Continuum Mechanics

— Shoaib Akhtar

Lec 1 (8^{th} Jan, 2021 ; 2:00 pm to 3:30 pm)

Continuum Description

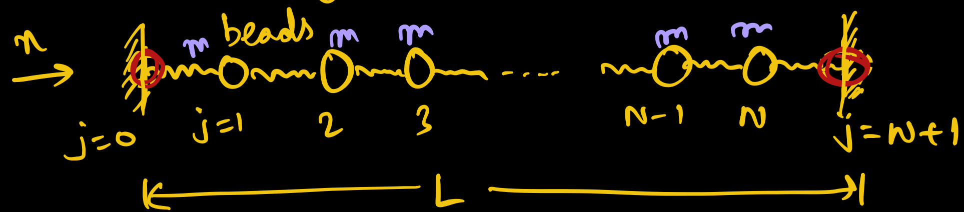
Classical Mechanics $\longrightarrow$ Continuum Mechanics.

$L(\{q_m, \dot{q}_m\}, t) \longrightarrow \mathcal{L}(q(x), q'(x), \dot{q}(x), t)$

x is continuous parameter.

Example Discrete system $\longrightarrow$ Continuum limit.

1 D string between walls.

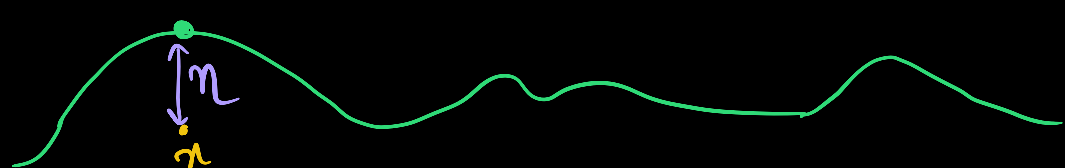


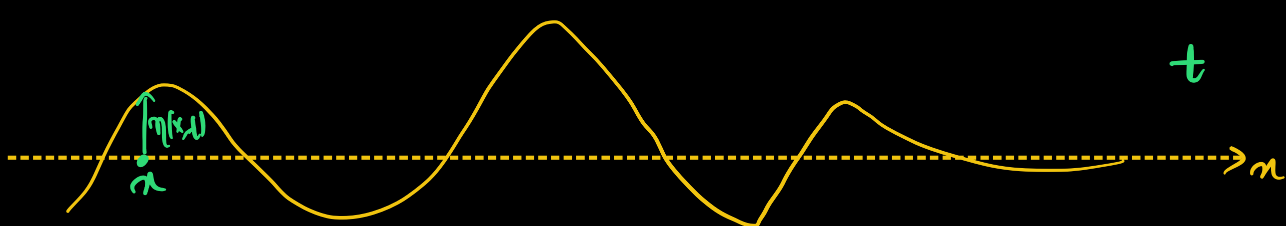
Discrete Counterpart of Wave Equation.

$$\partial_t^2 \eta(x,t) = c^2 \nabla^2 \eta(x,t)$$ Wave Equation

Solutions are well known.

Tells about movement of say string.





$\frac{1}{2}k(\Delta x)^2$ is potential energy of spring in terms of stretch (Δx).

Assume only longitudinal elongation.

length of whole chain $= L$.

l_0 = unstretched length of the Hookean spring.
a = spring length at Equilibrium. (Tension)

$x_j^{(eq)} = ja$: Equilibrium position of beads. ; $j \in \mathbb{Z}$

we have total $(N+1)$ springs.

so; $(N+1)\, l_0 = L_0$ = unstretched length.

for simplicity, take $\underline{l_0 \approx a}$. (No tension or compression in string at equilibrium)

Using newtons law:

$x_0 = 0, \quad x_{N+1} = L$

$$m\ddot{x}_3 = k(x_4 - x_3) - k(x_3 - x_2)$$

We will try to do this by lagrangian.

Now we disturb the string;

$$x_j = x_j^{(eq)} + \eta_j(t)$$

$$L = \underbrace{\frac{m}{2}\sum_{j=1}^{N} \dot{\eta}_j^2}_{\text{Kinetic}} - \underbrace{\frac{k}{2}\sum_{j=0}^{N} (x_{j+1} - x_j - l_0)^2}_{\text{Potential.}}$$

$$L = \frac{m}{2} \sum_{j=1}^{N} \dot{\eta}_j^2 - \frac{k}{2} \sum_{j=0}^{N} (\eta_{j+1} - \eta_j)^2 \quad \Bigg\} \text{ for } \underline{a = l_0} \text{ case}$$

$$\begin{aligned}
&\eta_{j+1} - \eta_j - l_0 \\
&= (j+1)a + \eta_{j+1} - ja - \eta_j - l_0 \\
&= (\eta_{j+1} - \eta_j) + (a - l_0)
\end{aligned}$$

so; $$L = \frac{m}{2} \sum_{j=1}^{N} \dot{\eta}_j^2 - \frac{k}{2} \sum_{j=0}^{N} \big((a - l_0) + (\eta_{j+1} - \eta_j) \big)^2$$

a = eq^m length (no motion; but string could be stretched or compressed)

so;
$$L = \frac{m}{2} \sum_{j=1}^{N} \dot{\eta}_j^2 - \frac{k}{2} \sum_{j=0}^{N} (\eta_{j+1} - \eta_j)^2 + \text{constant.}$$

The linear term looks like.

$$\frac{k}{2} \, 2(a - l_0) \sum_j (\eta_{j+1} - \eta_j)$$

$$= k(a - l_0)\underbrace{(\eta_{N+1} - \eta_0)}_{L} = \underbrace{k(a - l_0)L}_{\text{This is also then constant.}}$$

Now, we apply Lagrange's Eqn of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_j}\right) = \frac{\partial L}{\partial \eta_j}$$

Microscopic eqn of motion

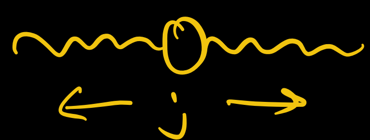
$$\Rightarrow \boxed{m\ddot{\eta}_j = k\left(\eta_{j+1} - 2\eta_j + \eta_{j-1}\right)}$$

Discrete Equation of Motion.

(can also be written by just using Newtons Laws of Motion)

$$\ddot{\eta}_j = \omega^2\left(\eta_{j+1} - 2\eta_j + \eta_{j-1}\right)$$

$$\omega = \sqrt{\frac{k}{m}}\,.$$



$$m\ddot{\eta}_j = k\left(\eta_{j+1} - \eta_j\right) - k\left(\eta_j - \eta_{j-1}\right)$$
$$= k\left(\eta_{j+1} - 2\eta_j + \eta_{j-1}\right)$$

Our microscopic model.

m, k, a are microscopic quantities.

Continuum Limit $(N \to \infty)$

Whole string has mass M,
Young's Modulus Y,
length L

} Macroscopic quantities.
(finite quantities)

The connection between m, k, a with M, Y, L is with N.

$\underline{mN = M}$, $\underline{aN = L}$ (for equilibrium)

$l_0 N = L_0$ for unstretched

$\boxed{\frac{k}{N} = Y}$ Need to derive.

$\rightarrow$ This tells $k = YN$

finite

so: $N \to \infty \Rightarrow k \to \infty$.

$a, m \propto \frac{1}{N}$

so; $a, m \to 0$ for $N \to \infty$

but $k \propto N$ so; $k \to \infty$ for $N \to \infty$.

divide EOM by a.

$$\frac{m}{a} \cdot \ddot{\eta}_j = a \cdot \frac{k}{a^2} \cdot (\eta_{j+1} - 2\eta_j + \eta_{j-1})$$

$\frac{m}{a} = \mu$ = mass density $= \frac{M}{L}$ = Finite quantity

$$\mu \cdot \ddot{\eta}_j = a \cdot \frac{k}{a^2} \cdot (\eta_{j+1} - 2\eta_j + \eta_{j-1})$$

In the continuum limit

$\eta_j(t) \xrightarrow{\quad\quad} \eta(x, t)$ ie; at every point x we have particle.

$j \in \mathbb{Z}$

ie; $\eta_x(t) \equiv \eta(x,t)$

$x \in \mathbb{R}$

$$\mu\,\ddot{\eta}(x,t) = ka \, \frac{\left[\dfrac{(\eta_{j+1} - \eta_j)}{a} - \dfrac{(\eta_j - \eta_{j-1})}{a}\right]}{a}$$

$$\boxed{\frac{m}{a}\cdot\ddot{\eta}_j = \frac{k}{a}\left(\eta_{j+1} - 2\eta_j + \eta_{j-1}\right)}$$

$\hookrightarrow \dfrac{\eta'(x+a) - \eta'(x)}{a}$

$$\frac{\eta_{j+1} - \eta_j}{a} \longrightarrow$$ in continuum limit, it becomes spatial derivative.

$$= \eta'(x)$$

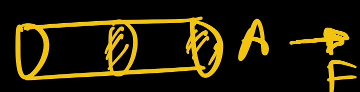
double derivative.

$$\mu\,\ddot{\eta}(x,t) = Y\,\eta''(x,t)$$

$c \equiv \sqrt{Y/\mu}$ ie; $c^2 = \dfrac{Y}{\mu}$

nothing to divide

Youngs Modulus $= \dfrac{\text{Force /area}}{\text{strain.}}$



$$= \frac{\left(k(\eta_{j+1} - \eta_j)\,/\,1\right)}{\underbrace{\dfrac{(\eta_{j+1} - \eta_j)}{a}}_{\text{strain}}}$$

(No area in 1d line. so; area shrinks to point)

Youngs Modulus $= ka \Rightarrow \boxed{Y = ka}$

$$\Rightarrow \quad k = \frac{Y}{a} \quad \Rightarrow \quad \boxed{k \propto N}$$

$$N \longrightarrow \infty$$
$$m \longrightarrow 0$$
$$a \longrightarrow 0$$

such that $\quad Na = L, \quad mN = M \quad$ finite.

} Continuum limit.

Forward derivative

$$\eta'_j = \frac{\eta_{j+1} - \eta_j}{a}$$

Backward Derivative

$$\eta'_j = \frac{\eta_j - \eta_{j-1}}{a}$$

Here, we are using this, Backward Derivative. while defing first derivative.

While writing double derivative ; we used Forward derivative.

$$\frac{d}{dx}\left(\frac{d\eta}{dx}\right) = \frac{d}{dx}\left(\frac{\eta_{j+1} - \eta_j}{a}\right)$$

using forward derivative.

$$= \frac{\eta'_{j+1} - \eta'_j}{a}$$

using forward.

$$= \frac{(\eta_{j+2} - \eta_{j+1}) - (\eta_{j+1} - \eta_j)}{a^2}$$

$$= \frac{\eta_{j+2} - 2\eta_{j+1} + \eta_j}{a^2}$$

In continuum limit; we could shift particle by 1 because there are ∞ particle.

so; $$\frac{\eta_{j+2} - 2\eta_{j+1} + \eta_j}{a^2} \rightarrow \underbrace{\frac{\eta_{j+1} - 2\eta_j + \eta_{j-1}}{a^2}}$$

This is more symmetric definition.

So we work with this.

B; Backward derivative, F; Forward derivative.

for double derivative, we could use $\underbrace{BB, FF, BF, FB}$ all equivalent in continuum limit.

$$L = \frac{m}{2} \sum_{j=1}^{N} \dot{\eta}_j^2 - \frac{k}{2} \sum_{j=0}^{N} (\eta_{j+1} - \eta_j)^2$$

should be finite quantity

$$L = \frac{a}{2} \left[\frac{m}{a} \sum \dot{\eta}_j^2 - ka \sum \left(\frac{\eta_{j+1} - \eta_j}{a} \right)^2 \right]$$

Taking continuum limit.

$$L = \frac{a}{2}\sum\left[\mu \cdot \dot{\eta}^2(x,t) - Y\cdot\left(\frac{d\eta(x,t)}{dx}\right)^2\right]$$

a is actually Δx

so; The summation becomes integral.

$$L = \Delta x \sum \frac{1}{2}\left[\mu \cdot \dot{\eta}^2(x,t) - Y\cdot\left(\frac{d\eta}{dx}\right)^2\right]$$

$$= \int dx \cdot \underbrace{\left[\frac{\mu}{2}\cdot\dot{\eta}^2(x,t) - \frac{Y}{2}\cdot\left(\frac{d\eta}{dx}\right)^2\right]}_{\mathcal{L}(\eta,\,\eta',\,\dot{\eta},\,t)\ :\ \text{Lagrangian density}}$$

$\eta' = \frac{\partial \eta}{\partial x}$, $\dot{\eta} = \frac{\partial \eta}{\partial t}$,

$$L = \int dx\, \mathcal{L}(\eta, \eta', \dot{\eta}, t)$$

$$\mathcal{L} = \frac{\mu}{2}\dot{\eta}^2(x,t) - \frac{Y}{2}\eta'^2(x,t)$$

x, t are independent variables.

$$\frac{\partial \eta}{\partial x} = \frac{d\eta}{dx} \quad , \quad \frac{\partial \eta}{\partial t} = \frac{d\eta}{dt}$$

because x & t are independent variables.

if $\eta(x(t), t)$

Then $\dfrac{d\eta}{dt} \neq \dfrac{\partial \eta}{\partial t}$

but for $\eta(x, t) \Rightarrow \dfrac{d\eta}{dt} = \dfrac{\partial \eta}{\partial t}$

$$\frac{d\eta}{dx} = \frac{\partial \eta}{\partial x}$$

$$L = \int \mathcal{L}(\eta(x,t), \eta', \dot{\eta}, t)\, dx$$

In Continuum Limit

$$L(\{q_j, \dot{q}_j\}, t) \longrightarrow \mathcal{L}(\eta(x,t), \eta', \dot{\eta}, t)$$

$$j \longrightarrow x$$

$$q_j(t) \longrightarrow \eta(x,t)$$

$$\dot{q}_j(t) \longrightarrow \frac{d\eta}{dx}, \quad \frac{d\eta}{dt}$$

$L(q, \dot{q}, t)$

t

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

Lagrange's Equation.

$\mathcal{L}(\eta, \eta', \dot{\eta}, x, t)$

x, t

$$\underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta'}\right) = \frac{\partial \mathcal{L}}{\partial \eta}}_{\text{Euler Lagrange Equations.}}$$

(Can be derived)

we have $\mathcal{L} = \frac{\mu}{2}\dot{\eta}^2 - \frac{Y}{2}\eta'^2 \Longrightarrow \underline{\mu \ddot{\eta} - Y\eta'' = 0}$ Wave Equation.

Continuum Mechanics — Shoaib Akhtar

Lec 2 (12th Jan, 2021 ; 2:00 pm to 3:30 pm)

Recap:

$\eta(0,t) = 0 = \eta(L,t)$ $\quad$ $\leftarrow a \rightarrow$ eqm position of spring.

$$m\ddot{\eta}_j = k(\eta_{j+1} - 2\eta_j + \eta_{j-1})$$

η_j are deviation from equilibrium position.

$$x_j = x_j^{(eq)} + \eta_j(t) \quad \text{ie;} \quad x_j - x_j^{(eq)} \equiv \eta_j(t)$$

If we have friction in the medium; then after long time the particle will come to rest at positions $x_j^{(eq)}$

$$x_j^{(eq)} = ja$$

Continuum Limit : $\quad \mu\, \partial_t^2 \eta(x,t) = Y\, \partial_x^2 \eta(x,t)$

$m \to 0$, $N \to \infty$, such that $mN = M = \mu . L$ = finite.
$a \to 0$

spring constant k ; $\quad ka = Y$ = finite

$$a \propto \frac{1}{N} \Rightarrow k \propto N$$

$\mathcal{L}(\eta, \dot{\eta}, \eta', x, t)$: Lagrangian Density.

Here $\mathcal{L}(\eta, \dot{\eta}, \eta', x, t) = \frac{\mu}{2}(\partial_t \eta)^2 - \frac{Y}{2}(\partial_x \eta)^2 \quad (*)$

(No explicit x and t dependence here)

$\eta_x \equiv \partial_x \eta$, $\eta_t \equiv \partial_t \eta$ (Notation)

Analogue of L - eqⁿ in discrete particle dynamics is called Euler's - Lagrange's eqn.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \eta_t}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta_x}\right) = \frac{\partial \mathcal{L}}{\partial \eta} \qquad \text{E - L Equation.}$$

Using the $\mathcal{L}$ (*), the E-L. eqⁿ gives.

$$\Rightarrow \quad \mu\, \partial_t^2 \eta = Y\, \partial_x^2 \eta .$$

Can have different boundary conditions:

ex) $\eta(0,t) = 0 = \eta(L,t)$

Boundary Condition

$\eta(0,t) = 0$

$$\left.\frac{\partial \eta}{\partial x}\right|_{x=L} = 0$$

$$m\ddot{\eta}_N = -k\,(\eta_N - \eta_{N-1})$$

$$= -ka\,\frac{(\eta_N - \eta_{N-1})}{a}$$

More general situation.

$$m\ddot{\eta}_N = -ka\cdot\left(\frac{\eta_N - \eta_{N-1}}{a}\right) + F$$

$\eta(0,t) = 0$

$$m\ddot{\eta}_N = -Y \cdot \frac{d\eta}{dx} + F \quad \Big| \quad \text{continuum limit}$$

$m\ddot{\eta}_N \longrightarrow 0$ in continuum limit, if $m = 0$

so;

$$\boxed{-Y \cdot \left.\frac{d\eta}{dx}\right|_{x=L} + F = 0}$$ Boundary Condition.

$$\boxed{F = Y \cdot \left.\frac{d\eta}{dx}\right|_{x=L}} \Rightarrow \boxed{\left.\frac{d\eta}{dx}\right|_{x=L} = \frac{F}{Y}}$$

Individual η_j are microscopic quantity.

so; difference is also microscopic.

$$\left.\begin{matrix}\dfrac{\overbrace{\eta_{j+1} - \eta_j}^{\rightarrow \text{ microscopic}}}{\underbrace{a}_{\rightarrow \text{ microscopic}}}\end{matrix}\right\} \text{ so can be finite.}$$

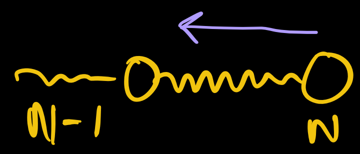
but for $\ddot{\eta}_N$, we dont expect this to go infinity.

for this to go to ∞, we need infinite force.

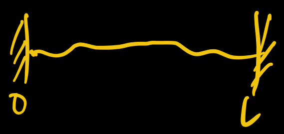
so; we dont expect $\dot{\eta}_N$ & $\ddot{\eta}_N$ to blow up.

$-k(\eta_j - \eta_{j-1}) \longleftarrow \quad \longrightarrow +k(\eta_{j+1} - \eta_j)$

for last particle



No mass at right.



$$\eta(x,t) = \sum A_n \sin k_n x \; e^{\lambda_n t}$$

$$\mu \ddot{\eta} = Y \eta'' \rightarrow \sin kx \cdot e^{\lambda t} \Rightarrow kL = n\pi \Rightarrow k_n = n\pi/L$$

$$\mu \lambda^2 = -Yk^2 \Rightarrow \lambda_n = i\sqrt{\frac{Y}{\mu}} \cdot k_n$$

so; $$\eta(x,t) = \sum A_n \sin(k_n x) \cdot e^{i\sqrt{\frac{Y}{\mu}} \cdot k_n \cdot t}$$

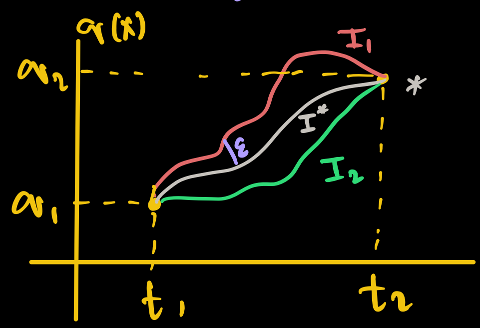
Using boundary conditions, we can find A_n's and hence $\eta(x,t)$ is obtained.

$$\begin{tikzcd}
\boxed{\text{Discrete Lagrangian}} \arrow[r, "\text{L-eq}^n"'] \arrow[d, "\text{continuum limit}"] & \boxed{\text{Discrete EOM}} \arrow[d, "\text{Continuum limit.}"] \\
\boxed{\text{Lagrangian Density}} \arrow[r, "\text{E-L eq}^n"'] & \boxed{\text{Wave equation}}
\end{tikzcd}$$

Continuum Limit

$$t \longmapsto x, t$$
$$j \longmapsto x$$
$$q_j(t) \longmapsto \eta(x,t)$$
$$\dot{q}_j(t) \longmapsto \dot{\eta}, \eta'$$

Principle of Least Action



$$I = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \qquad \text{Action.}$$

Which path $q(t)$ will extremize I ?
The physical path !

$I(q, \dot{q}, t)$ is a functional, depends on $q(t)$.

let $q(t)$ be the optimum path or the physical path, then we consider the deviation

$$q(t) \longrightarrow q(t) + \delta q(t).$$

$\delta I = I_1 - I_*$ should be zero

$$\delta I = O(\epsilon^2) + O(\epsilon)$$

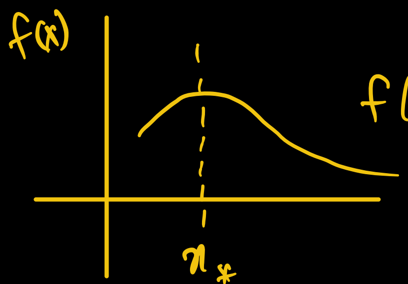
$$\left.\frac{\delta I}{\epsilon} \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0. \right\} \Rightarrow \text{that } * \text{ is the extremum path.}$$

$$\delta I = A\epsilon + B\epsilon^2 + \dots$$

$$\frac{\delta I}{\epsilon} = A + B\epsilon + \dots$$

in limit $\epsilon \to 0$ $\qquad \left.\frac{\delta I}{\epsilon}\right|_{\epsilon \to 0} = A$

so; for the path to be extremum, we will need $A = 0$.

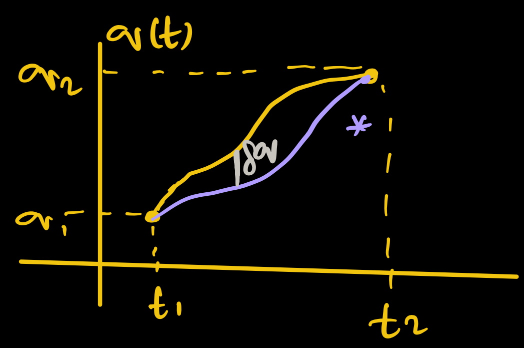


$$f(x_* + \epsilon) = f(x_*) + \epsilon f'(x_*) + \epsilon^2 f''(x_*) + \dots$$

$$\frac{f(x_* + \epsilon) - f(x_*)}{\epsilon} = \underbrace{f'(x_*)}_{\text{This has to be zero.}} + \epsilon f''(x_*) + \dots$$

$$q \longrightarrow q + \delta q \implies L(q) \longrightarrow L(q + \delta q)$$

$$\delta I = \int_{t_1}^{t_2} [\underbrace{L(q + \delta q)}_{q_1 \text{ path.}} - \underbrace{L(q)}_{q_* \text{ path.}}]\, dt$$



doing the analysis in path space.

$\delta q(t_1) = 0 = \delta q(t_2)$: End point variations are fixed.

$$\delta \dot{q} = \frac{d}{dt} \delta q \qquad q_1 = q + \delta q$$

Actually.

$$\delta I = \int_{t_1}^{t_2} \left[L(q+\delta q, \dot{q} + \delta \dot{q}, t) - L(q, \dot{q}, t) \right] dt$$

$$= \int_{t_1}^{t_2} \left[\cancel{L(q, \dot{q}, t)} + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} - \cancel{L(q, \dot{q}, t)} \right] dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] dt$$

$$= \int_{t_1}^{t_2} \left(\left[\delta q \cdot \frac{\partial L}{\partial q} \right] + \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \cdot \delta q \right] \right) dt$$

Note $\displaystyle\int_{t_1}^{t_2} dt \cdot \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) = \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{t_1}^{t_2}$

$$= \left. \frac{\partial L}{\partial \dot{q}} \right|_{t_2} \cdot \cancelto{0}{\delta q(t_2)} - \left. \frac{\partial L}{\partial \dot{q}} \right|_{t_1} \cdot \cancelto{0}{\delta q(t_1)}$$

$$= 0$$

$$\delta I = \int_{t_1}^{t_2} dt \cdot \delta q \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right]$$

$$\delta I = 0 \implies \frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0$$

Note| $\int_{x_1}^{x_2} f(x)\,dx = 0 \;\not\Rightarrow\; f(x) = 0$

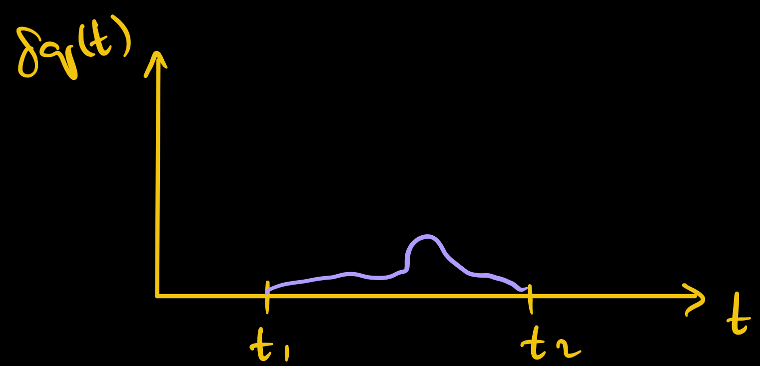
Here we could conclude because of δq.

and the deviation δq is arbitrary.

so; $\delta I = 0 \quad \forall \; \delta q$ arbitrary.

so; we get $\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = 0$

ex||



Derivation of Euler-Lagrange's Equation.

given $\mathcal{L}(\eta(x,t), \eta', \dot\eta, x, t)$ Lagrangian density

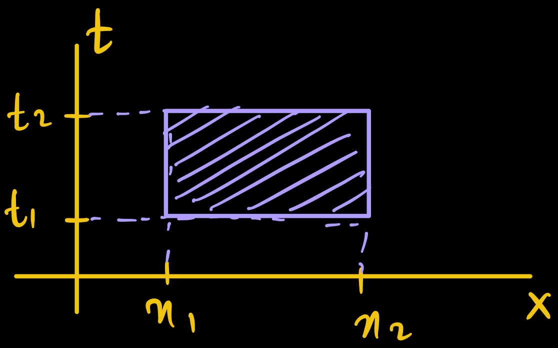
$L = \int \mathcal{L}\, dx$

$L \Rightarrow$ Lagrangian.

$\mathcal{L} \Rightarrow$ Lagrangian density.

Action $= \int L \cdot dt = \int \mathcal{L} \cdot dx\,dt = I$

Need to find δI.



$\delta\eta$

Surface $\eta(x,t)$ changed to $\eta(x,t) + \delta\eta(x,t)$.

$$\eta(x,t) \longrightarrow \eta(x,t) + \delta\eta(x,t)$$

$$\delta I = 0 \Longrightarrow \eta^*(x,t).$$

$$\delta\eta(x,t)\Big|_{\square} = 0$$

ie; $\delta\eta$ is zero on boundary of integration.

ie; $I = \int_{\Omega} dx\,dt\, \mathcal{L}$

so; $\delta\eta\big|_{\partial\Omega} = 0$.

$$\delta I = \int dx\,dt \left[\mathcal{L}(\eta + \delta\eta, \eta' + \delta\eta', \dot\eta + \delta\dot\eta) - \mathcal{L}(\eta, \eta', \dot\eta)\right]$$

$$= \int dx\,dt \left[\mathcal{L}(\eta, \eta', \dot\eta) + \frac{\partial\mathcal{L}}{\partial\eta}\delta\eta + \frac{\partial\mathcal{L}}{\partial\eta'}\delta\eta' + \frac{\partial\mathcal{L}}{\partial\dot\eta}\delta\dot\eta - \mathcal{L}(\eta, \eta', \dot\eta)\right].$$

we are doing double Taylor expansion.

