ADVANCED GENERAL RELATIVITY

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These notes are consequence of my self study; and are meant to complement the book **A Relativist's Toolkit** by *Eric Poisson* and are self made. Most of the material is inspired from his lecture which was given at Perimeter Institute. This remarkable notes were made during the Corona Pandemic.

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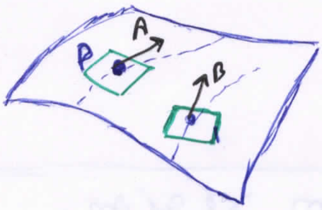
22/4/2020

A journey into the Black Holes!!
and beyond ....

Ch 1: Fundamentals

4D spacetime : $(\mathcal{M}, g)$ — $\mathcal{M}$ manifold, g metric

- coordinate system : x^α, $\alpha = 0, 1, 2, 3$.
- Vectors A^α (tangent to a curve at a point on $\mathcal{M}$)
- we cant integrate vectors on curved manifold because vectors at different points cant be compared.
- We can integrate scalars on manifold.



- Dual vector (on-form) P_α.
  (operations on vectors which gives number)
  A linear operation on vectors that yields numbers.

$$P_\alpha A^\alpha = \langle P, A \rangle \equiv \text{Inner Product}.$$

- Tensors $T^{\alpha\beta}$, $V^\alpha{}_\beta$, $W_{\alpha\beta}$.

- Metric tensor $g_{\alpha\beta}$. $ds^2 = g_{\alpha\beta} \, dx^\alpha dx^\beta$ = spacetime interval
  - convert coordinate differential into physical distances.
  - It also describes Gravity.

- Inverse metric $g^{\alpha\beta}$; $g^{\alpha\beta} \cdot g_{\beta\gamma} = \delta^\alpha{}_\gamma$

raise indices : $P^\alpha = g^{\alpha\beta} P_\beta$

lower indices ; $A_\alpha = g_{\alpha\beta} A^\beta$

Introduce Connection $\Gamma^\alpha_{\beta\gamma}$ (tensor)

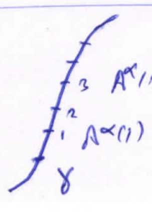
What we need in G.R ; — symmetric in lower indices.
— metric compatible.

$$\Rightarrow \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} \left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu} \right)$$

Christoeffl Symbol.

Covariant Differentiation.

Vector fields on a curve γ



Differentiate vector field A^α along the curve:

Primary description of γ: $\gamma^\alpha(\lambda)$; vector $A^\alpha(\lambda)$

Covariant derivative of A^α along curve

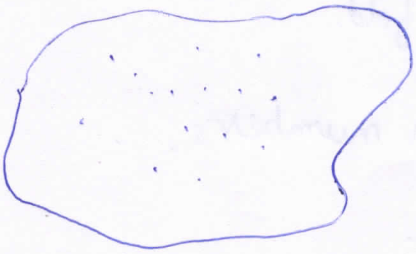
$$\frac{DA^\alpha}{d\lambda} \equiv \frac{\partial A^\alpha}{\partial\lambda} + \Gamma^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial\lambda}\frac{\partial x^\gamma}{\partial\lambda}$$

(notion of covariant derivative along a curve)

Tangent vector to γ: $t^\alpha \equiv \frac{\partial x^\alpha}{\partial\lambda}$

so, $$\frac{DA^\alpha}{d\lambda} = \frac{\partial A^\alpha}{\partial\lambda} + \Gamma^\alpha_{\beta\gamma}A^\beta t^\gamma .$$

Given a vector field $A^\alpha(x^\mu)$ everywhere in open region,



(Here, we can take derivative along any direction)
i.e., w.r.t. any coordinate direction.

$$A^\alpha_{;\beta} \equiv \nabla_\beta A^\alpha = \frac{\partial A^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma}A^\gamma$$
$$= A^\alpha_{,\beta} + \Gamma^\alpha_{\beta\gamma}A^\gamma .$$

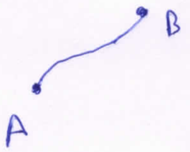
$$\frac{DA^\alpha}{d\lambda} = \frac{\partial A^\alpha}{\partial x^\beta}\frac{dx^\beta}{d\lambda} + \Gamma^\alpha_{\beta\gamma}\cdot\overset{A^\beta}{\cancel{\frac{\partial x^\beta}{\partial\lambda}}}\cdot\frac{\partial x^\gamma}{d\lambda}$$

$$\Rightarrow \frac{DA^\alpha}{d\lambda} = \frac{\partial A^\alpha}{\partial x^\beta}\cdot\frac{dx^\beta}{d\lambda} + \Gamma^\alpha_{\beta\gamma}A^\gamma\cdot\frac{dx^\beta}{d\lambda} = A^\alpha_{;\beta}\,t^\beta$$

Geodesics

$sig(g_{\alpha\beta}) = diag(-1, 1, 1, 1)$ (signature).

Timelike : curve in S.T. (spacetime) that extremizes proper time between two events.



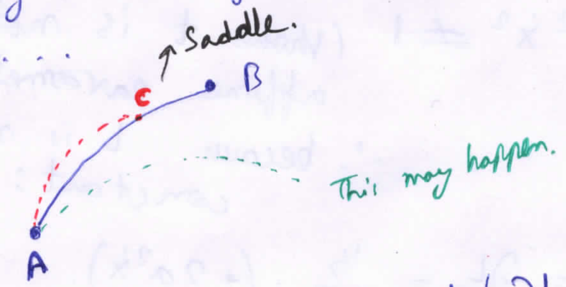
$$d\tau^2 = -ds^2 = -g_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$\Rightarrow \tau(A \to B) = \int_A^B \sqrt{-g_{\alpha\beta}\, dx^\alpha dx^\beta}$$

$$= \int_A^B \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \cdot \frac{dx^\beta}{d\lambda}} \cdot d\lambda$$

Here I am actually maximising . . .

. . . . . .



C ⇒ is called Conjugate point.

$\tau = \int_A^B L\, d\lambda$; extremum : $\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$; $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$

geodesic equation : $\left.\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = K \cdot \frac{dx^\alpha}{d\lambda}\right\}$ — This is more general! when we choose any general parameter λ.

$K = \frac{1}{L} \cdot \frac{dL}{d\lambda}$; $L = \sqrt{-g_{\alpha\beta} \cdot \frac{dx^\alpha}{d\lambda} \cdot \frac{dx^\beta}{d\lambda}}$

∴ when λ is proper time,

ie; $d\lambda = d\tau$; then $L = \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}} = 1$

so; $K = 0$ (when $d\lambda = d\tau$)

$K = 0$ is preserved by all transformation. $\lambda = a\tau + b$ (called affine connection)

called Affine parameter.

for all affine parameters; Geodesic equation is written with zero R.H.S.

ie;

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0.$$

non-affine parameters are useful for null geodesic.

Example of non-affine parametrization

F.R.W. : $ds^2 = -dt^2 + a^2(t)\cdot(dx^2+dy^2+dz^2)$

$$\Gamma^x_{tx} = \Gamma^y_{ty} = \Gamma^z_{tz} = \left(\frac{\dot a}{a}\right)$$

$$\Gamma^t_{xx} = \Gamma^t_{yy} = \Gamma^t_{zz} = a\dot a \qquad \dot{} = \frac{d}{dt}$$

Timelike geodesics moving along x-direction;

pick parameter $= t$ ~~proper time~~.

$x^\alpha(t) = [t, x(t), 0, 0]$; t is not proper time on the geodesic.

$$L = \sqrt{-g_{\alpha\beta}\cdot\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}} = \sqrt{1 - a^2\dot x^2} \neq 1$$

(shows t is not affine parameter) ∵ because L is not constant:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{\partial L}{\partial \dot x} = \text{constant}. \qquad ; p \equiv \frac{\partial L}{\partial \dot x} = \frac{1/2}{L}\cdot(+2a^2\dot x)$$

$$\boxed{p = \frac{a^2\dot x}{\sqrt{1-a^2\dot x^2}}} \qquad ; \text{solve for } \dot x \Rightarrow \dot x = \frac{p}{a\sqrt{p^2+a^2}}$$

} assuming particle is moving along positive x direction.

To calculate τ,

$$\frac{d\tau}{dt} = L = \sqrt{1-a^2\dot x^2} = \frac{a}{\sqrt{p^2+a^2}} \qquad \text{(easier to integrate)}$$

If we use τ as parameter;

$$\frac{dt}{d\tau} = \frac{\sqrt{p^2+a^2(t)}}{a(t)} \; ; \; \frac{dx}{d\tau} = \frac{p}{a^2(t)}$$

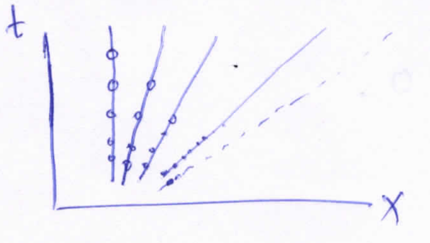
(little difficult to integrate, because a is known as function of t, not τ)

Lets take limit when particle is moving close to light.

→ ie: we want to go to null case.

so; The Null Limit $p \to \infty \Longrightarrow L = 0$.

$$\frac{dx}{dt} = \frac{1}{a}$$



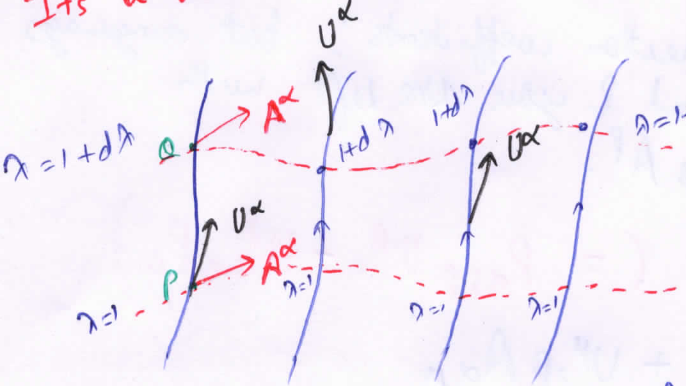
Lie Derivatives

vector : $\mathcal{L}_U A^\alpha = A^\alpha{}_{,\beta} U^\beta - U^\alpha{}_{,\beta} A^\beta$

$= -\mathcal{L}_A U^\alpha$

Its a vector

This notion does not require connections.



$$\mathcal{L}_U A^\alpha (P) \equiv \frac{A^\alpha(Q) - A'^\alpha(Q)}{d\lambda}$$ (reproduces the expression above)

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vector : $\mathcal{L}_U A^\alpha = A^\alpha{}_{,\beta} U^\beta - U^\alpha{}_{,\beta} A^\beta$

dual vector : $\mathcal{L}_U P_\alpha = P_{\alpha,\beta} U^\beta + U^\beta{}_{,\alpha} P_\beta$

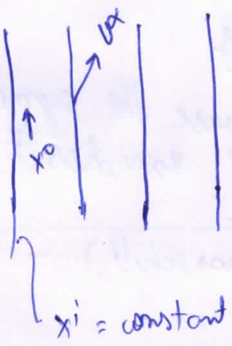
Tensor : $\mathcal{L}_U T^\alpha{}_\beta = T^\alpha{}_{\beta,\gamma} U^\gamma - U^\alpha{}_{,\gamma} T^\gamma{}_\beta + U^\gamma{}_{,\beta} T^\alpha{}_\gamma$

$\mathcal{L}_U V_{\alpha\beta} = V_{\alpha\beta,\gamma} U^\gamma + U^\gamma{}_{,\alpha} V_{\gamma\beta} + U^\gamma{}_{,\beta} V_{\alpha\gamma}$

Scalar : $\mathcal{L}_U f = f_{,\alpha} U^\alpha$

Vector A^α does not depend on x^0 : $\dfrac{\partial A^\alpha}{\partial x^0} \overset{*}{=} 0$

star to mention that it is valid in a particular coordinate system.



$U^\alpha = (1,0,0,0)$

$U^\alpha{}_{,\beta} \overset{*}{=} 0$

$$\frac{\partial A^\alpha}{\partial x^0} \overset{*}{=} 0 \overset{*}{=} A^\alpha{}_{,\beta} U^\beta$$

$$\mathcal{L}_U A^\alpha = A^\alpha{}_{,\beta} U^\beta - U^\alpha{}_{,\beta} A^\beta \overset{*}{=} 0$$

$= 0$ → in any coordinates.

$$\mathcal{L}_U A^\alpha = A^\alpha{}_{,\beta} U^\beta - U^\alpha{}_{,\beta} A^\beta \quad \text{(LIE DERIVATIVE)}$$

verify: $\mathcal{L}_U A^\alpha = A^\alpha{}_{;\beta} U^\beta - U^\alpha{}_{;\beta} A^\beta$ (but here, you bring in the connection coefficients; but anyways they cancel out at the end & you are left with $A^\alpha{}_{,\beta} U^\beta - U^\alpha{}_{,\beta} A^\beta$.

$$\mathcal{L}_U P_\alpha = P_{\alpha,\beta} U^\beta + U^\beta{}_{,\alpha} P_\beta \quad \left(= P_{\alpha;\beta} U^\beta + U^\beta{}_{;\alpha} P_\beta \right)$$

$$\mathcal{L}_U A_{\alpha\beta} = A_{\alpha\beta,\gamma} U^\gamma + U^\mu{}_{,\alpha} A_{\mu\beta} + U^\mu{}_{,\beta} A_{\alpha\mu}$$
$$= A_{\alpha\beta;\gamma} U^\gamma + U^\mu{}_{;\alpha} A_{\mu\beta} + U^\mu{}_{;\beta} A_{\alpha\mu}.$$

If in some coordinate system $A_{\alpha\beta}$ is independent of x^0, then $\mathcal{L}_U A_{\alpha\beta} = 0$; $U^\alpha = \delta^\alpha{}_0$.

