~ ~~to a~~ via a rotation.

(★2) The action of $SO(3)$ on S^2 is transitive.
(but it is not free)

Definition <u>Little group or Stabalizer Group</u>

Let G be a Lie group that acts on M. The Little group of $p \in M$ is defined to be the subgroup which leaves p invariant.

ie; Little group of p — $H(p) = \{ g \in G \mid \sigma(g, p) = p \}$

In other words; p is fixed point of the action of $H(p)$.

Example $SO(2) \subset SO(3)$. $SO(2)$ that leaves the north pole invariant is the little group of north pole.

~~Coset Manifold~~

~~Suppose $H \subset G$.~~

~~Consider the set $G/H = \{ Gh \mid h \in H \}$~~ ~~These are cosets~~

~~notation $Gh \to$ coset for the element h.~~

<u>Coset Manifold</u>

Suppose $H \subset G$, consider the set $G/H = \{ gH \mid g \in G \}$

Coset Space. These are cosets

if $g_1 = g_2 h$ (i.e. g_1 and g_2 are in same coset)

we say $g_1 \sim g_2$. (Equivalence relation)

If $g_1 \simeq g_2$ (ie: g_1 is near g_2), but $g_1 \not\sim g_2$ (ie: g_1 & g_2 are not in same coset)

Then ~~the~~ cosets corresponding to these group elements g_1 and g_2 are also nearby.

And therefore extend the structure of manifold on the coset space.

Example | $SO(3)$ acting on S^2 ; $SO(3)/SO(2)$.

$e_z \Rightarrow$ position vector pointing to the north pole.

If $g_1 \sim g_2 \implies g_1 e_z = g_2 e_z$.

Starting from the north pole e_z we can reach any other point by applying a representative of a coset such that each coset corresponds to a unique point on S^2.

So; we can think of S^2 as coset space

$$S^2 = SO(3)/SO(2)$$

Finite dimensional representation of SU(2)

$$[J_a, J_b] = i \varepsilon_{abc} J_c$$

we define ladder operators.

$$J_\pm = J_1 \pm J_2 \quad \sim \text{ladder operators}$$

These ladder operators are not Hermitian.

$$J_+^\dagger = J_- \quad ; \quad J_-^\dagger = J_+$$

$$[J_3, J_\pm] = \pm J_\pm$$

$$[J_+, J_-] = 2J_3$$

$$\text{Rank}(\mathcal{L}(SU(2))) = 1.$$

we choose $CSA = \{J_3\}$

Label vectors $J_3|m\rangle = m|m\rangle$

☆ $|m\rangle \Rightarrow$ a vector on which $\mathcal{L}(SU(2))$ acts.

☆ $m \Rightarrow$ not assuming this is an integer.

