Standard Model

— Shoaib Akhtar 1/6/2020

Lec 1 | Particle Content of the Standard Model, Renormalizability of massive gauge theories

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Fermions (spin 1/2)

		mass	charge
Quarks:	up (u)	2 MeV	$+2/3$
	charm (c)	1.3 GeV	$+2/3$
	top (t)	173 GeV	$+2/3$
	down (d)	5 MeV	$-1/3$
	strange (s)	95 MeV	$-1/3$
	bottom (b)	4.2 GeV	$-1/3$
Leptons: (charged leptons)	e (electron)	0.5 MeV	-1
	μ (muon)	106 MeV	-1
	τ (tau)	1.8 GeV	-1
Neutrinos	ν_e	≈ 0 *	0
	ν_μ	≈ 0 *	0
	ν_τ	≈ 0 *	0

* Neutrinos have small finite mass. But standard model is defined with only massless neutrinos.

Two ways to introduce mass into SM (Standard Model) Lagrangian for neutrino. (and we dont know which one is write; so we cant proceed further)

"Dont know yet which is correct.

Bosons (spin 0 or 1)

Gauge Bosons
(s = 1)
i.e. spin = 1

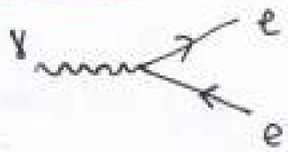
	Mediators of the type of force	Mass	Electric charge
photon (γ)	E.M. (Electromagnetic force)	0	0
Gluon (g)	Strong Force	0	0
$W^{\pm}$	Weak Force	80.4 GeV	$\pm e$
Z	Weak Force	91.2 GeV	0

(We have gauge bosons for every kind of forces we have: we have three kinds of forces in Standard Model)

	mass	charge
Higgs boson (h)	125 GeV	0

Forces mediated by gauge bosons.

① E.M. force (photon) : γ couples to electric charge



Feynman rule $\longrightarrow$ Columb potential ; $V(r) = \pm \frac{\alpha_{em}}{r}$

attractive or repulsive.

$\alpha_{em} \Rightarrow$ coupling constant

② Strong force (gluon)

g couples to fields carrying color charge

3 types of color : red (r), green (g), blue (b)
3 types of anti-color: antired ($\bar{r}$), antigreen ($\bar{g}$), antiblue ($\bar{b}$)

quark fields q ← color

* quark fields carry color
* Anti-quark fields carry anti-color.

we can think of quark as dirac field; but it ~~...~~ carries an additional index corresponding to its color.

so; because there are three types of color; we should think of it as three component object in color space.

so; we should think of it as vector.

$$\text{quark field } q = \begin{pmatrix} q^r \\ q^g \\ q^b \end{pmatrix}$$

$$\text{Anti-quark field } \bar{q} = \begin{pmatrix} \bar{q}^{\bar{r}} \\ \bar{q}^{\bar{g}} \\ \bar{q}^{\bar{b}} \end{pmatrix}$$

Antiquark field also has three indices, corresponding to the anti-colors.

Gluons carry color & anti-color, e.g. $r\bar{b}$, $r\bar{g}$, etc.
~~only 8 gluons~~ only 8 possible combinations → so 8 gluons.

quarks are in fundamental representation.

Anti-quarks " " anti-fundamental "

Gluons " " adjoint " , so gluons can carry both color & anti-color.

What happened to ~~...~~ 9th gluon: one particular combination is actually a singlet under the symmetry, so we will not include it in our listings. So this one gets cancelled. ∴ gets subtracted

$$9 - 1 = 8.$$

Feynman rules

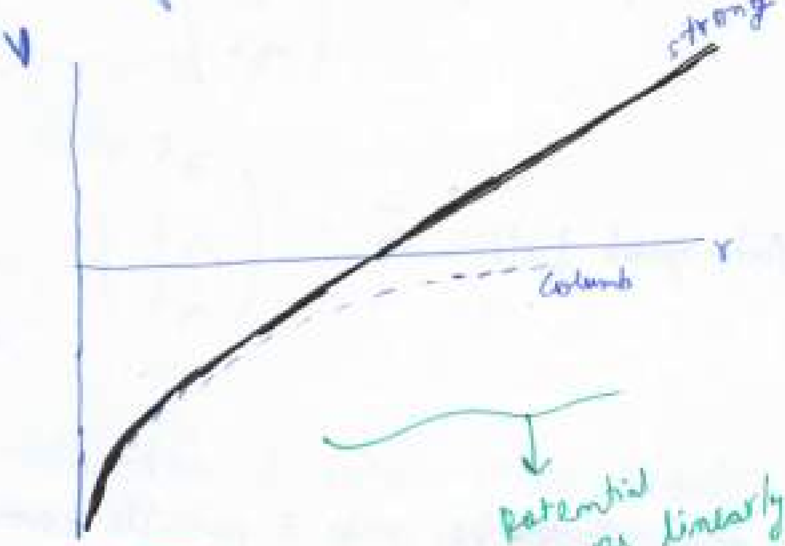
$q = u, c, t, d, s, b$

no counterpart in QED.

Potential between quark and anti-quark

$$V(r) \approx -\frac{\alpha_s}{r} + br$$

$\alpha_s \Rightarrow$ coupling constant for strong forces



Takes infinite amount of energy to seperate q and $\bar{q}$ to $r \to \infty$: So we say that there are no free quark.

potential looks like columb at small r

potential increases linearly for large r.

$\hookrightarrow$ no free quark

(this feature of strong force is known as <u>confinement</u>)

All quarks and anti-quarks must be bound into color-neutral objects (<u>hadrons</u>).

Meson: quark-antiquark pair ($q\bar{q}$) : $r\bar{r}$ = "white"

we can call something which is color neutral as white.

We can also form different types of objects.

Meson : quark - antiquark pair ($q\bar{q}$) : $r\bar{r}$ = "white".

Baryon : qqq ; rgb = "white"
(we can also form colour neutral from three q, as long as we have anti - symmetric combination of different colours)

e.g. proton (uud), neutron (udd)

$$Q_p = 2\left(\frac{2}{3}\right) + 1\left(\frac{-1}{3}\right) = +1$$ $\}\Rightarrow$ add up charges of quarks to get charge of proton.

$$Q_n = 1\left(\frac{2}{3}\right) + 2\cdot\left(\frac{-1}{3}\right) = 0$$

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③ Weak force / interaction.

"force for all fermions" : All fermions couple to weak force.

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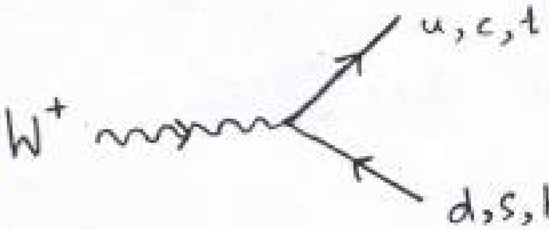


The interaction of Z boson with fermions are very similar to QED.
The dirac structure is little complicated
f = SM fermion
→ Same in as out.

cant have same type on fermion ↗ & ↙ ; because it will violate charge conservation.

~~f = SM fermion (same in as out)~~

We can have negatively charged lepton going into neutrino.

→ Incoming states same as outgoing states.

so: Interaction of Z bosons does not change fermions from one type or one flavour to another.

(B6)

$W^\pm$ boson is the only interaction that can change ~~fo~~ fermions from one type to another.

$W^\pm$ is only interaction to change fermions from one flavour to another.

If you cant change flavours, it would mean ~~&~~ that all flavours ~~are stable~~ would be stable; since no one flavour could decay into other flavour.

↳ Buts its because of the interaction with W, that most of the particles in S.M. are not stable.

(Every SM fermion would be stable if not for W boson)

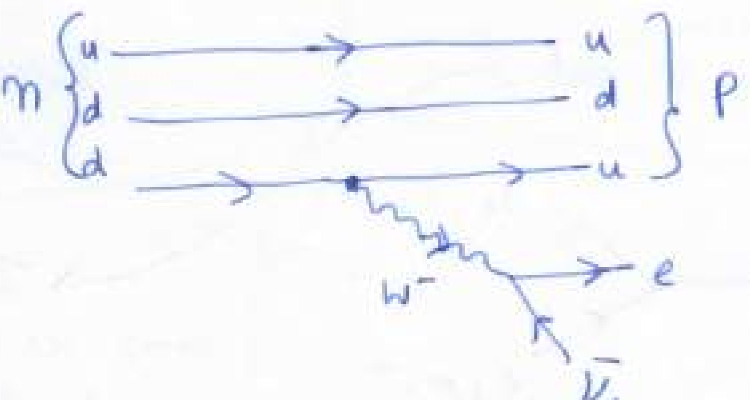
W bosons causes things to decay. These decays are known as <u>Weak Decay</u>.

① <u>Muon decay</u>:

$\mu \to e\,\nu_\mu\,\bar{\nu}_e$

② <u>Neutron decay</u>:

$n \to p\,e\,\bar{\nu}_e$



Till now; its an entire summary of Standard Model

Our goal is to write Lagrangian to describe all of these particles & all of degrees of freedom.

Goal: Write down Lagrangian to describe all* known particles and interactions.

* No dark matter, no gravity ....

Key Ingredients: * Renormalizability.
* Gauge ~~Invariance~~ Symmetry

We like Renormalizable theories. We could have non-renormalizable theory; its still predictive: its only predictive upto a certain energy scale.

So; in a sense there is a theoretical bias, that we want renormalizable theories so that we can make ~~predict~~ prediction at almost any scale we like.

Renormalizability. (want predictive theory valid at high energies)

~~Gauge Invariance~~
Gauge Symmetry (non abelian & abelian)

Higgs Mechanism and Spontaneous Symmetry Breaking

$W^{\pm}$, Z are massive gauge bosons, with chiral coupling to S.M. fermions (different couplings to LH & RH spinors)

$\Rightarrow$ inconsistent with Gauge Symmetry. & renormalizability.

QED like theory with massive photon A_μ:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 \cdot A_\mu A^\mu + \bar{\Psi}(i\not{D} - m_\Psi)\Psi$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$D_\mu = \partial_\mu + i(g_L P_L + g_R P_R) A_\mu$

we will take ~~m~~ fermion Ψ in this theory to be chiral so that it will have different gauge interactions for LH & RH spinors.

$P_{R,L} = \dfrac{1 \pm \gamma^5}{2}$; Right & Left ~~[illegible]~~ projector.

$g_{L,R}$ are gauge couplings.

Gauge Transformation: $A_\mu \to A_\mu + \partial_\mu \alpha(x)$

$$\Psi_{L,R} \to e^{-i g_{L,R} \alpha(x)} \cdot \Psi_{L,R}$$

* $F_{\mu\nu}$ and $\bar{\Psi}\not{D}\Psi$ are gauge invariant.

* $A_\mu A^\mu \longrightarrow A_\mu A^\mu + 2A_\mu \partial^\mu \alpha + (\partial_\mu \alpha)^2$ (not gauge invariant)

(mass term of photon is not gauge invariant)

* The mass term for spinors is also not gauge invariant.

$$\bar{\Psi}\Psi = \bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L \longrightarrow e^{i(g_L - g_R)\alpha} \cdot \bar{\Psi}_L \Psi_R + e^{-i(g_L - g_R)\alpha} \cdot \bar{\Psi}_R \Psi_L$$

$$\bar{\Psi}\Psi \longrightarrow e^{i(g_L - g_R)\alpha} \cdot \bar{\Psi}_L \Psi_R + e^{-i(g_L - g_R)\cdot\alpha} \cdot \bar{\Psi}_R \Psi_L$$

The reason of non-gauge invariance is ~~that~~ because we had these chiral ~~[illegible]~~ couplings g_L & g_R.

$\bar{\Psi}\Psi$ is gauge invariant only if $g_L = g_R$

There is no problem in ~~introduction~~ introducing a mass for electron in QED because Left & Right handed coupling are both just e.. So the mass term would be gauge invariant for QED ofcourse.