Notation: $\delta\eta' = \frac{d}{dx}\delta\eta$, $\delta\dot{\eta} = \frac{d}{dt}\delta\eta$

$$\delta I = \int dx\,dt \left[\frac{\partial \mathcal{L}}{\partial \eta}\delta\eta + \frac{\partial \mathcal{L}}{\partial \eta'}\delta\eta' + \frac{\partial \mathcal{L}}{\partial \dot{\eta}}\delta\dot{\eta}\right]$$

Note we also have terms like

$$\frac{\partial^2 \mathcal{L}}{\partial \eta' \partial \dot{\eta}}\delta\eta'\,\delta\dot{\eta} + \ldots$$

but we dont keep the higher order terms.

so;

$$\delta I = \int dx\,dt \left[\frac{\partial \mathcal{L}}{\partial \eta}\delta\eta + \frac{\partial \mathcal{L}}{\partial \eta'}\delta\eta' + \frac{\partial \mathcal{L}}{\partial \dot{\eta}}\delta\dot{\eta} + \theta(\delta\eta^2)\right]$$

$$= \int dx\,dt \left[\frac{\partial \mathcal{L}}{\partial \eta}\cdot\delta\eta + \underbrace{\frac{d}{dx}\left[\frac{\partial \mathcal{L}}{\partial \eta'}\delta\eta\right]} - \frac{d}{dx}\left[\frac{\partial \mathcal{L}}{\partial \eta'}\right]\delta\eta + \underbrace{\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\delta\eta\right]} - \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right]\cdot\delta\eta + \theta(\delta\eta^2)\right]$$

$$\int dx \rightarrow \frac{\partial \mathcal{L}}{\partial \eta'}\delta\eta\Big|_{x_1}^{x_2} = 0$$

doing just dx integral.

similarly this term drops out after dt integral.

$$\delta I = \int dx\,dt \cdot \delta\eta \left[\frac{\partial \mathcal{L}}{\partial \eta} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta'}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) + \theta(\delta\eta) \right]$$

$\delta\eta(x,t)$ is completely arbitrary.

so; $\delta I = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \eta} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta'}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) = 0$

Question) $\mathcal{L}(\eta, \eta', \eta'', x, t)$

$\mathcal{L}(\dot{\eta}, \ddot{\eta}, \eta', \ldots)$

when we have higher order derivatives.

Homework. Continuum Mechanics — Shoaib Akhtar

Given $d+1$ dimensional spacetime

$$\underbrace{\mu}_{} \in \{\underset{\text{time index}}{0}, \underbrace{1, 2, \ldots, d}_{\text{spatial index.}}\} \quad , \quad x^\mu$$

Given the Lagrangian Density

$$\mathcal{L}\left(\eta(x^\sigma), \partial_\mu \eta, \partial_\mu \partial_\nu \eta, \ldots, \partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_N}\eta, x^\mu\right)$$

We write the action

$$I[\eta] = \int_\Omega \mathcal{L} \cdot d^{d+1}x$$

Ω is the spacetime region over which we integrate.

Finding Generalized Euler Lagrange equation.

$$\eta \longrightarrow \eta + \delta\eta \qquad \text{s.t.} \qquad \delta\eta\big|_{\partial\Omega} = 0$$

under this infinitisimal transformation, lets find the change in action δI

$$\delta I = \int_\Omega \left[\frac{\partial \mathcal{L}}{\partial \eta}\delta\eta + \frac{\partial \mathcal{L}}{\partial \eta_{\mu_1}}\delta\eta_{\mu_1} + \frac{\partial \mathcal{L}}{\partial \eta_{\mu_1\mu_2}}\delta\eta_{\mu_1\mu_2} + \cdots + \frac{\partial \mathcal{L}}{\partial \eta_{\mu_1\mu_2\cdots\mu_N}}\cdot\delta\eta_{\mu_1\mu_2\cdots\mu_N}\right] d^{d+1}x$$

NOTATION :

$$\boxed{\eta_{\mu_1\mu_2\cdots\mu_j} \equiv \partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_j}\eta}$$

$$\delta I = \int_{\Omega} \left[\frac{\partial \mathcal{L}}{\partial \eta} \delta \eta + \sum_{j=1}^{N} \frac{\partial \mathcal{L}}{\partial \eta_{\mu_1 \cdots \mu_j}} \cdot \delta \eta_{\mu_1 \cdots \mu_j} \right] d^{d+1}x$$

consider the piece

$$\int_{\Omega} \frac{\partial \mathcal{L}}{\partial \eta_{\mu_1 \cdots \mu_j}} \cdot \delta \eta_{\mu_1 \cdots \mu_j} \cdot d^{d+1}x$$

$$= \int_{\Omega} \left[\partial_{\mu_1} \left(\frac{\partial \mathcal{L}}{\partial \eta_{\mu_1 \cdots \mu_j}} \cdot \delta \eta_{\mu_2 \cdots \mu_j} \right) - \partial_{\mu_1} \left(\frac{\partial \mathcal{L}}{\partial \eta_{\mu_1 \cdots \mu_j}} \right) \delta_{\mu_2 \cdots \mu_j} \right] d^{d+1}x$$

$$= \underbrace{\int_{\Omega} \partial_{\mu_1} \left(\frac{\partial \mathcal{L}}{\partial \eta_{\mu_1 \cdots \mu_j}} \cdot \delta \eta_{\mu_2 \cdots \mu_j} \right) d^{d+1}x}_{\Downarrow} - \int \partial_{\mu_1} \left(\frac{\partial \mathcal{L}}{\partial \eta_{\mu_1 \cdots \mu_j}} \right) \delta_{\mu_2 \cdots \mu_j} \cdot d^{d+1}x$$

This becomes the boundary integral over $\partial \Omega$

$\Rightarrow$ so it vanishes

if. $\delta \eta_{\mu_2 \cdots \mu_j} \Big|_{\partial \Omega} = 0$

$$= -\int_{\Omega} \partial_{\mu_1} \left(\frac{\partial \mathcal{L}}{\partial \eta_{\mu_1 \cdots \mu_j}} \right) \delta_{\mu_2 \cdots \mu_j} \eta \; d^{d+1}x$$

We again re-iterate the steps

$$= (-1)^j \int_\Omega \partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_j}\left(\frac{\partial \mathcal{L}}{\partial \eta_{\mu_1\cdots\mu_j}}\right)\cdot\delta\eta\cdot d^{d+1}x$$

NOTATION : $\partial_{\mu_1,\dots,\mu_j} = \partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_j}$

Using this we get

$$\delta I = \int_\Omega \left[\frac{\partial \mathcal{L}}{\partial \eta} + \sum_{j=1}^{N} (-1)^j \partial_{\mu_1\cdots\mu_j}\left(\frac{\partial \mathcal{L}}{\partial \eta_{\mu_1\cdots\mu_j}}\right)\right]\delta\eta\cdot d^{d+1}x$$

under the restriction.

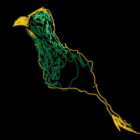
$$\delta\eta_{\mu_1\cdots\mu_j}\Big|_{\partial\Omega} = 0 \qquad \text{where } j \in \{1,\dots,N-1\}$$

and $$\delta\eta\Big|_{\partial\Omega} = 0$$

so $\delta I = 0$ for such arbitrary $\delta\eta$

gives us the Generalized Euler Lagrange Equation.

$$\boxed{\frac{\partial \mathcal{L}}{\partial \eta} + \sum_{j=1}^{N} (-1)^j \partial_{\mu_1\cdots\mu_j}\left(\frac{\partial \mathcal{L}}{\partial \eta_{\mu_1\cdots\mu_j}}\right) = 0}$$

Continuum Mechanics — Shoaib Akhtar 

Lec 3 (15th Jan, 2021 ; 2:00 pm to 3:30 pm)

given $\mathcal{L}(\eta, \eta_x, \eta_t, x, t) \rightarrow \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta_x}\right) + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \eta_t}\right) = \frac{\partial \mathcal{L}}{\partial \eta}$

$$\mathcal{L} = \frac{\mu}{2}(\partial_t \eta)^2 - \frac{Y}{2}(\partial_x \eta)^2$$

Independent variable x, t.

We could have $x, y, z, t \longrightarrow x_\mu \ : \mu = 0,1,2,3.$

$\vec{E}(\vec{x}, t)$, $\vec{B}(\vec{x}, t)$ $\qquad (\vec{x}, t) = x^\mu$

$$\begin{pmatrix} E_1(x^\mu) \\ E_2(x^\mu) \\ E_3(x^\mu) \end{pmatrix}$$

$x^\mu = x, y, z, t$ $\qquad \sum_\mu \frac{d}{dx_\mu}\left(\frac{\partial \mathcal{L}}{\partial \eta_\mu}\right) = \frac{\partial \mathcal{L}}{\partial \eta}$

notation: $\eta_\mu = \frac{\partial \eta}{\partial x^\mu}$ $\qquad$ (For now, dont differentiate between contravariant & covariant)

$x_\mu = x, t$, η_ρ $\qquad$ ρ stands for the components of the field η

$\rho = 1, 2, 3$

ie; η_1 is like E_1

η_2 " " E_2

η_3 " " E_3

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \equiv [\eta_\rho]$$

for each ρ, we have Euler-Lagrange equations.

for each ρ

$$\sum_\mu \frac{d}{dx_\mu}\left(\frac{\partial \mathcal{L}}{\partial \eta_{\rho,\mu}}\right) = \frac{\partial \mathcal{L}}{\partial \eta_\rho}$$

ρ stands for components of the vector.

$\mu = x, t$

NOTATION: $\eta_{\rho,\mu} \equiv \dfrac{\partial \eta_\rho}{\partial x_\mu}$

when $\mathcal{L}(\eta_\rho, \eta_{\rho,\mu}, x_\mu)$

$\mathcal{L}(\eta, \eta_x, \eta_{xx}, \dots)$

$-\frac{d}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial \eta_{,x,x}}\right)$ term comes on LHS

$\mathcal{L}(\dots, \eta_{xx} + \delta\eta_{xx}, \dots)$

expanding gives

$$\frac{\partial \mathcal{L}}{\partial \eta_{xx}}\delta\eta_{xx} = \underbrace{\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta_{xx}}\delta\eta_x\right)} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta_{xx}}\right)\delta\eta_x$$

This becomes boundary term after integral.

because it give $\left.\dfrac{\partial \mathcal{L}}{\partial \eta_{xx}} \cdot \delta\eta_x\right|_{\text{boundary}}$

$\delta\eta_x = 0$ on boundary.

$$-\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta_{xx}}\right)\delta\eta_x = \underline{-\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta_{xx}}\right)\delta\eta\right)} + \frac{d^2}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial \eta_{xx}}\right)\cdot\delta\eta$$

$\longrightarrow$ becomes boundary term after integration & vanishes. $\delta\eta = 0$ on boundary.

so; The extra term we get is

$$\int dx\,dt\left[\ldots\ldots + \frac{d^2}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial \eta_{xx}}\right) + \ldots\right]\delta\eta$$

ie; $$\int dx\,dt\left[\underbrace{\frac{\partial \mathcal{L}}{\partial \eta} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta_x}\right) + \frac{d^2}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial \eta_{xx}}\right)}_{\text{This is zero.}}\right]\delta\eta = 0$$

$$\frac{\partial \mathcal{L}}{\partial \eta} = \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta_x}\right) - \frac{d^2}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial \eta_{xx}}\right)$$

$\eta \rightarrow \eta + \delta\eta(x)$

Goldstein

$\eta + \alpha \cdot f(x)$

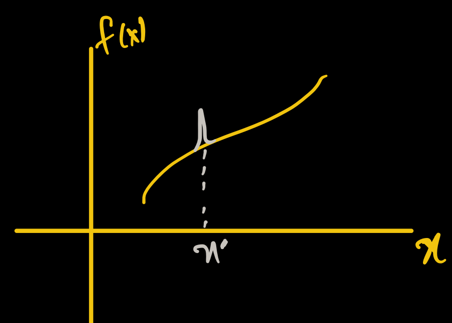
$\delta\eta(x) = \alpha \cdot f(x)$

α small number.

$I = I[f]$ Functional.

ex) $I[f] = \int_{1}^{2} f^2(x)\,dx$

we want to find $\dfrac{\delta I}{\delta f(x)} = g(x)$ → This will be some function of x.



$\mathcal{L}(\eta, \eta_x, \ldots)$

$\eta \rightarrow \eta + \delta\eta(x)$

$$\frac{\delta I[f]}{\epsilon} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int \left\{ [f(x) + \underbrace{\epsilon \cdot \delta(x-x')}_{\text{changing by small amount at a point } x'}]^2 - [f^2(x)] \right\} dx$$

definition of functional derivative $\frac{\delta I[f]}{\delta f(x')}$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dx [f^2 + 2\epsilon f(x) \cdot \delta(x-x') + \cancel{\epsilon^2 \cdot \delta^2(x-x')} - f^2]$$

neglect.

$$\int \delta(x-x')\, dx = 1$$

$$\int (\delta(x-x'))^2 dx = \int \delta(x-x')\, \delta(x-x')\, dx$$

$$= \delta(0)$$

so; $$\int \epsilon^2 \cdot \delta^2(x-x')\, dx = \epsilon^2 \delta(0)$$

$$\frac{1}{\epsilon} \int \epsilon^2 \cdot \delta^2(x-x')\, dx = \epsilon \cdot \delta(0)$$

$$\lim_{\epsilon \to 0} \epsilon \cdot \delta(0) = 0$$

ϵ is stronger 0 as compared to $\delta(0)$ being infinity.

choice of ϵ is arbitrary
so make it arbitrarily small
so that $\epsilon\, \delta(0) = 0$

$$\frac{\delta I[f]}{\epsilon} = 2 \cdot f(x') \quad \text{i.e.} \quad \frac{\delta I}{\delta f(x')} = 2 \cdot f(x')$$

$$\frac{\delta I}{\delta f(x')} = \lim_{\epsilon \to 0} \frac{I[f(x) + \epsilon\delta(x-x')] - I[f(x)]}{\epsilon}$$ Functional differentiation.

Example 1) $I = \int f^2 dx \longrightarrow \frac{\delta I}{\delta f(x')} = 2f(x')$

Example 2) $I = \int \left(\frac{df}{dx}\right)^2 dx \rightarrow \frac{\delta I}{\delta f(x')} = -2 \cdot \frac{d^2}{dx'^2} f(x')$

$$I[f+\epsilon\delta(x-x')] = \int \left(\frac{d}{dx}\left(f(x) + \epsilon\delta(x-x')\right)\right)^2 dx$$

$$\delta I = \int \left[\left(\frac{d}{dx}(f(x) + \epsilon\delta(x-x'))\right)^2 - \left(\frac{d}{dx} f(x)\right)^2\right] dx$$

$$= \int \left[(f'(x) + \epsilon\delta'(x-x'))^2 - (f'(x))^2\right] dx$$

$$= \int \left[f'(x)^2 + \epsilon^2(\delta'(x-x'))^2 + 2f'(x)\epsilon\delta'(x-x') - (f'(x))^2\right] dx$$

$$= \int \left[\epsilon^2 \cdot (\delta'(x-x))^2 + 2\epsilon f'(x)\delta'(x-x')\right] dx$$

$$\Rightarrow \frac{\delta I}{\delta f(x')} = \frac{1}{\epsilon}\delta I = \int 2f'(x)\delta'(x-x') dx$$

$$= 2\left[f'(x)\delta(x-x')\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f''(x) \cdot \delta(x-x') dx\right]$$

$$= 2 \cdot \left[0 - \frac{d^2 f}{dx^2}\Big|_{x'}\right]$$

$$= -2 \frac{d^2 f(x')}{dx'^2}$$

$$\int g(x)\,\delta'(x-x')\,dx = -g'(x')$$

$\mathcal{L}(\eta, \eta_x, x, t)$

$\delta I = 0$

$$\eta \longrightarrow \eta + \delta\eta$$

$$\eta \longrightarrow \eta + \epsilon\cdot\delta(x-x')$$

$$\delta I = \int\left(\frac{\partial\mathcal{L}}{\partial\eta}\delta\eta + \frac{\partial\mathcal{L}}{\partial\eta_x}\delta\eta_x\right)dx\,dt$$

$$\delta\eta = \epsilon\cdot\delta(x-x')$$

$$\delta I = \int \epsilon\left[\frac{\partial\mathcal{L}}{\partial\eta}\delta(x-x') + \frac{\partial\mathcal{L}}{\partial\eta_x}\cdot\delta'(x-x')\right]dx\,dt$$

$$= \epsilon\cdot\left[\frac{\partial\mathcal{L}}{\partial\eta(x')} - \frac{d}{dx'}\left(\frac{\partial\mathcal{L}}{\partial\eta_{x'}}\right)\right]$$

$$\frac{\delta I}{\epsilon} = \frac{\partial\mathcal{L}}{\partial\eta(x')} - \frac{d}{dx'}\left(\frac{\partial\mathcal{L}}{\partial\eta_{x'}}\right)$$

<u>Example 1)</u> $\left[\frac{-\hbar^2}{2m}\cdot\nabla^2 + V\right]\psi = i\hbar\cdot\frac{\partial\psi}{\partial t}$ Time dependent Schrodinger equation.

This equation follows from the Lagrangian density.

$$\mathcal{L} = \frac{\hbar^2}{2m}\,\bar{\nabla}\psi\cdot\bar{\nabla}\psi^* + V\psi^*\psi$$

$$+\frac{\hbar}{2i}\left(\psi^*\dot{\psi} - \psi\dot{\psi}^*\right)$$

current term.

Note, here we have ψ & ψ^*

There are actually two degrees of freedom

$\psi = \psi_R + i\psi_I$ $\qquad \psi_R, \psi_I$ independent or

$$\underbrace{\psi(x) = \psi_0(x)\, e^{i\eta(x)}}$$ → There are two fields.

or we can also think of ψ & ψ^* as independent degree of freedom.

It has 2 degrees of freedom.

$$\begin{array}{ccc} & (\psi, \psi^*) & \\ \swarrow\!\!\nearrow & & \searrow\!\!\nwarrow \\ (\psi_R, \psi_I) & \longleftrightarrow & (\psi_0, \eta) \end{array}$$

They convey same information.

$$\psi^* : \quad \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \psi^*_x}\right) + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \psi^*_t}\right) = \frac{\partial \mathcal{L}}{\partial \psi^*}$$

→ This gives Time dependent Schrodinger equation.

$$\psi : \quad \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \psi_x}\right) + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \psi_t}\right) = \frac{\partial \mathcal{L}}{\partial \psi}$$

⇒ This gives complex conjugate of Time dependent Schrodinger equation.

Example 2) Electromagnetic field Lagrangian
$\vec{E}, \vec{B}$

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad , \quad \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

We construct a Tensor quantity $F_{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \text{(Antisymmetric)}$$
$$= \frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu$$

$$\bar{\bar{A}} = (\vec{A}, i\phi)$$

four vector

$$\mathcal{L} = -\frac{1}{4c} F_{\mu\nu} F_{\mu\nu}$$

$$[F_{\mu\nu}]_{4\times4} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & & \\ +B_2 & & 0 & \\ iE_1 & & & 0 \end{bmatrix}$$

Equation for μ component of A four vector

$$\frac{d}{dx_\lambda}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_\mu}{\partial x_\lambda}\right)}\right) = \frac{\partial \mathcal{L}}{\partial A_\mu}$$

$\longrightarrow$ This is zero here

So;
$$\frac{d}{dx_\lambda}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_\mu}{\partial x_\lambda}\right)}\right) = 0$$

$\Rightarrow$ gives Maxwell equation for zero current & zero charge case.

we get $\partial_\mu F^{\mu\nu} = 0$ ✓

Continuum Mechanics — Shoaib Akhtar

Lec 4 (19th Jan, 2021 ; 3:30 pm to 5:00 pm)

Symmetries of $\mathcal{L}$

symmetries of $L(q, \dot{q}, t) \longrightarrow L + \frac{dF}{dt}(q, t) = L'$

This keeps L-eq^n unchanged.

i.e; $\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) = \frac{\partial L'}{\partial q}$ is satisfied.

$\frac{\partial L'}{\partial \dot{q}} \Rightarrow$ has an additional term

$$\frac{\partial}{\partial \dot{q}}\left(\frac{dF}{dt}\right) = \frac{\partial}{\partial \dot{q}}\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\cdot \dot{q}\right)$$

F not dependent explicitly on $\dot{q}$.

$$= \frac{\partial}{\partial \dot{q}}\left(\frac{\partial F}{\partial t}\right) + \frac{\partial}{\partial \dot{q}}\left(\frac{\partial F}{\partial q}\dot{q}\right)$$

This is independent of $\dot{q}$

$$= 0 + \cancel{\frac{\partial}{\partial \dot{q}}\left(\frac{\partial F}{\partial q}\right)}^{\,0}\cdot \dot{q} + \frac{\partial F}{\partial q}\frac{\partial \dot{q}}{\partial \dot{q}}$$

$$= \frac{\partial F}{\partial q}$$

$\frac{d}{dt}\left(\frac{\partial F}{\partial q}\right) \Big\} \Rightarrow$ extra term from LHS

We can show, that the RHS $\frac{\partial L'}{\partial q}$ will also generate same term.

$$\frac{\partial L'}{\partial q} \Rightarrow \frac{\partial}{\partial q}\left(\frac{dF}{dt}\right)$$

It can be shown

$$\frac{d}{dt}\left(\frac{\partial F}{\partial q}\right) = \frac{\partial}{\partial q}\left(\frac{dF}{dt}\right)$$ when F is not dependent explicitly on $\dot{q}$.

<u>Variational approach</u>

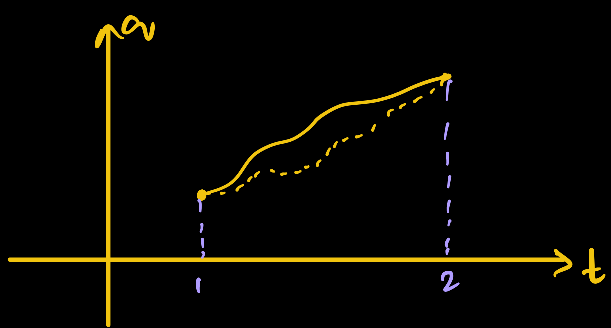
$$I' = \int_{①}^{②} L' \cdot dt = \int_{①}^{②} L \cdot dt + \int_{①}^{②} \frac{dF}{dt} \cdot dt$$

$$= \left[\int_{①}^{②} L \cdot dt\right] + \left(F(②) - F(①)\right)$$

$$\Rightarrow I' = I + \underbrace{\left(F(②) - F(①)\right)}_{\text{This is extra term.}}$$

when we do variation of I, we keep the end points fixed.

so; $$\delta\left(F(②) - F(①)\right) = 0$$



$\frac{dF}{dt}$ term is boundary term as far as the action is considered.

<u>Energy Conservation</u> $L = L(q, \dot{q})$ not explicitly dependent on time "t".

Then something is conserved (usually called Energy)

$$\frac{dL}{dt} = \underbrace{\frac{\partial L}{\partial q}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)} \cdot \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} + \frac{\partial L}{\partial t}$$

used Lagranges equation

$$\Rightarrow \frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \dot{q}\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \dot{q} - L\right) = -\frac{\partial L}{\partial t}$$

when $L = L(q, \dot{q})$

so $\frac{\partial L}{\partial t} = 0$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \dot{q} - L\right) = 0$$

This is called Hamiltonian.

$$H = \frac{\partial L}{\partial \dot{q}} \dot{q} - L$$

Analogy: for fields $\eta_\rho(x_\mu) \rightarrow x, y\ z, t$

vector component.

$$\sum_\nu \frac{d}{dx_\nu}\Big[\underbrace{\sum_\rho \left(\frac{\partial \mathcal{L}}{\partial \eta_{\rho,\nu}} \cdot \frac{\partial \eta_\rho}{\partial x_\mu}\right) - \mathcal{L} \cdot \delta_{\mu\nu}}_{T_{\mu\nu}}\Big] = 0$$

if $\frac{\partial \mathcal{L}}{\partial x_\mu} = 0$

so;
$$\frac{d}{dX_\nu} T_{\mu\nu} = 0$$

ie; divergence of $T_{\mu\nu}$ is zero.

$T_{\mu\nu} \Rightarrow$ Energy - Momentum Tensor.

ex 1  $\eta(x)$ $\rho = 1$ (only one component)

x, t independent variables.

$$\sum_\nu \frac{d}{dX_\nu} T_{\mu\nu} = 0 \Rightarrow \frac{d}{dt} T_{\mu 0} + \frac{d}{dx} T_{\mu 1} = 0$$

$\mu, \nu \in \{0, 1\}$ ← x - component.

"t": Time component

Analogue of " $L \longrightarrow L' = L + \frac{df}{dt}$ "

$$\boxed{\mathcal{L} \longrightarrow \mathcal{L}' = \mathcal{L} + \underbrace{\frac{d}{dX_\nu} F_\nu(\{\eta_\rho\}, \{X_\mu\})}_{\text{This is divergence term.}}}$$

F_ν should not be function of derivatives of fields

ie;
$$\frac{\partial F_\nu}{\partial\left(\frac{\partial \eta_\rho}{\partial X_\mu}\right)} = 0$$

This is divergence term.

F_ν is a vector of same dimension as the space - time.

Tensors

* Rotations are Linear Transformations.

$$\vec{A}' = R\vec{A}$$

rotated vector

Rotation.

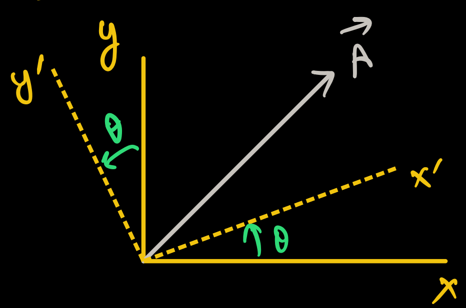
initial vector

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Passive Rotation (most used one)

Active Rotation.

Passive Rotation



vector does not move, only coordinate frame moves.

Object is not physically changed, only the coordinates are transformed.

$$\vec{A} = A_x\hat{x} + A_y\hat{y}$$

In new coordinate system

$$\vec{A} = A_x'\hat{x}' + A_y'\hat{y}'$$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} = A_x'\hat{x}' + A_y'\hat{y}'$$

$$\hat{x} = \cos\theta\, \hat{x}' - \sin\theta\, \hat{y}'$$
$$\hat{y} = \sin\theta\, \hat{x}' + \cos\theta\, \hat{y}'$$

so; $$A_x \hat{x} + A_y \hat{y} = A_x(\cos\theta\, \hat{x}' - \sin\theta\, \hat{y}') + A_y(\sin\theta\, \hat{x}' + \cos\theta\, \hat{y}')$$
$$= (A_x \cdot \cos\theta + A_y \cdot \sin\theta)\, \hat{x}' + (-A_x \cdot \sin\theta + A_y \cdot \cos\theta)\, \hat{y}'$$

so; $$A_{x'} = A_x \cdot \cos\theta + A_y \cdot \sin\theta$$
$$A_{y'} = -A_x \cdot \sin\theta + A_y \cdot \cos\theta$$

$$\Rightarrow \boxed{\begin{pmatrix} A_{x'} \\ A_{y'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}}$$

This is how vector transforms

Consider the moment of Inertia matrix I_{ij}, how it transforms ?

$$R : I_{ij} \longmapsto I'_{ij} = R\, I_{ij}\, R^{-1}$$

where R is Rotation matrix

i.e; $$\boxed{I' = R\, I\, R^{-1}}$$

This is how a matrix transform.

$R^{-1} = R^T$ for rotation matrices

General Transformation rules for Tensors

$$\bar{A}^{ijk} = \sum_{m,n,p} \underbrace{\frac{\partial \bar{x}^i}{\partial x^m} \cdot \frac{\partial \bar{x}^j}{\partial x^n} \cdot \frac{\partial \bar{x}^k}{\partial x^p}} \cdot A^{mnp}$$

if $m, n, p \in \{1, 2\}$, Then this has $2^3 = 8$ terms.

