

Lec 1 Brief Overview, Charged Particle on a circle surrounding a solenoid: Hamiltonian Quantization.

Chern-Simons Theory

Notes: www.physics.rutgers.edu/~gmoore/TASI-Chern-Simons-studentNotes.pdf

Chern-Simons Form related to Transgression (Chern, c. 1976).
→ 1975.

Anomaly descent formalism, early 1980's.

Deser, Jackiw, Templeton 1982.

1988, Witten (QFT & Jones Polynomial)

Let G be a Lie Group

Consider 3d Gauge Theory with Gauge Group = G .

Locally $U \subset M_3$, $A \in \Omega^1(U, \mathfrak{g})$, $A = A_\mu^a \text{ to } dx^\mu$

$\text{Lie}(G) = \mathfrak{g} = \text{mathfrak{frak}\{g\}}$

$d + A \cdot = g^{-1}(d + A)g$, $F = dA + A^2 \in \Omega^2(U, \mathfrak{g})$

Conjugation invariant Quadratic Polynomial $\mathcal{P}$

e.g. $\mathfrak{g}$ - Matrix Lie Algebra, $\mathcal{P}(X) = \text{tr } X^2$
← some representation.

$\mathcal{P}(F)$ 4-form

$\text{tr}(F^2) = d \text{tr}(A \cdot A + \frac{2}{3} A^3)$

$$d \in S_{\mathfrak{g}}(A) = \mathcal{P}(F) = \text{tr}(F^2) = d \left(\text{tr} \left[A \cdot A + \frac{2}{3} A^3 \right] \right)$$

Some measure $[dA]$ on A = space of ~~gauge~~ ^{all} gauge fields on M_3 . (192)

$[dA]$ pushed down to A/gf = space of all gauge-equivalent field configurations.

gf = Group of Gauge Transformations.

$$\sim \text{Map}(M_3 \rightarrow G)$$

$$Z(M_3) = \int_{A/\text{gf}} [dA] e^{i \int_{M_3} \text{CS}_\rho(A)}$$

M_3 is orientable.

$\int_{M_3} \text{CS}_\rho(A)$ gauge invariant?

$$\text{CS}_\rho(Ag) = \text{CS}_\rho(A) + \text{CS}_\rho(g^{-1}dg) + d(\dots)$$

even when $\partial M_3 = \emptyset$ (ie. M_3 don't have boundary)

drop it if $\partial M_3 = \emptyset$

If G has any non-trivial topology along with M_3 , then it can happen that

$$\int_{M_3} \text{CS}_\rho(g^{-1}dg) \neq 0$$

OK if $e^{i \int_{M_3} \langle \mathcal{S}_P(g^{-1} dg) \rangle} = 1 \quad \forall M_3,$
 $g: M_3 \rightarrow G$

~~We call this "Multivalued Action Principle"~~

We call this "Multivalued Action Principle" ~~and~~ or "Exponentiated Action Principle"

→ Action can fail to be Gauge Invariant, and still be good for physics; because only thing we need in path integral is Exponentiated Action.

This will happen for Quantized values of P .

$$\begin{array}{c} P \\ \downarrow \pi \\ M_3 \end{array}$$

P is Principle Bundle.

$$P \sim M_3 \times G$$

If P is trivializable,
 then

$$\text{Aut}(P) \cong \text{Map}(M_3 \rightarrow G)$$

An interesting class of observables,

"line defects" or "line operator"

Oriented loop $\gamma \hookrightarrow M_3$



Take a finite dimensional representation of the gauge group of G : R

Then we can write the ~~hitam~~ line defect as

$$W(R, \gamma) = \text{Tr}_R \left(\mathcal{P} \exp \oint_{\gamma} A \right)$$

(pg 3)

~~$\mathcal{P}$ is path ordering~~ $\rightarrow$ path ordered exponent

$\rightarrow$ This is gauge Invariant.

Think of $W(R, \gamma)$ as a function of A .

Now, we enhance the path integral little bit by imagining that there are bunch of loops (They might be linked, they might be knotted)

$$\langle \prod_{\alpha} W(R_{\alpha}, \gamma_{\alpha}) \rangle = \int [dA] \prod_{\alpha} W(R_{\alpha}, \gamma_{\alpha}) \cdot e^{i \int_{M_3} CS_P(A)}$$

A/g

Note that we did not use metric to define the action.

$\rightarrow$ So, we might hope naively that this theory is going to be Topological Field Theory, i.e. Correlation Function is going to be independent of Distance, metric.

And they are just going to define Topological invariants of M_3 and loops imbedded in M_3 .

He could imagine that these loops are linked or form ~~are~~ knots.



$$G = SU(2)$$

$$Q(x) = \frac{-k}{8\pi^2} \text{tr}_2(x^2) \quad k \in \mathbb{Z}$$

(pg 5)

Then, $\langle W(\underline{2}, \gamma) \rangle_{M_3 = S^3} = \text{polynomial in } q = e^{\frac{2\pi i}{k+2}}$
 $= \text{Jones Polynomial.}$

2 dimensional representation.

A Breakthrough (An important part for Witten's field Medal)

Jones polynomial is polynomial in q , that depends on the knot γ .

The idea was that, for every knot he would write down a polynomial in q . And then, if we deform the knot smoothly and recompute; it's the same polynomial.

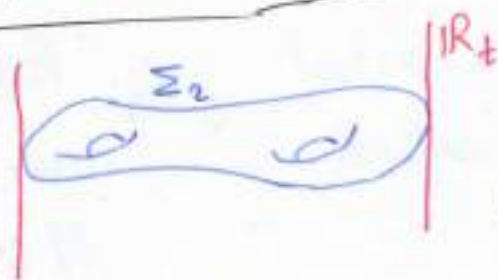
Applications

- ① Solvable, but non-trivial example of a Field Theory.
(topological Field Theory)
- ② "Holographic" relations to 2D^(Rational) Conformal Field Theory.

$$\text{tr}_R(x^2) = C_2(R) \text{tr}_{\text{adj}}(x^2)$$

$$\text{eg } \text{tr}_2(x^2) = \frac{1}{4} \text{tr}_{\text{adj}}(x^2)$$

$$M_3 = \Sigma_2 \times \mathbb{R}$$



$\mathcal{H}(\Sigma_2)$

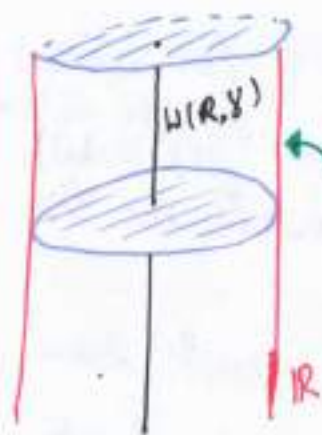
Then physics tells us that, there should be some Hilbert space.

$\mathcal{H}(\Sigma_2)$ finite dimensional

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$\sim$ V.S. of "Conformal Blocks" of a corresponding 2D CFT

V.S. $\equiv$ Vector space.



$A|_2 = ?$ (what boundary conditions to put)

With suitable boundary conditions,
The Gauge modes on the boundary
become Propagating fields of a CFT
(but this time, the CFT is
on the boundary of the solid cylinder)

We call it the "Edge Modes"

So there was 3D point of view on 2D CFT

3D view point gave a simple proof
of "Verlinde Formula"

Operator product
expansion coefficients



Response of correlators to
non-trivial diffeomorphism
of our surface.

If we take γ to be some loop inside the surface.

$$M_3 = \Sigma_2 \times \mathbb{R}$$



Then $W(R, \gamma)$ becomes an operator $\widehat{W(R, \gamma)}$ acting on the Hilbert space.

~~This is one of our~~ (We can write down the Algebra of Wilson line operators)

③ C-S for some non-compact groups might be related to 3d quantum gravity.

B.H. (Brown & Hanow)^{identified} edge modes of graviton with Virasoro symmetry. (This is one of the first manifestation of AdS-CFT correspondence).

④ Action $\frac{k}{8\pi^2} \int \text{tr} \left(A dA + \frac{2}{3} A^3 \right)$

one derivative

Effective action for FQHE (Fractional Quantum Hall Effect) (it uses emergent gauge field)

→ Then edge modes ~~become~~ becomes real physical mode that we can measure in lab.



$\Rightarrow$ nontrivial statistics

- "anyons".
- "Nonabelian".

⑤ Dynamics of Gauge Theories.

⑥ Supergravity: Chern Simons terms often appear in SUGRA actions.

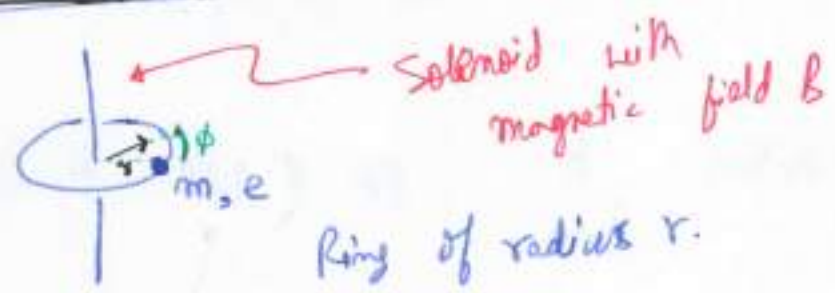
⑦ Topological String Theory: Target spacetime theories are often C-S type.

⑧ Fundamental Formulation of String Theory.

- String Field Theory: formulations often involve C-S type.

Quantum Mechanics

def of particle is the angle $\phi(t)$.



$$\phi(t) \sim \phi(t) + 2\pi$$

(ϕ is defined modulo 2π)

So; its better to think of $h(t) = e^{i\phi(t)}$

($h(t)$ is single valued ... good)

$$h: \mathbb{R}_t \longrightarrow U(1)_{\text{Target}}$$

h is map from time to target space (which happens to be $U(1)$)

We are really doing 0+1 dimensional Field Theory.

$$\begin{aligned} S &= \int \frac{1}{2} m \dot{x}^2 + \text{e.m. coupling.} \\ &= \int -\frac{1}{2} m \dot{x}^2 (h^{-1} \dot{h})^2 + \text{e.m. coupling.} \end{aligned}$$

→ This is a kind of action we see in non-linear ~~Sigma~~ Sigma Model.
(we have a map from spacetime (which is just time) to a group)

Lie algebra of $U(1) = i\mathbb{R}$

Lie algebra of $U(N) = \text{Anti-Hermitian matrices.}$

$$d + A\partial = g^{-1} (d + A) g$$
$$A \rightarrow A + i d\epsilon \quad ; \quad g = e^{i\epsilon(x)} \in U(1)$$

(since $\epsilon(x) \in \mathbb{R}$)

Here, we talk about charged particles. So we will be talking about Electromagnetism; There we take gauge group to be $U(1)$

$U(1)$ is topologically non-trivial.

It is not ~~contractible~~ contractible to a point.