(Invariance of $A_{\alpha\beta}$ in direction of vector U)

<u>Killing Vector</u>: a vector field ξ^α such that ~~$\mathcal{L}_\xi A$~~ $\mathcal{L}_\xi g_{\alpha\beta} = 0$

$$\hookrightarrow \quad 0 = \cancelto{0}{g_{\alpha\beta;\gamma}} \xi^\gamma + \xi^\mu{}_{;\alpha} g_{\mu\beta} + \xi^\mu{}_{;\beta} g_{\alpha\mu}$$

$$\Rightarrow \quad 0 = (g_{\mu\beta}\xi^\mu)_{;\alpha} + (g_{\alpha\mu}\xi^\mu)_{;\beta}$$

$$\Rightarrow \quad \boxed{\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0}$$ The Killing vector satisfies this property

$\Rightarrow \xi_{\alpha;\beta}$ is antisymmetric. (because the symmetric part vanishes)

Static, spherically symmetric spacetime (e.g. Schwarzschild)

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$
$$f = \left(1 - \frac{2m}{r}\right)$$

two obvious Killing vectors

$$\begin{array}{rccccc} & t & r & \theta & \phi \\ \xi^\alpha_{(t)} = (& 1, & 0, & 0, & 0 &) \\ \xi^\alpha_{(\phi)} = (& 0, & 0, & 0, & 1 &) \end{array}$$

symmetries in time

Invariance along rotation

(There are also other symmetry: rotation along other axis)

Killing Vectors can be used to construct constants of geodesic motion.

Geodesic that is affinely parametrized (say proper time τ)

velocity vector U^α : $U^\alpha{}_{;\beta} U^\beta = 0$

$$\Rightarrow U_\alpha U^\alpha = \text{constant}$$

Prove that $U^\alpha \xi_\alpha$ is a constant.

Proof: $\frac{d}{d\tau}(U^\alpha \xi_\alpha) = \frac{D}{d\tau}(U^\alpha \xi_\alpha) = \cancel{\frac{DU^\alpha}{d\tau}}^{\,0} \xi_\alpha + U^\alpha \frac{D\xi_\alpha}{d\tau}$

since we are dealing with scalar; we can use covariant derivative.

$$= U^\alpha \xi_{\alpha;\beta} U^\beta$$

$$= \underbrace{\xi_{\alpha;\beta}}_{\text{antisymmetric}} \underbrace{U^\alpha U^\beta}_{\text{symmetric}} = 0.$$

Static, Spherically symmetric spacetime.

$U^\alpha \xi_\alpha^{(t)} = -\tilde{E}$ put minus sign so that energy comes out to be constant.

$= -\frac{E}{m}$ (energy per unit mass)

$U^\alpha \xi_\alpha^{(\phi)} = \tilde{L} = L/m$ (angular momentum per unit mass)

"Energy"

① ⇒ Conserved energy ≡ Killing Energy ≡ energy at infinity

$\tilde{E} = -U^\alpha \xi_\alpha^{(t)}$

At infinity: $U^\alpha = \gamma(1, \dot{r}, 0, 0)$; $\gamma = \frac{1}{\sqrt{1-\dot{r}^2}}$

$U_\alpha U^\alpha = -1$

$$\boxed{\tilde{E} = \gamma}$$

② Locally measured energy by an observer in spacetime:

particle U^α

observer U^α_{obs} ; $\tilde{E}_{local} = -U_\alpha U^\alpha_{obs}$.

always positive

$\tilde{E}_{local} \neq \tilde{E}$.

not conserved — conserved — can be negative.

Metric Determinant

$g \equiv \det[g_{\alpha\beta}] < 0$; we need $\sqrt{-g}$

Invariant volume element $\equiv \sqrt{-g}\cdot d^4x$.

$$\Gamma^{\mu}_{\mu\alpha} = \frac{1}{\sqrt{-g}}\cdot\partial_\alpha\sqrt{-g}$$

$$A^{\alpha}{}_{;\alpha} = A^{\alpha}{}_{,\alpha} + \Gamma^{\alpha}_{\alpha\beta}A^{\beta}$$
$$= A^{\alpha}{}_{,\alpha} + \frac{1}{\sqrt{-g}}(\partial_\beta\sqrt{-g})A^{\beta}$$

$$A^{\alpha}{}_{;\alpha} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\cdot A^{\alpha})_{,\alpha}$$

For $B^{\alpha\beta}$ antisymmetric.

$$B^{\alpha\beta}{}_{;\beta} = \frac{1}{\sqrt{-g}}\cdot(\sqrt{-g}\cdot B^{\alpha\beta})_{,\beta}$$

Levi-Civita tensor : $\underbrace{\epsilon_{\alpha\beta\gamma\delta}}_{\text{Volume form in 4D spacetime.}} = \sqrt{-g}\cdot\underbrace{[\alpha\ \beta\ \gamma\ \delta]}_{\text{permutation symbol.}}$

$[0\,1\,2\,3] = 1$
$[1\,0\,2\,3] = -1$
$[2\,3\,0\,1] = 1$
$[2\,0\,3\,1] = +1$
$[2\,2\,3\,0] = 0$.

Curvature Tensor

★ Scalars : Covariant derivatives do commute ;

$$f_{;\alpha\beta} - f_{;\beta\alpha} = 0$$

★ Vectors & tensor : don't

$$A^{\mu}{}_{;\alpha\beta} - A^{\mu}{}_{;\beta\alpha} = -\underbrace{R^{\mu}{}_{\nu\alpha\beta}}_{\text{Reimann Tensor}}A^{\nu}$$

$$R \sim \partial\Gamma + \Gamma\Gamma - (\leftrightarrow)$$

$$P_{\mu;\alpha\beta} - P_{\mu;\beta\alpha} = +R^{\nu}{}_{\mu\alpha\beta}P_{\nu}$$

Symmetries ||

$$R_{\mu\nu\beta\alpha} = -R_{\mu\nu\alpha\beta} \qquad R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$$
$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$$
$$\cancel{R_{\alpha\beta\mu\nu} = +R_{\alpha\beta\mu\nu}}$$
$$R_{\mu\alpha\beta\gamma} + R_{\mu\gamma\alpha\beta} + R_{\mu\beta\gamma\alpha} = 0$$

4D ; R has 20 independent components.
2D ; R has 1 " " .

Ricci : $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$ (symmetric ; 10 independent components (in 4D))

Ricci scaler : $R^{\alpha}{}_{\alpha} = R$ (1 component)

Einstein Tensor : $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$

So called Bianchi ~~Bianchi~~ Identity ← $\left(G^{\alpha\beta}{}_{;\beta} = 0 \right.$ (obtained by contracting a Bianchi identity)

Einstein Field Equation : $G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta} \Rightarrow T^{\alpha\beta}{}_{;\beta} = 0$

Einstein Tensor. Energy Momentum Tensor (Energy momentum conservation)

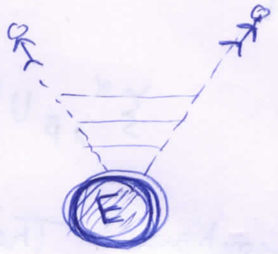
22/4/2020

Advanced General Relativity

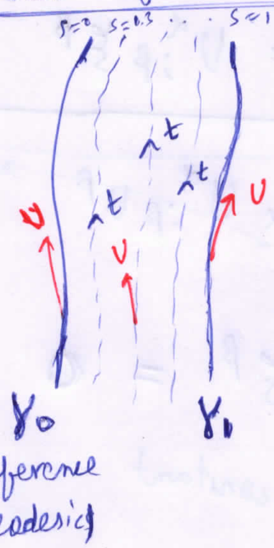
— Shoaib Akhtar

Lec 4 : Ch 1 Fundamentals.

Geodesic Deviation



Behavior of nearby geodesics



We want to describe behavior of γ_1 as seen from γ_0.

Sequences of geodesics $\gamma(s)$ such that
$\gamma(s=0) = \gamma_0$
$\gamma(s=1) = \gamma_1$

$s \Rightarrow$ is parameter that labels each geodesics

$t \Rightarrow$ running parameter on each $\gamma(s)$ (affine parameter)

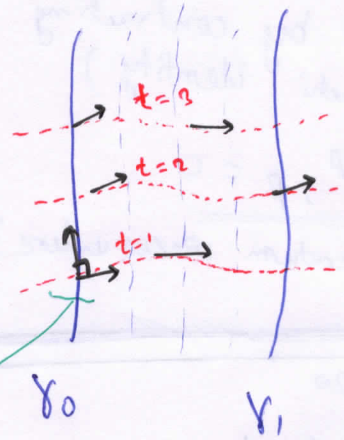
$x^{\alpha} = x^{\alpha}(s, t)$ } ⇒ parametric description of $\gamma(s)$

s ↓ selecting geodesics

t → selecting point on selected geodesics.

$$U^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_s = \text{Tangent vector field to each geodesic } \gamma(s)$$

so, $U^\alpha{}_{;\beta} U^\beta = 0$.



By keeping t fixed, and varying s in relation $x^\alpha(s,t)$ we define a family of cross curves that run from γ_0 to γ_1.

→ we can construct this by adjusting

Tangent vector to all cross cure is

$$\xi^\alpha \equiv \left(\frac{\partial x^\alpha}{\partial s}\right)_t$$

~~$U^\alpha{}_{;\beta}\xi^\beta = \frac{\partial U^\alpha}{\partial s} = \frac{\partial^2 x^\alpha}{\partial s \partial t}$~~

$$U^\alpha{}_{,\beta}\xi^\beta = \frac{\partial U^\alpha}{\partial s} = \frac{\partial^2 x^\alpha}{\partial s \partial t}$$

$$\xi^\alpha{}_{,\beta} U^\beta = \frac{\partial \xi^\alpha}{\partial t} = \frac{\partial^2 x^\alpha}{\partial t \partial s}$$

for smooth differentiable functions, They must be equal.

so, $\mathcal{L}_U \xi^\alpha = \xi^\alpha{}_{,\beta} U^\beta - U^\alpha{}_{,\beta}\xi^\beta$

$$\mathcal{L}_U \xi^\alpha = -\mathcal{L}_\xi U^\alpha = 0 \qquad \boxed{\xi^\alpha{}_{;\beta} U^\beta = U^\alpha{}_{;\beta}\xi^\beta}$$

$$\frac{\partial}{\partial t}(\xi_\alpha U^\alpha) = \frac{D}{dt}(\xi_\alpha U^\alpha) = (\xi_{\alpha;\beta}U^\beta)U^\alpha + \xi_\alpha \cancel{U^\alpha{}_{;\beta}U^\beta}$$

$$= (U_{\alpha;\beta}\xi^\beta)U^\alpha$$

$\xi_\alpha U^\alpha$ is constant as we move up along the congruence

$$= \frac{1}{2}\underbrace{(U_\alpha U^\alpha)}_{\text{This is constant}}{}_{;\beta}\,\xi^\beta = 0$$

by adjusting ξ & U are orthogonal.

~~we can~~

we can set $\xi_\alpha U^\alpha$ to be zero.

ξ^α can be decomposed as $\xi^\alpha = \lambda U^\alpha + \tilde{\xi}^\alpha$

$$\lambda = -\xi_\alpha U^\alpha = \text{(constant)} \rightarrow \text{as proved above.}$$

$$\tilde{\xi}_\alpha U^\alpha = 0$$

lets prove that we can arrange $\lambda = 0$.

ie: we can replace ξ^α by $\tilde{\xi}^\alpha$ without changing anything $\rightarrow$ the orthogonal piece w.r.t. U^α

Note $\underline{\mathcal{L}_U \tilde{\xi}^\alpha = 0}$

$$\tilde{\xi}^\alpha{}_{;\beta} U^\beta = (\xi^\alpha - \lambda U^\alpha)_{;\beta} U^\beta = \xi^\alpha{}_{;\beta} U^\beta - \cancel{}^{0}$$

$$U^\alpha{}_{;\beta} \tilde{\xi}^\beta = U^\alpha{}_{;\beta}(\xi^\beta - \lambda U^\beta) = U^\alpha{}_{;\beta}\xi^\beta$$

we can proceed $\tilde{\xi}$ instead of ξ — and hence we recognise that we have freedom to set $\lambda = 0$ so: $\xi^\alpha = \tilde{\xi}^\alpha$

so; we $\boxed{\text{Impose} \quad U_\alpha \xi^\alpha = 0}$ on γ_0.

$\dfrac{D^2\xi^\alpha}{dt^2}$: covariant :
$$\frac{D^2\xi^\alpha}{dt^2} = (\xi^\alpha{}_{;\beta} U^\beta)_{;\gamma} U^\gamma$$
$$= (U^\alpha{}_{;\beta}\xi^\beta)_{;\gamma} U^\gamma$$
$$= U^\alpha{}_{;\beta\gamma}\xi^\beta U^\gamma + U^\alpha{}_{;\beta}\xi^\beta{}_{;\gamma} U^\gamma$$
$$= (U^\alpha{}_{;\gamma\beta} - R^\alpha{}_{\mu\beta\gamma}U^\mu)\xi^\beta U^\gamma + U^\alpha{}_{;\beta} U^\beta{}_{;\gamma}\xi^\gamma$$

$$\Rightarrow \frac{D^2\xi^\alpha}{dt^2} = -R^\alpha{}_{\mu\beta\gamma}U^\mu\xi^\beta U^\gamma + \underbrace{U^\alpha{}_{;\gamma\beta}U^\gamma\xi^\beta}_{\Downarrow} + U^\alpha{}_{;\beta}U^\beta{}_{;\gamma}\xi^\gamma$$

$$\Big(\cancelto{0}{(U^\alpha{}_{;\gamma}U^\gamma)}_{;\beta}\xi^\beta\Big) \rightarrow \text{cancels due to geodesic equation}$$

$$- U^\alpha{}_{;\gamma}U^\gamma{}_{;\beta}\xi^\beta$$

$$\boxed{\frac{D^2\xi^\alpha}{dt^2} = -R^\alpha{}_{\mu\beta\gamma}U^\mu\xi^\beta U^\gamma}$$

Local Flatness

$\exists$ coordinate system such that, at any point P in spacetime.

$$g_{\alpha\beta}(P) = \eta_{\mu\nu} \quad ; \quad \partial_\gamma g_{\alpha\beta}(P) = 0$$
$$\text{or} \quad \Gamma^\alpha_{\beta\gamma}(P) = 0$$

we cant gaurantee for second derivative unless it is flat $\quad \boxed{\partial^2_{\gamma\delta} g_{\alpha\beta}(P) \neq 0} \quad$ Generic for curved spaces

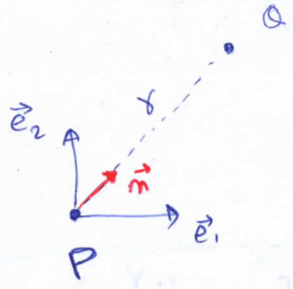
Reimann Normal Coordinates

$\exists$ coordinates X^α such that, around $P \in m$ (at which $X^\alpha = 0$)

$$g_{\alpha\beta} = \eta_{\alpha\beta} - \frac{1}{3} R_{\alpha\mu\beta\nu} X^\mu X^\nu + O(X^3)$$

$\hookrightarrow$ components of Reimann Tensor at P.

