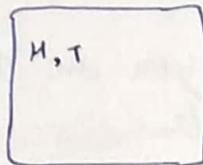


Lec 1 & Lec 2

example coin;



Pic of Law of motion



ex I

$$H \rightarrow H$$

$$T \rightarrow T$$

(12)

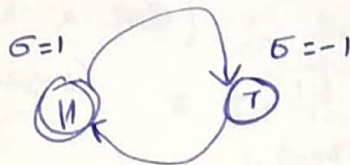
History of word if you start with H ; HH HH...
or T ; TTTT...

ex II

$$H \rightarrow T$$

$$T \rightarrow H$$

Pic of Law of motion



State Space

History of world if you start with H ; HTHT...
T ; THTH...

lets introduce mathematics

σ is a variable either equal 1 or -1 ; 1 for H
-1 for T

(now we have idea of configuration space with mathematical values)

★ $\sigma(t+1) = \sigma(t)$; law for example I

★ $\sigma(t+1) = -\sigma(t)$; law for example II

t = time
where we assume
time to be
integers for
simplicity.

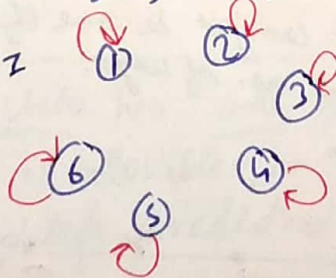
These are in fact equations of motion in our example.

★ For closed system, Laws of Physics are completely deterministic in classical mechanics.

example Die

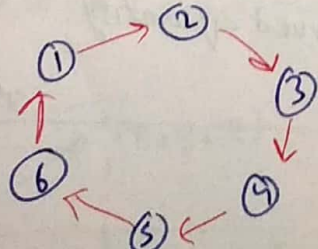
(has six states)

label states as ex I



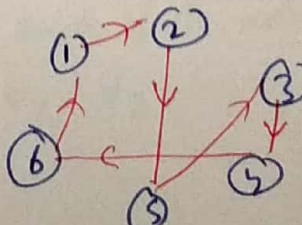
These are completely deterministic, wherever you begin.
lies for six different configuration
the die lie on the table.

ex II



History of world... if start with 1
then 1 2 3 4 5 6 1 2 3...

ex III



its just relabeling
and its just
a cycle as
previous.
(logically equivalent to II)

Here you dont have any conservation
law in II & III ... you go to all
states....

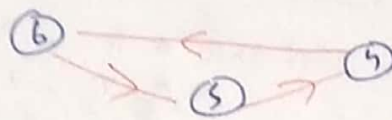
ex IV

Here you have conservation laws in II & V



completely deterministic

but not logically equivalent to previous examples II & III



In this law, you are trapped in one cycle.

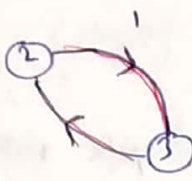
(~~state~~ ^{have} a name of this behaviour, Conserved Quantity)

i.e. label the cycles as C_1 & C_2

assign $C_1 := 1$; $C_2 := -1$;

ex V

0



There are three cycles now, if you start with any of them, you lie in them forever.

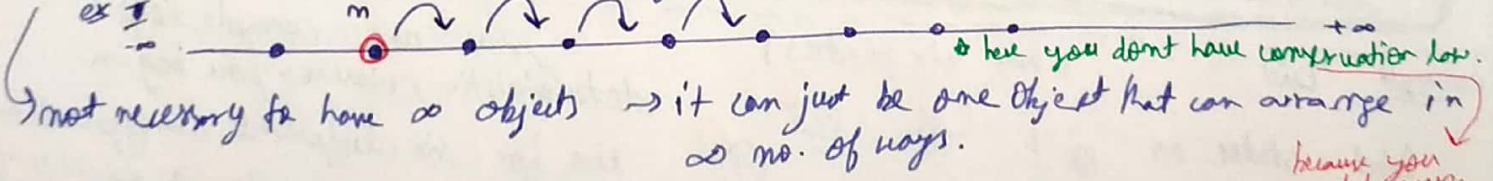


let 0, 1, 2 be the values of conserved quantity for the given loops

These are logically different evolutions

* distinctions are ~~not~~ of no. of distinct cycles .. and many more.

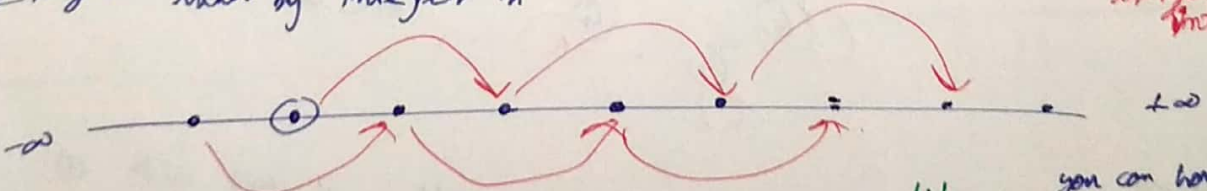
example consider ∞ integer line & a particle which sits at one of the integer. nothing prevents ^{us} from having ∞ no. of states. ~~impose~~ a law.



most necessary to have ∞ objects $\rightarrow$ it can just be one object that can arrange in ∞ no. of ways.

~~labeled by integer~~ laws by integer n

ex II



* here you have ~~conservation~~ conserved quantity.

because you passed to every distinguish integers.

you can have address & evenness to determine state of particle.

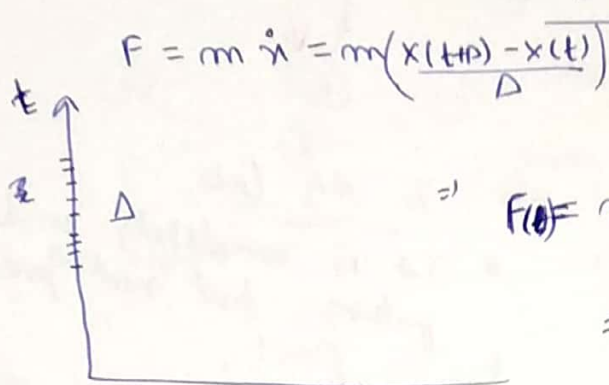
Aristotle Laws (Aristotle's Hypothesis)



$$\vec{F} = m \vec{v}$$

(we will see how this boils)

imagine one dimensional.



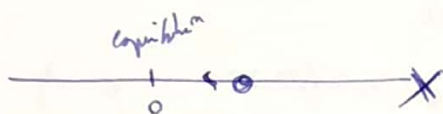
★ stroboscopic world where we break time in intervals

$$\Rightarrow F(t) = m \left(\frac{x(t+\Delta) - x(t)}{\Delta} \right)$$

$$\Rightarrow \frac{\Delta}{m} F(t) + x(t) = x(t+\Delta)$$

$\Rightarrow$ if you know at time t you know at $t+\Delta$ ~~then~~ future.

example



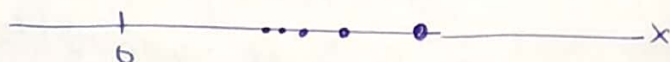
then $F(x) = -kx$

so: $-k \frac{\Delta}{m} x(t) + x(t) = x(t+\Delta)$

$$\Rightarrow x(t) \cdot \left[1 - \frac{k\Delta}{m} \right] = x(t+\Delta)$$

it also allows you to predict future.

so,



you can also write continuous form



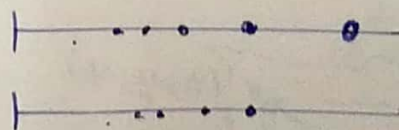
$$m \frac{dx}{dt} = -kx$$

$$\Rightarrow \frac{dx}{dt} = -\frac{k}{m} x$$

$$\Rightarrow x = x(0) e^{-k/m t} \text{ exponential...}$$

$$\Rightarrow x(t) = x(0) \cdot e^{-k/m t}$$

$\rightarrow$ now you can't predict past
★ although you know past where it will be in future but you don't know where it came from.



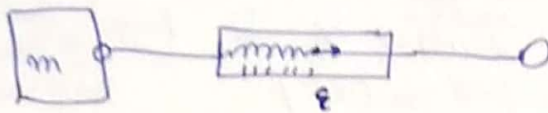
Aristotle law apart from being exponentially incorrect is also an irreversible law not - reversible law.

Newton's Law : $F = ma$

(There are reversible, you can predict future and also the past)

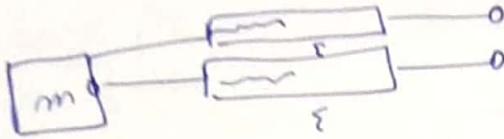
(pg 5)

$$F = m\ddot{x}$$



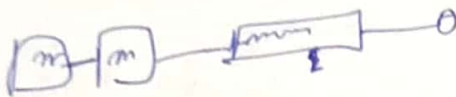
1 unit force

see the proportionality of mass



2 unit force.

; double acceleration.



1/2 acceleration

Is it predictive?

check; $\frac{x(t+\Delta) - x(t)}{\Delta} = v$

$$\frac{\frac{x(t+\Delta) - x(t)}{\Delta} - \frac{x(t) - x(t-\Delta)}{\Delta}}{\Delta} = \ddot{x}$$

$$\Rightarrow \ddot{x} = \frac{x(t+\Delta) + x(t-\Delta) - 2x(t)}{\Delta^2}$$

so; $\Delta^2 \frac{F}{m} = x(t+\Delta) + x(t-\Delta) - 2x(t)$

$\Rightarrow \Delta^2 \frac{F}{m} + 2x(t) - x(t-\Delta) = x(t+\Delta)$

$\downarrow$ present $\downarrow$ past $\downarrow$ future

; It is not enough to know where it is to predict future, but you also need to know where it was in the past.

Initial condition $\rightarrow x(t), \dot{x}(t)$ tells where it was a moment ago

$\downarrow$
 where it is now.

Momentum = $m\dot{x} = p$

mass of an object in Newton's theory is assumed to be constant.

$\therefore \vec{p} = m\vec{v}$

$F = \dot{p}$; $p = m\dot{x}$

is it predictive.

~~Yes~~ no;

$F = \frac{p(t+\Delta) - p(t)}{\Delta}$ } tells how p changes if you know F .

$p = m \cdot \left(\frac{x(t+\Delta) - x(t)}{\Delta} \right)$ } tells how x changes if you know p

You can draw trajectory in p & x space.
... it is completely predictive

$\therefore$ a given values of p will help in deciding x & F at that instant will decide p .

Phase - Space

Space in components of position & components of momentum

example :

$$\ddot{x} = -kx ; \text{ let } k=1 \Rightarrow \ddot{x} = -x$$

$$X = C \cdot \sin(t - t_0)$$

or use cosine

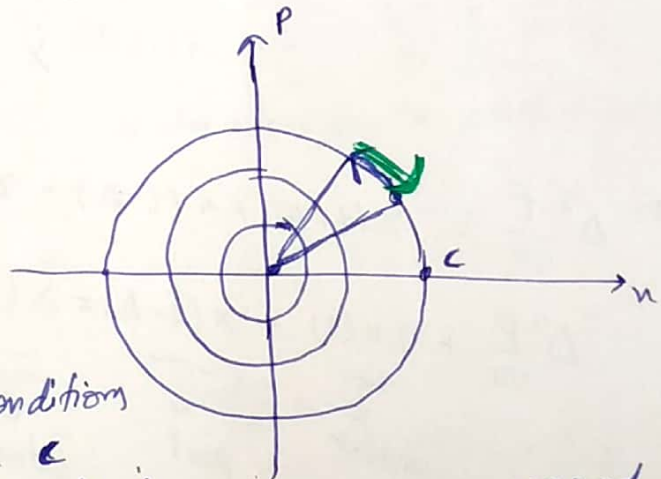
$$\text{or } X = C \cdot \cos(t - t_0)$$

$$\text{let } p = \dot{x} = -C \cdot \sin(t - t_0)$$

(let mass be 1)

$$\text{here; } x^2 + p^2 = C^2$$

(this is constant)



different initial conditions will set different C
(and thus which circle is it is in will be selected & will move around the same circle ~~with~~ of constant radius in phase space.

Here past and future are very alike
(both can be predicted)

- * For future move in clockwise in phase space
- * " " " " anticlockwise in phase space.

$\frac{dx}{dt} = F(t)$: change sign of time $T = -t$
it will then not remain same.

$-\frac{dx}{dT} = F(t)$

: the solutions will not remain same

This equation is not reversible as you change the sign of time.

where this equation does not remain same
→ if you run backward, it would not be a solution to the same equation as the previous forward time moving equation.

$\frac{d^2x}{dt^2} = F(t)$ $t \rightarrow -t$: don't change when you change the sign of time.
 $t^2 \rightarrow t^2$

Note When you a object has certain acceleration, & you take a movie of it and run the movie backward, what happens to the acceleration?

→ You run a movie of acceleration backwards, and it is exactly the same acceleration.

* whereas, if you run a movie backward of a velocity, the velocity changes sign

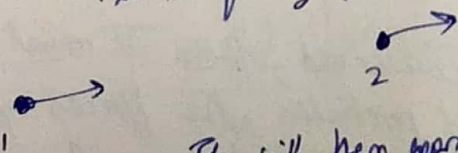
So, Newton's equations in that sense are reversible in that if you take a movie of possible motion, but solve Newton's equation, and run it backwards, it's another solution.

You can't tell which is the required solution, you ~~will~~ may need some kind of superposition along with initial conditions...

Conservation Laws.

Newton Third Law.

* let imagine the forces are in same direction.
(see the funny situation)

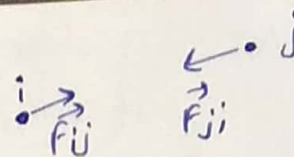


Assumption

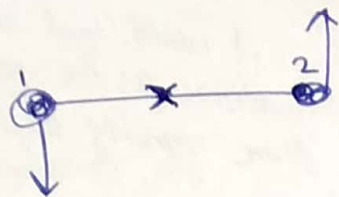
- * Forces add as vectors
- * Forces come in pairs.

It will then spontaneously ~~explode~~ take off...
and so in a way, they're very nice sort of energy
→ but doesn't happen from experience → There is no free lunch.

$\vec{F}_{ij} = -\vec{F}_{ji}$ along line of centres
 force on i due to j



or



but this will continue to rotate faster & faster
 but we know there is no perpetual motion machine

here must be something wrong.

The Alternative ~~is~~ forces are ~~the~~ along the lines of centres.



Newton's third Law has to consider with conservation of momentum... see how.

for a Closed System of some number of particles.

$$m_i \frac{d^2 \vec{r}_i}{dt^2} = \sum_{j \neq i} \vec{F}_{ij} \quad \Rightarrow \quad m_i \frac{d\vec{v}_i}{dt} = \sum_{j \neq i} \vec{F}_{ij}$$

(Total momentum is conserved as a consequence of Newton's principles)
 first Conservation Law.

$$\frac{d\vec{p}_i}{dt} = \sum_{j \neq i} \vec{F}_{ij}$$

$$\sum_i \frac{d\vec{p}_i}{dt} = \sum_i \left(\sum_{j \neq i} \vec{F}_{ij} \right) \quad (\text{Total})$$

$$\Rightarrow \frac{d}{dt} (\vec{p}_{\text{Total}}) = \sum_{\substack{i,j \\ (i \neq j)}} \vec{F}_{ij} = 0$$

from Newton's third Law.

In modern times physics, The conservation of momentum is more fundamental than Newton Law.

You could say, read the other way, "If momentum is conserved, then it must say, that for a pair of particles, the forces must be in the opposite directions and be going the opposite."

$F(x)$

one dimensional case.

Proof Energy Conservation

(139)

In one dimension you can always think of one function as being the derivative of some other function.

~~given~~ ~~$F(x)$~~ assuming, functions are continuous.

~~$F(x)$~~ so; $\exists V$ such that $F(x) = -\frac{dV(x)}{dx}$

(minus sign is convention for standard definition)

$$\int F(x) dx = -V(x)$$

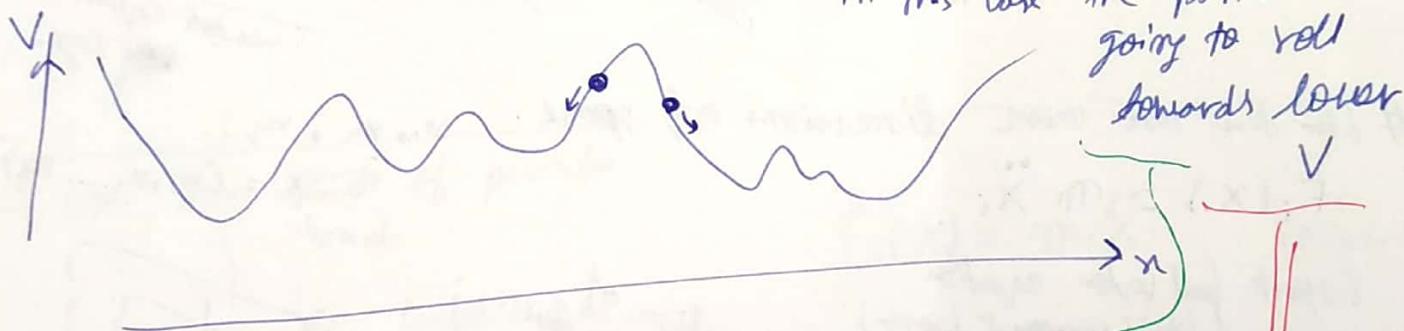
$$F(x) = -\frac{dV(x)}{dx}$$

reason for convention of -ve sign.

minus sign is used as convention

since it is easy to see that

in this case the particle is going to roll towards lower V



This convention says that there is a force which is ~~pushing~~ pushing you towards lower V .

if you would have not used minus sign it would roll up the potential hill.

$$\frac{1}{2}mv^2 + V(x)$$

This is called the Total Energy

$\hookrightarrow$ we are going to prove that it is conserved.

$$= \frac{1}{2}m\dot{x}^2 + V(x) \equiv E$$

$\hookrightarrow$ Kinetic Energy... usually labelled T

$$E = T + V$$

$\Rightarrow$ Prove this is constant with time using Newton Laws.

$$-\frac{dV(u)}{du} = m\ddot{x}$$

$$\therefore \frac{dE}{dt} = \frac{1}{2}m \cdot 2\dot{x} \cdot \ddot{x} + \frac{dV}{dt}$$

$\frac{dV(u)}{dt}$ is not zero $\rightarrow V(u)$ does not explicitly depend on time
 $\hookrightarrow$ but u can be written as a function of time
 $\hookrightarrow$ so implicitly $\dots V$ depends on time.

$$\frac{dE}{dt} = \frac{1}{2}m \cdot 2\dot{x} \ddot{x} + \frac{dV}{du} \cdot \frac{du}{dt}$$

$$= \dot{x} \cdot m\ddot{x} + \frac{dV}{du} \dot{x} = \dot{x} \left(m\ddot{x} + \frac{dV}{du} \right) = 0$$

reduces equations say

later there are more dimensions of space.

$$F_i(x) = m \ddot{x}_i$$

$$x_1, x_2, x_3 \dots$$

$$X = (x_1, x_2, \dots, x_n)^T$$

It is not something which can be derived.

(write full vector equation in component form)

Assumption of conservative forces.

We may like to write down

$$F_i(u) = - \frac{\partial V(u)}{\partial x_i}$$



It is law of Physics, from experience and experiment.

It's a law of physics is true. $\rightarrow$ there exist a V .
 here exist some $V(x)$ such that components of force equals the components of $-u$ of gradient.

This is very central as we move forward in course.

* In mathematical world it is not necessary that there always exist $V(x)$ such that ~~set~~ that component wise equality holds.

Question: Is it ~~the~~ true that if we take any set of functions, $f_1, f_2, \dots, f_n$, is there always a V whose partial derivatives gives you f_i 's.

Ans No,

$\Rightarrow$ so we see that in general there is no such things as Potential Energy which has property that its derivatives are equal to force.

$$\sum_i \frac{1}{2} m \dot{x}_i^2 + V(x) = E$$

called kinetic Energy: T

$$\begin{aligned} \therefore \frac{dE}{dt} &= \sum_i \frac{1}{2} m \cdot 2 \dot{x}_i \ddot{x}_i + \frac{dV}{dx} \dot{x} \\ &= \sum_i m \dot{x}_i \ddot{x}_i + \frac{\partial V}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial V}{\partial x_2} \frac{dx_2}{dt} \dots \\ &= \sum_i \left(m \dot{x}_i \ddot{x}_i + \frac{\partial V}{\partial x_i} \dot{x}_i \right) \\ \frac{dE}{dt} &= \sum_i \dot{x}_i \left(m \ddot{x}_i + \frac{\partial V}{\partial x_i} \right) = 0 \end{aligned}$$

Now for ~~group~~ bunch of particles.

$\left. \begin{matrix} x_1 \\ y_1 \\ z_1 \\ \vdots \\ x_N \\ y_N \\ z_N \end{matrix} \right\}$

X_i (relable term,
~~there are 3N~~
there are 3N terms

consider coordinates of N particles.

of course we know
not for a given particle
 i : m ~~in the~~ is same
 $\hookrightarrow$ but we just
relable them as
being different or equal
in general
(we just remember that
they were equal)

$$F_i(x) = m_i \ddot{X}_i \quad i \in \{1, 2, \dots, 3N\}$$
$$= - \frac{\partial V(x)}{\partial x_i}$$

by this way
we can conserve
energy for N
no. of particles.
(Same previous proof
....)
from assumption of
conservative forces.

Harmonic oscillator:

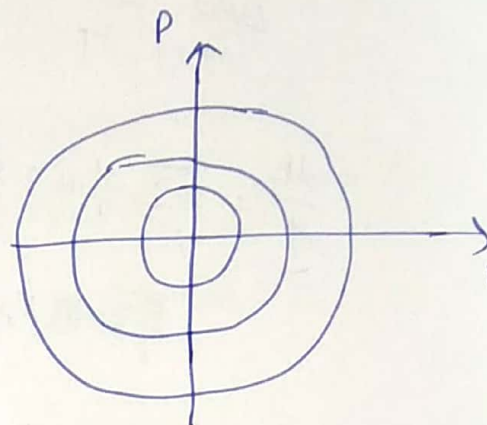
in our previous example
we kept $m=1, k=1$

$$F = -kx$$

$$V = \frac{k}{2} x^2 \quad \text{check: } -\frac{dV}{dx} = kx$$

$$\therefore \frac{m}{2} \dot{x}^2 + \frac{k}{2} x^2 = \frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2$$

this circle motion in
a given orbit has simple
demonstration of conservation of
energy. (staying in orbit of constant energy)



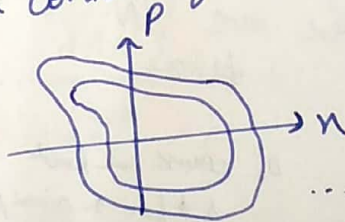
→ here; energy increases quadratically as you move
away from centre.

→ and the orbits are nothing but contours.
→ the contours of constant potential energy.

~~In general in
one dimension~~ ⇒

In general in one dimension

→ a particle will move on a contour of constant energy in
phase plane.



Principle of Least Action ^{Shoaib Akhtar}

Lec 3

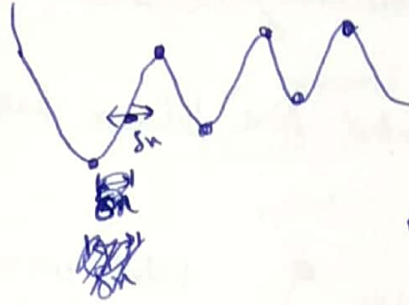
PJ13

(not quite correct.... will see later ... not always least, but more correctly stationary)

~~example~~ example: what are laws determining eq^m?
(when it does not move, when it is stationary)

ex 1-D case

$$\frac{dV(x)}{dx} = 0$$

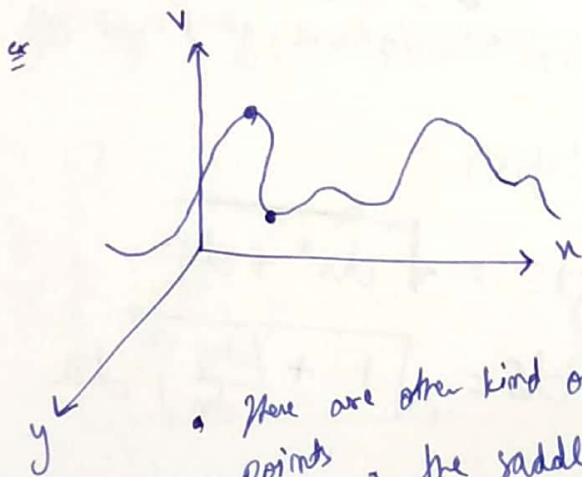


$$\delta V = \frac{dV}{dx} \delta x$$

$$\Rightarrow \delta V = \frac{dV}{dx} \delta x$$

(at eq^m... $\delta V = 0$)

* change in V is zero to the first order.

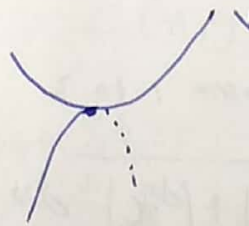


equilibrium when

$$\frac{\partial V(x,y)}{\partial x_i} = 0 \quad ; \quad \forall i$$

eq^m at top & bottom.

* There are other kind of stationary points, the saddle point



$$\delta V = \frac{\partial V}{\partial x} \delta x + \frac{\partial V}{\partial y} \delta y$$

for equilibrium $\delta V = 0$ for every possible variations of δx and δy .