$J_3|m\rangle = m|m\rangle$

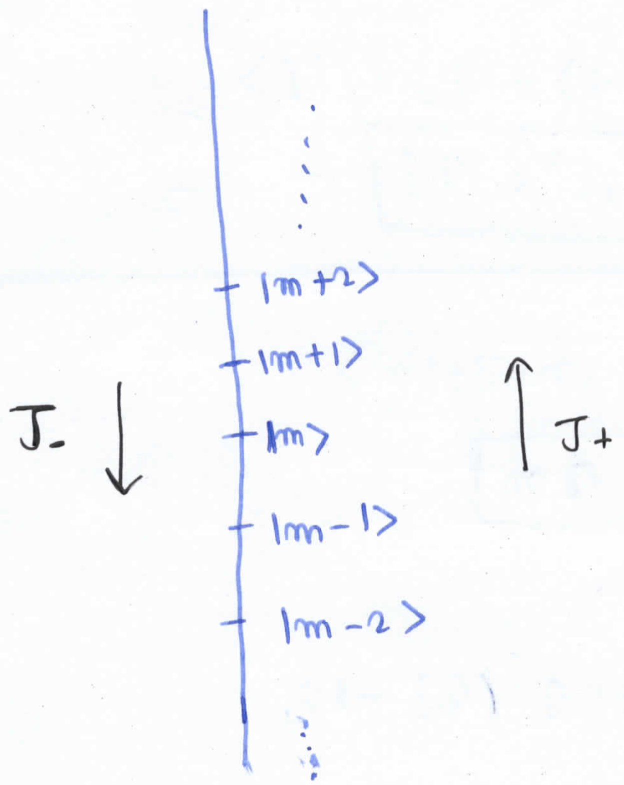
$J_+|m\rangle = ?$

so; $J_3(J_+|m\rangle) = J_+J_3|m\rangle + J_+|m\rangle$

$= J_+ m|m\rangle + J_+|m\rangle$

$= (m+1)(J_+|m\rangle)$

$J_3(J_-|m\rangle) = (m-1)(J_-|m\rangle)$



We can assume that states are normalized $\langle m|m\rangle = 1$.

$J_+|m\rangle = C_{m+1}|m+1\rangle$ $\cancel{C_{m+1}, D_{m-1}}$

$J_-|m\rangle = D_{m-1}|m-1\rangle$ C_{m+1}, D_{m-1} are constants to be determined.

$\langle m+1 | J_+ | m \rangle = \langle m+1 | C_{m+1} | m+1 \rangle = C_{m+1}$

$J_- | m+1 \rangle = D_m | m \rangle$

$\langle m+1 | J_+ = D_m^* \langle m |$

$\langle m+1 | J_+ | m \rangle = D_m^* \langle m | m \rangle = D_m^*$

$\Rightarrow \boxed{C_{m+1} = D_m^*}$

So; we can re write $J_- | m \rangle = C_m^* | m-1 \rangle$

Suppose $m = j$ is the highest possible eigenvalue of J_3

ie; $J_+ | j \rangle = 0$

So; starting with $J_+ J_- = J_- J_+ + 2 J_3$

$$\langle j | J_+ J_- | j \rangle = 0 + 2j \langle j | j \rangle$$

$$\Rightarrow C_j C_j^* \langle j-1 | j-1 \rangle = 2j \langle j | j \rangle$$

$$\Rightarrow \boxed{|C_j|^2 = 2j}$$

---

for generic m;

$\langle m | (J_+ J_- - J_- J_+) | m \rangle = 2 \langle m | J_3 | m \rangle$

$$\boxed{|C_m|^2 - |C_{m+1}|^2 = 2m}$$

$$|C_{j-1}|^2 = |C_j|^2 + 2(j-1)$$
$$= 2j + 2j - 1 = 2(2j-1)$$

$$|C_{j-2}|^2 = |C_{j-1}|^2 + 2(j-2)$$
$$= 2(3j - 1 - 2)$$

$$|C_{j-s}|^2 = 2((s+1)j - 1 - 2 - \dots - s),$$

s is positive integer.

$|C_{j-s}|^2 = 2(s+1)\,j - \frac{1}{2}s(s+1)$

$$\boxed{|C_{j-s}|^2 = (s+1)(2j-s)}$$

$\Rightarrow$ we can choose C_{j-s} to be.

$$C_{j-s} = \sqrt{(s+1)(2j-s)}$$

Now; if we are looking at finite dimensional representation. Then apply J_- ~~latter~~ ladder operator: at some point it should terminate;

which is possible only if one of the constant C_{j-s} is zero.

$$\Rightarrow \boxed{2j = s} \Rightarrow \boxed{j = \frac{s}{2}} \quad s \in \mathbb{N}.$$

$\Rightarrow$ $m \rightarrow$ is also integer or half integer.

$$\boxed{\begin{aligned} J_+|m\rangle &= C_{m+1}|m+1\rangle \\ &= \sqrt{(j-m)(j+m+1)}\,.\,|m+1\rangle \end{aligned}}$$

$$\boxed{J_-|m\rangle = \sqrt{(j+m)(j-m+1)}\,|m-1\rangle}$$

When you apply J_+ enough number of times so that $m = j$

lowest you have $-j$,

In total $(2j+1)$ steps in the ladder.

$$0 \xleftarrow{J_+} |j\rangle \underset{J_-}{\overset{J_+}{\rightleftarrows}} |j-1\rangle \underset{J_-}{\overset{J_+}{\rightleftarrows}} |j-2\rangle \cdots \underset{}{\overset{J_+}{\rightleftarrows}} |-j+1\rangle \underset{J_-}{\overset{J_+}{\rightleftarrows}} |-j\rangle \xrightarrow{J_-} 0$$

$J^2 = J_a J_a$

i.e. $\boxed{J_a J_a |m\rangle = j(j+1)|m\rangle}$ $\forall\, m$

$J_a J_a \Rightarrow$ Casimier invariant (that labels the whole representation)

$\langle m | J_a | m \rangle$

Find the matrices for $j=2$ representation.