$x^i \longrightarrow \bar{x}^i$ (coordinate frame tranformation)

$$\bar{x}^i = \bar{x}^i (x^m)$$

Rank = 0 are scalars $\quad \bar{\phi}(\bar{x}) = \phi(x)$

Rank = 1 are vector $\quad \bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^m} \cdot A^m$

we know that, under rotation vectors transforms as $\bar{A} = RA$ $\quad R = \left[\frac{\partial \bar{x}^i}{\partial x^m}\right]$

ie; $\bar{A}^i = \sum_m R^i{}_m A^m$

$$\bar{I}^{ij} = \frac{\partial \bar{x}^i}{\partial x^m} \cdot \frac{\partial \bar{x}^j}{\partial x^n} I^{mn}$$

For Rotation : $\frac{\partial \bar{x}^i}{\partial x^j} = \frac{\partial x^j}{\partial \bar{x}^i}$

$\frac{\partial \bar{x}^i}{\partial x^j}$: R matrix

$\frac{\partial x^j}{\partial \bar{x}^i}$: R^{-1} (Inverse matrix)

$R_{ij} = \frac{\partial \bar{x}^i}{\partial x^j}$ takes unprimed to prime

$$R^{-1}{}_{ji} = \frac{\partial x^j}{\partial \bar{x}^i}$$ takes primed to unprimed.

$$(R_{ij})^T = R_{ji} = (R^{-1})_{ji}$$

so, $$\bar{I} = R I R^{-1}$$

$$\bar{I}_{ij} = R_{im} I_{mn} (R^{-1})_{nj}$$

$$= \frac{\partial \bar{x}^i}{\partial x^m} I_{mn} \cdot \frac{\partial x^n}{\partial \bar{x}^j}$$

but $$\frac{\partial x^n}{\partial \bar{x}^j} = \frac{\partial \bar{x}^j}{\partial x^n}$$

so, $$\bar{I}_{ij} = \frac{\partial \bar{x}^i}{\partial x^m} \cdot \frac{\partial \bar{x}^j}{\partial x^n} I_{mn}$$

Rotational transformation of moment of inertia is compatible to rank-2 tensor transformation property

Contravariant

$$\boxed{\bar{A}^{ijk\cdots} = \frac{\partial \bar{x}^i}{\partial x^m} \cdot \frac{\partial \bar{x}^j}{\partial x^m} \cdot \frac{\partial \bar{x}^k}{\partial x^p} \cdots A^{mnp\cdots}}$$

Covariant

$$\boxed{\bar{A}_{ijk\cdots} = \frac{\partial x^m}{\partial \bar{x}^i} \cdot \frac{\partial x^n}{\partial \bar{x}^j} \cdot \frac{\partial x^p}{\partial \bar{x}^k} \cdots A_{mnp\cdots}}$$

Mixed Tensor: $$\bar{A}_i{}^{jk} = \frac{\partial x^m}{\partial \bar{x}^i} \cdot \frac{\partial \bar{x}^j}{\partial x^n} \cdot \frac{\partial \bar{x}^k}{\partial x^p} \cdot A_m{}^{np}$$

Example of Covariant vector

$$B_i = \frac{\partial \phi(x)}{\partial x^i}$$ This transforms in covariant way.

$$\bar{B}_i = \frac{\partial \bar{\phi}(\bar{x})}{\partial \bar{x}^i}$$

since $\phi(x)$ is scalar

so; $\phi(x) = \bar{\phi}(\bar{x})$

$$\Rightarrow \quad B_i = \frac{\partial \phi(x)}{\partial \bar{x}^i}$$

if a vector $\bar{B}_i = \frac{\partial \bar{\phi}(\bar{x})}{\partial x^m} \cdot \frac{\partial x^m}{\partial \bar{x}^i}$

$$= \frac{\partial \phi(x)}{\partial x^m} \cdot \frac{\partial x^m}{\partial \bar{x}^i}$$

$$\Rightarrow \quad \underbrace{\bar{B}_i}_{\text{new } B} = \frac{\partial x^m}{\partial \bar{x}^i} \cdot \underbrace{B_m}_{\text{old } B}$$

satisfies covariant transformation rule.

upper index $\Rightarrow$ Contravariant
lower index $\Rightarrow$ Covariant
Convention.

$B_{ij} = \frac{\partial A_i}{\partial x^j}$ does not follow transformation rule of rank 2 tensors.

B_{ij} → not tensor in general.

Continuum Mechanics — Shoaib Akhtar

Lec 5 (22th Jan, 2021 ; 3:30pm to 5:00 pm)

Rotation Matrices

① $R_{ij} = \dfrac{\partial \bar{x}^i}{\partial x^j} \equiv a_{ij}$ $\qquad R^{-1} = R^T \Rightarrow \dfrac{\partial \bar{x}^i}{\partial x^j} = \dfrac{\partial x^j}{\partial \bar{x}^i}$

② Isotropic : $A_{ij} = \lambda \cdot \delta_{ij}$ (does not change under rotation)

under rotation

These types of matrices are called Isotropic.

$A \longrightarrow A' = R A R^T$

if A is isotropic

Then $A' = R(\lambda \cdot I) R^T = \lambda \cdot R R^T = \lambda \cdot I$

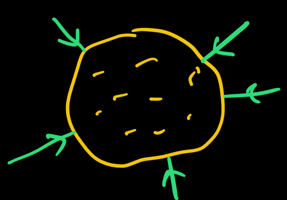
$\Rightarrow \quad A' = \lambda I = A \quad \Rightarrow \quad A' = A$

Stress : σ_{ij}

$\sigma_{ij} = -p \cdot \delta_{ij}$ → Isotropic Stress

$p \Rightarrow$ pressure

pressure is isotropic. (normal to surface)



we know, that if $A_{ijk\cdots}$ is tensor

then

$$A'_{ijk} = \frac{\partial \bar{x}^i}{\partial x^m} \cdot \frac{\partial \bar{x}^j}{\partial x^n} \cdot \frac{\partial \bar{x}^k}{\partial x^p} \cdots A_{mnp\cdots}$$

③ $\epsilon_{ijk\cdots}$

what is transformation property of Levi-Civita?

It is actually pseudo-tensor

$\epsilon_{ijk\ldots}$ does not transform exactly as tensor.

↪ fully antisymmetric w.r.t. any exchange of indices.

$$\epsilon_{ijkl} = -\epsilon_{ilkj} = -\epsilon_{lijk}$$

Any pair we change, get us negative sign.

$$\epsilon_{ijkl} \rightarrow \underbrace{ijlk}_{(-1)} \rightarrow \underbrace{iljk}_{(-1)} \rightarrow \underbrace{lijk}_{(-1)}$$

$(-1)^3 = (-1)$

so; $\epsilon_{ijkl} = -\epsilon_{lijk}$

(a) cyclic change may give sign change if even # of indices

(b) $\epsilon_{ijkl\ldots}$ all indices have to be different, otherwise it will be zero due to fully antisymmetric nature.

(c) Third Rank Levi-Cevita ϵ_{ijk}

ϵ_{ijk} $3!$ # of elements $= 6$ non-zero elements

$\longrightarrow$ 27 elements. (out of which $27-6=21$ are zero)

$i, j, k \in \{1, 2, 3\}$

- 3 of them is $+1$
- 3 of them is -1

declare $\epsilon_{123} = 1$

Then, the other components are fixed.

ie; $\epsilon_{132} = -\epsilon_{123} = -1$, etc.

$\epsilon_{112} = 0$, etc.

④ $(A\times B)_i = \epsilon_{ijk} A_j B_k$

⑤ Determinant :

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{23} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{23}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{23}a_{32} - a_{22}a_{31})$$

can be written compactly in terms of Levi-Cevita.

$$\boxed{|A| = \epsilon_{ijk}\, a_{1i}\, a_{2j}\, a_{3k}}$$

$$|A| = \epsilon_{123}\, a_{11} a_{22} a_{33} + \cdots$$

for $N\times N$ matrix A
we will have $N!$ terms in $|A|$

$$\boxed{|A| \equiv \det(A) = \epsilon_{i_1 i_2 \cdots i_N} \cdot a_{1i_1} a_{2i_2} \cdots a_{Ni_N}}$$

⑥ $\epsilon_{ijk}\,\epsilon_{imn} \equiv A_{jkmn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

(sum over i)

Einstein Notation: repeated indices are summed over.

Aside : Contraction reduces rank of a tensor (but it remains a <u>tensor</u> of lower rank)

<u>ex</u>

Consider, $A_{ij}B^j$ show that $A'_{ij}B^{j'} = \dfrac{\partial x^m}{\partial \bar{x}^i} \cdot A_{mn} B^n$

and hence we establish that $A_{ij} B^j$ is rank 1 object

$$A'_{ij} B'^j = \frac{\partial x^m}{\partial \bar{x}^i} \cdot \frac{\partial x^n}{\partial \bar{x}^j} A_{mn} \cdot \frac{\partial \bar{x}^j}{\partial x^p} \cdot B^p$$

$$= \frac{\partial x^m}{\partial \bar{x}^i} \cdot \underbrace{\left(\frac{\partial x^n}{\partial \bar{x}^j} \cdot \frac{\partial \bar{x}^j}{\partial \bar{x}^p} \right)}_{\frac{\partial x^n}{\partial x^p}} \cdot A_{mn} \cdot B^p$$

by chain rule

$$= \frac{\partial x^m}{\partial \bar{x}^i} \cdot \frac{\partial x^n}{\partial x^p} \cdot A_{mn} \cdot B^p$$

$$= \frac{\partial x^m}{\partial \bar{x}^i} \cdot \delta_{np} A_{mn} B^p$$

$$= \frac{\partial x^m}{\partial \bar{x}^i} \cdot (A_{mp} B^p)$$

Ex!!

$$(A \times B) \times C = (A \cdot C) B - (B \cdot C) A$$

prove using Levi-Civita

$$[(A \times B) \times C]_i = \epsilon_{ijk} (A \times B)_j C_k$$

$$= \epsilon_{ijk} (\epsilon_{jmn} A_m B_n) C_k$$

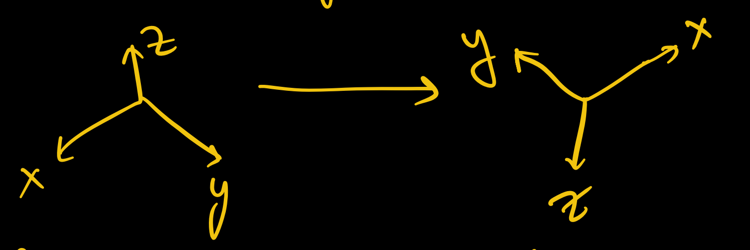
$$= \epsilon_{ijk} \epsilon_{jmn} A_m B_n C_k$$

Pseudo Tensor

Recall pseudo-vector, example $A = B \times C$ B, C vectors

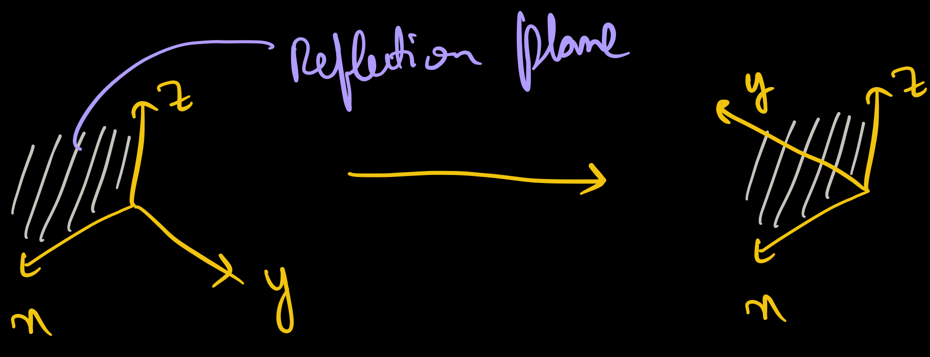
then A is pseudo vector or axial vector.

Inversion / reflections.



Right handed Left handed

$$\left.\begin{array}{l} x \to -x \\ y \to -y \\ z \to -z \end{array}\right\}$$ Inversion or Parity Transformation



} Reflection

$$\left.\begin{array}{l}
\text{Inversion : } R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
\text{Reflection : } R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{array}\right\}$$ satisfies orthogonality condition.

$$R^T = R^{-1}$$

→ Improper Rotations.

$$R R^{-1} = I$$

$$R^{-1} = R^T \quad \Rightarrow \quad R R^T = I$$

$$\Rightarrow \quad |R R^T| = |I| = 1$$

$$\Rightarrow \quad |R|\,|R^T| = 1$$

$$\Rightarrow \quad |R|\,|R| = 1 \quad \Rightarrow \quad |R|^2 = 1 \quad \Rightarrow \quad |R| = \pm 1$$

$|R| = 1$ (proper Rotation matrix)

$|R| = -1$ (improper Rotation matrix)

Given a vector $A = B \times C$, (B & C are vectors)

under Improper Rotation, ie; $(x, y, z) \longrightarrow (-x, -y, -z)$,

$$A = (-B) \times (-C) = B \times C$$

A remains same,

Then A is called Axial Vector or Pseudo Vectors,

B & C are usual vectors,
called Polar vectors.
Change sign under inversion.

Now, we discuss more general element,
Pseudo Tensor.

Transformation Rule.

let ϵ_{ijk} be pseudo tensor.
It transforms as

$$\bar{\epsilon}_{ijk} = |a| \cdot \frac{\partial \bar{x}^i}{\partial x^m} \cdot \frac{\partial \bar{x}^j}{\partial x^n} \cdot \frac{\partial \bar{x}^k}{\partial x^p} \cdot \epsilon_{mnp}$$

because of this factor, it is not truely a tensor.

$$|a| = \det\left(\frac{\partial \bar{x}^i}{\partial x^m}\right)$$ = determinant or Jacobian of the transformation

$\bar{\epsilon}_{ijk} = \epsilon_{ijk}$ under above transformation rule

(if we want to keep defination of levi-cevita same in any frame,
then we have to follow above transformation rule)

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{321} = 1$$
$$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$$

$$\bar{\epsilon}_{ijk} = |a|\, a_{im}\, a_{jn}\, a_{kp} \cdot \epsilon_{mnp}$$

$$\bar{\epsilon}_{123} = |a| \cdot a_{1m}\, a_{2n}\, a_{3p}\, \epsilon_{mnp}$$
$$= |a||a| = |a|^2 = 1 = \epsilon_{123}$$

$$\bar{\epsilon}_{213} = |a| \cdot a_{2m}\, a_{1n}\, a_{3p}\, \epsilon_{mnp}$$

m & n are dummy variable.
Just interchange them

$$= |a|\, a_{2n}\, a_{1m}\, a_{3p}\, \epsilon_{nmp}$$
$$= |a|\, a_{1m}\, a_{2n}\, a_{3p}\, \epsilon_{nmp}$$
$$= |a|\, a_{1m}\, a_{2n}\, a_{3p}\, (-\epsilon_{mnp})$$
$$= (-1)\, |a|\, |a| = (-1)\, |a|^2 = (-1) \cdot (1) = -1$$
$$= \epsilon_{213}$$

pseudo vector

$A = B \times C$, B & C polar. , Then A is axial

B & C change sign under inversion
A dont change sign under inversion.

axial = polar x polar
polar = polar x axial
axial = axial x axial

Pseudo Tensors

$$\underbrace{\epsilon_{ijk}}_{\text{pseudo tensor of rank 3}} \; \underbrace{B_j \, C_k}_{\text{Tensor of Rank 1}}$$

$\longrightarrow$ pseudo tensor of rank 1.

Product of odd no. of pseudo tensor is pseudo tensor.
" " even " " " " " tensor.

- If odd # of components are pseudo, then product is pseudo.

pseudo thing is defined w.r.t. inversion.

Under proper rotation, pseudo tensor transforms as usual tensor.

Symmetric / Anti-symmetric Tensor

defined w.r.t to pair of indices.

A_{ijkl}

consider a general matrix B_{ij}

B_{ij} can be written as

$$B_{ij} = A_{ij} + S_{ij}$$

where A_{ij} is antisymmetric

S_{ij} " symmetric.

$$A_{ij} = \frac{B_{ij} - B_{ji}}{2}, \quad S_{ij} = \frac{B_{ij} + B_{ji}}{2}$$

$$\Rightarrow B_{ij} = \left(\frac{B_{ij} - B_{ji}}{2}\right) + \left(\frac{B_{ij} + B_{ji}}{2}\right)$$

note $A_{ij} = -A_{ji}$, $S_{ij} = S_{ji}$

show that under rotation, anti-symmetric part transforms to itself; same for symmetric.

$A_{ij} \to A'_{ij}$

$S_{ij} \to S'_{ij}$

in general $A_{ij} \neq A'_{ij}$

$S_{ij} \neq S'_{ij}$

but $S'_{ij} = S'_{ji}$

$A'_{ij} = -A'_{ji}$

Example)

$$A_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \cdot \delta_{ik} \cdot \left(\underbrace{\frac{\partial v_l}{\partial x_l}} \right)$$

sum over l, so it is just $\bar{\nabla} \cdot \bar{v}$

(a) A_{ij} is symmetric

(b) $tr(A) = 0$

$$tr\, A = A_{ii} = \frac{\partial v_i}{\partial x_i} + \frac{\partial v_i}{\partial x_i} - \frac{2}{3} \delta_{ij} (\nabla \cdot v)$$

$$= \nabla \cdot v + \nabla \cdot v - \frac{2}{3} \cdot 3 \, (\nabla \cdot v)$$

$$= 2 \nabla \cdot v - 2 \nabla \cdot v = 0$$

Continuum Mechanics — Shoaib Akhtar

Lec 6 (29th Jan, 2021 ; 3:30 pm to 5:00 pm)

$$\mathcal{L} = \frac{-F_{\mu\nu}F^{\mu\nu}}{16\pi} + \frac{j_\alpha A_\alpha}{c} \qquad \text{where} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and $A_\mu = \begin{pmatrix} \vec{A} \\ i\phi \end{pmatrix}$, $x_\mu = \begin{pmatrix} \vec{x} \\ ict \end{pmatrix}$

$c \Rightarrow$ speed of light.

$$j_\mu = \begin{pmatrix} \vec{J} \\ ic\rho \end{pmatrix} \qquad \bar{B} = \bar{\nabla} \times \bar{A} \quad , \quad \bar{E} = -\bar{\nabla}\phi - \frac{1}{c}\cdot\frac{\partial \bar{A}}{\partial t}$$

for the limit $j_\alpha = 0$, ie; $\rho = 0$, $\vec{J} = 0$

Maxwell equation becomes
$$\bar{\nabla} \times \bar{B} = \frac{1}{c}\cdot\frac{\partial \bar{E}}{\partial t}$$
$$\bar{\nabla} \times \bar{E} = -\frac{1}{c}\cdot\frac{\partial \bar{B}}{\partial t}$$
$$\bar{\nabla}\cdot\bar{B} = 0$$
$$\bar{\nabla}\cdot\bar{E} = 0$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}$$

$$F_{13} = \partial_1 A_3 - \partial_3 A_1 = -(\bar{\nabla}\times\bar{A})_2 = -B_2$$

$$F_{34} = \frac{\partial A_4}{\partial x_3} - \frac{\partial A_3}{\partial x_4} = i\frac{\partial \phi}{\partial x_3} - \frac{\partial A_3}{ic\partial t} = i\left(\frac{\partial \phi}{\partial x_3} + \frac{\partial A_3}{c\partial t}\right)$$
$$= -iE_3$$

Euler Lagrange equation

$$\frac{d}{dx_\lambda}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\alpha)}\right) = \frac{\partial \mathcal{L}}{\partial A_\alpha}$$

$$\mathcal{L} = \frac{-1}{16\pi} F_{\mu\nu} F_{\mu\nu}$$

$$\frac{d}{dx_\lambda}\left[-\frac{1}{16\pi} \times 2 \times F_{\mu\nu} \cdot \frac{\partial F_{\mu\nu}}{\partial(\partial_\lambda A_\alpha)}\right] = 0$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\Rightarrow \frac{\partial F_{\mu\nu}}{\partial(\partial_\lambda A_\alpha)} = \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\lambda A_\alpha)} - \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\lambda A_\alpha)}$$

$$= \delta_{\mu\lambda}\delta_{\nu\alpha} - \delta_{\nu\lambda}\cdot\delta_{\mu\alpha}$$

$$\Rightarrow \frac{d}{dx_\lambda}\left[F_{\mu\nu}\cdot(\delta_{\mu\lambda}\cdot\delta_{\nu\alpha} - \delta_{\nu\lambda}\cdot\delta_{\mu\alpha})\right] = 0$$

$$\Rightarrow \frac{d}{dx_\lambda}\left[F_{\lambda\alpha} - F_{\alpha\lambda}\right] = 0 \Rightarrow \frac{d}{dx_\lambda}\left[F_{\lambda\alpha} + F_{\lambda\alpha}\right] = 0$$

$$\Rightarrow 2\cdot\frac{d}{dx_\lambda} F_{\lambda\alpha} = 0$$

$$\Rightarrow \boxed{\frac{d}{dx_\lambda} F_{\lambda\alpha} = 0}$$

$$\boxed{\frac{d}{dx_\lambda}(F_{\lambda\alpha}) = 0}$$ Euler Lagrange Equation for E.M. field.

for $\alpha = 3$, $\frac{dF_{13}}{dx_1} + \frac{dF_{23}}{dx_2} + \cancel{\frac{dF_{33}}{dx_3}}^{\,0} + \frac{dF_{43}}{dx_4} = 0$

$$\Rightarrow \frac{d}{dx}F_{13} + \frac{dF_{23}}{dy} + \frac{dF_{43}}{ic\,dt} = 0$$

Third column elements of F

$$\begin{matrix} -B_2 \\ B_1 \\ 0 \\ iE_3 \end{matrix}$$

$$\Rightarrow \frac{-d}{dx_1}B_2 + \frac{d}{dx_2}B_1 + \frac{i}{ic}\cdot\frac{dE_3}{dt} = 0$$

$$\Rightarrow -\left[\frac{dB_2}{dx_1} - \frac{dB_1}{dx_2}\right] + \frac{1}{c}\cdot\frac{dE_3}{dt} = 0$$

$$\Rightarrow -(\nabla \times B)_3 + \frac{1}{c}\cdot\frac{\partial E_3}{\partial t} = 0$$

for $\alpha = 4$, $\bar{\nabla}\cdot\bar{E} = 0$, for $\alpha = 1,2,3$

$$\Rightarrow \bar{\nabla}\times\bar{B} = \frac{1}{c}\cdot\frac{\partial \bar{E}}{\partial t}$$

$$\left.\begin{aligned} &\bar{\nabla}\cdot\bar{B} = 0 \\ &\bar{\nabla}\times\bar{E} + \frac{1}{c}\cdot\frac{\partial B}{\partial t} = 0 \end{aligned}\right\} \Rightarrow$$

follows from the identity

$$\frac{dF_{\mu\nu}}{dx_\lambda} + \frac{dF_{\lambda\mu}}{dx_\nu} + \frac{dF_{\nu\lambda}}{dx_\mu} = 0$$

$$\boxed{\begin{aligned} &F_{\mu\nu} \text{ is invariant under} \\ &A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda(\vec{x},t) \end{aligned}}$$

Gauge Symmetry.

Gauge invariance of Maxwell's Equations.

ie;

$$\left.\begin{aligned} &\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda \\ &i\phi \rightarrow i\phi + \frac{1}{ic}\cdot\frac{\partial\Lambda}{\partial t} \end{aligned}\right\}$$

Maxwell equations dont change under these transformations

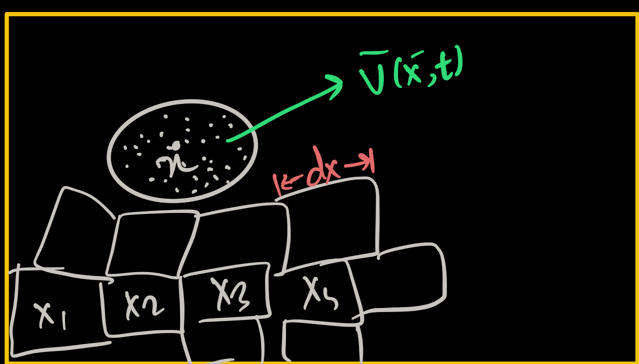
$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = (\partial_\mu A_\nu' - \partial_\nu A_\mu')$$

$$A_\mu' = A_\mu + \partial_\mu \Lambda$$

$$\Rightarrow \quad F'_{\mu\nu} = F_{\mu\nu} + \partial_\mu(\partial_\nu \Lambda) - \partial_\nu(\partial_\mu \Lambda)$$
$$= F_{\mu\nu} + [\partial_\mu, \partial_\nu]\Lambda = F_{\mu\nu}$$

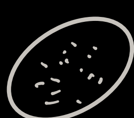
$$\Rightarrow \quad \boxed{F'_{\mu\nu} = F_{\mu\nu}}$$

Fluid Mechanics



we study a small blob of fluid. (still have macroscopic no. of particles inside the blob)

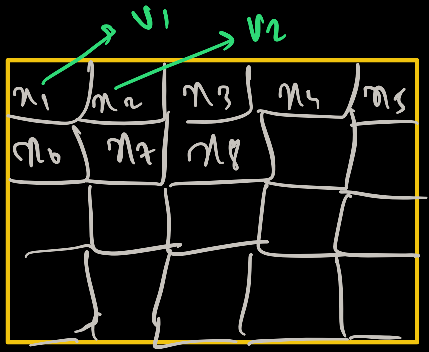
$\bar{v}$ = average velocity of all the particles in the small volume element (the blob)

The whole blob  has only one point, x.

The blobs (or the boxes) have the size dx.

These blobs are mesescopic object. (not microscopic nor macroscopic)

$\bar{v}$ $\Rightarrow$ mesoscopic variable (coarse grained over space)



$\bar{v}(\bar{x}, t)$

want to write equation which describes this mesoscopic variable, velocity.

Navier-Stokes equation

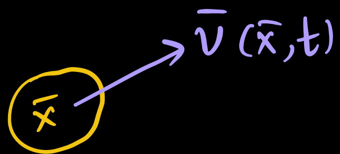
$$\boxed{\rho\left[\frac{\partial \bar{v}(\bar{x},t)}{\partial t} + (\bar{v}\cdot\bar{\nabla})\bar{v}\right] = -\bar{\nabla} p + \bar{f}_{ext} + \underbrace{\eta \nabla^2 \bar{v}(\bar{x},t)}_{\text{viscous dissipation term}}}$$

$f_{ext} \Rightarrow$ External force density

= force on unit volume

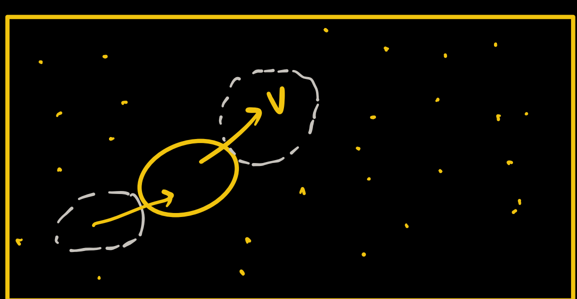
$\eta \Rightarrow$ viscosity.

$$\boxed{\rho \cdot \frac{D\bar{v}}{Dt} = \text{all forces}}$$



$\frac{D\bar{v}}{Dt}$ = acceleration of the blob.

$\left.\frac{\partial \bar{v}(x,t)}{\partial t}\right\}$ at fixed $\bar{x}$

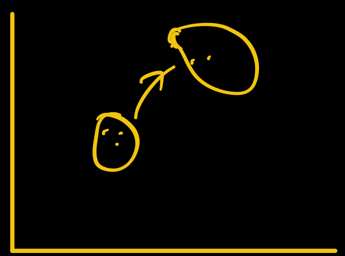


$$\frac{\partial \bar{v}(\bar{x},t)}{\partial t} = \lim_{\Delta t\to 0} \frac{v(x,t+\Delta t) - v(x,t)}{\Delta t}$$

When the sensor for measuring velocity is located at some fixed point $\bar{x}$

at time t & $t+\Delta t$, it measures velocity of different blobs.

but when we follow the blob we get the term $(\bar{v}\cdot\bar{\nabla})\bar{v}$



$\frac{D\bar{v}}{Dt}$ This is w.r.t. lab frame.