Condition of Equilibrium $\delta V = 0$ for every possible choice of variations.

General Law of Nature for Equilibrium

$$\delta V = 0$$

variation of V is zero for any little motion involved to first order in small quantities."

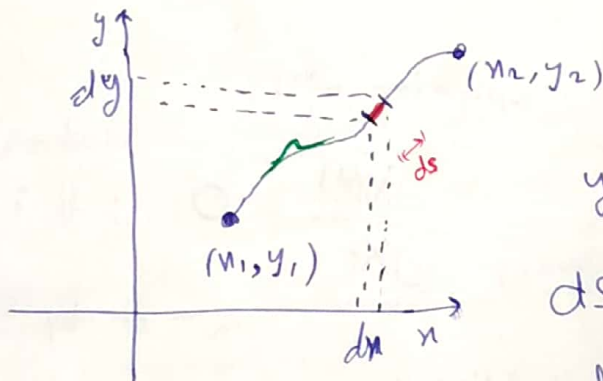
Principle of Stationary Action.

Question: What is the shortest line between two points.



What curves minimize the distance between them?

Setting the
Question.



s is used as a symbol for distance along a curve.

$$ds = \sqrt{dx^2 + dy^2}$$
$$ds = \sqrt{1 + \left(\frac{dy}{dn}\right)^2} dn$$

S_{12} := distance from 1 to 2

$$S_{12} = \int \sqrt{1 + \left(\frac{dy}{dn}\right)^2} dn$$

$$\delta \int \sqrt{1 + \left(\frac{dy}{dn}\right)^2} dn = 0$$

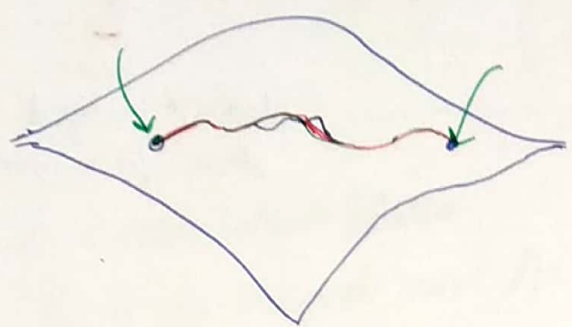
$$\delta S_{12} = 0$$

any little change in the function will increase the length.

what we gonna change
as we will change the
function.

Solving an equation like this means to reduce this to a ^{simple} differential equation similar to Newton's equation.

ex.



could ask shortest distance on a surface.

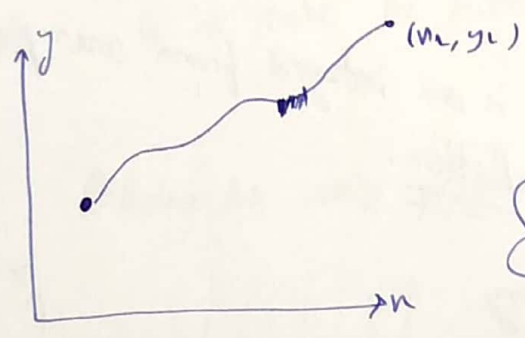
physically interpreting
 $\therefore$ tie a string, pull string as tight as possible
 which define the shortest distance between the points.
 $\therefore$ In general it will not be straight line, time surface is curved.

Geodesic: The shortest path between two points on a curve.

ex geodesic on a flat plane is straight line.

ex light ray $\delta t = 0$ ~~$\delta t \neq 0$~~

Fermat's Principle (a physical law for light)



$$\delta \left(\int_0^t \frac{\sqrt{dn^2 + dy^2}}{c(n, y)} \right) = 0$$

$\Rightarrow$ and from variational principle ~~$\delta t \neq 0$~~ $\delta t = 0$

$$\therefore \text{so: } \delta \left(\int \frac{\sqrt{1 + (dy/dn)^2}}{c(n, y)} dn \right) = 0$$

This is the way light moves through a medium of variable index of refraction.

...defines the path of shortest time.

ex Special mechanics problem (1-D) dimensional.

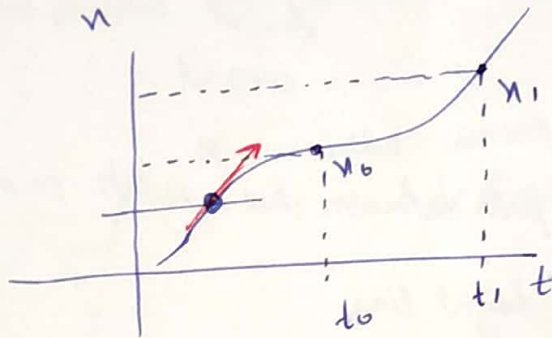
(independent variable) $t \uparrow$

* plot x on y -axis
... * time flows upward

Notoka...

physicist realized
time flows upward.

for ~~constant~~ x

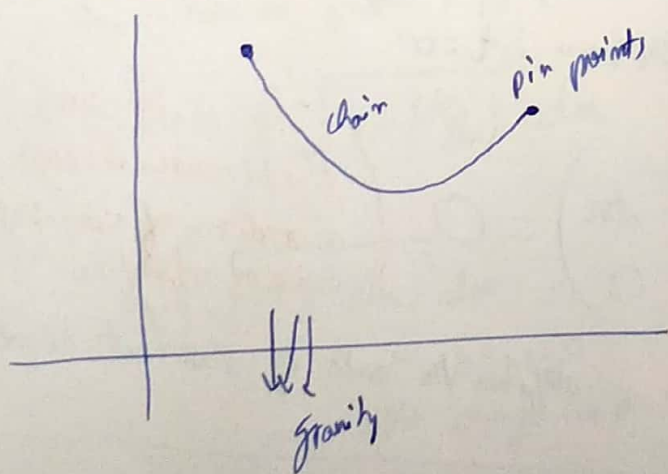


★ "The answer in all problems of Classical Physics is to minimize something associated with the paths between these two points."

↳ & in fact ^{that} is something is an integral from one point to another and is called the Action.

↳ when we say minimize we mean stationary.

ex



here we minimize potential energy

$$A = \int_{\text{initial}}^{\text{final}} dt L(v, \dot{v})$$

There is a quantity A say it action.

(PS17)

$$A = \int_{t_0}^{t_1} dt \mathcal{L}(x, \dot{x})$$

$\mathcal{L}$ is Lagrangian.

~~A is some integral between~~

A is an integral over the trajectory.

→ Integral of something that depends on trajectory.

What is that something?
Ans → We gonna call it a Lagrangian.

$\mathcal{L}$ depends on position along the trajectory (x) and velocity along the trajectory ($\dot{x}$).

such problems can always be reduced to a differential eqn which we gave a name Euler-Lagrange equation.

* Problem is to make the Action stationary.

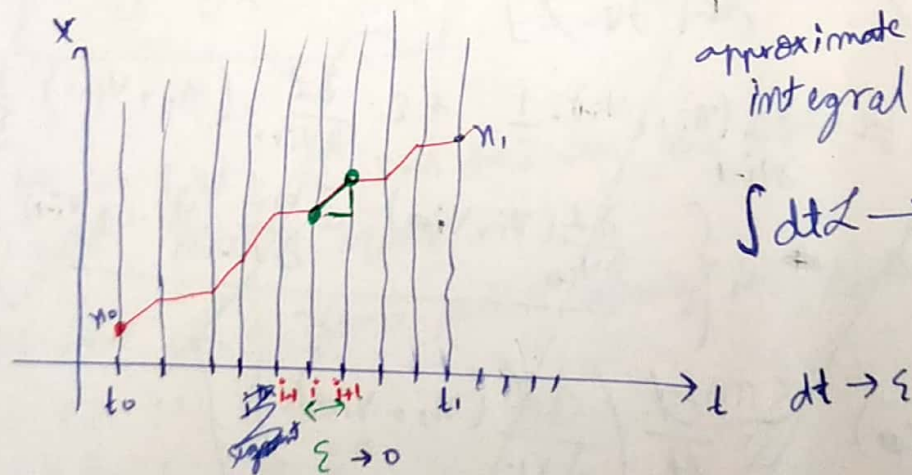
ie:

$$\delta \int_{t_0}^{t_1} dt \mathcal{L} = 0$$

$$\delta A = 0$$

Solution: Replace this precise problem by an approximate problem.

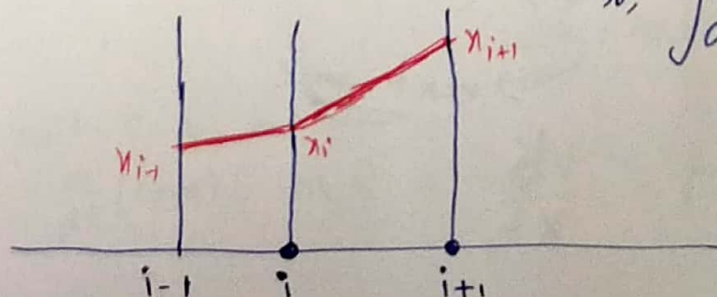
approximate theory replaces the integral by a sum.



$$\int dt \mathcal{L} \rightarrow \sum_i \epsilon \mathcal{L}(x_i, \frac{y_{i+1} - y_i}{\epsilon})$$

$$\dot{x} \rightarrow \frac{\Delta x}{\Delta t}$$

So, $\int dt \mathcal{L} \rightarrow \epsilon \sum_i \mathcal{L}(x_i, \frac{y_{i+1} - y_i}{\epsilon})$



~~I labeled it as belonging to i~~

$$V_{ij} = \frac{n_{i+} - n_i}{\Sigma}$$

label as belonging to i

$$\frac{\partial \mathcal{L}}{\partial \mu_i} \left(\mu_i, \frac{\mu_{i+1} - \mu_i}{\epsilon} \right) + \frac{\partial \mathcal{L}}{\partial \nu_i} \left(\mu_{i-1}, \frac{\mu_i - \mu_{i-1}}{\epsilon} \right)$$

differentiate this function w.r.t. x_i *note: $\frac{dx_{i-1}}{dx_i} = 0$

$$\begin{aligned}
 & \epsilon \frac{\partial \mathcal{L}}{\partial x_i}(x_i, v_{i+1}) + \epsilon \cdot \frac{\partial \mathcal{L}}{\partial v_i} \cdot \frac{1}{\epsilon} + \epsilon \cdot \left(-\frac{1}{\epsilon}\right) \frac{\partial \mathcal{L}}{\partial v_{i+1}} \\
 &= \epsilon \left[\frac{\partial \mathcal{L}}{\partial x_i}(x_i, v_i) \Rightarrow \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial v} \right) \right] \\
 & \epsilon \cdot \frac{\partial \mathcal{L}}{\partial x_i}(x_i, v_{i+1}) + \epsilon \cdot \frac{\partial \mathcal{L}}{\partial v_{i-1}}(x_{i-1}, v_{i-1}) \cdot \frac{1}{\epsilon} + \epsilon \cdot \frac{\partial \mathcal{L}}{\partial v_{i+1}}(x_i, v_{i+1}) \cdot \frac{1}{\epsilon} \\
 &= \epsilon \left[\frac{\partial \mathcal{L}}{\partial x_i}(x_i, v_{i+1}) \Rightarrow \left\{ \frac{\frac{\partial \mathcal{L}}{\partial v_i}(x_i, v_{i+1}) - \frac{\partial \mathcal{L}}{\partial v_{i-1}}(x_{i-1}, v_{i-1})}{\epsilon} \right\} \right] \\
 &= \epsilon \left[\frac{\partial \mathcal{L}}{\partial x_i}(x_i, v_{i+1}) - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial v_i}(x_i, v_{i+1}) \right) \right]
 \end{aligned}$$

→ next page

$$\sum \mathcal{L}(n_i, \frac{n_{i+1} - n_i}{\epsilon}) + \sum \mathcal{L}(n_{i-1}, \frac{n_i - n_{i-1}}{\epsilon}) \quad \left\{ \begin{array}{l} \text{manipulating} \\ \text{in pairs.} \end{array} \right.$$

new differential w.r.t. n_i

note; $\frac{dn_{i-1}}{dn_i} = 0$

$V_i := \frac{n_{i+1} - n_i}{\epsilon}$

$$\sum \left[\frac{\partial \mathcal{L}}{\partial n_i}(n_i, V_i) + \frac{1}{\epsilon} \frac{\partial \mathcal{L}}{\partial V_{i-1}}(n_{i-1}, V_{i-1}) - \frac{1}{\epsilon} \frac{\partial \mathcal{L}}{\partial V_i}(n_i, V_i) \right] \quad \text{treated as belonging to } i \dots$$

$$= \sum \left[\frac{\partial \mathcal{L}}{\partial n_i}(n_i, V_i) - \left\{ \frac{\frac{\partial \mathcal{L}}{\partial V_i}(n_i, V_i) - \frac{\partial \mathcal{L}}{\partial V_{i-1}}(n_{i-1}, V_{i-1})}{\epsilon} \right\} \right]$$

$$= \sum \left[\frac{\partial \mathcal{L}}{\partial n_i}(n_i, V_i) - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial V_i}(n_i, V_i) \right) \right]$$

$$\delta A \propto \left(-\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial v} \right) + \frac{\partial \mathcal{L}}{\partial x} \right) = 0$$

$$\boxed{\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}}$$

Euler Lagrangian Equation.
(General form)

We can find $\mathcal{L}$ such that when you write E.L. equations, you get Newton's Equation.

ex; what $\mathcal{L}$ is for a particle moving in Potential energy.

P.E. $:= V(x)$

K.E. $:= \frac{1}{2} m \dot{x}^2$

$\mathcal{L} = \frac{m \dot{x}^2}{2} - V(x)$

$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m \dot{x} := \text{momentum};$

Write $\mathcal{L}$ as

$\therefore \frac{d}{dt}(m \dot{x}) = m \ddot{x} = -\frac{dV}{dx} = F(x)$

for many particles.

$$A = \int_{t_1}^{t_2} dt \mathcal{L}(\mathbf{r}, \dot{\mathbf{r}})$$

to

depends on all coordinates & all velocities

i now stands for particles & direction of space.

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial \mathcal{L}}{\partial x_i}$$

$$\mathbf{X} = \{x_i\}$$

$$\dot{\mathbf{X}} = \{\dot{x}_i\}$$

$$P.E. = V(\mathbf{X})$$

$$KE = \sum \frac{m}{2} \dot{x}_i^2$$

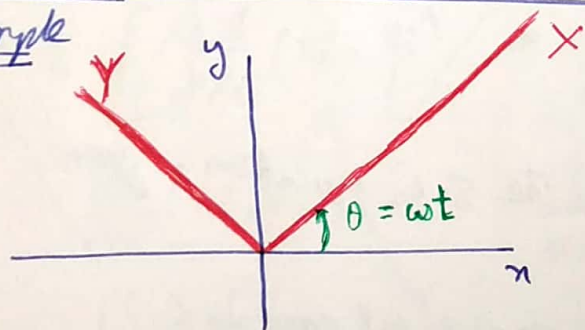
$$m \ddot{x}_i = -\frac{\partial V}{\partial x_i} = F_i(\mathbf{x})$$

// Note // $\mathcal{L}$ is one function that packages the entire laws of physics for that system.

$\mathcal{L}$ is extremely good tool for changing variables or coordinates.

★ Action principle is a statement that does not depend on coordinate you use

example



Two frames of reference
one stationary inertial reference
& the other rotating with time.

→ Hence newtons equations apply ; $\vec{F} = 0$; $m \ddot{\mathbf{x}} = 0$
 $m \ddot{\mathbf{y}} = 0$

without $\mathcal{L}$... ; simply ... & complicated method

$$x = X \cos \omega t + Y \sin \omega t$$

$$y = -X \sin \omega t + Y \cos \omega t$$

$$A = \int \mathcal{L} dt$$

$$\mathcal{L} = \frac{m}{2} (\dot{x}^2 + \dot{y}^2)$$

→ apply $\Sigma \mathcal{L} = 0$ → set $\ddot{x} = 0$
(no potential energy) $\ddot{y} = 0$

... → get in XY coordinate

$$\dot{x} = \dot{X} \cos \omega t - \omega X \sin \omega t + \dot{Y} \sin \omega t + \omega Y \cos \omega t$$

$$\dot{y} = -\dot{X} \sin \omega t - \omega X \cos \omega t + \dot{Y} \cos \omega t - \omega Y \sin \omega t$$

$$\mathcal{L} = \frac{m}{2} (\dot{x}^2 + \dot{y}^2)$$

(P521)

$$= \frac{m}{2} \left\{ (\dot{X}^2 + \dot{Y}^2) (\sin^2(\omega t) + \cos^2(\omega t)) + \omega^2 (X^2 + Y^2) + \omega (\dot{X}Y - \dot{Y}X) \right\}$$

$$\mathcal{L} = \frac{m}{2} \left\{ (\dot{X}^2 + \dot{Y}^2) \right\} + \frac{m\omega^2}{2} (X^2 + Y^2) + \frac{m\omega}{2} (\dot{X}Y - \dot{Y}X)$$

remember we normally write

$$\mathcal{L} = T - V$$

has a look of Potential Energy
it's a fictitious potential energy.

$$V_{\text{fict}} = -\frac{\omega^2 m}{2} (X^2 + Y^2)$$

generates centrifugal force

this term generates Coriolis force.

$$\frac{\partial \mathcal{L}}{\partial \dot{X}} = m \dot{X} + \frac{m\omega}{2} Y \quad ; \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{X}} \right) = m \ddot{X} + \frac{m\omega}{2} \dot{Y}$$

$$\frac{\partial \mathcal{L}}{\partial X} = m\omega \dot{Y}$$

$$= \frac{\partial \mathcal{L}}{\partial X} = \omega^2 m X - \frac{m\omega}{2} \dot{Y}$$

$$\therefore m a_x := m \ddot{X} = \omega^2 m X - m\omega \dot{Y}$$

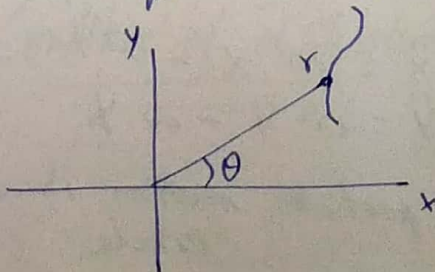
component of acceleration

called centrifugal force.

velocity dependent force
... called Coriolis force.

$$m a_y := m \ddot{Y} = \omega^2 m Y + m\omega \dot{X}$$

example in ordinary Polar Coordinates. for a particle ~~where~~
~~potential depends on radius.~~



$$\begin{aligned} \frac{1}{2} m (\dot{X}^2 + \dot{Y}^2) & \quad X = r \cos \theta \quad Y = r \sin \theta \\ & \quad \dot{X} = \dot{r} \cos \theta - r \sin \theta \dot{\theta} \\ & \quad \dot{Y} = \dot{r} \sin \theta + r \cos \theta \dot{\theta} \\ & = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) \end{aligned}$$

$\therefore$ case when potential energy depends on distance from origin.

$$\mathcal{L} = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$$

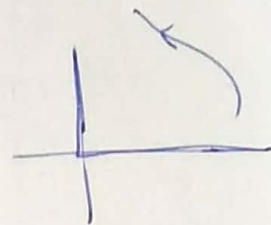
$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \right) = m \ddot{r} = m r \dot{\theta}^2 - \frac{dV(r)}{dr}$$

$\downarrow$
r component
of acceleration

$\downarrow$
centrifugal
force.

$\downarrow$ r component of
force.

(because of particle moving around)



$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = \frac{d}{dt} (m r^2 \dot{\theta}) = 0 \quad \therefore \Rightarrow \text{so we get some conserved quantity ... Angular Momentum.}$$

Conservation of Angular Momentum

$\hookrightarrow$ follows because of the assumption that V does not depend on θ .

G $L = m r^2 \dot{\theta} = \text{constant}$
 $\dot{\theta} = \frac{L}{m r^2}$
 & does not change with time.
 (determined by initial condition)

★ Cyclic Coordinates: coordinates who don't appear directly in the Lagrangian. ~~there is a conserved~~

whenever there is a Cyclic Coordinate, there is a Conservation Law.

Σ - L -eqⁿ are invariant under coordinate transformation.
 $\hookrightarrow$ i.e. they take the same form.

the components of forces due to Coriolis forces -

$$F_x = m \omega \dot{y} \quad , \quad F_x \propto \dot{y}$$

$$F_y = -m \omega \dot{x} \quad ; \quad F_y \propto -\dot{x}$$

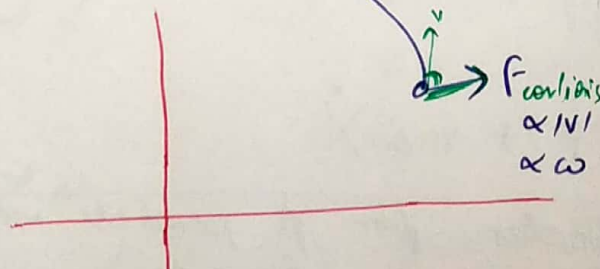
$\therefore$ i.e. this F is $\perp$ to velocity.

$$\therefore F_x \dot{x} + F_y \dot{y} = 0$$

prove; that one of the trajectory of

the system is moving on a circle. $\ddot{x} = \omega^2 x - \omega \dot{y}$
 $\ddot{y} = \omega^2 y - \omega \dot{x}$

$\hookrightarrow$ gives rise to particles motion in a circle.



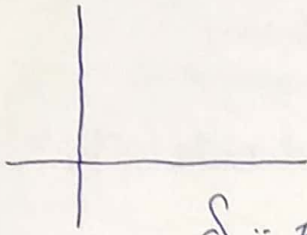
Symmetry & Conservation Laws.

★ Conservation Laws are always connected with symmetries.

Pg 23

Lec - 4

- Shoaib Akhtar



$F(x, y)$

$$\delta F(x, y) = \frac{\partial F}{\partial x} \delta x + \frac{\partial F}{\partial y} \delta y$$

change in F when you make a small shift proportional to the quantity δ .

δ := Infinitesimally small quantity : $\delta \rightarrow 0$; $\delta \neq 0$

Mathematical rule for an infinitesimally small quantity.

$$\delta \pm \delta^2 \approx \delta$$

$$c\delta = c\delta$$

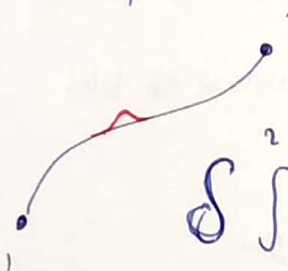
later in course we changed $\delta x \propto \epsilon$, $\delta y \propto \epsilon$ for convenience & to escape confusion

$$\delta x \propto \delta$$

$$\delta y \propto \delta$$

where δx & δy are proportional to the universal small quantity δ .

This is called the First Order Change in F .



$$\delta \int_1^2 dt \mathcal{L}(x, \dot{x}) = 0$$

first order change in action is zero.

- ★ x is set of coordinates x_i
- ★ $\dot{x}$ is set of $\dot{x}_i$'s ...

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

This is the contribution by shifting the i th coordinate.

because of Principle of Stationary Action.

★ This is the first order change in Action.

for General Curvilinear coordinates : $q = \{q_i\}$

$\dot{q}_i$:= i th component of velocity.

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i$$

:= conjugate momentum conjugate to q_i

ex $\mathcal{L}$ for particle: $\mathcal{L} = \frac{m\dot{x}^2}{2} - V(x)$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

(This is from where the name comes from)
 $\hookrightarrow$ the name from momentum.

Canonical Momentum conjugate to the coordinate q_i 's

$$:= p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Definition $\rightarrow$

you know from experience why this is defined so.

so; equation becomes; $\frac{d}{dt} p_i = \frac{\partial \mathcal{L}}{\partial q_i}$

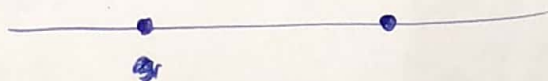
for cyclic coordinates q_i $\frac{\partial \mathcal{L}}{\partial q_i} = 0 \Rightarrow \frac{d}{dt} p_i = 0 \Rightarrow p_i$ is conserved.

Symmetry Transformation

example $\mathcal{L} = \frac{\dot{q}_1^2 + \dot{q}_2^2}{2} - V(q_1 - q_2)$

set the mass equal to 1.

now if you think q_1 & q_2 as position in line.



now writing eqⁿ of motions

$$\dot{p}_1 = -V'(q_1 - q_2)$$

$$\dot{p}_2 = -V'(q_1 - q_2) \cdot (-1) = +V'(q_1 - q_2)$$

$$\frac{d}{dt} (p_1 + p_2) = 0$$

and hence we have a conservation law for $p_1 + p_2$.

$$V(\xi)$$

$\therefore$ then $V'(\xi)$ represent the derivative of V w.r.t. argument of function (in this case ξ)

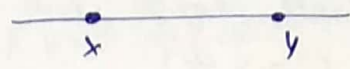
ex now; let; ~~V~~ $V(aq_1 + bq_2)$

so; eqⁿs of motions are: $p_1 = -aV'$ $\Rightarrow \frac{d}{dt} \left(\frac{p_1}{a} - \frac{p_2}{b} \right) = 0 \Rightarrow \frac{d}{dt} (bp_1 - ap_2) = 0$

$$p_2 = -bV'$$

$\hookrightarrow$ so we again have a conservation law.

$$\mathcal{L} = \frac{m\dot{x}^2}{2} + \frac{M\dot{y}^2}{2} - V(x-y)$$



x, y are along same direction of space for different particles of masses m & M respectively.