(pg 10)

$$\pi_1(U(1)) \cong \mathbb{Z}$$

Here, the gauge field is $A = \frac{B}{2\pi} d\phi$

The coupling of charged particle to e.m. field is

$$\exp\left(ie \int A_\mu(x(t)) \frac{dx^\mu}{dt} dt\right)$$

So, The action is

$$S = \int \frac{1}{2} I \dot{\phi}^2 + \int e \phi^*(A)$$

$I \neq$ moment of inertia.

$$= \int \frac{1}{2} I \dot{\phi}^2 + \underbrace{\frac{eB}{2\pi} \dot{\phi} \cdot dt}_{\text{"}\theta\text{" term}}$$

(θ terms are related to interesting aspects of topology in field theory)

Consider 1+1 dim. E.M.



The only gauge invariant degree of freedom here is,

The holonomy of our gauge field around the circle.

~~$h(t) = \exp(i \oint \sigma A_\sigma)$~~

$$h(t) = \exp(i \oint \sigma A_\sigma)(t) = e^{i\phi(t)}$$

$$S = \int \frac{1}{e^2} F * F + \int \frac{\theta}{2\pi} F$$

$$\frac{\theta}{\pi} = 2B$$

In 4D

$$S = \int \frac{1}{2e^2} \text{tr}(F * F) + \frac{\theta}{(2\pi)^2} \text{tr}(F \wedge F)$$

θ terms are Total derivatives, and so don't affect the equations of motion;

nevertheless, They matter in Quantum Mechanics.

Quantize

$$L = \frac{\delta S}{\delta \dot{\phi}} = I \dot{\phi} + \frac{eB}{2\pi} \quad (\text{Angular momentum})$$

↓
 $-i\hbar \frac{\partial}{\partial \phi}$

$$H = \frac{1}{2I} \left(L - \frac{eB}{2\pi} \right)^2$$

~~$H = \frac{L^2}{2I} - \frac{eB}{2\pi} L + \frac{e^2 B^2}{8\pi^2 I}$~~

Spectrum is periodic in β .

(19/13)

$\exists$ U unitary on $L^2(S^1)$

such that $U H_\beta U^{-1} = H_{\beta+1}$

Classical Symmetry : Symmetries of eqs of motion
(so they don't know about θ -term)

$$\ddot{\phi} = 0$$

$$R(\alpha) : \phi \rightarrow \phi + \alpha, \quad e^{i\alpha} \in U(1) \cong SO(2)$$

$$P : \phi \rightarrow -\phi$$

$$h(t) : \mathbb{R}_t \rightarrow U(1)_t$$

so: P is really a charge conjugation.

$$P = C, \quad \text{because } P h(t) = h(t)^*$$

complex conjugate.

$$\text{ie: } h(t) \xrightarrow{P} h(t)^*$$

$$P R(\alpha) P = R(-\alpha) \Rightarrow \underline{O(2)}, \quad \text{The group we get.}$$

Wigner's Theorem There is going to be Unitary or Anti-unitary operators for all symmetries of our classical system.

However, the relations that are satisfied classically, will in general only be satisfied up to ~~the~~ phase in the Quantum System.

$$\begin{array}{ccc} g & \xrightarrow{\text{Wigner Theorem}} & U(g) \\ \text{symmetry of} & & \text{(unitary or anti-unitary operator)} \\ \text{classical system} & & \end{array}$$

If $g_1 \cdot g_2 = g_3$ classically.

(Pg 15)

Then in Quantum Mechanics $U(g_1)U(g_2) = c(g_1, g_2)U(g_3)$
 $\rightarrow$ phase term.

For our system in concern here,

Quantum: $R(\alpha) \cdot \Psi_m = e^{im\alpha} \Psi_m$

When $2B \in \mathbb{Z}$,

Then $\mathcal{C} \cdot \Psi_m = \Psi_{2B-m}$

$$\mathcal{C} R(\alpha) \mathcal{C} = (e^{i\alpha})^{2B} R(-\alpha)$$

① $2B = \frac{\theta}{2\pi} \notin \mathbb{Z}$ then $O(2) \Rightarrow SO(2)$
(~~for~~ $O(2)$ is quantum mechanically broken down to $SO(2)$)

② $2B = \frac{\theta}{2\pi} \in 2\mathbb{Z}$ then $O(2) \Rightarrow O(2)$
($O(2)$ remains quantum symmetry)

③ $2B = \frac{\theta}{2\pi} \in 2\mathbb{Z} + 1$, then $O(2) \Rightarrow \text{Pin}^+(2)$

$\text{Pin}^+(2)$ is double cover (central extension) $O(2)$

$$\begin{array}{c} \text{Pin}^+(2) \\ \downarrow 2:1 \\ O(2) \end{array}$$

- Group Theory notes.
- Appendix D.

Lec 2: Charged Particle on a circle surrounding solenoid: Path Integral;
 Gauging the global $SO(2)$ symmetry & Chern Simons terms, U(1) C-S
 theory, Quantization of Phase space & Hamiltonian Reduction.


 $\hbar = 1$

consider $\langle \phi_2 | e^{-\beta H_B} | \phi_1 \rangle$ matrix element
 of (The euclidean time propagator)

H_B is hamiltonian, β is euclidean.

(we diagonalize the hamiltonian
 and call it H_B)

By sticking in the complete sets of eigenstates we get

$$\langle \phi_2 | e^{-\beta H_B} | \phi_1 \rangle = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{-\frac{\beta}{2I} (m - B)^2 + im(\phi_1 - \phi_2)}$$

||

$$Z(\phi_2, \phi_1 | \beta) = \int_{\phi(0)=\phi_1}^{\phi(\beta)=\phi_2} [d\phi(t)] \cdot \exp \left(- \int_0^\beta \left(\frac{1}{2} I \dot{\phi}^2 + i B \dot{\phi} \right) dt \right)$$

A term (does not affect equations of motion)

Euclidean Path integral.

Like Rotated Action.

This path integral is quadratic.

$\Rightarrow$ So, semiclassical approximation is exact.

(So, we sum over the accessible solutions of the euclidean equations of motion, and multiply by one loop determinant)

Equation of motion: $\ddot{\phi} = 0$

$$\phi(t) = \phi_1 + \left(\frac{\phi_2 - \phi_1 + 2\pi W}{\beta} \right) t, \quad W \in \mathbb{Z} \quad (Pg 16)$$

ϕ is defined modulo 2π , hence we can have the W piece.

$$\phi(t) = \phi_1 + \left(\frac{\phi_2 - \phi_1 + 2\pi W}{\beta} \right) t, \quad W \in \mathbb{Z}$$

$$h(t) = e^{i\phi(t)} \quad \} \rightarrow \text{This has to be single valued.}$$

$$Z = \sqrt{\frac{I}{2\pi\beta}} \cdot \sum_{W \in \mathbb{Z}} e^{-\frac{2\pi^2}{\beta} \left(W + \frac{\phi_2 - \phi_1}{2\pi} \right)^2 - 2\pi i \beta \left(W + \frac{\phi_2 - \phi_1}{2\pi} \right)}$$

RMS of $\langle \phi_2 | e^{-\beta H_\beta} | \phi_1 \rangle$ terms are exponentially convergent for $\beta \rightarrow +\infty$
 " " " " " " " " " " " " $\beta \rightarrow 0$
 " " " " " " " " " " " "

But They are same

(Physical explanation: They are same because, they are same quantity)

(Mathematical explanation: Both are special cases of Riemann's ζ function)

ζ function plays an important role in discussing the spaces of states in Chern-Simons Theory

$$\vartheta \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (z|\tau) \equiv \sum_{n \in \mathbb{Z}} e^{i\pi\tau (n+\theta)^2 + 2\pi i (n+\theta)(z+\phi)}$$

Absolutely convergent holomorphic function for $\text{Im } z > 0$.

Entire function for z .



Properties of ϑ function

① Modular Property

$$\vartheta \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] \left(-\frac{z}{\tau} \mid -\frac{1}{\tau} \right) = (-i\tau)^{1/2} \cdot e^{2\pi i \theta \phi} \cdot e^{i\frac{\pi z^2}{\tau}} \cdot \vartheta \left[\begin{smallmatrix} -\phi \\ \theta \end{smallmatrix} \right] (z|\tau)$$

(can prove this using Poisson Summation Formula (P.S.F.))
(sum of plane waves is a periodic
Sum of Delta functions)

If 2β is odd,

then as $\beta \rightarrow +\infty$

$$\langle \phi_2 | e^{-\beta H} | \phi_1 \rangle = 2 e^{\frac{i}{2}(\phi_1 - \phi_2)} \cdot \cos\left(\frac{\phi_1 - \phi_2}{2}\right) \cdot e^{-\beta E_g} \cdot (1 + \text{exp small})$$

E_g is ground state energy.

~~2nd Remark~~

General Remark: $\mathcal{T}$ theory with G -symmetry, then we can gauge it,

"Gauging" (i) Make the theory $\mathcal{T}$ by equivariantizing by coupling to external G -gauge fields.

(iii) Sum over gauge inequivalent G -gauge fields.

~~If we~~

If we view n -dimensional field theory as a functor.

$$F : \mathcal{B}_{\langle n-1, n \rangle}^{\text{str}} \longrightarrow \mathcal{T}$$

(i) $F^{\text{equiv}} : \mathcal{B}^{\text{str}} + \left\{ \begin{array}{l} \text{Principle } G\text{-bundles} \\ \text{with connection} \end{array} \right\} \longrightarrow \mathcal{T}$

(ii) Summing over isomorphism classes of ~~principle~~ principle G bundles with connection.

$$\phi(t) \longrightarrow \phi(t) + \alpha(t)$$

$$g = e^{i\alpha} \in U(1)$$

To make equivariant, we allow α to depend on time. and we introduce an external gauge field $A^{(e)}$

$$A^{(e)} \longrightarrow A^{(e)} + i g^{-1}(t) \cdot d_t \cdot g(t)$$

↑
not maxwell
field.

$$S_E = \int \frac{1}{2} \mathbb{I} (\dot{\phi} + A^{(e)})^2 + i \mathbb{B} (\dot{\phi} + A_t^{(e)}) dt$$

↑
Euclidean
action.

↘ In Euclidean space, the
B term remains purely imaginary.

$$2B = \frac{\theta}{\pi}$$

(pg 19)

$A_t^{(0)}$ spoils periodicity

We can restore it partially by ~~repeating~~ adding on one dimensional Cherns Simon term.

$$e^{-S_E} \cdot e^{ik \int A_t^{(0)} dt}$$

with some constant k .

So, it should be manifest now that

$$\mathcal{I}(B, k) = \mathcal{I}(B+1, k+1)$$

Theory with B
& k

→ Theory with $B+1$ & $k+1$.

Under gauge transformation.

$$e^{ik \int_{t_1}^{t_2} A_t^{(0)} dt} \rightarrow e^{-i\alpha(t_2)k} \cdot e^{ik \int_{t_1}^{t_2} A_t^{(0)} dt} \cdot e^{ik\alpha(t_1)}$$

(depends on gauge transformation at the end points)

- If $\mathbb{R}_t$, and $\alpha(t) \rightarrow 0$, Then its gauge invariant.
- If $G_{ext} = \mathbb{R}$, yes, even on S_t^1 (euclidean circle)

$$e^{ik \oint_{S_t^1} A_t^{(0)} dt}$$

~~Milnor~~
- If $G_{ext} = U(1)$, Then $\alpha(t+\beta) = \alpha(t) + 2\pi u$, $u \in \mathbb{Z}$
 because only $e^{i\alpha(t)}$ is periodic.

$$\Rightarrow (\cdot) e^{i 2 \pi n k}$$

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for this to be 1
 $\Rightarrow k \in \mathbb{Z}$

Another way to try to define:

$$e^{i k \oint_{S'_t} A_t^{(e)} dt}$$



$$\partial \Sigma = S'_t$$

Imagine that we could extend the gauge field to the rest of the surface.