Lagrangian Velocity

Euler Velocity

let us look at any quantity Q (could be scalar or vector)

$$\frac{DQ(\bar{x},t)}{Dt} = \lim_{\Delta t\to 0} \frac{Q(\bar{r}+\bar{v}\Delta t,\, t+\Delta t) - Q(\bar{r},t)}{\Delta t}$$

$$= \frac{Q(\bar{r},t) + \frac{\partial Q}{\partial \bar{r}}\cdot\bar{v}\Delta t + \frac{\partial Q}{\partial t}\cdot\Delta t - Q(\bar{r},t)}{\Delta t}$$

$$\frac{\partial Q}{\partial \bar{r}} = \nabla Q$$

$$\Rightarrow \quad \frac{DQ}{Dt} = (\bar{\nabla} Q)\cdot \bar{V} + \frac{\partial Q}{\partial t}$$

$$\Rightarrow \quad \boxed{\underbrace{\frac{DQ}{Dt}}_{\text{Material Derivative}} = \underbrace{\frac{\partial Q}{\partial t}}_{\text{Eulerian Derivative.}} + (\bar{v}\cdot\bar{\nabla})Q}$$

if $\bar{Q}$ is vector

$$\frac{D\bar{Q}}{Dt} = \underbrace{\frac{\partial \bar{Q}}{\partial t}}_{\text{sitting at fixed point \& measuring } \bar{Q}} + \underbrace{(\bar{v}\cdot\bar{\nabla})\bar{Q}}_{\text{following the fluid \& measuring } \bar{Q}.}$$

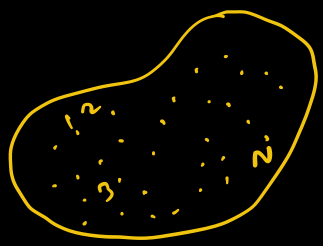
practically, we measure $\frac{\partial Q}{\partial t}$ (just the Eulerian derivative w.r.t. time)

If $\bar{Q} = \bar{V}(\bar{x}, t)$

Then $\frac{D\bar{V}(\bar{x},t)}{Dt} = \frac{\partial \bar{V}(\bar{x},t)}{\partial t} + (\bar{V}(\bar{x},t)\cdot\bar{\nabla})\bar{V}(\bar{x},t)$

$\rho \cdot \frac{D\bar{V}}{Dt}$ is rate of change of momentum. (also for compressible fluid)

(dont think that ρ should go inside derivative, $\frac{D}{Dt}(\rho\bar{V})$ is not needed) when the blob has particle constituent of same mass.



Blob with N particles of mass m

$$m\frac{dv_1}{dt} = \dots$$

$$m \cdot \frac{dv_2}{dt} = \dots$$

$$\vdots$$

$$m \cdot \frac{dv_N}{dt} = \dots$$

$$\bar{v} = \frac{1}{N}\sum_{n=1}^{N} \bar{v}_n$$

$$m \sum_j \frac{d\bar{v}_j}{dt} = \text{Total force.}$$

$$\Rightarrow \quad N m \cdot \frac{d}{dt}\left(\frac{1}{N}\sum_j \bar{v}_j\right) = \text{Total force.}$$

$$\Rightarrow \quad (Nm) \cdot \frac{d}{dt}\bar{v} = \text{Total force}$$

$Nm \Rightarrow$ total mass of blob.

divide by volume V

$$\frac{(Nm)}{V} \cdot \frac{d}{dt}\bar{v} = \frac{\text{force}}{V} = \text{force density}$$

Note $\quad \rho = \dfrac{Nm}{V}$

$$\Rightarrow \quad \rho \cdot \frac{d}{dt}\bar{v} = \text{force density.}$$

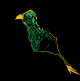
so; although ρ is function of $\bar{x}$ & t, $\rho = \rho(\bar{x}, t)$,

it does not go inside $\frac{D}{Dt}$.

Coarse grained : observing with lower resolution.

Continuum Mechanics — Shoaib Akhtar

Lec 7 (9th Feb, 2021 ; 3:30 pm to 5:00 pm)



$$\underbrace{\rho \cdot \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \bar{\nabla}) \bar{v} \right]} = -\bar{\nabla} P + \eta \nabla^2 \bar{v} + \bar{f}_{ext}(\bar{x})$$

$\frac{D\bar{v}}{Dt}$: material derivative or Time derivative in Lagrangian frame.

$\frac{\partial \bar{v}}{\partial t}$: Eulerian derivative

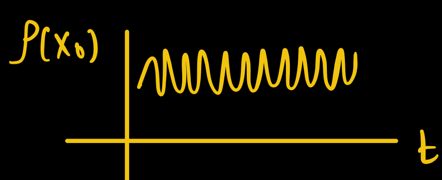
$\frac{D}{Dt}$: hard to calculate; need to follow the blob.

① $\rho = \rho_0$: constant : incompressible

ex) Water is practically incompressible

$\rho(\bar{x})$ density oscillates at sound speed.

$f \cdot \lambda = c$ c = sound speed.



$t_v \Rightarrow$ velocity time scale

if $f >> (t_v)^{-1} \Rightarrow \rho$ is practically constant.

Mach number $= \frac{\bar{v}}{c} << 1$ $\Rightarrow$ density fluctuation is fast as compared to velocity time scale.

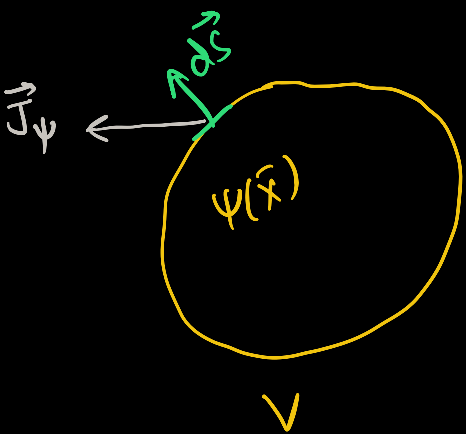
$\Rightarrow$ and hence practically constant.

$c \Rightarrow$ property of material.

② Continuity Equation for property ψ.

ψ could be mass, energy, entropy, etc.

$\psi(\bar{x})$



Content of the property inside volume V is $\int \psi(x)\, dV$

$\vec{J}_\psi \Rightarrow$ flux of ψ

$$\frac{\partial}{\partial t} \int \psi(x)\, dV = -\oint \vec{J}_\psi \cdot d\vec{s} + \underbrace{\int g_\psi\, dV}$$

rate of generation or destruction of ψ inside volume

$$\frac{\partial}{\partial t} \int \psi(x)\, dV = -\int \bar{\nabla} \cdot \vec{J}_\psi\, dV + \int g_\psi\, dV$$

$$\Rightarrow \boxed{\int_V dV \left[\frac{\partial \psi}{\partial t} + \bar{\nabla} \cdot \vec{J}_\psi - g_n \right] = 0}$$

for any arbitrary volume.

for $V \to 0$ (volume) $\Rightarrow$ $\boxed{\frac{\partial \psi}{\partial t} + \vec{\nabla}\cdot\vec{J}_\psi - g_\psi = 0}$ Continuity Equation

for $g_\psi = 0$ $\Rightarrow$ $\boxed{\frac{\partial \psi}{\partial t} = -\vec{\nabla}\cdot\vec{J}_\psi}$

$$\boxed{\vec{J}_\psi = \psi \vec{v}}$$

ψ: amount of ψ leaving

$\vec{v}$: velocity with which it is leaving

Continuity Equation : a conservation law if $g_\psi = 0$

$$\frac{\partial \psi}{\partial t} = -\vec{\nabla}\cdot(\psi \vec{v})$$

If $\psi = \rho(x,t)$ density

Then $\boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot(\rho \vec{V})}$

If density $\rho = \rho_0$ $\Rightarrow$ $0 = -\vec{\nabla}\cdot(\rho\vec{V})$

$$= -\rho\vec{\nabla}\cdot\vec{V} - \vec{V}\cdot\cancel{(\vec{\nabla}\rho)}^{\,0}$$

$\Rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{V} = 0}$

Incompressibility $\Rightarrow$ $\vec{\nabla} \cdot \vec{v} = 0$

Recall $\rho \left[\frac{\partial \bar{v}}{\partial t} + \underbrace{(\bar{v} \cdot \bar{\nabla}) \bar{v}}_{\text{non linear term}} \right] = -\bar{\nabla} p + \eta \nabla^2 \bar{v} + \bar{f}_{ext}$

let $\bar{f}_{ext} = 0$

If ρ is incompressible, then p determined by $\vec{v}$, but not converse.

($\Rightarrow$ these types of quantities are called Passive Quantities; i.e; it does not decide $\vec{v}$, but get decided by $\vec{v}$)

$$\text{Reynolds Number} = \frac{\rho (\vec{v} \cdot \vec{\nabla}) \vec{v}}{\eta \nabla^2 \vec{v}}$$

$$\approx \frac{\left(\rho \cdot \frac{V_0^2}{L} \right)}{\left(\frac{\eta V_0}{L^2} \right)} = \frac{\rho V_0 L}{\eta}$$

L is the scale of the flow

$$\rho (\vec{v} \cdot \vec{\nabla}) \vec{v} \approx \rho \cdot V_0 \cdot \frac{1}{L} \cdot V_0$$

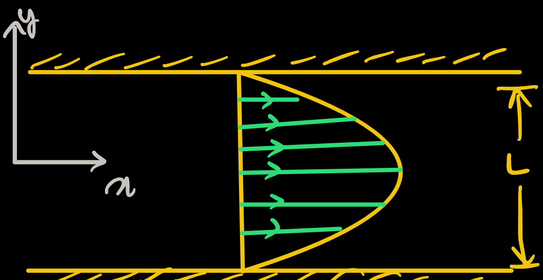
$$\eta \cdot \nabla^2 \vec{v} \approx \eta \cdot \frac{1}{L} \cdot \frac{1}{L} \cdot V_0$$

$\eta \Rightarrow$ dynamic viscosity, $\nu \Rightarrow$ kinematic viscosity

$$\boxed{R_e = \left(\frac{\rho}{\eta}\right) \cdot V_0 L = \frac{V_0 L}{\nu}}$$

Reynolds Number

$$\boxed{\nu = \frac{\eta}{\rho}}$$ Kinematic viscosity



Parabolic flow profile

$V_0 \sim \frac{1}{L} \cdot \int_0^L v_x \cdot dy$

L is the width over which v is varying.

If $Re \ll 1 \Rightarrow$ Laminar flow. (smooth), ignore the non-linear term.

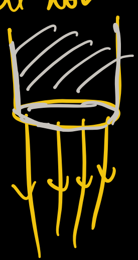
ex

: Chaotic or non-laminar flow

If $Re \gg 1 \Rightarrow$ Turbulant flow (chaotic)

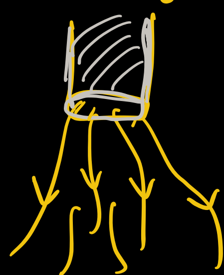
Tap

at low speed



$Re \ll 1$

at high speed.



$R_e \gg 1$

$\rho \cdot \frac{\partial v}{\partial t} + \dots = \dots$ so; $\left[\rho \cdot \frac{\partial v}{\partial t}\right] = [\eta \nabla^2 v]$ dimensional analysis

$\Rightarrow [\eta] = \frac{[\rho]}{[t]}[L]^2 = \frac{M}{L^3} \cdot \frac{L^2}{T} = \frac{M}{LT}$

$$\Rightarrow \quad [\eta] = M L^{-1} T^{-1}$$

and $\quad [\nu] = L^2 T^{-1}$ $\quad$ Kinematic Viscosity.

$$\eta_{water} = 10^{-3} \frac{kg}{m \cdot s} = 10^{-3}\ Pa \cdot s = 0.01 \frac{gm}{cm \cdot sec}$$

$$\eta_{honey} = 850 \cdot \eta_{water}$$

$$\eta_{mercury} = 1.5 \cdot \eta_{water}$$

$$\text{Pascal} = \frac{N}{m^2}$$

$N \Rightarrow$ Newton

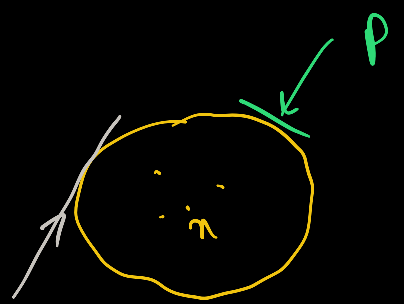
Astrophysical $\Rightarrow$ V_0 high $\Rightarrow$ Re high.

Ocean $\Rightarrow$ L high $\Rightarrow$ Re high.

Blood flow is also Turbulent.

Terms on the RHS of N.S. equation.

$\eta \nabla^2 \bar{V}$ (a kind of force) : Viscous term.



pressure acts perpendicular to surface.

forces on the blob are due to neighbours.

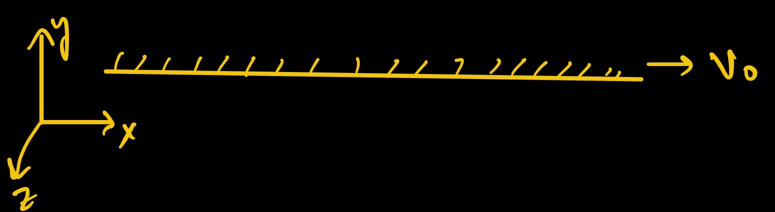
There are other forces which work tangentially to blob (viscous force / shear force) $\eta \nabla^2 V$

When fluid is compressible, there are other forces than pressure which act perpendicular to surface.

for incompressible fluid, pressure is the only force $\perp$ to surface

$\Rightarrow$ & viscous term is tangential to surface.

Shear flow



consider flow between two plates.

- upper plate moving.
- lower plate at rest.

find profile of the flow.

Can think of ∞ x-z plane. For simplicity being suppress z direction.

Assumptions

1) ∞ plate in x direction $\Rightarrow$ V is not function of x
2) steady flow (pattern of flow does not change with time)

$$\frac{\partial v}{\partial t} = 0$$ at a given $\bar{x}$, velocity is not changing.

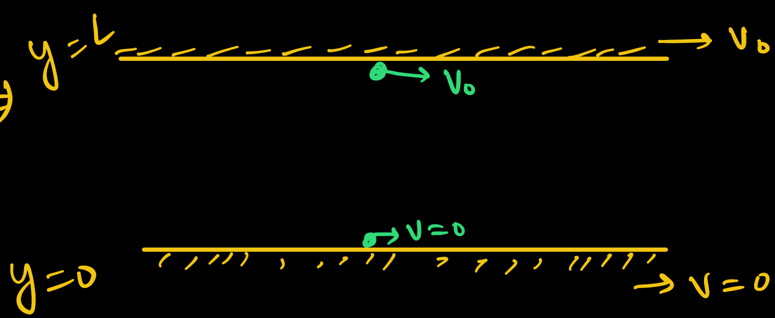
So; $\bar{v}(x,y)$ Ignore z, can be uniform along that also.

$\bar{v}(x,y) = [v_x(y), v_y(y)]$ because of (1), ie; Translational invariance.

3) No slip Boundary Condition.

 → fluid which is in contact with wall, is at rest w.r.t. wall.

fluid element / blob which is in contact with the wall is always at rest w.r.t wall.

⇒ 

4) Incompressible fluid

$$\Rightarrow \quad \bar{\nabla}\cdot\bar{v} = 0 \quad \Rightarrow \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$v_x = v_x(y) \quad \Rightarrow \quad \frac{\partial v_x}{\partial x} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial v_y}{\partial y} = 0}$$

$$\Rightarrow \quad \boxed{v_y = \text{constant}}$$

$$\Rightarrow \quad v_y = v_y(y=0) = v_y(y=L) = 0$$

$$\Rightarrow \quad \boxed{v_y = 0}$$

v_y is zero at the two walls.

$$\Rightarrow \quad \boxed{\bar{v} = v_x(y)\,\hat{x}}$$

5) Non turbulent / Laminar / slow flow

$$\Rightarrow \quad (\bar{v} \cdot \bar{\nabla}) \cdot \bar{v} \simeq 0$$

but here, we can see $(\bar{v} \cdot \bar{\nabla}) \cdot \bar{v} = 0$ exactly zero

$$\left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} \right) v_x = v_x \cancelto{0}{\frac{\partial v_x}{\partial x}} + \cancelto{0}{v_y} \cdot \frac{\partial v_x}{\partial y} = 0$$

since $v_y = 0 \Rightarrow \left(v_x \frac{\partial}{\partial x} + v_y \cdot \frac{\partial}{\partial y} \right) v_y = 0$

Here, the non-linearly term is any way exactly zero.

So, we can ignore this non-linear term from begining.

Now we solve the N-V equation.

$$\left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \bar{\nabla}) \bar{v} \right] = -\frac{\bar{\nabla} p}{\rho_0} + \frac{\eta}{\rho} \bar{\nabla}^2 v + \cancelto{0}{f_{ext}}$$

x equation :

$\frac{\partial \bar{v}}{\partial t} = 0$ because of steady flow.

$(\bar{v} \cdot \bar{\nabla}) \bar{v} = 0$ anyway

$$\Rightarrow \quad \boxed{\frac{\partial p}{\partial x} = \eta \cdot \frac{\partial^2 v_x(y)}{\partial y^2}}$$

$$\nabla^2 v_x = \cancelto{0}{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} = \frac{\partial^2 v_x}{\partial y^2}$$

P can only be function of y

so; $\frac{\partial p}{\partial x} = 0$

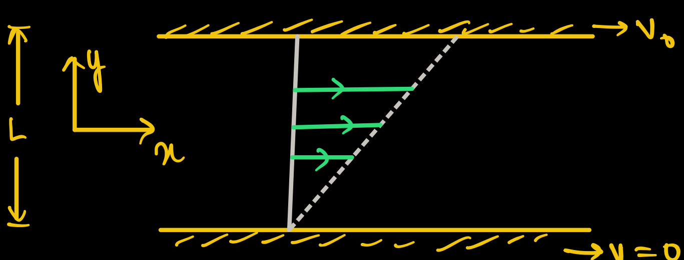
$$\Rightarrow \quad \boxed{\frac{\partial^2 V_x(y)}{\partial y^2} = 0}$$

$$\Rightarrow \quad V_x = Ay + B$$

B.C. : $V_x(0) = 0$, $V_x(L) = V_0$

y-equation: $\frac{\partial p}{\partial y} = \cancel{\eta(\ )V_y} \rightarrow 0$

$$\Rightarrow \quad \frac{\partial p}{\partial y} = 0 \quad \Rightarrow \quad \boxed{P = \text{constant}}$$



linear profile.

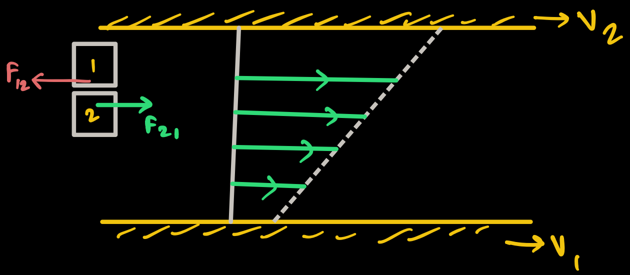
$$\Rightarrow \quad \boxed{V_x(y) = \frac{V_0}{L} y}$$

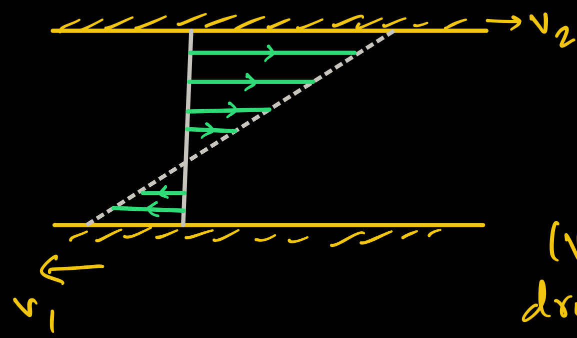
P = constant.

So what deriving the flow? Tangential force.
so called Shear Flow.

Continuum Mechanics — Shoaib Akhtar

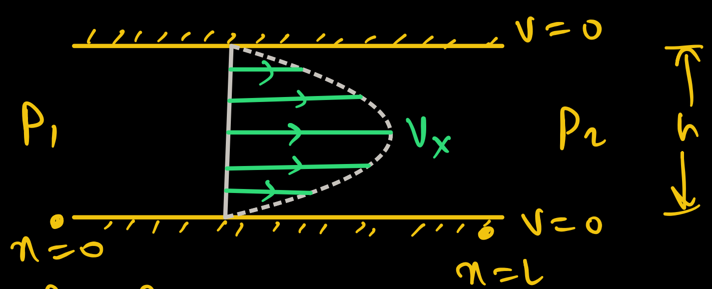
Lec 9 (12th Feb, 2021 ; 3:30 pm to 5:00 pm)





Couette Flow (viscosity driven flow)

Poiseuille Flow (pressure driven flow)



$P_1 > P_2$

Assumption

① $\frac{\partial V}{\partial t} = 0$, $(\bar{V}\cdot\bar{\nabla})\bar{V} = 0$ (slow flow), $\rho = \rho_0$

② No slip boundary condition at the walls.

pressure difference breaks the invariance in x direction.

$P(x)$

$V_x(y)$

$V_y(y)$

For now, just ask can x independent solution exist?

Since it is incompressible fluid; so even the invariance in x-direction is broken, we get velocities to be independent of x.

for $L >> h$, what happens at boundary $y = 0$ or $y = h$ will not affect much in the middle.

at $x=0$; pressure is same

so; $V_y(x=0,y)=0$

so; pressure cant be function of y

$$\frac{\partial v_x}{\partial x}+\frac{\partial V_y}{\partial y}=0 \qquad \text{if } V_x=V_x(y)$$

$$\Rightarrow \frac{\partial V_y}{\partial y}=0$$

so; V_y = constant

at; $h=0$ & $h=L$, $V_y=0$

$$\Rightarrow \boxed{V_y=0}$$

So, we look steady state solution $V_x=V_x(y)$

N-V. equation

steady state

$Re \ll 1$, slow flow

$$\cancel{\frac{\partial \bar{V}}{\partial t}}+\cancel{(\bar{V}\cdot\bar{\nabla})\cdot\bar{V}}=-\bar{\nabla}P+\eta\nabla^2\bar{V}$$

$(v_x\cdot\partial_x V_x)=0$ exactly here. (no need to truely worry about low or high Re)

$$\bar{\nabla}P=\eta\nabla^2 V$$

$x:$ $$\frac{\partial p}{\partial x} = \eta \cdot \frac{\partial^2}{\partial y^2} V_x$$

$y:$ $$\frac{\partial p}{\partial y} = 0 \implies p = p(x)$$ p is function of x only

$$\underbrace{\frac{\partial p(x)}{\partial x}}_{\text{function of } x} = \eta \cdot \underbrace{\frac{\partial^2 V_x(y)}{\partial y^2}}_{\text{function of } y}$$

LHS : function of x
RHS : " " y
$\Big\} \Rightarrow$ They have to be constant.

$$\Rightarrow \quad \frac{\partial p}{\partial x} = \text{constant}$$

so; $$\boxed{p = p_2 + \frac{x}{L} \cdot (p_2 - p_1)}$$

$$\Rightarrow \frac{\partial p}{\partial x} = \frac{p_2 - p_1}{L}$$

so; $$\eta \cdot \frac{\partial^2 V_x(y)}{\partial y^2} = \frac{p_2 - p_1}{L}$$

$$\Rightarrow \frac{\partial^2 V_x(y)}{\partial y^2} = \frac{1}{\eta} \cdot \left(\frac{p_2 - p_1}{L}\right)$$

$$\Rightarrow \boxed{V_x(y) = \frac{1}{2\eta} \cdot \left(\frac{P_2 - P_1}{L}\right) \cdot y^2 + Ay + B}$$

$$V_x(y=0) = 0 \Rightarrow B = 0$$

$V_x(y=h) = 0 \Rightarrow$ gives us A

$$\frac{1}{2\eta} \cdot \frac{(P_2 - P_1)}{L} \times h^2 + Ah = 0$$

$$\Rightarrow A = -\frac{1}{2\eta} \cdot \frac{h}{L} \cdot (P_2 - P_1)$$

$$\boxed{V_x(y) = \frac{1}{2\eta} \cdot \left(\frac{P_2 - P_1}{L}\right) \cdot \left(y^2 - hy\right)}$$

$$P_1 - P_2 = \Delta P > 0$$

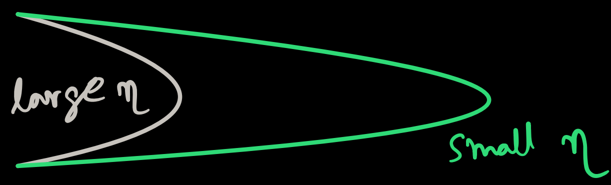
$$\boxed{V_x(y) = -\frac{\Delta P}{2\eta L} \cdot \left[y^2 - hy\right]}$$

This flow is driven by pressure difference.

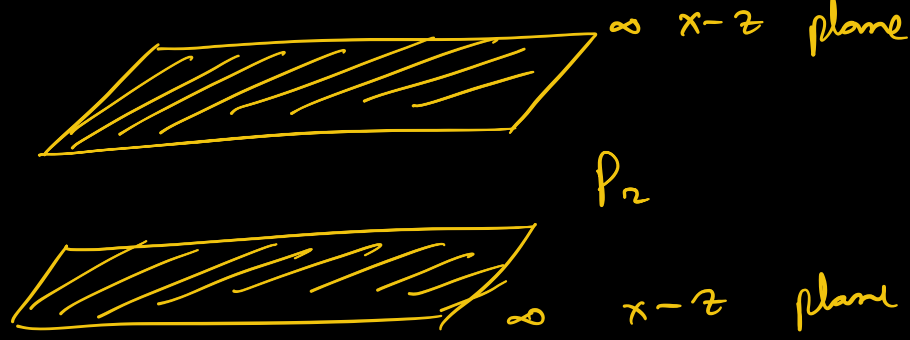
$$V_x = \frac{-1}{2\eta} \cdot \left(\frac{\Delta P}{L}\right) \cdot \left(y^2 - hy\right)$$

$$\frac{dP}{dx} = \frac{-\Delta P}{L}$$

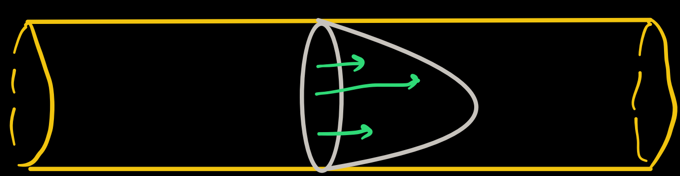
so, $$\boxed{V_x = -\frac{dp}{dx}\cdot\frac{y\cdot(y-h)}{2\eta}}$$ flow under constant pressure gradient.



ex



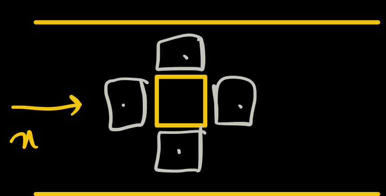
ex



circular cross section.

use cylindrical coordinates.

Paraboloid flow profile



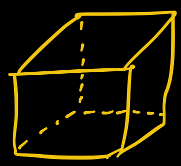
we got steady non accelerating solution along x.

$$V_x = V_x(y)$$

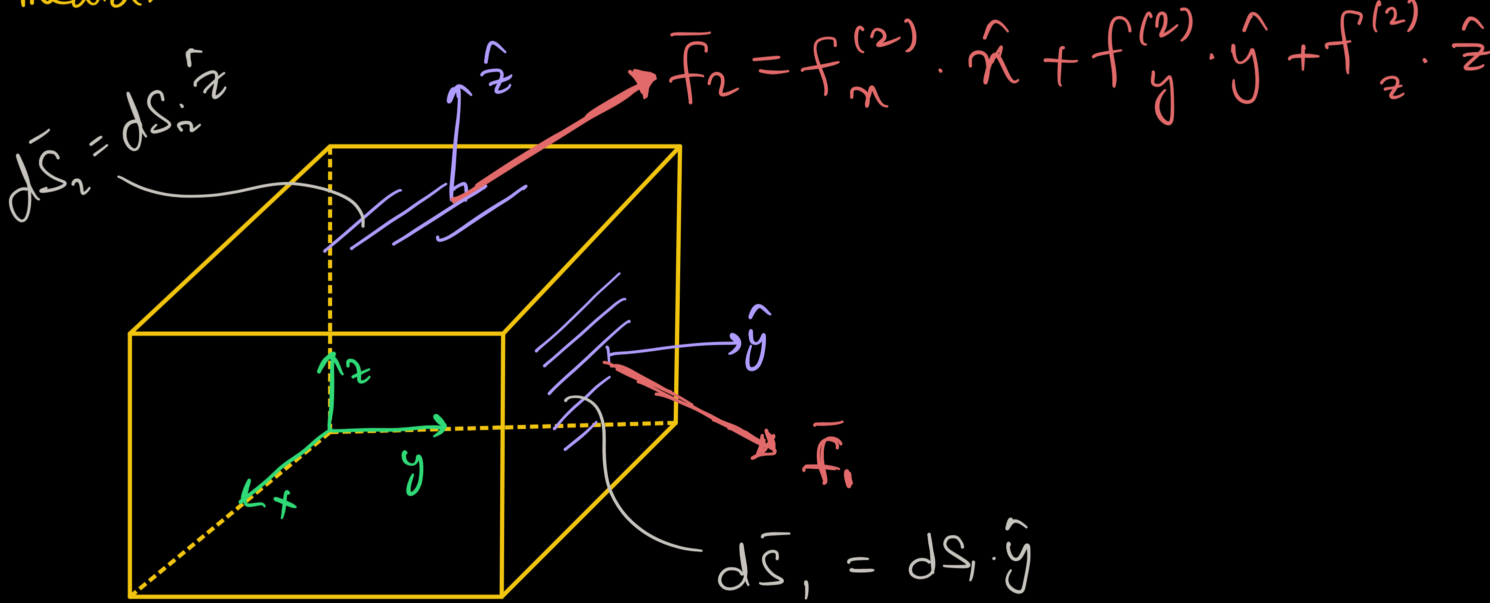
so; net force on fluid element is zero actually.

find force due to neighbours.... Drag.