What else goes ~~so~~ wrong? Not renormalizable.

<u>Feynman propagator for massless photon</u>

$$D_{\mu\nu}(x,y) = \langle T(A_\mu(x)\,A_\nu(x))\rangle \xrightarrow[\text{transform}]{\text{fourier}} \text{momentum basic} \quad D_{\mu\nu}(k) = \frac{-i\eta_{\mu\nu}}{k^2+i\varepsilon}$$

Start with the mode expansion.

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2E_k} \sum_{\substack{\text{polarization}\\ j}} \left(\varepsilon_\mu^{(j)}(k)\, a_k^{(j)} e^{-ik\cdot x} + \varepsilon_\mu^{(j)*}(k)\, a_k^{(j)\dagger} e^{ik\cdot x}\right)$$

so,

$$D_{\mu\nu}(x,y) = \int \frac{d^3k}{(2\pi)^3}\cdot\frac{1}{2E_k}\int\frac{d^3k'}{(2\pi)^3}\cdot\frac{1}{2E_{k'}} \sum_{j,\ell} \varepsilon_\mu^{(j)}(k)\,\varepsilon_\nu^{(\ell)*}(k')\,\langle a_k^{(j)} a_{k'}^{(\ell)\dagger}\rangle \times$$
$$\times\, e^{-i(k\cdot x - k'\cdot y)}$$

other term $\langle 0|\, a^\dagger a\,|0\rangle$ is zero......

$$\langle a_k^{(j)} a_{k'}^{(\ell)\dagger}\rangle \equiv \langle 0|\, a_k^{(j)} a_{k'}^{(\ell)\dagger}\,|0\rangle = 2E_k\cdot(2\pi)^3\,\delta^3(\underline{k}-\underline{k}')\,\delta_{j\ell}$$

$\underline{k}$ is vector notation: $\underline{k}$ is same as $\bar{k}$ or say $\vec{k}$. (what you used in high school)

$$\langle a_k^{(j)} a_{k'}^{(\ell)\dagger}\rangle = 2E_k\cdot(2\pi)^3\cdot\delta_{j\ell}\cdot\delta^3(\underline{k}-\underline{k}')$$

$$D_{\mu\nu}(x,y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \sum_j \varepsilon_\mu^{(j)}(k)\, \varepsilon_\nu^{(j)}(k)^* \cdot e^{-ik\cdot(x-y)}$$

now; we make an argument, that for massless photon sum of polarization vector is just replaced by the metric tensor (because it can have only two polarizations)

i.e:

$$\boxed{\sum_j \varepsilon_\mu^{(j)}(k)\, \varepsilon_\nu^{(j)}(k)^* = -\eta_{\mu\nu}}$$

Massive Photon :

3 polarizations. (unlike the massless photon which only had 2)

Massive photon has mass; so we can describe those polarizations at rest.

At rest :

$$\left.\begin{aligned} \varepsilon_\mu^{(1)} &= (0,1,0,0) \\ \varepsilon_\mu^{(2)} &= (0,0,1,0) \\ \varepsilon_\mu^{(3)} &= (0,0,0,1) \end{aligned}\right\} \text{Choose a basis}$$

now; we can compute $\sum_j \varepsilon_\mu^{(j)}(k)\, \varepsilon_\nu^{(j)}(k)^*$ for our massive photon.

Since its at rest; we can put zero momentum there.

$$\sum_{j=1}^{3} \varepsilon_\mu^{(j)}(0)\, \varepsilon_\nu^{(j)*}(0) = \begin{pmatrix} 0 & & & \mathbf{0} \\ & 1 & & \\ & & 1 & \\ \mathbf{0} & & & 1 \end{pmatrix}_{\mu\nu} = \mathrm{diag}(0,1,1,1)$$

But we dont want just zero momentum. We want it as function of momentum because we have to integrate.

∴ ~~So;~~ So; we just need to Boost to momentum

$$k^\mu = (E_k, 0, 0, k)$$

Boost along z direction.

Boost Matrix $\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}$; $\gamma = \frac{E_k}{m_A}$

$\beta = \frac{k}{E_k}$

$$\sum_{j=1}^{3} \varepsilon_\mu^{(j)}(0)\, \varepsilon_\nu^{(j)}(0)^* \xrightarrow{\text{Boost}} \sum_{j=1}^{3} \varepsilon_\mu^{(j)}(k)\, \varepsilon_\nu^{(j)*}(k)$$

$$= \left[\Lambda \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Lambda \right]$$

$$= \begin{pmatrix} k^2/m_A^2 & 0 & 0 & \frac{k \cdot E_k}{m_A^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{k E_k}{m_A^2} & 0 & 0 & \frac{E_k^2}{m_A^2} \end{pmatrix}_{\mu\nu}$$

$$= -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{m_A^2}$$

massless vector boson :

$$\sum_i \varepsilon_\mu^{(i)} \varepsilon_\nu^{(i)*} = -\eta_{\mu\nu}$$

massive vector boson :

$$\sum_i \varepsilon_\mu^{(i)} \varepsilon_\nu^{(i)*} = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{m_A^2}$$

Massive vector boson propagator

$$D_{\mu\nu}(k) = \frac{-i\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m_A^2}\right)}{k^2 - m_A^2 + i\varepsilon}$$

Lec 2 | Spontaneously broken global symmetries

$$D_{\mu\nu}(k) = \frac{-i}{k^2 - m_A^2 + i\varepsilon} \cdot \left(\eta_{\mu\nu} - \underbrace{\frac{k_\mu k_\nu}{m_{A^2}}} \right)$$

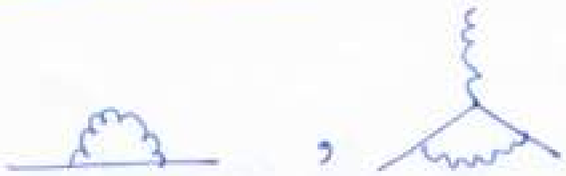
This is the term which will cause problem.

$\big\downarrow$ for large k

$$\frac{i\, k_\mu k_\nu}{m_A^2\, k^2}$$

(so it has completely different behavior at large k compared to massless gauge boson) $\Rightarrow$ This will spoil renormalizability of the theory.

Divergences in QED

 are divergences.

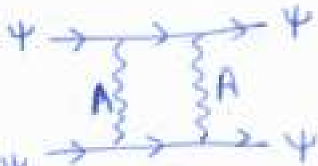
These are absorbed by redefining parameters of the theory into them.

So, Renormalize the wavefunction, electric charge, electron mass.

$\longrightarrow$ we can absorb all the divergences no matter how complicated the diagram is into redefinations of these finite set of things.

Consider 4-point function in massive QED at 1 loop

$\psi\psi \rightarrow \psi\bar{\psi}$



Box diagram

$$\sim g^4 \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \cdot \left(\frac{1}{\not{k}}\right)^2 \cdot \left(\frac{k_\mu k_\nu}{k^2}\right)^2$$

Λ = momentum cut-off

$\sim g^4 \Lambda^2$

$\rightarrow$ keep divergent term

Other diagrams cancel quadratic divergences.

→ one of the propagator flip sign ... and they cancel each other

... this sign flip cancels the leading quadratic divergence; but still leaves something which is log divergent.

We dont have linear divergent part due to symmetry (odd no. of k's; so becomes zero)

⇒ so, the leading divergent is log divergence.

ie:

$$\frac{-\not{k} + m_\psi}{k^2 - m^2_\psi} \qquad \frac{+\not{k} + m_\psi}{k^2 - m^2_\psi}$$

* The $\not{k}$ do not contribute to the divergence because they cancel out.
* The linear term divergence comes from ~~other p~~ other part of propagator; but it is zero due to symmetry.

So; we only have $\log \Lambda^2$ divergence.

 $\propto g^4 m^2_\psi \cdot \log \Lambda^2$

So, 4 point functions have divergence. We cannot ~~absorb it~~ absorb it into any other parameter of the theory. So if we want to renormalize the theory; we need to introduce a counter term in the Lagrangian that cancels this divergence.

<u>Divergence absorbed by counter-term:</u>

Need dimension - 6 term in $\mathcal{L}$:

$$\mathcal{L} = c \cdot \bar{\Psi}\Psi\bar{\Psi}\Psi$$

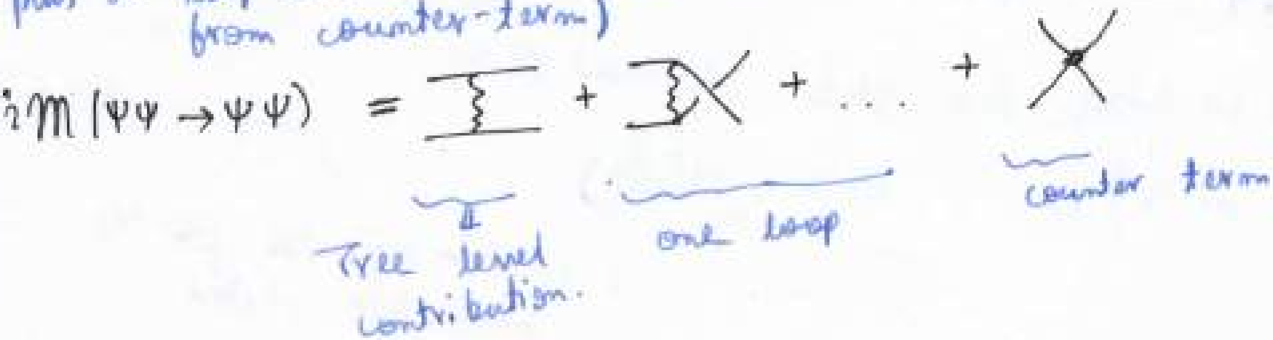
(four fermion operator because we want to cancel the logarathmic divergence of <u>4 point functions</u>)

We can absorb the divergence by renormalizing c.

c has mass dimension -2.

So; even though we started out with a theory in which all of the couplings of interactions has mass dimensions positive or zero: I must introduce higher dimensional operators in my theory to absorb divergences that appear at one loop.

(Then; if we want to compute maxtrix element for scattering process; we must include tree level contributions plus a loop contribution; & also ~~we~~ has to introduce contribution from counter-term)



It's not possible to predict magnitude of scattering process from first principle ~~because~~ based on the parameters of Lagrangian we had before: because we have this new parameter c. And we have to fix that by doing experimental measuring that will allow to determine the coupling c.

Spontaneous Symmetry Breaking

Violating gauge symmetry explicitly (at Lagrangian level) introduces problems. However symmetry can be spontaneously broken even if the Lagrangian is symmetric.

"We will violate the symmetry in such a way that will avoid the problem we introduced in last few pages."

Start with a theory that is invariant under a symmetry (such as gauge symmetry). Ground state where symmetry is broken may have lower energy compared to state where symmetry is un-broken.

Let, U be a symmetry transformation that leaves the hamiltonian H invariant.

$$H \longrightarrow U H U^{\dagger} = H$$

suppose we have two states $|A\rangle$ and $|B\rangle$.

$$|A\rangle = a^{\dagger}_{A}|0\rangle \quad , \quad |B\rangle = a^{\dagger}_{B}|0\rangle$$

creation operator for the respective states.

~~Think about under what cond cond~~ condition

Want $|A\rangle, |B\rangle$ related by U : $U|A\rangle = |B\rangle$

First suppose $U a_A^\dagger U^\dagger = a_B^\dagger$

$\hookrightarrow$ is it sufficient to ensure $U|A\rangle = |B\rangle$

lets check...

Then $U|A\rangle = U a_A^\dagger \underbrace{U^\dagger U}_{\mathbb{1}} |0\rangle = a_B^\dagger U|0\rangle$

if we make further assumption that vaccum is invariant under symmetry transformation ; i.e; $U|0\rangle = |0\rangle$
Assumption!