Then
$$e^{i k \oint_{S'_t} A_t^{(e)} dt} := e^{i k \int_{\Sigma} F}$$

LHS is gauge invariant for $k \in \mathbb{Z}$

but RHS seems to be gauge invariant for any k (if we fix Σ)
 What's going on?

F is locally dA
 $F = dA$ locally.

Now, it is nice because F is gauge invariant.

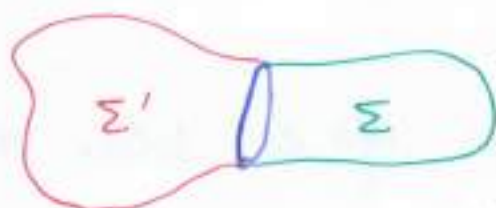
"but we will have to make a choice for Σ "
 $\Rightarrow$ If we make two different choices of Σ , we might ~~even~~ get different answers; so would actually depend on higher dimensional theory (which from the point of view we are doing here is fictitious)

"Extension is independent of Σ only if $k \in \mathbb{Z}$ "

because $F \in \Omega^2_{2\pi\mathbb{Z}}$

(pg 21)

F is closed 2 form, with 2π period.



Two different extensions Σ & Σ'

The difference of the integral, over the closed manifold, of F is $2\pi \times (\text{Integer})$.

(because $\frac{F}{2\pi}$ represents first Chern class)

What data goes into a C-S theory?

① Lie group G

② Level. Topology gives $G \rightsquigarrow H^4(BG, \mathbb{Z})$ finite dimensional Abelian group

$k \in$ satisfies a non-degeneracy condition.

~~M_3 oriented~~

- M_3 oriented (+ P, - structure)
- time defects becomes Ribbons

Best understood for G compact
(Theory is best understood...)

G compact Lie group.

Then we ask about connected components

We have: $G_i =$ connected component with 1_G (identity)

$$G_1 \triangleleft G \Rightarrow G/G_1 \cong \Pi_0(G)$$

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← any finite group.

$$G_1 \cong (G_1^{ss} \times U(1)^d) / \mathbb{Z}$$

$\mathbb{Z} \subset \mathbb{Z}(G_1^{ss} \times U(1)^d)$ finite abelian group.

$G_1^{ss} \neq (G_1 \text{ semi-simple})$

$$G_1^{ss} = \prod G_i^s, \quad G_i^s \text{ Cartan: } A \rightarrow G_i$$

$$G = U(1), \quad H^4(\underbrace{BU(1)}_{\cong \mathbb{CP}^\infty}, \mathbb{Z}) \cong \mathbb{Z} \Rightarrow k \neq 0.$$

$$A^g = A - ig^{-1} dg \quad ; \quad g: M_3 \rightarrow U(1)$$

Small gauge transformation: $g(x) = e^{i\epsilon(x)}$ single valued.
 $\mathbb{R}$ -valued function on M_3 .

Large gauge transformation: $\exists$ loop in M_3 $\oint g^{-1} dg \neq 0$
 (period not zero)

$A \rightarrow A + \omega$; ω is closed 1-form with
 $2\pi\mathbb{Z}$ period.

$$\exp \left(i \frac{k}{2\pi} \int_{M_3} AF \right)$$

looking at Gauge Invariance
of $U(1)$ Chern Simons.

(Pg 23)

$$F = dA$$

gauge invariant
iff $k \in \mathbb{Z}$

$$AF \longrightarrow AF + \omega F$$

$\downarrow$ $\searrow$
 $2\pi\mathbb{Z}$ $2\pi\mathbb{Z}$

$$\int_{\Sigma_2} F \in 2\pi\mathbb{Z}$$

(closed 2 cycle)

$\frac{F}{2\pi}$ is called Chern-Weyl (C-W)
representative of $C_1(L)$.

$\exp \left(i \frac{k}{2\pi} \int_{M_3} AF \right)$ is gauge invariant iff $k \in \mathbb{Z}$,

because $\exists M_3, P \rightarrow M_3$ s.t. $\int_{M_3} AF \in (2\pi)^2 \times (\text{Any integer})$
 (principal
 $U(1)$ bundle
 over the 3-manifold
 M_3)

$$\text{i.e. } \int_{M_3} AF \in (2\pi)^2 \times (\text{Any integer})$$

This fact forces k to be an integer
for gauge invariance.

(1925)

$$\left. \begin{aligned} e^{i S_{\text{Lorentz}}} &= e^{i \int \frac{1}{8\pi^2 e^2} F \wedge F} \\ e^{-S_{\text{Euclidean}}} &= e^{-\int \frac{1}{8\pi^2 e^2} F \wedge F} \end{aligned} \right\} \text{Maxwell Theory.}$$

Now we can add $A \wedge A$, The Chern-Simons term.
(we can only do it when K is integer)

$$e^{i S_{\text{Lorentz}}} = e^{i \int_{M_3} \frac{1}{8\pi^2 e^2} F \wedge F + \frac{K}{2\pi} A \wedge A}$$

$$e^{-S_{\text{Euclidean}}} = e^{-\int_{M_3} \frac{1}{8\pi^2 e^2} F \wedge F + i \frac{K}{2\pi} \int_{M_3} A \wedge A}$$

depend on a
metric; not on
orientation

independent of
metric; and
depend on
orientation.

$$d^3x \cdot \frac{1}{e^2} \sqrt{\det g} \cdot g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} F_{\mu_1 \mu_2} F_{\nu_1 \nu_2}$$

Consider (1,2) Minkowski Space $M^{1,2}$.

Equations of motion

$$d \star F - m F = 0$$

$$\text{where } m = 4\pi k e^2$$

(in 2+1 dimensions; electromagnetic coupling
constant square, has the units of mass)

looking at plane waves solution.

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$$A_i \sim \epsilon_i(p) e^{ip \cdot x}$$

plugging this in equation of motion, leads to the following dispersion relation $p_0^2 \cdot (p_0^2 - \vec{p}^2 - m^2) = 0$.

$$p_0^2 = 0$$

or

$$p_0^2 - \vec{p}^2 - m^2 = 0$$

Dispersion relation of
Massive scalar.

$$\Rightarrow p_0 = 0$$

$$\Leftrightarrow F_{12}(p) = 0$$

$\Rightarrow$ flat gauge field.

Solutions of equations of motion factorize.

$$\mathcal{S}_{\text{flat}} \times \mathcal{S}_{\text{non-flat}}$$

↑
Space of flat
solution.

Space of non-flat solution.
"describes the degrees of freedom
of a massive scalar particle in
2+1 dimensions"

Correlation Functions

$$\langle F_{\mu\nu}(x_1) \dots F_{\mu\nu}(x_n) \rangle$$

$$\langle W(m_i, Y_i) \dots W(m_s, Y_s) \rangle$$

Take the metric dependent term

$$\int d^3x \frac{1}{e^2} \sqrt{\det g} \cdot g^{\mu_1 \mu_2} g^{\nu_1 \nu_2} F_{\mu_1 \mu_2} F_{\nu_1 \nu_2}$$

$\downarrow$
 Ω^2 volume?

$\downarrow$
 $\frac{1}{\Omega^2}$

when we look at longer

distance scale

$$g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$$

(Ω constant: just rescaling distance by factor of Ω)

so: scaling distance by factor of Ω

$\equiv$ equivalent scaling coupling constant by factor of Ω .

$g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$

$e^2 \rightarrow \Omega e^2$

$\Rightarrow$ but $m \propto e^2$
 $\uparrow$ mass of particle.

so: if $\Omega \rightarrow \infty$

$\Rightarrow m \rightarrow \infty$ (Then mass can't propagate)

~~At large~~ At large distances or large masses

$$\langle F_{\mu_1 \nu_1}(x_1) \dots F_{\mu_n \nu_n}(x_n) \rangle \sim e^{-|m| \cdot \min \{ |x_i - x_j| \}} \rightarrow 0$$

for $\langle W(n_1, x_1) \dots W(n_g, x_g) \rangle$ this correlator



nothing can propagate from here to there.
→ so we expect to get topological invariants.

In this limit, The action degenerates into Chern Simons action.

Remark

① Formal metric independence.

② In C-S, $k \sim 1/h$ so $k \rightarrow \infty$ is

Semi-classical (S.C.) limit.

③ Theory is Gaussian : No excuse, have to understand everything about the Theory.

Take $M_3 = \Sigma_2 \times \mathbb{R}$, $G = U(1)$

There should be a Hilbert space.

$$\mathcal{H}(\Sigma_2, \mathcal{O}(\Sigma_2), k) = ?$$

(The theory depends on orientation of M_3 but $\mathcal{O}(M_3) \cong \mathcal{O}(\Sigma_2)$ equivalent)

When we have spacetime decomposition like $\Sigma_2 \times \mathbb{R}$

Then we can write A as

$$A = A_S + A_0 dx^0$$

$$\Rightarrow \int A dA = - \int dx^0 \left(\int_{\Sigma_2} A_S \partial_0 A_S \right) - \int dx^0 \left(\int_{\Sigma_2} A_0 F_S \right)$$

First order in time derivatives

(review lesson) $S = \int \lambda_i(x) \frac{d x^i}{dt} dt$

↔ The integrate path integral ~~over~~ as paths in phase space

(so path integrals over the spaces of p_s & q_s , (not just q))

A_0 does not appear with derivative

↪ so: its functioning as Lagrange multiplier

↪ so $F_S = 0$ flat gauge field on Σ_2 .

(We can hope that path integral is going to localize to the space of flat gauge fields on Σ_2 (generalizing the plane waves in Minkowski space))

Phase space = Symplectic manifold

$(\mathcal{P}, \omega)$

↑
phase space

↑
2 form

which is closed & non degenerate.

$d\omega = 0$ ~~non~~
non-degenerate.

$$\omega = \frac{1}{2!} \omega_{ij}(x) dx^i dx^j$$

↖ always
~~invertible~~ invertible

$$\delta S = \int \underbrace{(\partial_i \lambda_j - \partial_j \lambda_i)}_{\omega_{ij}} dx^j \frac{dx^i}{dt}$$

In C.S. theory,

Phase Space = Space of Gauge fields on Σ_2

Σ_2 is some topological surface, with orientations and with some holes.



$$\mathcal{P} = \mathcal{A}(\mathcal{P} \rightarrow \Sigma_2)$$

Phase space $\mathcal{P}$ is equal to (here) space of all connections on principle $U(1)$ bundles $\mathcal{P}$ over Σ_2 .

(we got rid of third dimension, by taking Hamiltonian point of view)

What does it mean to quantize $(\mathcal{P}, \omega)$ (phase space)?