$|-2\rangle, |-1\rangle, |0\rangle, |+1\rangle, |+2\rangle$

SU(2) tensor and Young tableau

$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$

in group theory langauge we write as $2 \otimes 2 = 3 \oplus 1$

refers to the fact that spin $\frac{1}{2}$ is 2 dimensional

$1/2$ spin is 2 dimensional.

spin 1 is 3 dimensional.

---

2 of SU(2)

$\psi_a = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2$

if you have Lie Algebra $[T_a, T_b] = i\,\varepsilon_{abc} T_c$

$\tilde{T}_a \equiv (-T_a)^*$

$\Rightarrow$ we can check $[\tilde{T}_a, \tilde{T}_b] = i\,\varepsilon_{abc}\tilde{T}_c$

$\tilde{T}_a$ satisfy the same Lie Algebra.

$\tilde{T}_a$ form Conjugate representation. (Anti fundamental)

~~However~~ However for SU(2) $\tilde{T}_a = C\, T_a\, C^{-1}$

where $C = i\,\sigma_2$

for $SU(2)$; the fundamental & anti-fundamental representation are related by similarity transformations.

The vector space on which Anti-fundamental representation acts on : 2^*

Their elements are ψ_a^*.

for $SU(2)$; $2 \cong 2^*$ (They are isomorphic)

$SU(n)$, $n \geq 3$;

$$n \neq n^*$$

fundamental representation

Anti-fundamental representation.

---

for $SU(2)$ 2 & 2^* are same vector spaces.

---

① Fundamental Representation: 2

Under $SU(2)$ $\psi_a \longrightarrow U_a{}^b \psi_b$

Invariant Tensors.

① ε_{ab} , ε^{ab}

Under $SU(2)$

$$\varepsilon_{ab} \rightarrow U_a{}^{a'} U_b{}^{b'} \varepsilon_{a'b'}$$

$$= \det(U) \cdot \varepsilon_{ab}$$

$$= 1 \cdot \varepsilon_{ab} = \varepsilon_{ab}$$

Now, we pretend that 2^* is different representation for the sake of good notations.

2^* : $\psi^a = \psi_a^*$ ZEE book.

Under a SU(2) transformation.

$$\psi_a \rightarrow (U^+)_{a'}{}^{a} \psi^{a'}$$

$$\varepsilon^{ab} \longrightarrow (U^+)_{a'}{}^{a} (U^+)_{b'}{}^{b} \varepsilon^{a'b'}$$

$$= \varepsilon^{ab}$$

χ_a , ψ^b

we can contract to make singlet out of them; and get a representation which is trivial.

Contracting a downstair index with upstair index is done by the tensor $\delta^a{}_b$

which should transform as $\delta^a{}_b \rightarrow U_b{}^{b'} (U^+)_{a'}{}^{a} \delta^{a'}_{b'}$

$$= U_b{}^{b'} (U^+)^a{}_{b'}$$

$$= \delta^a{}_b$$

So; $\delta^a{}_b$ is also invariant tensor of SU(2)

Lecture 6 | SU(2) and SU(3) tensors, Young's Tableau

Pg 43

—Shoaib Akhtar 15/6/2020

Correcting mistake of lecture 5

$g = e^{i\xi T}$

then $dg = e^{i\xi T}\, d\xi T$ (This is not true)

because $(\xi T)(d\xi T)$ dont commute.

$g^{-1}dg \in T^*(M) \otimes \mathcal{L}(G)$ This is true

ex in G.R. $T_\mu{}^\nu \in T^*(M) \otimes T(M)$

SU(2)

$2 = \psi_a \;;\; a = 1, 2$

$2^* = \psi_a^* \equiv \psi^a$

2 and 2^* are related

ie; $2 = 2^*$

$\varepsilon^{ab}, \varepsilon_{ab}, \delta^a{}_b$ are invariant tensor under SU(2).