Flat Space



Decompose $\vec{n}$ into the basis

$$\vec{n} = n^j \vec{e}_j$$

we can write the position vector of Q as

$$\vec{x} = r\vec{n} = r n^j \vec{e}_j$$

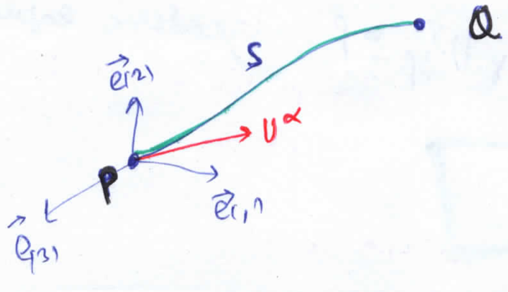
Assign to Q the coordinates $x^j = r n^j$

$\implies$ Metric is δ_{jk} (due to construction)

$\hookrightarrow$ Construction of ~~setting~~ setting up cartesian coordinates in flat space.

Curved Space

(use geodesis instead of straight line)



There is an assumption here that geodesic is unique

$\hookrightarrow$ This is always true if Q is close to P.

(If you go far away, you can have different geodesics;

eg )

$$U^\alpha = U^{(P)} e^\alpha_{(P)}$$

I dont have notion of position vector; but we can directly go away by declaring my coordinates RNC (Reiman Normal Coordinates)

$$\text{RNC}: \quad X^\mu \equiv S\, U^{(P)}$$

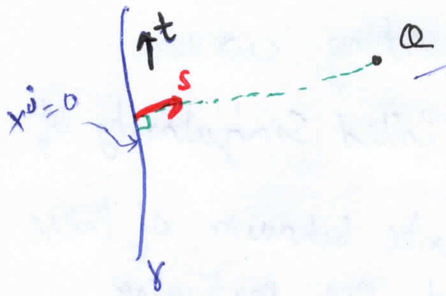
$\Rightarrow$ mathematically it is straight line because $U^{(P)}$ is constant.

$\hookrightarrow$ If you calculate metric in this; you will get

$$g_{\alpha\beta} = \eta_{\alpha\beta} - \frac{1}{3} R_{\alpha\mu\beta\nu} X^\mu X^\nu + O(X^3)$$

Fermi Normal Coordinates

Timelike geodesic as spacial origin.



To find spacial coordinates of Q, we will draw geodesic from Q to γ; which has the property that ~~them~~ it crosses γ at right angle.

$$\boxed{\begin{aligned} g_{tt} &= -1 - R_{tatb}(t)\,x^a x^b + \ldots \\ g_{ta} &= -\frac{2}{3} R_{tbac}(t)\,x^b x^c + \ldots \\ g_{ab} &= \delta_{ab} - \frac{1}{3} R_{acbd}(t)\,x^c x^d + \ldots \end{aligned}}$$

This says that, we can select a geodesic and can extend local flatness along that geodesic (but if you go away from that geodesic, ~~your~~ our metric will deviate away from local flatness)

$\Rightarrow$ This is actually the statement, that if we happen to be freely falling in spacetime and if we follow ourselves along a time-like geodesic, we ~~ca~~ see spacetime being locally flat, i.e. ~~our~~ in our immediate neighbourhood ~~you~~ our time-like spacetime is locally flat everywhere along geodesic $\Longrightarrow$ That means, for us in free fall gravity does not exists (i.e. gravity does not exist locally)

The convergence property of different geodesic convinces that infact there is something like gravity. Global property of geodesics converging at a non-linear rate.

Chapter 1 Completed

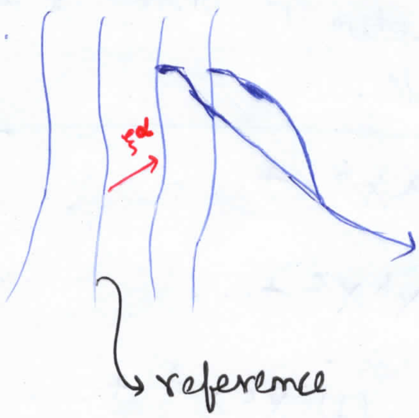
Advanced General Relativity 22/4/2020

Lec 5 : Ch 2 Geodesic Congruences — Shoaib Akhtar

Congruence ⇒ family of curves that dont intersects.

Congruence is a family of non-intersecting curves.

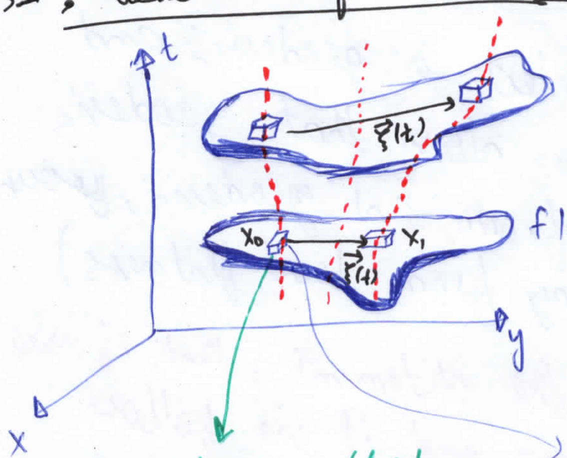
If we have point of intersection, that is called Singularity of congruence.



We will describe behavior of these curves w.r.t. one particular curve. (through behavior of deviation vector ξ^α) → neighbouring geodesics

to be continued.....

3D, Newtonian fluid mechanics



$\vec{\xi}(t)$: displacement between a given fluid element & reference fluid element.

$$\vec{\xi} = \vec{X}_1 - \vec{X}_0$$ (This we cant do in curved Spacetime)

Relative velocity between fluid elements $\frac{d\vec{\xi}}{dt} = \vec{V}(\vec{x}_1) - \vec{V}(\vec{x}_0)$

doing taylor expansion for nearby points.

$$\frac{d\xi^j}{dt} = V^j(\vec{x}_0 + \vec{\xi}) - V^j(\vec{x}_0) = \underbrace{V^j{}_{,k}}_{\text{evaluated at } \vec{x}_0} \xi^k + O(\xi^2)$$

$$\therefore \frac{d\xi^j}{dt} = B^j{}_k \xi^k$$, where $B^j{}_k(t) = V^j{}_{,k}(t, \vec{x}_0)$ (evaluated at refererence fluid element)

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$$\frac{d\xi^j}{dt} = B^j{}_k \xi^k$$

$$B_{jk}(t) = V_{j,k}(t, \vec{x}_0)$$

matrix without symmetries; we will decompose it into irreducible pieces.

$$B_{jk} = \frac{1}{3}\delta_{jk}\Theta + \sigma_{jk} + \omega_{jk}$$

The trace part
Trace part
(Expansion)

The symmetric trace free part
Symmetric trace free part
(Shear Tensor)

Antisymmetric part
(Rotation Tensor)

$$\Theta = \delta^{jk} B_{jk}$$

$$\sigma_{jk} = B_{(jk)} - \frac{1}{3}\delta_{jk}\Theta$$

$$\omega_{jk} = B_{[jk]}$$

$$\frac{1}{3}\delta_{jk}\Theta = \begin{pmatrix} \frac{1}{3}\Theta & & 0 \\ & \frac{1}{3}\Theta & \\ 0 & & \frac{1}{3}\Theta \end{pmatrix}$$ (1 component)

$$\sigma_{jk} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$$ (5 component)

has to be $-(\sigma_{11} + \sigma_{22})$ so that trace vanishes.

$$\omega_{jk} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}$$ (3 components)

9 components total.

Expansion $\Theta = \delta^{jk} V_{j,k} = \vec{\nabla}\cdot\vec{V}$ = ~~Rate of~~ Fractional rate of expansion of volume of fluids.

Mass conservation (expressed by continuity equation)

$\rho = \frac{\delta m}{\delta V}$ mass density

$\rho\vec{V}$ ≡ mass current density

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \vec{\nabla}\cdot\vec{V} + \vec{V}\cdot\vec{\nabla}\rho = 0$$

$$\Rightarrow \underbrace{\frac{\partial \rho}{\partial t} + \vec{V}\cdot\vec{\nabla}\rho}_{} = -\rho\,\vec{\nabla}\cdot\vec{V}$$

↳ Total time derivative of density $\frac{d\rho}{dt}$ as we follow motion of fluid element.

* As we follow a given fluid element;

at t : $\rho(t, \vec{x})$

at $t+dt$: $\rho(t+dt, \vec{x}+d\vec{x})$

Total change $d\rho = \rho(t+dt, \vec{x}+d\vec{x}) - \rho(t, \vec{x})$

$$= \frac{\partial \rho}{\partial t}dt + d\vec{x}\cdot\vec{\nabla}\rho$$

$$\Rightarrow \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{V}\cdot\vec{\nabla}\rho = \text{Convective derivative of density.}$$

↳ If we dont follow the fluid element; Then this is just the rate of change of density at a point $\vec{x}$.

for as $\frac{d\rho}{dt}$ is approximation for ρ differentiated in the direction of V^α four velocity of the fluid element;

$V^\alpha \simeq (1, \vec{v})$; non relativistic approximation.

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{V}\cdot\vec{\nabla}\rho \simeq \rho_{,\alpha}V^\alpha$$

$$\vec{\nabla}\cdot\vec{V} = -\frac{1}{\rho}\frac{d\rho}{dt}$$

(within a volume element, the mass is unchanged; so density can change only if volume changes)

$$= \frac{1}{\delta V}\frac{d}{dt}\delta V$$

= Fractional rate of change of volume of fluid element.

$$\Theta = \frac{1}{\delta V}\frac{d}{dt}(\delta V)$$

(corresponds to scaling of the fluid)

So, our expansion parameter is fractional rate of change of volume.

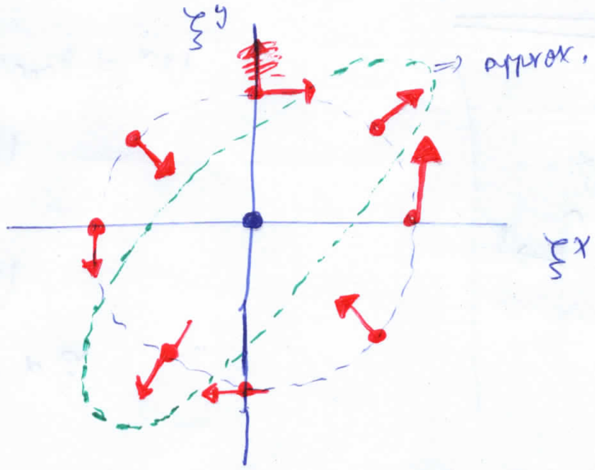
Shear set $\theta = 0 = \omega_{jk}$

$$\frac{d\xi^j}{dt} = \sigma^j{}_k \xi^k$$

take σ_{12} is only non-vanishing component. ; lets call $\sigma = \sigma_{12}$

$$\frac{d\xi^x}{dt} = \sigma \xi^y$$

$$\frac{d\xi^y}{dt} = +\sigma \xi^x$$



This gives the effect of shear tensor.

"Volume inside the shape is preserved"

"Deformation in shape without change in volume"

* refrence fluid element at centre

* neighbouring " " in the shape of circle at initial time.

So; The effect is to shear the circle;

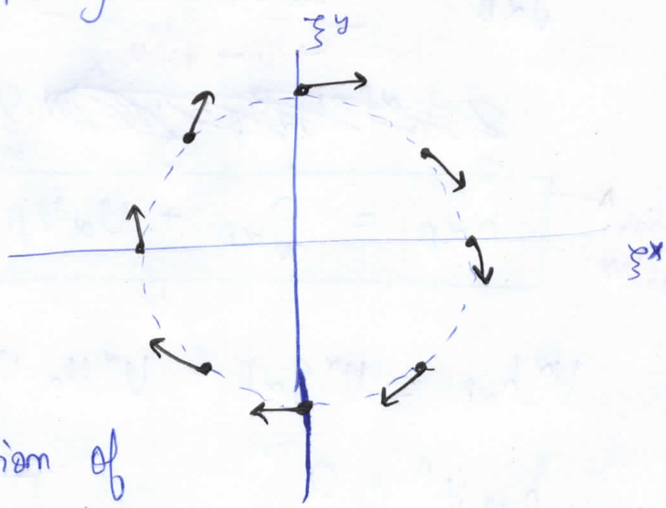
- squeeze one direction.
- stretch the other direction.