∴ let: $\sqrt{m} X = q_1$ (change of coordinates)
 $\sqrt{M} Y = q_2 \Rightarrow \sqrt{m} \dot{x} = \dot{q}_1$
 $\sqrt{M} \dot{y} = \dot{q}_2$

so; $\mathcal{L} = \frac{\dot{q}_1^2}{2} + \frac{\dot{q}_2^2}{2} - V\left(\frac{q_1}{\sqrt{m}} - \frac{q_2}{\sqrt{M}}\right)$

Alternate; let: $\sqrt{m} X = q_1$
 $-\sqrt{M} Y = q_2$ so; $\mathcal{L} = \frac{\dot{q}_1^2}{2} + \frac{\dot{q}_2^2}{2} - V\left(\frac{q_1}{\sqrt{m}} + \frac{q_2}{\sqrt{M}}\right)$

∴ The Conservation Laws are connected with symmetries.

ex simple symmetry. $\mathcal{L} = \frac{\dot{q}^2}{2}$ (only one q ...)

What do we mean by symmetry: ⇒ a change that you can make in q, a coordinate change in q, that does not affect the Lagrangian.

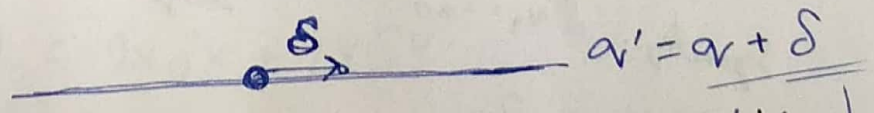
You can think in two ways: • Actively • Passively.

[one way → change your coordinate frame, shift origin. Passively
 [other way → stand at same place and move the object (for coordinate frame) Actively.
 → Active Coordinate Transformation.

Passive Coordinate Transformation

* in this case, the result ahead does not really depend which way you transform.

$$\delta q = \delta$$



change in q
 ∴ δ a small number
 (in this case, it not depends whether δ is small or not)

Note: Change q, means same shift at all times
 ∴ nothing happens to $\dot{q}$
 since δ is constant
 $\dot{q}' = \dot{q}$

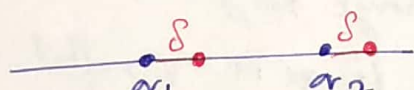
under $\delta q = \delta$; $\delta \mathcal{L} = 0$ (The variation in $\mathcal{L}$ is 0)
 ⇒ It is symmetry.

A coordinate change which may or may not involve the small number, the change in the Lagrangian is 0.
 that's called Symmetry

And what it means is that $\mathcal{L}$ does not care, the $\mathcal{L}$ has same form if we shift coordinate or push the particle. And, the $\mathcal{L}$ does not change, it is symmetric.

* in this example it is translational symmetry.
 (translation of space)

in our example $\vec{p} = 0$

ex 
$$\begin{aligned} r_1' &= r_1 + \delta \\ r_2' &= r_2 + \delta \end{aligned}$$

$\therefore V(r_1 - r_2)$ - it also does not change.

here also there is symmetry & there is one conservation law as we have seen before
 $p_1 + p_2 = \text{constant}$

ex
$$\begin{aligned} r_1' &= r_1 + b\delta \Rightarrow \dot{r}_1' = \dot{r}_1 \\ r_2' &= r_2 - a\delta \Rightarrow \dot{r}_2' = \dot{r}_2 \end{aligned}$$

 i.e. $\delta r_1 = b\delta$
 $\delta r_2 = -a\delta$
 symmetry.
 So; T of $\mathcal{L}$ does not change

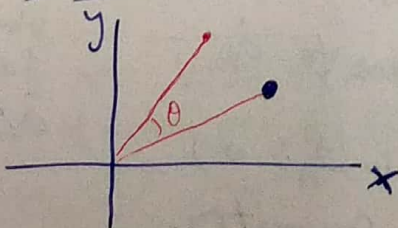
$\therefore$ let's see V

$\therefore V(a r_1 + b r_2)$

$$a r_1 + b r_2 + b\delta + b r_2 - a\delta$$

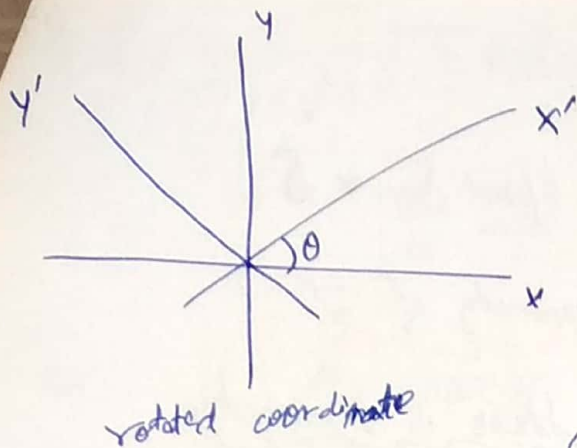
 $= a r_1' + b r_2'$

ex Rotational Symmetry



$$\mathcal{L} = \frac{m}{2} [\dot{x}^2 + \dot{y}^2] - V(x^2 + y^2)$$

invariant in rotation of coordinate system.



$$\therefore x' = x \cos \theta + y \sin \theta$$

$$y' = -x \sin \theta + y \cos \theta$$

if we can prove that $\mathcal{L}$ does not change if we make a small change in angle, then we can also prove that it does not change when we make a large change in angle.

(simple, think a large change in angle as a sequence of small change in angle.)

The small parameter is angle θ ; keep $\theta = \delta$ ~~keep the small~~

$$\cos \theta = 1 \quad ; \quad (\cos \delta = 1)$$

$$\cos \theta = 1 - \frac{\delta^2}{2} + \dots = 1$$

zero...

(keep the small change in the parameter)

Initially when non-rotated θ has 0

$$\text{so; } \delta \theta = \theta - 0 = \delta$$

$$\sin \theta \approx \delta$$

$$\text{so; } x' = x + y\delta$$

$$y' = -x\delta + y$$

summarize as

$$\delta x = y\delta$$

$$\delta y = -x\delta$$

lets check that $\mathcal{L}$ does not change.

$$\begin{cases} \delta x = y\delta \\ \delta y = -x\delta \end{cases}$$

$$\begin{cases} \delta x = y\delta \\ \delta y = -x\delta \end{cases}$$

This is our infinitesimally small rotation.

$$\therefore \text{check, } x^2 + y^2 = r^2$$

$$\delta(x^2 + y^2) = 2x\delta x + 2y\delta y$$

$$= 2x y\delta + 2y(-x)\delta = 0$$

this is the small change in the square of the radius.

$$\hookrightarrow \text{same argument for } \delta(x'^2 + y'^2) = \dots = 0$$

$\hookrightarrow$ so this is also an example of Symmetry.

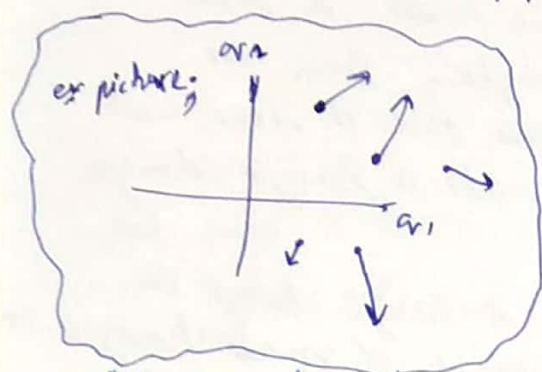
Symmetry :

we have

q_i 's

$q = \{q_i\}$

$$\delta q_i = f_i(q) \cdot \delta \quad (\text{first } \delta q_i \propto \delta)$$



shift may depend on where you are ... although it is proportional to δ .

Is this a symmetry of $\mathcal{L}$?

let suppose this is symmetry.

is

also;

$$\delta \dot{q}_i = \frac{d}{dt} \delta q_i$$

that is we can write down the condition when the $\mathcal{L}$ does not change.

$$\delta \mathcal{L}(q, \dot{q}) = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} \delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

$\mathcal{L}$ changes because each one of q_i 's & $\dot{q}_i$'s changes.

* let's assume motion of ~~equation~~ system satisfies the equations of motion, we are assuming that the system is moving according to the laws governed by Lagrangian,

$$\therefore \text{sim } p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$\delta \mathcal{L}(q, \dot{q}) = \sum_i \frac{\partial \mathcal{L}}{\partial q_i} \delta q_i + p_i \delta \dot{q}_i$$

$$\text{but from } \Sigma - \mathcal{L} \text{ eqn: } \frac{d}{dt} (p_i) = \frac{\partial \mathcal{L}}{\partial q_i}$$

$$\text{so; } \delta \mathcal{L}(q, \dot{q}) = \sum_i (p_i \delta q_i + p_i \delta \dot{q}_i)$$

$$= \sum_i \frac{d}{dt} (p_i \delta q_i) = \frac{d}{dt} \left(\sum_i p_i \delta q_i \right)$$

$$\delta \mathcal{L}(q, \dot{q}) = \frac{d}{dt} \sum_i P_i \delta q_i$$

$$= \frac{d}{dt} \sum_i P_i f_i(q) \delta$$

Now; if this change is symmetry. it says
that $\delta \mathcal{L} = 0$

(we haven't proved there is symmetry, ... we assumed there is symmetry)

$$\text{Then; } \delta \mathcal{L} = \frac{d}{dt} \left(\sum_i P_i f_i(q) \cdot \delta \right) = 0$$

$$\Rightarrow \frac{d}{dt} \left(\sum_i P_i f_i(q) \right) = 0$$

δ is just a constant number.

$\therefore$ we now have a conservation law,

i.e. $\frac{d}{dt}$ of something is equal to zero.
 $\hookrightarrow$ that something is made up of velocities & coordinates)

$$\frac{d}{dt} \left(\sum_i P_i f_i(q) \right) = 0$$

$\hookrightarrow$ say Q is a conserved quantity (for quantity)

If there is symmetry of type $\delta q_i = f_i(q) \delta$

then there is a conservation law

~~we learn, if there is symmetry of type $\delta q_i = f_i(q) \delta$~~

we learn: if there is symmetry of type $\delta q_i = f_i(q) \delta$

then there is automatically a conservation law.

The conservation is the conservation of

$$Q = \sum_i P_i f_i(q)$$

is coefficient in the small change of the q^{th} coordinate.

contribution from the each coordinate $\rightarrow$ canonical momentum conjugate to that coordinate.

examples

$$\delta q_1 = \delta \quad \therefore f_1 \text{ \& } f_2 \text{ are 1}$$

$$\delta q_2 = \delta$$

$$\text{So; } Q = p_1 \cdot 1 + p_2 \cdot 1 = p_1 + p_2$$

(momentum conservation as a consequence of translation invariance)

ex 2

$$\delta q_1 = b \delta$$

$$\delta q_2 = -a \delta$$

$$\therefore f_1 = b$$

$$f_2 = -a$$

$$\therefore Q = \underline{b p_1 - a p_2}$$

ex 3 Rotation of Coordinates

$$\delta x = y \delta$$

$$\delta y = -x \delta$$

$$\therefore f_x = y$$

$$f_y = -x$$

so; $Q = \cancel{p_x y} \quad Q = \sum_i p_i f_i$

there $Q = \underline{p_x y - p_y x}$

it is conserved

as a consequence of symmetry ... & what is that symmetry,

the Symmetry of Rotational Invariance.

★ Deep connection between Invariances, or Symmetry rules and Conservation laws.

★ we are talking about the cases where the Symmetry parameters are time independent.

ex Harmonic Oscillator.

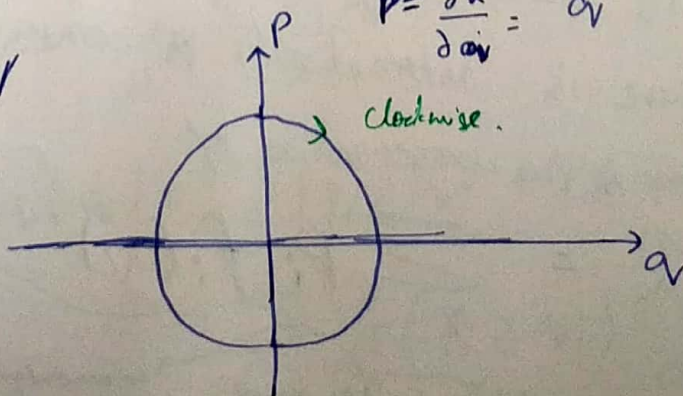
set $m, k=1$ for simplicity.

$$L = \frac{\dot{q}^2}{2} - \frac{q^2}{2} \quad ; \quad \frac{dp}{dt} = -q \quad \text{no: } \dot{q}$$

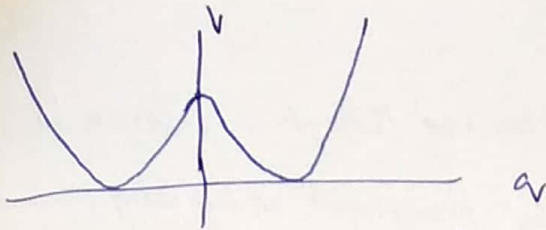
$$\frac{dp}{dt} = -q$$

$$\frac{dq}{dt} = p$$

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$$



#



This problem has a symmetry
 $q \rightarrow -q$

also says $\dot{q} \rightarrow -\dot{q}$

If the K.E. does not change

& if the P.E. has this reflection property
 then it also does not change.

~~there is no motion of building~~

* Here we don't have any notion of infinitesimal transformation

* the flipping cannot be built up by successive infinitesimal transformations.

→ This type of Symmetry in Classical Mechanics does not have Conservation Law associated with it. (Discrete Symmetry)
 (there is one in Quantum mechanics.)

* If you want to derive a conservation law from symmetry it should be kind of symmetry that you can build up by little infinitesimally small motions; and in fact what you really need is infinitesimally version of symmetry transformation. (Continuous Symmetry)

* Rotation, Translation you can build by little infinitesimal thing
 * Reflection you can't.

Discrete Symmetry

Continuous Symmetry

* If the thing is invariant under small transformation, then iterate it ~~and~~ for some finite transformation, that you can (there is a point to prove)

that why the higher order changes don't build up to finite quantity by successive addition.

Think it in terms of equation ... an object moving through a viscous fluid.

$$(m\ddot{x} = -c\dot{x})$$
 ; is this eqⁿ invariant under any symmetry?

$\downarrow$
viscous drag force.

Yes: $x \rightarrow x + \delta$

~~$\dot{x} \rightarrow \dot{x}$~~
 $\ddot{x} \rightarrow \ddot{x}$

where we p from momentum only
(no link that it come from Lagrangian)

$$\dot{p} = -c\dot{x} = -\frac{c \cdot p}{m}$$

$$m\ddot{x} = p \Rightarrow \dot{x} = p/m$$

Solution $p \propto e^{-\frac{c}{m}t}$ (clearly p is not conserved)
 $\hookrightarrow$ ~~but~~ ^{yet} it is translational invariant.

• What is wrong, is that this equation can never come from Principle of Least Action.

It is not possible for a fundamental law coming from a Principle of Least action.

It is the Statistical law of an object moving through a viscous fluid, constantly being bombarded by molecules such that on a statistical average, (there are many-many molecules hitting it),

The Force is $-c\dot{x}$.

$\hookrightarrow$ It is kind of effective description of not just the stone but also all of water molecules.

$\hookrightarrow$ if we take into account everything, stone, earth, water molecules, then momentum would be conserved.

~~But generally, that is an equation, not a law, it is not possible as a fundamental law coming from a principle of least action.~~

It is not the kind of fundamental equation which does come from the Pr. of L. Action. It is not correct to say Symmetry & Conservation are inextricably connected to each other.

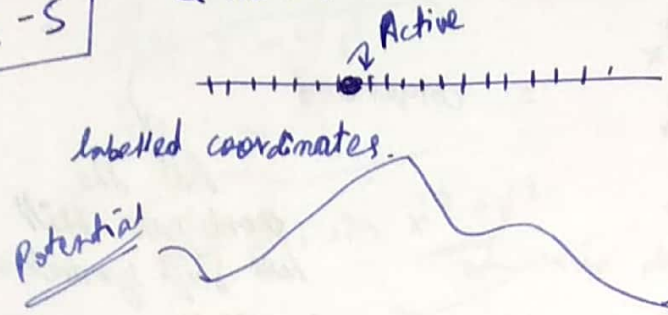
★ Symmetry & Conservation are connected through the Principle of Least Action" or through the Principle of Euler Lagrangian equation.
 • Principle of Stationary Action comes from Feynmann path integral.

★ Symmetry may or may not exist in general case

ex
lec-5

Perform the transformation $x \rightarrow x+1$

- Shazib Akhtar



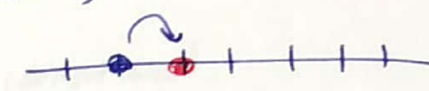
it could mean two things -
i) ~~Re-label~~ the coordinate, (Positive) $x=0 \Rightarrow x'=1$

~~(Potential energy does not change)~~ $x=1 \Rightarrow x'=2$

ii) or take the particle, where it is, and make it move 2 unit.

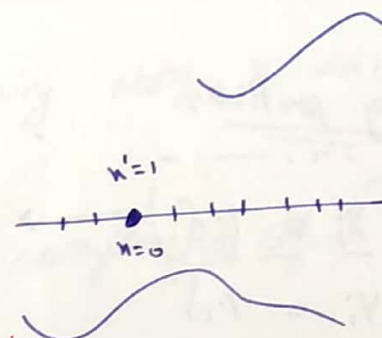
$\hookrightarrow$ here potential energy may change.

* since Active transformation moves the particle, it may change Potential Energy.



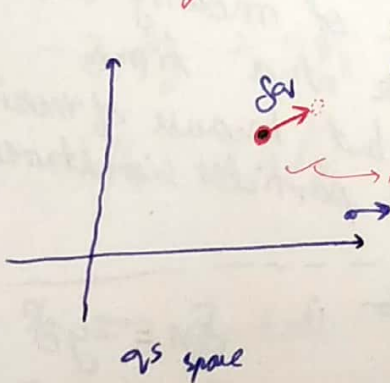
* Passive Transformation, Potential energy does not change.

(Particle is not moved physically, just the coordinates are relabelled)



when we shift thing, we shift them by amount which don't depend on time.

ex



* A configuration is a point in q^s space.

$\hookrightarrow$ move to new position, (Active transformation)

δq may depend where we are.

so: $\delta q_i = f_i(q) \delta$

* Assume $f_i(q)$ are independent of time.

$\hookrightarrow$ when we do so, the Potential energy may change

(since this is Active transformation)

$\hookrightarrow$ It may even be that K.E. transforms.

$\hookrightarrow$ or else think in term of active transformation.

ex Kinetic Energy in some coordinate might have the form.

Refer to R. Douglas Gregory.

$T(q) \dot{q}^2$

* * When small change in coordinate does not lead to a comparable change, (in other words first order change) in Lagrangian, then that's the Symmetry.

$$\# Q = \sum_i P_i f_i(q)$$

ex Translation Symmetry $\delta x = 1 \cdot \delta$
 $\delta y = 1 \cdot \delta$
 $\delta z = 1 \cdot \delta$

Momentum Conservation follows from Shift Symmetry.

Note in the above example,

to conserve P_x
 look at $\delta x = 1 \cdot \delta$
 $\delta y = 0$
 $\delta z = 0$
 $\Rightarrow P_x = \text{conserved}$
 ... & similarly for others...

P_x
 P_y
 P_z
 = conserved

All the coordinates will have shift symmetries.
 $\Rightarrow$ All the components of momentum are conserved.

We can imagine full translation invariance - no matter where you place the system.

imagine many particles

$$\delta x_i = 1 \cdot \delta$$

$$\delta y_i = 1 \cdot \delta$$

$$\delta z_i = 1 \cdot \delta$$

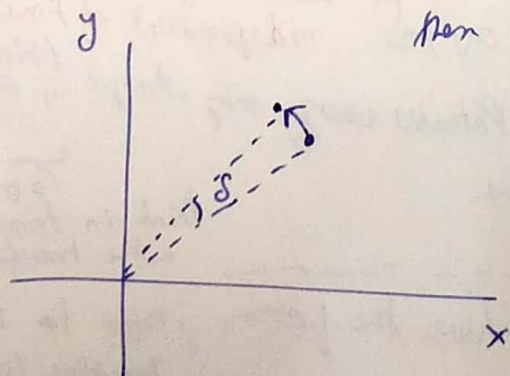
i labels particle. $\Rightarrow$ (assuming Potential depends on distance between particles)
 $\Rightarrow$ only depend on relative position.

$$\sum_i P_i = \text{conserved.}$$

here symmetry is not because of moving one particle at a time
 $\Rightarrow$ but because of moving all the particles simultaneously.

* Translation Invariance means Momentum Conservation.

ex



then coordinate transformation is

$$\delta x = -y \delta$$

$$\delta y = +x \delta$$

does this infinitesimal transformation change the distance from the origin or the square of distance from the origin.

$$r^2 = x^2 + y^2$$

$$\delta AB = (\delta A) \cdot B + A \delta B$$

$$\delta r^2 = 2x \delta x + 2y \delta y = -2xy \delta + 2xy \delta = 0$$

assuming the $\mathcal{L}$ does not change.

P935

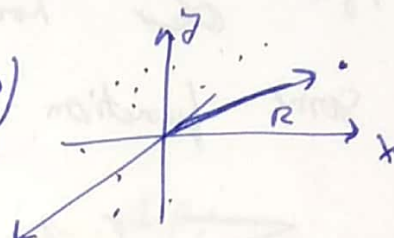
$$\delta x = -y \delta$$

$$\delta y = x \delta$$

so; if P.E. only depends on distance from origin $\mathcal{L}$ does not change

feel this in $\frac{x^2 + y^2}{2}$ KE does not change

$$Q = -y p_x + x p_y = \frac{(\vec{r} \times \vec{p})_z}{z}$$



in 3 dimensions

i.e. z - component of angular momentum is conserved.

$\Rightarrow$ we could also do the same by looking along x axis, & rotating $y-z$ plane.

if we did so; ... , we find all components of $(\vec{r} \times \vec{p})$ is conserved.

$$-z p_y + y p_z$$

$$-x p_z + z p_x$$

If Potential depends on radial distance

$\hookrightarrow$ Angular Momentum Vector is conserved.

* Rotational Invariance means Angular Momentum Conservation.

$\therefore$ in many particles

$$\sum_i z_i p_{y_i} + y_i p_{z_i}$$

General Rule

$$\delta \dot{q} = \frac{d}{dt}(\delta q) \iff \delta \left(\frac{dq}{dt} \right) = \frac{d}{dt}(\delta q)$$

if

$$T := \sum_{ij} T_{ij}(v) \dot{v}_i \dot{v}_j \quad \} \Rightarrow \text{Quadratic form in velocities}$$

$\hookrightarrow$ Kinetic Energy

(And this is possible, ... refer to R. Douglas' concept)

$$\mathcal{L} = \sum_{i,j} T_{ij}(q) \dot{q}_i \dot{q}_j - V(q)$$

When do we say terms in $\mathcal{L}$, T or V

Only if $\mathcal{L}$ has the form of some function something quadratic form in velocities.

$$\sum_{i,j} T_{ij}(q) \dot{q}_i \dot{q}_j = V(q)$$

The thing which is quadratic in velocities is called Kinetic Energy

& then minus something which depends only depends on q 's.

& the thing which does not contain velocities all together is called Potential Energy.

→ ~~then~~ only when $\mathcal{L}$ ~~separates~~ ^{separates} like the form above then only it is represented as $T - V$.

→ but there are situations where you can have mixed dependencies.

ex ~~things~~ things involving $\dot{q}$ times functions of q .

$$\therefore \mathcal{L} = \sum_{i,j} T_{ij}(q) \dot{q}_i \dot{q}_j - V(q) + \sum_i \dot{q}_i g_i(q)$$

Then you have to only think $\mathcal{L}$ as simply a function of q and $\dot{q}$.

(most profitable to try to break up it up into T or V)

→ This raises the question what we mean by energy then?

→ (will be covered later)

~~ex~~ $\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i} = - \frac{\partial V}{\partial q_i}$

(1937)

at equilibrium configuration: $L = T - V = -V$

$(T=0)$

(in situations where you can divide in terms of P.E. & so forth...)

$\therefore \frac{dp_i}{dt} = - \frac{\partial V}{\partial q_i}$

Assumption: when you expect to be static: then $\frac{dp_i}{dt} = 0$

so: $\Rightarrow \frac{\partial V}{\partial q_i} = 0 \rightarrow$ so this means that you are at stationary point of potential.

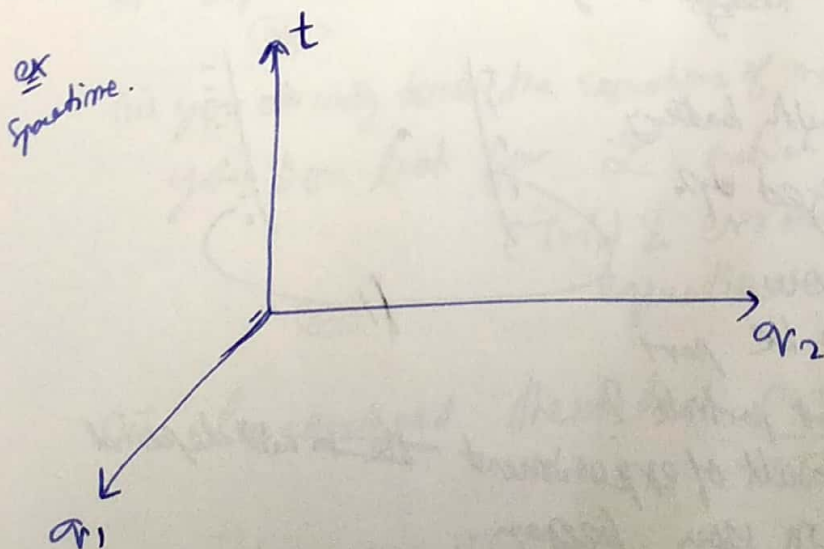
~~Assume~~ L has the form $T - V$.