Then $\boxed{U|A\rangle = |B\rangle}$

~~It has~~ If vaccum is not invariant under U, then you cannot ~~read~~ relate the two states with one another.

$U|A\rangle = |B\rangle$ only if vaccum is invariant under U

If this is true ; then $E_A = E_B$

$E_A = \langle A|H|A\rangle = \langle B|\underbrace{UHU^\dagger}_{H}|B\rangle = \langle B|H|B\rangle = E_B$.

If ground state preserves the symmetry ; then the symmetry will be manifest in the spectrum of states.

If ground state violates symmetry ; the symmetry is no longer manifest in spectrum.

However; there will be symmetry relations that reflect hidden symmetry of the orignal theory.

So; if your are violating symmetry here ; its because the vaccum state itself violates the symmetry. ~~So~~ ; ~~but~~ ~~its~~ But its not the same thing as if you violate the symmetry by hand. All the parameters that appear in the theory remember that they ultimately come from a theory that was originally symmetric.

Discrete Symmetry

Theory with real scalar field ϕ and $\mathbb{Z}_2$ symmetry ($\phi \to -\phi$)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4$$

($\mathbb{Z}_2$ symmetry forbids linear & cubic term)

No linear / cubic term.

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}|\vec{\nabla}\phi|^2 + V(\phi)$$: Hamiltonian density.

where; $V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4$

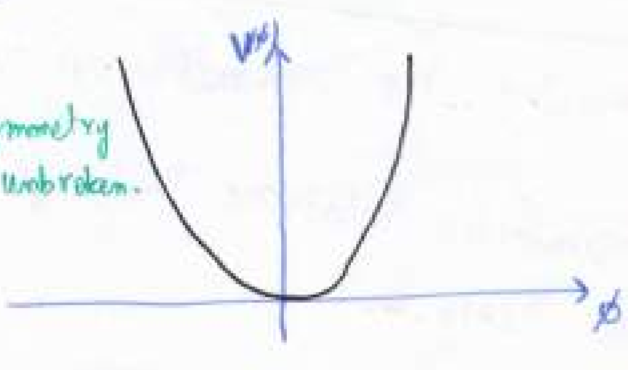
~~to minimize~~ Energy is minimized if ϕ = constant

(ie; $\dot{\phi}, \vec{\nabla}\phi = 0$)

now; we just need to worry about minimizing potential $V(\phi)$

$\lambda > 0$ (otherwise $\mathcal{H}$ unbounded from below)

μ^2 can have either sign.

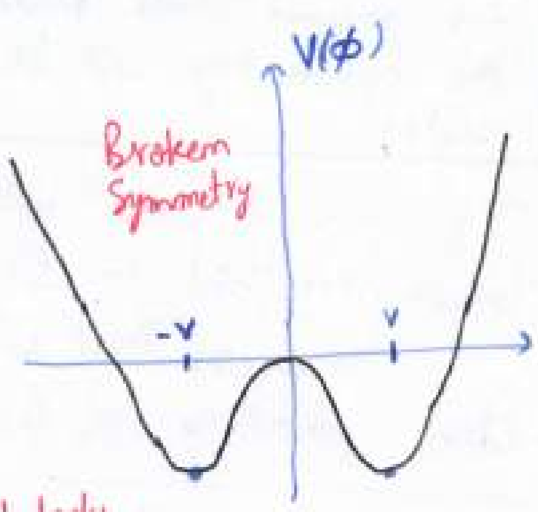


Symmetry is unbroken.

$\mu^2 > 0$

minimum at $\phi = 0$

$\langle 0|\phi|0\rangle = 0$



Broken Symmetry

potential looks different on both side for both the minima point.

$\mu^2 < 0$

minimum at $\phi = v \neq 0$.

i.e; $\langle 0|\phi|0\rangle = v$

$$\left.\frac{\partial V}{\partial \phi}\right|_{\phi = v} = 0 \Rightarrow \mu^2 v + \lambda v^3 \Rightarrow v = \pm\sqrt{\frac{-\mu^2}{\lambda}}$$

v = vacuum expection value.

Physical states are small oscillations about the ground state.

Expand around $\phi = v$:

Define a shifted fields: $\phi = v + \phi'$ (Broken case)

write theory in terms of ϕ'

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi')^2 - V(v + \phi')$$

$$= \frac{1}{2}(\partial_\mu \phi')^2 - \frac{1}{2}\mu^2 (v + \phi')^2 - \frac{1}{4}\lambda \cdot (v + \phi')^4$$

$$= \frac{1}{2}(\partial_\mu \phi)^2 - (-\mu^2)\phi'^2 - \lambda \cdot v \cdot \phi'^3 - \frac{\lambda}{4}\phi'^4$$

we no longer have $\mathbb{Z}_2$ symmetry for our new field.

① No $\mathbb{Z}_2$ symmetry; is broken for ϕ'

→ we see not all parameters are independent from one another. And thats how the symmetry is remembered.

② Relations between parameters are remnant of original symmetry.

Mass of ϕ' : $m^2_{\phi'} = -2\mu^2 = 2\lambda \cdot v^2$

Still only two parameters. (λ, μ^2) or $(m^2_{\phi'}, v)$

$\Rightarrow$ 3 interactions in $\mathcal{L}$ defined in terms of 2 parameters. This is how symmetry is remembered.

Abelian Global Symmetry

Complex scalar ϕ : $\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2$

$\mathcal{L}$ is invariant under U(1) symmetry: $\phi \to e^{i\alpha} \cdot \phi$

Assume $\mu^2 > 0 \Rightarrow$ so we have spontaneous symmetry breaking.

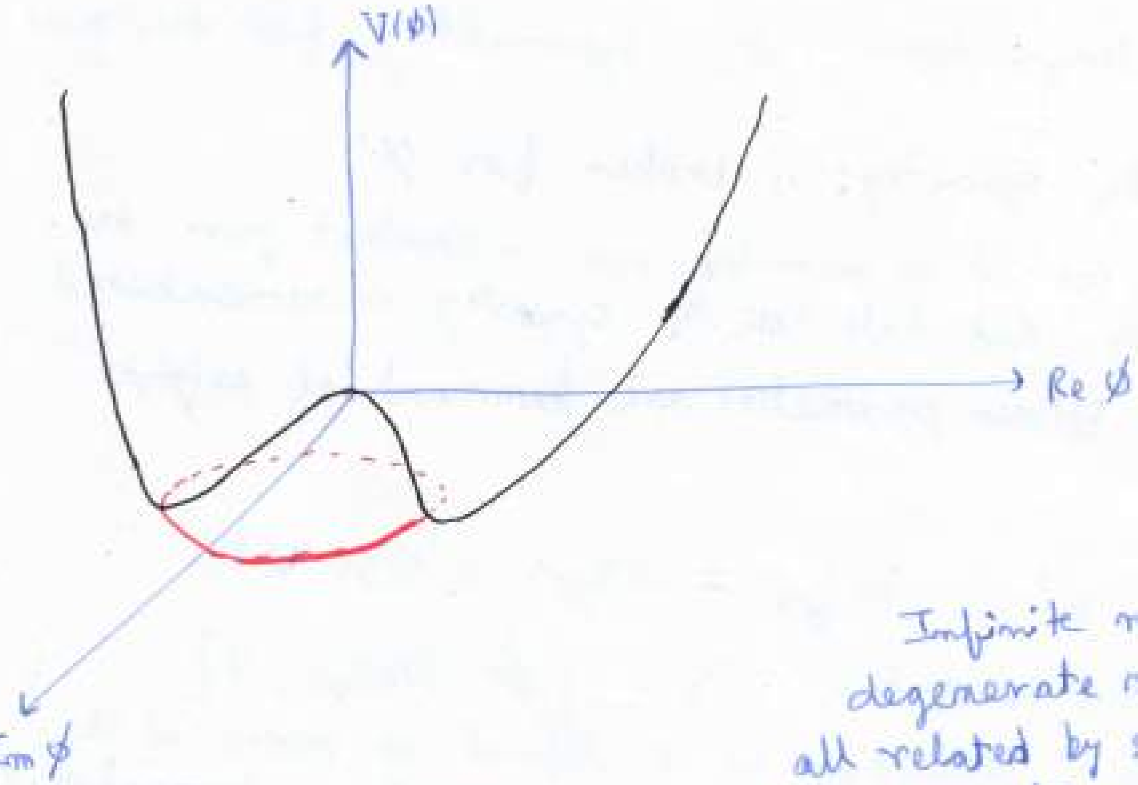
$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda \cdot (\phi^\dagger \phi)^2$$

$$\frac{\partial V}{\partial \phi} = (-\mu^2 + 2\lambda \phi^\dagger \phi)\phi^\dagger = 0$$

$\Rightarrow |\phi| = \sqrt{\frac{\mu^2}{2\lambda}}$ $\Rightarrow$ This will be our vacuum expectation value.

$\rightarrow$ lets normalize it by $\sqrt{2}$. (convention)

$$|\phi| = \sqrt{\frac{\mu^2}{2\lambda}} = \frac{v}{\sqrt{2}}$$



Infinite number of degenerate minima all related by symmetry transformation.

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$$

free to assume only ϕ_1 gets vacuum expectation value (VEV) because we are free to use symmetry transformation.

$$\langle 0|\phi_1|0\rangle = \frac{v}{\sqrt{2}} = \sqrt{\frac{\mu^2}{\lambda}}$$

Expand $\mathcal{L}$ in terms of shifted fields:

$\phi_1 = v + \phi_1'$; $\phi_2 = \phi_2'$
(no need to shift because vev of ϕ_2 is zero)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1')^2 + \frac{1}{2}(\partial_\mu \phi_2')^2 - \mu^2 \phi_1'^2 - \lambda \cdot v\, \phi_1' (\phi_1'^2 + \phi_2'^2) - \frac{\lambda}{4}(\phi_1'^2 + \phi_2'^2)^2$$

$m^2_{\phi_1'} = 2\mu^2$

$m^2_{\phi_2'} = 0$; ϕ_2' is massless. No energy cost to travel along minimum.

Goldstone's Theorem

For every symmetry that's broken spontaneously there is one massless degree of freedom (Goldstone Boson) for every generator that is no longer a symmetry

ex U(1) had one generator $\longrightarrow$ one Goldstone boson.

Standard Model

Lec 3 | Abelian Higgs Model

Abelian Higgs Model

Complex scalar ϕ with local $U(1)$ gauge symmetry.

$$\mathcal{L} = (D_\mu \phi^\dagger)(D^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

$$D_\mu = \partial_\mu + igA_\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2$ has minimum as

$$|\phi| = \sqrt{\frac{\mu^2}{2\lambda}} = \frac{v}{\sqrt{2}}$$

Write $\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$ and shift the field around the vev.

$$\phi_1 = v + \phi_1' \; ; \; \phi_2 = \phi_2' .$$

Covariant derivative term:

$$\mathcal{L} \supset |D_\mu \phi|^2 = \left| (\partial_\mu + igA_\mu)\left(\frac{v + \phi_1' + i\phi_2'}{\sqrt{2}}\right)\right|^2$$

$$= \frac{1}{2}\left| \partial_\mu \phi_1' + i\partial_\mu \phi_2' + ig \cdot vA_\mu + i \cdot g A_\mu \phi_1' - gA_\mu \phi_2' \right|^2$$

$$\mathcal{L} \supset |D_\mu \phi|^2 = \frac{1}{2}\left|\partial_\mu \phi_1' + i\partial_\mu \phi_2' + igv \cdot A_\mu + igA_\mu \phi_1' - gA_\mu \phi_2'\right|^2$$

$$= \frac{1}{2}(\partial_\mu \phi_1')^2 + \frac{1}{2}(\partial_\mu \phi_2')^2 + g \cdot v A^\mu \cdot \partial_\mu \phi_2' + \underline{\frac{1}{2} g^2 v^2 A_\mu A^\mu} + \ldots \text{cubic \& quadratic terms}$$

↳ vev for ϕ generates mass term for photon.

$$m_A^2 = g^2 v^2$$

(So: we obtained mass term for photon not by explicitly writing it down; but by spontaneous symmetry breaking through scalar field.)