Rotation set $\theta = 0 = \sigma_{jk}$

$\omega_{12} = \omega$ only non-vanishing component.

$$\frac{d\xi^x}{dt} = \omega \xi^y$$

$$\frac{d\xi^y}{dt} = -\omega \xi^x$$

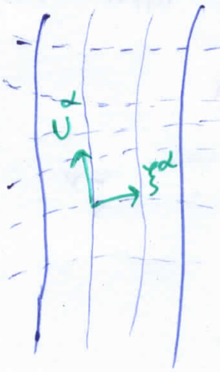


Rotation of orignal shape without distortion.

Note, the analysis is actually not relying on material part of fluid. We can just keep the curves...
... and build the theory for G.R.

→ because we erased out mass... we finally worked with volumes
or $\theta = \frac{1}{\delta V}\frac{d}{dt}(\delta V)$

Congruences



U^α ⇒ tangent vector field to congruence

$$U^\alpha{}_{;\beta} U^\beta = 0$$

$$U^\alpha U_\alpha = -1$$

ξ^α : deviation vector field tangent to cross curve.

time like geodesic

using time as parameter

$$\mathcal{L}_U \xi^\alpha = \mathcal{L}_\xi U^\alpha$$

$$\Rightarrow \quad \xi^\alpha{}_{;\beta} U^\beta = U^\alpha{}_{;\beta}\xi^\beta$$

also: $\xi^\alpha U_\alpha = 0$.

Decomposition of $g_{\alpha\beta}$:

$$g_{\alpha\beta} = \underbrace{-U_\alpha U_\beta}_{\text{Time piece}} + \underbrace{h_{\alpha\beta}}_{\text{Spacial piece.}}$$

→ longitudinal metric

~~$h^{\alpha\beta} U_\alpha U_\beta = 0$~~ shows $h_{\alpha\beta}$ is spacial piece

$h_{\alpha\beta} U^\beta = 0$

Projection operator

$$\boxed{h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta}$$

Transverse metric.

$$U^\alpha h_{\alpha\beta} = U^\alpha g_{\alpha\beta} + U^\alpha U_\alpha U_\beta = U_\beta + (\underbrace{U^\alpha U_\alpha}_{=-1}) U_\beta = 0$$

$$h_{\alpha\beta} U^\beta = 0$$

$$\boxed{h^\alpha{}_\gamma h^\gamma{}_\beta = h^\alpha{}_\beta}$$

h is actually projection operator

$$\boxed{h^\alpha{}_\alpha = 3}$$

~~Local~~ In a local Lorentz frame momentarily co-moving with reference geodesics

$$U^\alpha \overset{*}{=} (1, 0, 0, 0)$$

$$g_{\alpha\beta} \overset{*}{=} \begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{pmatrix}$$

$$h_{\alpha\beta} \overset{*}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta$$

Behavior of neighbouring geodesic relative to reference geodesic is captured by $\xi^\alpha{}_{;\beta} U^\beta = B^\alpha{}_\beta \xi^\beta$

$$B_{\alpha\beta} = U_{\alpha;\beta}$$

$B_{\alpha\beta} U^\beta = 0 = U^\alpha B_{\alpha\beta}$; so ; $B_{\alpha\beta}$ is purely spacial.

$B_{\alpha\beta} U^\beta = 0 = U^\alpha B_{\alpha\beta}$

* $B_{\alpha\beta} U^\beta = U_{\alpha;\beta} U^\beta = 0$ (by virtue of geodesic equation)

* $U^\alpha B_{\alpha\beta} = U^\alpha U_{\alpha;\beta} = \frac{1}{2} (U^\alpha U_\alpha)_{;\beta} = 0$.

The time entry of B is trivial ; because B is spacial in nature.

Decomposition of $B_{\alpha\beta}$.

$$B_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}\, \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

Θ → Expansion, $\sigma_{\alpha\beta}$ → Shear, $\omega_{\alpha\beta}$ → Rotation.

(all tensors orthogonal to U^α) ⟹ We can easily check this in co-moving local frame.

$h_{\alpha\beta}$ → Build it with only spacial ; because anyways B is spacial in nature.

$$\boxed{\Theta \equiv h^{\alpha\beta} B_{\alpha\beta}} \boxed{= g^{\alpha\beta} B_{\alpha\beta}} \boxed{= U^\alpha{}_{;\alpha}}$$

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$$\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{3} h_{\alpha\beta}\, \theta$$

$$\omega_{\alpha\beta} = B_{[\alpha\beta]}$$

Same interpretation as fluid: $\Theta = \frac{1}{\delta V} \frac{d}{d\tau} \delta V$

H/W ||

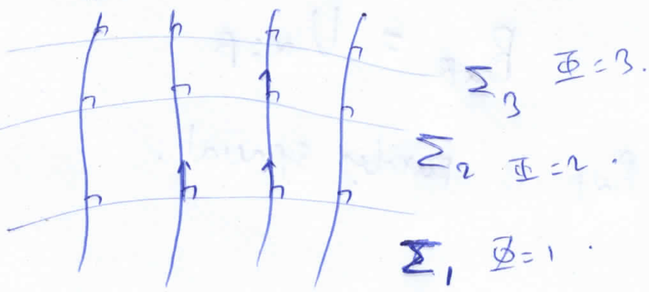
1.13 : #3, #6, #9

2.5 : #1, #3

Lec : 6 || 23/4/2020

Frobenius Theorem The congruence is hypersurface orthogonal iff $\omega_{\alpha\beta} = 0$.

we have a family of hypersurfaces such that normal vector is everywhere aligned with U^α.



(In general, the condition is $U_{[\alpha} U_{\beta,\gamma]} = 0$)

when we restrict to geodesics; this reduces to $\omega_{\alpha\beta} = 0$

A hypersurface is described by

$\Phi(x^\alpha) =$ constant.

ex 2 sphere in flat 3D ; $\Phi(x,y,z) = x^2 + y^2 + z^2 = R^2$

Normal to hypersurface $n_\alpha \propto \partial_\alpha \Phi$

$n_\alpha = -\mu\, \partial_\alpha \Phi$ (minus sign is there because we are dealing with Lorentz manifold)

μ is scalar function for normalization. $\mu \geq 0$.

$n_\alpha n^\alpha = -1$

minus sign means that vector is future pointing.

we have choosen a sensible convention that Φ is increasing forward in time

The Congruence is hypersurface orthogonal when

$U^\alpha = n^\alpha \quad ; \quad U_\alpha = -\mu \partial_\alpha \Phi .$

$$U_{\alpha;\beta} = -\mu \Phi_{;\alpha\beta} - \partial_\alpha \Phi \, \partial_\beta \mu$$

$$\omega_{\alpha\beta} = U_{[\alpha,\beta]} = -\frac{1}{2}\left\{ \partial_\alpha \Phi \, \partial_\beta \mu - \partial_\alpha \mu \, \partial_\beta \Phi \right\}$$

$$= \frac{1}{2\mu}\left(U_\alpha \cdot \partial_\beta \mu - \partial_\alpha \mu U_\beta \right)$$

$$\omega_{\alpha\beta} = \frac{1}{2\mu}\left(U_\alpha \partial_\beta \mu - U_\beta \partial_\alpha \mu \right)$$

$$0 = U^\alpha \omega_{\alpha\beta} = \frac{1}{2\mu}\left(-\partial_\beta \mu - U_\beta \left(U^\alpha \partial_\alpha \mu \right) \right) = 0$$

$$\Rightarrow \boxed{\partial_\beta \mu = -\left(U^\alpha \partial_\alpha \mu \right) U_\beta}$$

μ can vary only along normal direction.

$\Rightarrow$ so; if this is true; ie; is $\partial_\beta \mu \propto U_\beta$

Then $\omega_{\alpha\beta}$ has to vanish.

~~Focusing Theorem~~

Till now we have just describe relative velocity of neighbouring geodesics

$\hookrightarrow$ effectively we just did Kinematics.

Rayachaudhuri's Equation

Now lets do Dynamics.

so; we want to derive evaluation equation for θ, σ, ω that describes gradient field

We want to access dynamics for neighbouring geodesics.

$\therefore$ Evolution equation for $B_{\alpha\beta}$:

$\left.\dfrac{DB_{\alpha\beta}}{d\tau}\right\} \Rightarrow$ covariant derivative of B along congruence.

$$\frac{DB_{\alpha\beta}}{d\tau} = B_{\alpha\beta;\mu} U^\mu = U_{\alpha;\beta\mu} U^\mu$$

$$\frac{DB_{\alpha\beta}}{d\tau} = \left(U_{\alpha;\mu\beta} + R^{\nu}{}_{\alpha\beta\mu}U_\nu\right)U^\mu$$

$$= \underbrace{(U_{\alpha;\mu}U^\mu)_{;\beta}}_{0} - \underbrace{U_{\alpha;\mu}}_{B}\,\underbrace{U^\mu{}_{;\beta}}_{B} + R_{\alpha\nu\beta\mu}U^\nu U^\mu$$

$$\Rightarrow \frac{DB_{\alpha\beta}}{d\tau} = -B_{\alpha\mu}B^\mu{}_\beta + R_{\alpha\mu\beta\nu}U^\mu U^\nu$$

$$\boxed{\frac{DB_{\alpha\beta}}{d\tau} = -B_{\alpha\mu}B^\mu{}_\beta + R_{\alpha\mu\beta\nu}U^\mu U^\nu}$$ Evolution equation for B.

Taking trace $\quad B = B^\alpha{}_\alpha\,(=\theta)\quad \frac{D}{d\tau}B = \frac{d}{d\tau}B$

$$\Rightarrow \frac{dB}{d\tau} = -B^{\alpha\mu}B_{\mu\alpha} + R_{\mu\nu}U^\mu U^\nu$$

$$B^{\alpha\mu}B_{\mu\alpha} = \left(\tfrac{1}{3}h^{\alpha\mu}\theta + \sigma^{\alpha\mu} + \omega^{\alpha\mu}\right)\left(\tfrac{1}{3}h_{\mu\alpha}\theta + \sigma_{\mu\alpha} + \omega_{\mu\alpha}\right)$$

(Since each piece is irreducible, it means that they dont interfere with one another; and so the calculation will become simple.

$$\Rightarrow B^{\alpha\mu}B_{\mu\alpha} = \frac{1}{3^2}\cdot 3\cdot\theta^2 + \sigma^{\alpha\mu}\sigma_{\alpha\mu} - \omega^{\alpha\mu}\omega_{\alpha\mu}$$

So. $$\boxed{\underbrace{\frac{d\theta}{d\tau}}_{\text{Rate of change of }\theta\text{ along congruence.}} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\mu}\sigma_{\alpha\mu} + \omega^{\alpha\mu}\omega_{\alpha\mu} + R_{\mu\nu}U^\mu U^\nu}$$

Rayachaudhri Equation.

Though we call θ to be expansion; it is actually rate of expansion $\frac{1}{\delta V}\frac{d}{d\tau}(\delta V)$

since all our tensor ω, σ is spacial

so;

$$\left.\begin{array}{l} \theta^2 \geq 0 \\ \sigma^{\alpha\beta}\sigma_{\alpha\beta} \geq 0 \\ \omega^{\alpha\beta}\omega_{\alpha\beta} \geq 0 \end{array}\right\}$$ Because tensors are spacial.

so; focusing Theorem will come out if we kill $\omega^{\alpha\beta}\omega_{\alpha\beta}$ and put some restrictions of $R_{\mu\nu}U^\mu U^\nu$.

∴ Why we do care about hypersurface orthogonal congruence? Because we want to kill out ω.

∴ Why do we care about Energy Conditions? Because we want to control $R_{\mu\nu}U^\mu U^\nu$

We will impose condition that $R_{\mu\nu}U^\mu U^\nu \geq 0$ so we have focusing theorem that $\frac{d\theta}{d\tau}$ is negative.

Focusing Theorem

- Let congruence be hypersurface orthogonal ($\omega_{\alpha\beta} = 0$)
- Impose strong energy condition: $R_{\alpha\beta}U^\alpha U^\nu \geq 0$

(using Einstein field Equation. $(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta})U^\alpha U^\beta \geq 0$

Then

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - R_{\alpha\beta}U^\alpha U^\beta \leq 0$$

so; $$\boxed{\frac{d\theta}{d\tau} \leq 0}$$

Cosmology

Cosmological fluid is a congruence

U^α, θ, $\sigma_{\alpha\beta}$, $\omega_{\alpha\beta}$.

we also have ρ, P

so; we can form Energy momentum tensor:

$$T^{\alpha\beta} = \underbrace{\rho U^\alpha U^\beta}_{\text{mass density term in longitudinal direction}} + \underbrace{P(g^{\alpha\beta} + U^\alpha U^\beta)}_{\text{pressure term in transverse direction.}}$$

Also, in cosmology; we have to postulate an equation of state

$$P = w\rho \quad ; \quad w = \text{constant}.$$

* $w=0$ ($P \ll \rho$; for matter dominated universe) $P=0$
* $w=\frac{1}{3}$ (radiation dominated universe)
* $w=-1$ (Dark energy)

$$T^{\alpha\beta}{}_{;\beta} = 0 \implies \frac{d\rho}{d\tau} + (P+\rho)\, U^\alpha{}_{;\alpha} = 0$$

$$(\rho+P)\frac{DU^\alpha}{d\tau} + (g^{\alpha\beta} + U^\alpha U^\beta)\partial_\beta P = 0$$

assume fluid moves along geodesic so; $\frac{DU^\alpha}{d\tau} = 0$

assume pressure is uniform; $\partial_\beta P = 0$ $\hookrightarrow$ to be consistent with $\frac{DU^\alpha}{d\tau} = 0$

$$\boxed{\frac{d\rho}{d\tau} + \underbrace{(P+\rho)}_{(1+w)\rho}\theta = 0}$$

$$\Rightarrow \boxed{\frac{d\rho}{d\tau} + (1+w)\rho\,\theta = 0}$$

Assumptions:

1. geodesic motion
2. Equation of State.
3. No shear; $\sigma_{\alpha\beta}=0$
4. No rotation; $\omega_{\alpha\beta}=0$
5. Assume E.F.E.; $R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right)$

$$R_{\alpha\beta}U^{\alpha}U^{\beta} = 4\pi(\rho+3p) = 4\pi(1+3w)\rho$$

$$\boxed{\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - 4\pi(1+3w)\rho}$$

$\theta = \frac{1}{\delta V}\frac{d}{d\tau}\delta V$; we can define scale factor as follows.

$$\delta V \propto a^3(t)$$

cross sectional volume.

$$\theta \equiv \frac{1}{a^3}\frac{d}{d\tau}a^3 = 3\frac{\dot{a}}{a}$$

$$= 3H$$

$$\Rightarrow \boxed{\theta = 3H}$$

Lec 7 ||

23/4/2020

Congruences of timelike geodesics.