Question: Is Energy conservation associated with symmetry?

$\hookrightarrow$ this will bring us to the notion of Hamiltonian.

Ans Yes. But it is not a symmetry of the type discussed earlier. (It is not due to shift of q 's.)

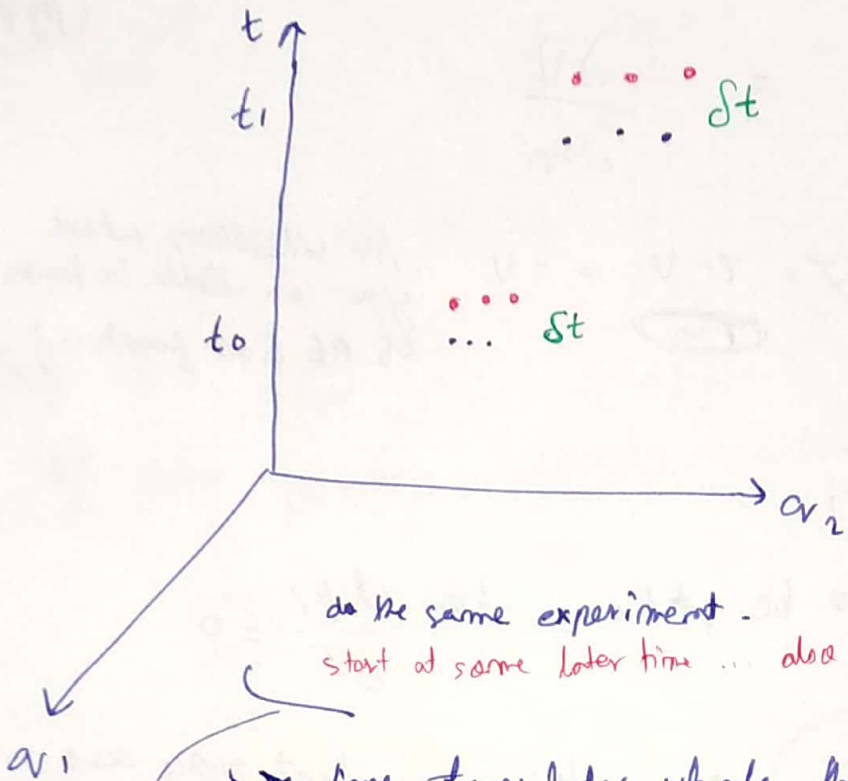
* Assume: L has the form $T - V$.



Translations & Rotational invariance means sliding or rotating the system in the q 's plane.

(w.r.t. t axis)

* Now we will discuss time translational invariance.



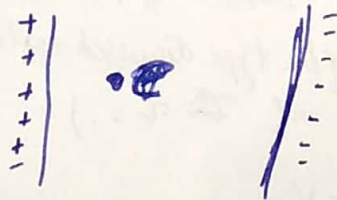
experiment started with some initial conditions at t_0 (the input)
 $\Rightarrow$ & at t_1 we have result of the experiment.

do the same experiment -
 start at same later time ... also end at same later time.

$\rightarrow$ time translates whole thing ... actually mean active transformation.

$\rightarrow$ ~~are~~ is the result of experiment same?
 is it time translational invariant?

ex not invariant
 capacitor

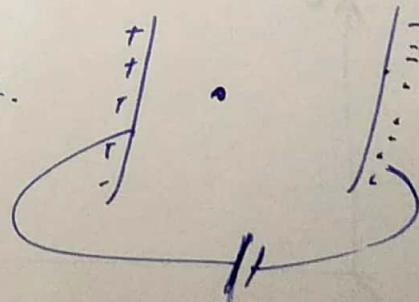


if charges on plates are constant with time $\rightarrow$ then it is invariant.

There will be a $\mathcal{L}$ for ~~things~~ charged particle e .

now,

suppose those plates are connected with battery & being charged up.



if we have complete control over capacitor, we don't need it to be part of system. ... What about particle?

$\rightarrow$ now result of experiment ~~the~~ will depend when you began.

$\rightarrow$ but if we do entire experiment including capacitor in system then we could get time translational invariant ... but let's not do that.

In order to understand what we mean by
to have Time Non-translational invariance system.

(1839)

ii: $\implies$ Some parameter of the system (it could be
electric field, varying spring constant with time, etc.)
dependent on time.

★ Time Translational Invariance $\implies$ Energy Conservation.

~~that~~ then what that does mean for when you don't have
Time Translational Invariance, ...

... there is exchange back & forth between the
thing you called the system & the rest of it, the energy.

replace δ by ϵ for simplicity: ie: $\boxed{\delta a_i = f_i(r) \epsilon}$

Motivation

where ϵ is small quantity
as δ we
thought earlier.

From where you could get Lagrangians —

$L(r, \dot{r})$

(i) Experiment
(or)

(ii) you already know the equations of motion from somewhere &
you then look for L which gives that.
(Trial & error until you match the
equations of motion)

(or)

(iii) Preconceived theoretical prejudices about the way ~~the~~
(पहिली) thing should look.

(A Theoretical Prejudices)

Assume there is a Lagrangian.

lets talk about what it means for a L to have time translational invariance.

* Harmonic Oscillator $L = \frac{1}{2} m \dot{x}^2 - \frac{k}{2} x^2$;

• if k is constant, we say system has time translational invariance

• if k varies with time, it does not.

General if $L(q, \dot{q})$ is only functions of q & $\dot{q}$ and does not have additional explicit dependence on time $\rightarrow$ then it is Time Translational Invariance.

* $L(q, \dot{q}, t)$ $\Rightarrow$ if any parameter has time dependent.

$\rightarrow$ ie; you know everything about L if you know q & $\dot{q}$ at a given instant (you dont need explicit time... then).
(actually for each instant of time)

Suppose you thought L was Energy.

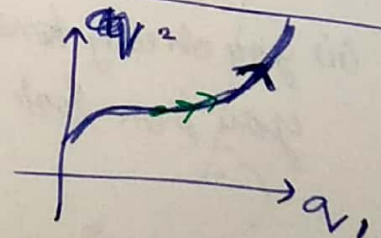
$$L(q, \dot{q})$$

$\therefore$ lets calculate: $\frac{dL}{dt}$;

... and then we would conclude something else.

The system moves through the coordinate space & as it moves through the coordinate space the P.E & K.E. change with time

Question is As you move along a trajectory is the value of L changing or not.



$$\frac{d\mathcal{L}(q, \dot{q})}{dt} = \sum_i \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i$$

pg 91

assume motion along q space satisfies equations of motions.

$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \equiv p_i$ definition of
Canonical
Momentum Conjugate to
 q_i 's.

$$\ddot{q}_i \equiv (\dot{\dot{q}}) \quad \text{Notation.}$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \dot{p}_i \quad (\text{from } \mathcal{L} - \mathcal{L} \text{ equations})$$

$$\text{so; } \frac{d\mathcal{L}}{dt} = \sum_i \dot{p}_i \dot{q}_i + p_i \ddot{q}_i$$

$$= \frac{d}{dt} \sum_i p_i \dot{q}_i = \sum \frac{d}{dt} p_i \dot{q}_i$$

$$\text{so; } \frac{d\mathcal{L}}{dt} = \sum_i \frac{d}{dt} (p_i \dot{q}_i)$$

$$\text{he found } \frac{d}{dt} \mathcal{L} = \frac{d}{dt} \left(\sum_i p_i \dot{q}_i \right)$$

In general the $\mathcal{L}$ is not conserved, ... in general the right hand side is not zero.

$$\text{now; } \frac{d}{dt} \left(\mathcal{L} - \sum_i p_i \dot{q}_i \right) = 0$$

H for Hamiltonian.

$$\mathcal{L} - \sum_i p_i \dot{q}_i \equiv -H$$

This is conserved and has
got a name.

so; we write H (which is a conserved quantity) ... as follows.

$$H = \sum p_i \dot{q}_i - L$$

ex $L = \frac{m}{2} \dot{x}^2 - V(x)$: particle moving in a P.E. $V(x)$,

$$p = m\dot{x} \quad ; \quad \text{circled } \dot{x} \rightarrow q_i$$

$\downarrow$
 p_i

$$H = (m\dot{x})(\dot{x}) - \left\{ \frac{1}{2} m \dot{x}^2 - V(x) \right\}$$

$$\Rightarrow H = \frac{1}{2} m \dot{x}^2 + V(x) = E$$

but this is a conserved quantity

$\hookrightarrow$ this is what we usually call as energy

* more generally, whether or not the L divides up in T or V , nevertheless H is conserved.

{ since we proved the above fact ~~by~~ without assuming that L has any particular form, (except for explicit time dependence) }

$$\# L(q, \dot{q}, t)$$

for explicit time dependence.

$$\frac{dL}{dt} = \left(\sum \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt} \left(L - \sum p_i \dot{q}_i \right) = \frac{\partial L}{\partial t} \quad \text{cancel } \frac{dH}{dt}$$

$\downarrow$
total derivative w.r.t.
along the trajectory
as the particle moves or
as the system moves
 $\hookrightarrow$ it contains not only $\frac{\partial}{\partial t}$
but also the fact that q 's are
changing.

$\downarrow$
This you get
by just differentiating
the parameters in
the L .

$$\Rightarrow -\frac{dH}{dt} = \frac{\partial L}{\partial t}$$

we have a theorem then.

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}$$

In other words, energy would not be conserved if L had explicit time dependence.

(13/11)

ex $L = \frac{1}{2} m \dot{x}^2 - \frac{k(t)}{2} x^2$

so: $\frac{\partial L}{\partial t} = -\frac{\dot{k}(t)}{2} x^2$

$-\frac{dH}{dt} = -\frac{\dot{k}(t)}{2} x^2$

$$\Rightarrow \boxed{\frac{dH}{dt} = \frac{1}{2} \dot{k}(t) x^2}$$



This occurs because you are looking at a particular thing & ignoring the thing which are causing the change of parameters.

↳ if you include all system, the whole thing, then Energy would be conserved, i.e; Hamiltonian will be conserved.

$L(q, \dot{q}, t)$

Euler-Lagrange Equations of ~~motion~~

~~ex~~ $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$\dot{p}_i = \frac{\partial L}{\partial q_i}$

Definition

★ If there is Symmetry

(ordinary symmetry, ... not time translational symmetry)

of the form $\delta q_i = f_i(q) \epsilon \Rightarrow Q = \sum_i p_i f_i(q), \dot{Q} = 0$

(means to say Q is conserved)

★ ~~we have~~

Generally:
$$\frac{d\mathcal{L}}{dt} = \left(\sum_i \frac{d}{dt} (p_i \dot{q}_i) \right) + \frac{\partial \mathcal{L}}{\partial t}$$

Time derivative of the $\mathcal{L}$ along the trajectory of the system

plus ... the explicit time dependence.

~~or if we transpose~~
 or if we transpose to the other side of equation we get.

$$-\frac{dH}{dt} = \frac{\partial \mathcal{L}}{\partial t} \quad \text{where ; } H := \sum_i p_i \dot{q}_i - \mathcal{L}$$

or
$$\frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t}$$

~~where $\mathcal{L}$ is the Lagrangian and H is the Hamiltonian~~

* $\mathcal{L}$ emphasises the stationary action idea.

* H principle are more abstract, but very easier.

(☺ Hamiltonian Formulations played a very central role in Quantum Mechanics)

Note Things which are made up of ordinary particles, then here Lagrangian can be written as $T - V$.

(~~it's~~ non relativistic, ... no magnetic fields ... cases)

- ~~Elastic~~ collision happens because there is repulsive force between particles.
- Inelastic collisions involve many many particles.



(collision between lots & lots of particles ...)

(~~some parts~~) particles can get excited, ... internal motion ...
 or can heat up or can radiate or something else.

ex non-standard Lagrangian (but it is a perfectly good $\mathcal{L}$) (PS 45)

$$\mathcal{L} = \overbrace{q^2 \dot{q}^2}^T - \overbrace{\frac{q^2}{2}}^V$$

$$p = q^2 \dot{q}$$

$$\dot{p} = q^2 \ddot{q} + 2q\dot{q}^2 = \overbrace{q(\dot{q}^2 - 1)}^{\text{friction}}$$

$$\Rightarrow q^2 \ddot{q} + q\dot{q}^2 = -q \quad \cdot \quad q\ddot{q} + \dot{q}^2 = -1$$

$$H = (q^2 \dot{q}) \dot{q} - \left(\frac{q^2 \dot{q}^2}{2} - \frac{q^2}{2} \right)$$

$$H = q^2 \dot{q}^2 - \frac{q^2 \dot{q}^2}{2} + \frac{q^2}{2} = \frac{q^2 \dot{q}^2}{2} + \frac{q^2}{2} = T + V$$

Note At the level of elementary particles, there is no such thing as Friction.

Friction is actually statistical effect of lots & lots of particles that you just can't see (🌀 😊)

Experiment
Guess
Symmetry } These are the places from where we get pieces of theory...

Now is it that real classical nature confirms to this abstract system $\leadsto$ Quantum Mechanics...

* Quantum Mechanics logically begins with Hamiltonian, and then you work it through $\mathcal{L}$... $\rightarrow$ In Quantum Mechanics, Hamiltonian is much much more intuitive concept.

Hamiltonian in classical mechanics is deeply abstract thing, whereas in Quantum Mechanics it is really very intuitive.

[Faint, illegible handwritten text, likely bleed-through from the reverse side of the page.]

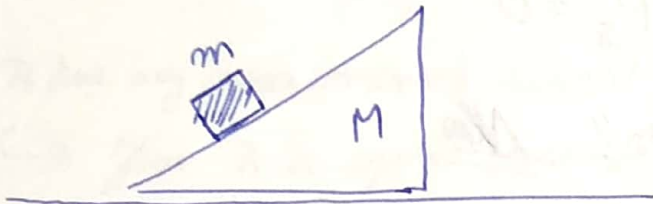
★ Power of Lagrangian for solving simple problems.

(Solving problem does not mean necessarily solving equations of motion, it actually means finding the equations of motion.)

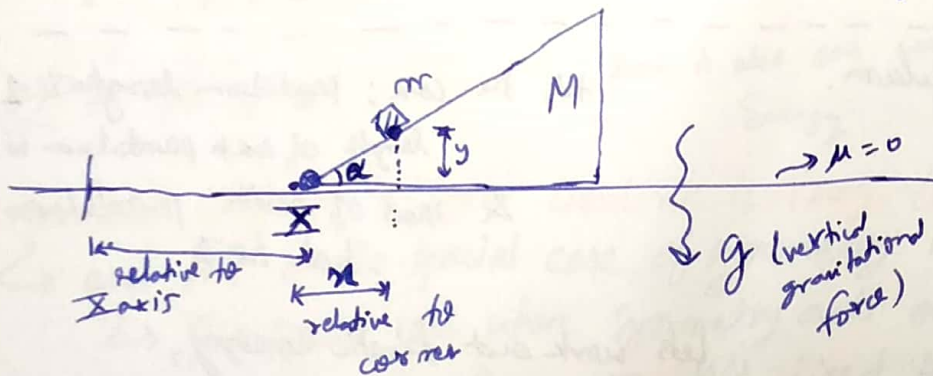
- once the eqⁿ of motions are found, you can put them onto computer & numerically solve them)

ex

find the motion of this system



Step 1: assign a set of coordinates. ~~state~~ ^{make sure they are sufficient to encode the instantaneous state of the system}



$$\frac{y}{x} = \tan \alpha$$

$$\Rightarrow y = (\tan \alpha) x$$

take $\alpha = 45^\circ$

Step 2: Write down the Lagrangian $\mathcal{L} = T - V$.

$$\underline{y = x}$$

$$\mathcal{L} = \underbrace{\frac{M}{2} \dot{X}^2 + \frac{m}{2} ((\dot{X} + \dot{x})^2 + (\dot{x})^2)}_T - \underbrace{m \cdot g \cdot x}_V$$

$$(V_m)_x = \dot{X} + \dot{x}$$

$$V = m \cdot g \cdot x$$

$$(V_m)_y = \dot{y} = \dot{x}$$

$$P_X = \frac{\partial \mathcal{L}}{\partial \dot{X}} = M \dot{X} + m(\dot{X} + \dot{x}) = (M+m) \dot{X} + m \dot{x}$$

$$= M \dot{X} + (m)(\dot{X} + \dot{x})$$

↳ momentum conjugate to x .

$$P_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m(\dot{X} + \dot{x} + \dot{x}) = m(2\dot{x} + \dot{X})$$

Is there any symmetry?

It has a translational symmetry.

little x does not change when you translate X

$$\delta X = \epsilon \quad ; \quad \delta x = 0$$

$\therefore$ now lets work out the conserved quantity.

$$Q = \frac{d}{dt} (P_X \cdot 1) = P_X$$

check// X is cyclic variable $\Rightarrow \frac{d}{dt} P_X = 0$

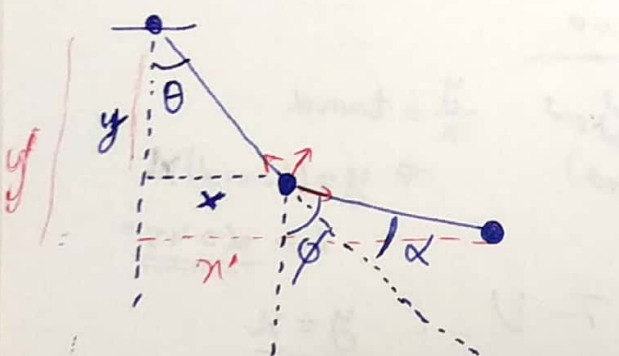
• Is there symmetry w.r.t. vertical motion? No

Step 3; Write down eqns of motion

$$\dot{P}_X = 0 \quad ; \quad \dot{P}_x = -mg$$

example Double Pendulum.

take the case; pendulum length is 1.
length of each pendulum be 1
& mass of both pendulum
bob be 1



(θ, α) or (θ, ϕ) this gives easier answers.

lets work out kinetic energy,
by starting in Cartesian
Coordinates, and then
re-write it in terms of the
angles.

$\hookrightarrow$ so it is often a good thing to
realize that your objects are moving around in
Cartesian space and then the Kinetic Energy is just
 $\frac{1}{2} m v^2$.

• Just for a moment as a little overkill step here. introduce the
 x & y of the end of the pendulum bob here.

$$\begin{aligned} x &= \sin \theta \\ y &= \cos \theta \end{aligned} \quad \left\{ \begin{array}{l} \text{actually } x \text{ \& } y \text{ are not new variables...} \\ \dot{x} = \dot{\theta} \cos \theta \\ \dot{y} = -\dot{\theta} \sin \theta \end{array} \right.$$

angle relative to vertical for ' is $\alpha + \theta$

$$x' = x + \sin(\theta + \alpha) = \sin \theta + \sin(\theta + \alpha)$$

$$y' = y + \cos(\theta + \alpha) = \cos \theta + \cos(\theta + \alpha)$$

$$\dot{x}' = \dot{\theta} \cos \theta + (\dot{\theta} + \dot{\alpha}) \cos(\theta + \alpha)$$

$$\dot{y}' = -\{\dot{\theta} \sin \theta + (\dot{\theta} + \dot{\alpha}) \sin(\theta + \alpha)\}$$

$$T = \frac{\dot{\theta}^2 + (\dot{\theta} + \dot{\alpha})^2}{2} + \dot{\theta}(\dot{\theta} + \dot{\alpha}) \cos \alpha$$

Is there any ~~conserved~~ conserved quantity? (for the ^{case} when there is no gravitation)

→ There is a cyclic coordinate: θ

So; $\dot{P}_\theta = 0 \Rightarrow P_\theta$ is conserved. (in this case angular momentum conservation)

$$V = -m g \cos \theta - m g \{\cos(\theta + \alpha) + \cos \alpha\}$$

$$\mathcal{L} = T - V$$

∴ there is also one conservation law in this case
Energy $T + V$

• Momentum Conjugate to Cyclic Coordinate is Always Conserved.

→ and this is the special case of symmetry idea.

2 → the special case where symmetry acts on only one coordinate, all others are left fixed by symmetry.

in above example, symmetry operation ^{acts} on θ , but keeps α unchanged.

→ so try to define your coordinates in clever way so ^{that} only one of them shifts under the symmetry operation (that's when you have the cyclic ~~coord~~ coordinates)

↳ But in general to solve, you don't need to be clever in choosing coordinates.

absence of gravitational field.

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 2\dot{\theta} + (\dot{\theta} + \dot{\alpha}) + (2\dot{\theta} + \dot{\alpha}) \cos \alpha$$

$$\therefore \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = 2\ddot{\theta} + (\ddot{\theta} + \ddot{\alpha}) + (2\ddot{\theta} + \ddot{\alpha}) \cos \alpha - (2\dot{\theta} + \dot{\alpha}) \dot{\alpha} \sin \alpha = \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

Hamiltonian



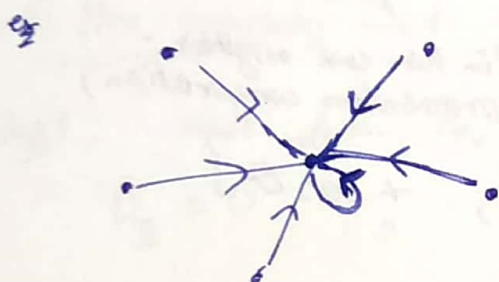
Good
loop



There is a
conservation
law depending
on which cycle
you are
in.

- Formulations of
mechanics
- i) Newton's Equations
 - ii) Principle of Stationary Action.
(Euler Lagrangian Formulation)
 - iii) Hamiltonian Formulation.
- (H. formulation is equivalent to
ii) & (iii) ... but it looks
different)
... and has enormous power.

Analogy of ~~bad~~ bad laws are forbidden in physics.
(also in Hamiltonian physics ...)



There are more arrows pointing in
than pointing out
→ that's the kind of forbidden thing.
(opposite is also ~~not allowed~~ forbidden)

In these bad laws there is
often a sense of convergence.

namely a divergence.

★ Neither a divergence or convergence
laws are allowed if Laws of Physics are intended to be
reversible.

Keep this idea in your head.

The flow associated with the Law of Physics
should not have divergence or convergence.

→ This is the ultimate meaning of reversibility.

Hamilton Equations of motion.

(PS 51)

The number of things you need to know to get a mechanical system started are twice the no. of degrees of freedom. (twice the no. of q 's)

↳ it could be q 's or $\dot{q}$'s.

↳ or it could be q 's or p 's.

In general for good mechanical system.

$$\dot{q}_i = \dot{q}_i(q, p)$$

i.e. $\dot{q}_i$ can always be thought as function of q 's & p 's.

example $H = \sum_i \dot{q}_i p_i - \mathcal{L}$

now: since $\dot{q}_i = \dot{q}_i(q, p)$; $H = H(p, q)$ so thing Hamiltonian as a function of p & q

≡ harmonic oscillator (in suitable coordinates...)

$$\mathcal{L} = \underbrace{\frac{1}{2\omega}}_{T} \dot{q}^2 - \underbrace{\frac{\omega}{2}}_{V} q^2$$

$$p = \frac{\dot{q}}{\omega} \Rightarrow \boxed{\dot{q} = \omega p}$$

↳ & now rewrite the Hamiltonian

$$H = \omega p^2 - \left\{ \frac{1}{2\omega} p^2 \omega^2 - \frac{\omega}{2} q^2 \right\} = \frac{\omega p^2}{2} + \frac{\omega q^2}{2}$$

$$H = \frac{\omega}{2} (q^2 + p^2)$$

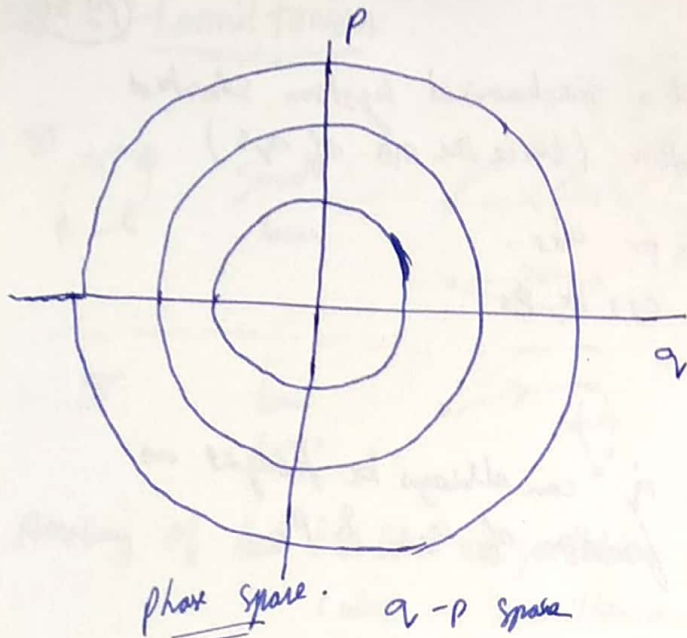
When you are doing Hamiltonian version of Classical mechanics, you get rid of $\dot{q}$'s by solving for them in terms of q 's & p 's.

↳ but some time you can't solve equations, → not because you are not proper but to solve the eq'n, but because the equations don't have solutions.