Stress Tensor $\tilde{\sigma}_{ij}$

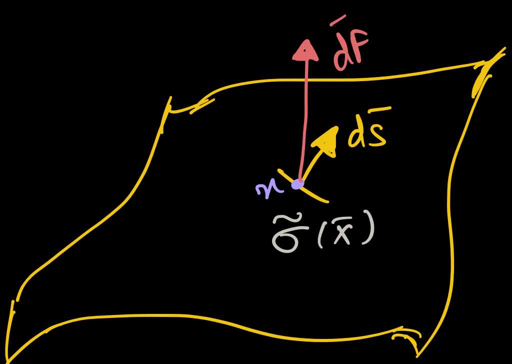


consider any volume element in continuum media.

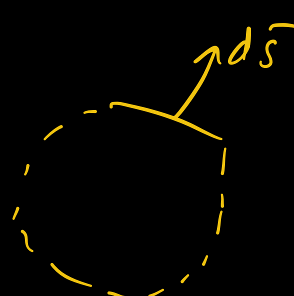


$\bar{f} = \tilde{\sigma} \cdot d\bar{s}$ $\qquad$ σ is a matrix

$$= \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix}$$



$$d\bar{F} = \tilde{\sigma} \cdot d\bar{s}$$

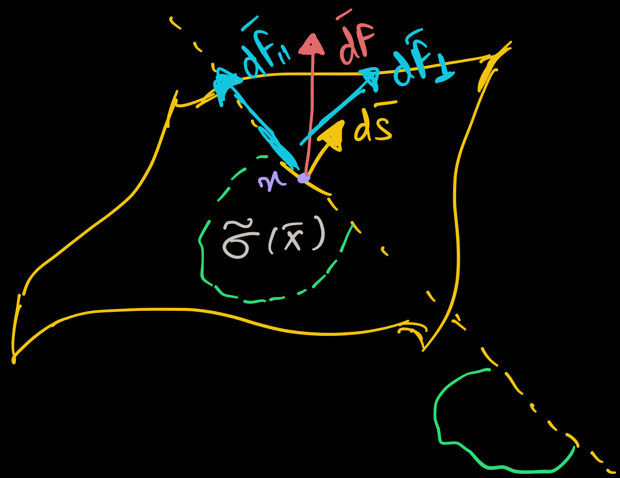


This surface belongs to some volume.

$d\bar{F} \cdot d\bar{S} > 0 \Rightarrow$ neighbours are pulling the volume element along the same direction as $d\bar{S}$.

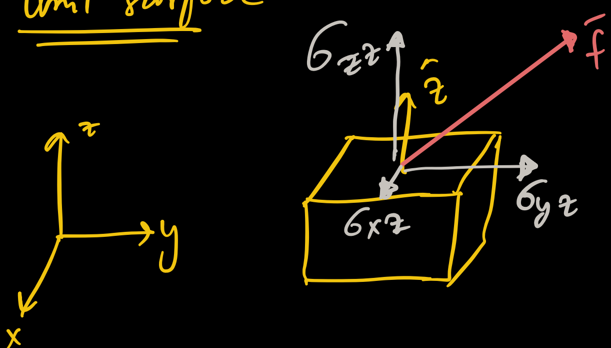
(ie; outward normal)

$d\bar{F} \cdot d\bar{S} < 0 \Rightarrow$ neighbours are pressing or compressing the volume element.



$$d\bar{F} = \underbrace{dF_\parallel}_{\text{shear force}} \cdot \hat{t} + \underbrace{dF_\perp}_{\substack{\text{extensile (when } >0) \\ \text{or} \\ \text{compressive (when } <0)}} \cdot \hat{n}$$

ex. Unit surface



$$\bar{f} = \sigma \cdot \hat{z}$$

$$= \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{bmatrix}$$

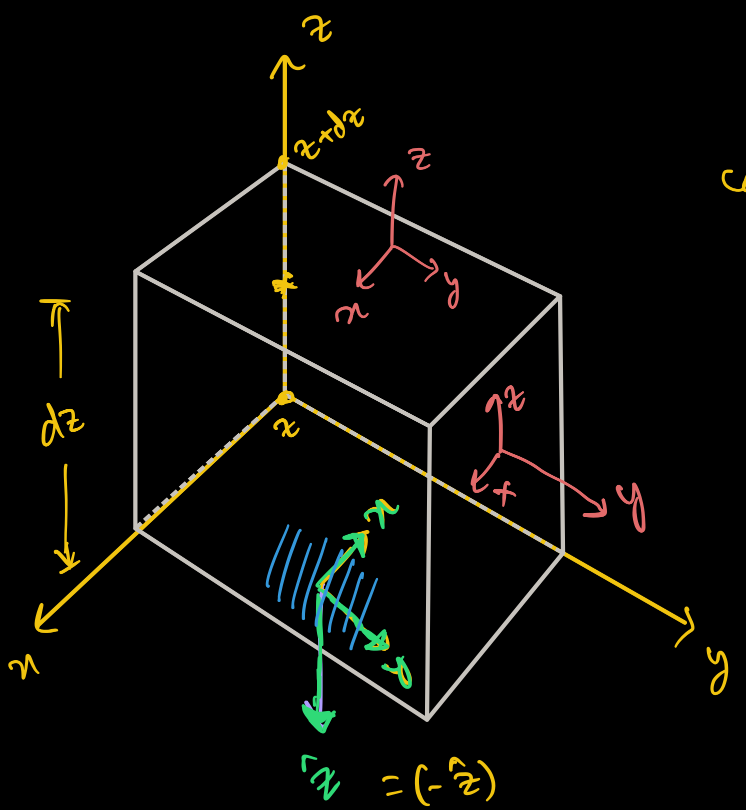
We chose unit surface, and now directly interpreting σ_{ij} as forces.

σ_{ij} = force on unit area along $\hat{i}$ direction on the j^{th} surface

σ_{ij}

i → direction of force

j → surface.



can take origin at midpoint.

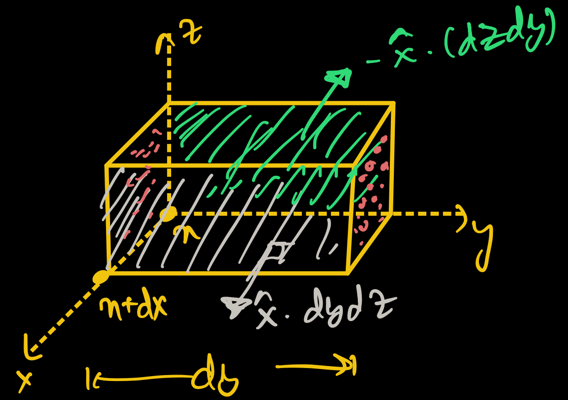
if all elements of $\tilde{\sigma}$ are positive ⇒ system is under stretch from all side.

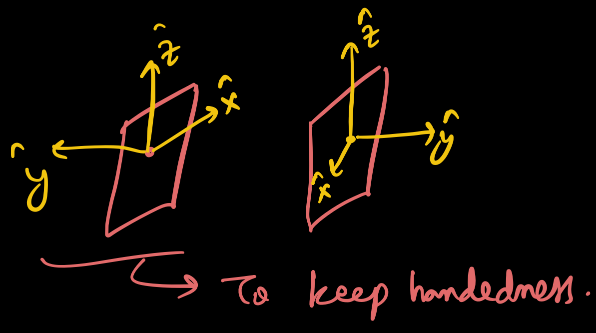
$$(-\sigma_{ij})\cdot\hat{z} = -ve$$

Here we have 6 faces

Then,

total force along x-direction.





$$F_x = \left(\sigma_{xx}(x+dx) - \sigma_{xx}(x)\right)dy\,dz$$
$$+ \left(\sigma_{xy}(y+dy) - \sigma_{xy}(y)\right)dx\,dz$$
$$+ \left(\sigma_{xz}(z+dz) - \sigma_{xz}(z)\right)dx\,dy$$

$\dfrac{F_x}{dx\,dy\,dz}$ = Force per unit olume

$$\frac{F_x}{dx\,dy\,dz} = \frac{\sigma_{xx}(x+dx) - \sigma_{xx}(x)}{dx} + \frac{\sigma_{xy}(y+dy) - \sigma_{xy}(y)}{dy}$$
$$+ \frac{\sigma_{xz}(z+dz) - \sigma_{xz}(z)}{dz}$$

$$= \frac{d\sigma_{xx}}{dx} + \frac{d\sigma_{xy}}{dy} + \frac{d\sigma_{xz}}{dz}$$

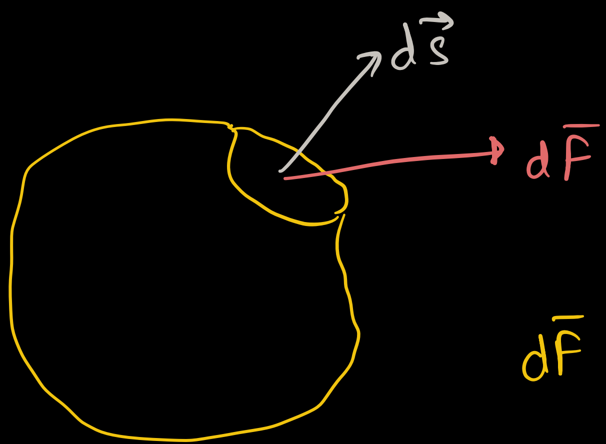
$$= \frac{\partial}{\partial x_i}\tilde{\sigma}_{xi} = f_x$$ $\longrightarrow$ Force density.

$$f_y = \frac{\partial}{\partial x_i} \tilde{\sigma}_{yi}$$

$$f_z = \frac{\partial}{\partial x_i} \tilde{\sigma}_{zi}$$

$$\boxed{f_i = \frac{\partial}{\partial x_j} \tilde{\sigma}_{ij}}$$

$$\bar{f} = \bar{\nabla} \cdot \tilde{\sigma}$$



$$d\bar{F} = \tilde{\sigma} \cdot d\bar{s}$$

Total force on the volume element V

$$\bar{F} = \oint_{\partial V} \tilde{\sigma} \cdot d\bar{s} \overset{\text{Div}}{\underset{\text{Theorem}}{=}} \oint_{V} \bar{\nabla} \cdot \tilde{\sigma} \, dV$$

$$\underbrace{\bar{F}}_{\text{net force}} = \int_V \underbrace{\bar{f}}_{\text{force density}} dV = \oint_V \bar{\nabla} \cdot \tilde{\sigma} \, dV$$

$$\boxed{\bar{f} = \bar{\nabla} \cdot \tilde{\sigma}}$$

$$\boxed{\bar{F} = \int_V \bar{f} \, dV}$$ defining equation for force density $\bar{f}$.

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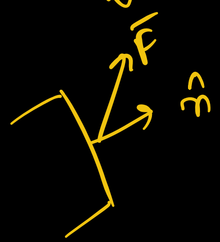
Lec 10 (16th Feb, 2021 ; 3:30 pm to 5:00 pm)

Identify the surface by identifying normal $\hat{n}$



$$d\bar{s} = ds\,\hat{n}$$

$$\bar{F} = \tilde{\sigma} \cdot d\bar{s} \quad \text{Force away from the volume element}$$

$$= \begin{pmatrix} \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} \vdots \end{pmatrix}$$

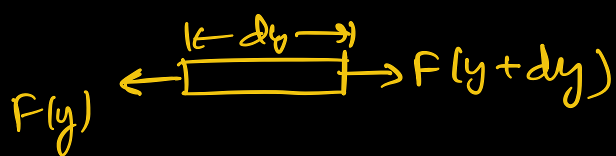
$$\begin{pmatrix} \sigma_{11}(x) & \sigma_{12}(x) & \sigma_{13} \\ \sigma_{21}(x) & \sigma_{22}(x) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{pmatrix}$$

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{pmatrix}$$

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{pmatrix}$$

Ex 1

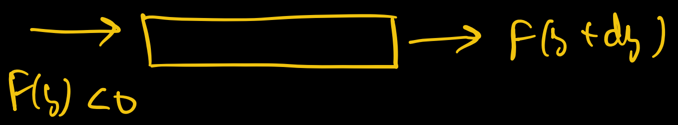
1-D



In 1-D, only longitudinal force.

Tangential force at not there

If F is positive definite $\Rightarrow$ Stretched

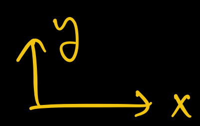


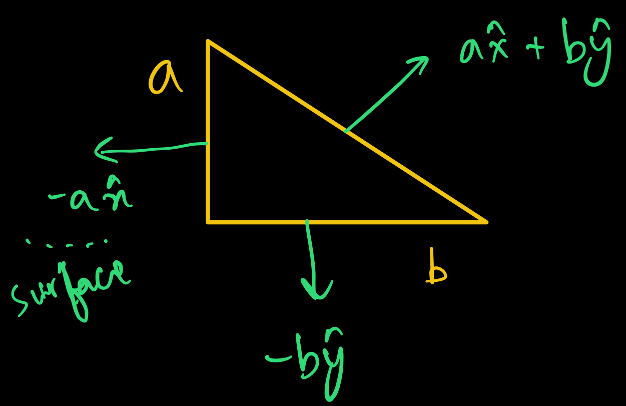
Compressed if $F(y) > F(y+dy)$

Stretched if $F(y) < F(y+dy)$

Still 2-D

let stress matrix be positive





$$\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

constant, positive definite.

Net force on triangle

$$\tilde{\sigma}(-a\hat{x}) + \tilde{\sigma}(-b\hat{y}) + \tilde{\sigma}(a\hat{x} + b\hat{y})$$

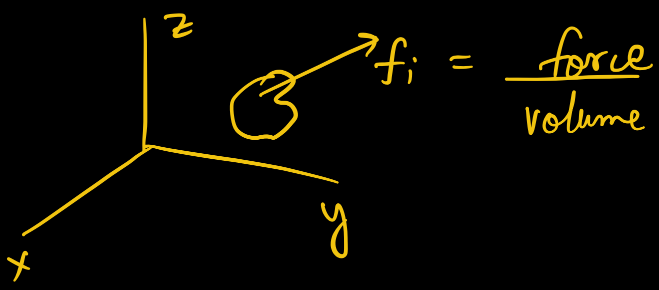
$$= \tilde{\sigma}(-a\hat{x} - b\hat{y} + a\hat{x} + b\hat{y})$$ (because $\tilde{\sigma}$ is constant vector)

$$= \tilde{\sigma}(\vec{0}) = \vec{0}$$

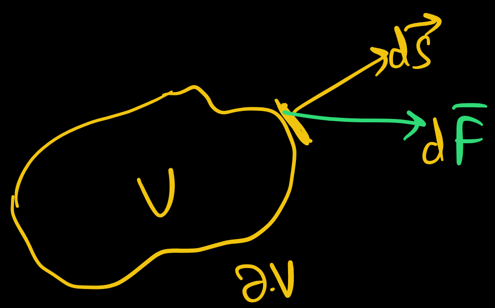
This could not be done if σ varied in space.

So, net force on the triangle is zero.

We had $f_i = \dfrac{\partial \sigma_{ij}}{\partial x_j}$



We are only thinking of contact forces for now; while thinking stress tensor.



$$\bar{F} = \oint_{\partial V} d\bar{F} = \oint_{\partial V} \tilde{\sigma} \cdot d\bar{s} \overset{\text{Divergence Theorem}}{=} \int_V (\bar{\nabla} \cdot \tilde{\sigma})\, dV$$

In the absence of external force.

$\bar{F} = \int_V \bar{f} \cdot dV$ is the defining eqn for force density $\bar{f}$

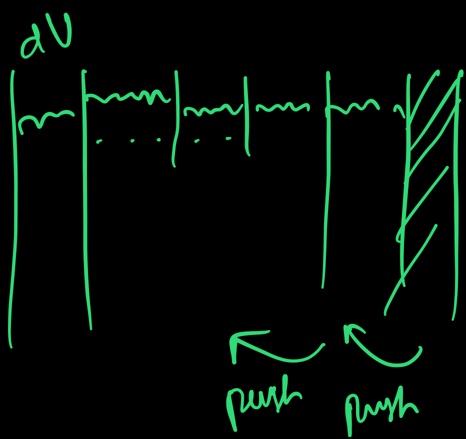
$$\Rightarrow \int_V \bar{f}\, dV = \int_V (\bar{\nabla} \cdot \tilde{\sigma})\, dV$$

$$\Rightarrow \boxed{\bar{f} = (\bar{\nabla} \cdot \tilde{\sigma})}$$

when we have only surface forces. (no forces like electrostatic, gravity, ... etc)

$\int_V dV \cdot \{\bar{f} - \bar{\nabla} \cdot \tilde{\sigma}\} = 0$ for arbitrary V

$\exists\ \boxed{\bar{f} = \bar{\nabla}\cdot\tilde{\sigma}}$



So $\bar{f}$ is a local expression, because forces are short distance forces.

How does $\tilde{\sigma}$ looks like?

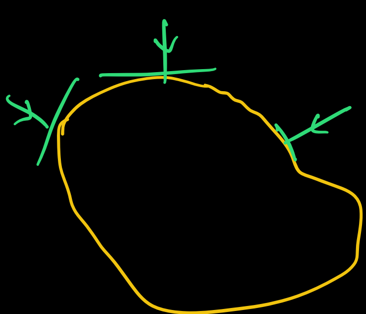
This term will always be there

For a fluid, $\sigma_{ij}(\bar{x}) = \overbrace{-p(\bar{x})\delta_{ij}} + \underbrace{\sigma'_{ij}(\bar{x})}$

→ stress due to non ideal fluid. (ex, fluid with friction) (friction between volume element of fluids)

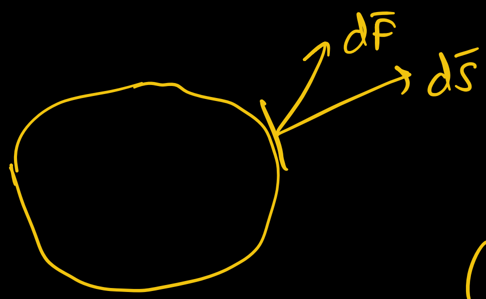
$p \equiv$ conventional pressure

Consider a volume element



$-p\,\delta_{ij}$ ensures that pressure always act normal to surface.

Our notion of pressure (always $\perp$ to surface, & compresses the volume element)



$$d\bar F = \tilde\sigma_p \cdot d\bar S$$

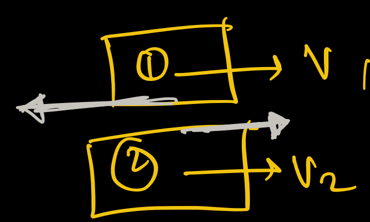
$$(\tilde\sigma_p)_{ij} = -p \cdot \delta_{ij}$$

$$dF_i = -p \cdot \delta_{ij} \cdot dS_j = -p \cdot dS_i$$

$\Rightarrow \boxed{d\bar F \propto d\bar S}$ ie; $\boxed{d\bar F = -p\, d\bar S}$

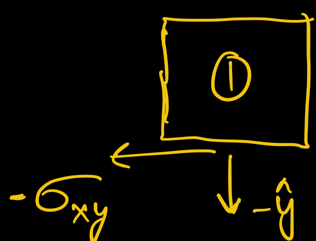
Now σ'_{ij}

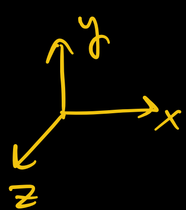
① $\sigma'_{ij}(x) \propto \dfrac{\partial v_i}{\partial x_j}$



let $v_1 > v_2$

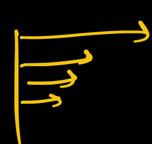
So there will be friction



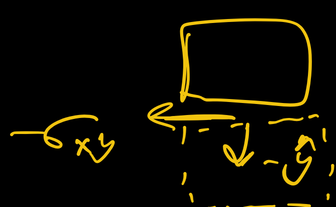


example To check $\sigma'_{ij} \propto \dfrac{\partial v_i}{\partial x_j}$ or $\sigma'_{ij} \propto -\dfrac{\partial v_i}{\partial x_j}$

let $\dfrac{\partial v_x}{\partial y} > 0$



$\sigma_{xy} = \dfrac{\partial v_x}{\partial y}$



so, $\boxed{\sigma'_{ij} \propto \dfrac{\partial v_i}{\partial x_j} \quad ✓ \text{ correct.}}$

We could choose higher order term

$$\sigma_{ij} \propto \sum_k \frac{\partial v_i}{\partial x_k} \frac{\partial v_j}{\partial x_k}$$

We cant have $\sigma_{ij} \propto v_i v_j$

because for uniform v also we will get friction.

so $v_i v_j$ is not possible as friction term.

$\sum_k \alpha_k v_i v_j v_k$ is possible.

What about $\sigma_{ij} \propto v_i \cdot \frac{\partial v_j}{\partial x_k} \cdot v_k$

This is possible, but it is higher order in v.

σ'_{ij} should depend on gradients & has to be 2^{nd} order.

② $\sigma'_{ij} \propto$ Terms symmetric in i & j

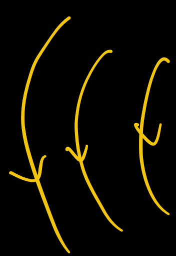
like $\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$

$$\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

Due to global rotational symmetry, only symmetric combinations are allowed.

Rotating bucket of water.

$$\bar{v} = \bar{\omega}_0 \times \bar{x}$$ $\Rightarrow$ This corresponds to uniform rotation with $\bar{\omega}_0$.



move at same rotational speed, so no relative motion $\Rightarrow$ so no friction.

so: $\sigma'_{ij} = 0$

and it is only possible if

$$\sigma'_{ij} \propto \text{sym}\left(\frac{\partial v_i}{\partial x_j}\right)$$

i.e.: $\sigma'_{ij} \propto \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$

$$\sigma'_{ij} = \underbrace{\eta}_{\text{some constant.}} \cdot \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

Should take antisymmetric part, because that will not go to zero for cases like $\bar{v} = \bar{\omega}_0 \times \bar{x}$.

Still can have, $\lambda \cdot \delta_{ij} \cdot \left(\frac{\partial x_l}{\partial x_l} \right)$

$$= \lambda \, \delta_{ij} \, (\bar{\nabla} \cdot \bar{v})$$

$$\sigma'_{ij} = \eta \cdot \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \lambda \cdot \delta_{ij} \cdot \bar{\nabla} \cdot \bar{v}$$

Conventionally written in following form

$$\sigma'_{ij} = \eta \cdot \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \cdot \frac{\partial v_l}{\partial v_l} \right) + \lambda' \cdot \delta_{ij} \cdot \frac{\partial v_k}{\partial x_k}$$

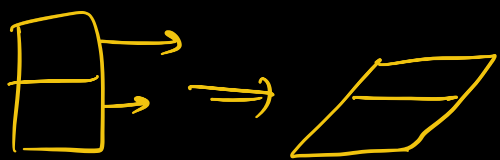
λ piece is split in this form, so that the trace of $\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \cdot \frac{\partial v_l}{\partial v_l} \right)$ vanishes.

often written as

$$\sigma'_{ij} = \eta \cdot \underbrace{\left(\frac{\partial v_i}{\partial x_j} + i \leftrightarrow j - \frac{2}{3}\delta_{ij} \cdot \frac{\partial v_l}{\partial v_l}\right)} + \underbrace{\lambda \cdot \delta_{ij} \cdot \frac{\partial v_k}{\partial x_k}}$$

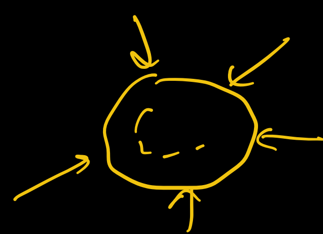
has no change in volume. ; but does volume deformation.

This associated to volume change



$\eta \equiv$ Dynamic Viscosity.

$\lambda \equiv$ Viscosity due to compressibility



The term $\frac{\partial v_k}{\partial x_k}$ is due to compression of the fluid, and contribute to non-ideal part of fluid.

Application

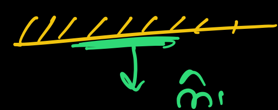
①

what are kind of forces acting on solid-liquid interface.

$$d\bar{F} = \tilde{\sigma}\cdot\hat{n}$$

= force on the fluid element below due to the solid.

Force on the solid due to liquid (Reaction force) → called Drag.



$$d\bar{F}_1 = \tilde{\sigma}\cdot\hat{n}_1$$

$$= -\tilde{\sigma}\cdot\hat{n}$$

note $\hat{n}_1 = -\hat{n}$

Application

①

Parabolic profile

$$v_x = -\left(\frac{p_1 - p_2}{2\eta L}\right) y\cdot(y-h)$$

Drag Force on solid (at $y=h$)

$$= \bar{F}_h = \tilde{\sigma} \cdot \hat{n}$$

for incompressible fluid

$$\tilde{\sigma} = \eta \cdot \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \text{pressure term.}$$

$$\boxed{F_i = \left[\eta \cdot \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - p \cdot \delta_{ij} \right] \cdot \hat{n}_j}$$

here,

$$\hat{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\hat{y}$$

$$F_x = \eta \cdot \left[\underbrace{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_x}{\partial x} \right) \cancel{n_x}^{\,0}}_{j=x} + \underbrace{\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) n_y}_{j=y} \right]$$

$$= -\eta \cdot \left(\frac{\partial v_x}{\partial y} + \cancel{\frac{\partial v_y}{\partial x}}^{\,0} \right)$$

$$= -\eta \cdot \frac{\partial v_x}{\partial y}$$

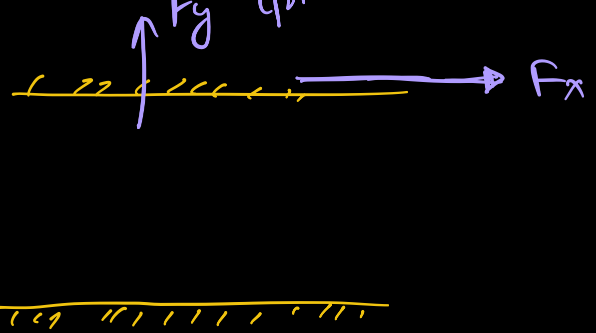
note

$$\left. \frac{\partial v_x}{\partial y} \right|_{y=h} < 0$$

$$\Rightarrow \quad F_x = -\frac{(P_1 - P_2)}{2 \eta L} \cdot (2y - h)(-1) \Big|_{y=h}$$

$$\Rightarrow \quad \boxed{F_x = \frac{(P_1 - P_2)h}{2\eta L}}$$

F_y (pressure) : comming due to pressure term.



What is F_y on upper plate.

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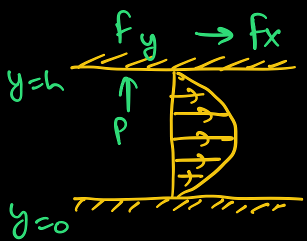


$\bar{F} = -\tilde{\sigma} \cdot \hat{n}$

volume is not changed under pure shear.

Has zero Trace

$$\rho \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \bar{\nabla}) \bar{v} \right] = -\bar{\nabla} p + \eta \nabla^2 \bar{v} + \bar{f}_{ext}$$

$$\sigma_{ij} = -p \cdot \delta_{ij} + \eta \cdot \overbrace{\left(\frac{\partial v_i}{\partial x_j} + i \leftrightarrow j - \frac{2}{3} \delta_{ij} \cdot \frac{\partial v_k}{\partial x_k} \right)}$$

$$+ \xi \cdot \delta_{ij} \cdot \frac{\partial v_k}{\partial x_k}$$

shear viscosity

for incompressible fluid.

Bulk viscosity

(not physically a friction...)

* related to expansion & compression of volume.