★ Example of diverging congruence in flat space.

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

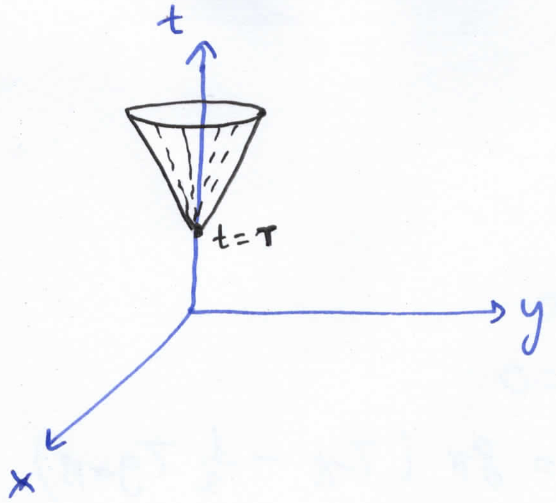
time like geodesics $\quad t = T + \lambda$ parameter along congruence.

$$r = V\lambda$$

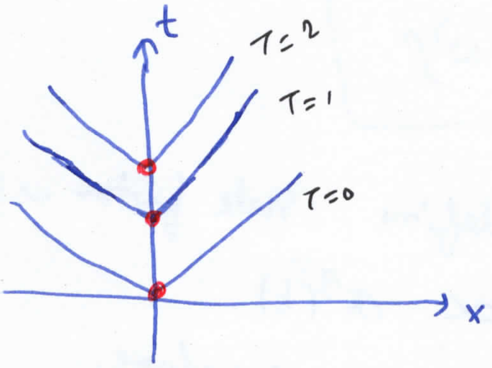
constant

θ = constant along a congruence $= \theta_0$

ϕ = " " " " $= \phi_0$



each geodesic is labelled by (T, θ_0, ϕ_0) and λ is an affine parameter on each geodesic.



excluding the crossing points on axis — which is singularity of congruence.

Flat spacetime: ξ, $\dot{\xi} \neq 0$ (generically) ; $\underbrace{\ddot{\xi} = 0}_{\text{from geodesic deviation equation.}}$

$$\theta \sim \frac{1}{\xi^3}\frac{d}{d\lambda}\xi^3 = 3\frac{\dot{\xi}}{\xi}$$

θ → Rate of expansion

→ Its non linear in ξ.

$$\frac{d\theta}{d\lambda} \sim \cancel{3\frac{\ddot{\xi}}{\xi}}^{\,0} - 3\left(\frac{\dot{\xi}}{\xi}\right)^2$$

→ non-zero.

→ and it gaurantees to give us minus sign for $\frac{d\theta}{d\lambda}$

~~Stuff Espal~~

Tangent vector field : $U^\alpha = \frac{dx^\alpha}{d\lambda} = (1, v, 0, 0)$

$$g_{\alpha\beta} U^\alpha U^\beta = -(1-v^2) \quad \text{(Not normalized)}$$

$$\neq 0 \; ; \; \lambda \neq \tau .$$

$$U_\alpha = g_{\alpha\beta} U^\beta = (-1, v, 0, 0)$$

$$= -\partial_\alpha (t - vr) \longrightarrow \text{normal to hypersurface}$$

when $U_\alpha \propto$ gradient $\Rightarrow$ $\Phi = t - vr =$ constant

Congruence is hypersurface orthogonal.

$$B_{\alpha\beta} = U_{\alpha;\beta} = \text{diag}\left[0, 0, rv, rv \cdot \sin^2\theta\right]$$

$\therefore$ clearly; ω vanishes. ; $B_{\alpha\beta} = B_{\beta\alpha} \Rightarrow \omega_{\alpha\beta} = 0$

$$h_{\alpha\beta} = g_{\alpha\beta} + \underbrace{\frac{U_\alpha U_\beta}{|U_\mu U^\mu|}}_{\text{for normalizing.}} \Rightarrow h_{\alpha\beta} U^\beta = U_\alpha + U_\alpha(-1) = 0$$

$$B_{\alpha\beta} = \frac{1}{3} h_{\alpha\beta}\theta + \sigma_{\alpha\beta}$$

$$\Rightarrow \theta = h^{\alpha\beta} B_{\alpha\beta} = g^{\alpha\beta} B_{\alpha\beta} = U^\alpha{}_{;\alpha} = 2v/r$$

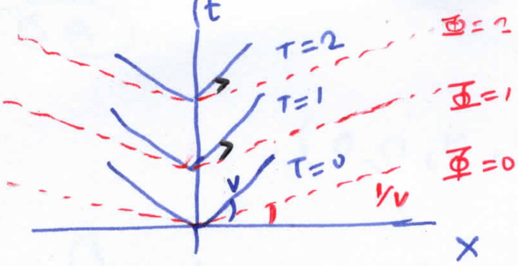
$$\sigma_{\alpha\beta}\sigma^{\alpha\beta} = \frac{2}{3}\frac{v^2}{r^2}$$

$$\frac{d\theta}{d\lambda} = \theta_{,\alpha} U^\alpha = \theta_{,r} v^r = v\frac{d\theta}{dr} = -\frac{2v^2}{r^2}$$

by Rayachaudhuri;

$$\frac{d\theta}{d\lambda} = -\frac{1}{3}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} \quad \checkmark \text{ check}$$

agreed..



$\Phi = t - vr =$ constant.

↖ hypersurfaces.

On each hypersurface ; $t = vr +$ constant

$$\Rightarrow \underline{dt = v\,dr}$$

Indused metric on each hypersurface

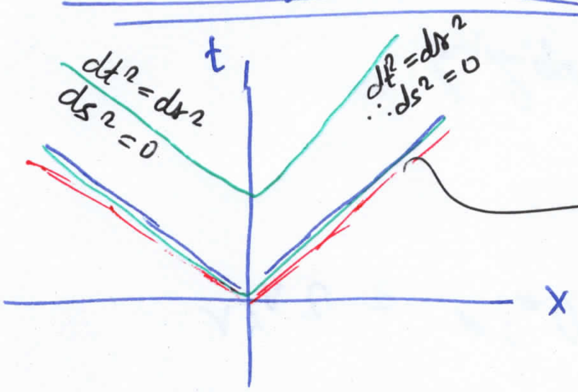
$$ds^2 = -(v\,dr)^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\boxed{ds^2\big|_{\text{hypersurface}} = (1-v^2)dr^2 + r^2 d\Omega^2}$$

$$R^{\theta\phi}{}_{\theta\phi} = -\frac{v^2}{(1-v^2)r^2}$$ (has intrinsic curvature)

↳ is singular at $r=0$.

lets take light like limit.



In the limit $v \to 1$

hyper surface & null geodesics.

~~* null geodesics are orthogonal to~~

~~* " " "~~

* null geodesics are orthogonal to hypersurfaces.

* " " " tangent " hyper surfaces.

In the null limit $U_\alpha U^\alpha = 0$.

$$U_\alpha = -\partial_\alpha \Phi$$

$$ds^2 \longrightarrow r^2 d\Omega^2$$

metric becomes degenerate (we looses a dimension)

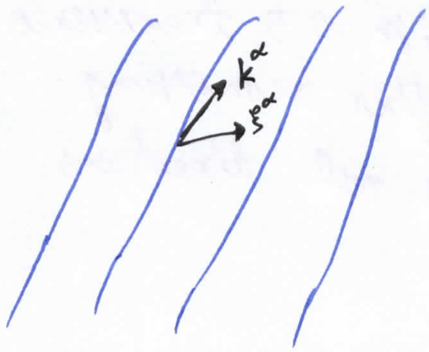
Metric effectively becomes 2-dimensional.

we have two null directions along which $ds^2 = 0$

2D transverse space

$ds^2 \neq 0$

Null Geodesics



k^α : tangent vector field.

$k_\alpha k^\alpha = 0$

$k^\alpha{}_{;\beta} k^\beta = 0$

(assume affine parameter)

$k^\alpha{}_{;\beta} \xi^\beta = \xi^\alpha{}_{;\beta} k^\beta$

(Lie derivative property)

$\therefore$ we can choose $\xi_\alpha k^\alpha = 0$.

→ But this does not kill a component of ξ^α along k^α in null geodesic case.

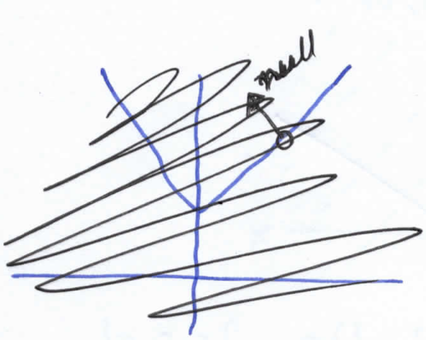
Transverse metric

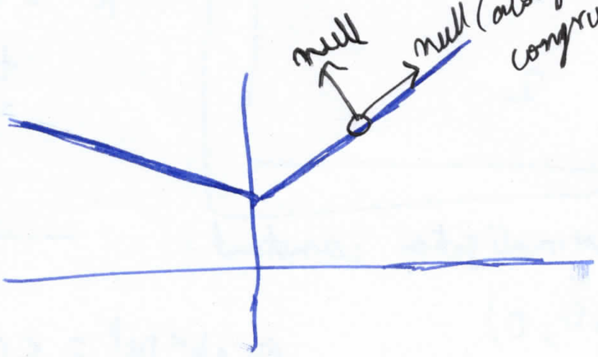
Time like (TL) : $h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta$

here; try : $h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha k_\beta$?

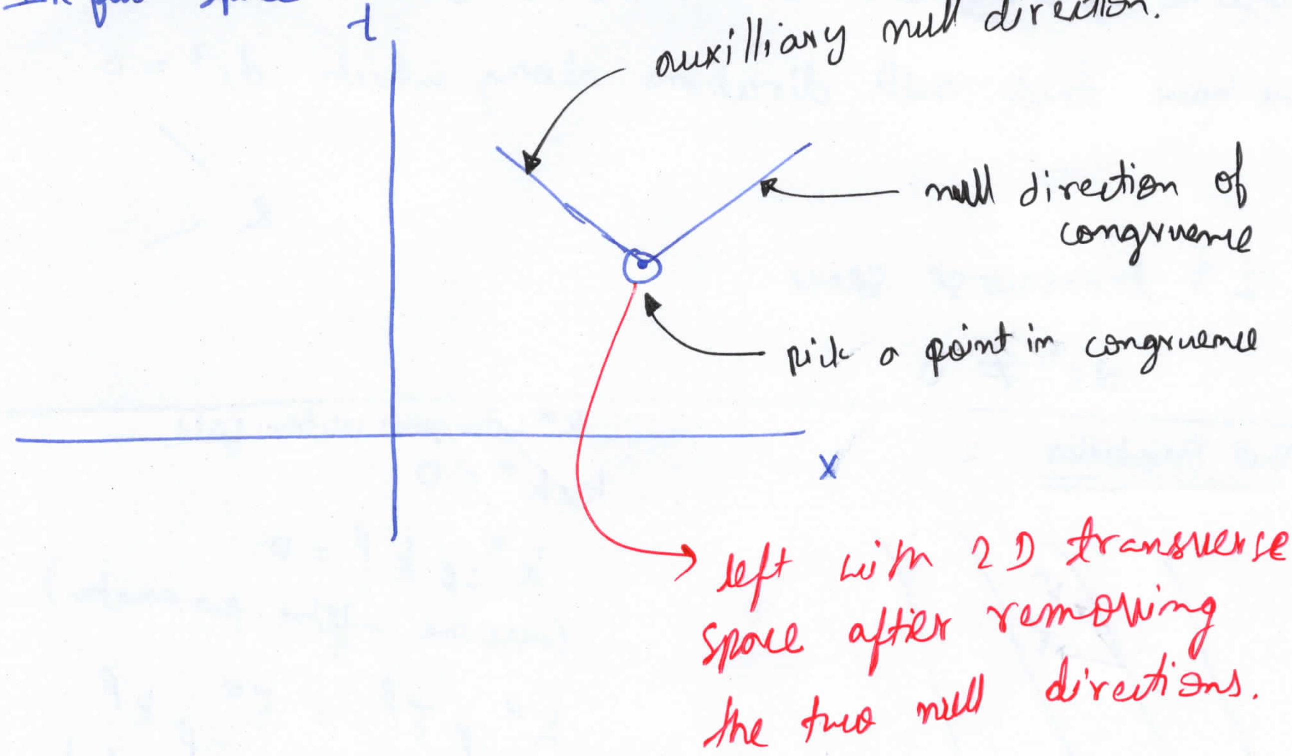
$h_{\alpha\beta} k^\beta = k_\alpha + 0 \neq 0$.