↳ You can write down $\mathcal{L}$ for which there will not be solutions

↳ They are Bad systems → Systems which cannot come from Quantum Mechanics

↳ So unless it is possible to solve for velocities in terms of the momenta & q 's, this is not considered a legitimate Lagrangian. → even though there may be sensible Euler-Lagrange Equations.



~~Harmonic oscillator~~
~~The system moves ~~in circles~~ through~~
~~trajectories in this case~~
 phase with time

The Harmonic oscillator system moves through phase space with time on circular trajectories.

~~Energy must be conserved in~~
~~because it must be conserved.~~
 because it must be conserved in this case.

Derive Hamilton's Equation

$$\delta F(q, p) = \frac{\partial F}{\partial q} \delta q + \frac{\partial F}{\partial p} \delta p$$

later put $\sum$ for n numbers of q_i 's.

not considering changing with time $\Rightarrow$ but just because slight movement in phase space.

if someone told you

$$\delta F(q, p) = A \delta q + B \delta p$$

$\hookrightarrow$ you could read off that A must be $\frac{\partial F}{\partial q}$
 & B must be $\frac{\partial F}{\partial p}$

now let's look δH

~~definition~~ from $H = \sum \dot{q}_i p_i - L$

$$\delta H = \sum_i \dot{q}_i \delta p_i + \delta \dot{q}_i p_i - \delta L$$

$$\therefore \text{but: } \delta L = \sum \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial q_i} \delta q_i$$

$$\Rightarrow \delta H = \sum_i \left(\dot{q}_i \delta p_i + \delta \dot{q}_i p_i - \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i - \frac{\partial L}{\partial q_i} \delta q_i \right)$$

$$\Rightarrow \delta H = \sum_i \left(\dot{q}_i \delta p_i - \frac{\partial L}{\partial q_i} \delta q_i \right)$$

$$\Rightarrow \delta H = \sum_i \left(\dot{q}_i \delta p_i - \dot{p}_i \delta q_i \right)$$

$$\delta H = \sum_i (\dot{q}_i \delta p_i - \dot{p}_i \delta q_i)$$

(PSS)

lets compare with $\delta H = \sum_i \frac{\partial H}{\partial q_i} \delta q_i + \frac{\partial H}{\partial p_i} \delta p_i$; $H = H(q, p)$

so; from here we could read off.

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad \text{and} \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

→ these are Hamilton equations of motion.

Observation: how parallel q & p come into equations apart from a sign change

ex apply H. eqn on Harmonic oscillator.

$$H = \frac{\omega}{2} (p^2 + q^2)$$

$$\frac{\partial H}{\partial p} = \omega p = \dot{q}$$

} this is just the statement the momentum is proportional to velocity.

$$\frac{\partial H}{\partial q} = \omega q = -\dot{p}$$

⇒ these two equations are equations of motion for Harmonic oscillator.

If there are n coordinates there are n E-L equations

→ but $2n$ Hamiltonian equations.

Hamiltonian equations are first order in time & there are twice as many of them,

but Lagrange equations are of second order & their are half as many of them.

→ you may like to think, ... that this is somewhat like definition .. that comes from Hamiltonian formulations.

→ lets see how these are familiar equations of motion (our familiar equations are of second order in time)

→ so differentiate the first equation to get some familiar equation.

gives as $\ddot{q} = (\text{acceleration}) = \omega \dot{p}$

$$\Rightarrow \ddot{q} = -\omega^2 q$$

→ This of course is standard Harmonic oscillator equation of motion for an oscillator with frequency ω .

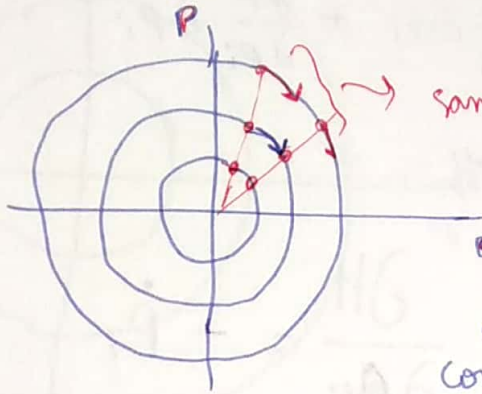
→ This is the usual $F = ma$.

→ This is the form how Euler Lagrange equations come out.

As in the form $\omega p = \dot{q}$
 $\omega q = -\dot{p}$

& you go back to circles in phase space.

(P2)

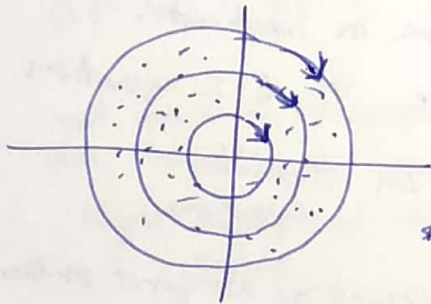


same angular velocity independent of which circle you are in.

↪ tells you that thing moves around the circle in phase space with constant ~~angular velocity~~ ω .

↪ & that ω is independent of distance from origin.

↪ all of them rotate around with frequency ω . independent of where you are.



Point

Thinking simultaneously in terms of p & q is very powerful way about motion.

* Think about motion of fluid, ... all bunches filled up with points

↪ the points on the diagram represent starting (possible starting point)

↪ the fluid just rotates around the centre here. with constant velocity.

⇒ That's the character of Hamilton equations phase

exactly what ~~how~~ they do is

they tell you, wherever you are at $(q, p) \rightarrow$ the next place of q is determined by ~~where you are at (q, p)~~ how Hamiltonian depends on p

↪ & vice versa.

↪ That's Hamiltonian mechanics.

↪ later we will see how the fictitious fluid move.

Harmonic Oscillator in usual coordinates.

$$\mathcal{L} = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}$$

units
 $[x] = L$
 $[m] = M$
 $[t] = T$
 $[\mathcal{L}] = [\text{Energy}]$

& everything is dimensionally consistent.

$$\omega = \sqrt{\frac{k}{m}}$$

if you solve equations of motion you find frequency of oscillation is $\sqrt{k/m}$.

→ now change definition of coordinates.

~~defining~~ ~~q~~ ~~such that~~ defining q such that

$$\frac{\omega}{2} q^2 = \frac{k}{2} x^2$$

... here we are simply forcing it.

$$\omega = \sqrt{\frac{k}{m}}$$

what we get $q^2 = \frac{k}{\omega} x^2 \Rightarrow q^2 = \sqrt{\frac{m}{k}} x^2$
 $q^2 = \sqrt{km} \cdot x^2$

→ this is the definition of q

$$q := (km)^{1/4} x$$

→ this ensures that $\frac{\omega}{2} q^2 = \frac{k}{2} x^2$

→ now no longer q has units of length.

note

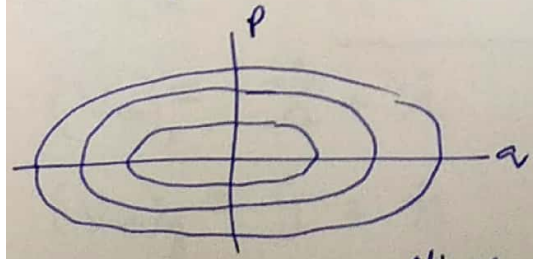
$\dot{q}^2 = \sqrt{(km)} \dot{x}^2 \Rightarrow$ so; the kinetic energy term becomes

$$\frac{m}{2} \left(\frac{\dot{q}^2}{\sqrt{km}} \right) = \frac{1}{2} \left(\sqrt{\frac{m}{k}} \right) \dot{q}^2$$

$$= \frac{1}{2} \cdot \frac{1}{\omega} \dot{q}^2$$

in ordinary p & x

$$H = \frac{p^2}{2m} + \frac{k}{2} x^2$$



we will have ellipse instead of circles.

$$\mathcal{L} = \frac{1}{2\omega} \dot{q}^2 + \frac{\omega}{2} q^2$$

$$\frac{d}{dt} \left(\frac{\dot{q}}{\omega} \right) = \omega q$$

$$\Rightarrow \frac{\ddot{q}}{\omega} = -\omega q$$

$$\Rightarrow \underline{\underline{\ddot{q} = -\omega^2 q}}$$

... of the ...
... of the ...
... of the ...

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→ Hamiltonian equations (which can be thought as a re-packaging of 2 equations) leads to Energy Conservation. → which means the conservation of H itself.

~~Q, P~~

We suppose we have some system characterised by bunch of q_s & bunch of p_s

q_i, p_i

for each q_i there is a p_i
 & for each p_i there is a q_i

In Hamiltonian, the laws governing ~~governing~~ laws of motion are packaged in H as a function of q_s & p_s

$H(q, p, t)$

this is the basic object of theory.

→ it is connected to the equations of motion through Hamilton's equations,

namely;

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \text{and} \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\therefore \text{now, } \frac{dH}{dt} = \sum_i \left[\frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial q_i} \dot{q}_i \right] = \sum_i \left[-\frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} \right] + \frac{\partial H}{\partial t}$$

total derivative of H w.r.t. of t

H may vary because either
 or varies with time or
 p varies with time
 or both varies with time.

$$\text{if } H(p, q, t) \\ \text{then: } \frac{dH}{dt} = 0 + \frac{\partial H}{\partial t}$$

H might depend on time, due to some parameter depending on time.

Important point

- If the system is translationally invariant w.r.t. time, (which is another way of saying that parameters of problem don't change with time)
- For a closed system where everything is being taken into account, we expect that the parameter don't change with time
- then in that case $\frac{\partial H}{\partial t} = 0$;

* if Hamiltonian has explicit time dependence,
 then $\frac{dH}{dt} = \frac{\partial H}{\partial t} \rightarrow$ then of course, the
 energy of the system would not be conserved.

$\hookrightarrow$ This emphasize over & over, that the meaning ^{of that} is
 usually that there is some other second system that is
 moving in such a way as to share & exchange energy back &
 forth with the system we are talking about.

* Let's focus on simple case; when H does not depend on time,
 when it is time Translational invariant.

$H(p, q) = E$ (time Hamiltonian is conserved ~~constant~~ quantity
 $\hookrightarrow$ let the constant be E)

$\hookrightarrow$ to say that

$$H(q_i, p_i)$$

$\hookrightarrow$ this is the space here

of some number of dimensions, if no. of q 's is N

$\hookrightarrow$ then the total no. of dimensions
 of space is $2N$

& write down the equation $H(q_i, p_i) = E$ (where E is just a
 number)

$\hookrightarrow$ then the solutions of the equation $H = E$ are just ^{some} contours
 or just some surfaces in the phase space. (the p - q space)

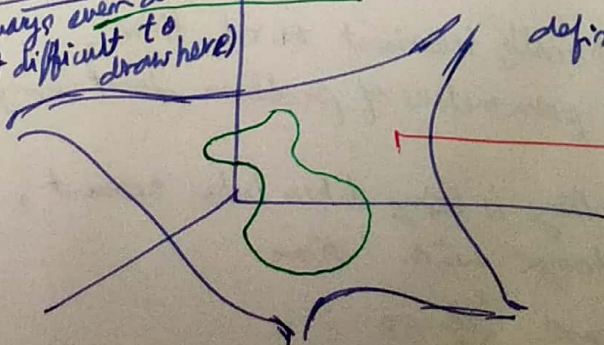
$\hookrightarrow$ the surface of constant energy, .. the core equation in an
 $2N$ dimensional space. $\rightarrow$ gives you a surface of 1 less dimension

Phase space
 always even dimensional ($2N$)
 (but difficult to draw here)

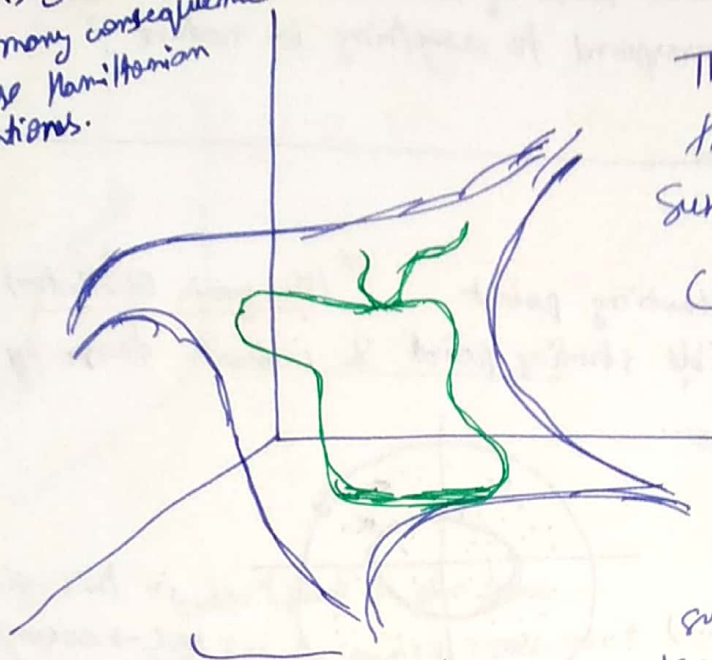
defines some surface
 in space.

"o; $2N-1$ $\begin{matrix} 2N \rightarrow n \\ \text{say} \\ n-1 \end{matrix}$

$\rightarrow$ dimension of surface is one less than
 the total no. of dimension.
 (that is what writing one equation does)



This is one of the many consequences of these Hamiltonian equations.



The ~~way~~ motion through the phase space stays on the surfaces of constant H

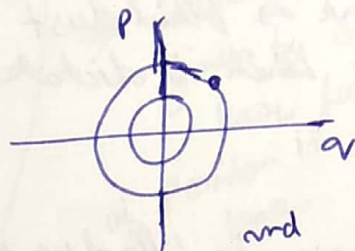
→ the trajectories in the phase space stick to the surface of constant H .

There can be many surfaces of constant H as we vary the value of H ... we make different surfaces.

→ not necessarily to form a loop ... can be open or closed or even crossing over.

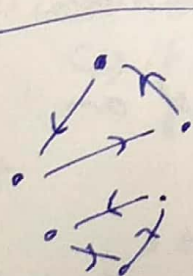
ex

$H = \omega(p^2 + q^2) \rightarrow$ defines circle ... of 1 dimension (i.e. $2-1=1$)



(we have different circles corresponding to different total energy)

and The way the system moves or behaves with time is wherever you start the system, it will stay on the same circle.



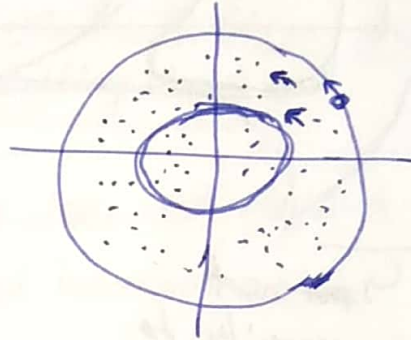
The loop conservation can be thought as an analogy for the general case being trapped in a n dimensional surface ... embedded in some higher dimensional space.

Note We can write many-many other kinds of Hamilton.
and the question is do they correspond to anything in nature?
and the answer is often yes.

• Idea of flow in phase space.

ex instead of given any particular starting point ^{or} (Harmonic Oscillator)
↳ let's imagine all possible starting points & indicate them by points on this plane

↳ ~~sprinkling~~
↳ sprinkling of points
(uniformly distributed.... points...)

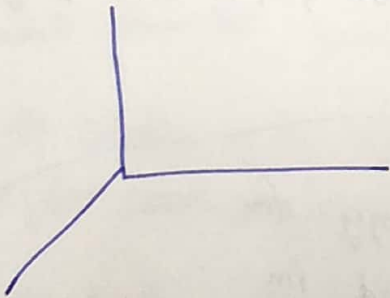


in this simple case → everything is rigidly moving around



↳ and in the back of our mind, we are thinking that this sprinkling of point can be thought of as fluid → not a real fluid → but an imaginary fluid composed of a huge huge dust. → think as fluid dust where all particles in the dust move in a way which is dictated by Hamilton equations. collection of dust.

#



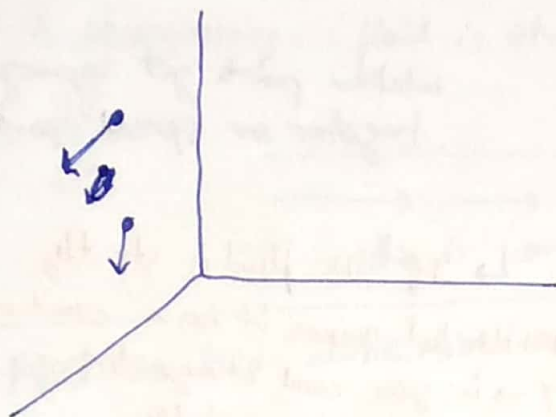
The phase space is filled up with dust
↳ the dust correspond to every possible starting point → & then just start the clock & let it go → and the dust moves through the phase space

↳ & defines a flow of an imaginary fluid.

Arbitrary phase space

(always even dimensional)

↳ we are interested in one property of this imaginary fluid, very specific property → the property called having divergence or convergence of a flow.



• What are the components of position in ~~phase~~ phase space
 $\rightarrow q_i, p_i$

• What are the components of velocity in phase space

~~$\dot{q}_i, \dot{p}_i$~~ $\dot{q}_i, \dot{p}_i$

2N

motion of points in phase space.
 $\rightarrow$ where the particle moves next only depends where they are on the phase space.

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{p}_1 \\ \dot{p}_2 \\ \vdots \end{pmatrix}$$

= This velocity vector can be thought of as a vector in the phase space.
 $\rightarrow$ showing the direction of flow.

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{p}_1 \\ \dot{p}_2 \\ \vdots \end{pmatrix}$$

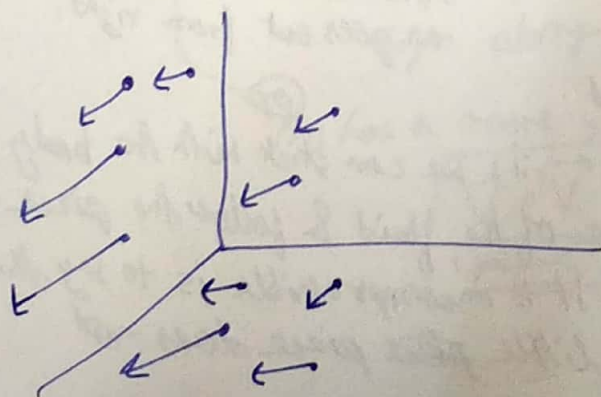
$\Rightarrow$ This velocity vector is given by entirely known functions of position in phase space.

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$\rightarrow$ This means whenever a point in the flow gets ~~sent~~ to a particular given point $\rightarrow$ it always have same velocity

$\rightarrow$ these are known functions of position in phase space

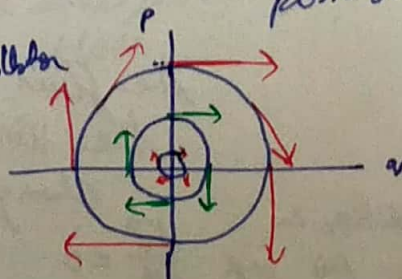
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so we can fill the phase space with a vector field.

(a vector at every position)

ex Harmonic oscillator

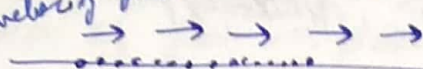


Q) whether this fluid is compressible or not?

P96

ex 1 Dimensional fluid.

velocity field (can this vary along axis)



collection of points moving along one dimensional axis.

↳ ~~the only way~~

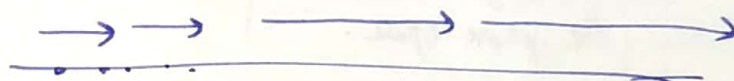
whether points get squeezed together or spread apart?

↳ If the fluid is strictly incompressible that means it has a constant density $\rightarrow$ i.e. you can't change distance between neighbouring particles.

(also can't stretch them)

can this hold for incompressible fluid

No



The only way to have a incompressible fluid in one dimension is for the velocity to be absolutely constant independent of where you are.

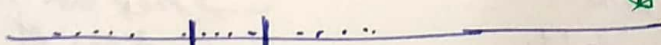
$\rightarrow$ it could depend on time

(but we are further assuming it does not depend on time ...)

↳ here they will be moving apart.

↳ the points will have to be spreading apart here.

in this small volume



no. of molecules coming in here = no. of molecules going out there.

(otherwise something have to be compressed or de-compressed)



We can think in two ways -

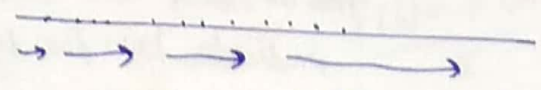
i) Stay in fluid at some fixed little element there, & require that the same amount comes in from left as goes out from right.

or

ii) we can stick with the body of the piece of the fluid & follow the piece of the fluid & then to say it is incompressible is to say that the volume of that little piece does not change.

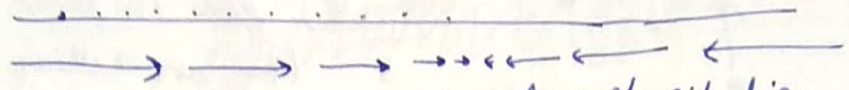
condition in this case $\frac{dV}{dM} = 0$

if $\frac{dv}{dn} > 0$



it is decompressing ... fluid is stretching out.

if $\frac{dv}{dn} < 0$

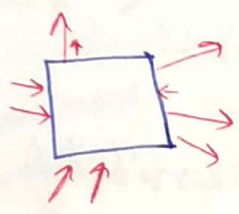


... this is compressible type of situation.

Fluid in two dimension. (incompressible)



the volume or the area of that little piece remains the same.



flow vector components

$$V_x(x, y)$$

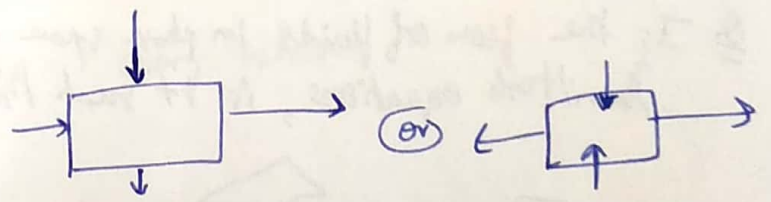
$$V_y(x, y)$$

Condition that fluid does not build up in the region.

but this can stretch, as long as from the y direction it is compressing.

$$\frac{\partial V_x}{\partial x}$$

this is some measure how the fluid is stretching in x direction



In two dimensions $\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$

$$\text{so: } \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0 \quad \left\{ \text{in three dimension} \right.$$

what we need is the combined sum of compression with sign along each direction is zero

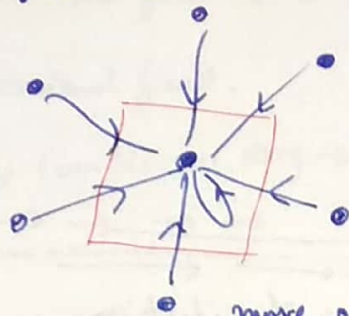
this has a name, Divergence of Vector field.

$$\nabla \cdot \vec{V} = 0$$

$\nabla \cdot \vec{V}$ is called the divergence of flow.

ex die of seven faces

at some bad law



more arrows come in than go out

→ that's nature of irreversible law.

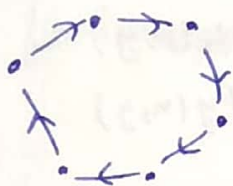
Space of configurations

the arrows are used to indicate how the system moves
 → the arrows in some sense are very similar to the idea of flow in phase space.
 (they are telling you where to go next ... this is what they are doing)

for good laws



ex



so; the law of reversibility in this discrete system almost sounds like same as the fluid is incompressible.

Q Is the flow of fluids in phase space dictated ~~by phase space~~ by Hamilton's equations, is it such that it is incompressible?



→ it is consequence of Hamilton's equations.

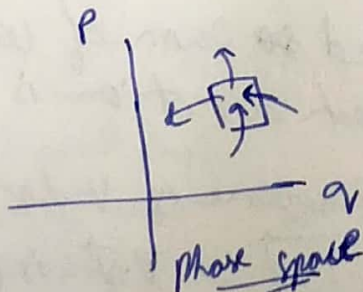
(Divergence of flow vector is zero)

Liouville's Theorem: Flow in Phase space is incompressible

by Hamilton's equations

ex $1p$ & $1q$

$$\begin{pmatrix} p \\ q \end{pmatrix}$$



$$\begin{pmatrix} v_p \\ v_q \end{pmatrix} = \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -\frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

by definition

now; calculate divergence of flow:

$$\frac{\partial v_p}{\partial p} + \frac{\partial v_q}{\partial q} = -\frac{\partial}{\partial p} \left(\frac{\partial H}{\partial q} \right) + \frac{\partial}{\partial q} \left(\frac{\partial H}{\partial p} \right) = 0$$

• partial derivatives are independent of order of differentiations for many good functions.

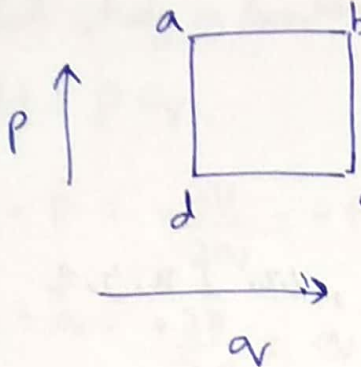
this is Liouville's Theorem.

lets prove

$$\frac{F(b) - F(a)}{\epsilon}$$

(omit $\epsilon \dots$)

P365



$$\frac{\partial F}{\partial q} = \frac{F(c) - F(d)}{\epsilon}$$

(omit $\epsilon \dots$)

$$\frac{\partial}{\partial p} \frac{\partial F}{\partial q} = (F(b) - F(a)) - (F(c) - F(d))$$

$$= (F(b) + F(d)) - (F(a) + F(c))$$

↙ tells you move vertically
↘ tells you move horizontally.