$\varepsilon^{ab}\psi_a$: $\psi_a \to U_a{}^b \psi_b$

$\varepsilon^{ab}\psi_a \longrightarrow (U^\dagger)_{b'}{}^b (U^\dagger)_{a'}{}^a U_a{}^c \psi_c \cdot \varepsilon^{b'a'}$

$= (U^\dagger)_{b'}{}^b \delta_{a'}^c \psi_c \varepsilon^{b'a'}$

$= \varepsilon^{b'a'} (U^\dagger)_{b'}{}^b \delta_{a'}^c \psi_c$

$= (U^\dagger)_{b'}{}^b \varepsilon^{b'c} \psi_c$

$\Rightarrow (\varepsilon^{ab}\psi_a) \longrightarrow (U^\dagger)_{b'}{}^b (\varepsilon^{b'c}\psi_c)$

define $\chi^b = \varepsilon^{ba}\psi_a$

then $\chi^b \longrightarrow (U^\dagger)_{b'}{}^b \chi^{b'}$

if $\psi_a = 2$, then $\varepsilon^{ab}\psi_b = 2^*$

SU(2) tensor with 2 indices ψ_{ab}

Then $\psi_{ab} = \psi_{[ab]} + \psi_{(ab)}$

We can check that under SU(2), $\psi_{[ab]}$ & $\psi_{(ab)}$ dont mix

So, they actually form irreducible representation.

$\psi_{abcd} \Rightarrow$ not in general ~~an~~ irreducible.

$$\underset{2 \otimes 2^*}{\chi_a \psi^b} \xRightarrow{\text{contract}} \underbrace{\delta^a_b \chi_a \psi^b}_{1}$$

This transforms as a singlet.

$\boxed{1}$ $\psi_{ab} = \chi_a \psi_b$

$\chi_a \psi_b = 2 \otimes 2$

$$\chi_a \psi_b = \chi_{[a}\psi_{b]} + \chi_{(a}\psi_{b)}$$

$\chi_{[a}\psi_{b]}$: This has one component

$\chi_{(a}\psi_{b)}$: This has 3 components

So; $2 \otimes 2$ breakes up into $1 \oplus 3$

ie; $\chi_a \psi_b = \chi_{[a}\psi_{b]} + \chi_{(a}\psi_{b)} = 1 \oplus 3$

$\Rightarrow 2 \otimes 2 = 1 \oplus 3$

This we know from Quantum Mechanics; when you combine two spin 1/2 particles, you get a vector ~~a~~ representation & a scalar representation.

A singlet does not change under SU(2) $\phi \to \phi$

A fundamental under SU(2) ; $\psi_a \to U_a{}^b \psi_b$

Young Tableau

a fundamental $\psi_a = 2 = \boxed{a}$

Rank 2 tensor with two indices which is anti-symmetry $\psi_{[ab]} = 1 = \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \Big\}$ antisymmetrized.

$$\psi_{(ab)} = 3 = \begin{array}{|c|c|}\hline a & b \\ \hline \end{array}$$

So; The statement $2 \otimes 2 = 1 \oplus 3$ in the box language can be written as

$$2 \otimes 2 = 1 \oplus 3$$

$$\square \times \square = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} + \begin{array}{|c|c|}\hline & \\ \hline \end{array}$$

for SU(2) $\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \Big\} \Rightarrow$ something like this cannot exist.

ex1) $2 \otimes 2 \otimes 2 = ?$

ie; $\square \times \square \times \square = ?$

$= (\square \times \square) \times \square$

we know

$\square \times \square = \begin{array}{|c|c|}\hline & \\ \hline \end{array} + \underbrace{\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}}_{\text{singlet}}$