$\rho \frac{D\bar{v}}{Dt}$ = Force on fluid element which we are following in space.

$$\int \rho \cdot \frac{D\bar{v}}{Dt} \cdot dV = \oint \tilde{\sigma} \cdot d\bar{s} + \int f_{ext} \cdot dV$$

surface

volume.

$$= \int (\bar{\nabla} \cdot \tilde{\sigma}) dV + \int f_{ext} \, dV$$

$$\boxed{\rho \frac{D\bar{v}}{Dt} = \bar{\nabla} \cdot \tilde{\sigma} + \bar{f}_{ext}}$$

$$\rho \frac{Dv_i}{Dt} = \frac{\partial}{\partial x_j} \tilde{\sigma}_{ij} + \bar{f}_{ext\, i}$$

lets work only with $\rho \frac{Dv_i}{Dt} = \frac{\partial}{\partial x_j} \tilde{\sigma}_{ij}$

$$= \frac{\partial}{\partial x_j} \left[-p \delta_{ij} + \eta \cdot \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \cdot \frac{\partial v_k}{\partial x_k} \right) + \xi \cdot \delta_{ij} \cdot \frac{\partial v_k}{\partial x_k} \right]$$

$$\Rightarrow \rho \cdot \frac{Dv_i}{Dt} = -\frac{\partial p}{\partial x_i} + \eta \cdot \left(\sum_j \frac{\partial^2 v_i}{\partial x_j^2} + \frac{\partial}{\partial x_i} \left(\frac{\partial v_j}{\partial x_j} \right) + \left(\xi - \frac{2\eta}{3} \right) \frac{\partial}{\partial x_i} \left(\frac{\partial v_k}{\partial x_k} \right) \right)$$

$$\rho \cdot \frac{D\bar{v}}{Dt} = -\bar{\nabla} p + \eta \cdot \left(\nabla^2 \bar{v} + \bar{\nabla} (\bar{\nabla} \cdot \bar{v}) \right)$$

$$+ \left(\xi - \frac{2\eta}{3} \right) \cdot \bar{\nabla} (\bar{\nabla} \cdot \bar{v})$$

for incompressible fluid $\bar{\nabla} \cdot \bar{v} = 0$

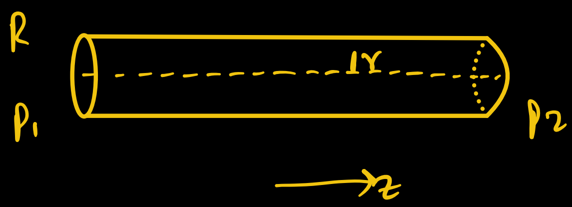
$$\boxed{\rho \cdot \frac{D\bar{v}}{Dt} = -\bar{\nabla} p + \eta \nabla^2 \bar{v} + \left(\xi + \frac{\eta}{3} \right) \bar{\nabla} (\bar{\nabla} \cdot \bar{v}) + \bar{f}_{ext}}$$

Full Navier Stokes Equation.

ex 11 for fluid in gravity $\bar{f}_{ext} = -\rho g \hat{z}$

---

ex 11



$P_1 > P_2$

Symmetry $\Rightarrow$ $V_z = V_z(r)$

$V_\phi = 0$

$V_r = 0$

$V_z(r = R) = 0$ (No slip boundary condition)

The continuity equation;

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}(r \cdot V_r) + \frac{1}{r} \cdot \cancelto{0}{\frac{\partial V_\phi}{\partial \phi}} + \cancelto{0}{\frac{\partial V_z}{\partial z}} = 0$$

$V_z = V_z(r)$ due to translational symmetry along z axis.

$$\Rightarrow \frac{\partial}{\partial r}(r \cdot V_r) = 0$$

$\Rightarrow$ $r \cdot V_r$ = constant

constant $\neq 0$ $\Rightarrow$ $\boxed{V_r(R) \neq 0}$ contradicts no slip boundary condition.

$\Rightarrow$ There is no radial velocity

so; constant $= 0$, and we get $V_r = 0$.

$V_z(r)$, $p(r)$

r equation : $0 = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial r} + \phi$ $\Rightarrow$ p not function of r

ϕ equation: $0 = -\frac{1}{\rho \cdot r} \cdot \frac{\partial p}{\partial \phi} = 0 \Rightarrow p$ is not function of ϕ.

z equation: $\cancel{V_r} \cdot \frac{\partial}{\partial r} V_z = \underbrace{-\frac{1}{\rho} \cdot \frac{\partial p(z)}{\partial z}}_{\text{function of } z} + \underbrace{\frac{\nu}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial V_z}{\partial r}\right)}_{\text{function of } r}$ $\qquad \boxed{\nu = \frac{\eta}{\rho}}$

$\Rightarrow$ They are constant.

$$\Rightarrow \underbrace{\frac{1}{\rho} \cdot \frac{\partial p(z)}{\partial z}}_{z} = \underbrace{\frac{\nu}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial V_z(r)}{\partial r}\right)}_{r}$$

So; They are some constant C_1

$\frac{1}{\rho} \cdot \frac{\partial p}{\partial z} = C_1$ and $\frac{\eta}{\rho \cdot r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial V_z}{\partial r}\right) = C_1$

$p(z)$ linear function of z.
with p_1 & p_2 at $z = 0$ & L respectively.

$$\frac{\eta}{\rho}\left(r \cdot \frac{\partial V_z}{\partial r}\right) = C_1 \cdot \frac{r^2}{2} + C_2$$

$$\frac{\partial V_z}{\partial r} = \frac{\rho}{\eta}\left[C_1 \cdot \frac{r}{2} + \frac{C_2}{r}\right]$$

$$\Rightarrow V_z = \frac{\rho}{\eta}\left[C_1 \cdot \frac{r^2}{4} + C_2 \cdot \ln r\right] + C_3$$

C_2 must be zero, because then

$V_z(r=0) \rightarrow -\infty$

To avoid ∞ at $r=0$, $C_2 = 0$

$$\Rightarrow \quad V_z = \frac{\rho}{\eta} \cdot C_1 \cdot \frac{r^2}{4} + C_3$$

$$\frac{1}{\rho} \cdot \frac{\partial p}{\partial z} = C_1$$

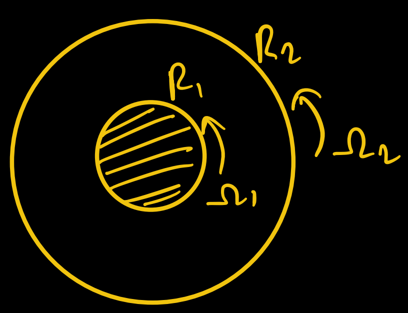
C_1 is known from here

$$V_z(R) = 0 \quad \Rightarrow \quad V_z(r) = \frac{C_1 \rho}{4\eta}(r^2 - R^2)$$

$$\boxed{V_z(r) = \frac{C_1 \rho}{4\eta}(r^2 - R^2)}$$

Parabolic flow

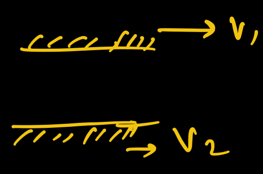
Example



concentric circular cylinders (rotating)

This is viscous driven force

Polar analogue of Quet flow.



$v_1 \neq v_2$

∞ cylinders ; z ranges over $(-\infty, \infty)$

$V_\phi(r)$, $P(r)$

r equation : $\quad -\dfrac{V_\phi^2}{r} = -\dfrac{1}{\rho} \cdot \dfrac{\partial p}{\partial r}$

ϕ equation : $\quad V_\phi \cdot \dfrac{1}{r} \cdot \dfrac{\partial V_\phi}{\partial \phi} = \nu \left[\dfrac{1}{r} \cdot \dfrac{\partial}{\partial r}\left(r \cdot \dfrac{\partial V_\phi}{\partial r} \right) - \dfrac{V_\phi}{r^2} \right]$

solve this & get V_ϕ

$$\frac{\partial V_\phi}{\partial r} + r \cdot \frac{\partial^2 V_\phi}{\partial r^2} - \frac{V_\phi}{r} = 0$$

$$V_\phi = r^n$$

$$n r^{n-1} + r \cdot (n)(n-1) \cdot r \cdot r^{n-2} - r^{n-1} = 0$$

$$\Rightarrow \quad n + n(n-1) - 1 = 0$$

$$\Rightarrow \quad n + n^2 - n - 1 = 0$$

$$\Rightarrow \quad \boxed{n = \pm 1}$$

$$\boxed{V_\phi = Ar + \frac{B}{r}}$$

$V_\phi(R_2) = \Omega_2 R_2$

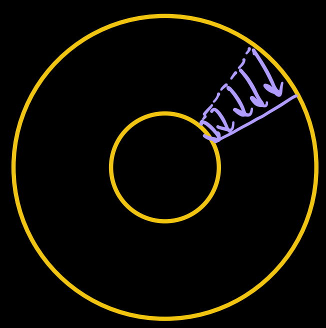
$V_\phi(R_1) = \Omega_1 R_1$ $\longrightarrow$ get A and B.

$$\boxed{V_\phi(r) = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \cdot r + \frac{\Omega_1 - \Omega_2}{R_2^2 - R_1^2} \cdot R_1^2 \cdot R_2^2 \cdot \frac{1}{r}}$$

Limiting cases

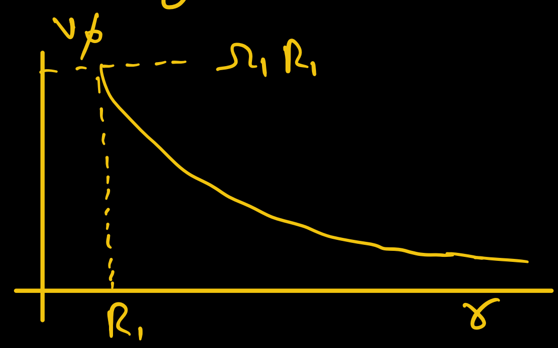
① $\Omega_1 = \Omega_2 \equiv \Omega$

Then $\boxed{v_\phi(r) = \Omega r}$ linear profile in r

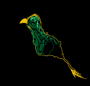


② $\Omega_2 = 0$, $R_2 \longrightarrow \infty$ (no outer boundary)

$$\boxed{v_\phi(r) = (\Omega_2 - \Omega_1) R_1^2 \cdot \frac{1}{r}}$$



Continuum Mechanics — Shoaib Akhtar



Lec 12 (5th March, 2021 ; 3:30 pm to 5:00 pm)

$$\rho\left[\frac{\partial v_i}{\partial t} + (\bar{v}\cdot\bar{\nabla})v_i\right] = -\nabla_i P + \rho g_i + \eta \nabla^2 v_i + (\xi + \tfrac{1}{3})\bar{\nabla}(\bar{\nabla}\cdot\bar{v})$$

under some force.

$\bar{f} = -\bar{\nabla}\phi$ (potential force)

① when $\bar{v}$ small ($Re \ll 1$) + steady state
drop the LHS $\Rightarrow$ Stokes equation.

② Incompressible $\Rightarrow$ $\bar{\nabla}\cdot\bar{v} = 0$

③ $\eta = 0$, $\xi = 0$ $\Rightarrow$ Ideal fluid. $\Rightarrow$ Euler Equation.

Here (a) Incompressible (b) Ideal $\Rightarrow$ η & ξ are zero.

$$\bar{v}\times(\bar{\nabla}\times\bar{v}) = \bar{\nabla}\left(\frac{v^2}{2}\right) - (\bar{v}\cdot\bar{\nabla})\bar{v}$$

$$\rho\left[\frac{\partial\bar{v}}{\partial t} - \bar{v}\times(\bar{\nabla}\times\bar{v})\right] = -\rho\bar{\nabla}\left(\frac{v^2}{2}\right) - \bar{\nabla}P + \rho\bar{g}$$

divide by ρ

$$\frac{\partial\bar{v}}{\partial t} - \bar{v}\times(\bar{\nabla}\times\bar{v}) = -\bar{\nabla}\left(\frac{v^2}{2}\right) - \bar{\nabla}\mathbb{P} + \bar{g}$$

where $\bar{\nabla}\mathbb{P} \equiv \dfrac{\bar{\nabla}P}{\rho}$

for incompressible $\bar{\nabla}\mathbb{P} = \bar{\nabla}\left(\dfrac{P}{\rho}\right)$

$\Rightarrow \mathbb{P} = \dfrac{P}{\rho}$

$$\left[\frac{\partial\bar{v}}{\partial t} - \bar{v}\times(\bar{\nabla}\times\bar{v})\right] = -\bar{\nabla}\left[\frac{v^2}{2} + \frac{P}{\rho_0} + z\cdot g\right]$$

$\longrightarrow \dfrac{\phi(\bar{r})}{\rho_0}$ in general.

Now, if steady state + Curl free flow (Irrotational flow))

$\frac{\partial \bar{v}}{\partial t} = 0$ $\qquad$ $\bar{\nabla} \times \bar{v} = 0$

Then $\quad 0 = -\bar{\nabla}\psi(\vec{x},t)$ → since steady state, it can't be t dependent

$$\Rightarrow \quad 0 = -\bar{\nabla}\psi(\vec{x})$$

where $\psi(\vec{x}) = \frac{v^2}{2} + \frac{P}{\rho_0} + z.g$

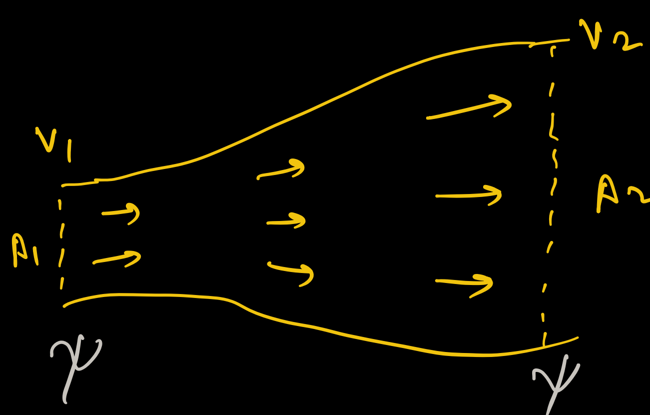
The function $\psi(\vec{x})$ is called Stream Function.

Stream Function: $\quad \psi = \frac{v^2}{2} + \frac{P}{\rho_0} + \frac{\Omega}{\rho_0}$

where $\quad \bar{f} = -\bar{\nabla}\Omega$

since $\bar{\nabla}\psi = 0 \Rightarrow \psi$ does not vary in space

i.e; ψ is constant.



$A_1 V_1 = A_2 V_2$ from continuity.

$$\psi = \frac{V_1^2}{2} + \frac{P_1}{\rho_0} + \frac{\Omega_1}{\rho_0} = \frac{V_2^2}{2} + \frac{P_2}{\rho_0} + \frac{\Omega_2}{\rho_0}$$

let $\Omega_1 = \Omega_2 \Rightarrow \frac{V_1^2}{2} + \frac{P_1}{\rho_0} = \frac{V_2^2}{2} + \frac{P_2}{\rho_0}$

$$\psi_1 = \frac{v_1^2}{2} + \frac{P}{\rho_0} = \psi_2 = \frac{v_2^2}{2} + \frac{P_2}{\rho_0}$$

$$\psi_1 = \frac{v_1^2}{2} + \frac{P_1}{\rho_0} + \frac{\Omega_1}{\rho_0} = \psi_2 = \frac{V_2^2}{2} + \frac{P_2}{\rho_0} + \frac{\Omega_2}{\rho_0}$$

Continuum Mechanics — Shoaib Akhtar

Lec 13 (9th March, 2021 ; 3:30 pm to 5:00 pm)

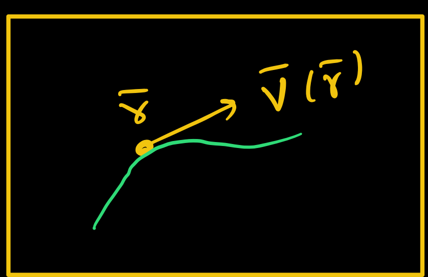
$$\frac{\bar{\nabla} \times \bar{v}}{2} = \bar{\omega} \quad \text{(vorticity)}$$

$$\frac{\partial \bar{v}}{\partial t} - 2\bar{v} \times \bar{\omega} = -\bar{\nabla} \psi$$

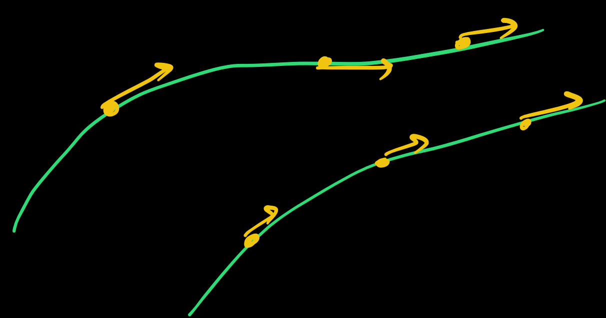
$$\psi = \text{stream function} = \frac{v^2}{2} + \frac{P}{\rho_0} + \Omega$$

At steady state $2\bar{v} \times \bar{\omega} = \bar{\nabla} \psi$

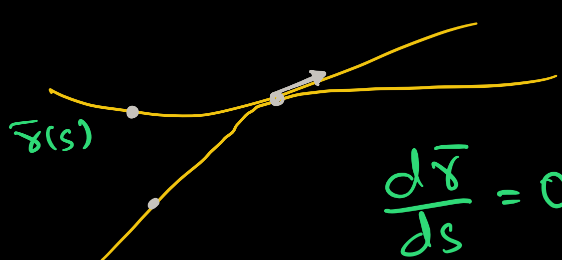
Stream Line : A curve at a given time such that its tangent at any point gives, the direction of $\bar{v}$.



at a given time



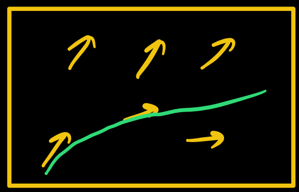
streamlines can't cross at a point otherwise we will have two velocity for a particle ; which is unphysical.

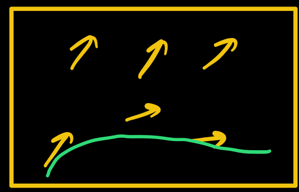


$$\frac{d\bar{r}}{ds} = \alpha \cdot \bar{v}(\bar{r})$$

$$\left(\frac{d\bar{r}_s(s)}{ds} = \alpha \, \bar{v}(\bar{r}) \right)$$

$\bar{r}_s$: s is for stream line.





Two possibility ?? which one should we take.

check that the points on a streamline have same α.
α charaterizes the stream line.

Streamlines can change with time.

$$2\bar{v} \times \bar{\omega} = \bar{\nabla}\psi \quad , \quad \psi = \frac{v^2}{2} + \frac{P}{\rho_0} + \Omega.$$

Along this stream line, how much ψ change

$$d\psi = \sum_i \frac{\partial \psi}{\partial x_i} \cdot dx_i = \bar{\nabla}\psi \cdot d\bar{r}$$

Now $d\bar{r}$ is movement along strealine.

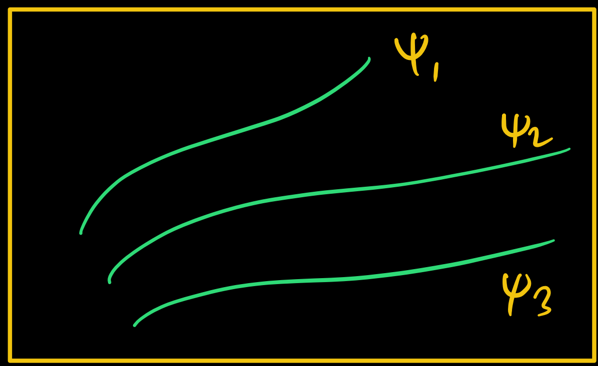
$$d\bar{r} = \alpha\, \bar{v}(\bar{r})\, ds \qquad \left(\frac{d\bar{r}}{ds} = \alpha \bar{v} \right)$$

$$\Rightarrow \quad d\psi = \bar{\nabla}\psi \cdot (\alpha \cdot \bar{v}(\bar{r}) ds)$$

$$\Rightarrow \quad d\psi = (2\bar{v} \times \bar{\omega}) \cdot (\alpha \bar{v}\, ds)$$
$$= 0$$

$$\Rightarrow \quad d\psi = 0 \text{ along streamline.}$$

ψ does not change along streamline. Stream function ψ remains constant on a Stream line.



$$\psi = \frac{v^2}{2} + \frac{P}{\rho_0} + \Omega(\vec{r})$$

Bernoulli's Theorem

$$\boxed{\frac{f}{\rho_0} = -\bar{\nabla}\Omega}$$

Force per unit mass.

ex $f = -\rho_0 g$

So; $\Omega = gz$

$\Omega = gz$

① Body Forces are conservative.

② Incompressible.

(because then $-\frac{\bar{\nabla} P}{\rho} = -\bar{\nabla}\left(\frac{P}{\rho}\right)$

only for $\rho = \rho_0$ constant

③ Ideal fluid: $\begin{cases} \eta = 0 \\ \xi = 0 \end{cases}$

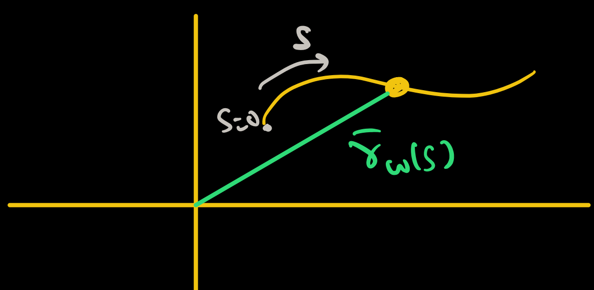
④ Steady flow $\frac{\partial v}{\partial t} = 0$

$\Rightarrow \psi$ is constant along stream lines.

Vortex Lines: tangent to this line gives direction of $\bar{\omega} = \left(\frac{\bar{\nabla} \times \bar{v}}{2}\right)$

$$\frac{d\bar{\gamma}_\omega(s)}{ds} = \beta\bar{\omega}$$

$\bar{\gamma}_\omega$ is position vector along the vortex line.



$d\psi = \bar{\nabla}\psi \cdot (\beta \bar{\omega}\, ds)$: Change of ψ along the Vortex.

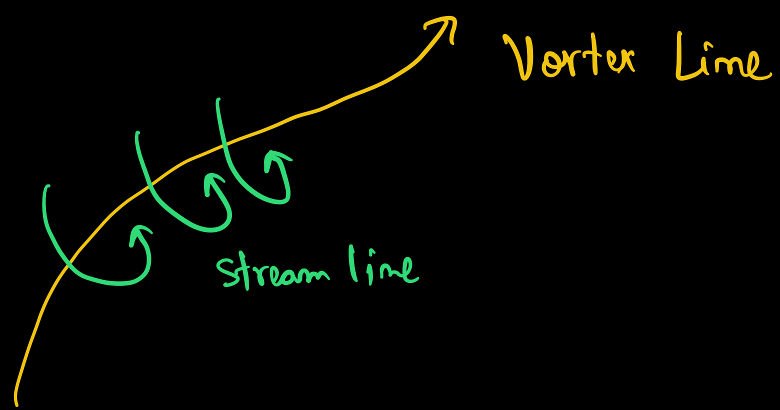
$= (2\bar{V} \times \bar{\omega}) \cdot (\beta \bar{\omega}\, ds)$

$= 0$

$d\psi = 0$ along vortex line.

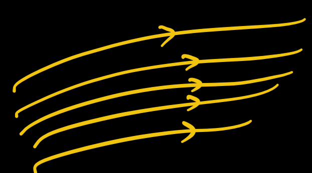
ψ does not change along vortex line.

ψ is constant along streamline & Vortex line.

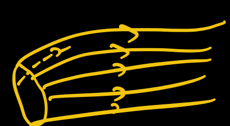


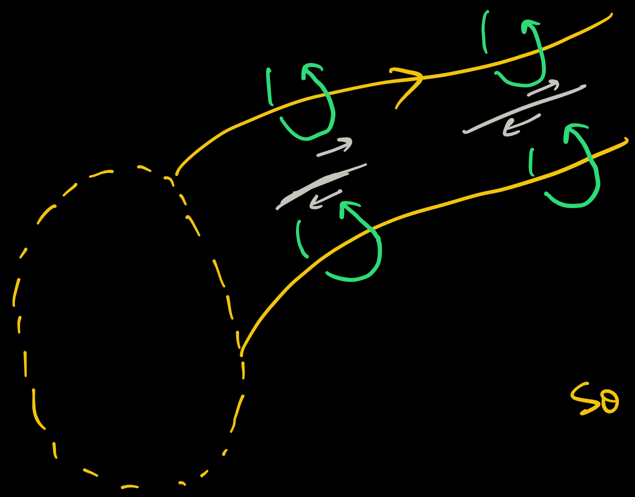
Stream lines go around the vortex line.

Vortex Sheet (when many vortex lines are parallel to each other, and form a sheet)



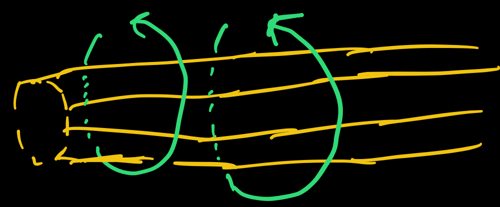
Vortex Tube (Vortex sheet curl up to form tube)





3 $\Rightarrow$ These transverse motion along the surface is zero.

so :



ie; The fluid is going round the vortex tube.

Stream Line : $\dfrac{d\bar{r}_s(s)}{ds}$ — s stands for stream.

$$\frac{d\bar{r}_s(s)}{ds} = \alpha\,\bar{V}$$

so; $$\frac{dx_s}{ds} = \alpha V_x, \quad \frac{dy_s}{ds} = \alpha V_y, \quad \frac{dz_s}{ds} = \alpha V_z$$

so; $$\boxed{\frac{dx_s}{V_s} = \frac{dy_s}{V_y} = \frac{dz_s}{V_z} = \alpha \cdot ds}$$

---

lets say $(V_x, V_y, V_z) = \bar{V}(\bar{r})$ given to us

Now draw the stream lines. (at a given time)

$$V_x = \frac{x}{1+t}, \quad V_y = \frac{y}{1+2t}, \quad V_z = 0$$

$$\frac{dx_s}{ds} = \alpha V_x = \frac{\alpha x_s}{1+t}$$

$$\frac{dy_s}{ds} = \alpha V_y = \frac{\alpha y_s}{1+2t}$$

let $\alpha = 1$ $\quad \frac{dx_s}{ds} = \frac{x_s}{1+t}, \quad \frac{dy_s}{ds} = \frac{y_s}{1+2t}$

Note that time is fixed here

$$\int_{x_0}^{x_s} \frac{dx_s}{x_s} = \int_0^s \frac{ds}{1+t}$$

$$\Rightarrow \ln\left(\frac{x_s}{x_0}\right) = \frac{s}{1+t}$$

Similarly $\ln\left(\frac{y_s}{y_0}\right) = \frac{s}{1+2t}$

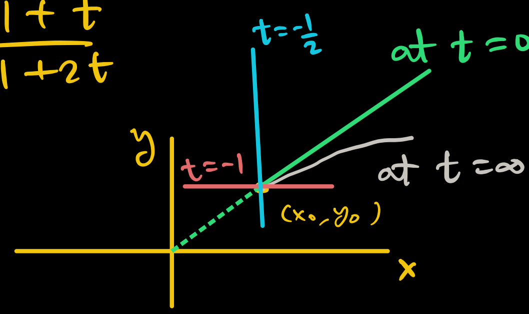
$\Rightarrow$ now eliminate s & get equation between y_s & x_s including a parameter t.

$$\Rightarrow \frac{y_s}{y_0} = \left(\frac{x_s}{x_0}\right)^n \quad \text{where } n = \frac{1+t}{1+2t}$$

alternate: $\frac{dx_s}{dy_s} = \frac{x_s}{y_s} \cdot \frac{(1+2t)}{(1+t)} \Rightarrow \int_{x_0}^{x_s} \frac{dx_s}{x_s} = \frac{1}{n} \int_{y_0}^{y_s} \frac{dy_s}{y_s}$

$$\Rightarrow \ln\left(\frac{y_s}{y_0}\right) = n \cdot \ln\left(\frac{x_s}{x_0}\right) \Rightarrow \left(\frac{y_s}{y_0}\right) = \left(\frac{x_s}{x_0}\right)^n$$

so; $$\frac{y}{y_0} = \left(\frac{x}{x_0}\right)^n \quad \text{where} \quad n = \frac{1+t}{1+2t}$$



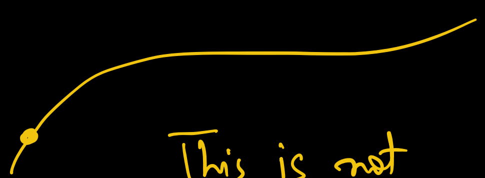
at $t=0$ $\ni$ $n=1$ $\Rightarrow$ $\dfrac{y}{y_0} = \dfrac{x}{x_0}$

at $t=\infty$, $n=\frac{1}{2}$ $\quad \dfrac{y}{y_0} = \left(\dfrac{x}{x_0}\right)^{1/2}$

at $t=-1$, $n=0$ $\quad y=y_0$

at $t=\frac{-1}{2}$, $n=\infty$ $\quad x=x_0$

Path Line : actual path travelled by an individual fluid particle.