∴ now think about other order of differentiation

$$\frac{\partial}{\partial q} \left(\frac{\partial F}{\partial p} \right) = \dots = \text{get the same as above.}$$

for n^{th} dimensions; put i every where ...
any no. of direction.

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad ; \quad \frac{\partial V_{p_i}}{\partial p_i} + \frac{\partial V_{q_i}}{\partial q_i} = 0$$

$$\dot{q} = \frac{\partial H}{\partial p}$$

↳ multiple version of same thing in any no. of dimensions.

↳ In each pair of directions → in each q_i & p_i

↳ the flow along one axis is compensated by the flow along the other axis.

★★
★ That ensures incidentally that the points in the phase space cannot come together and lose memory of which point was which.
↳ They maintain their integrity ... they don't stay the same distance apart (not necessarily)
↳ but each one occupies a little volume, which is sort of sacred & other ones can't penetrate, and can't lose their individuality.

this is special kind of compressibility

→ where the terms cancel in pairs

→ ultimately $\nabla \cdot \vec{v} = 0$

→ It is stronger kind of compressibility.

Notes: Phase space are always taken as x grid spaces, like x, y, z cartesian coordinates. (for simplicity...)

● Liouville's Theorem is independent of the fact whether Hamiltonian has time independent or not.

• ex imagine a box of gas → expand box of gas (almost like universe is expanding)

A fluid expanding in an ordinary q space means that it is contracting in p space

→ so; as to retain the fact that in $p-q$ (phase) space ^{that} they are incompressible.

for the ordinary fluid, in ~~real space~~ the position of molecules in real space, they are obviously spreading out of each other.

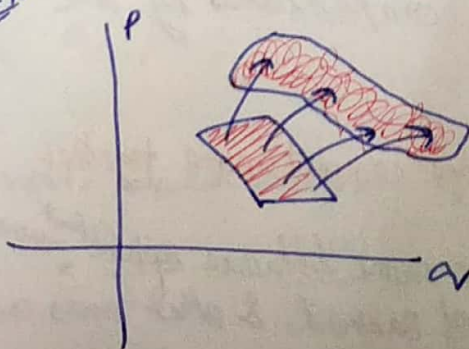
→ this suggested that in momentum space the points must be getting closer to each other.

→ when you squeeze the gas, exactly the opposite happens.

66 Phase - Space volume is conserved"

→ shape is not ~~changing or may not~~

Note //



- Take a patch region of phase space
- Every point in this phase space goes somewhere after a certain period of time ↑.
- The patch will move.
- The patch moves in a way which preserves volume in phase space.

if you change coordinates ... then also the theorem will be true.

(P367)

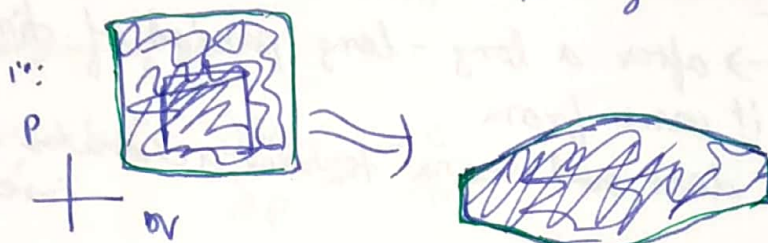
ex: Let's study a Hamiltonian, (it's perfectly a good Hamiltonian for some system)

$$H = p^2 + q^2$$

$$V_p = \dot{p} = -\frac{\partial H}{\partial q} = -p$$

$$V_q = \dot{q} = +\frac{\partial H}{\partial p} = q$$

contracting in one direction
 $\dot{p} = -p \Rightarrow p \propto e^{-t}$ (shrinking exponential)
 $\dot{q} = q \Rightarrow q \propto e^t$ (expanding exponential)
 expanding in the other direction.

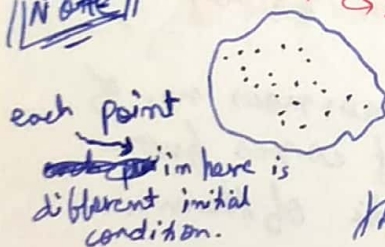


(Volume remains the same ... then also)

$\nabla \cdot \vec{V} = -1 + 1 = 0$; Divergence of the imaginary fluid is 0.

→ the two kinds of motion cancel each other.

Note ★ Blob of phase space you draw & you are following it.



the volume can't go like this

can't the volume in phase space go to infinite which is like in breakable

→ Can a closed volume in phase space go to broken pieces of volume. ... Answer is NO

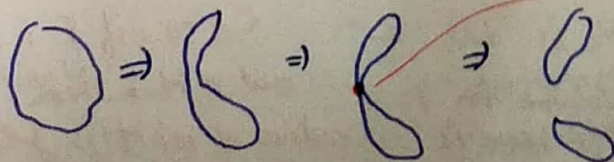
→ The other part of the theorem to prove.

that

→ The topology of the region of phase space does not change.

(can be proved)

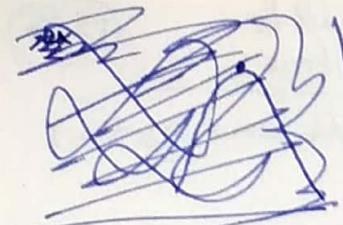
idea



initial

final

→ this is a common point & where it goes is definite.
 → so this does not allow them to split.



ex System which does not conserve the volume in phase system.

(so, I will have to take a system which does not have a Hamiltonian, ... which does not have a Lagrangian ... & which therefore violates ~~the~~ the rule)

Damped Harmonic Oscillator

: runs down with time
position & velocity goes to zero with time.

→ after a long-long period of time you can't tell where it came from

→ it does ~~not~~ seem to have a character of ~~the~~ irreversibility.

lets work out the equations...

$$m\ddot{x} = -kx - c\dot{x}$$

$\underbrace{m\ddot{x}}$ as the restoring force of harmonic oscillator
 $\underbrace{-kx}$ viscous drag force.

} equation of motion of Damped Harmonic Oscillator.

let $\dot{x} = p$

even though, there is no Lagrangian to derive it from ~~the rest~~

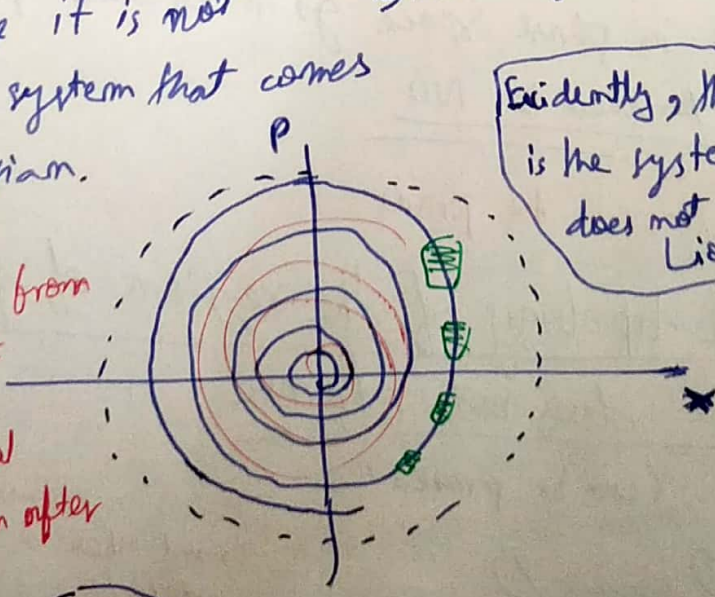
↳ in this case it is not a Legitimate system that comes from Lagrangian.

• ~~the~~ Viscosity cannot be derived from Hamiltonian, → where does it really come from?

→ it comes from the enormous no. of particles colliding, & it comes from ~~statistic of them~~ statistic of them..

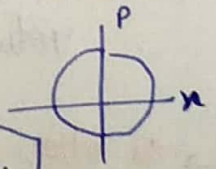
it does matter from where you start

→ you spiral into the origin after long time.



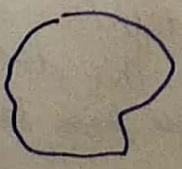
without damping term

Evidently, this is the system which does not have Liouville Theorem.



• the viscous drag force sucks energy out of it → (remember energy $\propto v^2 \propto p^2$)

→ so, if E is sucked out of it, then radius of orbit is getting smaller with time.



after long time

• Volume in phase space is not conserved.

so; ~~the~~ $\dot{x} = P/m = V_x$
 $(-kn - \frac{cP}{m}) = (m\ddot{x}) = \dot{P} = V_P$

$$\Rightarrow \dot{x} = V_x = \frac{P}{m}$$

$$\dot{P} = V_P = -kn - \frac{cP}{m}$$

now: calculate divergence of flow

$$\frac{\partial V_x}{\partial x} = 0; \text{ but: } \frac{\partial V_P}{\partial P} = -\frac{c}{m}$$

$$\text{so; } \frac{\partial V_x}{\partial x} + \frac{\partial V_P}{\partial P} = -\frac{c}{m} < 0$$

(so, this means flow vectors are kind of pointing inwards)
 $\hookrightarrow$ they are converging.

$\hookrightarrow$ This was an example of the system $\rightarrow$ where you really do lose information.

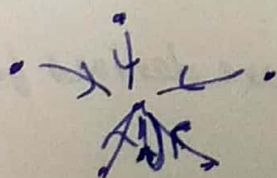
$\hookrightarrow$ you started going & after a while
 $\hookrightarrow$ the phase points lose their integrity.

$\hookrightarrow$ and they all get ~~squeezed~~
 squeezed ~~squeezed~~ to the same small volume

(the state of being whole & undivided...
 ... state of their unique identity)

$\hookrightarrow$ so it gets harder & harder to distinguish the points of phase space.

This example is playing the same role as



$\hookrightarrow$ this is sort of system that cannot be a consequence of fundamental laws of motion.

This actually does not violate Liouville's theorem
 $\hookrightarrow$ it actually fails to satisfy the conditions in the statement of the theorem.

NOTE It is possible to have a divergence of zero for a system without Hamiltonian. . . . Yes

note: The Hamiltonian construction is not necessary & sufficient for Liouville's condition.

↳ it is sufficient for it ... but not necessary.

• not any mechanical system ~~works out~~ (usually not) that works out the above way.

↳ but yes a fluid does.

ex ~~a fluid~~ a real genuine incompressible fluid is 3-dimensional

↳ it can't come from a Hamiltonian because the phase space of a Hamiltonian's system is always even dimensional.

↳ so it just can't.

↳ but ~~if~~ still, if the fluid was made up of lot of billiard balls that were stuck together & incompressible
↳ each billiard ball would have an integrity & ~~it~~ could not get lost and so would have character of being not losing information.

Note Damped Harmonic Oscillator is described by Newton's equation, but it is not described by a force which is derivative of Potential Energy.

• It is incorrect that Lagrangian Equation or Hamiltonian equations are equivalent to Newton's equation.

↳ they are same as ~~the~~ $F = ma$ if the forces are derived from potential.

Incompressibility

does not imply

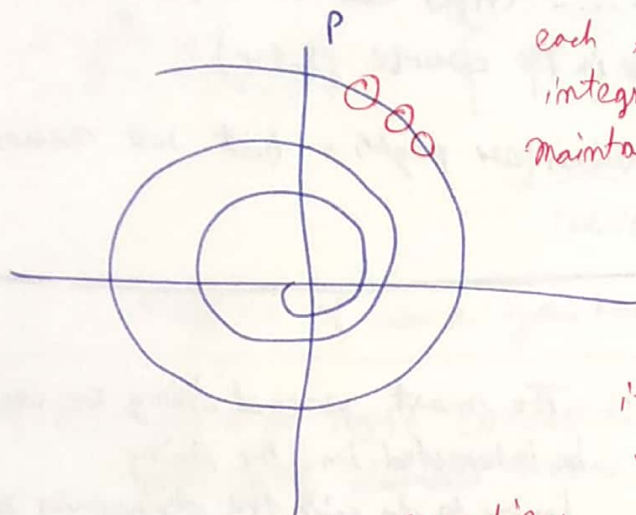
Hamilton's Equations

Hamilton's equations

imply

Incompressibility.

ex Harmonic Oscillator Damped.



each little cell of phase space is not maintaining its integrity in the sense that it is not maintaining the same volume in phase space

↳ if you had ∞ (infinite) precision and you know exactly where that point was, & exactly what velocity it had $\rightarrow$ you could ~~reconstruct~~

reconstruct the past from the

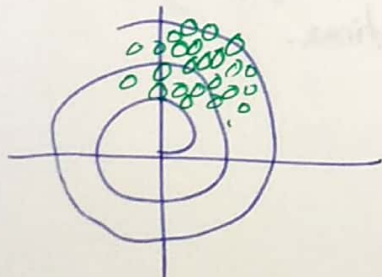
equations ... but in practice all of the phase

points are getting ~~squeezed~~ ^{squeezed} together and losing

their individuality $\rightarrow$ you need a better & better microscope to disentangle what's happening near the origin.

In Quantum mechanics we have smallest unit of phase space

↳ that's what quantised.



note • The flow is along the surface...

ex



↳ ~~maintaining its volume~~ maintaining its volume...

(They are not so much related)

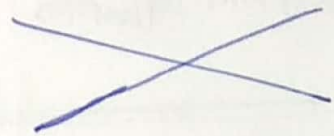
Notice that in this case the shape of the region is also conserved or say maintained.

↳ here the small volume you took includes several different energies, and you follow it $\rightarrow$ it maintains its volume.

- Liouville's Theorem preserves the topology of the phase space, but it does not preserve angle.

ex $\Pi = p q$

take $\times \Rightarrow$ evolve it



- Legendre Transformation is a useful trick (right now not important)
 (when it will be needed, we will develop in the course later)

• actually we used it over & over in previous few pages, but we never actually called it that.

Poisson's Bracket.

suppose you have a function $F(q, p)$

How does $F(q, p)$ change with time?

- why it changes with time?

it changes because p & q changes with time as you follow the trajectory through phase space.

The most general thing we could be interested in the thing having to do with the dynamics of the system. $\rightarrow$ is to know how a function changes with time.

$\hookrightarrow$ since F is a function of p & q ,

$\hookrightarrow$ so figuring out how F changes with time is figuring out how p & q changes with time.

$$\begin{aligned} \frac{dF(q, p)}{dt} &= \sum_i \frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i \\ &= \sum_i \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \end{aligned}$$

Definition Poisson bracket

Given any two functions: F and G

$$\{F, G\} = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right)$$

so: a new way of writing Hamilton equation & everything about the system. p. 73

$\Rightarrow$ so we have a new formulation of mechanics,
in which the time derivative of anything
is given as

$$\dot{F}(q, p) = \{F, H\}$$

where: H is the
Hamiltonian.

Poisson bracket of F with the Hamiltonian.

~~•~~ • of course you have to know what Hamiltonian is.

★ (i) This is the most condensed version of Classical Mechanics what
~~prof.~~ prof. Leonard knows later.

$\hookrightarrow$ he can also make it ~~so~~ more condensed.

$$(\dot{F}(q, p) - \{F, H\})^2 = 0$$

$\hookrightarrow$ call this thing here as A

and say $A=0$... that's Suskind equation ...
Misha ...

JOKE

... of the ...

... of the ...

... of the ...

$$F(0,0) = \frac{1}{2}$$

... of the ...

... of the ...

... of the ...

... of the ...

... of the ...

let $F(p, q)$ be any arbitrary

function of p & q $\Rightarrow$ when we say arbitrary in this context, we mean smooth enough & differentiable enough

$$\dot{F}(p, q)$$

$$= \sum_i \frac{\partial F}{\partial p_i} \dot{p}_i + \frac{\partial F}{\partial q_i} \dot{q}_i$$

$$= \sum_i \frac{\partial F}{\partial p_i} \left(-\frac{\partial H}{\partial q_i} \right) + \frac{\partial F}{\partial q_i} \left(\frac{\partial H}{\partial p_i} \right)$$

$$= \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

$$\dot{F}(p, q) = \sum_i \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} = \{F, H\}$$

$\Rightarrow$ This kind of expression comes

over and over in classical mechanics

$\Rightarrow$ not only in the context of time derivative but it is in a context of all kind of situations where you have two functions of p s & q s

so; $\dot{F}(p, q) = \{F, H\}$

~~now let imagine time to be~~ is being

now let imagine time being broken up in ϵ little intervals.
and then

$$\frac{\delta F}{\epsilon} = \dot{F}(p, q) = \{F, H\}$$

Classical mechanics can be beautifully formulated in terms of Poisson brackets

* There is an Algebra for Poisson brackets.

Rules of Poisson Bracket (they follow from definition)

1) $\{A, B\} = -\{B, A\}$

2) $\{A+B, C\} = \{A, C\} + \{B, C\}$

3) $\{\lambda A, B\} = \lambda \{A, B\}$

4) $\{AB, C\} = \{A, C\}B + A\{B, C\}$

★ Special Cases

$$1) \{q_i, q_j\} = 0 = \{p_i, p_j\} \quad ; \forall i, j$$

$$2) \{q_i, p_j\} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} = \delta_{ij}$$

$$3) \dot{F} = \{F, H\}$$

examples of motion for harmonic oscillators

$$H = \frac{p^2}{2m} + \frac{k}{2} q^2$$

$$H(p, q)$$

let oscillator has mass m

$$\text{we know : } \dot{F} = \{F, H\}$$

$$\dot{q} = \{q, H\}$$

$$= \{q, \frac{p^2}{2m} + \frac{k}{2} q^2\}$$

$$= \{q, \frac{p^2}{2m}\} + \{q, \frac{k}{2} q^2\}$$

Poissonating

Poissonating q with Hamiltonian, (means {q, H})

requires only that we poissonate

q with $\frac{p^2}{2m}$ (means {q, p²/2m}), because

q poissonates with q² to zero.

but we will now proceed as - ~~we~~ don't make any reference to Lagrangian

not any reference to Hamilton's equation

(although new eqⁿ went into the rule while

using $\dot{F} = \{F, H\}$)

→ now we don't care where it came from

→ in fact we don't even need to know that $\dot{F} = \{F, H\}$ certainly contains Hamilton's equations.

→ let's forget it & just use rules.

$$\text{so } \dot{q} = \frac{1}{2m} \{q, p^2\} = \frac{1}{2m} \{q, p \cdot p\}$$

$$= \frac{1}{2m} \cdot (2 \cdot p \{q, p\})$$

$$= \frac{p}{m} \{q, p\} = \frac{p}{m} \cdot 1$$

=> so, first equation of motion gives

$$\dot{q} = \frac{p}{m}$$

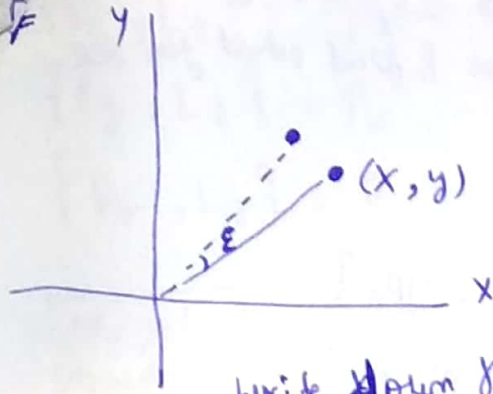
$$\text{also } \dot{p} = \{p, H\} = \{p, \frac{k}{2} q^2\} = \frac{k}{2} \{p, q^2\} = \frac{k}{2} \cdot 2 \cdot q \{p, q\}$$

$$= k \cdot q \{p, q\} = k \cdot q (-1) = -kq$$

Angular Momentum

→ when you start thinking about Angular Momentum

→ you start seeing Power of Poisson Brackets.



$$\begin{aligned} \delta x &= -y\epsilon \\ \delta y &= +x\epsilon \\ \delta z &= 0 \end{aligned} \left\{ \begin{array}{l} \text{checking} \\ \text{rotational} \\ \text{symmetry} \\ \text{along } z \text{ axis.} \\ \text{(rotate } x-y \\ \text{plane)} \end{array} \right.$$

Q is conserved if there is symmetry
 $\therefore$ the type of symmetry being $\delta \varphi_i = f_i(\varphi) \epsilon$ which does not change the Lagrangian.

write down the conserved quantity. $\rightarrow$ and the conserved quantity is called Angular Momentum & labelled with L .

$\hookrightarrow$ If there is symmetry there is a conserved quantity

$$Q = \sum_i p_i f_i(\varphi)$$

or also

$$\sum_i p_i \delta \varphi_i \text{ is}$$

also conserved (just multiplied by a small constant factor ϵ).

so, what is L

$$L = -p_x y + p_y x$$

$$L = x p_y - y p_x$$

$\rightarrow$ this is actually angular momentum of particle moving in two dimension. (z-component)

$$L_z = x p_y - y p_x$$

& similarly

$$L_x = y p_z - z p_y$$

$$L_y = z p_x - x p_z$$

$$\vec{L} = \vec{r} \times \vec{p}$$

their conservation correspond to or ~~rotation~~ is a consequence of symmetry due to rotation about three axis.

• if Q is conserved

$\hookrightarrow$ then so is ϵQ

$\hookrightarrow$ what was the symmetry?

$\Rightarrow$ An active ~~trans~~ transformation which does not change the Lagrangian.

$\hookrightarrow$ for finding angular momentum in any general direction

$\hookrightarrow$ dot it with that direction (the above expression... you know how) $\odot$

$\rightarrow$ and that gives you the angular momentum

which is the angular momentum conserved by the virtue of rotational invariance about that axis.

Think the below as ~~exp~~ exploration try to explore & evaluate some values & find what you see. (P2)

$$\begin{aligned} \{x, L_z\} &= \{x, x p_y - y p_x\} \\ &= \{x, x p_y\} - \{x, y p_x\} = -\{x, y p_x\} \\ &= -y \{x, p_x\} \\ &= -y \end{aligned}$$

so; $\{x, L_z\} = -y$

$$\begin{aligned} \{y, L_z\} &= \{y, x p_y - y p_x\} = \{y, x p_y\} - \{y, y p_x\} \\ &= x \{y, p_y\} = x \end{aligned}$$

$$\{z, L_z\} = 0$$

so; $\{x, L_z\} = -y$
 $\{y, L_z\} = x$
 $\{z, L_z\} = 0$

remember ; $\left. \begin{aligned} \delta x &= -y \\ \delta y &= x \\ \delta z &= 0 \end{aligned} \right\} \otimes \epsilon$

what we notice, that taking the Poisson bracket ~~of~~ with respect to L_z gives you the small change in the variable when you rotate a little bit.

$$\begin{aligned} \delta p_x &= -p_y \\ \delta p_y &= p_x \\ \delta p_z &= 0 \end{aligned} \left. \right\} \otimes \epsilon$$

what are the changes in p s

when you rotate the coordinates.

this is justified.

they all do the same thing.

exactly the same structure.

$\vec{p}$ is a vector & the p_x, p_y, p_z are just the components.

we same. $\left. \begin{aligned} \delta x &= -y \\ \delta y &= x \\ \delta z &= 0 \end{aligned} \right\} \otimes \epsilon$

with suitable replacement.

2. the above is just the way of talking about the components of a vector change when you rotate axis.

it does not matter whether it is position vector or the velocity vector or the force vector

$$\begin{aligned}\{P_x, L_z\} &= -P_y \\ \{P_y, L_z\} &= P_x \\ \{P_z, L_z\} &= 0\end{aligned}$$

we get the same structure as previous...

we found,

Poisson bracket of something with angular momentum is just a small change in that something when you make a rotation.

(PS 24)

→ this is actually much more general.

$$\text{look at: } \dot{F} = \{F, H\} = \frac{\delta F}{\delta t}$$

taking the poisson bracket of something with H gives you small change in that something when you change a time little bit

remember what hamiltonian was,

→ It was the conserved quantity that was conserved by the virtue of Time translational invariance

(in other words ... shifting time little bit)

"Taking the Poisson Bracket with Hamiltonian gives you something proportional to the change in that something when you shift the time little bit"

let's do translation Momentum is conserved by virtue of translational invariance.
small translation; means $\delta q = \epsilon$, here $f=1$

so; the conserved quantity is $P \cdot 1$

lets see $\{F(q), P\}$

any function of q

the conserved quantity... (in this case)

$$\delta F = \epsilon \cdot \frac{dF}{dq}$$

Question is whether $\{F(q), p\} = \frac{dF}{dq}$

(138)

★ let's try for $F(q) = q^n$

$$\text{so: } \{q^n, p\} = \{ \underbrace{q(q \dots q)}_{n \text{ times}}, p \} = q \{q^{n-1}, p\} + q^{n-1} \{q, p\} \\ = q \{q^{n-1}, p\} + q^{n-1}$$

now: use Induction ... assume for $n-1$... & see whether you get n .