- $\omega^{(k)} = k \omega$, $k \in \mathbb{Z}$
- $\mathcal{H}_k =$ Hilbert space (for each k)
- $f \in \text{Fun}(\mathcal{P} \rightarrow \mathbb{C}) \rightsquigarrow Q^{(k)}(f)$ $\mathbb{C}$ -linear operator on $\mathcal{H}_k$.

We need some conditions on $Q^{(k)}$.

- ① $\lim_{k \rightarrow \infty} k \| Q^{(k)}(f) Q^{(k)}(g) - Q^{(k)}(fg) \| < \infty$
- ② $[Q^{(k)}(f), Q^{(k)}(g)] = -\frac{i}{k} Q^{(k)}(\{f, g\}) + \mathcal{O}\left(\frac{1}{k^2}\right)$
- ③ $(Q^{(k)}(f))^\dagger = Q^{(k)}(f^*)$

$\exists$ formal procedure (Fedosov, Kontsevich (Cattaneo-Felder)) (1930)
gives formal series in $1/k$.

① Global separation of Darboux coordinates,

$P = T^*M$, M Riemannian,

$$H = L^2(M), \quad P_i = -i\hbar \left(\frac{\partial}{\partial q_i} + \dots \right)$$

② Kähler Quantization

P has a complex structure.

$$J_\mu^\nu(\phi), \quad J^2 = -1$$

suppose the complex structure is compatible with symplectic structure.

~~$$J_\mu^\nu(\phi) \omega_{\mu\nu}(\phi) = g_{\mu\nu}(\phi) > 0$$~~

$$\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$$

$$\Rightarrow g(v_1, v_2) := \omega(Jv_1, v_2) > 0$$

$\Rightarrow$ Kähler metric

With complex structure, we can choose local complex coordinates $dz^i, d\bar{z}^{\bar{j}}$, and can form Kähler form

$$\omega = \frac{i}{2} g_{i\bar{j}} \cdot dz^i d\bar{z}^{\bar{j}}$$

We have to assume that there is complex structure.

$\rightarrow$ Here we gave an extra piece of data;
The data about Complex Structure (one extra data)

(other piece of extra data)

$L_i \longrightarrow P$ holomorphic line bundle with a positive curvature.
 $\longleftarrow s$ (section, not necessarily holomorphic)

If we have a section s ; Then, we have a Hermitian metric on our line bundle $h(s)$

Then
$$\frac{i}{2\pi} \partial \bar{\partial} \log(h(s)) = \omega = \frac{i}{2\pi} \partial \bar{\partial} K$$

Kähler potential
 (locally defined real analytic function)

Given these extra data.

$\mathcal{H}_k := \text{Hol}(L^{\otimes k})$ (Holomorphic section of k^{th} tensor power of L)

Inner product is,

$$\langle \psi_1, \psi_2 \rangle_{\mathcal{H}_k} = \int_P e^{-kK} \cdot \underbrace{\psi_1^*(z, \bar{z}) \psi_2(z, \bar{z})}_{\text{Kähler potential}} \frac{(w/2\pi)^n}{n!}$$

where $\dim_{\mathbb{R}} P = 2n$
 $\dim_{\mathbb{C}} P = n$

using local trivialization of our line bundle
 (Taking it as a local holomorphic function of z and $\bar{z}$)

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$$\langle \psi_1 | Q^{(k)}(f) | \psi_2 \rangle = \int e^{-kK} \psi_1^* f(z, \bar{z}) \psi_2(z) \dots$$

Ex 1 $\mathcal{P} = \mathbb{R}^2 \cong \mathbb{C}$

$$\frac{\omega}{2\pi} = \frac{dp \wedge dq}{2\pi} = \frac{i}{2\pi} dz \wedge d\bar{z}$$

$$L \rightarrow \mathbb{C}, \quad L^{\otimes k} = \mathbb{C} \times \mathbb{C}$$

$\text{Hol}(L^{\otimes k}) =$ squarely normalizable, entire functions.

$$\langle \psi_1, \psi_2 \rangle = \int e^{-k|z|^2} \psi_1^* \psi_2 \cdot \frac{\omega}{2\pi}$$

define $a^+ := \sqrt{k} Q^{(k)}(z)$, $a = \sqrt{k} Q^{(k)}(\bar{z})$

Bergman representation.

quantization of z

quantization of $\bar{z}$

We can check a^+ & a satisfies usual harmonic oscillator commutation relation.

Ex 2 $\mathcal{P} = \mathbb{C} \mathbb{P}^1 = S^2$

$$L \rightarrow \mathbb{C} \mathbb{P}^1$$

↓ stereograph.
 $\mathbb{C}$

$$\text{Hof } c_1(L) = +1$$

Magnetic monopole bundle

(This is the space where the wave function of an electron in the presence of a magnetic monopole which is sitting inside S^2 is valued)

so, $\mathcal{H}_k = \text{Hol}(L^{\otimes k}) = \text{Holomorphic section of } L^{\otimes k}$ (pg 33)

$$\langle \psi_1, \psi_2 \rangle = \frac{i}{2\pi} \int e^{-kK} \psi_1^* \psi_2 \cdot \frac{dz d\bar{z}}{(1+|z|^2)^2}$$

$$= \frac{i}{2\pi} \int_{\mathbb{C}} \psi_1^* \psi_2 \cdot \frac{dz d\bar{z}}{(1+|z|^2)^{k+2}}, \quad k > 0.$$

so: $1, z, z^2, \dots, z^k$ normalizable. (but not beyond)

(we get different set of holomorphic functions which are normalizable depending on k)

different k is corresponding to the different line bundles, gives very different spaces of states.

$k+1$ dimensional spaces of states.

(In fancy cases, we could not just stereographically project to the complex plane; and ~~we need~~ so we need to work with line bundles)

Define $J^i := (j+1) Q^{(k)}(x^i)$, where $j = k/2$

~~[scribble]~~

Think of $S^2 \subset \mathbb{R}^3 \ni (x^1, x^2, x^3)$

Then $[J^i, J^j] = \sqrt{1} \cdot \epsilon^{ijk} J^k$

$\iff$ so $\mathcal{H}_k$ is representation of $SU(2)$.

$1, z, z^2, \dots, z^k$ are actually eigenstates of J^3 .

$\mathcal{H}_k \cong$ irrep of $SU(2)$ of spin $j = k/2$

$$\dim_{\mathbb{C}} H^k = k+1 = k(1 + \frac{1}{k}) \quad \hbar = 1/k \quad (1939)$$

Bohr Sommerfeld rule, for each unit of phase ~~space~~ space area we should get a quantum state in the semi-classical limit.

$$\int_{\mathbb{C}P^1} \frac{\omega^{(k)}}{2\pi} = k$$

but we got

$$k+1$$

→ This plus 1 is quantum correction of what we expect from Bohr Sommerfeld rule.

In general, if L is sufficiently positive, then,

$$\dim_{\mathbb{C}} H^0(P, L^{\otimes k}) = \sum_{i=0}^{\dim P} (-1)^i \dim H^i(P, L^{\otimes k})$$

$$= \text{Index}(\bar{\partial}_L) = \int_P e^{k c_1(L)} Td(T^{1,0}P)$$

Borel - Weil - Bott Theorem

$$= k^{\dim P/2} \cdot \text{Symplectic volume} (1 + O(1/k))$$

(so we see, The curvature either adds or subtracts quantum states)

Lec 1

$\mathcal{C}$ manifold, $\dim \mathcal{C} = \infty$

$\hookrightarrow$
 $\mathcal{G}$ Lie group, $\dim \mathcal{G} = \infty$

$F: \mathcal{C} \longrightarrow \mathcal{V} \quad ; \mathcal{G} : \text{representation.}$

F is $\mathcal{G}$ -equivariant $F(m \cdot g) = g^{-1} F(m)$

$F^{-1}(0)$

$\hookrightarrow$
 $\mathcal{G}$

$$\mathcal{M} = F^{-1}(0)/\mathcal{G}$$

$$\mathcal{M}^* = \{ m \in F^{-1}(0) \mid \text{Stab}(m) = \{1\} \} / \mathcal{G}$$

Assume $\mathcal{M}^*$ is smooth, compact, oriented manifold of $\dim = d < \infty$.

$$[\mathcal{M}^*] \in H_d(B^*; \mathbb{Z}) \quad (\text{homology class})$$

$$\text{where } B^* = \mathcal{C}^*/\mathcal{G}$$

Now we want to extract numbers out of it.

$$0 \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}_1 \longrightarrow \{1\}$$

finite dimensional Lie group.

$$\hat{\mathcal{M}}^* = F^{-1}(0)/\mathcal{G}_0$$

$$\mathcal{G}_1 \hookrightarrow \hat{\mathcal{M}}^* \downarrow \mathcal{M}$$

$\leadsto$ Characteristic classes $\eta \in H^d(\mathcal{M}^*, \mathbb{Z})$ (homology class)

$$\int_M \eta$$

"invariant"

(pg 2)

Lee 1+2: Bundles, Connection, Curvature.

Lee 3+4: Tools from analysis.

(Fredholm maps, Elliptic: PDEs, ...)

Lee 5+6: Seiberg-Witten gauge Theory

Vector Bundles

family of vector spaces E_m , parametrized by points in manifold M (more generally in topological space)

$$\{E_m \mid m \in M\}$$

assumed to be locally trivial.

$$\text{ie: } \forall m \in M, \exists U; \psi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$$

$$\{E_m \mid m \in M\} \xrightarrow{\pi} M \quad (\text{Natural projection map})$$

$$\pi^{-1}(U) \xrightarrow{\psi} U \times \mathbb{R}^k$$

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times \mathbb{R}^k \\ \searrow \pi & & \swarrow \text{pr}_1 \\ & U & \end{array}$$

This diagram commutes.

SECTION MAP

$$S: M \rightarrow E \quad \text{sections, if } \pi \circ S = \text{id}_M.$$

Smooth.

We can construct $\Lambda^p E, E^*, \text{End}(E), \dots$

sections of TM are called vector fields.
 " " $\Lambda^1 T^*M$ " differential 1-forms

Def 11 $\Gamma(E)$ space of all sections

then $\nabla: \Gamma(E) \rightarrow \Gamma(T^*M \otimes E)$ is an $\mathbb{R}$ linear map

$$\nabla(f \cdot s) = df \otimes s + f \nabla s$$

given " ∇ " a connection,

we can construct another connection ∇'

$$\nabla' = \nabla + \alpha, \text{ where } \alpha \in \Omega^1(M, \text{End}(E))$$

Lemma Let $A: \Gamma(E) \rightarrow \Omega^1(M)$ be an $\mathbb{R}$ -linear map, which is also $C^\infty(M)$ -linear, i.e., $A(fs) = fA(s) \forall f \in C^\infty(M) \text{ and } \forall s \in \Gamma(E)$.
Then $\exists \alpha \in \Omega^1(\text{Hom}(E, E))$ s.t. $A(s) = \alpha \cdot s$

$\downarrow$
 α is one form in M ,
with values in $\text{End}(E)$.