(does not work, because ~~to~~ we loose a dimension)





In flat space



Given k^α

pick N^α (auxilliary null vector)

$N_\alpha N^\alpha = 0$; $\underline{\underline{N_\alpha k^\alpha = -1}}$ (for a chosen normalization of N^α)

claim: $\boxed{h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta}$ Not unique! Because N^α is not unique.

$$h_{\alpha\beta} k^\beta = k_\alpha + k_\alpha(-1) + 0 = 0.$$

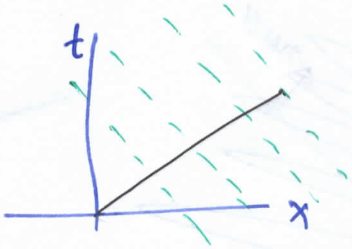
$$h_{\alpha\beta} N^\beta = 0.$$

$$\boxed{\begin{aligned} h^\alpha{}_\beta h^\beta{}_\gamma &= h^\alpha{}_\gamma \\ h^\mu{}_\mu &= 2 \end{aligned}}$$

Flat Space

$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$

$k^\alpha = (1,1,0,0)$



normalization constant.

$N^\alpha = c(1,-1,0,0)$

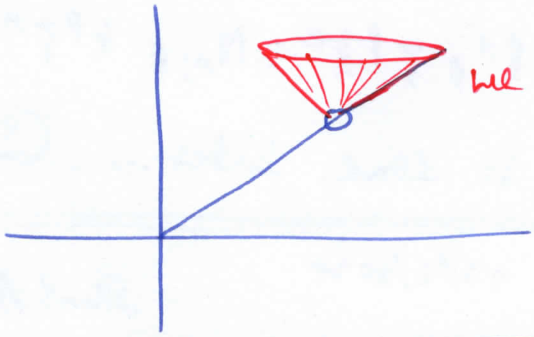
$g_{\alpha\beta} k^\alpha N^\beta = c(-1-1) = -2c = -1$

$N^\alpha = \frac{1}{2}(1, -1, 0, 0)$

$h_{\alpha\beta} = \mathrm{diag}\,[0, 0, r^2, r^2\sin^2\theta]$

This is the transverse 2D space we were looking for.

~~N^α~~ N is not unique.



we have whole null cone.

$B_{\alpha\beta} = k_{\alpha;\beta}$ $\quad$ $\boxed{\xi^\alpha{}_{;\beta}\, k^\beta = B^\alpha{}_\beta\, \xi^\beta}$

One think we have to work on is the fact that ~~..~~, ~~..~~ we should select a deviation vector between neighbouring geodesics that will be purely transverse; which ~~..~~ will have no component along K & N.

Project ξ^α into transverse ~~..~~ subspace:

$$\tilde{\xi}^\alpha = h^\alpha{}_\beta\, \xi^\beta$$

Transverse projection operator.

lets find evolution equation for $\tilde{\xi}^\alpha$.

$$\tilde{\xi}^\alpha = (\delta^\alpha{}_\mu + k^\alpha N_\mu + N^\alpha k_\mu)\xi^\mu$$

$$= \xi^\alpha + (N_\mu \xi^\mu) k^\alpha$$

$$\Rightarrow \boxed{\tilde{\xi}^\alpha = \xi^\alpha + (N_\mu \xi^\mu) k^\alpha}$$

$\tilde{\xi}^\alpha$ is orthogonal to both K & N.

so: $\tilde{\xi}$ is transverse.

$\tilde{\xi}^\alpha$ is transverse deviation vector.

Rate of change of transverse deviation vector.

$$\tilde{\xi}^\alpha{}_{;\beta} k^\beta = \xi^\alpha{}_{;\beta} k^\beta + (N_\mu \xi^\mu)_{;\beta} k^\beta k^\alpha + (N_\mu \xi^\mu) \cancel{k^\alpha{}_{;\beta} k^\beta}^{\,0}$$

$$(N_{\mu;\beta} k^\beta) \xi^\mu + N_\mu \xi^\mu{}_{;\beta} k^\beta$$

$$= B^\alpha{}_\beta \xi^\beta + N_\mu B^\mu{}_\beta \xi^\beta k^\alpha + N_{\mu;\beta} k^\beta \xi^\mu k^\alpha$$

.... will be done later...

Lec-8|| 23/4/2020   Shoaib Akhtar

Flat space-time $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$.

$k^\alpha = (1, 1, 0, 0)$

$k_\alpha = (-1, 1, 0, 0) = -\partial_\alpha \Phi$; $\Phi = t - x$

use Φ as coordinate ; $dt = d\Phi + dx$

so; $ds^2 = -d\Phi^2 - 2\, d\Phi\, dx + dy^2 + dz^2$

This is the full metric of flat space-time expressed in a new coordinate system that's adapted to my adaptive null direction.

∴ so; If we want to know induced metric on Φ = constant ; just set $d\Phi = 0$

$$\Rightarrow \; ds^2 \Big|_{\Phi = c} = dy^2 + dz^2$$

Now we dont expect to have cross-sectional volume ; but cross sectional area.

$N^\alpha = \frac{1}{2}(1, -1, 0, 0)$; $h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$.

$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$$

$$\xi^\alpha{}_{;\beta} k^\beta = B^\alpha{}_\beta \xi^\beta \quad ; \quad B_{\alpha\beta} = k_{\alpha;\beta}$$

$$k^\alpha B_{\alpha\beta} = 0 = B_{\alpha\beta} k^\beta = 0 .$$

$$\therefore \quad k^\alpha k_{\alpha;\beta} = \frac{1}{2}(h^\alpha h_\alpha)_{;\beta} = 0$$

$$k_{\alpha;\beta} k^\beta = 0 \quad (\text{geodesic})$$

But: $\left.\begin{array}{l} B_{\alpha\beta} N^\beta \neq 0 \\ N^\alpha B_{\alpha\beta} \neq 0 \end{array}\right\}$

$$\tilde{\xi}^\alpha = h^\alpha{}_\beta \xi^\beta = \text{transverse part of } \xi^\alpha .$$

$$\tilde{\xi}^\alpha{}_{;\beta} k^\beta = (h^\alpha{}_\mu \xi^\mu)_{;\beta} k^\beta$$

$$= \underbrace{h^\alpha{}_{\mu;\beta} k^\beta \xi^\mu} + h^\alpha{}_\mu B^\mu{}_\beta \xi^\beta$$

$$(\delta^\alpha{}_\mu + k^\alpha N_\mu + N^\alpha k_\mu)_{;\beta} k^\beta \xi^\mu$$

$$= k^\alpha (N_{\mu;\beta} k^\beta) \xi^\mu + N^\alpha (\underbrace{\quad}_{=0})$$

$$= k^\alpha (N_{\mu;\beta} k^\beta) \xi^\mu$$

$$\Rightarrow \tilde{\xi}^\alpha{}_{;\beta} k^\beta = h^\alpha{}_\mu B^\mu{}_\beta \xi^\beta + (\xi^\mu N_{\mu;\beta} k^\beta) k^\alpha$$

(we are trying to define transverse velocity between two neighbouring geodesics.)

The transverse relative velocity.

$$(\widetilde{\tilde{\xi}_{;\beta} k^\beta}) \equiv h^\alpha{}_\mu (\tilde{\xi}^\mu{}_{;\beta} k^\beta)$$

$$= h^\alpha{}_\mu h^\mu{}_\nu B^\nu{}_\beta \xi^\beta$$

$$= h^\alpha{}_\nu B^\nu{}_\beta \xi^\beta$$

$$= h^\alpha{}_\nu B^\nu{}_\beta \tilde{\xi}^\beta$$

$$= h^\alpha{}_\nu B^\nu{}_\beta \left(h^\beta{}_\gamma \tilde{\xi}^\gamma \right)$$

$$= h^\alpha{}_\nu h^\beta{}_\gamma B^\nu{}_\beta \tilde{\xi}^\gamma$$

$$= \tilde{B}^\alpha{}_\gamma \tilde{\xi}^\gamma$$

so;

$$\boxed{\begin{aligned} \widetilde{\left(\tilde{\xi}^\alpha{}_{;\beta} k^\beta \right)} &= \tilde{B}^\alpha{}_\beta \tilde{\xi}^\beta \\ \tilde{B}_{\alpha\beta} &= h_\alpha{}^\mu h_\beta{}^\nu B_{\mu\nu} \end{aligned}}$$

→ Transverse relative velocity in null case

$$B_{\mu\nu} = k_{\mu;\nu}$$

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$ (irreducible parts)

$$\theta = h^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} \tilde{B}_{\alpha\beta} = \dots \dots = k^\alpha{}_{;\alpha}$$

$$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta} \theta$$

$$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$$

$$\theta = \frac{1}{\delta A} \frac{d}{d\lambda} \delta A$$; $\delta A \equiv$ cross-sectional area!

$$\tilde{B}_{\alpha\beta} = (\delta^\mu{}_\alpha + k^\mu N_\alpha + N^\mu k_\alpha) \underbrace{(\delta^\mu{}_\beta + k^\mu N_\beta + N^\mu k_\beta) B_{\mu\nu}}_{B_{\mu\beta} + k_\beta N^\nu B_{\mu\nu}}$$

$$= B_{\alpha\beta} + k_\beta N^\nu B_{\alpha\nu} + k_\alpha N^\mu B_{\mu\beta} + k_\alpha k_\beta N^\mu N^\nu B_{\mu\nu}$$

$$\Rightarrow \boxed{\begin{aligned} \tilde{B}_{\alpha\beta} = B_{\alpha\beta} &+ k_\alpha (N^\mu B_{\mu\beta}) + k_\beta (N^\nu B_{\alpha\nu}) \\ &+ k_\alpha k_\beta (N^\mu N^\nu B_{\mu\nu}) \end{aligned}}$$

$$\Rightarrow g^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} B_{\alpha\beta} = g^{\alpha\beta} k_{\alpha;\beta} = k^\alpha{}_{;\alpha}$$

The choice of N is not unique because it has four components & we only give two constraints.

$\therefore$ as a result ; transverse metric & selection of transverse space is not unique

$\tilde{B}$ is not unique

but $\tilde{B}^{\alpha}{}_{\alpha}$ is unique.

$\therefore$ θ is independent of choice of N.

$\therefore$ σ & ω depend on choice of N. ; but there will be still some invariance properties.

Frobenius: Congruence is hypersurface "orthogonal" (surface forming)

if $k_\alpha = -\mu \, \partial_\alpha \Phi$

surface of constant Φ.

$$k_{\alpha;\beta} = -\mu \Phi_{;\alpha\beta} - \partial_\alpha \Phi \, \partial_\beta \mu$$

$$B_{\alpha\beta} = -\mu \Phi_{;\alpha\beta} + \frac{1}{\mu} k_\alpha \partial_\beta \mu$$

$$\cancel{B_{\alpha\mu} N^\mu = -\mu \Phi_{;\alpha\mu} N^\mu + \frac{1}{\mu} k_\alpha (\partial_\mu \beta N^\beta)}$$

$$B_{\alpha\mu} N^\mu = -\mu \Phi_{;\alpha\mu} N^\mu + \frac{1}{\mu} k_\alpha \partial_\mu \mu N^\mu$$

$$N^\mu B_{\mu\beta} = -\mu \Phi_{;\mu\beta} N^\mu - \frac{1}{\mu} \partial_\beta \mu$$

$$B_{\mu\nu} N^\mu N^\nu = -\mu \Phi_{;\mu\nu} N^\mu N^\nu - \frac{1}{\mu} (\partial_\nu \mu N^\nu)$$

$$\tilde{B}_{\alpha\beta} = -\mu \Phi_{;\alpha\beta} + \frac{1}{\mu} k_\alpha \partial_\beta \mu$$
$$- \mu k_\alpha \Phi_{;\nu\beta} N^\nu - \frac{1}{\mu} k_\alpha \partial_\beta \mu$$
$$- \mu k_\beta \Phi_{;\alpha\mu} N^\mu + \frac{1}{\mu} k_\alpha k_\beta (\partial_\mu \mu N^\mu)$$

$$-\mu k_\alpha k_\beta \Phi_{;\mu\nu} N^\mu N^\nu - \frac{1}{\mu} k_\alpha k_\beta (\partial_\mu \mu N^\nu)$$

$$\Rightarrow \tilde{B}_{\alpha\beta} = -\mu \Phi_{;\alpha\beta} - \mu k_\alpha \Phi_{;\mu\beta} N^\mu - \mu k_\beta \Phi_{;\alpha\mu} N^\mu$$
$$- \mu k_\alpha k_\beta \Phi_{;\mu\nu} N^\mu N^\nu$$

$$= \tilde{B}_{(\alpha\beta)} \qquad \text{so; } \omega_{\alpha\beta} = 0$$

Hypersurface orthogonality $\Longleftrightarrow$ $\omega_{\alpha\beta} = 0$.

k^α is both normal & tangent to hypersurface

Raychaudhri Equation

From time like case $\longrightarrow$ null case

$$\frac{d\theta}{d\lambda} = -B^{\alpha\beta} B_{\beta\alpha} - R_{\alpha\beta} k^\alpha k^\beta$$

we can check; $\tilde{B}^{\alpha\beta} \tilde{B}_{\beta\alpha} = B^{\alpha\beta} B_{\beta\alpha}$

$$\Rightarrow \boxed{\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta} k^\alpha k^\beta}$$

~~Situations where null surface is~~

~~Focusing Theorem~~

Hypersurface $\Phi = \text{constant}$

normal to hypersurface $\propto \partial_\alpha \Phi$

null surface $\equiv$ normal is null

: $g^{\alpha\beta}\partial_\alpha \Phi \partial_\beta \Phi = 0$

Definition.

Focusing Theorem

① Congruence that's hypersurface orthogonal $\Longleftrightarrow$ $\omega_{\alpha\beta} = 0$

② Null Energy Condition $R_{\alpha\beta} k^\alpha k^\beta \geq 0$

$$R_{\alpha\beta} k^\alpha k^\beta = 8\pi \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) k^\alpha k^\beta$$
$$= 8\pi T_{\alpha\beta} k^\alpha k^\beta$$

$\Rightarrow \frac{d\theta}{d\lambda} \leq 0$

θ initially < 0 $\Longrightarrow$ ~~cross~~ crossing (Caustic formation)

θ is independent of N

σ & ω are not independent of N

$\sigma^{\alpha\beta}\sigma_{\alpha\beta}$, $\omega^{\alpha\beta}\omega_{\alpha\beta}$ are independent of N.