Induction hypothesis: $(n-1) q^{n-1} + q^{n-1} = n \cdot q^{n-1}$

↳ this is actually $\frac{d q^n}{dq}$

∴ now, check for $n=1 \rightarrow \{q, p\} = 1 = \frac{dq}{dq} = 1$

so, we found for the function of the form $F = q^n$

$$\{F, p\} = \frac{dF}{dq}$$

Induction hypothesis assumes: $\{q^{n-1}, p\} = (n-1)q^{n-2}$

$$q \{q^{n-1}, p\} = (n-1)q^{n-1}$$

$$\Rightarrow \{q^n, p\} - q^{n-1} = (n-1)q^{n-1}$$

$$\Rightarrow \{q^n, p\} = (n-1)q^{n-1} + q^{n-1} = nq^{n-1}$$

what about $F = \sum_n a_n q^n$

↳ it also follows for an arbitrary F of the form $F = \sum_n a_n q^n$

↳ it says that when you poissonate something with momentum it gives you the small change in that something when you translate q ... (note that p is conserved due to translational invariance in q)

⇒ till now, we have at least three ~~laws of conservation laws~~ ^{examples of symmetries} being connected to Poisson brackets.

↳ "so: symmetry operation is defined or is constructed by taking the poisson bracket with the quantity which is expected to be conserved. by virtue of symmetry"

↳ In other words, we take that thing which is conserved & take its poisson bracket $\rightarrow$ & that gives us small change.

↳ In other words it gives us what the transformation is.

so if I didn't know what the transformation (PS81) was associated with the Angular momentum conservation
 $\rightarrow$ all I have to do is to take the Poisson bracket with angular momentum of all the things in the theory & that will tell me ~~the~~ what the small change are.

note 11

- The generator of translation is momentum.
- The generator of time translation is the Hamiltonian.

$\rightarrow$ And to compute how something changes under a small shift or whatever you're doing, you take the Poisson Bracket.

~~the point we come to Hamiltonian Relating Conservation~~
Relating Conservation & Symmetry.

lets give name to various generators, i.e. to L for rotation,
 P for translation, H for time translation.

$\rightarrow$ lets call them generators and call them G .

G is functions of p & q s. $G(p, q)$

~~$\rightarrow$ i.e. $G(p, q)$ generates~~

$\rightarrow$ i.e. G generates same symmetry.
 (by using Poisson brackets)

★ Lets start with G being conserved

$\rightarrow$ it says $\dot{G} = 0$ $\therefore$ but $\dot{G} = \{G, H\}$

$\rightarrow$ so if G is conserved it says $\{G, H\} = 0$

$\rightarrow$ this says G is conserved.

lets turn it around $\{H, G\} = 0$

$\rightarrow$ this equation says that when you do small transformation associated with G
 $\rightarrow H$ does not change.

(H does not change when you do transformation)

• H does not change when system evolves by transformation associated with G .

Every ~~transformation~~ that you could write
 ★ For every G that you write down, we can define an infinitesimal transformation.
 (that's all it says)

$\rightarrow$ if that G is conserved it says symmetry.

$$\{G, H\} = 0$$

$\rightarrow$ this equation says G does not change as the system evolves.

• if G was L_z , it says that H does not change when you rotate the system.

• if G was p , it says that H does not change when you translate the system.

→ it says there is symmetry.

~~"H does not change under an operation which is generated by G"~~
"H does not change under an operation which is generated by G"

★ → This emphasises once more the connection between conservation laws ($\{G, H\} = 0$) and symmetries ($\{H, G\} = 0$) in Poisson Bracket language.

→ Quite beautiful packaging of that connection.

• For every G that you write down, we can define an infinitesimal transformation.

→ and if that G is conserved, it says $\{G, H\} = 0$

but if we then write back the other way $\{H, G\} = 0$ it says H

does not change when we do G transformation.
(H does not change when system evolves by transformation associated with G)

so this relates conservation to symmetry (by some equation)

evidently here G is conserved like H .
(we know that for H)

$\{H, G\} = 0 \Rightarrow$ this says that H does not change when system evolves by transformation associated with G .
(indeed the infinitesimal transformation)

lets ask now what is the poisson bracket of one component of angular momentum with other.

$$\{L_x, L_z\} = \{(y p_z - z p_y), (x p_y - y p_x)\}$$

$$= p_z x - z p_x = -L_y$$

What would I have guessed for $\{L_x, L_z\}$

$\hookrightarrow -L_y$ $\rightarrow$ but for entirely different reason.

★ L is a vector, whenever you poissonate anything, ~~component~~ any component of vector (x, y, z) with L_z , you get the minus of y component.

Some rule $\{L_y, L_z\} = +L_x$

(we can find all other ones by cycling)

lets adjust the sign,

$$\{L_z, L_x\} = L_y$$

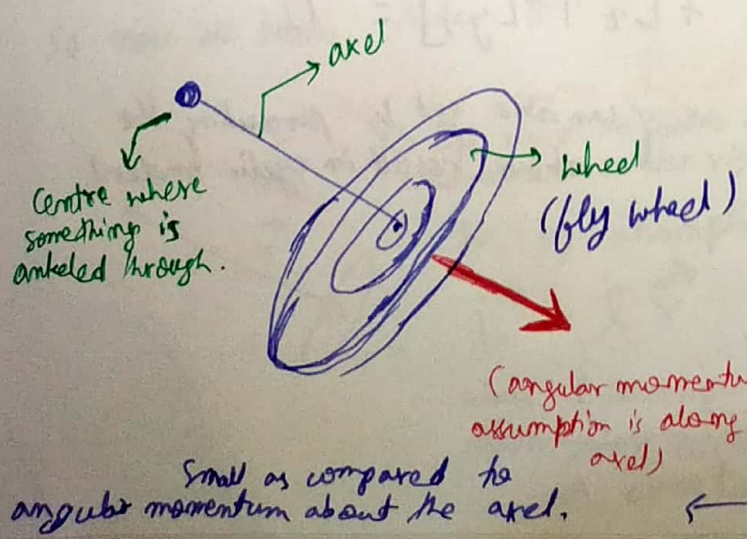
$$\{L_x, L_y\} = L_z$$

$$\{L_y, L_z\} = L_x$$

we are changing the order.

Basic poisson brackets between components of Angular momentum

ex Gyroscope



thing about Gyroscope

- The flywheel is rotating so fast, that the only angular momentum that you have to worry about is the angular momentum about the axle.

$\hookrightarrow$ (the axle may be rotating little bit... and there may be some other component ... but they are assumed to be very small)

now, what is the Hamiltonian of it

→ well it does have some energy by virtue of its rotation around that axis → & it's a large energy

→ it is proportional to the square of the angular momentum about that axis → (in fact it is proportional to the square of the total angular momentum)

• The magnitude of the angular momentum does not change much.
→ (assume that...)

→ We have just change in directions.

I := moment of inertia.

• What is the energy?

or first for free space.

(no gravitational field)
→ no ~~torque~~ torque on this gyroscope → it just will continue to point in same direction → let's prove it.

$$\frac{1}{2} \frac{(L_x^2 + L_y^2 + L_z^2)}{I}$$

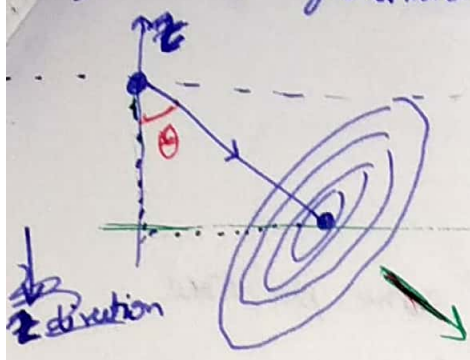
energy by just virtue of the fact that it's rotating.

$$\begin{aligned} \dot{L}_x &= \frac{1}{2I} \{ L_x, L_x^2 + L_y^2 + L_z^2 \} \\ &= \frac{1}{2I} \{ \cancel{L_x, L_x^2} + \{ L_x, L_y^2 \} + \{ L_x, L_z^2 \} \} \\ &= \frac{1}{2I} \{ 2L_y \{ L_x, L_y \} + 2L_z \{ L_x, L_z \} \} \\ &= \frac{1}{I} [L_y \{ L_x, L_y \} + L_z \{ L_x, L_z \}] \\ &= \frac{1}{I} [L_y L_z + L_z (-L_y)] = 0 \end{aligned}$$

$$\text{Similarly } \dot{L}_y = 0 ; \dot{L}_z = 0$$

(can also get by permuting the above result in cyclic order)

Let take gravitational force into account.

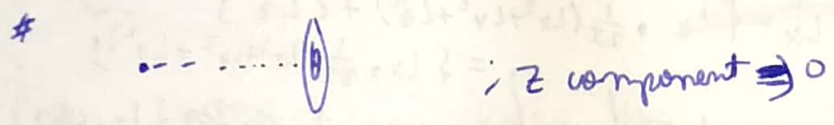


$z \propto L_z$

take this to be reference for zero potential.

Mg (contributes gravitational potential energy Mg times height)

What is the height ; height $\propto$ (z component of angular momentum)

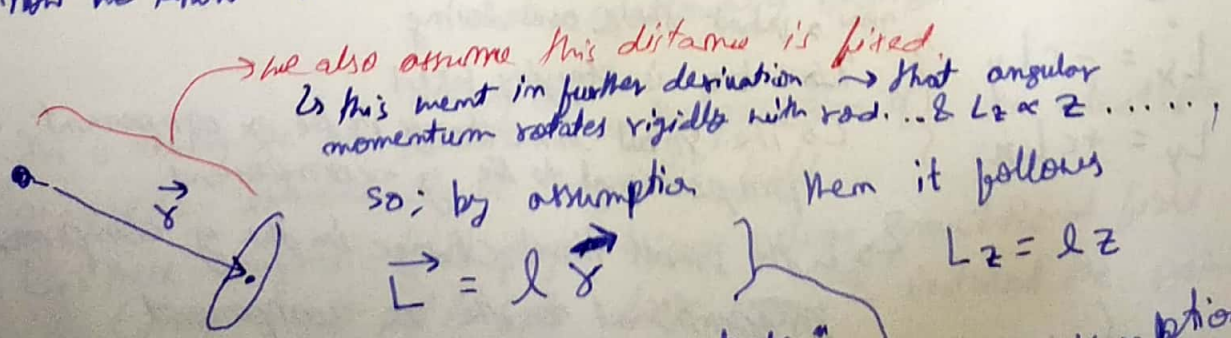


so; Potential energy is proportional to ~~minus~~ the z component of angular momentum.

$\hookrightarrow$ minus because the force is downward

$V = + c L_z$, where c is some constant.

so now we know there is



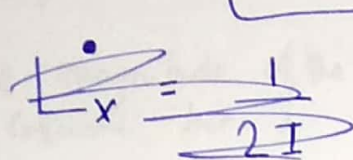
$\therefore$ where l is ~~constant~~ is of angular momentum a constant

$\rightarrow$ true by assumption that angular momentum along z axis is very large.

- All of the components of angular momentum are proportional to all the components of position vector here.

Potential energy = $+mgz$ ~~or~~ $+L_z$

so; $V = +cL_z$ where c is some positive constant.



→ This is the Gravitational Potential Energy.

So now we know that there is some other contribution in the energy which is proportional to L_z

actually doing $\dot{L}_x = \{L_x, \frac{1}{2I}(L_x^2 + L_y^2 + L_z^2) + cL_z\}$
 so; $L_x = \{L_x, +cL_z\} \Rightarrow \dot{L}_x = \{L_x, cL_z\}$

now we can calculate what components of L do.

$\dot{L}_x = \{L_x, \frac{1}{2I}(L_x^2 + L_y^2 + L_z^2) + cL_z\}$
 $= \{L_x, cL_z\}$
 $\dot{L}_x = -cL_y$
 $\dot{L}_y = \{L_y, cL_z\} = +cL_x$
 $\dot{L}_z = \{L_z, cL_z\} = 0$

* $\dot{L}_z = \{L_z, cL_z\} = 0$ (vertical component of angular momentum does not change) $\Rightarrow$ i.e. the θ mentioned in the diagram does not change $\hookrightarrow$ i.e., it stays at the same angle.

so; $\dot{L}_z = 0$
 $\dot{L}_x = -cL_y$
 $\dot{L}_y = +cL_x$

now, what these are doing
 $\hookrightarrow L_z$ is staying fixed
 $\hookrightarrow$ The small time change in the x component is proportional to the y -component
 $\hookrightarrow$ & the small time change in the y component is proportional to the x -component.

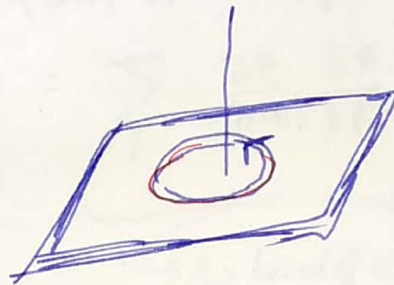
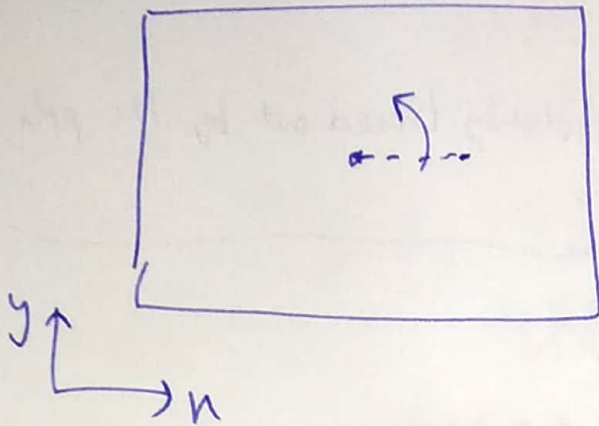
$\hookrightarrow$ so; it's rotating in x - y plane.

So: if we look down over x-y plane

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↳ we look the component of angular momentum along x-z plane
the angular momentum

↳ Its ~~precessing~~ ^{precessing} around here  with an angular velocity ω .



• but once you know this, you know exactly the same thing would be true for any angular momentum

↳ wherever it came from → whatever is the source of angular momentum (it might be a zillion electrons spin all aligned in cap) , will exactly do the same thing



if

↳ So the result is quite general.

• You could not call it equilibrium

↳ but you could call it steady state motion.
or stationary motion.

Note II This is exactly the same ~~precession~~ ^{precession}, as the ~~precession~~ ^{precession} of an electron ~~mag~~ magnetic moment in a magnetic field.

↳ here magnetic field plays the same role as gravitational field (they try to pull the spin down way, & instead the spin precesses ...)

ex  it could be nuclear spin also, ... will work same way.
(Nuclear Magnetic Resonance)

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Precession: the slow movement of the axis of a spinning body around another axis due to a torque acting to change the direction of the first axis.

or It is seen in the circle slowly traced out by the pole of a spinning gyroscope.



Vector Calculus

What is field?

A physical quantity which depends on space & time.

ex wind velocity is a vector field,

ex temperature is a scalar field.

 $\vec{\nabla}$ is a kind of fake vector, ... and have some fake components.

here, talking about 3 dimensional case.

$$\vec{\nabla} \rightarrow \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$\text{ex } \vec{\nabla} S(x, y, z) = \left(\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z} \right) \quad \left. \vphantom{\frac{\partial S}{\partial x}} \right\} \text{action of } \vec{\nabla} \text{ on scalar } S$$

• Scalars has no directionality, (the $\vec{\nabla}$ is to be thought of as a vector in some fictitious sense), but, the action of $\vec{\nabla}$ on a scalar gives a vector.

$$\text{ex } \vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

: $\vec{A}$ is real vector field.

$$\begin{aligned} \text{note } \otimes \quad (\vec{\nabla} \times \vec{A})_x &= V_y A_z - V_z A_y \\ (\vec{\nabla} \times \vec{A})_y &= V_z A_x - V_x A_z \\ (\vec{\nabla} \times \vec{A})_z &= V_x A_y - V_y A_x \end{aligned}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

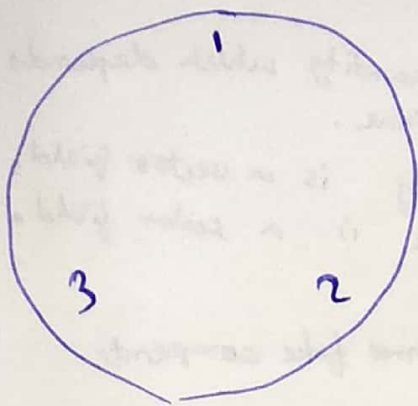
Kronecker symbol

Levi-Civita symbol.

$$\epsilon_{ijk} = 0 \quad \text{if any two of the indices are the same or three of them are the same.}$$

ϵ_{ijk} is only non-zero when all i, j, k are different

$$\epsilon_{123}, \epsilon_{231}$$



$$\epsilon_{ijk}$$

ex ~~$\epsilon_{1,2,3}$~~

(in this case $\epsilon_{1,2,3}$)

if sequence i, j, k corresponds to clockwise permutation, then the ϵ symbol is 1.

so:

$$\epsilon_{1,2,3} = 1 \quad \epsilon_{1,3,2} = -1$$

$$\epsilon_{2,3,1} = 1 \quad \epsilon_{2,1,3} = -1$$

$$\epsilon_{3,1,2} = 1 \quad \epsilon_{3,2,1} = -1$$

$\hookrightarrow$ if they correspond anticlockwise then -1.

$$\epsilon_{1,3,2} = -1$$

$$\epsilon_{3,2,1} = -1$$

$$\epsilon_{2,1,3} = -1$$

$\therefore$ Notice: ϵ_{ijk} (the Levi-Civita) is antisymmetric.

- if you ~~change any~~ interchange any two of the indices $\rightarrow$ the sign changes.

★ so ϵ_{ijk} symbol is antisymmetric with respect to the exchange of any two indices. $\rightarrow$ together with magnitude 1, this impact is complete enough to determine ϵ 's entries.

$$(\vec{V} \times \vec{A})_i = \epsilon_{ijk} V_j A_k$$

Definition of cross product.

★ (use Einstein summation convention)

pick $i=1$; (in other words if $i=1$) , then j & k must not be equal to one & equal to each other.

ex $i=1$; two possibilities

$$\epsilon_{ijk}$$

$$\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{matrix}$$

$$(\vec{V} \times \vec{A})_1 = 1 \cdot V_2 A_3 + (-1) V_3 A_2$$

other way to write it

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$$(V_i A_j - V_j A_i) = \epsilon_{ijk} (\vec{V} \times \vec{A})_k$$

check

now;

$$(\vec{V} \times \vec{A})_i = \epsilon_{ijk} \frac{\partial A_k}{\partial x_j}$$

$$\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) = \epsilon_{ijk} (\vec{V} \times \vec{A})_k$$

Curl

think cross product as giving you a vector

$(\vec{V} \times \vec{A})$ and this has three components

$$(\vec{V} \times \vec{A})_x = V_y A_z - V_z A_y$$

and you can think of it ~~as a vector~~ is a thing with two components.

↳ Its antisymmetric object

↳ a 3×3 matrix so to speak (antisymmetric matrix)

0	1	2
-1	0	3
-2	-3	0

↳ so now, how many independent components does anti-symmetric matrix have, || three ||

↳ so amount of number of components of an antisymmetric matrix in three dimensions is also three.

↳ that matches the three components of vector.
(that's why you can identify them)

★ now lets go to four dimensions.

• How many components of vector are there $\Rightarrow$ Ans Four

• How many components are there to an antisymmetric matrix of four dimension $\Rightarrow$ Ans Six

0	.	.	.
	0	-	.
		0	.
			0

2 $\rightarrow$ so in other dimensions, you cannot identify antisymmetric matrices with vectors.

(That's a ~~special~~ special to three dimensions)

$\Rightarrow$ on the other hand, the antisymmetric matrix of components like $(\mathbf{v} \times \mathbf{A})_x = v_y A_z - v_z A_y$ this always exists

2 $\rightarrow$ & its generalization of the cross product.

• Cross Product in other dimensions is not itself a vector.

Properties

$$\nabla \cdot (\vec{\nabla} \times \vec{A}) = 0$$

$$\nabla \times (\vec{\nabla} \cdot \vec{S}) = 0$$

Magnetic Fields.

• of course a magnetic field is defined by defination through its action on charged particles or charged currents or flow of charge and so forth,

but we are just going to say there is a thing called magnetic field, labelled ~~for~~ as $\vec{B}$.

• and it has property that its divergence is always equal to 0

$$\therefore \nabla \cdot \vec{B} = 0$$

What is divergence of an electric field
 $\rightarrow$ its charge density by ϵ_0

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$\rightarrow$ so wherever there is charge, the field spreads out from that charge & divergence is nothing but the response to the charges & the field flowing out of it.

(and the charge you call, the source of the field)

* Magnetic Field have no sources (at least magnetic field have no sources as confirmed by experiment)

$\rightarrow$ If magnetic monopoles are discovered, then we will have to back track on this $\rightarrow$ & everybody fully expects that they do exist

$\rightarrow$ but they are probably so heavy that they are out of range of any experiment $\rightarrow$ (are u sure ? flus)

$\rightarrow$ so for the ~~ordinary~~ ordinary electromagnetic theory, which does not involves magnetic monopoles

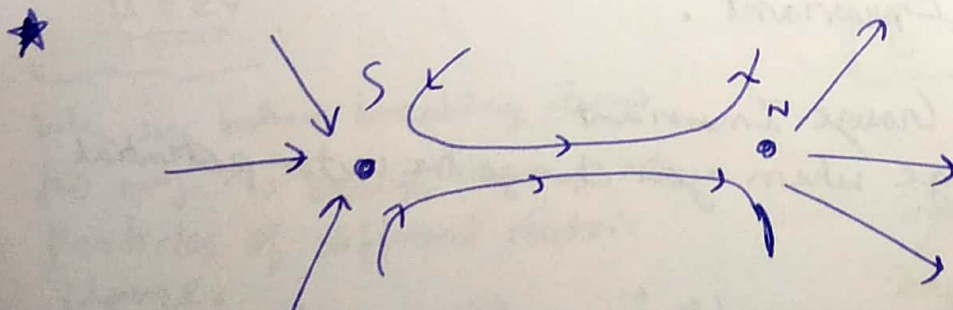
$\rightarrow$ this means source of magnetic field is 0

$\rightarrow$ & this means divergence of magnetic field is 0.

$$\nabla \cdot \vec{B} = 0$$

Magnetic field is divergence free.

Himansh
Rathore



since $\nabla \cdot \vec{B} = 0 \Rightarrow$ so there exist $\vec{A}$, such that $\vec{B} = \nabla \times \vec{A}$

we can always write $\vec{B}$ as.

a curl of something.

(here $\vec{A}$ is not unique)

- Magnetic field is real thing & you can measure it locally.
 - $\vec{A}$ is not a real thing & you can't measure it locally.
- one way to observe that is by realizing that $\vec{A}$ is not unique.

Suppose; you take $\vec{A} + \nabla S := \vec{A}'$ (add gradient of anything)

here also; $\nabla \times \vec{A}' = B$

→ this means that $\vec{A}$ is undefined or illdefined upto the addition of the gradient of a scalar.

(That's a kind of transformation of $\vec{A}$ field which should have no physical affect on the motion of charged particles)

→ because the motion of charged particles is ~~not~~ determined by magnetic field alone. & also the electric field.

(let's ~~to~~ leave out electric fields for now)

→ ~~That's a kind~~ That transformation, a transformation like this is called Gauge Transformation of the vector potential by an arbitrary function S .

→ & the ~~word~~ word that goes with it is that is the Physical Forces on the charged particles are Gauge Invariant.

- The magnetic field is Gauge Invariant

→ it does not change when you change the vector potential this way.

⇒ $\vec{A}$ is sometime called Gauge Field.

→ "Fields that you can transform in this way without changing the physical character of the influence of the field"

- Gauge Transformations are applied to Gauge Fields.

Question If this $\vec{A}$ field is so non-unique, really does not have any physical meaning in a localized region of space, then why do we introduce it?

Answer We introduce it, because without it we could not describe the motion of charged particles using the Principle of Least Action, Lagrangian, Hamiltonians & so forth.

Goal: we want to write a theory based on the Principle of Least Action, based on Lagrangian mechanics, & so forth, which gives rise to forces which are written down as follows -

(forces are by definition the left hand side of $F = ma$)

→ This force is called Lorentz Force (the same Lorentz is Lorentz Transformations)

$$\vec{F} = e \left(-\nabla V(\vec{r}) + \frac{\vec{v}}{c} \times \vec{B} \right) = ma$$

Force on a charged particle the electric charge scalar potential gradient of scalar potential the fact that there is a velocity of light in here means the particle has to be going damn fast in order

* let's use symbol U for potential energy.

$$U = eV$$

why we bother breaking it up this way → because there are particles of different electric charge.

$$\vec{F} = e \left(+\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)$$

→ so; magnetic forces are associated with motion somehow.

~~we have seen one example like this;~~

• V is not quite the potential energy.

$$\vec{E} := -\nabla V$$

to experience any significant amount of magnetic force.

History: Lorentz was around 50 year old when Einstein was young.

Before Lorentz, it was known that forces on conducting wires with charge flowing through them.

↳ and Lorentz figured out correctly that in order to get that kind of force on wires with currents flowing through them, the force on an individual electron for example will have to be governed as Lorentz force which he found.

★ (i) we have seen something like Magnetic force...

Yes, Coriolis Force.

$$\cancel{F_{\text{Coriolis}}} = 2m \vec{V} \times \vec{\omega}$$

if you are on a rotating platform, then there is an $\vec{\omega}$ (which is $\perp$ to the plane...)

↳ similar, if we have uniform Magnetic field pointing same upward, ... ~~Coriolis~~

ex 

$$\begin{aligned} B_z &= b \\ B_x &= 0 \\ B_y &= 0 \end{aligned}$$

a uniform magnetic field about z axis.