Eliminate the mixing term $\sim (\partial_\mu \phi_2') A^\mu$ to put $\mathcal{L}$ in canonical form.

Expand ϕ in polar form : $\phi = \frac{1}{\sqrt{2}} (v + h(x)) \cdot e^{i \xi(x)/v}$

Shift in magnitude

So: The fields are here $\xi(x)$ & $h(x)$.

we divide by v in exp( ) so that $\xi(x)$ has mass dimension 1

$$\phi = \frac{1}{\sqrt{2}} (v + h(x)) \cdot e^{i \xi(x)/v}.$$

For small fluctuations we can expand linearly.

$$\phi = \frac{1}{\sqrt{2}} (v + h + i \xi + \dots)$$

for small fluctuation we have h & ξ as real & imaginary part

Make ϕ purely real by gauge transformation.

$$\phi \to e^{-i\xi/v} \cdot \phi = \frac{1}{\sqrt{2}} (v + h(x))$$

The gauge in which ϕ is purely real is called the Unitary gauge.

$$A_\mu \to A_\mu' = A_\mu + \frac{1}{gv} \cdot \partial_\mu \xi$$

In this gauge choice; expand $\mathcal{L}$.

$$D_\mu \phi \to (\partial_\mu + i g A_\mu') \frac{1}{\sqrt{2}} (v + h)$$

Eliminated ξ from $\mathcal{L}$. Expand covariant derivative term, actually lets expand the entire Lagrangian.

$$\mathcal{L} = \frac{1}{2}\left|(\partial_\mu + igA_\mu)(v+h)\right|^2 + \frac{\mu^2}{2}(v+h)^2 - \frac{\lambda}{4}(v+h)^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

(drop prime)

$$\Rightarrow \mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}g^2v^2A_\mu A^\mu + g^2 v A_\mu A^\mu h + \frac{1}{2}g^2h^2A_\mu A^\mu}_{\text{This is kinetic term expansion.}}$$

$$+ \left(\frac{\mu^2}{2} - \frac{3}{2}\lambda v^2\right)h^2 - \lambda v \cdot h^3 - \frac{\lambda}{4}h^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Recall; $v^2 = \dfrac{\mu^2}{\lambda}$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 - \mu^2 h^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}g^2v^2A_\mu A^\mu + \mathcal{L}_{int}$$

$m_h = \sqrt{2\mu^2}$; $m_A = g \cdot v$

$$\mathcal{L}_{int} = g^2 v \cdot A_\mu A^\mu h + \frac{1}{2}g^2h^2A_\mu A^\mu - \lambda v h^3 - \frac{\lambda}{4}h^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

There is no symmetry manifest. But the coefficient of each of interaction terms are described in terms of fewer number of fundamental parameters (This remembers the symmetry)

Interactions fixed in terms of few parameters (g, μ^2, λ) or (m_h^2, m_A^2, v)

~~[illegible]~~ Once you fix 3 parameters ; you fix all the terms.

Degrees of freedom:

massless A_μ (2 dof because of 2 polarization) + complex ϕ (2 dof) } Initially we started out with this.

And we ended with a theory with.

massive A_μ + Real scalar
(3 dof) (1 dof)

so; counting of dof is consistent .. we always ~~have~~ have 4.

massless A_μ + complex ϕ $\longrightarrow$ massive A_μ + real scalar h
(2 dof) + (2 dof) = (3 d.o.f) + (1 d.o.f)

We see now, that why one of the d.o.f. of ~~unbroken~~ unbroken theory; the phase of ϕ gets removed from the Lagrangian. $\Rightarrow$ It becomes the degree of freedom corresponding to the longitudinal polarization of the gauge field.

Field ξ gets "eaten" by gauge field to become longitudinal polarization.

(This is would-be Goldstone; except its get eaten away)

The real scalar field left over is Higgs Field.

Fermions: If ψ_L, ψ_R transform differently under local U(1) then $\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$ is forbidden.

Use spontaneous symmetry breaking (to generate mass term for fermion as well)

Notation: $g_L = g \cdot Q_L$ ← charge
$g_R = g \cdot Q_R$

($Q_L \neq Q_R$ → Chiral gauge symmetry)

Q_L & Q_R represents quantum number under gauge groups.

	Quantum Numbers
ϕ :	$Q_\phi = +1$
Ψ_L :	Q_L
Ψ_R :	Q_R

Gauge transformation

$$\phi \to e^{-igQ_\phi \cdot \alpha}\phi$$

$$\Psi_{L,R} \to e^{-igQ_{L,R}\cdot\alpha}\cdot\Psi_{L,R}$$

In principle these three charges Q_ϕ, Q_L, Q_R can be any number. (Its like particles have different electric charges)

If we make a very particular choice of these charges, then we can write another interaction term in the $\mathcal{L}$; the Yukawa Interaction.

If $Q_L = Q_R + Q_\phi$; then ~~we can write another~~ Yukawa Interaction is gauge invariant.

$$\mathcal{L}_{Yukawa} = -y\bar{\Psi}_L \Psi_R \phi + h.c.$$

means Hermitian conjugate.

$y \Rightarrow$ Yukawa Coupling.

i.e; $\mathcal{L}_{Yukawa} = -y\bar{\Psi}_L\Psi_R\phi - y\phi^\dagger\bar{\Psi}_R\Psi_L$

$\Big\downarrow$ gauge transformation

$$e^{ig(Q_L - Q_R - Q_\phi)\cdot\alpha}\cdot\bar{\Psi}_L\Psi_R\phi$$

It is invariant only if $Q_L = Q_R + Q_\phi$

$\therefore$ Now, we assume $Q_L = Q_R + Q_\phi$ to be true; and ask what happens to the interaction after symmetry breaking.

again working in Unitary gauge

$$\mathcal{L}_{Yukawa} = \frac{-y}{\sqrt{2}} \bar{\psi}_L \psi_R \cdot (v+h) + h.c. = -\frac{yv}{\sqrt{2}} \bar{\psi}\psi - \frac{y}{\sqrt{2}} \cdot h \cdot \bar{\psi}\psi$$

$\Rightarrow$ fermion mass $m_\psi = \dfrac{yv}{\sqrt{2}}$

Spontaneous Symmetry breaking generates term in $\mathcal{L}$ that violates gauge symmetry.

Consequences

① Not all interaction coefficients are independent

So the theory remembers that it came from an originally symmetric theory, so there are relationships between coefficients of different interactions in the ~~Lag~~ Lagrangian. They can be described by fewer parameters, than you would naively write down if you had just violated the symmetry explicitly.

② Real scalar field h is left over after symmetry breaking

Renormalizability:

$\psi\psi \to \psi\psi$

Box diagram

The divergences come from leading term of the propagator

$$\frac{-i}{k^2-m_A^2} \cdot \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m_A^2}\right) \xrightarrow[\text{large } k]{} \frac{k_\mu k_\nu}{m_A^2 k^2}$$

$$\sim g^4 \int \frac{d^4k}{(2\pi)^4} \cdot \left(\frac{\not{k} + m_\psi}{k^2}\right)^2 \left(\frac{k_\mu k_\nu}{m_A^2 k^2}\right)^2$$

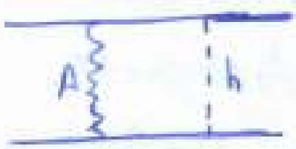
$$\sim \frac{g^4 m_\psi^2}{m_A^4} \log \Lambda^2 \sim \frac{g^4 y^2 v^2}{g^4 v^4} \log \Lambda^2$$

(log divergence) (g's cancel out)

$$\sim \frac{y^2}{v^2} \log \Lambda^2 .$$

(we still have divergence)

Extra contributions:



$$+ \text{ permutations } \sim g^2 y^2 \underbrace{\int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k}\right)^2 \left(\frac{k_\mu k_\nu}{k^2 m_A^2}\right) \frac{1}{k^2}}_{\text{at large } k}$$

$m_A = g \cdot v$

$$\sim \frac{g^2 y^2}{m_A^2} \log \Lambda^2$$

$$= \frac{y^2}{v^2} \log \Lambda^2$$

We are effectively doing dimensional analysis and which coupling constant appears.

But the argument is that:

Because we have this additional degree of freedom h, and because the coupling are all related to each other; the divergence that appear "  " here is exactly the same what needs to be cancelled by the divergence which occur "  " here.

Divergences cancel out since masses m_A, m_ψ ~~and~~ and

~~g, y are related~~ couplings g, y are related to each other.

In a sense; it was obvious that they should cancel out because we started with a theory which was invariant under gauge symmetry. The divergences are sensitive to high momenta: much larger than ~~say~~ vev lets say.

And in that limit, when momentum is very large; it does not matter whether the theory is spontaneously broken or not

So, we could have done calculation of this box diagrams in unbroken theory as far as computing the divergences.

And because that unbroken theory was renormalizable, we are gauranteed that no new divergences will appear from this box diagram

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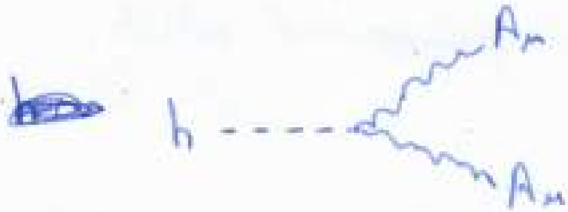
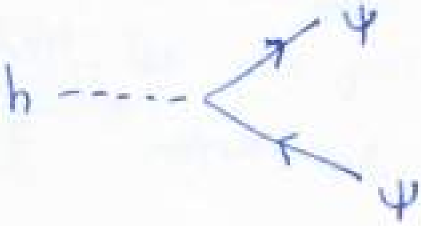
~~Prediction~~:
Prediction: If gauge theory is spontaneously broken; we have extra scalar h whose couplings are fixed by masses



$$= \frac{2 i m_A^2}{v} \eta_{\mu\nu}$$



$$= \frac{i m_\psi}{v}$$

So, the spontaneously broken theory predicts that there should be this extra scalar: the Higgs boson. And the couplings of that Higgs boson to any of the other field ~~that~~ that it gives mass to are related to the mass itself.

So: Its giving mass to gauge boson: The coupling proportional to mass.

giving mass to fermions: coupling is proportional to mass.

So, its a predictive theory. Once we measure masses of these different particles, i.e. fermions & gauge bosons; all of the couplings of Higgs boson are predicted once we know the vev.

Non-abelian Higgs model

ex) SU(2) gauge theory + complex scalar field (which will be in fundamental representation of gauge group) so it will be a doublet ϕ_1 & ϕ_2.

$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ doublet of complex field ϕ_1, ϕ_2

$$\mathcal{L} = (D_\mu \Phi^\dagger)(D^\mu \Phi) - V(\Phi) - \frac{1}{2} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})$$

→ Gauge Kinetic part.

$$D_\mu \Phi = (\partial_\mu + i g T^a A_\mu^a)\Phi \quad , \quad T^a = \frac{\sigma^a}{2}$$

(Generators)

$\sigma^a \Rightarrow$ Pauli matrices.

$\cancel{F^a_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a}$

$$F^a_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\, \varepsilon^{abc} A_\mu^b A_\nu^c$$

Potential : $V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$

we want to minimize the potential : $\Phi^\dagger \Phi = \frac{v^2}{2}$

where : $v = \sqrt{\frac{\mu^2}{\lambda}}$

~~free to make gauge transformation~~

Expand the scalar field as follows

$$\Phi(x) = \exp\left(i T^a \xi^a(x)/v\right) \cdot \begin{pmatrix} 0 \\ \frac{v + h(x)}{\sqrt{2}} \end{pmatrix}$$

Three $\xi^a(x)$ s
$a = 1,2,3$
(3 d.o.f)

+

$h(x)$
(2 d.o.f)

$= 2 \times (2 \text{d.o.f})$

since ϕ_1 & ϕ_2 are complex; so each have 2 d.o.f ; so Total $2+2=4$

This will end up being the physical higgs field that is left over

ξ^1, ξ^2, ξ^3, h are real fields.