Statement of hyper surface orthogonality is independent of N ; $\omega^{\alpha\beta} = 0$ independent of N.

$\omega^2 = 0$ if $\omega = 0$ (this is independent of N)

Chapter 2 ends

Advanced General Relativity — 23/4/2020 — Pg 38

Lec 9: Ch 3 Hypersurfaces — Shoaib Akhtar

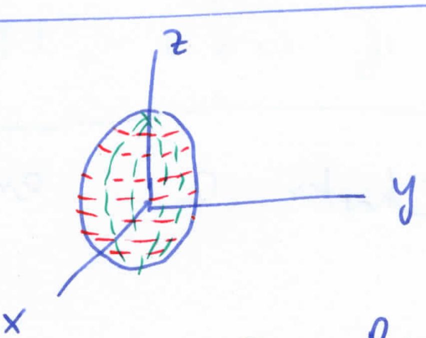
H/w 2: 2.5: #6, #8
3.13: #1, #2

Hypersurfaces

- description
- integration
- Intrinsic & extrinsic geometric
- Initial value - problem

⋮

2 sphere in flat space



$$\Phi(x,y,z) = x^2+y^2+z^2-R^2 = 0.$$

$$\vec{x} = [R\sin\theta\cos\phi,\ R\sin\theta\sin\phi,\ R\cos\theta] \sim$$ Parametric representation (requires choice of intrinsic coordinates)

normal: $\vec{n} \propto \vec{\nabla}\Phi = [2x, 2y, 2z]$

$\vec{n} = [x/R,\ y/R,\ z/R]$ (normalize it)

tangent vectors:

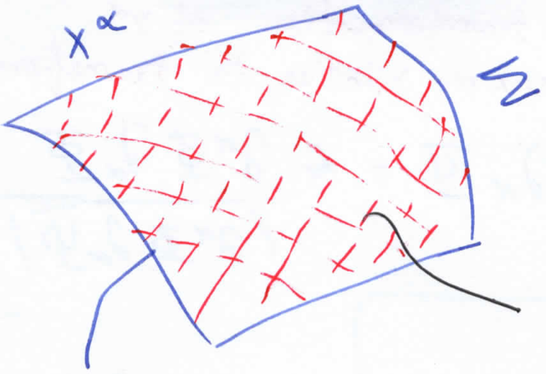
$$\vec{e}_\theta = \frac{\partial \vec{x}}{\partial\theta} = [R\cos\theta\cos\phi,\ R\cos\theta\sin\phi,\ -R\sin\theta]$$

$$\vec{e}_\phi = \frac{\partial \vec{x}}{\partial\phi} = [-R\sin\theta\sin\phi,\ R\sin\theta\cos\phi,\ 0]$$

we dont normalize it because there inner product gives us metric

Spacetime Hypersurface in spacetime

$$\Phi(x^\alpha) = 0$$



$y^a \equiv$ intrinsic coordinates

$\Phi = 0$

$\cancel{x^\alpha = x^\alpha(y^a)}$

$x^\alpha = x^\alpha(y^a)$; parametric description in terms of intrinsic coordinates

Spacelike or Timelike hyper surface

$n_\alpha \propto \partial_\alpha \Phi \Rightarrow$ Then normalize.

by convention, n^α points in the direction of increasing Φ

$\therefore n^\alpha \partial_\alpha \Phi > 0$

$$\Rightarrow \quad n_\alpha = \frac{\epsilon\, \partial_\alpha \Phi}{|g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi|} \qquad \therefore \cancel{\epsilon = n_\alpha n^\alpha = \pm 1}$$

$$\varepsilon \equiv n_\alpha n^\alpha = \begin{cases} -1 & \text{for } \Sigma \text{ spacelike (normal timelike)} \\ +1 & \text{for } \Sigma \text{ timelike (normal spacelike)} \end{cases}$$

Tangent vectors: $e^\alpha_a = \dfrac{\partial x^\alpha}{\partial y^a}$

Displacements on Σ : $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$

$$\rightarrow g_{\alpha\beta} \left(\frac{\partial x^\alpha}{\partial y^a} dy^a\right)\left(\frac{\partial x^\beta}{dy^b} dy^b\right)$$

$$= (g_{\alpha\beta}\, e^\alpha_a e^\beta_b)\, dy^a dy^b$$

Induced metric on Σ : h_{ab}

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$h_{ab} = g_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b}$ → scalars relative to transformation of x^{α}
→ tensor relative to transformation of y^a.

$$n^{\alpha} = \frac{\epsilon \nabla^2 \Phi}{| \quad |} \Rightarrow n^{\alpha}\partial_{\alpha}\Phi = \epsilon \frac{\partial^{\alpha}\Phi\, \partial_{\alpha}\Phi}{|\partial^{\alpha}\Phi\, \partial_{\alpha}\Phi|} > 0$$

Check ~~both~~ for both cases.

Basis of vectors : n^{α}, e^{α}_{a}

$$g_{\alpha\beta} = \underbrace{\varepsilon n_{\alpha} n_{\beta}}_{\text{a piece normal to hypersurface } \Sigma} + \underbrace{h_{\alpha\beta}}_{\text{a piece tangent to } \Sigma}$$ (we decompose metric into two pieces)

$h_{\alpha\beta} n^{\alpha} = 0.$

$$g_{\alpha\beta} n^{\alpha} n^{\beta} = \varepsilon \underbrace{(n_{\alpha} n^{\alpha})}_{\varepsilon} \underbrace{(n_{\beta} n^{\beta})}_{\varepsilon} + \cancel{h_{\alpha\beta} n^{\alpha} n^{\beta}}^{0}$$

$= \varepsilon$ → for sign. ∴ we need it.

∵ because $h_{\alpha\beta}$ is tangent to Σ ; there must be its decomposition in the direction of e^{α}_{a}'s.

$h^{\alpha\beta} = A^{ab} e^{\alpha}_{a} e^{\beta}_{b}$ } → so this must hold.

we will show; A^{ab} is inverse of h_{ab}

Claim // $A^{ab} = h^{ab}$; $h^{ab} h_{bc} = \delta^{a}_{c}$

↑ Inverse intrinsic metric

Proof // $g^{\alpha\beta} e_{m\alpha} e_{n\beta} = \underline{h_{mn}} = (\varepsilon n^{\alpha} n^{\beta} + h^{\alpha\beta}) e_{m\alpha} e_{n\beta}$
$= h^{\alpha\beta} e_{m\alpha} e_{n\beta}$

∴ also ; $h_{mn} = (A^{ab} e^{\alpha}_{a} e^{\beta}_{b}) e_{m\alpha} e_{n\beta}$

$$= A^{ab} \underbrace{(e^\alpha_a e_{m\alpha})}_{\hookrightarrow h_{am}} \underbrace{(e^\beta_b e_{n\beta})}_{\hookrightarrow h_{bn}}$$

$$h_{mn} = \underbrace{(A^{ab} h_{am})}_{\hookrightarrow \delta^b_m} h_{bn} \Rightarrow \text{so; } \underline{A^{ab} = h^{ab}}.$$

$$\boxed{h^{\alpha\beta} = h^{ab} e^\alpha_a e^\beta_b}$$

$$g^{\alpha\beta} = \varepsilon n^\alpha n^\beta + h^{\alpha\beta}$$

$$\therefore h^{\alpha\beta} = h^{ab} e^\alpha_a e^\beta_b$$

$$h^{ab} h_{bc} = \delta^a_c$$

analogous to completeness relation in Q.M.

so; called Completeness relation.

Null Surface

$\Phi(x^\alpha) = 0$; $x^\alpha(y^a) = x^\alpha$

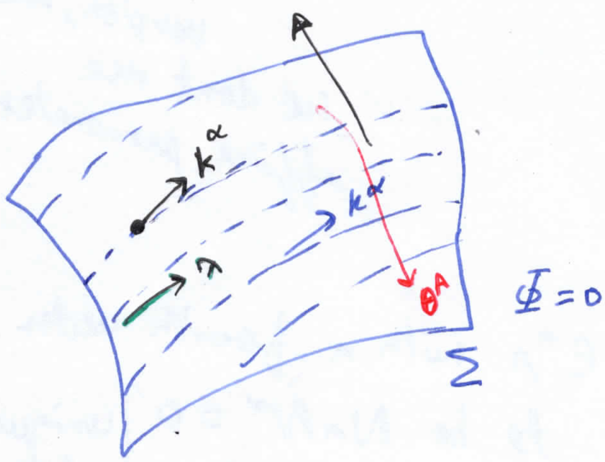
$\partial_\alpha \Phi$ a null vector $\Rightarrow$ cannot obtain a unit normal.

normal : $k_\alpha = -\partial_\alpha \Phi$; (Φ increases towards future)

$\hookrightarrow$ so; k^α is future pointing.

$k^\alpha k_\alpha = 0$

$\hookrightarrow$ k^α is also tangent to hypersurface.



k^α is tangent to null curves in Σ (geodesics!)

$\hookrightarrow$ form congruence of hypersurface orthogonal null geodesics

$\hookrightarrow$ null generators

we will pick intrinsic coordinates y^a, that are adapted to the network of null curves.

$y^1 \equiv \lambda \equiv$ running parameter on each curve

$y^2, y^3 \equiv \theta^2, \theta^3 \to \theta^A$. (These are constant on each null curve)

$y^a = (\lambda, \theta^A)$

$$e^\alpha{}_1 = \frac{\partial x^\alpha}{\partial y^1} = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_{\theta^A} = k^\alpha$$

This is actually the defination for λ.

$$e^\alpha{}_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_\lambda$$

we also impose;

$$\underline{k_\alpha e^\alpha{}_A = 0}$$

↳ restriction of the freedom to choose θ^A.

~~we impose~~ $\boxed{\cancel{k^\alpha e^\alpha_A = 0}}$

↳ If we adopt this, then we find all ~~s~~ subtlties about null is simplified.

$a^\alpha = k^\alpha{}_{;\beta} k^\beta$; the curve is geodesic if $a^\alpha = 0$ or $\underline{a^\alpha \propto k^\alpha}$ (for generalized version, when we dont use affine parameters)

as we calculate rate of change of k; along k, we are actually moving on the hypersurface.

* complete the vectorial basis k^α, $e^\alpha{}_A$ with a fourth vector.

N^α ; we choose it to be

$$\left.\begin{aligned} N_\alpha N^\alpha &= 0 \\ N_\alpha k^\alpha &= -1 \\ N_\alpha e^\alpha{}_A &= 0 \end{aligned}\right\} \text{unique solution.}$$

$a^\alpha = \underbrace{k^\alpha{}_{;\beta} k^\beta}$ → contains information from hypersurface ↳ but it can have component out of it.

$a^\alpha = \kappa k^\alpha + a^A e^\alpha_A + b N^\alpha$

$K_\alpha = -\partial_\alpha \Phi$

$a_\alpha k^\alpha = -b$; b is component of a along k.

so; $-b = k_{\alpha;\beta} k^\alpha k^\beta$

$= \frac{1}{2} (k_\alpha k^\alpha)_{;\beta} k^\beta = 0$

so; $b = 0$.

$\therefore a_\alpha e^\alpha_B = a^A \underbrace{(e^\alpha_B e_{A\alpha})}_{\neq 0}$

$\Rightarrow a_\alpha e^\alpha_\beta = k_{\alpha;\beta} e^\alpha_\beta k^\beta = a^A (e^\alpha_\beta e_{A\alpha})$

$k_{\alpha;\beta} = -\Phi_{;\alpha\beta} = -\Phi_{;\beta\alpha} = k_{\beta;\alpha}$

so; $a^A (e^\alpha_\beta e_{A\alpha}) = k_{\beta;\alpha} k^\beta e^\alpha_\beta = \underbrace{\frac{1}{2} (k_\beta k^\beta)_{;\alpha} e^\alpha_A}_{=0}$

$\Rightarrow \boxed{a^A \cdot (e^\alpha_\beta e_{A\alpha}) = 0}$

also; $a_\alpha N^\alpha = -\kappa$

$= k_{\alpha;\beta} N^\alpha k^\beta$

$= k_{\beta;\alpha} k^\beta N^\alpha$

$= \frac{1}{2} (k_\beta k^\beta)_{;\alpha} N^\alpha$

$= \frac{1}{2} \underbrace{(\partial_\beta \Phi \partial^\beta \Phi)}_{;\alpha} N^\alpha$

Here we can't put this to be zero; although it is zero on Σ

↳ but we don't know what happens when we go off the Σ

So; we found that.

$$\boxed{k^\alpha{}_{;\beta}\, k^\beta = \kappa\, k^\alpha}$$

if $\kappa \neq 0$; then λ is not affine parameter

$$\kappa = -\frac{1}{2}(\partial_\beta \Phi\, \partial^\beta \Phi)_{;\alpha} N^\alpha$$

if $\kappa = 0$; then λ will be affine.