①

$$\begin{aligned} A_z &= 0 \\ A_x &= 0 \\ A_y &= bx \end{aligned}$$

this is a vector potential which generates uniform magnetic field along z axis.

lets check $\vec{\nabla} \times \vec{A}$

$$\begin{aligned} \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} &= b \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} &= b \\ \underline{\quad 0 \quad} & \\ (\vec{\nabla} \times \vec{A})_z &= b \\ \Rightarrow \vec{B}_z &= b \end{aligned}$$

②

$$\begin{aligned} A_z &= 0 \\ A_x &= -by \\ A_y &= 0 \end{aligned}$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = b$$

how ① & ② related $\Rightarrow$ the difference between them is related by a gradient.

$\vec{E} = -\nabla V$; here also there is ambiguity upto a additive constant. (1997)
 ↳ this is electrostatic potential.

↳ So when people started thinking of ~~Heat or~~ Potential
 ↳ they gave it a name Potential, since the real ~~ph.~~ field was invariant upto something additive quantity.

Q Can Lorentz Force law be derived from Action Principle?

Ans Yes, (but at the cost that you have to introduce Vector Potential)

- Electric part is easy, (that's standard) the particle has a potential energy & put $\mathcal{L}$ as $T - V$.
- The non-standard is magnetic force (lets concentrate on this)

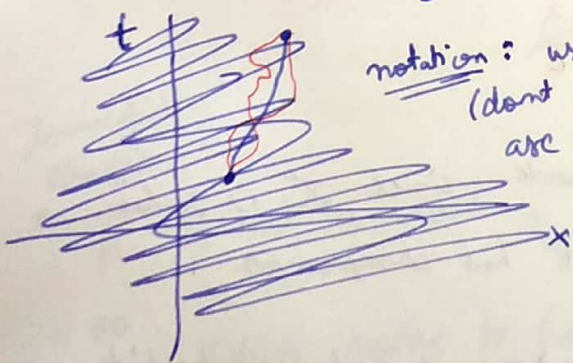
^{lets}
 # Think about things which we can put in Lagrangian.

★ Lets start with a particle with no forces.

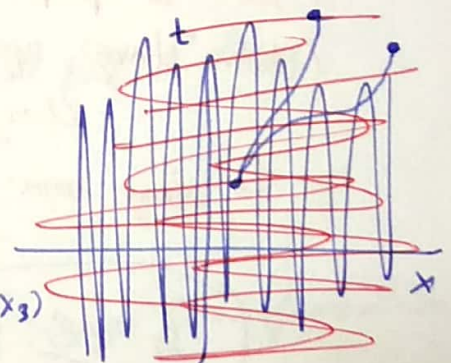
we are going to build Action Principle.

Action Principle ; that there exist an Action (A)

$$A = \int \mathcal{L}(x, \dot{x}) dt$$



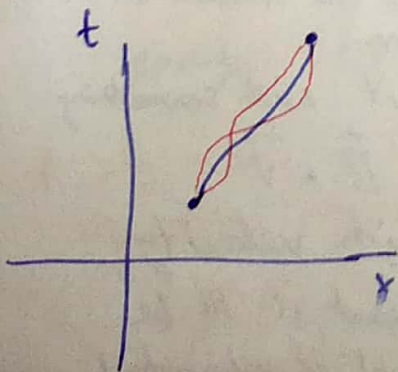
notation : we use x_1, x_2, x_3 or x, y, z
 (dont we cvs...) because we
 are now really talking about
 x, y, z . $x := (x_1, x_2, x_3)$



Action Principle

There exists an expression called the Action,
 its an integral along the trajectory ; & the
 rule is, that the true trajectory should
 minimize or stationarize the Action along the
 trajectory.

- we hold the two end points fixed. $\rightarrow$ & then vary the trajectory in different ways $\rightarrow$ & then ask for the trajectory of least action $\rightarrow$ that defines true trajectory by using Euler Lagrange equations.



what kind of things we could write in Lagrangian?

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- first of all, if we turn off all fields, we would expect this $\mathcal{L}$ to be kinetic energy.

so, begin as

$$\dot{\mathbf{x}}^2 = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2$$

$$A = \int \frac{m}{2} \dot{\mathbf{x}}^2 + \left(\cancel{\frac{q}{c} \mathbf{E} \cdot \mathbf{x}} + \frac{q}{c} \mathbf{B} \cdot \mathbf{x} \right) dt$$

~~for magnetic~~

- now we want to add something which gives rise to new kind of force.
(forget electric force now)

[for Electric forces simply subtract the electrostatic potential energy]

- Now we want to reproduce magnetic term

- first of all it's going to be something which we go to add

- it should be proportional to electric charge

(electric charge appears in formula, so slightly to have electric charge in it)

→ always come in divided by c .

$$\frac{e}{c}$$

$$\left(\frac{q}{c} \mathbf{B} \right)$$

this will not work since $\mathcal{L}$ is scalar.

$\vec{B}$ has components....

we expect something related to magnetic field should appear.

→ so try to make it scalar.

→ so dot it with something

$$\vec{B} \cdot \vec{v}$$

(dot it with velocity, because we want it to be proportional to velocity)

actually in final answer instead of $\vec{B}$, $\nabla \times \vec{A}$ appears.

but this gives wrong answer.

~~write answer.~~

write answer.

(pg 99)

$$A = \int \left[\frac{m \dot{x}^2}{2} + \frac{e}{c} \overbrace{A_i \dot{x}_i}^{\substack{\text{vector potential} \\ \text{dotted velocity}}} \right] dt$$

vector potential
dotted velocity.

actually first of all without solving before
→ we have confusion what should we dot
with $\vec{A}$ or $\vec{B}$

→ actually first put $\vec{B}$

→ then you find you get force as $\vec{F} = \frac{e}{c} \vec{v} \times (\nabla \times \vec{A})$

→ so now you say, hey wait, I will put $\vec{A}$ there
such that $\vec{B} = \nabla \times \vec{A}$.

Immediate
Question

• $\vec{A}$ is not unique, $\vec{A}$ is subject to gauge transformation.

→ What's going to happen if we change $\vec{A}$ from one
gauge to another.

★ Why making a gauge transformation does not affect the
equations of motions, (it has a small effect on Action,
but it's the effect that does not change the equations
of motions)

→ next page.

~~lets~~ ~~check~~
 * lets check
 * summation convention used.

$$\int A_i \frac{dx_i}{dt} dt = \int A_i dx_i$$

How does it change if we do a gauge transformation.

→ lets do a gauge transformation & compare the transformational result with above.

change $\vec{A}$ by gauge transformation.

$$\int \left(A_i + \frac{\partial S}{\partial x_i} \right) \frac{dx_i}{dt} dt = \int (A_i + \frac{\partial S}{\partial x_i}) dx_i$$

So; the action changed by

$$\int \frac{\partial S}{\partial x_i} dx_i$$

this much
 (also there is some multiplicative factor)
 ... actually $\frac{e}{c}$.

$$\Delta A = \left(\int \frac{\partial S}{\partial x_i} dx_i \right) \frac{e}{c}$$

→ change in Action due to Gauge Transformation

(ie; change in Action under a Gauge Transformation)

∴ actually;

$$\frac{dS}{dx_i} dx_i = (\nabla S) \cdot d\vec{x}$$

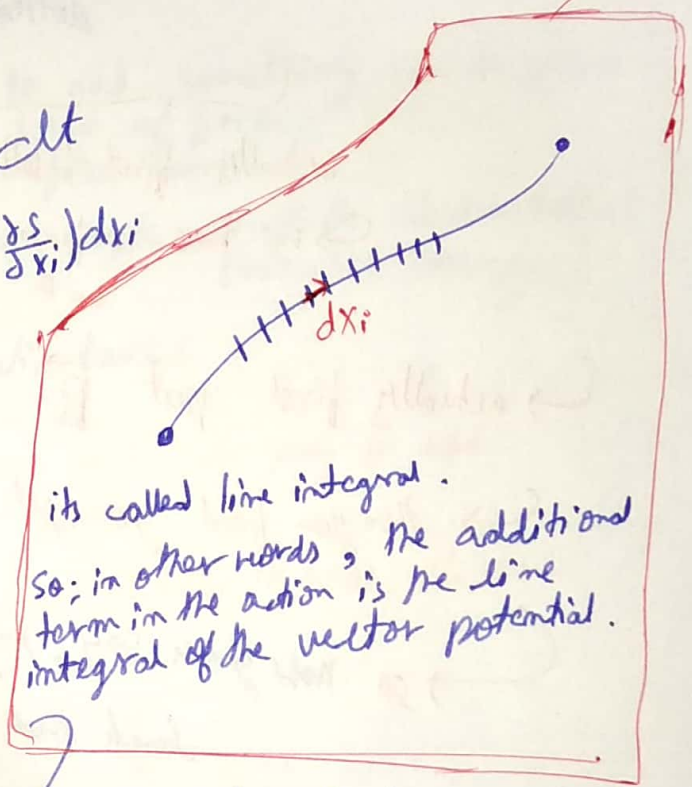
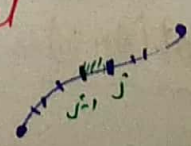
$$\cancel{S} \Rightarrow dS = (\nabla S) \cdot d\vec{x}$$

→ if you add them all together, they will all ~~cancel out~~ cancel out except the ones on the end points.

$$\frac{\partial S}{\partial x_i} dx_i = S_j - S_{j-1}$$

one end of the small interval

other end of small interval



its called line integral.
 So; in other words, the additional term in the action is the line integral of the vector potential.

→ ii; Integral of components of vector potential along the trajectory integrated from the beginning to the end of the trajectory.

together....

PS101

$$\Delta A = \int_{t_i}^{t_f} \frac{\partial S}{\partial x_i} dx_i = S(x_f) - S(x_i)$$

so the whole thing only depends on the ends of trajectory
→ but end points are fixed.

→ so whatever S you pick, so this is contribution to the action which really does not depend on trajectory at all. (depends on end points) → but using action principle we keep end points fixed. → result is, that the equations of motion and the choice of correct trajectory ~~that~~ will not change when you do a Gauge Transformation.

Note • It is not quite true that the Action is Gauge Invariant.
→ but the equations of motion that follow from it & the trajectory that follows from it are Gauge Invariant, → the Physics here is Gauge invariant.

→ The actual solution of Action Principle is Gauge Invariant

$$\vec{B}(x)$$

$$B_i(x) = B_x, B_y, B_z$$

Rule: no sources of the magnetic field
 $\hookrightarrow$ i.e. no magnetic monopoles

In future ~~this~~ may be proved wrong

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times [\nabla S] = 0$$

$$\vec{B} = \nabla \times \vec{A}$$

~~it is possible~~

- Assume time independent Electric & Magnetic field.

* Vector potential is unique upto a additive gradient.

$\hookrightarrow$ This is the only ambiguity

$\hookrightarrow$ it is possible to prove that there is no other ambiguity in the vector potential other than adding to it the gradient of a scalar.

$$\vec{A}' = \vec{A} + \nabla S$$

$\therefore$ Any two vector potentials, which give rise to the same magnetic field $\hookrightarrow$ the difference between them, will automatically be a gradient
 Mathematical Theorem.

$\vec{A}' = \vec{A} + \nabla S$ } Vector Potential is ambiguous upto Gauge Transformation.

Gauge Transformation.

* Magnetic Field is Gauge Invariant.

~~First~~ (lots of other things that we might measure ~~in laboratory~~ directly in laboratory are Gauge Invariant.)

$$\begin{aligned} B_z &= b \\ B_x &= 0 \\ B_y &= 0 \end{aligned}$$

First Gauge

$$\begin{aligned} \text{I} \quad A_x &= 0 \\ A_y &= bx \\ A_z &= 0 \end{aligned}$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = b$$

II

$$\begin{aligned} A_x &= -by \\ A_y &= 0 \\ A_z &= 0 \end{aligned}$$

$$\Rightarrow (B)_z = b$$

$$S = cxy \quad \therefore \nabla_x S = cy$$

$$\nabla_y S = cx$$

$$\nabla_z S = 0$$

$$\begin{aligned} \text{if } c = -b/2 \quad A_x &\rightarrow -b/2 y \\ A_y &\rightarrow b/2 x \\ A_z &\rightarrow 0 \end{aligned}$$

I + ∇S ; so;

$$\begin{aligned} A_x &= 0 \rightarrow cy \\ A_y &= bx \rightarrow bx + cx \\ A_z &= 0 \rightarrow 0 \end{aligned}$$

choose $c = -b$

$$\begin{aligned} \text{then; } A_x &\rightarrow -by \\ A_y &\rightarrow 0 \\ A_z &\rightarrow 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} A_x &\rightarrow -by \\ A_y &\rightarrow 0 \\ A_z &\rightarrow 0 \end{aligned}} \right\} \text{ got II}$$

... a particular choice of $\vec{A}$ is called Gauge.

- The actual physical phenomenon that take place should be independent of the particular gauge (that's called Gauge Invariance)

- You can prove that there always does exist a gauge in which the x component is equal to zero.

(not just for uniform magnetic field $\rightarrow$ but in general)

$\Rightarrow$ Then you could speak about the gauge in which x component is equal to zero.

or ~~the~~ the gauge in which y component is zero.

* You may use different gauges to study different properties as per your ease.

ie; • You may use one gauge to study energy conservation, & other gauge to study Lorentz Invariant.

Lagrangian for a particle moving in a static magnetic field.

$$\mathcal{L} = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{e}{c} \vec{A} \cdot \vec{v}$$

you could change the definition of electric charge ~~e~~

$e' := \frac{e}{c}$; & would get rid of speed of light if you want.

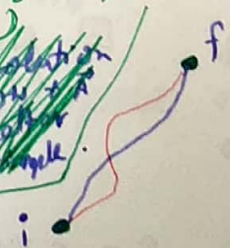
we don't subtract this quantity because that will only correspond to changing sign of $\vec{A} \Rightarrow$ so you just replace $\vec{A}$ by $-\vec{A}$ $\rightarrow$ formula will stay same, just $-\vec{A}$ will appear

gives simple result with no much twist of signs...

... to;

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{e}{c} (A_x \dot{x} + A_y \dot{y} + A_z \dot{z})$$

~~solve for $\vec{A}$ & later after solving, & ~~knowing~~ the solution known $\vec{A}$ is better & simple.~~



change in ^{Action} $\mathcal{L}$ when we make ~~any~~ a gauge transformation

$$= \frac{e}{c} \int \frac{\partial S}{\partial x_i} \dot{x}_i dt = \frac{e}{c} (S(x_f) - S(x_i))$$

Proof; $\frac{e}{c} \int \frac{\partial S}{\partial x_i} \frac{dx_i}{dt} dt = \frac{e}{c} \int \frac{\partial S}{\partial x_i} dx_i = \frac{e}{c} \int ds = \frac{e}{c} (S(x_f) - S(x_i))$

• Action is not Gauge Invariant.

• Exploring the trajectories keeping fixed the end points does gives rise to Gauge Invariant answer $\rightarrow$ The trajectory which minimizes the Action will not depend on Gauge Transformation.

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i}$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i}$$

Two different notions of momentum -

- (1) one of them is old fashion momentum, $\underline{mv}$ (its called the Mechanical Momentum)

$$m \vec{v} := \text{Mechanical Momentum}$$

- (2) the other one is,

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} := \text{Canonical momentum}$$

(in general Mechanical momentum & Canonical momentum are not same,)
~~but they~~ (may be ~~same~~ same in some cases)

$$\dot{p}_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + \frac{e}{c} A_x$$

↪ Canonical momentum

~~$$\dot{p}_x = m\dot{x} + \frac{e}{c} A_x$$~~

$$\dot{p}_x = m\ddot{x} + \frac{e}{c} \left(\frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_x}{\partial y} \dot{y} + \frac{\partial A_x}{\partial z} \dot{z} \right)$$

rate of change in A_x
 with time as the
 particle moves.

~~$$\vec{A}(\vec{x})$$~~

now; $\vec{x}$ is
 position vector
 $\vec{x} = (x_1, x_2, x_3)$

When you ask that the momentum
 is conserved by virtue of symmetry,
 you are talking about
Canonical Momentum.

now

~~$\vec{A}$~~ $\vec{A}$ is just a function of
 position $\rightarrow$ it depends
 on where the particle is

↪ at the position of particle
 the $\vec{A}$ will vary with time
 even though there may
 be ~~no~~ explicit dependence
 on time (the field may be
 completely constant at a given
 point ~~at~~ with time)

↪ because as particle moves
 $\vec{x}$ changes & thus $\vec{A}$
 changes.

$$\dot{p}_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{e}{c} \left[\frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_x}{\partial y} \dot{y} + \frac{\partial A_x}{\partial z} \dot{z} \right]$$

so;

$$m\ddot{x} + \frac{e}{c} \left[\frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_x}{\partial y} \dot{y} + \frac{\partial A_x}{\partial z} \dot{z} \right] = \frac{e}{c} \left[\frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_y}{\partial x} \dot{y} + \frac{\partial A_z}{\partial x} \dot{z} \right]$$

$\ddot{x} := a_x$ (a component of acceleration familiar name)

P31

~~$m a_x = \frac{e}{c} \left[\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \dot{y} + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \dot{z} \right]$~~

$$m a_x = \frac{e}{c} \left[\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \dot{y} + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \dot{z} \right]$$
$$= \frac{e}{c} \left[B_z \dot{y} - B_y \dot{z} \right]$$

Observation:

- does something, it creates a force, the RHS is not zero
- That force is Gauge Invariant, it does not involve the ambiguity in A .
- and finally whatever it is, ... can be written as a component of ~~$\vec{v} \times \vec{v}$~~ $\vec{v} \times \vec{B}$

$$\Rightarrow m a_x = \frac{e}{c} (\vec{v} \times \vec{B})_x$$

→ do exactly the same thing for other components.

& finally what you get is Lorentz Force law.

$$m \vec{a} = \frac{e}{c} (\vec{v} \times \vec{B})$$

Newton

Lorentz

Newton - Lorentz's Equations.

Theorem: You cannot express the motion of charged particle in an Action form (in the form of Lagrangian equation of motion) without introducing the vector potential. (pg 107)

Is there any expression for Energy & can we use it as Hamiltonian?

$$P_x = m\dot{x} + \frac{e}{c} A_x$$

$$P_y = m\dot{y} + \frac{e}{c} A_y$$

$$P_z = m\dot{z} + \frac{e}{c} A_z$$

$$H = \sum_i P_i \dot{x}_i - \mathcal{L}(x, \dot{x})$$

↳ we need H as a function of x & P

so:
$$\frac{(P_x - \frac{e}{c} A_x)}{m} = \dot{x}$$

$$H = P_x \dot{x} + P_y \dot{y} + P_z \dot{z} - \frac{m}{2} \dot{x}^2 - \frac{m}{2} \dot{y}^2 - \frac{m}{2} \dot{z}^2 - \frac{e}{c} [A_x \dot{x} + A_y \dot{y} + A_z \dot{z}]$$

↳ substitute all $\dot{x}_i$

$$H = \frac{(P_x - \frac{e}{c} A_x)^2}{2m} + \frac{(P_y - \frac{e}{c} A_y)^2}{2m} + \frac{(P_z - \frac{e}{c} A_z)^2}{2m}$$

it is just $\frac{1}{2} m v^2$

↳ energy is exactly the same function of velocity as it was without the magnetic field. (this does not tell us very much)

↳ the total Hamiltonian is just $\frac{1}{2} m v^2$ & it is conserved.

↳ tells that a particle moving in magnetic field has constant speed.

Uniform magnetic field. (pointing in z direction)

$$\begin{aligned} A_z &= 0 \\ A_x &= 0 \\ A_y &= Bx \end{aligned}$$

lets put this in Lagrangian.

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{e}{c} Bx \dot{y}$$

two cyclic coordinates

~~lets~~ # lets put in Lagrangian

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{e}{c} b x \dot{y}$$

here we have two cyclic coordinates
y & z

↳ so corresponding momentum is conserved.

(as a consequence of symmetry,

what's the symmetry?

→ shifts of cyclic coordinate

(if they don't enter, it does not matter if you shift them))

$$z \rightarrow z + \epsilon$$

$$y \rightarrow y + \epsilon$$

$$p_x = m \dot{x}$$

$$p_y = m \dot{y} + \frac{e}{c} b x$$

$$p_z = m \dot{z}$$

$$\dot{p}_z = 0 \quad \text{or} \quad a_z = 0$$

$$\dot{p}_y = 0$$

$$\Rightarrow \frac{d}{dt} (m \dot{y} + \frac{e}{c} b x) = 0$$

$$\Rightarrow m a_y + \frac{e}{c} b v_x = 0$$

$$\Rightarrow m a_y = -\frac{e}{c} \cdot \frac{b}{m} v_x$$

Lorentz Force law
for the special case of constant magnetic field.

↳ This equation incidentally is
Gauge Invariant of ~~force~~
↳ it does not have vector potential in
it → It is real equation about
real things.

* you can work out equations of motion for x : i.e. $\dot{x} = \frac{\partial L}{\partial p_x}$

↳ but the other way is to select & change gauges.

• earlier we chose a particular gauge which was very good for figuring out ~~by~~ what y motion was doing → because y was not appearing in the Lagrangian.

↳ lets use the other gauge

$$\left. \begin{array}{l} A_z = 0 \\ A_x = -by \\ A_y = 0 \end{array} \right\} \rightarrow \text{gives same } \vec{B}$$

So, new: L'_{new} is : $L_{\text{new}} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{e}{c} (-b x \dot{y})$

So, in new $\mathcal{L}$; i.e. $\mathcal{L}_{\text{new}}$ which coordinate is cyclic? (pg 109)
a coordinate & z .
 $\Rightarrow$ Important for us now.

So, $(P_x)_{\text{new}}$ is conserved.

but this $(P_x)_{\text{new}}$ is not same as initial P_x which was written.

$$(P_x)_{\text{new}} = m\dot{x} = \frac{e}{c} b y$$

$$\hookrightarrow m\ddot{x} - \frac{e}{c} b \dot{y} = 0 \Rightarrow \boxed{m a_x = \frac{e}{c} b \dot{y}}$$

So, we got

$$m a_y = -\frac{e}{c} b \dot{x}$$

$$m a_x = \frac{e}{c} b \dot{y}$$

$$m a_z = 0$$

● Canonical momentum is not something you measure.
~~It is not invariant under gauge transform.~~
 $\rightarrow$ They are most Gauge Invariant.

$$\# \quad m(\vec{a})_x = \frac{e}{c} (\vec{v} \times \vec{B})_x$$

$$m(\vec{a})_y = \frac{e}{c} (\vec{v} \times \vec{B})_y$$

$$m\vec{a} = \frac{e}{c} (\vec{v} \times \vec{B})$$



$$|\vec{a}| = \frac{v^2}{r}$$

$$\therefore m|\vec{a}| = \frac{e}{c} v b = m \frac{v^2}{r} \Rightarrow c \cdot \frac{mv}{r} = eb$$

$$\Rightarrow \boxed{r = \frac{mv}{eb}}$$

but where is centre of circle?

$\hookrightarrow$ can be any where.

$\therefore \infty$ circles possible...

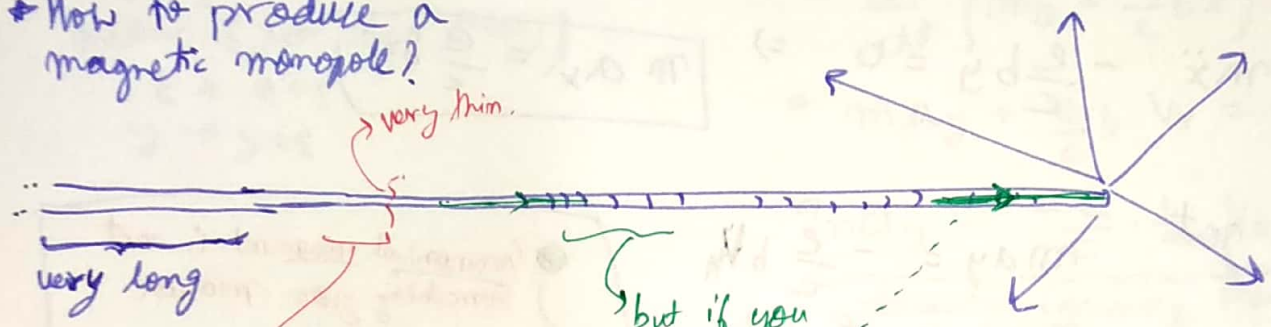
#★ What are the legitimate Lagrangians?

Ans and they are basically those that leads to sensible Hamiltonian, that don't lead to ambiguity in the equations.

- In order to deal with maxwell equations, we have to add to the Action the things which generate maxwell equations.

Take a very long bar magnet. (or it could be a solenoid)

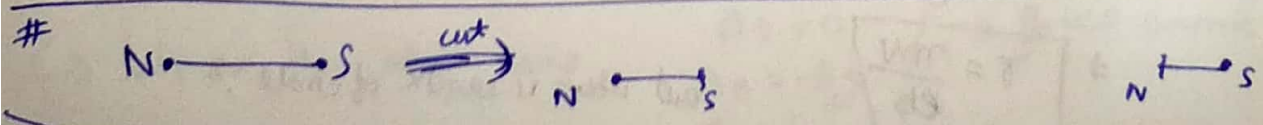
★ How to produce a magnetic monopole?



but if you look very carefully here, you would discover that there was also magnetic field in there.

if this is ~~really~~ really very very thin, $\rightarrow$ perhaps it does not affect the motion of particles & so forth (unless the particle ~~just~~ by accident just happens to pass through this solenoid) which would be very improbable thing if solenoid is thin enough

$\rightarrow$ then you manufacture yourself a monopole.



★ Pauli matrices :... (extra) ...