Theorem The space of all connections $A(E)$ is an affine space modelled on $\Omega^1(\text{End}(E))$

$$\begin{array}{ccccccc} 0 \rightarrow \Omega^0(E) & \xrightarrow{\nabla = d_\nabla} & \Omega^1(E) & \xrightarrow{d_\nabla} & \Omega^2(E) & \xrightarrow{d_\nabla} & \dots \\ \parallel & & & & & & \\ \Gamma(E) & & & & & & \end{array}$$

$$\alpha \in \Omega^1(E)$$

α can be locally written as $\alpha = \omega \otimes s$

ω $\nearrow$ k form on m

s $\nwarrow$ a section on E

define $d_\nabla \alpha = d\omega \otimes s + (-1)^{\deg \omega} \omega \wedge \nabla s$

$$d_\nabla \circ d_\nabla = F_\nabla \in \Omega^2(\text{End}(E))$$

$\hookrightarrow$ Curvature of ∇

If we choose local coordinates on M , then we can ask what is $\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} S - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} S$ (pg 4)

$(x_1, \dots, x_m)$ local coordinates on M

$$\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} S - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} S = F_{\nabla} \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right)$$

So; Curvature is measure of non ~~comm~~ commutativity of ~~partial~~ partial covariant derivatives.

locally, $\nabla = d + A$, where $A \in \Omega^1(M, \mathfrak{g}_k(\mathbb{R}))$

Then curvature form F_{∇} is locally given as $F_{\nabla} = dA + A \wedge A$

$F_{\nabla+a} = F_{\nabla} + d_{\nabla} a + a \wedge a$

~~Suppose~~ let $\mathcal{G}(E)$ denote

$$\mathcal{G}(E) = \{ g \in \text{End}(E) \mid g(m) \in GL(E_m) \}$$

It is called Gauge Group.

so; $\nabla^g s := g^{-1} \nabla(g \cdot s)$

∇ and ∇^g are gauge equivalent.

Exercise] Compute F_{∇^g}

Principle Bundle

$$P \xrightarrow{\pi} M \quad (\text{projection})$$

manifold P
base manifold M .

G Lie group, $G \subset P$ (G acts on P)

- $\pi(p \cdot g) = \pi(p)$ (π is G invariant)
- $\forall m \in M, \pi^{-1}(m) \hookrightarrow G$ free transitive
- locally trivial

(Here we have family of Lie groups parametrized by point of Base Manifold).

If we have Vector Bundle, then there is canonical way to construct Principle Bundle.

E is vector bundle.

$Fr(E_m) = \{ \text{set of all bases of } E_m \}$

Defⁿ $Fr(E) = \bigsqcup_{m \in M} Fr(E_m) = \bigsqcup_{m \in M} \{ \text{set of all isomorphism from } \mathbb{R}^k \text{ to } E_m \}$

Frame bundle of E

$\mathbb{R}^k \xrightarrow{\cong} E_m$

P principle G bundle.

associated vector Bundle

$\rho: G \longrightarrow GL_k(\mathbb{R})$

Then $P \times \mathbb{R}^k / G$: $(p, x) \cdot g = (p \cdot g, \rho(g^{-1})x)$

$E := P \times \mathbb{R}^k / G$ is a vector bundle; There is natural projection to M ; $E \longrightarrow M$.

Defⁿ A connection on P is $A \in \Omega^1(P; \mathfrak{g})$

$\mathfrak{g} = \text{Lie}(G)$ s.t.

(A is 1-form on P , whose value is in $\mathfrak{g}$)

- A is G equivariant,

$R_g^* \omega = \text{ad}_{g^{-1}} \omega$

$\therefore \omega$ takes values in Lie Algebra $\mathfrak{g}$

$$\bullet A(K_\xi) = \xi \quad \forall \xi \in \mathfrak{g}$$

↑ infinitesimal action of G on P .

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Vector Bundle	Principle bundles
vector bundle E	$Fr(E)$
$E := P \times_{G, \rho} \mathbb{R}^k$	P
S section	$\hat{S}: P \rightarrow \mathbb{R}^k$ G -equivariant.
∇ connection	$A \in \Omega^1(P, \mathfrak{g})$
$S(P, \mathcal{A}) = [S(P), \mathcal{A}]$	$S(P, \mathcal{A}) = [\hat{S}(P), \mathcal{A}]$
$S(P, \mathcal{A}) = [S(P), \mathcal{A}]$	
$\mathcal{G}(E)$ gauge group	$\mathcal{G}(P) = \{ \varphi: P \rightarrow P \mid \varphi(p \cdot g) = \varphi(p) \cdot g \}$
	$\varphi(p \cdot g) = g^{-1} \varphi(p) g$
	$\mathcal{G}(P) = \{ \Psi: P \rightarrow G \mid \Psi(p \cdot g) = g^{-1} \Psi(p) g \}$
F_∇ curvature	$F_A = dA + \frac{1}{2} [A \wedge A]$

Theorem (Bianchi Identity)

$$d_\nabla(F_\nabla) = 0 \quad \Longleftrightarrow \quad dF_A = -[A \wedge F_A]$$

Ex) $P \rightarrow M$ $U(1)$ - bundle

$$\langle F \rangle = \frac{\int \mathcal{D}[\phi] F(\phi) \exp\left(\frac{i}{\hbar} S[\phi]\right)}{\int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)}$$

$$Z = \int \mathcal{D}\phi \cdot \exp\left(\frac{i}{\hbar} S[\phi]\right) \quad \text{Partition Function.}$$

We have to Study Propagators.

$$\Delta_F(x-y) = \Theta(x^0 - y^0) \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle + \Theta(y^0 - x^0) \langle 0 | \bar{\psi}(y) \psi(x) | 0 \rangle$$

$$= \langle 0 | T(\psi(x) \bar{\psi}(y)) | 0 \rangle$$

↑
Feynman Propagator.

For simplicity, we also write as $G(x, y)$

(usually $G(x, y) = G(x - y)$)

Generality

Suppose $S[\phi] = \frac{i}{2} \int d^4x \phi L \phi$

note path order
formalism takes
into account
ordering of
operators.

Then

$$\langle \phi(x_1) \phi(x_2) \rangle = \int \mathcal{D}[\phi] F(\phi) \quad L \text{ is an operator.}$$

Then

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{\int \mathcal{D}[\phi] \phi(x_1) \phi(x_2) \cdot \exp\left(\frac{i}{\hbar} \int d^4x \frac{1}{2} \phi(x) L \phi(x)\right)}{\int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar} \int d^4x \frac{1}{2} \phi(x) L \phi(x)\right)}$$

$$= (L^{-1})_{x_1 x_2} \quad \Rightarrow \text{This is a continuum limit for } N \text{ dimensional Gaussian integral.}$$

L^{-1} is the inverse of the operator $\frac{iL}{\hbar}$

FOR BOSONIC
FIELD VARIABLES.

where $L^{-1}(x_1, x_2)$ satisfies

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$$\frac{iL}{\hbar} (L^{-1}(x_1, x_2)) = \delta^{d+1}(x_1 - x_2)$$

→ This is the meaning of $L^{-1}(x_1, x_2)$ being
inverse of $\frac{iL}{\hbar}$ (RHS is continuum analogue of identity)

~~$L^{-1}(x_1, x_2)$~~

$$L(L^{-1}(x_1, x_2)) = -i\hbar \delta^{d+1}(x_1 - x_2)$$

$L^{-1}(x_1, x_2) = L^{-1}(x_1 - x_2)$ This is the equation which is needed to be solved.

lets call $L^{-1}(x_1 - x_2)$ to be $G(x_1 - x_2)$

So, we have

$$\langle \phi(x_1) \phi(x_2) \rangle = G(x_1 - x_2)$$

This is to be solved to get propagator.

where $G(x_1 - x_2)$ satisfies

$$L(G(x_1 - x_2)) = -i\hbar \delta^{d+1}(x_1 - x_2)$$

1 Klein Gordon Field.

$$\eta_{\mu\nu} = \text{diag}(+1, \underbrace{-1, -1, \dots, -1}_d \text{ times})$$

Action for Klein Gordon Field is

$$\square \equiv \partial_\mu \partial^\mu$$

$$S = \int d^{d+1}x \left[\frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \right]$$

$$= \int d^{d+1}x \left[\frac{1}{2} \partial_\mu \partial^\mu \phi^2 - \frac{1}{2} m^2 \phi^2 \right] + \int d^{d+1}x \left[\frac{1}{2} \partial_\mu (\phi \partial^\mu \phi) \right]$$

$$\text{we } \partial_\mu (\phi \partial^\mu \phi) = \partial_\mu \phi \partial^\mu \phi + \phi \partial_\mu \partial^\mu \phi$$

$$\Rightarrow (\partial_\mu \phi) (\partial^\mu \phi) = \partial_\mu (\phi \partial^\mu \phi) + \phi \square \phi$$

$$S = \int d^4x \cdot \frac{1}{2} \phi [+\square + m^2] \phi + \int d^4x \cdot \frac{1}{2} \partial_\mu (\phi \partial^\mu \phi)$$

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→ This is surface integral term.

(Taking the fields to vanish at boundary (at ∞); This term drops out)

$$\Rightarrow S = \int d^4x \cdot \frac{1}{2} \phi [+\square + m^2] \phi$$

~~$L_{K.G.} = \square + m^2$~~ ^{SB} $L_{K.G.} = \square + m^2$

let $G_{K.G.}(x_1 - x_2)$ be the Greens function or better say Propagator for K.G. field.

Then $(+\square + m^2) G_{K.G.}(x_1 - x_2) = -i \delta^4(x_1 - x_2)$

→ To solve this; we resort back to Fourier Transforms.

$$\delta^4(x_1 - x_2) = \frac{1}{(2\pi)^4} \int d^4k \cdot e^{ik^\mu (x_{1\mu} - x_{2\mu})}$$

~~$G_{K.G.}(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} e^{ik^\mu x_{1\mu}} \hat{G}_{K.G.}(k)$~~

$$G_{K.G.}(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} e^{ik^\mu (x_1 - x_2)_\mu} \hat{G}_{K.G.}(k^\mu)$$

Substituting This in the equation gives.

$$\int \frac{d^4 k}{(2\pi)^4} e^{ik^\mu (x_1 - x_2)_\mu} \cdot [-k_\mu k^\mu + m^2] \hat{G}_{k.G.}(k^\mu) = \int \frac{d^4 k}{(2\pi)^4} \cdot (-i\hbar) e^{ik^\mu (x_1 - x_2)_\mu} \quad (Pg 4)$$

$$\Rightarrow (k_\mu k^\mu - m^2) \hat{G}_{k.G.}(k) = i\hbar$$

$$\Rightarrow \hat{G}_{k.G.}(k) = \frac{i\hbar}{(k^2 - m^2)}$$

$$\langle \phi(x_1) \phi(x_2) \rangle \Big|_{k.G.} = G_{k.G.}(x_1 - x_2) = \int \frac{d^4 k}{(2\pi)^4} \frac{i\hbar \cdot e^{ik^\mu (x_1 - x_2)_\mu}}{(k^\mu k_\mu - m^2)}$$