Tracking a particular particle.

This is not streamline.

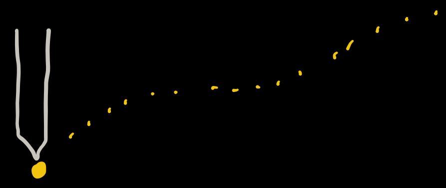
The particle moves on path line as time passes.

Streak line at a particle point (x_0, y_0)

all the particles which pass through (x_0, y_0) before t, where they are now.

Streak line is the present locus of particles which have passed through (x_0, y_0) at earlier times.

hold a color dye at a point. The fluid parcel takes away the color and you can track them.



(x_0, y_0)

for steady flow

path line $\equiv$ stream line

Chandrasekhar book (Astrophysical calculation)

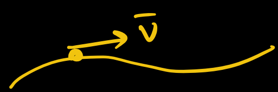
↪ some stability calculation.

Continuum Mechanics — Shoaib Akhtar

Lec 14 (12th March, 2021 ; 3:30 pm to 5:00 pm)

Second Quiz : fluid mech only ; polar & spherical polar (specifically)

① Streamline



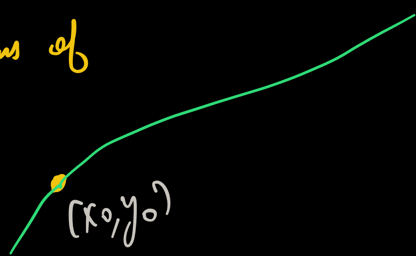
snapshot at fixed t.

② Pathline (tag a particle & follow it in time)

for a given particle as time progress

③ Streak line

gives present locus of all the particles which pass through (x_0, y_0) at $\tau < t$.



all particles passed through (x_0, y_0) before t

Example 1 Velocity given $u = \dot{x}$, $v = \dot{y}$ $\quad \bar{v} = (u, v)$

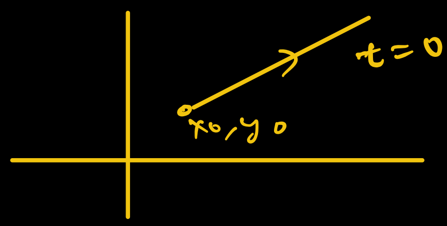
Streamline $\bar{\gamma}(s)$ (a space curve) $\quad \dfrac{dx}{ds} = u$, $\dfrac{dy}{ds} = v$

$$u = \frac{x}{1+t} \qquad v = \frac{y}{1+2t} \qquad \text{given.}$$

$$x = c_1 \cdot e^{s/1+t} \quad , \quad y = c_2 \cdot e^{s/1+2t}$$

$$\Rightarrow \quad y = y_0 \cdot \left(\frac{x}{x_0}\right)^m \qquad \text{where} \quad m = \frac{1+t}{1+2t}$$

at $t=0$,



<u>path line</u> $\frac{dx}{dt} = u$, $\dot{y} = v$ $\Rightarrow$ $\frac{dx}{dt} = \frac{x}{1+t}$

$\Rightarrow \ln\left(\frac{x}{c_1}\right) = \ln(1+t)$

$\Rightarrow \boxed{x = c_1 \cdot (1+t)}$

at $t=0 \Rightarrow x = c_1$

but $x(t=0) = x_0 \Rightarrow c_1 = x_0$

$$\Rightarrow \boxed{x = x_0 \cdot (1+t)}$$

$\frac{dy}{dt} = \frac{y}{1+2t} \Rightarrow \frac{dy}{y} = \frac{dt}{1+2t} = \frac{d\tau}{2}\left(\frac{1}{1+\tau}\right)$ $\tau = 2t$

$\Rightarrow \ln\left(\frac{y}{c_2}\right) = \frac{1}{2}\ln(1+\tau)$

$\Rightarrow \frac{y}{c_2} = (1+2t)^{1/2}$

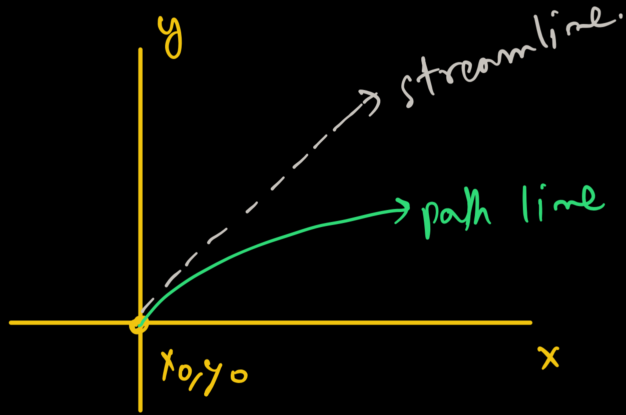
$t=0$, $y=y_0 \Rightarrow c_2 = y_0$

$$\Rightarrow \boxed{y = y_0 \cdot (1+2t)^{1/2}}$$

eliminate t & get trajectory $f(x,y)=0$

$$\Rightarrow \quad y = y_0 \cdot \left[1 + 2\left(\frac{x}{x_0} - 1\right)\right]^{1/2}$$

at $(t=0, (x,y) = (x_0, y_0))$



Streak line

$$x = c_1 (1+t), \quad y = c_2 \cdot (1+2t)^{1/2}$$

All those particles which passed through (x_0, y_0) at $\tau < t$

$$x_0 = c_1 (1+\tau), \quad y_0 = c_2 (1+2\tau)^{1/2}$$

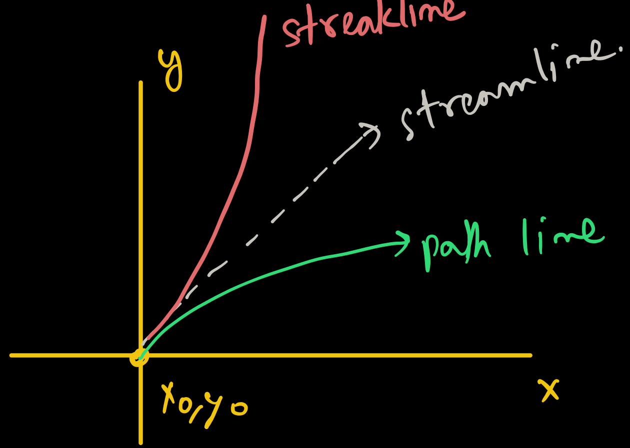
$$c_1 = \frac{x_0}{1+\tau}, \quad c_2 = \frac{y_0}{(1+2\tau)^{1/2}}$$

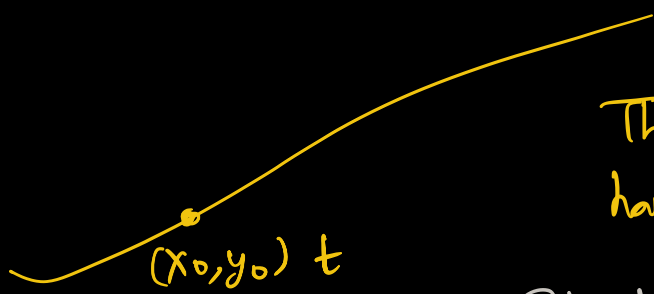
$$\boxed{x = \frac{x_0}{1+\tau} \cdot (1+t)} \Rightarrow \tau = \frac{x_0}{x} \cdot (1+t) - 1$$

$$y = \frac{y_0}{(1+2\tau)^{1/2}} (1+2t)^{1/2} \Rightarrow \boxed{y = \frac{y_0}{(1+2\tau)^{1/2}} \cdot (1+2t)^{1/2}}$$

$$\Rightarrow \boxed{\left(\frac{y_0}{y}\right)^2 = \frac{1+2t}{1 + 2(t+1)\left(\frac{x_0}{x}\right) - 1}}$$

at $t = 0$ $\quad \dfrac{y}{y_0} = \dfrac{1}{\left[1 + 2\left(\frac{x_0}{x} - 1\right)\right]^{1/2}}$ } streakline





This means, that all these particles have passed (x_0, y_0) before t.

Streak line contains information about the particle which crossed (x_0, y_0) before t.

Where as stream line tells the future of particle which crossed (x_0, y_0) at t.

Time Reversal Symmetry is broken in Navier's Stokes Equation.

$$\rho \cdot \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \bar{\nabla}) \bar{v} \right] = -\bar{\nabla} p + \eta \nabla^2 \bar{v}$$

Breaks $t \to -t$ symmetry

because $\bar{v} \to -\bar{v}$ and so $\dfrac{\partial \bar{v}}{\partial t} \to \dfrac{\partial \bar{v}}{\partial t}$

$$(\bar{v}\cdot\bar{\nabla})\bar{v} \rightarrow (\bar{v}\cdot\bar{\nabla})\bar{v}$$

but $\nabla^2\bar{v} \longrightarrow -\nabla^2 v$

so; $LHS = \rho\left[\frac{\partial\bar{v}}{\partial t} + (\bar{v}\cdot\bar{\nabla})\bar{v}\right]$ is time reversal invariant

$$\eta\nabla^2 v \longrightarrow -\eta\nabla^2 v$$

so; on RHS $\eta\nabla^2 v$ term breaks the Time Reversal Symmetry.

$$\bar{\nabla}p \rightarrow \bar{\nabla}p$$

The viscosity is not symmetric under $t \rightarrow -t$. and this causes the Navier's Stokes equation to break time reversal symmetry.

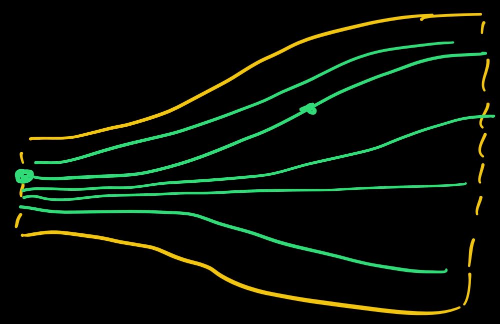
solve it.

$$\boxed{\begin{aligned} P_x &= -\frac{\rho_0 x^2}{2(1+t)^2} \\ P_y &= -\frac{\rho_0 y^2}{2(1+2t)^2} \end{aligned}}$$

$$\frac{\partial v_x}{\partial t} = \frac{-x}{(1+t)^2}$$

$$\frac{\partial v_y}{\partial t} = \frac{-y}{(1+2t)^2}\times 2$$

For time independent flows, streamline & pathline are same. & streakline



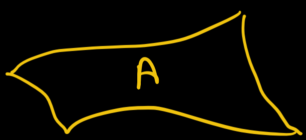
For time independent flows.

and streakline & pathline grows with time and follow the streamline.

Streakline & Pathline grows with time and will follow streamline.

Continuum Mechanics — Shoaib Akhtar

Lec 15 (16th March, 2021 ; 3:30 pm to 5:00 pm)



$E = \sigma A$

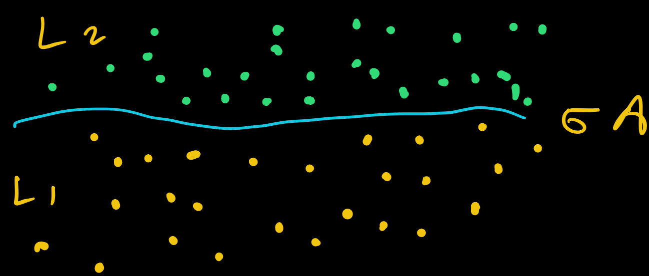
area

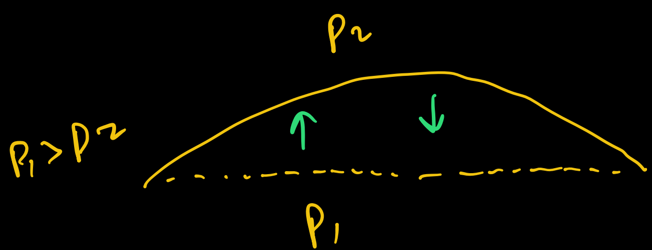
energy

surface tension

surface stretching energy $K(A - A_0)^2$

Interface



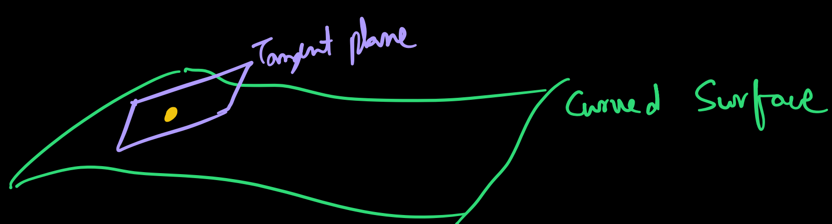


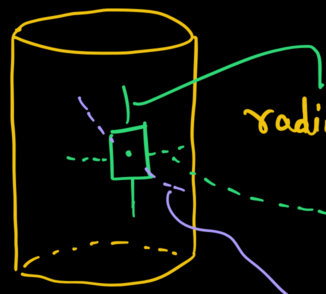
curved interface

(has more energy, due to morea area locally)

$$\Delta P = \sigma \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

Principle Curvatures





radius of curvature along the flat direction $R_1 = \infty$

but $R_2 = R$ finite.

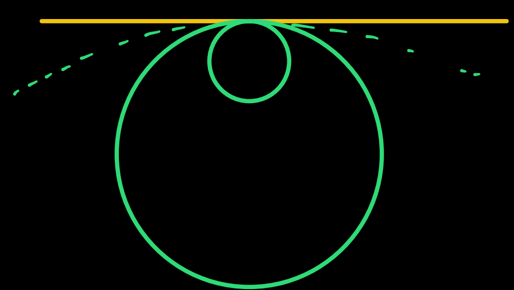
curvature between R_1 & R_2

at a point we can draw many circles.

Principle Curvature : The maximum and minimum raddii of the circles that we can draw through a point tangent to the local tangent plane.

$R_1 = \max \quad , \quad R_2 = \min$

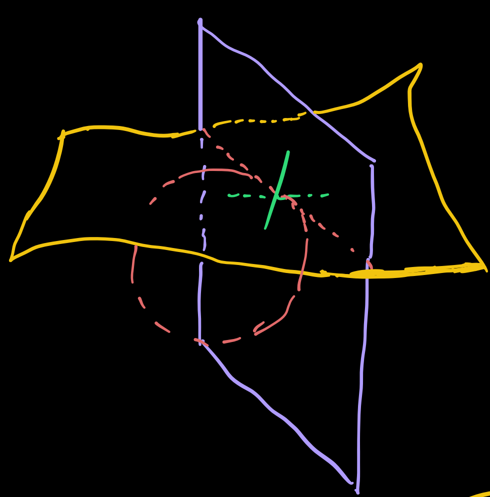
ex



straight line.

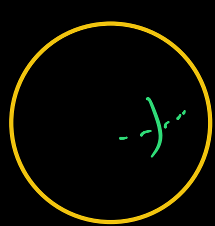
$R_1 = \infty \quad , \quad R_2 = 0$

ex



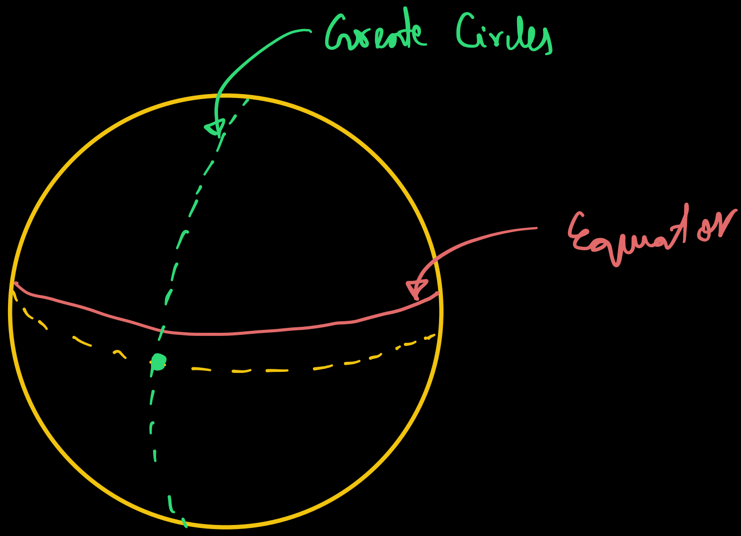
example!! sphere

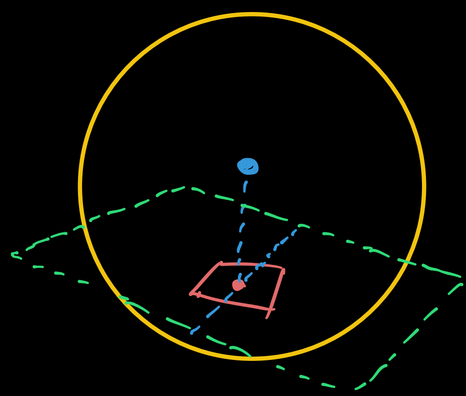
R



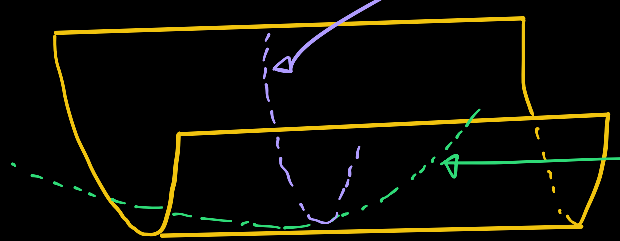
both circles have curvature $1/R$

both principle curvature are same





ex

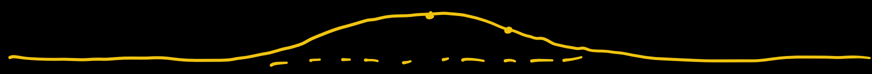


$$\Delta P = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{2\sigma}{R}$$

for spherical interface (locally the area will be a path of a sphere)

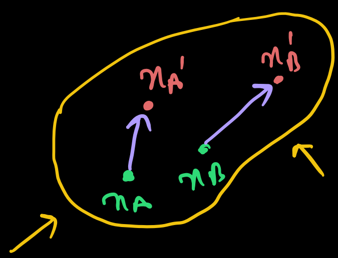
Laplace Pressure (comes at interface)

The pressure needed to prevent curvature of the surface.



ELASTICITY

when released, it comes back.



$x_A \longrightarrow x_A'$

$x_B \longrightarrow x_B'$

displacement field $\quad \bar{u} = x' - x$

analogue of x_i in "this" case

$$(x_A' - x_B') = (x_A + u_A) - (x_B + u_B)$$
$$= (x_A - x_B) + (u_A - u_B)$$

Continuum : x_A & x_B are close by.

then, $\quad x_A' - x_B' = dx'$

$$x_A - x_B = dx$$

and $\quad u_A - u_B = du$

$$\Rightarrow \quad dx' = dx + du$$

$$\boxed{dx_i' = dx_i + du_i}$$ putting indices for vector components.

$$du_i = \frac{\partial u_i}{\partial x_j} \cdot dx_j$$

$$\Rightarrow \quad dx_i' = dx_i + \frac{\partial u_i}{\partial x_j} dx_j$$

$$\boxed{dx_i' = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j} \right) dx_j}$$

Assume: $u_i(\bar{x})$ is a smooth function.

Change in distance:

How does the distance change; $dx_i'^2 = ?$

$$dx_i' = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j} \right) dx_j \qquad \frac{\partial u_i}{\partial x_j} \equiv u_{i,j} \text{ or } u_{ij}$$

Now we write the line element.

$$dx_i'^2 = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j} \right) dx_j \cdot \left(\delta_{in} + \frac{\partial u_i}{\partial x_n} \right) dx_n$$

$$= \underbrace{\left(\delta_{ij}\, dx_j\, \delta_{in}\, dx_n \right)}_{dx_i^2} + \delta_{in} \frac{\partial u_i}{\partial x_j} dx_j\, dx_n$$

$$+ \delta_{ij} \cdot \frac{\partial u_i}{\partial x_n} \cdot dx_j \cdot dx_n$$

$$+ O\left(\left(\frac{\partial u_i}{\partial x_j} \right)^2 \right)$$

if u is smooth, then its derivative is small number.

Assumption that; u is small field.

$\Leftrightarrow$ so its gradient is small.

$$dx_i'^2 = dx_i^2 + \frac{\partial v_k}{\partial x_j} dx_k \cdot dx_j + \underbrace{\frac{\partial v_j}{\partial x_k} \cdot dx_j\, dx_k}$$

$$\frac{\partial v_k}{\partial x_j} dx_k\, dx_j$$

(renaming dummy index)

$$\boxed{dx_i'^2 = dx_i^2 + 2\, dx_i\, dx_j \cdot \frac{\partial u_i}{\partial x_j}}$$

$$2 \cdot \frac{\partial u_i}{\partial x_j} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$= \underbrace{S_{ij}}_{\text{symmetric}} + \underbrace{A_{ij}}_{\text{antisymmetric.}}$$

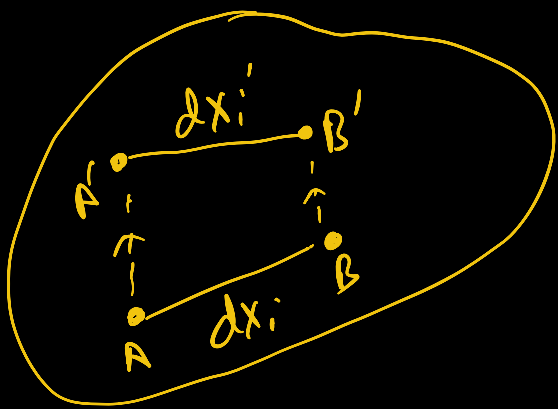
$$dx_i'^2 = dx_i^2 + dx_i dx_j \,(S_{ij} + A_{ij})$$

$$= dx_i^2 + dx_i dx_j\, S_{ij} + \cancel{dx_i dx_j A_{ij}} \rightarrow 0$$

$$\boxed{dx_i'^2 = dx_i^2 + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) dx_i\, dx_j}$$

NOTATION : $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv e_{ij}$ STRAIN TENSOR

$$\boxed{dx_i'^2 = dx_i^2 + 2\,dx_i\,dx_j \cdot e_{ij}}$$



$$dx_i'^2 = dx_i^2 + 2e_{ij}\,dx_i\,dx_j$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + i \leftrightarrow j\right)$$

strain : $\frac{dl}{l}$

Go to a frame, in which e_{ij} is diagonal.

(so, will need to choose different frames at different point to make it diagonal)

here sum over i.

$$d\tilde{x}_i'^2 = d\tilde{x}_i^2 + 2 \cdot \tilde{e}_{ii}\,d\tilde{x}_i^2$$

i^{th} element ; no sum over i

$$e_{ij} \longrightarrow \delta_{ij}\,\tilde{e}_{ij} \quad \text{(in diagonal frame)}$$

(no sum over i)

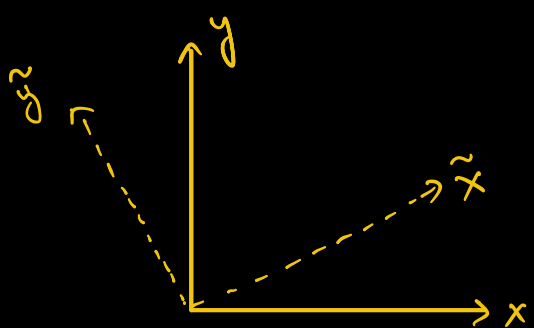
$$\tilde{e} = \begin{bmatrix} \tilde{e}_{11} & 0 & 0 \\ 0 & \tilde{e}_{22} & 0 \\ 0 & 0 & \tilde{e}_{33} \end{bmatrix}$$

$$d\tilde{x}_i'^2 = d\tilde{x}_i^2 \left(1 + 2\tilde{e}_{ii}\right)$$ → no sum over i.

Because e_{ij} was real symmetric,

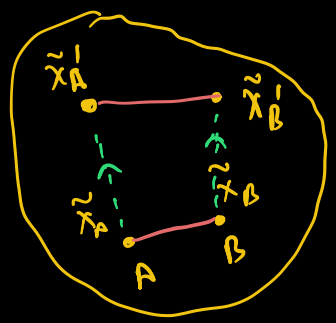
it can always be diagonalized by a rotation; and also all the eigenvalues are real & eigenvectors are orthogonal.

$d\tilde{x}_i$ is along that orthogonal eigenvectors.



$\Downarrow$

$$\boxed{\begin{aligned} d\tilde{x}_i' &= d\tilde{x}_i \cdot \sqrt{1 + 2\tilde{e}_{ii}} \\ &\simeq d\tilde{x}_i \cdot (1 + \tilde{e}_{ii}) \end{aligned}}$$



Continuum Mechanics — Shoaib Akhtar

Lec 16 (19th March, 2021 ; 3:30 pm to 5:00 pm)

$$d\tilde{x}_i' = d\tilde{x}_i \cdot (1 + \tilde{e}_{ii})$$

change in volume $dx\,dy\,dz \longrightarrow dx'\,dy'\,dz'$

omitting the tilde.

$$dx'\,dy'\,dz' = dx(1+\tilde{e}_{11})\cdot dy(1+\tilde{e}_{22})\,dz(1+\tilde{e}_{33})$$

$1, 2, 3 \equiv x, y, z$

$$e_{ij} \longrightarrow \tilde{e}_{ii}$$

$$\downarrow$$

$$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + i \leftrightarrow j\right)$$

u_i are small, $\frac{\partial u_i}{\partial x_j}$ even smaller

$$\Rightarrow dx'\,dy'\,dz' \approx dx\,dy\,dz\,(1 + e_{11} + e_{22} + e_{33}) + \theta(e^2)$$

$$dV' = dV\cdot(1 + \mathrm{Tr}(\tilde{e}))$$

$\mathrm{Tr}(e)$ does not change under rotation of frame.

$$\mathrm{Tr}[e_{ij}] = \mathrm{Tr}[\tilde{e}_{ij}]$$

so,

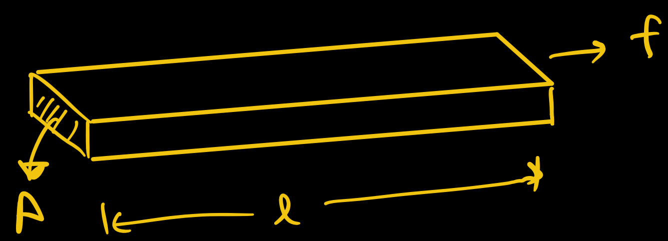
$$\boxed{dV' = dV\cdot(1 + \mathrm{Tr}(e))}$$

Tr(e) is fractional volume change

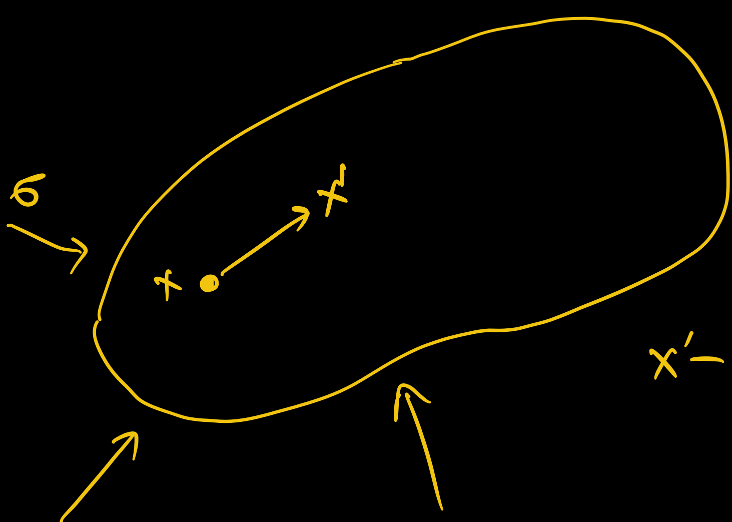
$$Tr(e) = \frac{dV' - dV}{dV}$$

Connecting Stress to Strain
(Constitutive Relation)

$$\sigma_{ij} = f(e_{ij})$$



$$\sigma = \frac{f}{A} = Y \cdot \frac{\Delta l}{l}$$



$x' - x = u(x)$ displacement

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + i \leftrightarrow j\right)$$

Strain is effect.
Stress is cause.