Make gauge transformation : $\Phi(x) \to e^{-i T^a \xi^a / v} \cdot \Phi = \begin{pmatrix} 0 \\ \frac{v + h(x)}{\sqrt{2}} \end{pmatrix}$

so, we just made a gauge transformation to a gauge where these fields are purely real, hence Unitary Gauge. And its only non-zero in its second component.

Lec 4| Non abelian Higgs model, Standard Model Higgs Sector.

Non-abelian Higgs model

$$\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi) \quad ; \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \text{complex doublet.}$$
$$-\frac{1}{2} Tr[F_{\mu\nu} F^{\mu\nu}]$$

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$$

$$D_\mu \Phi = (\partial_\mu + ig T^a A^a_\mu)\Phi$$

So; we minimize the potential. We want $\Phi^\dagger \Phi = \frac{v^2}{2}$;

v is vev ; $v = \sqrt{\frac{\mu^2}{\lambda}}$

Write $\Phi = \exp(i T^a \xi^a(x)/v) \cdot \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix}$; $a = 1,2,3$

So; we have four real fields ; $h(x)$, $\xi^{(a)}(x)$; $a = 1,2,3$

Gauge transformation : $\Phi \longrightarrow \exp(-iT^a\xi^a(x)/v) \cdot \Phi = \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix}$

Covariant derivative term:

$$\mathcal{L} \supset (D_\mu \Phi)^\dagger (D^\mu \Phi)$$

$$= \frac{1}{2}(0, v+h)(\overleftarrow{\partial}_\mu - ig T^a A^a_\mu)(\overrightarrow{\partial}_\mu + ig T^b A^b_\mu)\begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

$$= \frac{1}{2}(\partial_\mu h)^2 + \frac{g^2}{8}(0, v+h)\begin{pmatrix} A^3_\mu & A^1_\mu - iA^2_\mu \\ A^1_\mu + iA^2_\mu & -A^3_\mu \end{pmatrix}^2 \begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

$$= \frac{1}{2}(\partial_\mu h)^2 + \frac{g^2}{8}(v+h)^2 \underbrace{\left((A^1_\mu)^2 + (A^2_\mu)^2 + (A^3_\mu)^2 \right)}_{}$$

all three gauge bosons get mass

$$= \frac{1}{2}(\partial_\mu h)^2 + \frac{m_A^2}{2}\left((A^1_\mu)^2 + (A^2_\mu)^2 + (A^3_\mu)^2\right) \cdot \left(1 + \frac{h}{v}\right)^2 \quad ; \quad m_A = \frac{gv}{2}$$

So, we have given mass to all three gauge bosons in the theory.

SU(2) has 3 generators, so have 3 gauge fields.

∴ look at propagator of gauge fields in fourier space.
You have p^2 sign opposite to m^2 term.

So; it turns out if you look at kinetic term of gauge bosons you have $-p^2$; so we want sign of of mass term to be $+m_A^2$

$-p^2 + m_A^2$; so the pole is at ~~p~~
$p^2 = m_A^2$.

Unbroken Theory: 2×3 gauge degrees of freedom
for A^1 & A^2 & A^3
& for each of them we have two polarization.

We also have complex scalar doublet: $2 \times 2_{\mathbb{C}}$ because each one of them is complex.

∴ 2×3 gauge d.o.f + 2×2 real scalar d.o.f. = 10

Broken Theory: 3×3 gauge d.o.f + 1 scalar d.o.f = 10
(f)

2×3 d.o.f + 2×2 d.o.f. = 3×3 gauge d.o.f + 1 scalar d.o.f.

Almost like S.M.

Here 3 massive gauge bosons A^1, A^2, A^3 ; $A^1 = A^2 = A^3$
SM — " — " — " — W^+, W^-, Z ; $M_W \neq Z$

In S.M. we also have massless gauge field photon; whereas; here we dont have any massless gauge field ~~left~~ left over in this theory.

So, we need to extend our Gauge group if we want to describe S.M.

Standard Model Lagrangian

Gauge Group : (W, Z, γ + gluons)

D.o.f : (fermions)

What are quantum numbers for fermions/scalar.

This is what we need to know first to build lagrangian.

Gauge group : $G = \underbrace{SU(3)_c}_{QCD} \times \underbrace{SU(2)_L \times U(1)_Y}_{\text{Electroweak}}$

c= color

L = left , Y = hypercharge.

(Three different gauge groups; and the total group is the product of these)

Fields	Quantum Numbers $(SU(3)_c, SU(2)_L, U(1)_Y)$
$Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix}$ (Left handed quark field) doublet under $SU(2)_L$ has two components under $SU(2)_L$; i.e: U_L & d_L	$(3, 2, \frac{1}{6})$
U_R^i (Right handed up quark field)	$(3, 1, \frac{2}{3})$
d_R^i (Right handed down quark field)	$(3, 1, -\frac{1}{3})$

All of the quarks are in fundamental representation of SU(3).

Only left handed fields transform under SU(2) part of gauge group.

; i = 1, 2, 3 labels generation.

$L_L^i = \begin{pmatrix} \nu_L^i \\ e_i \end{pmatrix}$ (left handed lepton doublet)

$(1, 2, -\frac{1}{2})$ — singlet under SU(3); doublet under SU(2); hypercharge $-1/2$

$\nu_L \Rightarrow$ neutrino

$e_L \Rightarrow$ charged lepton; the electron.

e_R^i

$(1, 1, -1)$ — " SU(3); singlet under SU(2); hypercharge -1.

In S.M.; we won't have any neutrino masses. So, only neutrino that would matter if its massless is just the left handed component ν_L.

If we did want to include neutrino masses (remember mass term in Lagrangian pairs up left handed & right handed fields); so we would need to introduce Right Handed Neutrino in the theory as well.

So; If we want to include; it must be

$$(\nu_R \qquad (1, 1, 0))$$

→ for the momentum, lets through it away

The S.M. is defined in sense of massless neutrino just historically; just because we dont know how to include contribution of mass into the Lagrangian for this particular field. → so; we neglect it for now.

$H = \begin{pmatrix} H^+ \\ H^- \end{pmatrix}$ $\quad (1, 2, 1/2)$ Complex doublet scalar Higgs field (sames as Φ)

Gauge Bosons

$SU(3)_c$: gluon fields g_μ^A $\quad$ $A \Rightarrow$ group index.

(gluon field is in adjoint representation of $SU(3)$ and there are 8 generators for it for $SU(3)$)
so; $(A = 1, 2, \ldots 8)$

$SU(2)_L$: W_μ^a $\quad$ $a = 1, 2, 3$

(This will be in adjoint representation of $SU(2)$)
so; $a = 1, 2, 3$

$U(1)_Y$: B_μ

Each of the gauge group will have there own gauge coupling.

		couplings
so; $SU(3)_c$	:	g_s
$SU(2)_L$	:	g
$U(1)_Y$	:	g'

These choices works to describe phenomenological choices; so we choose these values

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{fermion} + \mathcal{L}_{scalar} + \underbrace{\mathcal{L}_{Yukawa}}$$

we will see that this will be necessary to give masses to ~~different~~ different fermions.....

~~$\mathcal{L}_{gauge} = -\frac{1}{2}\mathrm{Tr}(g_{\mu\nu}g^{\mu\nu}) - \frac{1}{2}\mathrm{Tr}(\ldots)$~~

The gauge part is exactly what we expect for gauge theory. We have different kinetic term for each of the gauge group.

$$\mathcal{L}_{gauge} = -\frac{1}{2}\mathrm{Tr}(g_{\mu\nu}g^{\mu\nu}) - \frac{1}{2}\mathrm{Tr}(W_{\mu\nu}W^{\mu\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

$g_{\mu\nu}$ is field strength for $SU(3)_c$

$W_{\mu\nu}$ " " " " $SU(2)_L$

$B_{\mu\nu}$ " " " " Hypercharge $U(1)$

$$\mathcal{L}_{fermion} = \sum_{fermion\ \Psi} \bar{\Psi}\, i \not{D} \Psi$$

→ dirac lagrangian for massless fermion

(sum over all different fermion species)

(mass term here are forbidden by ~~mass~~ explicitly writing them here by hand here; Mass terms according to gauge symmetry are forbidden: They are forbidden because I have chiral interactions of gauge field; left & right ~~hand~~ handed field transform different as we can see because they have different quantum numbers) ... but of course we still have the kinetic term (mass term forbidden due to chiral gauge interaction)

~~$\mathcal{L}_{scalar} = (D_\mu H)^\dagger (D_\mu \ldots$~~

$$\mathcal{L}_{scalar} = (D_\mu H)^\dagger (D^\mu H) - V(H)$$

where; $V(H) = -\mu^2 H^\dagger H + \lambda \cdot (H^\dagger H)^2$

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{fermion} + \mathcal{L}_{scalar} + \mathcal{L}_{Yukawa}.$$

$$\mathcal{L}_{gauge} = -\frac{1}{2}\mathrm{Tr}(g_{\mu\nu}g^{\mu\nu}) - \frac{1}{2}\mathrm{Tr}(W_{\mu\nu}W^{\mu\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

$$\mathcal{L}_{fermion} = \sum_{\text{fermion } \psi} \bar{\psi}\, i \not{D}\, \psi$$

$$\mathcal{L}_{scalar} = (D_\mu H^\dagger)(D^\mu H) - V(H)$$

where: $V(H) = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2$

Now, first consider scalar part of lagrangian to see how we get masses for gauge fields.

Scalar Lagrangian

Work in unitary gauge : $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}$; $v = \sqrt{\frac{\mu^2}{\lambda}}$

$$D_\mu H = \Big(\partial_\mu + \frac{ig}{2}\underbrace{\sigma^a}_{\text{generator } \frac{\sigma^a}{2}} \underbrace{W^a_\mu}_{\text{gauge field}} + ig'\underbrace{\tfrac{1}{2}}_{\text{quantum number for charge of this field under } U(1)_Y} \underbrace{B_\mu}_{\text{gauge field}}\Big) H$$

Its complicated than non-abelian Higgs model that we did before. There we had only one gauge group to worry about the SU(2). Here we have two gauge ~~to worry~~ groups to worry about since Higgs field transform under both $SU(2)_L$ & $U(1)_Y$. So; we need to write down two gauge interaction in the covariant derivative.

So: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}$

So:

$$D_\mu H = \left(\partial_\mu + \frac{ig}{2}\sigma^a W^a_\mu + ig'\cdot\frac{1}{2}B_\mu\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

~~$D_\mu H = \left(\partial_\mu + \frac{i}{2}\left(gW^3_\mu + g'B_\mu \quad g(W^1_\mu - iW^2_\mu) \quad g(W_\mu^1 + iW_\mu^2)\right)\right)$~~

now, we expand $D_\mu H$ as a matrix acting on the doublet $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\v+h\end{pmatrix}$

$i\frac{g'}{2}B_\mu$ acting as a matrix on $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\v+h\end{pmatrix}$

is just $i\frac{g'}{2}B_\mu\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

so:

Hypercharge guage fields lives on diagonal

$$D_\mu H = \left(\partial_\mu + \frac{i}{2}\begin{bmatrix} gW^3_\mu + g'B_\mu & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -gW^3_\mu + g'B_\mu\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\v+h\end{pmatrix}$$

$SU(N) = \mathbf{1}$: trivial representation.

N : Fundamental "

$U(1) = 0$: trivial representation

Q : charge Q.

so; if something is purely a singlet under everything; like Right Handed neutrino field, it will be $(1, 1, 0)$ That would be trivial under everything.