[2] Dirac Field (Spin 1/2)

$$\mathcal{L}_D = \bar{\Psi} (i \gamma^\mu \partial_\mu - m) \Psi$$

$$(\mathcal{L}_D = \bar{\Psi}_a (i \gamma^\mu_{ab} \partial_\mu - m \mathbb{I}_{ab}) \Psi_b)$$

Feynman Slash
Notation

Hence the action is

$$S_D = \int d^4 x \cdot \bar{\Psi} (i \gamma^\mu \partial_\mu - m) \Psi$$

$$\gamma^\mu F_\mu = \not{F}$$

$$\Leftrightarrow \text{Dirac Equation: } \boxed{(i \not{\partial} - m) \Psi(x) = 0}$$

let $G_D(x_1, x_2)$ be the propagator for fermionic field

~~$$\text{Then Note: } \langle 0 | T \Psi(x_1) \bar{\Psi}(x_2) | 0 \rangle = G_D(x_1, x_2)$$~~

Note that $\Psi(y)$, $\bar{\Psi}(x)$ are Fermionic Grassmann field variables.

$$\langle 0 | T \bar{\Psi}(x_1) \Psi(x_2) | 0 \rangle = G_D(x_1, x_2)$$

$$(G_D(x_1 - x_2))_{ab}$$

Hence the equation for the propagator is

$$\frac{i}{\hbar} [i \not{\partial} - m] G_D(x_1 - x_2) = \delta^{d+1}(x_1 - x_2)$$

$$\Rightarrow [i \not{\partial} - m] G_D(x_1 - x_2) = -i \hbar \delta^{d+1}(x_1 - x_2)$$

$$G_D(x_1 - x_2) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot e^{-ik^\mu \cdot (x_1 - x_2)_\mu} \cdot \hat{G}_D(k)$$

$$\delta^{d+1}(x_1 - x_2) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{-ik^\mu \cdot (x_1 - x_2)_\mu}$$

$$\Rightarrow \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot e^{-ik^\mu (x_1 - x_2)_\mu} [\gamma^\mu k_\mu - m] \hat{G}_D(k) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot e^{-ik^\mu (x_1 - x_2)_\mu} (-i \hbar)$$

$$\Rightarrow [K - m] \hat{G}_D(k) = -i \hbar$$

$$\Rightarrow \hat{G}_D(k) = \frac{-i \hbar}{(K - m)}$$

$$\Rightarrow G_D(x_1 - x_2) = -i \hbar \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot \frac{e^{-ik \cdot (x_1 - x_2)}}{(K - m)}$$

Note The propagator $\Delta_{\text{fermion}}(x_1 - x_2)$ is defined as

$$\Delta_{\text{fermion}} = \langle 0 | T \psi(x_1) \bar{\psi}(x_2) | 0 \rangle$$

from anticommutation of Grassman field variables ψ & $\bar{\psi}$

$$\Rightarrow \Delta_{\text{fermion}}(x_1 - x_2) = - \langle 0 | T \bar{\psi}(x_2) \psi(x_1) | 0 \rangle$$

but $\langle 0 | T \bar{\Psi}(x_2) \Psi(x_1) | 0 \rangle = G_D(x_2 - x_1)$

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$$\Rightarrow \Delta_{\text{fermion}} = - G_D(x_2 - x_1)$$

$$= - \left(-i\hbar \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot \frac{e^{-ik^\mu(x_2-x_1)_\mu}}{k - m} \right)$$

$$\Delta_{\text{fermion}}(x_1 - x_2) = i\hbar \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot \frac{e^{ik^\mu(x_1-x_2)_\mu}}{k - m}$$

Note $k k = \frac{1}{2} (k k + k k) = \frac{1}{2} (\gamma_\mu k^\mu \gamma_\nu k^\nu + \gamma_\nu k^\nu \gamma_\mu k^\mu)$

$$= \frac{1}{2} (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) k^\mu k^\nu$$

we know $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$

$$\Rightarrow k k = \frac{1}{2} \cdot 2g_{\mu\nu} k^\mu k^\nu = k^\mu k_\mu = k^2$$

also: $(k - m)(k + m) = k^2 + k m - k m - m^2$

$$= k^2 - m^2$$

Multiply denominator & numerator of the integrand of $\Delta_{\text{fermion}}(x_1 - x_2)$ by $(k + m)$

we get

$$\Delta_{\text{fermion}}(x_1 - x_2) = i\hbar \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \cdot \frac{(k + m)}{(k^2 - m^2)} \cdot e^{ik^\mu(x_1-x_2)_\mu}$$

3) U(1) Chern Simons Theory.

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Action
$$S = \frac{k}{8\pi} \int_{M_3} d^3x \epsilon^{\lambda\mu\nu} A_\lambda \partial_\mu A_\nu$$

finding eqn of motion

$$\Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right) = \frac{\partial \mathcal{L}}{\partial A_\nu} \Rightarrow \partial_\mu (\epsilon^{\lambda\mu\nu} A_\lambda) = \epsilon^{\lambda\mu\sigma} \cdot \delta_{\lambda\nu} \cdot \partial_\mu A_\sigma$$

$$\Rightarrow \partial_\mu (\epsilon^{\lambda\mu\nu} A_\lambda) = \epsilon^{\nu\mu\sigma} \partial_\mu A_\sigma = \epsilon^{\nu\mu\lambda} \partial_\mu A_\lambda = \epsilon^{\nu\mu\lambda} \partial_\lambda A_\mu = -\epsilon^{\lambda\mu\nu} \partial_\lambda A_\mu = \epsilon^{\lambda\mu\nu} \partial_\lambda A_\mu$$

$$\Rightarrow \epsilon^{\lambda\mu\nu} (\partial_\mu A_\lambda - \partial_\lambda A_\mu) = 0$$

$$\boxed{\epsilon^{\lambda\mu\nu} F_{\mu\lambda} = 0}$$

$F_{\mu\mu} = 0$ due to antisymmetry

$$\boxed{F_{\mu\lambda} = 0}$$

$\Rightarrow$ Equation of Motion corresponds to Flat Gauge Field.

Now, we compute Propagators.

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{\int \mathcal{D}[A] \exp \left(\frac{ik}{8\pi\hbar} \int_{M_3} d^3x \epsilon^{\lambda\mu\nu} A_\lambda \partial_\mu A_\nu \right) A_\mu(x) A_\nu(y)}{\int \mathcal{D}[A] \exp \left(\frac{ik}{8\pi\hbar} \int_{M_3} d^3x \epsilon^{\lambda\mu\nu} A_\lambda \partial_\mu A_\nu \right)}$$

$$\equiv G_{\mu\nu}(x-y)$$

$$\frac{iS}{\hbar} = \frac{1}{2} \int_{M_3} d^3x A_\lambda \left[\frac{ik}{4\pi\hbar} \epsilon^{\lambda\mu\nu} \partial_\mu \right] A_\nu$$

Hence $[G]$ is inverse of the operator

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$$[L] = \frac{ik}{2\pi\hbar} [\epsilon^{\mu\sigma\nu} \partial_\sigma] \quad [.] \text{ represents matrix}$$

$$\Rightarrow [L][G(x-y)] = \mathbb{1}_{3 \times 3} \cdot \delta^3(x-y)$$

$$\Rightarrow \frac{ik}{2\pi\hbar} \cdot \epsilon^{\mu\sigma\nu} \partial_\sigma [G_{\nu\lambda}(x-y)] = g^\mu_\lambda \cdot \delta^3(x-y)$$

$$\Rightarrow \boxed{\epsilon^{\mu\sigma\nu} \partial_\sigma G_{\nu\lambda}(x-y) = -\frac{i2\pi\hbar}{k} g^\mu_\lambda \cdot \delta^3(x-y)} \quad (*)$$

→ Solving this equation gives our propagator $G_{\nu\lambda}$.

$$\delta^3(x-y) = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot (x-y)}$$

$$G_{\mu\nu}(x-y) = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot (x-y)} \hat{G}_{\mu\nu}(p)$$

Hence we get

$$\Rightarrow \epsilon^{\mu\sigma\nu} \cdot i p_\sigma \cdot \hat{G}_{\nu\lambda}(p) = -\frac{i2\pi\hbar}{k} g^\mu_\lambda$$

$$\Rightarrow \epsilon^{\mu\sigma\nu} \cdot p_\sigma \cdot \hat{G}_{\nu\lambda}(p) = -\frac{2\pi\hbar}{k} g^\mu_\lambda$$

multiply this equation by $\epsilon_{\mu\alpha\beta}$

Using $\epsilon_{\mu\alpha\beta} \epsilon^{\mu\epsilon\nu} = \delta_{\alpha}^{\epsilon} \delta_{\beta}^{\nu} - \delta_{\alpha}^{\nu} \delta_{\beta}^{\epsilon}$

$$\Rightarrow (\delta_{\alpha}^{\epsilon} \delta_{\beta}^{\nu} - \delta_{\alpha}^{\nu} \delta_{\beta}^{\epsilon}) P_{\epsilon} \hat{G}_{\nu\lambda}(p) = -\frac{4\pi\hbar}{k} \epsilon_{\mu\alpha\beta} g^{\mu\lambda}$$

$$\Rightarrow P_{\alpha} \hat{G}_{\beta\lambda} - P_{\beta} \hat{G}_{\alpha\lambda} = -\frac{4\pi\hbar}{k} \epsilon_{\lambda\alpha\beta}$$

$$P_{\alpha} \hat{G}_{\beta\lambda} - P_{\beta} \hat{G}_{\alpha\lambda} = -\frac{4\pi\hbar}{k} \epsilon_{\lambda\alpha\beta}$$

holds trivially for $\alpha = \beta$.

~~NOT able to solve;~~
~~because the operator has non~~
~~trivial kernel. (Not easy to~~
~~invert)~~

Seems problematic to solve

Get back to the eqn without fourier transformation; but multiplied with $\epsilon_{\mu\alpha\beta}$.

$$\partial_{\alpha} G_{\beta\lambda} - \partial_{\beta} G_{\alpha\lambda} = \frac{-i4\pi\hbar}{k} \epsilon_{\lambda\alpha\beta} \delta^3(x-y)$$

Note that $\epsilon_{\alpha\beta\gamma} \partial^{\gamma} (\epsilon^{\alpha\beta\lambda} \partial_{\lambda}) = 2 \partial^{\alpha} \partial_{\alpha}$

~~$\epsilon_{\alpha\beta\gamma} \partial^{\gamma} (\epsilon^{\alpha\beta\lambda} \partial_{\lambda}) = 2 \partial^{\alpha} \partial_{\alpha}$~~

← Laplacian.

$$\Rightarrow \partial^2 = \frac{1}{2} (\epsilon_{\alpha\beta\gamma} \partial^{\gamma} (\epsilon^{\alpha\beta\lambda} \partial_{\lambda}))$$

∂^2 is a kind of square of our operator whose greens function we want to evaluate.

$$G_{\partial^2} \sim L^{-2}$$

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L is our operator in concern.

Then $L(L^{-2}) \sim L^{-1}$

$\Rightarrow L^{-1}$ is our greens function.