So; The non-trivial part is

$\begin{array}{|c|c|}\hline & \\ \hline \end{array} \times \square = ?$

So; $\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \times \square = \square$

because $\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$ is like 1

ie; like multiplying ~~scalar~~ scalar & fundamental) for $(\square\square \times \square)$ lets switch to tensor notation

then $3\,\chi_{(a}\varphi_{b)} \cdot \psi_c \in (\square\square \times \square)$

$$= (\chi_{(a}\varphi_{b)}\psi_c + \chi_{(b}\varphi_{c)} + \chi_{(c}\varphi_{a)}\psi_b)$$
$$+ (\chi_{(a}\varphi_{b)}\psi_c - \chi_{(b}\varphi_{c)}\psi_a)$$
$$+ (\chi_{(a}\varphi_{b)}\cdot\psi_c - \chi_{(c}\varphi_{a)}\psi_b)$$

expand last line

$$\frac{1}{2}[\chi_a\varphi_b\psi_c + \chi_b\varphi_a\psi_c + \chi_b\varphi_c\psi_a + \chi_c\varphi_b\psi_a + \chi_c\varphi_a\psi_b + \chi_a\varphi_c\psi_b] \longrightarrow \text{symmetric in } a, b, c$$

so; have 4 components in $SU(2)$.

$$\boxed{\psi_{(a_1,\dots,a_n)} = n+1}$$

expand second line

$$\frac{1}{2}[\chi_a\varphi_b\psi_c + \chi_b\varphi_a\psi_c - \chi_b\varphi_c\psi_a - \chi_c\varphi_b\psi_a]$$

expand third line ... (same as second line; just different label)

$\longrightarrow$ antisymmetric in c & a, and contract with ε^{ca} (we dont loose any information because its already anti-symmetric in c and a)

ie $$\frac{1}{2}[\chi_a\varphi_b\psi_c + \chi_b\varphi_a\psi_c - \chi_b\varphi_c\psi_a - \chi_c\varphi_b\psi_a]\varepsilon^{ca} = \eta_b$$

call it η_b; because the only free index now left is b.

~~but $\eta_b = 2$~~ but $\eta_b = 2$

So; That second line object is a $\underline{\underline{2}}$.

suppose $A_{cab} = -A_{acb}$ $\quad$ A anti-symmetry in 2 indices

121	211
122	212

only 2 independent components.

for SU(2) ; $\boxed{a_1}\boxed{a_2}\boxed{\;}\cdots\boxed{a_n} = n+1$

$\square \times \square \times \square = \square\square\square + \square + \square$

So;

$$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2$$

note; The second & third line live in same vector space so they add up

So; ~~the~~ the second & third line just sum up to $\square$

for $\underline{\text{SU(3) tensors}}$:

$3 = \psi_a$: fundamental.

$3^* = \psi_a^* = \psi^a$: anti-fundamental. $\quad ; \tilde{T}_a = -T_a^*$

$\nexists$ any C s.t. $\tilde{T}_a = C T_a C^{-1}$

Invariant tensors ε^{abc}, ε_{abc}, $\delta^a{}_b$.

Consider, $\varphi_{[ab]}$ has 3 independent components in 3 dimensions.

$$\varphi_{[ab]} = 3$$

then $3^* = \psi^a = \varepsilon^{abc}\,\varphi_{[bc]}$

This motivates $3 = \boxed{a} = \psi_a$

but $3^* = \psi^a = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$ $\longrightarrow$ motivation of $\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$ from $\psi^a = \varepsilon^{abc}\varphi_{[bc]}$ because something is being anti-symmetrized.

ex11 ε^{abc}

$\varepsilon^{abc}\,\varphi_{[bc]} \longrightarrow U$

note; $\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$ was 1 for $SU(2)$

but here; $1 = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}$ in $SU(3)$.

Examples

① $3 \otimes 3 = ?$

$$\psi_a \chi_b = \underbrace{\psi_{[a}\chi_{b]}}_{3} + \underbrace{\psi_{(a}\chi_{b)}}_{6}$$

$\Rightarrow \cancel{3 \otimes 3 = 3^* \oplus 6}$

$$\square \times \square = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} + \begin{array}{|c|c|}\hline & \\ \hline \end{array}$$

$$3 \otimes 3 = 3^* \oplus 6$$

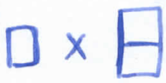
---

② $3 \otimes 3^* = ?$

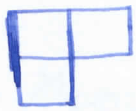
$\psi_a = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$; we are taking a fundamental of triplet of quarks ; and 3^* is then anti-quarks.

we will find $\psi^a = \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix}$

$$\psi_a \psi^b = \underbrace{\left(\psi_a \psi^b - \tfrac{1}{3}\delta^b_a \psi_c \psi^c\right)}_{8} + \underbrace{\tfrac{1}{3}\delta^b_a \psi_c \psi^c}_{1}$$