Higgs field : $H = (1, 2, 1/2)$

hypercharge $\frac{1}{2}$

doublet under $SU(2)$

singlet under $SU(3)$

$$(D_\mu H^\dagger)(D^\mu H) = \frac{1}{2}(\partial_\mu h)^2 + \cancel{\frac{g^2 v^2}{8}} \cdots \left(\begin{array}{c}\text{Mass term} \\ \text{for gauge} \\ \text{fields}\end{array}\right)$$

Kinetic terms for higgs field.

$\therefore \frac{1}{2}(\partial_\mu h)^2$: Kinetic term for Higgs field.

$$(D_\mu H^\dagger)(D^\mu H) = \frac{1}{2}(\partial_\mu h)^2 + \frac{g^2 v^2}{8}\left((W^1_\mu)^2 + (W^2_\mu)^2\right)\cdot\left(1+\frac{h}{v}\right)^2$$
$$+ \frac{1}{8}v^2\cdot\left(g\cdot W^3_\mu - g'B_\mu\right)^2\cdot\left(1+\frac{h}{v}\right)^2$$

$$\boxed{\begin{aligned}(D_\mu H^\dagger)(D^\mu H) = \frac{1}{2}(\partial_\mu h)^2 + \frac{g^2 v^2}{8}\left((W^1_\mu)^2 + (W^2_\mu)^2\right)\cdot\left(1+\frac{h}{v}\right)^2 \\ + \frac{1}{8}v^2\cdot\left(g\cdot W^3_\mu - g'\cdot B_\mu\right)^2\cdot\left(1+\frac{h}{v}\right)^2\end{aligned}}$$

→ now we want effectively a mass matrix or mass terms that are ~~diagd~~ diagonal in gauge fields.

But, here we see they are not diagonal.
there are terms like factor B_μ & W^3_μ like $(gW^3_\mu - g'B_\mu)^2$

↳ So; in order to figure out masses; we want to diagonalize these mass terms.

So; it is convenient for us to make a field redefination.
The mass eigenstate of these theory will not be W_μ s and B s.
But will be linear combination of these fields.

So, we need to do rotations

We define new fields Z_μ and A_μ

Rotation ... rotating two components by an angle, denoted by θ_W

W ⇒ Weak Mixing Angle.

and also have another orthogonal component A_μ.

$$\boxed{\begin{aligned} Z_\mu &= \cos\theta_W \cdot W_\mu^3 - \sin\theta_W \cdot B_\mu \\ A_\mu &= \sin\theta_W \cdot W_\mu^3 + \cos\theta_W \cdot B_\mu \end{aligned}}$$

θ_W = weak mixing angle.

notation: $C_W = \cos\theta_W$
$S_W = \sin\theta_W$

---

we can see that W_μ^1 and W_μ^2 have same mass, as they pair up in this way $\frac{g^2 v^2}{8}\left((W_\mu^1)^2 + (W_\mu^2)^2\right)\cdot\left(1+\frac{h}{v}\right)^2$

Here we have two real gauge fields.
We can think of them as a single complex gauge field.

We can define complex gauge fields

$$W_\mu^{\pm} = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp i W_\mu^2\right)$$

---

we can show;

$$\boxed{\tan\theta_W = \frac{g'}{g}}$$

mixing angle is given in terms of gauge coupling.

$$(D_\mu H^\dagger)(D^\mu H) = \frac{1}{2}(\partial_\mu h)^2 + \frac{g^2 v^2}{4} W_\mu^+ W^{-\mu} \cdot \left(1+\frac{h}{v}\right)^2$$

$$+ \frac{1}{8}\frac{g^2 v^2}{c_W^2} Z_\mu Z^\mu \cdot \left(1+\frac{h}{v}\right)^2$$

$$m_W = \frac{gv}{2}$$

(no mass term for A_μ).

A_μ dont get mass $\Rightarrow$ it is the photon

$$m_Z = \frac{gv}{2}\cdot\frac{1}{c_W}$$

$m_A = 0$ (seems massless $\Rightarrow$ photon)

Three parameters m_W, m_Z, $\cos\theta_W$

(By measurment of coupling (independent of masses) we can measure θ_W)

Prediction]

$$\boxed{\rho = \frac{m_W^2}{m_Z^2 c_W^2} = 1}$$

We can measure each of m_W, m_Z & c_W independently & form the ratio ρ and should be equal to 1.

(we can test this)

If you had other sources of ~~static~~ electroweak symmetry breaking. Say we introduce new higgs fields with other quantum numbers which could also get vev. Addition contribution to parameter ρ comes out.

Electro weak symmetry breaking

$SU(2)_L \times U(1)_Y \longrightarrow U(1)_{em}$

$W_\mu^{1,2,3} + B_\mu \longrightarrow A_\mu$

~~goto ot~~
So; Electroweak symmetry group gets broken into U(1) of electromagnetism (since we are left with one massless A_μ at last)

where; gauge field of photon A_μ is linear combination of W_μ^3 and hypercharge gauge field B_μ.

Fermion terms

$$\mathcal{L}_{fermion} = \sum_\psi \bar{\Psi} i \not{D} \Psi$$

$$= \sum_{i=1}^{3} \bar{Q}_L^i \cdot i \left(\not{\partial} + i\frac{g}{2}\sigma^a \not{W}^a + ig' \cdot \frac{1}{6}\not{B}_\mu + ig_s \frac{\lambda^A}{2} \cdot \not{g}^A\right) Q_L^i$$

+ Also for others

(sum over generational indices)

one term for ~~one~~ ~~option~~ covariant derivative for one of quark field.

left handed quark field.

$$\mathcal{L}_{fermion} = \sum_{i=1}^{3} \underbrace{\bar{Q}_L^i}_{\text{left handed quark field}} \cdot i\left(\not{\partial} + i\cdot\frac{g}{2}\sigma^a \not{W}^a + i\cdot g' \cdot \frac{1}{6}\not{B}_\mu + i\cdot g_s \cdot \frac{\lambda^A}{2}\cdot \not{g}^A\right) Q_L^i$$

up type quarks $\{ + \bar{u}_e^i \cdot i\left(\not{\partial} + ig'\cdot\frac{2}{3}\cdot\not{B} + ig_s\cdot\frac{\lambda^A}{2}\cdot\not{g}^A\right) u_e^i$

down type quark $\{ + \bar{d}_e^i \cdot i \cdot \left(\not{\partial} + i\cdot g'\cdot\left(\frac{-1}{3}\right)\cdot\not{B} + ig_s\cdot\frac{\lambda^A}{2}\cdot\not{g}^A\right)\cdot d_e^i$

Left handed lepton. $\{ + \bar{L}_L^i \cdot i \cdot \left(\not{\partial} + ig'\cdot\left(\frac{-1}{2}\right)\not{B} + i\cdot\frac{g}{2}\cdot\sigma^a \not{W}^a\right) L_L^i$

Right handed lepton $\{ + \bar{e}_R^i \cdot i \cdot \left(\not{\partial} - ig'\cdot(-1)\not{B}\right) e_R^i$

$\not\partial$ terms: $\bar{Q}^i_L \cdot \not\partial Q^i_L + \bar{U}_R{}^i \cdot \not\partial U^i_R + \bar{d}_R{}^i \cdot \not\partial d_R{}^i$

we can expand $\bar{Q}^i{}_L$ into left handed up & down quark.
↪ So we multiply in SU(2) space.

... we get.

$$= \bar{U}_L{}^i \not\partial U^i_L + \bar{d}_L{}^i \not\partial d^i_L + \bar{U}_R{}^i \not\partial U^i_R + \bar{d}_R{}^i \not\partial d^i{}_R$$

$$= \bar{u}^i \not\partial u^i + \bar{d}^i \not\partial d^i$$

(so; even we started with a chiral theory; we can write a straightforwardly as four component dirac fermions)

∴ Works same way for gluons.

⟶ gluon terms also pair up in terms of four component objects.

gluon terms: $-g_s G^A_\mu \left(\bar{u}^i \gamma^\mu \frac{\lambda^A}{2} u^i + \bar{d}^i \gamma^\mu \cdot \frac{\lambda^A}{2} d^i\right)$

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Lec 5] Yukawa Interactions, CKM matrix

Electroweak Interactions $(W^\pm, Z, \gamma)$

Charged current interactions $(W^\pm)$

$$\mathcal{L}_{cc} = i\,\bar{Q}_L^i \cdot \left(\frac{ig}{2}\right) \cdot \begin{bmatrix} 0 & W^1 - iW^2 \\ W^1 + iW^2 & 0 \end{bmatrix} Q_L^i$$

$$+ i\,\bar{L}_L^i \cdot \left(\frac{ig}{2}\right) \begin{bmatrix} 0 & W^1 - iW^2 \\ W^1 + iW^2 & 0 \end{bmatrix} L_L^i$$

$$= \frac{-g}{\sqrt{2}} (\bar{u}_L^i, \bar{d}_L^i) \cdot \begin{pmatrix} 0 & W^+ \\ W^- & 0 \end{pmatrix} \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix} - \frac{g}{\sqrt{2}} (\bar{\nu}_L^i, \bar{e}_L^i) \cdot \begin{pmatrix} 0 & W^+ \\ W^- & 0 \end{pmatrix} \begin{pmatrix} \nu_L^i \\ e_L^i \end{pmatrix}$$

$\Rightarrow$

$$\mathcal{L}_{cc} = \frac{-g}{\sqrt{2}} \bar{u}_L^i W^+ d_L^i - \frac{g}{\sqrt{2}} \bar{\nu}_L^i W^+ e_L^i + \text{h.c.}$$

This will be W^- terms

Charged current interaction.

Neutral Current Interactions (Z, γ)

$$\mathcal{L}_{NC} = -\bar{Q}_L^i \cdot \left(\frac{g}{2} \begin{bmatrix} W^3 & 0 \\ 0 & -W^3 \end{bmatrix} + \frac{g'}{6} \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \right) Q_L^i$$

$$- \bar{u}_R^i \cdot \left(\frac{2}{3} \cdot g' \cdot B\right) u_R^i - \bar{d}_R^i \cdot \left(-\frac{1}{3} \cdot g' \cdot B\right) d_R^i$$

$$- \bar{L}_L^i \cdot \left(\frac{g}{2} \cdot \begin{bmatrix} W^3 & 0 \\ 0 & -W^3 \end{bmatrix} - \frac{g'}{2} \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \right) L_L^i$$

$$- \bar{e}_R^i \cdot (-g' \cdot B) \cdot e_R^i$$

NC for general fermion ψ_L, ψ_R.

we will use the substitution.

$W_\mu^3 = c_W Z_\mu + s_W \cdot A_\mu$

$B_\mu = -s_W Z_\mu + c_W A_\mu$

$\mathcal{L}_{NC}$ ~~[illegible]~~

Neutral Current Interaction in general.

; we have

$$\mathcal{L}_{NC} = -\bar{\Psi}_L \left(g\, T_{3L} \cdot (c_W \not{Z} + s_W \not{A}) + g' Y_L (-s_W \not{Z} + c_W \not{A}) \right) \Psi_L$$

$$- \bar{\Psi}_R \left(g' \cdot Y_R (-s_W \not{Z} + c_W \not{A}) \right) \Psi_R$$

$Y_{L,R}$ → hypercharges (under $U(1)$)

ex if $\Psi = u$ (up type quark)

then $Y_L = \frac{1}{6}$; $Y_R = \frac{2}{3}$; $T_{3L} = \frac{1}{2}$

ex if $\Psi = d$

then $T_{3L} = -\frac{1}{2}$

T_{3L} ⇒ eigenvalues of the generator T_{3L}

(it is $\pm 1/2$; if upper component of $SU(2)$ doublet then $T_{3L} = 1/2$

if lower component of $SU(2)$ doublet, then $T_{3L} = -1/2$)

$g' = g \cdot \tan\theta_W$

⇓

$$\mathcal{L}_{NC} = -\bar{\Psi} \Big\{ g \cdot s_W \not{A} \left((T_{3L} + Y_L) P_L + Y_R P_R \right) + \frac{g}{c_W} \not{Z} \cdot \left([T_{3L} - (T_{3L} + Y_L) s_W^2] P_L + (-Y_R s_W^2) P_R \right) \Big\} \Psi$$

we chose hypercharges; such that for any fermion $T_{3L} + Y_L = Y_R$ is true.