Normalization of L^{-1} can be adjusted from (*)

Greens function for ∂^2 is proportional to $\frac{1}{|x|}$

$$G_{\partial^2} \propto \frac{1}{|x-y|}$$

~~$$G_{L^{-1}}(x-y) = \sum \epsilon_{\mu\sigma\nu} \partial^\sigma \left(\frac{1}{|x-y|} \right)$$~~

$$G_{L^{-1}}(x-y) = \sum L \left(\frac{1}{|x-y|} \right) = \sum \epsilon_{\mu\sigma\nu} \partial^\sigma \left(\frac{1}{|x-y|} \right)$$

↑
Normalization constant.

(to be found by putting this $G_{L^{-1}}(x-y)$ in (*))

$$\Rightarrow G_{L^{-1}}(\vec{x}-\vec{y}) = \sum \epsilon_{\mu\sigma\nu} \partial^\sigma \left(\frac{1}{\sqrt{(x-y) \cdot (x-y)}} \right)$$

$$= -\frac{1}{2} \sum \epsilon_{\mu\sigma\nu} \frac{(x-y)^\sigma}{(\sqrt{(x-y) \cdot (x-y)})^3}$$

$$\Rightarrow \boxed{G_{L^{-1}}(x-y)_{\mu\nu} = - \sum \epsilon_{\mu\sigma\nu} \frac{(x-y)^\sigma}{|x-y|^3}}$$

$$\epsilon^{\mu\sigma\nu} \partial_\sigma \left(-\xi \epsilon_{\nu\theta\lambda} \frac{(x-y)^\theta}{|x-y|^3} \right)$$

$$= -\xi (\epsilon^{\mu\sigma\nu} \epsilon_{\nu\theta\lambda}) \partial_\sigma \left(\frac{(x-y)^\theta}{|x-y|^3} \right)$$

$$= -\xi (\delta_\theta^\mu \delta_\lambda^\sigma - \delta_\theta^\sigma \delta_\lambda^\mu) \partial_\sigma \left(\frac{(x-y)^\theta}{|x-y|^3} \right)$$

$$\delta^\mu_\nu = g^\mu_\nu$$

$$= -\xi \partial_\lambda \left(\frac{(x-y)^\mu}{|x-y|^3} \right) + \xi \cdot \partial_\theta \left(\frac{(x-y)^\theta}{|x-y|^3} \right) \cdot \delta^\mu_\nu$$

→ No need to worry about this; can be adjusted by gauge fixing.

→ This is laplacian of $\frac{1}{|x-y|}$ and hence give the delta function piece.

$$\partial^\theta \left(\frac{1}{|x-y|} \right) = -\frac{1}{2} \frac{x^\theta \cdot (x-y)}{|x-y|^3}$$

$$\Rightarrow \partial_\theta \partial^\theta \left(\frac{1}{|x-y|} \right) = -\partial_\theta \left(\frac{(x-y)^\theta}{|x-y|^3} \right)$$

This is $-4\pi \delta^3(x-y)$
(previous result)

Hence equating the dirac delta terms.

$$\xi \cdot 4\pi \delta^3(x-y) \delta^\mu_\nu = -\frac{i 4\pi \hbar}{k} \delta^\mu_\nu \cdot \delta^3(x-y)$$

$$\Rightarrow \boxed{\xi = \frac{-i \hbar}{k}}$$

Hence

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$$G_{L-1}(x-y)_{\mu\nu} = \frac{i\hbar}{k} \epsilon_{\mu\sigma\nu} \frac{(x-y)^\sigma}{|x-y|^3}$$

→ This is the required Propagator $\langle A_\mu(x) A_\nu(y) \rangle$

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{i\hbar}{k} \epsilon_{\mu\sigma\nu} \frac{(x-y)^\sigma}{|x-y|^3}$$

$$G_{\mu\nu}(x-y) = -\frac{i\hbar}{k} \epsilon_{\mu\nu\sigma} \frac{(x-y)^\sigma}{|x-y|^3}$$

Consider the Wilson loop variables

$$A = A_\mu(x) dx^\mu$$

$$W_m(\gamma) = \exp\left(im \oint_\gamma A\right)$$

(take the irreducible representation of $U(1)$)

$$W_m(\gamma) = \sum_{n=0}^{\infty} \frac{(-in)^n}{n!} \left(\oint_\gamma A_\mu(x) dx^\mu \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-in)^n}{n!} \underbrace{\oint_\gamma \oint_\gamma \dots \oint_\gamma}_{n \text{ times}} A_{\mu_1}(x_1) \dots A_{\mu_n}(x_n) dx_1^{\mu_1} \dots dx_n^{\mu_n}$$

$$\langle W_m(\gamma) \rangle = \sum_{n=0}^{\infty} \frac{(-in)^n}{n!} \underbrace{\oint_\gamma \dots \oint_\gamma}_{n \text{ times}} \langle A_{\mu_1}(x_1) \dots A_{\mu_n}(x_n) \rangle dx_1^{\mu_1} \dots dx_n^{\mu_n}$$

In this theory, we have propagator of the kind $\langle A_\mu(x) A_\nu(y) \rangle = G_{\mu\nu}(x-y)$.

Use Wick's Theorem.

$$\langle A_\mu(x) A_\nu(y) \rangle = \begin{array}{c} x \\ \bullet \\ \mu \end{array} \xrightarrow{\quad} \begin{array}{c} y \\ \bullet \\ \nu \end{array}$$

note $\langle A_{\mu_1}(x_1) A_{\mu_2}(x_2) \dots A_{\mu_{2m+1}}(x_{2m+1}) \rangle = 0$

ex) $\langle A_{\mu_1}(x_1) A_{\mu_2}(x_2) A_{\mu_3}(x_3) A_{\mu_4}(x_4) \rangle =$

$\begin{array}{c} 1 \\ \bullet \\ 2 \end{array} \begin{array}{c} 4 \\ \bullet \\ 3 \end{array}$

+

$\begin{array}{c} 1 \\ \bullet \\ 2 \end{array} \begin{array}{c} 4 \\ \bullet \\ 3 \end{array}$

+

$\begin{array}{c} 1 \\ \bullet \\ 2 \end{array} \begin{array}{c} 4 \\ \bullet \\ 3 \end{array}$

We see that

$$\oint \dots \oint \langle A_{\mu_1}(x_1) \dots A_{\mu_{2m}}(x_{2m}) \rangle dx_1^{\mu_1} \dots dx_{2m}^{\mu_{2m}} = C_{2m} \left(\oint \oint \langle A_\mu(x) A_\nu(y) \rangle dx^\mu dx^\nu \right)^m$$

where C_{2m} is the associated combinatorial factor.

C_{2m} is the no. of ways of joining $2m$ elements into pairs

$$C_{2m} = \frac{(2m)!}{(2^m m!)}$$

$$\Rightarrow \langle W_m(\gamma) \rangle = \sum_{m=0}^{\infty} \underbrace{\frac{(-i)^{2m}}{(2m)!}}_{\frac{(-m^2)^m}{(2m)!} \cdot \frac{(2m)!}{2^m m!}} C_{2m} \left(\oint \oint \langle A_\mu(x) A_\nu(y) \rangle dx^\mu dx^\nu \right)^m$$

$$\langle W_m(\gamma) \rangle = \sum_{m=0}^{\infty} \frac{(-m^2/2)^m}{m!} \left(\oint \oint \langle A_\mu(x) A_\nu(x) \rangle dx^\mu dx^\nu \right)^m$$

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$$\langle W_m(\gamma) \rangle = \exp \left(-\frac{n^2}{2} \oint \oint_{\gamma \gamma} \langle A_\mu(x) A_\nu(y) \rangle dx^\mu dy^\nu \right)$$

$$\Rightarrow \langle W_m(\gamma) \rangle = \exp \left(-\frac{n^2}{2} \oint \oint_{\gamma \gamma} \epsilon_{\mu\nu} (x-y) dx^\mu dy^\nu \right)$$

plugging $\epsilon_{\mu\nu}(x-y)$ which we found.

$$\langle W_m(\gamma) \rangle = \exp \left(\frac{n^2}{2} \frac{i\hbar}{k} \oint \oint_{\gamma \gamma} \epsilon_{\mu\nu\sigma} \frac{(x-y)^\sigma}{|x-y|^3} dx^\mu dy^\nu \right)$$

Now we use the Gauss Linking Number Formula.

$$lk(\gamma_\alpha, \gamma_\beta) = \frac{1}{4\pi} \int_{\gamma_\alpha} dx^\mu \int_{\gamma_\beta} dy^\nu \cdot \epsilon_{\mu\nu\sigma} \cdot \frac{(x-y)^\sigma}{|x-y|^3}$$

$$\Rightarrow \langle W_m(\gamma) \rangle = \exp \left(\frac{2\pi i\hbar}{k} \cdot n^2 \cdot lk(\gamma, \gamma) \right)$$

$lk(\gamma, \gamma)$ is self linking number.

$$\prod_\alpha W_{m_\alpha}(\gamma_\alpha) = \prod_\alpha \exp \left(i m_\alpha \oint_{\gamma_\alpha} A \right)$$

$$= \exp \left(i \sum_\alpha m_\alpha \oint_{\gamma_\alpha} A \right)$$

$$\exp(i \sum_{\alpha} \eta_{\alpha} \oint A_{\mu}(x) dx^{\mu})$$

$$= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \left(\sum_{\alpha} \eta_{\alpha} \oint A_{\mu}(x) dx^{\mu} \right)^m$$

$$= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \left(\sum_{\alpha_1, \dots, \alpha_m} \eta_{\alpha_1} \dots \eta_{\alpha_m} \oint_{\gamma_{\alpha_1}} \dots \oint_{\gamma_{\alpha_m}} A_{\mu_1}(x_1) \dots A_{\mu_m}(x_m) dx_1^{\mu_1} \dots dx_m^{\mu_m} \right)$$

define $W(\alpha^N) = \prod_{\alpha=1}^N W_{\eta_{\alpha}}(\gamma_{\alpha})$

Then by taking expectation value; and using the fact that expectation value of odd no. of fields is zero gives:

$$\langle W(\alpha^N) \rangle = \sum_{m=0}^{\infty} \frac{(-i)^m}{2^m m!} \left(\sum_{\alpha_1, \dots, \alpha_{2m}} \eta_{\alpha_1} \dots \eta_{\alpha_{2m}} \oint_{\gamma_{\alpha_1}} \dots \oint_{\gamma_{\alpha_{2m}}} \langle A_{\mu_1}(x_1) \dots A_{\mu_{2m}}(x_{2m}) \rangle dx_1^{\mu_1} \dots dx_{2m}^{\mu_{2m}} \right)$$

$$\langle A_{\mu_1}(x_1) \dots A_{\mu_{2m}}(x_{2m}) \rangle = (2^m m!) \langle A_{\mu_1}(x_1) A_{\mu_2}(x_2) \rangle \dots \langle A_{\mu_{2m-1}}(x_{2m-1}) A_{\mu_{2m}}(x_{2m}) \rangle$$

$$\langle W(\alpha^N) \rangle = \sum_{m=0}^{\infty} \frac{(-i)^m}{m! 2^m} \left(\sum_{\alpha_1, \dots, \alpha_{2m}} \left\{ \eta_{\alpha_1} \eta_{\alpha_2} \oint_{\gamma_{\alpha_1}} \oint_{\gamma_{\alpha_2}} \langle A_{\mu_1}(x_1) A_{\mu_2}(x_2) \rangle dx_1^{\mu_1} dx_2^{\mu_2} \right\} \times \dots \right)$$

$$\times \dots \times \left\{ \eta_{\alpha_{2m-1}} \eta_{\alpha_{2m}} \oint_{\gamma_{\alpha_{2m-1}}} \oint_{\gamma_{\alpha_{2m}}} \langle A_{\mu_{2m-1}}(x_{2m-1}) A_{\mu_{2m}}(x_{2m}) \rangle dx_{2m-1}^{\mu_{2m-1}} dx_{2m}^{\mu_{2m}} \right\}$$