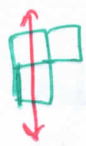


Now to represent 8

Answer 

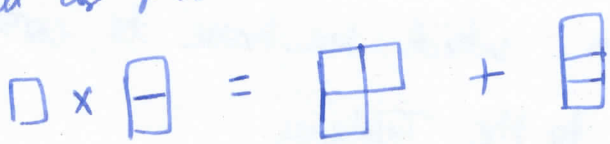
$$\cancel{\psi_a \psi^b} \quad \psi^b \psi_a = \underbrace{\varepsilon^{bcd}\varphi_{[cd]}}\ \underbrace{\psi_a}$$

i.e. 

here indices along column is antisymmetrized 

but nothing along row.

This tell us that



$$3 \otimes 3^* = 8 \oplus 1$$

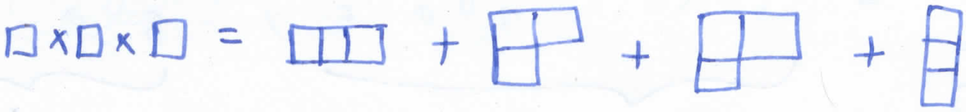
read Jones book for more details.

$$P_a^{\ b} = \psi_a \psi^b - \frac{1}{3}\delta^b_a \psi_c \psi^c$$

it turns out that ---- we have wave functions of mesons.

$$P^b_a = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix}$$

$3 \otimes 3 \otimes 3$

We can show 

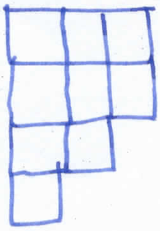
$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

Young's Tableau

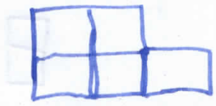
Refer to Jones Book.

~~① A legal Young's~~

① In a legal Young's Tableau, the length of any row cannot be larger than the row above.



allowed



Not allowed

② The number of boxes in a column for ~~SU(n)~~ $SU(n)$ cannot be more than n due to antisymmetry.

③ Suppose we have a Young's Tableaux ~~corresponding to a rank n tensor~~ with n boxes. We have a rank n tensor from which we have to construct the irrep. corresponding to the Tableau.

a_1	a_2	a_3	a_4
a_5	a_6	a_7	
a_8			

$\psi_{a_1 \dots a_8}$

Then

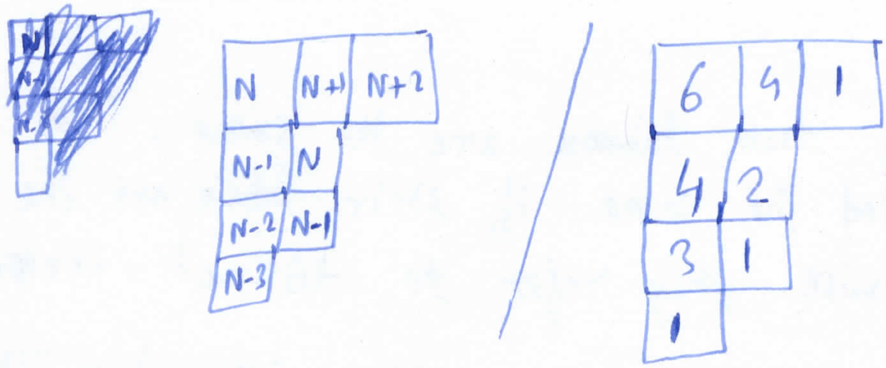
③ a) We first symmetrize the indices which lie on each row. $\psi_{(a_1 a_2 a_3 a_4)(a_5 a_6 a_7) a_8}$

③ b) Antisymmetrize the columns.

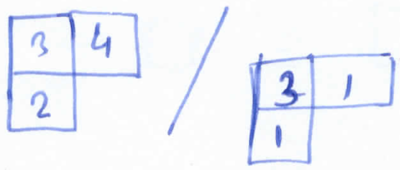
$\psi_{(a_1 \_\_), (\_\_) a_8} \longrightarrow$ Antisymmetrize

(4) How to compute the dimension of a tableaux.