(assume linear response)

Stress $\longleftrightarrow$ Strain ?

$\sigma_{ij}(\bar{x})$ $e_{ij}(\bar{x})$

e_{ij} is symmetrized form of $\frac{\partial u_i}{\partial x_j}$

$$\sigma_{ij}(\bar{x}) = 2\mu\, e_{ij}(\bar{x}) + \lambda \cdot \underline{\delta_{ij}}\, e_{ll}$$

Isotropic

Assuming linearity in simple way.

the $2\mu \cdot e_{ij}(\bar{x})$ term

(we could have $A_{ijkl}\, e_{kl}$)

μ and λ are Lame Constants.

① σ_{ij} is a linear function of e_{ij} (because e_{ij} is small)

② σ_{ij} should not have constant on RHS.

if $\sigma_{ij} = f_{ij}(e) + C$

because if $\sigma_{ij} = 0 \Rightarrow f_{ij}(e) = -C$

i.e; $e_{ij} \neq 0$

③ can't have $u_i\, u_j$ on the RHS, since u_i = constant

i.e; uniform translation

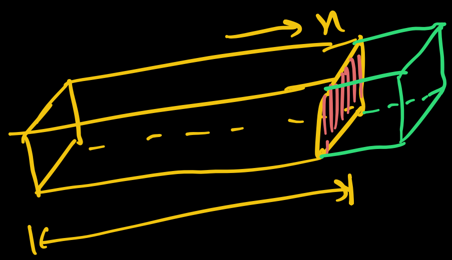
we will have stress.

so we have to always use their gradient.

although ∇u_i is small,

but $u_i u_j$ can be large

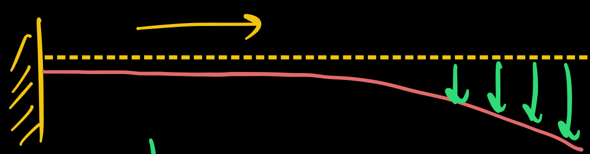
ex)



$u_x = x' - x$

u_L is large, but $\dfrac{\partial u}{\partial x}$ is still small.

ex)



θ_L

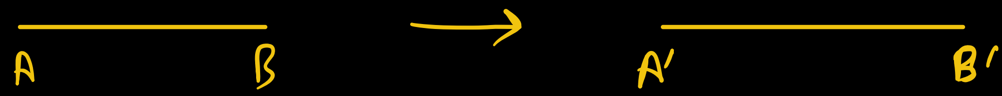
bending due to gravity.

$\dfrac{d\theta}{dx} << 1$, but θ_L may not be small.

σ_{ij} must be symmetric for angular momentum conservation (when there is no external torque)

This is why we dont just add $\dfrac{\partial u_i}{\partial x_j}$ in RHS of σ_{ij}; but $\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)$; so we indeed use e_{ij}

ex// For a Bar



$$x_{A}' - x_{B}' = (x_A - x_B) + (u_A - u_B)$$

ie; $dx' = dx + du$

ie; $l' = l + du$

$$du = \Delta l$$

$$\Rightarrow \quad \frac{\Delta u}{l} = \frac{du}{l} = \frac{du}{dx}$$ Fractional Change of length.

$$\boxed{\frac{F}{A} = \sigma = Y \cdot \frac{\Delta l}{l}}$$ found before.

$$\sigma_{ij} = Y \cdot \left(\frac{\partial u_i}{\partial x_j} + i \leftrightarrow j \right)$$

$$\sigma_{ij} = 2\mu \cdot e_{ij} + \lambda \cdot \delta_{ij} \cdot e_{ll}$$

$$Tr(e) = e_{ll} = \frac{\partial u_l}{\partial x_l}$$

$$\sigma_{ij} = 2\mu \, e_{ij} + \lambda \cdot \delta_{ij} \cdot \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right)$$

$$dV' = dV \cdot (1 + e_{ll})$$

$$\Delta \equiv e_{ll} = \frac{dV' - dV}{dV}$$

$$\underbrace{e_{ij}}_{\text{has Tr()} = \Delta} = \underbrace{\left(e_{ij} - \frac{1}{3}\cdot\Delta\cdot\delta_{ij}\right)}_{\text{has Tr( ) = 0}} + \underbrace{\frac{1}{3}\cdot\Delta\cdot\delta_{ij}}_{\text{has Tr( ) = }\Delta}$$

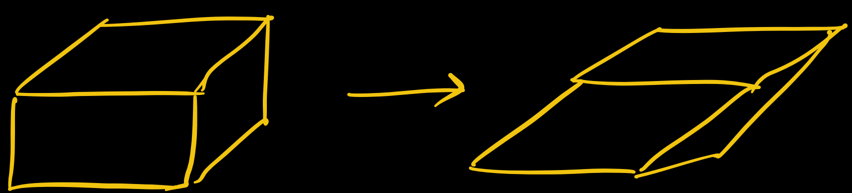
$$\boxed{Tr(e_{ij}) = \Delta}$$

Traceless part of Strain Tensor.

This does not lead to volume change.;

but it changes the shape of the object.

ex 3D



Shape change; but not volume change due to traceless part of strain.

so, $\frac{1}{3}\Delta\cdot\delta_{ij}$ leads to volume change.

$$\sigma_{ij} = 2\tilde{\mu}\cdot\left(e_{ij} - \frac{\Delta}{3}\cdot\delta_{ij}\right) + K\cdot\delta_{ij}\cdot\Delta$$

Shear Modulus

Bulk Modulus.

2D 

Show: $\tilde{\mu} = \mu$, but $\lambda = -\frac{2}{3}\mu + K$

$\Delta \cdot K \cdot \delta_{ij}$ leads to no shape change, but only volume change.

Imagine pure strain of type $e_{ij} = \frac{1}{3} \cdot \delta_{ij} \cdot \Delta$

$\Rightarrow$ $e_{ii} = \frac{\Delta}{3}$ (no sum over i)

& $e_{ij} = 0$ for $i \neq j$

so; $\frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial y} = \frac{\partial u_z}{\partial z} = \frac{\Delta}{3} = c$

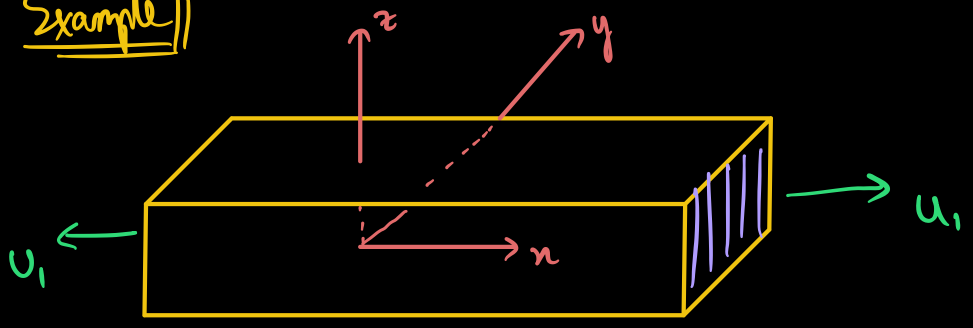
$$\Rightarrow \left.\begin{aligned} u_x &= c \cdot x \\ u_y &= c \cdot y \\ u_z &= c \cdot z \end{aligned}\right\} \Rightarrow$$
so, we have just scale transformation in all direction, by the scale factor c.

any possible constants like

$U_x = c \cdot x + c_1$

c_1 in this can be gotten rid of (This just means all points shift in x direction (so no shape change for sure); just pure translation) by shifting x-origin.

Example ||



put force in x direction.

displacement field.

$$u_1 = a x_1$$
$$u_2 = b x_2$$
$$u_3 = b x_3$$

y & z directions are similar. (Assume symmetry in y-z)

Strain Field: $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$$[e_{ij}] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}$$

$$\sigma_{ij} = 2\mu \cdot \left(e_{ij} - \delta_{ij} \cdot \frac{\Delta}{3}\right) + k \cdot \Delta \cdot \delta_{ij}$$

$$\Delta = a + b + b = a + 2b$$

$$[\sigma_{ij}] = 2\mu \cdot \begin{pmatrix} a - \frac{(a+2b)}{3} & 0 & 0 \\ 0 & b - \frac{(a+2b)}{3} & 0 \\ 0 & 0 & b - \frac{(a+2b)}{3} \end{pmatrix}$$

Diagonal

$$+ k \cdot \begin{pmatrix} a+2b & 0 & 0 \\ 0 & a+2b & 0 \\ 0 & 0 & a+2b \end{pmatrix}$$

$$\sigma_{11} = K\cdot(a+2b) + \frac{4\mu}{3}\cdot(a-b)$$

$$\sigma_{22} = K\cdot(a+2b) - \frac{2\mu}{3}\cdot(a-b) = \sigma_{33}$$

No stress on y and z surfaces. $\Rightarrow$ $\sigma_{22} = 0$, $\sigma_{33} = 0$
($\hat{y}$) ($\hat{z}$)

since σ_{22} or σ_{33} does not depend on x
and we satisfy boundary condition
$\Rightarrow$ $\sigma_{22} = 0$ & $\sigma_{33} = 0$

$$\Rightarrow K = \frac{2\mu}{3}\cdot\frac{(a-b)}{(a+2b)}$$

minus; because we want to keep σ positive for normal material.

$$\Rightarrow \frac{b}{a} = \frac{1}{2}\cdot\left(\frac{2\mu - 3K}{\mu + 3K}\right) = -\sigma$$

σ is called Poisson Ratio.

if $3K > 2\mu$ then $b < 0 \Rightarrow$ shrinks along y and z direction.
(if expand along x)

If volume has to be conserved, ie; incompressible
we need $K = \infty$ (K is bulk Modulus)

for large K, the cost of expansion or contraction is large.

$\frac{\mu}{K} \to 0$ $\Rightarrow$ $\frac{b}{a} = -\frac{1}{2}$

(K is small for compressible)

$\Rightarrow$ $\sigma = 1/2$

Usually; when material expand in one direction, it gets shrinked in other.

If shear modulus $= \mu \to \infty \Rightarrow \dfrac{b}{a} = 1 \Rightarrow \sigma = -1$

so range for $\sigma \in [-1, 1/2]$

Regular materials have $\sigma > 0 \Rightarrow$ shrinks along y, z if pulled along x.

Can calculate σ_{11}

$$\sigma_{11} = K(a + 2b) + \frac{4\mu}{3} \cdot (a - b)$$

using relation $\dfrac{b}{a} = \dfrac{1}{2} \cdot \dfrac{(2\mu - 3K)}{(\mu + 3K)}$

$$\Rightarrow \boxed{\sigma_{11} = a \cdot \left(\frac{9\mu \cdot K}{3K + \mu} \right)}$$

stress along x direction

strain

$$\Rightarrow \boxed{\frac{9\mu K}{3K + \mu} \equiv Y}$$ Youngs Modulus

for simple linear elasticity.

Volume Change

$$\Delta = \mathrm{Tr}(e_{ij}) = a + 2b \quad = \begin{cases} 0 & \text{only for } K \to \infty \\ a+2b \neq 0 & \text{in general.} \end{cases}$$

$$a + 2b = a \cdot \left(1 + \frac{2\mu - 3K}{\mu + 3K}\right)$$

$$= a \cdot \left(\frac{\mu + 3K + 2\mu - 3K}{\mu + 3K}\right)$$

$$= a \cdot \left(\frac{3\mu}{\mu + 3K}\right)$$

$$\boxed{a + 2b = a \cdot \frac{3\mu}{\mu + 3K}}$$

$$= \frac{\sigma_{11}}{3K} \quad \text{(volume change)}$$

When incompressible, we have $K \to \infty$.

In fluids

$$\rho\left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \bar{\nabla})\bar{v}\right] = -\bar{\nabla} p + \eta \nabla^2 \bar{v} + \xi \bar{\nabla}(\bar{\nabla} \cdot \bar{v}) + f_{ext}$$

shear viscosity

Bulk Viscosity (resist compression or expansion)

(for incompressible fluid, $\xi \to \infty$)

↳ so system chooses to have $\bar{\nabla} \cdot \bar{v}$, otherwise equation becomes singular)

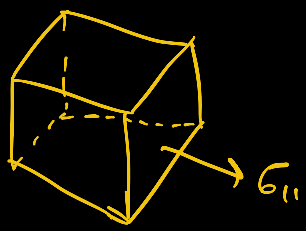
for incompressible, this resistance is ∞.

so: $\xi = \infty$.

Continuum Mechanics — Shoaib Akhtar

Lec 17 (30th March, 2021 ; 3:30 pm to 5:00 pm)

Shear Stress



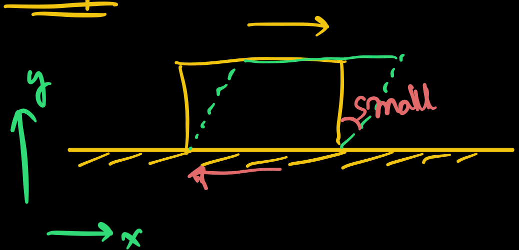
σ_{11} , σ_{22}, $\sigma_{33} \neq 0$

$\sigma_{i \neq j} = 0$ (no shear)

$$a + 2b = \frac{\sigma_{11}}{3K} = \text{fractional volume change} = \Delta = \frac{\delta V}{V}$$

$$a + 2b = \frac{\sigma_{11}}{3K} \xrightarrow[k \to \infty]{} 0$$

example



$u_x = \gamma . y$

$u_y = 0$

assume no volume (area) change,

i.e; $\sigma = \frac{1}{2}$

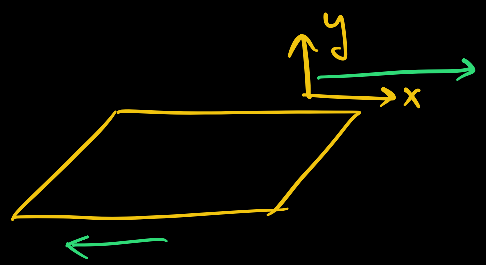
$$e_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \cdot \frac{1}{2}$$

$$[e_{ij}] = \begin{bmatrix} 0 & \gamma/2 \\ \gamma/2 & 0 \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix}$$ Strain Matrix

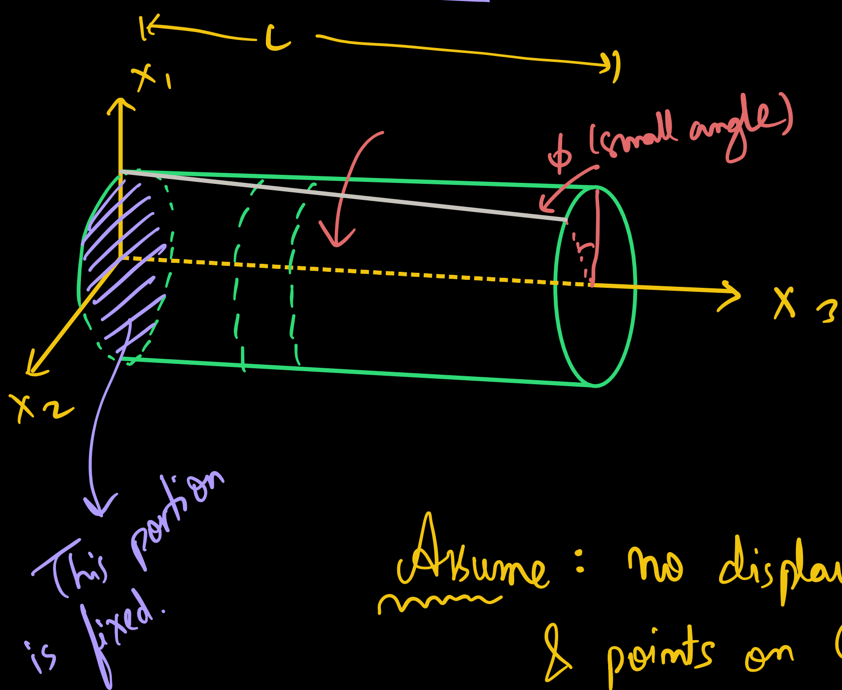
$$\sigma_{ij} = 2\mu \cdot \left(e_{ij} - \frac{\delta_{ij}}{3} \cdot \Delta \right) + K \cdot \Delta \cdot \delta_{ij}$$ here $\Delta = 0$

$$\Rightarrow \quad \sigma_{ij} = 2\mu \cdot e_{ij}$$

$$\Rightarrow \quad [\sigma] = \begin{bmatrix} 0 & \mu \cdot \gamma \\ \mu \cdot \gamma & 0 \end{bmatrix}$$



Torsion in a Beam

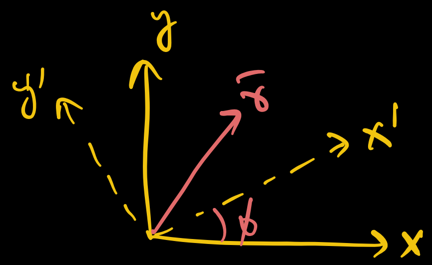


There is shear force between every layer.

Assume: no displacement along $\hat{x}_3$.
& points on circular cross section rotates by small angle ϕ.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \xrightarrow{\text{after rotating}} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

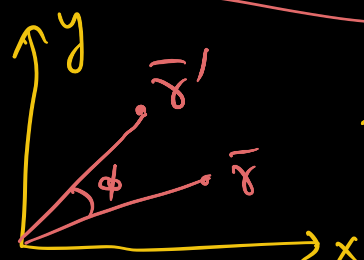
we are moving vector;
so active rotation.
(not rotating the frame)



<u>Passive Rotation</u> (because the frame rotated; not the vector)

$$\vec{r} \longrightarrow \vec{r}\,' = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \vec{r}$$ → equation among components.

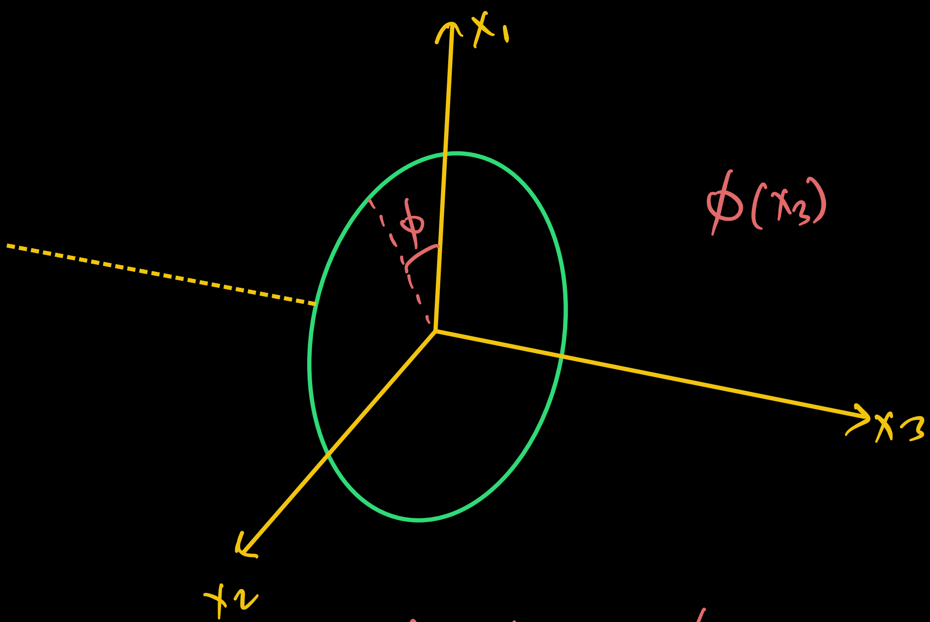
$\vec{r}\,' = \vec{r}$ → equal as vector.



<u>Active Rotation</u> (the vector rotates)

$\vec{r}\,' \neq \vec{r}$ not equal as vector.

$$\vec{r}\,' = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \vec{r}$$



$\phi(x_3)$

Assumption $\phi \propto x_3$

ie; $\phi = \alpha \cdot x_3$

↖ some elastic modulus.
(not Bulk Modulus K)

$$\sigma_{ij} = K \cdot \Delta \cdot \delta_{ij} + \left(\sigma_{ij} - \frac{\Delta \cdot \delta_{ij}}{3}\right) 2\mu$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\alpha x_3 & -\sin\alpha x_3 & 0 \\ \sin\alpha x_3 & \cos\alpha x_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} x_1 \cdot \cos\alpha x_3 - x_2 \cdot \sin\alpha x_3 \\ x_1 \cdot \sin\alpha x_3 + x_2 \cdot \cos\alpha x_3 \\ x_3 \end{pmatrix}$$

$$u_1 = x_1' - x_1 = x_1 \cdot (\cos\alpha x_3 - 1) - x_2 \cdot \sin\alpha x_3$$

$$u_2 = x_2' - x_2 = x_2 \cdot (\cos\alpha x_3 - 1) + x_1 \cdot \sin\alpha x_3$$

$$u_3 = x_3' - x_3 = 0$$

Assume α is small.

$$\cos\alpha x_3 - 1 = -\frac{(\alpha x_3)^2}{2}$$

So; $\boxed{\begin{aligned} u_1 &\cong -x_2 \cdot \alpha x_3 \\ u_2 &\cong x_1 \cdot \alpha x_3 \\ u_3 &= 0 \end{aligned}}$ $\Rightarrow$ $\boxed{\begin{aligned} u_1 &\cong -x_2 x_3 \cdot \alpha \\ u_2 &\cong x_1 x_3 \cdot \alpha \\ u_3 &\cong 0 \end{aligned}}$

$$[\alpha] = L^{-1}$$

α small means $\alpha << L^{-1}$

L is length of beam.

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial x_j}{\partial x_i}\right)$$

$$\frac{\partial u_i}{\partial x_j} = \begin{pmatrix} 0 & -\alpha x_3 & -\alpha x_2 \\ \alpha x_3 & 0 & \alpha x_1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[e_{ij}] = \begin{pmatrix} 0 & 0 & -\frac{\alpha x_2}{2} \\ 0 & 0 & \alpha x_1/2 \\ -\frac{\alpha x_2}{2} & \frac{\alpha x_1}{2} & 0 \end{pmatrix}$$

$\Delta = tr(e_{ij}) = 0 \Rightarrow$ no volume change.

$$\sigma_{ij} = 2\mu\left(e_{ij} - \cancelto{0}{\frac{\Delta \cdot \delta_{ij}}{3}}\right) + \cancelto{0}{K \cdot \Delta \frac{\delta_{ij}}{3}}$$

$$\sigma_{ij} = 2\mu e_{ij}$$

$$[\sigma] = \begin{pmatrix} 0 & 0 & -\mu\alpha x_2 \\ 0 & 0 & \mu\alpha x_1 \\ -\mu\alpha x_2 & \mu\alpha x_1 & 0 \end{pmatrix}$$

force density: $f_i = \frac{\text{force}}{\text{volume}} = \frac{\partial}{\partial x_j}\sigma_{ij}$

$$f_1 = \frac{\partial}{\partial x_1}\sigma_{11} + \frac{\partial}{\partial x_2}\sigma_{12} + \frac{\partial}{\partial x_3}\sigma_{13}$$

$$= 0 + 0 + 0$$

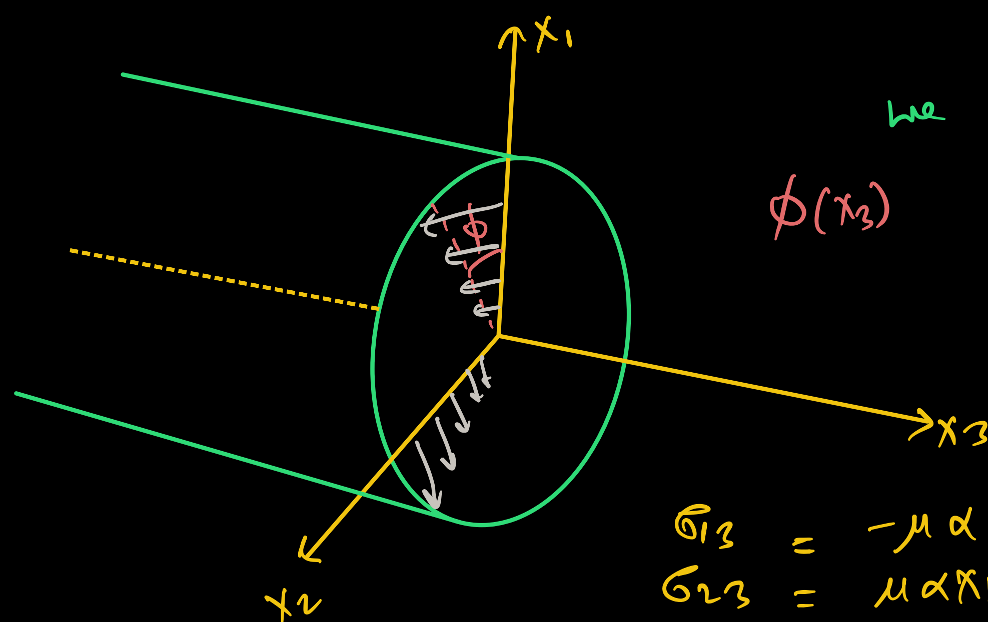
$$f_2 = 0$$

$$f_3 = 0$$

$$\bar{f} = 0$$

force per unit volume is zero everywhere (and it is consistent; because the body is at equilibrium)

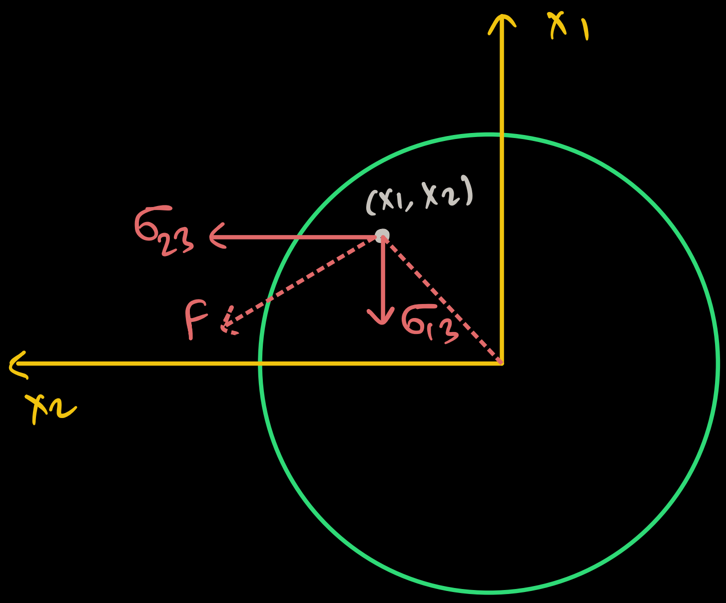
→ we applied torque & had hold it. given point are at rest; & internal stresses have developed.



we want to find forces on surface.

$\phi(x_3)$

$$\sigma_{13} = -\mu\alpha x_2$$
$$\sigma_{23} = \mu\alpha x_1$$
$$\sigma_{33} = 0$$



Force / Area

$$\bar{F} = \mu\alpha(-x_2\,\hat{i} + x_1\,\hat{j})$$

$$\bar{r} = (x_1\,\hat{i} + x_2\,\hat{j})$$

$$\ni \quad \bar{F} \perp \bar{r}$$

Torque per unit area $\bar{r} \times \bar{F}\, dA$

$$= \sqrt{x_1^2 + x_2^2} \times \mu \cdot \alpha \sqrt{x_2^2 + x_1^2} \cdot dA \; \hat{z}$$

$$= \mu\alpha\,(x_1^2 + x_2^2)\, dA\; \hat{z}$$

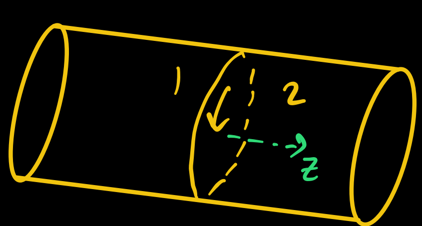
$$= \mu\alpha\, r^2 \cdot dA \cdot \hat{z}$$

$\mu\alpha r^2 \hat{z}$ = Torque per unit area.

Torque on the cross section $= \int_0^{2\pi}\int_0^R \mu\cdot\alpha\cdot r^2 \cdot r \cdot dr\, d\theta$

$$= \mu\cdot\alpha \cdot 2\pi \int_0^R r^3\, dr$$

$$= \frac{\mu\alpha\pi R^4}{2}$$



Torque applied by section 2 on section 1