$\Rightarrow$

$$\langle W(\alpha^N) \rangle = \sum_{m=0}^{\infty} \frac{(-1/2)^m}{m!} \left[\sum_{\alpha_1, \dots, \alpha_{2m}} \left\{ \prod_{k=1}^m \eta_{\alpha_{2k-1}} \eta_{\alpha_{2k}} \oint_{\gamma_{\alpha_{2k-1}}} \oint_{\gamma_{\alpha_{2k}}} \langle A_{\mu_{2k-1}}(x_{2k-1}) A_{\mu_{2k}}(x_{2k}) \rangle dx_{2k-1}^{\mu_{2k-1}} dx_{2k}^{\mu_{2k}} \right\} \right]$$

Hence closing up ; and we get exponential

(Pg 16)

$$\langle W(\alpha_i^N) \rangle = \exp \left[\frac{-1}{2} \sum_{\alpha, \beta} \eta_\alpha \eta_\beta \oint_{\gamma_\alpha} \oint_{\gamma_\beta} \langle A_\mu(x) A_\nu(y) \rangle dx^\mu dx^\nu \right]$$

$$\langle W(\alpha_i^N) \rangle = \exp \left[\frac{i\hbar}{2k} \sum_{\alpha, \beta} \eta_\alpha \eta_\beta \oint_{\gamma_\alpha} \oint_{\gamma_\beta} \epsilon_{\mu\nu\sigma} \frac{(x-y)^\sigma}{|x-y|^3} dx^\mu dx^\nu \right]$$

U(1) CHERN SIMON

$$\langle W(\{\alpha_n\}_{n=1}^N) \rangle = \exp \left[\frac{2\pi i\hbar}{k} \sum_{\alpha, \beta} \eta_\alpha \eta_\beta \text{L.k.}(\gamma_\alpha, \gamma_\beta) \right]$$



Shoaib Akhtar

An intense mathematical calculation regarding propagators ; especially for U(1) Chern Simon in complete generality is potrait.

2nd September, 2020

$$F = dA \quad , \quad F \wedge F = d(A \wedge dA)$$

(1+2) dimensions

Considering a 4d path integral with $\partial M_3 = M_3$,

Then we worry about $\int_{M_3} A \wedge dA$.

$$A \wedge F \longrightarrow A \wedge F + d(\epsilon F) \quad \Rightarrow \text{small gauge transformation.}$$

$$A \wedge F \longrightarrow A \wedge F + \omega \wedge F \quad \Rightarrow \text{large gauge transformation.}$$

$$S = \int_{M_3} \frac{-1}{8\pi^2 e^2} dA \wedge * dA + \frac{k}{2\pi} A \wedge dA$$

Chern Simons Action: $S_{CS} = \frac{k}{2\pi} \int_{M_3} A \wedge dA$

not gauge invariant.

$$\boxed{F = dA}$$

But, The thing which matters is $e^{iS_{CS}}$

$$e^{iS_{CS}} = e^{i \frac{k}{2\pi} \int_{M_3} A \wedge dA} \longrightarrow e^{\frac{ik}{2\pi} \int A \wedge dA + \frac{ik}{2\pi} \int \omega \wedge F}$$

$$\Rightarrow \boxed{e^{\frac{ik}{2\pi} \int \omega \wedge F} = 1} \Rightarrow \text{This must hold, for path integral measure to be gauge invariant.}$$

$\hookrightarrow$ This leads to quantization condition, & sets $k \in \mathbb{Z}$

Let R be a finite-dimensional representation of the gauge group $U(1)$, and $\gamma \subset M_3$ is a closed oriented loop; Then the chemical observable is

$$W(R, \gamma) := \text{Tr}_R \exp \left(i \oint_{\gamma} A \right)$$

$$A = A_{\mu} dx^{\mu}.$$

"For $U(1)$, every finite dimensional representation is fully reducible to a sum of one-dimensional irreducibles & the irreducible representations are $\rho_m(z) = z^m$ for $m \in \mathbb{Z}$."

Using

• Peter Weyl Theorem

• Schur's Lemma.

Hence, in general we can consider

$$W(m, \gamma) := \exp \left(i m \oint_{\gamma} A \right)$$

The $e^2 \rightarrow \infty$ limit of the correlators will only depend on γ up to "isotopy."

So, we have to compute the following.

$$\left\langle \prod_{\alpha} W(m_{\alpha}, \gamma_{\alpha}) \right\rangle = \int_{A/U(1)} [dA] \cdot e^{i \frac{k}{2\pi} \int_{M_3} A \wedge dA} \cdot \prod_{\alpha} W(m_{\alpha}, \gamma_{\alpha})$$

Chern Simons Action

$$S_{c.s.} = \frac{k}{4\pi} \int_M \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

Here we consider M to be a 3 dimensional manifold. (denote the dimensionality by the symbol M_3)
(Because in higher dimensions, every knot is an unknot, and in lower dimensions we can't form a knot (just unknot))

$$S_{c.s.} = \frac{k}{4\pi} \int_{M_3} \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

Consider the Wilson loop $W_R(K)$

$$W_R(K) = \text{Tr}_R \mathcal{P} \exp \left(i \oint_K A \right)$$

$K \subset M_3$ be a closed loop (a knot)

R be a finite dimensional representation of $U(1)$ gauge group.

Here we are dealing with Abelian gauge group, so, we can remove path ordering in the definition of $W_R(K)$

$$W_R(K) = \text{Tr}_R \exp \left(i \oint_K A \right)$$

$A = A_\mu dx^\mu$ one form

$A_\mu = A_\mu^a T_a$, with $a=1, 2, \dots, n^2-1$

Here A_μ is $n \times n$ matrix

Concerning the Representation.

We have to compute the correlators.

$$\langle \prod_{\alpha} W_{R_{\alpha}}(K_{\alpha}) \rangle = \int_{A/U(1)} [dA] \cdot e^{\frac{ik}{4\pi} \int_{M_2} \text{tr}(A dA + \frac{2}{3} A A A)} \cdot \prod_{\alpha} W_{R_{\alpha}}(K_{\alpha})$$

Since $U(1)$ is compact, its (continuous, complex, finite-dimensional) representations are unitary and thus the direct sum of irreps by the Peter Weyl Theorem.

By Schur Lemma, such irreps are all 1-dimensional.
ie: They are given by $\chi(t) = t^n$ (identifying $U(1)$ with the unit circle in $\mathbb{C}$) for some integer n .

The representations of $U(1)$ are thus given by
 $t \rightarrow (t^{n_1}, \dots, t^{n_k})$ over some basis of $\mathbb{C}^k$
for $n_i \in \mathbb{Z}$.

In general, we can consider one-dimensional irreducible representations $\chi(t) = t^n \quad \forall n \in \mathbb{Z}$

and find following correlators.

$$\langle \prod_{\alpha} W_{n_{\alpha}}(K_{\alpha}) \rangle = \int_{A/U(1)} [dA] e^{\frac{ik}{4\pi} \int_{M_2} \text{tr}(A dA + \frac{2}{3} A A A)} \cdot \prod_{\alpha} W_{n_{\alpha}}(K_{\alpha})$$

$n_{\alpha} \in \mathbb{Z}$

where, $W_{n_{\alpha}}(K_{\alpha}) = \exp\left(i \oint_{K_{\alpha}} n_{\alpha} A\right) = \exp\left(i n_{\alpha} \oint_{K_{\alpha}} A\right)$

$g \in U(1)$, Then $g = e^{i\theta}$, $\theta \in \mathbb{R}$
 $\theta \in [0, 2\pi)$

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Then in the one-dimensional irreducible representation
 $g \longmapsto g^n$ it becomes g^n .

$$\text{i.e. } \chi_n(g) = g^n = e^{i\theta n}$$

Hence the generator is n

so: $A_\mu dx^\mu$ in representation $\chi_n(g)$
 becomes $A_\mu n dx^\mu$

$\exp\left(i n \oint_k A_\mu dx^\mu\right)$ becomes the thing which we need.

$$\text{Tr}_{\chi_n(g)} \left(\exp i \oint_k A \right) = \exp \left(i n \oint_k A \right)$$

because one dimensional representation has only one element (a scalar); so taking trace is a trivial operation)

$$\Rightarrow W_{n\alpha}(K_\alpha) = \exp \left(i n_\alpha \oint_{K_\alpha} A \right)$$

The $U(1)$ Chern - Simons action over three dimensional Manifold is

$$S_{\text{c.s.}} = -\frac{k}{4\pi} \int_{M_3} d^3x \epsilon^{\lambda\mu\nu} A_\lambda \partial_\mu A_\nu$$

$\mu, \lambda, \nu \in \{0, 1, 2\}$

The $AAAA$ piece is zero

because; Wedge

product of one form

with itself is zero.

Hence we simplify our problem;
and need to evaluate following loop
correlators.

$$\langle \prod_{\alpha} W_{m_{\alpha}}(K_{\alpha}) \rangle = \int_{A/U(1)} [dA] \cdot e^{\frac{ik}{4\pi} \int_{M_3} \epsilon^{\lambda\mu\nu} \cdot A_{\lambda} \cdot \partial_{\mu} A_{\nu}} \cdot \prod_{\alpha} W_{m_{\alpha}}(K_{\alpha})$$

Quoting answer from Edward Witten's Paper "Quantum Field Theory and the Jones Polynomial)

$$\langle \prod_{\alpha} W_{m_{\alpha}}(K_{\alpha}) \rangle = \exp \left(\frac{i}{2k} \sum_{\alpha, \beta} m_{\alpha} m_{\beta} \cdot \int_{K_{\alpha}} dx^{\mu} \int_{K_{\beta}} dy^{\nu} \cdot \epsilon_{\mu\nu\lambda} \cdot \frac{(x-y)^{\lambda}}{|x-y|^3} \right)$$

Now, we use the Gauss linking Number formula.

$$lk(K_{\alpha}, K_{\beta}) = \frac{1}{4\pi} \int_{K_{\alpha}} dx^{\mu} \int_{K_{\beta}} dx^{\nu} \cdot \epsilon_{\mu\nu\lambda} \cdot \frac{(x-y)^{\lambda}}{|x-y|^3}$$

Then we get

$$\langle \prod_{\alpha} W_{m_{\alpha}}(K_{\alpha}) \rangle = \exp \left(\frac{2\pi i}{k} \sum_{\alpha, \beta} m_{\alpha} m_{\beta} \cdot lk(K_{\alpha}, K_{\beta}) \right)$$