SU(N)



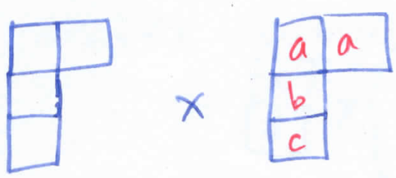
SU(3)



$$= \frac{3\times 4\times 2}{3\times 1\times 1} = 8$$

---

(5) How to expand the product of 2 ~~tableau~~ tableau as a sum of tableau.



T_1 T_2

(i) Label the rows of T_2 by a for the first row,
b " " second ",
c " " third

(ii) Add one box from T_2 to T_1 at a time starting from the top row making sure the following is satisfied. (at each step)

~~(a) T_1 must be a legal~~

(a) Each tableaux must be a legal Young Tableau at each stage.

(b) More than one box with the same lable must not appear in the same column.

(c) For each box assign a number n_a = # of a_s above and right to it.

Similarly assingn a number n_b = # of b_s above and right to it.

$n_a \geq n_b \geq n_c \geq \ldots$

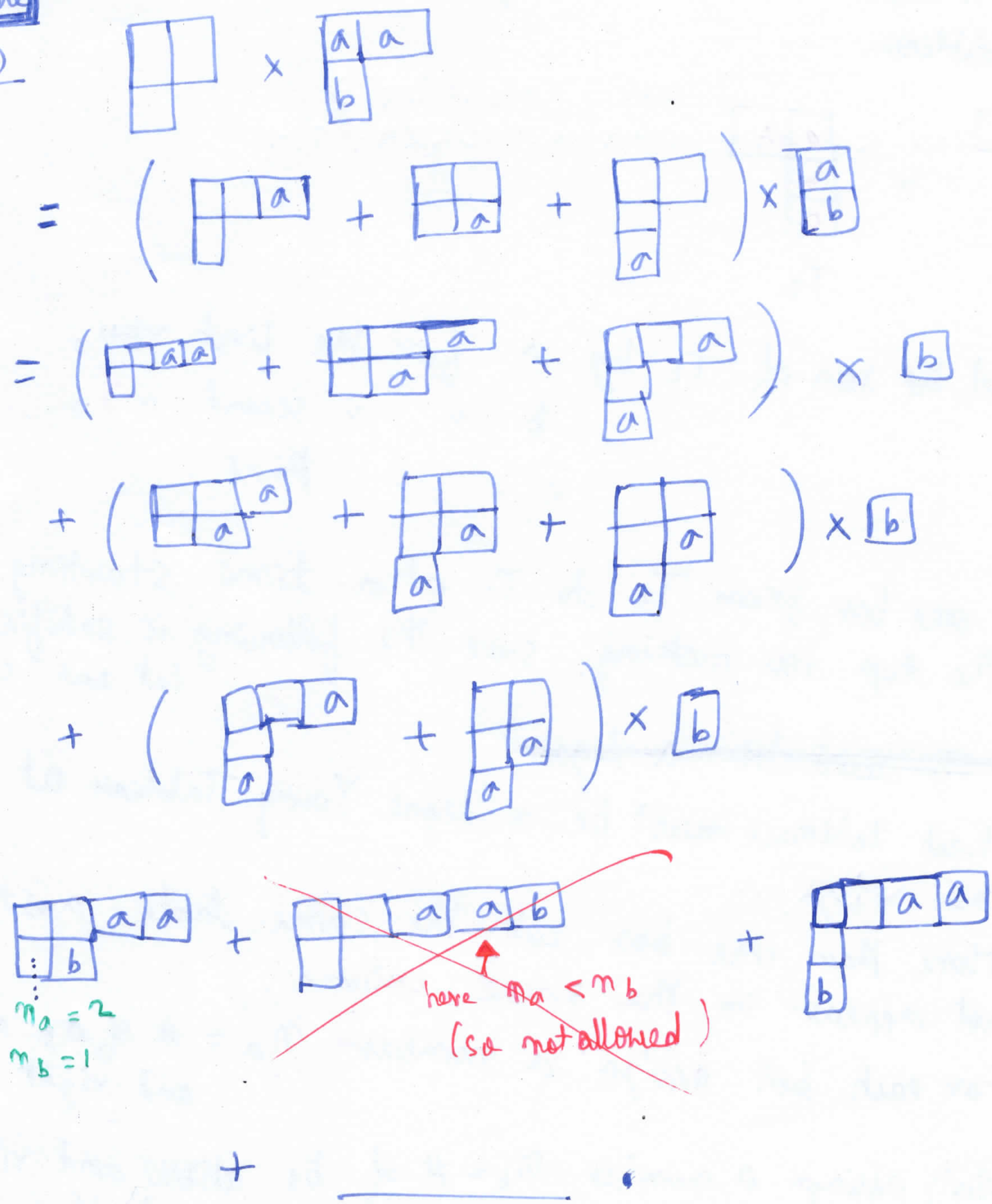
(iii) If two tableau are the same, then they are counted the same if their labels are the same. Otherwise they refer to distinct irreps.