$u_L : T_{3L} + Y_L = \frac{2}{3}$; $T_{3L} = \frac{1}{2}$; $Y_L = \frac{1}{6}$

$u_R : \quad Y_R = \frac{2}{3}$

$d_L : \quad -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}$

$d_R : \quad -\frac{1}{3}$

Chosen Hyperchanges || $T_{3L} + Y_L = Y_R$

$\left.\begin{array}{l} u_L : T_{3L} + Y_L = 2/3 \\ u_R : \quad Y_R = \frac{2}{3} \end{array}\right\} Q_u$ $\quad$ $\left.\begin{array}{l} d_L : -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3} \\ d_R : -\frac{1}{3} \end{array}\right\} Q_d$

$\left.\begin{array}{l} e_L : -\frac{1}{2} - \frac{1}{2} = -1 \\ e_R : -1 \end{array}\right\} Q_e$ $\quad$ $\nu_L : T_{3L} + Y_L = \frac{1}{2} - \frac{1}{2} = 0 \} Q_\nu$

These are electric charges for each of different fermions. And they are same for left & right handed components because we chose particular values of hyperchanges.

$$\mathcal{L}_{NC} = -eQ_\psi \bar{\Psi} A\!\!\!/ \Psi - \frac{g}{c_W} \bar{\Psi} Z\!\!\!/ (T_{3L} P_L - Q_\psi \cdot s_W^2) \Psi$$

$e = g s_W$

fundamental electric charge.

its no longer chiral interaction. It couples equally to both left & right handed components.

Coupling of Z remains chiral.

This part couples only to left handed part of fermion field.

This part is same for both left & right handed components

Fermion masses

$\bar{\Psi}\Psi$ terms are forbidden, because gauge interaction is chiral.
∴ We want to generate these mass terms through Yukawa interaction; through couplings of the Higgs fields & fermions.
And once the Higgs field gets ~~vev~~ vev it will generate the mass term (like we saw with Abelian Higgs model)

SU(2) group theory: Suppose we have two fundamental representations of SU(2) η, ξ. (2 component objects which transform under fundamental representation of SU(2))

Two ways to contract η, ξ in SU(2) invariant way:

① $\eta^\dagger \xi$ or ② $\eta^T \varepsilon \xi$

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ Antisymmetric tensor.

In general; SU(N):

Invariant tensors (two we can write down)

① δ_{ij} (tensor): $\mathbb{1} \to U^\dagger \mathbb{1} U = \mathbb{1}$

② $\underbrace{\varepsilon_{ij\ldots k}}_{N}$ (Antisymmetric tensor)

$$\varepsilon_{ij\ldots k} \to U_{ii'} U_{jj'} \ldots \varepsilon_{i'j'\ldots}$$
$$= \det(U)\, \varepsilon_{ij\ldots k}$$
$$= 1 \cdot \varepsilon_{ij\ldots k} = \varepsilon_{ij\ldots k}$$
$$\det(U) = 1 \text{ if } U \in SU(N)$$

for case of SU(2) ε has just two indices.

Valid Yukawa Interactions

$H^\dagger \bar{d}_R Q_L$, $\bar{u}_R Q_L^T \varepsilon H$, $H^\dagger \bar{e}_R L_L$

(These are the different Yukawa interactions we can write down using different ways to write down SU(2) invariant terms)

because we want to contract other indices. (spinor indices)

~~we can convert H & $H^\dagger$ in above.~~

$\left(\bar{d}_R Q_L^T \varepsilon H \right.$, $H^\dagger \bar{u}_R Q_L$, $\left. \bar{e}_R \bar{L}_L^T \varepsilon H \right) \rightarrow$ Invalid Interactions

→ spinor & color part is OK. SU(2) part is also OK.

but Hypercharge !

→ looks fine except for $U(1)_Y$

forbidden by $U(1)_Y$.

In order for $\bar{d}_R \bar{Q}_L \varepsilon H$ to be allowed by hypercharge we need :

$$Y_{d_R} = Y_{Q_L} + Y_H$$

$$-\frac{1}{3} \neq \frac{1}{6} + \frac{1}{2} \quad \text{(does not work out)}$$

but in $H^\dagger \bar{d}_R Q_L$

$$y_H + y_{d_R} = y_{Q_L}$$

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6} \quad \text{(consistent)}$$

General form

here we can allow interaction between generators.

$$\cancel{\mathcal{L}_{Yukawa} = -Y^{(u)}_{ij}\,\bar{u}^i_R\, Q^{j\,T}_L \cdot \varepsilon H - Y^{(d)}_{ij} H^\dagger \bar{d}^i_R Q^j_L - Y^{(d)}_{ij} H^\dagger \bar{d}^i_R Q^j_L}$$

$$\mathcal{L}_{Yukawa} = -Y^{(u)}_{ij} \cdot \bar{u}^i_R\, Q^{j\,T}_L \cdot \varepsilon H - Y^{(d)}_{ij} \cdot H^\dagger \bar{d}^i_R Q^j_L - Y^{(e)}_{ij}\, H^\dagger \cdot \bar{e}^i_R L^j_L \quad + h.c.$$

~~Each~~ of We can write couplings of any two generations together.
so; i, j.
↳ and each coupling between any two pairs can have its own coupling strength.

We need to write down different Yukawa coupling for every pair of generation we have.
∴ we can think of it as matrix of couplings labelled by generation i, & j. $Y^{(u)}_{ij}$, $Y^{(d)}_{ij}$, $Y^{(e)}_{ij}$

expand this out by letting Higgs getting its v.e.v. ; $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}$

$$\boxed{\begin{aligned}\mathcal{L}_{Yukawa} = &-Y^{(u)}_{ij} \cdot \frac{v}{\sqrt{2}} \cdot \bar{u}^i_R u^j_L \cdot \left(1+\frac{h}{v}\right) - Y^{(d)}_{ij} \cdot \frac{v}{\sqrt{2}} \cdot \bar{d}^i_R \cdot d^j_L \cdot \left(1+\frac{h}{v}\right) \\ &- Y^{(e)}_{ij} \cdot \frac{v}{\sqrt{2}} \cdot \bar{e}^i_R\, e^j_L \cdot \left(1+\frac{h}{v}\right) \quad + h.c.\end{aligned}}$$

So; we have generated some mass term.
(but they are not diagonal)
↳ we want to make it diagonal so that we can put the free ~~of~~ part of Lagrangian into canonical form.

Diagonalize mass terms by 3×3 rotations among generations.
Diagonalize any complex square matrix M using a biunitary transformations; where the resulting matrix is diagonal (real, positive entries)

proof// Hermitian matrix can be diagonalized by unitary transformation.

consider $MM^\dagger$ (it is hermitian)

so; $MM^\dagger$ can be diagonalize by using Unitary transformation

$$UMM^\dagger U^\dagger = M_d^2$$ = diagonal matrix, real positive eigenvalues.

we can define M_d as square root of all the entries of M_d^2

$$M_d \equiv \sqrt{M_d^2}$$

∴ now:
$$\underbrace{UMM^\dagger U^\dagger M_d^{-1}}_{\text{call it } V} = M_d$$

So; now; we have a transformation. $UM \cdot V = M_d$ ← a diagonalized matrix with real positive values on diagonal.

This proves that we can diagonalize M by using bi-unitary transformation.

V is unitary.

But we must also show, V is unitary.

$$V^\dagger V = (M_d^{-1} U \cdot M)(M^\dagger U^\dagger M_d^{-1})$$
$$= M_d^{-1} \cdot M_d^2 \cdot M_d^{-1} = \mathbb{1}$$

so: V is also unitary ...

In the proof; we assumed M_d^{-1} exists.

ie; no zero eigenvalues. of $MM^\dagger$

(It is fine for SM; because all of the masses in SM are non-zero, except Neutrinos)

We are not here including Yukawa interaction for neutrinos so far.

We only have Yukawa interaction for

- up type quark
- down " "
- charge leptons

} all of them have non-zero masses.

We can generalize our above proof even if we do have zero masses in our ~~mass~~ mass matrix.

$$\mathcal{L}_{\text{Yukawa}} = -\bar{u}_R M^{(u)} u_L \left(1+\frac{h}{v}\right) - \bar{d}_R M^{(d)} \cdot d_L \cdot \left(1+\frac{h}{v}\right)$$
$$- \bar{e}_R M^{(e)} e_L \cdot \left(1+\frac{h}{v}\right) + \text{h.c.}$$

$$M^{(u,d,e)}_{ij} = Y^{(u,d,e)}_{ij} \cdot \frac{v}{\sqrt{2}}$$

This is the arbitrary complex 3×3 mass matrix.

(now we want to diagonalize it by bi-unitary transformation)

we have not written components for or generation label for fermion fields.

~~Think~~ Think of it as three component objects in flavour space.

ex $U_{L,R} = \begin{pmatrix} u^1_{L,R} \\ u^2_{L,R} \\ u^3_{L,R} \end{pmatrix}$ in generation space.

$$\mathcal{L}_{Yukawa} = -\bar{u}_R M^{(u)} \cdot u_L \cdot \left(1+\frac{h}{v}\right) - \bar{d}_R M^{(d)} \cdot d_L \cdot \left(1+\frac{h}{v}\right) - \bar{e}_R \cdot M^{(e)} \cdot e_L \cdot \left(1+\frac{h}{v}\right) + h.c.$$ (19.55)

Biunitary transformation.

$$U^{\dagger}_{u_R} \cdot M^{(u)} U_{u_L} = m^{(u)} = \mathrm{diag}\left(m_1^{(u)}, m_2^{(u)}, m_3^{(u)} \right)$$

U_{u_L}, U_{u_R} are unitary matrices and in general they are different.

$$m^{(u)} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$$

for up, charm and top.

We should think of this rotation as change of basis.

We started out in original basis $\longrightarrow$ Transformed to new basis, the Mass basis. in which all of the mass terms are diagonal.

Now, we define new fields in this mass basis.

New basis (mass basis) for fermions fields u_L', u_R'

$$u_{L,R} = U_{u_{L,R}} \cdot u'_{L,R}$$

(original basis)

$\hookrightarrow$ mass basis.

$$\Rightarrow \quad \bar{u}_R M^{(u)} u_L = \underbrace{\bar{u}'_R \, m^{(u)} \cdot u'_L}$$

$\hookrightarrow$ now mass term are in canonical form,

all real and diagonal.

We do similar transformation for all of the fermion fields.

Each field in general has its own seperate transformation matrix

$$d_{L,R} = U_{d_{L,R}} \cdot d'_{L,R}$$

$$e_{L,R} = U_{e_{L,R}} \cdot e'_{L,R}$$

$$\nu_L = U_{e_L} \cdot \nu_L'$$

for neutrino, there was no neutrino mass matrix. We are not including neutrino masses in S.M.
By defination SM does not include neutrino masses.
Because we have no Yukawa matrix or matrix for neutrino, We are free to rotate those in any basis we like.
↳ we gonna rotate them using same rotation matrix that I have for left handed charged lepton fields.