(iv) Cancel columns with n boxes for $SU(n)$ because they are trivial.

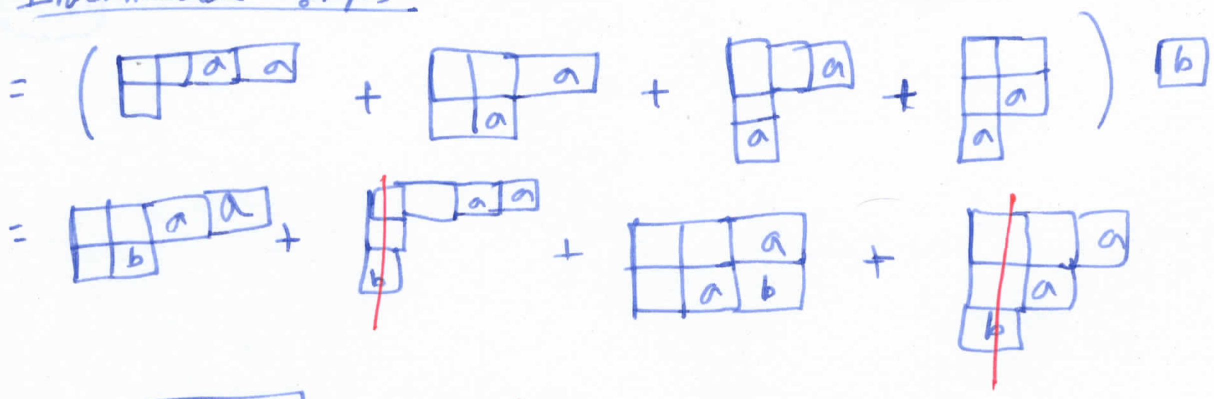
ex for $SU(2)$

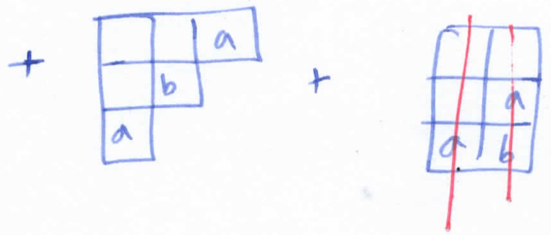
Example

$SU(3)$

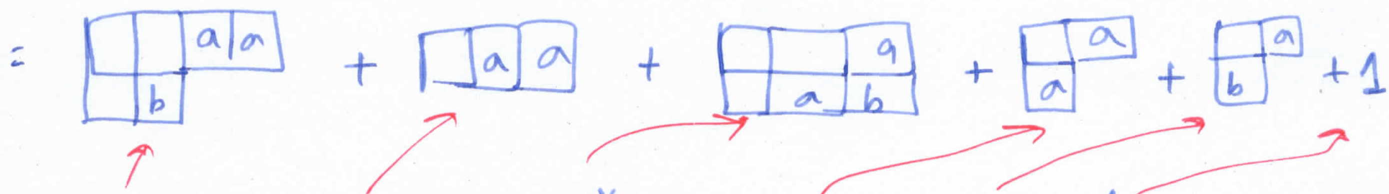


Intermediate steps





cancel three columns because we are doing SU(3)



$$= 27 \oplus 10 \oplus 10^* \oplus 8 \oplus 8 \oplus 1$$

1954

LIE GROUPS DON'T LIE.

THANK YOU

Shoaib Akhtar