Now after doing this transformation the Yukawa term is simple.

$$\mathcal{L}_{Yukawa} = -\sum_{\psi} m_\psi \cdot \bar{\Psi} \cdot \Psi \cdot \left(1 + \frac{h}{v}\right)$$

(only fermions actually get mass; so no neutrinos in this sum)

What happens to other interations with new basis.

$$\mathcal{L}_{NC} = -e Q_\psi \bar{\Psi} A\!\!\!/ \Psi - \frac{g}{c_W} \bar{\Psi} Z\!\!\!/ (T_{3L} P_L - Q_\psi s_W^2) \Psi$$

↳ general form ~~for~~ for NC interaction for ~~fermions~~ fermions.

for $\Psi = u$:

$$\mathcal{L}_{N.C.} = -e Q_u \cdot \left(\bar{u}'^i_L \not{A} u'^i_L + \bar{u}_R \not{A} u'^i_R \right)$$

$$-\frac{g}{c_W} \left(\bar{u}'^i_L \not{Z} \cdot (T_{3L} - Q_u s^2_W) u'^i_L - \bar{u}'^i_R \not{Z} Q_u s^2_W \cdot u'^i_R \right)$$

(This is in original basis)

we have reseperate in terms of left handed & right handed components becausing we are rotating differently in left & right handed components.

Transform to mass basis :

$$\bar{u}'^i_L \gamma^\mu u'^i_L = \bar{u}'^j_L \underbrace{(U^\dagger_{u_L})_{ji} \cdot \gamma^\mu \cdot (U_{u_L})_{ik}}_{\text{They are numbers.}} u'^k_L$$

$$= \bar{u}'^j_L \underbrace{(U^\dagger_{u_L})_{ji} (U_{u_L})_{ik}}_{\delta_{jk}} \cdot \gamma^\mu u'^k_L$$

$$= \bar{u}'_i \gamma^\mu u'_i \qquad \text{invariant under change of basis.}$$

we can see N.C. interaction always behave this way.
(left, right components ; Z , γ^μ) because those interactions are always proportional to identity in generation space.
$\hookrightarrow$ so this change of basis has no effect.

$$\boxed{\text{NC is unchanged under change of basis}}$$

Now lets look at $\mathcal{L}_{cc}$.

$$\mathcal{L}_{cc} = \frac{-g}{\sqrt{2}} \left(\bar{u}'^i_L \cdot \not{W}^+ P_L \cdot d'^i_L + \bar{\nu}'^i_L \cdot \not{W}^+ e'^i_L \right) + h.c.$$

now we go to mass basis.

$$= -\frac{g}{\sqrt{2}} \Big(\bar{u}'^{j}_{L} \cdot (U^{\dagger}_{u_L})_{ji} \, W^{+} \cdot (U_{d_L})_{ik} \, d'^{k}_{L}$$

$$+ \bar{\nu}'^{j}_{L} \cdot (U^{\dagger}_{e_L})_{ji} \, W^{+} (U_{e_L})_{ik} \, e'^{k}_{L} \Big) + h.c.$$

for the lepton term; this is why we wanted to rotate the neutrino fields with exactly the same rotation matrix as for the charged left ~~handed~~ handed terms. :

so cc for leptons is diagonal.

This is still diagonal : $\bar{\nu}'^{i}_{L} \, W^{+} \, e'^{i}_{L}$

We are free to choose any basis we like for neutrino. But why not to choose the basis, to be the basis in which flavour of neutrino is exactly corresponds to mass eigenstate of that particular charged lepton..

but for the quark it is not as nice;

we have $(U^{\dagger}_{u_L})_{ji}$ $(U_{d_L})_{ik}$

different unitary matrix.

lets call $V_{ij} = (U^{\dagger}_{u_L})_{ik} (U_{d_L})_{kj}$

i.e. define $V = U^{\dagger}_{u_L} U_{d_L}$

so;

$$\mathcal{L}_{cc} = -\frac{g}{\sqrt{2}} \left(\bar{u}'^{i}_{L} \, V_{ij} \, W^{+} d'^{j}_{L} + \bar{\nu}'^{i}_{L} \, W^{+} e'^{i}_{L} \right)$$

~~V is arbitrary 3x3 matrix in flavour space.~~

$$V = U^{\dagger}_{u_L} U_{d_L} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

V is arbitrary 3×3 matrix in flavour space.

$$V = U^{\dagger}_{u_L} U_{d_L} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

Cabibbo - Kobayashi - Maskawa (CKM) Matrix.

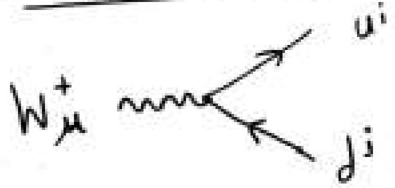
We should think of entries of V to be parameters of theory; that one would want to measure experimentally.

(But yes; V is unitary $V^{\dagger}V = 1$)

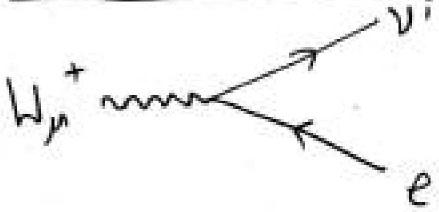
Feynman Rules

(to summarize interactions)

In general we can couple any three generations using CKM matrix



$$= -\frac{ig}{\sqrt{2}} V_{ij} \cdot \gamma^\mu \cdot P_L$$



$$= -\frac{ig}{\sqrt{2}} \delta_{ij} \gamma^\mu P_L$$

If we did include neutrino masses; we would include another mixing matrix here.

But for moment we are considering massless neutrinos.

Lec 6) Weak Interactions, Fermi Theory

CC Interaction ($W^{\pm}$).

$$\mathcal{L}_{CC} = -\frac{g}{\sqrt{2}} \cdot \left(\bar{u}^i_L V_{ij} W^+ d^j_L + \bar{\nu}^i_L W^+ e^i_L \right) + h.c.$$

(we have droped the prime here) ; we are working in mass basis.

If we included, mass for neutrino;
then we would get.

$$\mathcal{L}_{CC} = \frac{-g}{\sqrt{2}} \left(\bar{u}^i_L V_{ij} W^+ d^j_L + \bar{\nu}^i_L U_{ij} W^+ e^j_L \right) + h.c.$$

V = CKM Matrix
U = PMNS Matrix

U, V are unitary.

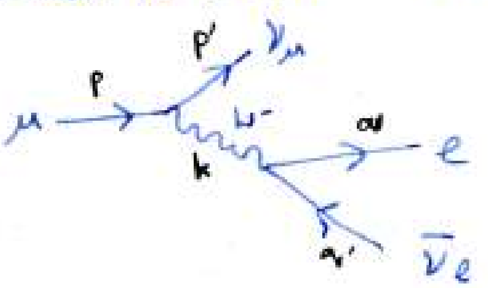
Effective Theory for Weak Interaction

Weak processes with energy $E \ll M_{W,Z}$ ⟶ weak scale ; set by the mass of W and Z.

we dont need ; or say we can integrate out the heavy degrees of freedom to get an effective lagrangian.
(heavy compared to energy scale)

Example) Muon Decay

we want to calculate $i\mathcal{M}(\mu \to \nu_\mu e \bar{\nu}_e)$



$$k = p - p' = q + q'$$

$$i\mathcal{M}(\mu \to \nu_\mu e \bar{\nu}_e) = \bar{u}_e \cdot \left(\frac{-ig}{\sqrt{2}}\right) \gamma^\mu P_L \cdot V_{\bar{\nu}_e} \cdot \frac{-i}{k^2 - m_W^2} \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m_W^2} \right) \times \bar{u}_{\nu_\mu} \cdot \left(\frac{-ig}{\sqrt{2}} \right) \cdot \gamma^\mu P_L \cdot U_\mu$$

Consider $k_\mu k_\nu$ term and ~~consider~~ use Dirac equation.

$$\bar{u}_e \not{k} \cdot P_L V_{\nu_e} = \bar{u}_e \cdot (\not{p} + \not{p}') P_L V_{\nu_e}$$

$$= m_e \bar{u}_e P_L V_{\nu_e} \quad + \quad m_{\nu_e} \rightarrow \text{zero.}$$

similarly; for muon contracted part

$$\bar{u}_{\nu_\mu} \cdot \not{k} \cdot P_L \cdot u_\mu = m_\mu \cdot \bar{u}_{\nu_\mu} \cdot P_R \cdot u_\mu$$

$\frac{m_e m_\mu}{m_W^2}$ is very small number.

$$\left(\frac{m_e m_\mu}{m_W^2}\right) \sim 10^{-8} \quad \text{(so we can through away these terms)}$$

Calculation

$$\bar{u}_e \not{k} P_L V_{\nu_e} = \bar{u}_e \cdot (\not{p} + \not{p}') P_L V_{\nu_e}$$

$$= \bar{u}_e \not{p} P_L V_{\nu_e} + \bar{u}_e P_R \not{q}' V_{\nu_e}$$

$$= m_e \bar{u}_e P_L V_{\nu_e} + \left(\text{term proportional to mass of neutrino}\right)$$

use dirac equation

$$\not{p}\, u(p) = m\, u(p)$$

$$i\mathcal{M}(\mu \to \nu_\mu e \bar{\nu}_e) = \frac{i g^2}{8 m_W^2} \bar{u}_e \gamma^\alpha (1-\gamma^5) V_{\nu_e} \quad \bar{u}_{\nu_\mu} \gamma_\alpha (1-\gamma^5) u_\mu$$

in $\frac{-i}{k^2 - m_W^2}$ we can through away k^2

k^2 is set by muon mass

so; $k^2 <<< m_W^2$

($m_W >>$ all momenta scale i.e; k)

Now, lets ask what term in Lagrangian we could have written (and this will be effective lagrangian) to generate exactly the same matrix Element. This is process of integrating out heavy degrees of freedom.

This can be expressed by operator.

$$\mathcal{L}_{eff} = \frac{G_F}{\sqrt{2}} \cdot \bar{e}\gamma^\alpha \cdot (1-\gamma_5) \nu_e \; \bar{\nu}_\mu \gamma_\alpha (1-\gamma_5)_\mu + h.c.$$

where : $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 m_W^2}$ ∴ G_F = Fermi's constant.

Historically) People were able to deduce that $\mathcal{L}_{eff}$ was correct lagrangian for weak interaction (~~....~~ describing muon decay & other weak decays)
And then, they tried to do inverse process,
What high energy theory would give rise to this effective lagrangian once you integrate it out. And that's how SM was written.

$$m_W = \frac{gv}{2} \Rightarrow \frac{G_F}{\sqrt{2}} = \frac{1}{2v^2} \Rightarrow \boxed{G_F = \frac{1}{\sqrt{2} \cdot v^2}}$$

This is how we know that their is vev in SM; its directly related to Fermi's constant.

Higgs v.e.v. $v = (\sqrt{2} \cdot G_F)^{-1/2} \simeq 246$ GeV fixed by Fermi's Constant

we had the convention, expanding Higgs in $H = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix}$ gauge.

$$\begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} \xrightarrow[\text{convention}]{\text{Alternative}} H = \begin{pmatrix} 0 \\ v + \frac{h}{\sqrt{2}} \end{pmatrix}$$

here: $v \approx 174$ GeV

We can generate larger set of interactions by integrating out Ws. Because W couples to quarks in addition to leptons.

Charged current interactions described by (at low energy): (pg 65)

$$\mathcal{L}_{eff}^{cc} = \sum_{i,j} \frac{G_F}{\sqrt{2}} \bar{e}^i \gamma^\mu (1-\gamma^5) \nu^i \bar{\nu}^j \gamma_\mu (1-\gamma^5) e^j + h.c.$$

(Leptonic charged current interactions.

$$+ \frac{G_F}{\sqrt{2}} \sum_{ijk} V_{ij} \bar{u}^i \gamma^\mu (1-\gamma^5) d^j e^{-k} \gamma_\mu (1-\gamma^5) \nu^k + h.c.$$

Semi leptonic

$$+ \frac{G_F}{\sqrt{2}} \sum_{ijk} V_{ij} V_{kl}^* \bar{u}^i \gamma^\mu (1-\gamma^5) d^j \bar{d}^l \gamma_\mu (1-\gamma^5) u^k + h.c.$$

hadronic cc. interaction.