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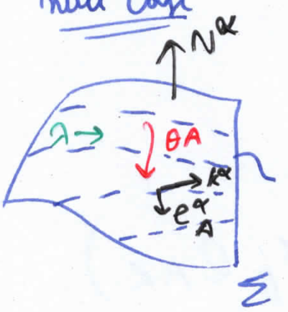
null case



$y = (\lambda, \theta^A)$

$(N^\alpha, k^\alpha, e^\alpha{}_A)$

↳ Any vector can be ~~deco~~ decomposed into these

Displacement on Σ:

$$dx^\alpha = \frac{\partial x^\alpha}{\partial \lambda} d\lambda + \frac{\partial x^\alpha}{\partial \theta^A} d\theta^A$$

$$= k^\alpha d\lambda + e^\alpha{}_A\, d\theta^A$$

For these; $ds^2 = g_{\alpha\beta}(k^\alpha d\lambda + e^\alpha{}_A d\theta^A) \times (k^\beta d\lambda + e^\beta{}_B d\theta^B)$

$$\Rightarrow ds^2 = \underbrace{(g_{\alpha\beta}\, e^\alpha{}_A e^\beta{}_B)\, d\theta^A d\theta^B}$$

↳ Something nice happened here what we would had expected; the three dimensional coordinate system that we placed on the null hypersurface; only displacement along two of these matter; & the induced metric on null Σ is effectively two dimensional.

$$\sigma_{AB} = g_{\alpha\beta}\, e^\alpha{}_A\, e^\beta{}_B$$

= Induced metric (2D)

$$ds^2 = \sigma_{AB}\, d\theta^A d\theta^B$$

where; $\sigma_{AB} = g_{\alpha\beta}\, e^\alpha{}_A\, e^\beta{}_B$

Completeness relation for metric.

$$g^{\alpha\beta} = -k^\alpha N^\beta - N^\alpha k^\beta + \underbrace{\sigma^{AB} e^\alpha{}_A e^\beta{}_B}$$

Transverse part of metric

$$\sigma^{AB}\sigma_{BC} = \delta^A{}_C .$$

~~check~~ check!! $g^{\alpha\beta} k_\alpha N_\beta = -1$; .. also others ...

In the ~~contex~~ context of congruences;

$$h^{\alpha\beta} = g^{\alpha\beta} + k^\alpha N^\beta + N^\alpha k^\beta$$
$$= \sigma^{AB} e^\alpha{}_A e^\beta{}_B .$$

Integration ~~for time like & space like Σ!~~

In spacetime

$$dV = \sqrt{-g}\, d^4x$$

Gauss Theorem

$$\int_V \vec{\nabla}\cdot\vec{A}\, d^3x = \oint_{\partial V} \vec{A}\cdot\underbrace{\vec{n}\, da}_{d\vec{a}}.$$

$d\vec{a}$ direct surface element.

convention

~~on~~

on Hypersurface. Σ

we have induced metric h_{ab}:

$$d\Sigma = \sqrt{|h|}\, d^3y$$

Undirected surface element

put absolute sign so as to capture both space like & timelike surfaces.

Directed surface element: $n_\mu d\Sigma$ $\quad (n_\mu n^\mu = \varepsilon)$

n_μ is normal to Σ.

Directed surface element in all cases.

$$d\Sigma_\mu = \varepsilon_{\mu\alpha\beta\gamma}\, e^\alpha{}_1 e^\beta{}_2 e^\gamma{}_3\, d^3y$$

$\Rightarrow$ has good null limit.

$$\epsilon_{\mu\alpha\beta\gamma} = \sqrt{-g}\,[\mu\ \alpha\ \beta\ \gamma]$$

↑ volume four form for our space time

claim ; for timelike & spacelike case
$d\Sigma_\mu$ should point along normal.

check ; $d\Sigma_\mu e^\mu{}_j = 0$; $j = 1, 2, 3$.

so; $d\Sigma_\mu = f \cdot n_\mu$
↑ some scalar quantity.

$f = \varepsilon \cdot \varepsilon_{\mu\alpha\beta\gamma}\, n^\mu e^\alpha{}_1 e^\beta{}_2 e^\gamma{}_3\, d^3y$

note
$\frac{1}{\varepsilon} = \varepsilon$
Hahaha.

ex
suppose ; $ds^2 = -dt^2 + h_{ab}\,dy^a dy^b$

~~take~~ take Σ : $t = \text{constant}$; $n^\alpha = (1, 0, 0, 0)$

$x^0 = t$, $x^a = y^a$ (choose this coordinate system)

$$e^\alpha{}_1 = \frac{\partial x^\alpha}{\partial y^1} = \frac{\partial y^\alpha}{\partial y^1} = (0, 1, 0, 0)$$

$$\sqrt{-g} = \sqrt{h}$$

so; we get ;

$$f = -\sqrt{-g} \cdot [\mu\ \alpha\ \beta\ \gamma]\, n^\mu e^\alpha{}_1 e^\beta{}_2 e^\gamma{}_3 d^3y$$

$$= -\sqrt{h}\, \underbrace{[0\ 1\ 2\ 3]}_{2}\, d^3y$$

$$= -\sqrt{h}\, d^3y$$

For timelike or spacelike Σ,

$$\boxed{d\Sigma_\mu = \varepsilon \cdot n_\mu \sqrt{|h|}\, d^3y}$$

Null case

$$d\Sigma_\mu = \varepsilon_{\mu\alpha\beta\gamma} e^\alpha{}_1 e^\beta{}_2 e^\gamma{}_3 \, d^3y$$

apply this to adapted coordinate system $y^a = (\lambda, \theta^A)$.

$$e^\alpha{}_1 = k^\alpha$$

$$d^3y = d\lambda \, d^2\theta$$

so; $d\Sigma_\mu = \varepsilon_{\mu\alpha\beta\gamma} k^\alpha e^\beta{}_2 e^\gamma{}_3 \, d\lambda \, d^2\theta$

$$= \underbrace{(\varepsilon_{\mu\alpha\beta\gamma} e^\beta{}_2 e^\gamma{}_3 \, d^2\theta)}_{dS_{\mu\alpha}} k^\alpha d\lambda$$

integrating along the generators.

something along null direction.

This is infact notion of surface element on a two-dimensional surface.

$$\boxed{\begin{aligned} d\Sigma_\mu &= dS_{\mu\alpha} k^\alpha \, d\lambda \\ dS_{\mu\nu} &= \varepsilon_{\mu\nu\beta\gamma} e^\beta{}_2 e^\gamma{}_3 \, d^2\theta \end{aligned}}$$

an element of 2-surface in transverse subspace.

$dS_{\mu\nu}$ is antisymmetric

& $e^\mu{}_2 \, dS_{\mu\nu} = 0$

$e^\mu{}_3 \, dS_{\mu\nu} = 0$

so; $dS_{\mu\nu}$ cannot point in directions of e_2 or e_3.

so; $dS_{\mu\nu} = f \, k_{[\mu} N_{\nu]}$

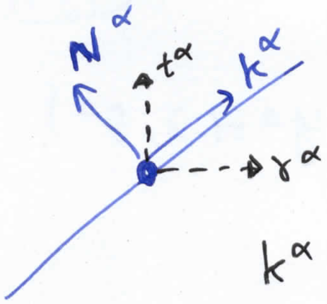
$$f = N^\mu k^\nu dS_{\mu\nu}$$

$\sigma \equiv \det[\sigma_{AB}]$

Claim!! $f \propto \sqrt{\sigma}$

so:

$$\boxed{dS_{\mu\nu} = 2 k_{[\mu} N_{\nu]} \sqrt{\sigma} \, d^2\theta}$$



k^α, N^α are null normals to transverse manifold.

$$k^\alpha = \frac{1}{\sqrt{2}}\left(t^\alpha + r^\alpha\right)$$
$$N^\alpha = \frac{1}{\sqrt{2}}\left(t^\alpha - r^\alpha\right)$$

t^α, r^α are more conventional time like & spacelike normals.

$$d\Sigma_\mu = (k_\mu N_\nu - N_\mu k_\nu)\sqrt{\sigma}\, d^2\theta\, k^\nu d\lambda$$
$$= -k_\mu \sqrt{\sigma}\, d^2\theta\, d\lambda$$

so; $$d\Sigma_\mu = -\underbrace{k_\mu}_{\text{direction.}} \sqrt{\sigma}\, \underbrace{d^2\theta}_{\text{integral over 2d subspaces}} \cdot \underbrace{d\lambda}_{\text{integration over the generators themselves running up along the generators.}}$$

Gauss Theorem in 4D spacetime

$$\boxed{\int_V A^\alpha{}_{;\alpha}\sqrt{-g}\, d^4x = \oint_{\partial V} A^\alpha\, d\Sigma_\alpha}$$

$\Sigma = \partial V$

↳ a clever proof in the book. The authors thesis advisor taught him this proof. (Historical fact)

Stokes Theorem:

$$\int_\Sigma B^{\alpha\beta}{}_{;\beta}\, d\Sigma_\alpha = \frac{1}{2}\oint_{\partial\Sigma} B^{\alpha\beta}\, dS_{\alpha\beta}$$

→ because symmetric part will be killed here.

$B^{\alpha\beta} \equiv$ antisymmetric.

Conservation Statement

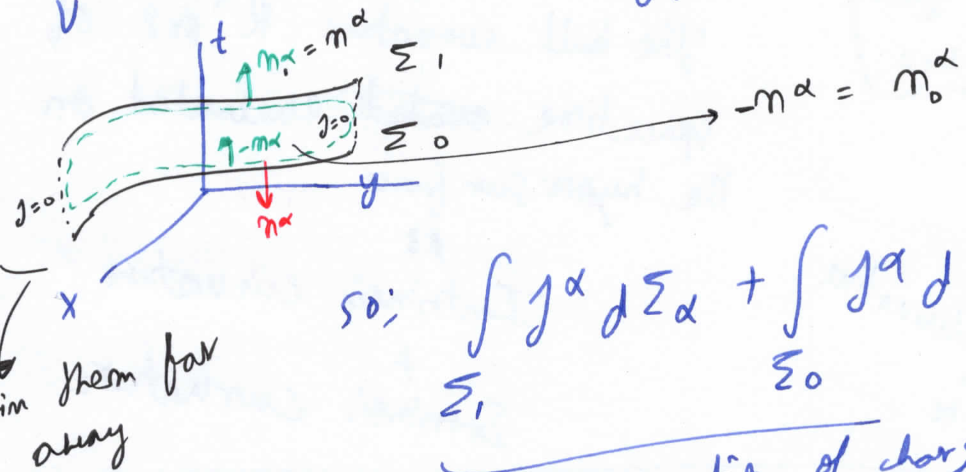
$$\boxed{J^{\alpha}{}_{;\alpha} = 0}$$

if $j^{\alpha} = T^{\alpha}{}_{\beta}\,\xi^{\beta}$; (ξ^{α} killing vector)

$$J^{\alpha}{}_{;\alpha} = \cancel{T^{\alpha}{}_{\beta;\mu}}^{\,0}\,\xi^{\beta} + T^{\alpha}{}_{\beta}\underbrace{\xi^{\beta}{}_{;\alpha}}_{0} = 0$$

due to conservation of energy momentum tensor

so; $$\int_V J^{\alpha}{}_{;\alpha}\sqrt{g}\,d^4x = 0 = \oint_{\partial V} J^{\alpha}\,d\Sigma_{\alpha}$$



so; $$\underbrace{\int_{\Sigma_1} J^{\alpha}\,d\Sigma_{\alpha} + \int_{\Sigma_0} J^{\alpha}\,d\Sigma_{\alpha} = 0}$$

equation of charge conservation

$$\cancel{0 = \int_{\Sigma_1} J^{\alpha}(-n_{1\alpha}\sqrt{h}\,d^3y)}$$

+

$$0 = \int_{\Sigma_1} J^{\alpha}(-n_{1\alpha}\sqrt{h}\,d^3y) + \int_{\Sigma_0} J^{\alpha}(+n_{0\alpha}\sqrt{h}\,d^3y)$$

$$\boxed{Q = \int_{\Sigma} J^{\alpha}\,n_{\alpha}\sqrt{h}\,d^3y}$$

Independent of Σ ; & so Q conserved.

There is no version for $T^{\alpha\beta}{}_{;\beta} = 0$ turning into global conserved quantity in G.R. (analogous to charge)

↪ you need killing vector.

Lec 11

23/4/2020

— Shoaib Akhtar.

Intrinsic & extrinsic geometries of hypersurfaces

$y^a, h_{ab} \to \Gamma^a_{bc}$

$\downarrow$

$R^a{}_{bcd}$

$D_a h_{bc} = 0$

$h_{bc|a} = 0$

notation for covariant derivative w.r.t. Γ^a_{bc}

$x^\alpha ; g_{\alpha\beta} \to \Gamma^\alpha_{\beta\gamma} \to R^\alpha{}_{\beta\gamma\delta}$

↪ $D_\alpha g_{\beta\gamma} = 0$ or $g_{\beta\gamma;\alpha} = 0$

The bulk curvature $R^\alpha{}_{\beta\gamma\delta}$ of spacetime ~~evaluat~~ evaluated on the hypersurface

$\Downarrow$

Intrinsic curvature

$+$

Extrinsic curvature.

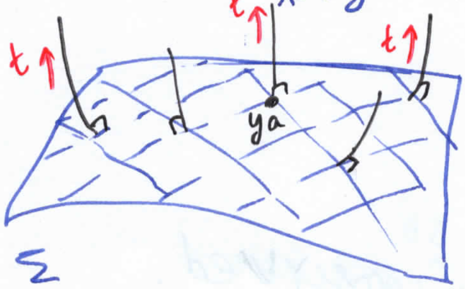
Assume Σ is spacelike (for concreteness)

Σ not null.

Gaussian normal coordinates (x^α)

(covariant w.r.t. $y^a \to y'^a$)

$x^a = y^a$. (we do)



~~taking~~ launch a curve orthogonal to Σ at each point y^a.

Each curve carries the label y^a; and t = proper time on each curve, with $t = 0$ set at Σ.

G.N.C. $\equiv x^\alpha$: $\begin{cases} x^0 = t \\ x^a = y^a \end{cases}$

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$
$$= -dt^2 + g_{ab}(t, x^a)\, dx^a dx^b$$

(dont have cross term $dt\,dx^c$ because the curves  were orthogonal to Σ)

$$g_{ab}(t=0, x^a) = h_{ab}$$

$$\boxed{\begin{aligned} ds^2 &= -dt^2 + g_{ab}(t, x^a)\, dx^a dx^b \\ g_{ab}&(t=0, x^a) = h_{ab} \end{aligned}}$$

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & g_{ab} \end{pmatrix} \; ; \; g^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & g^{ab} \end{pmatrix}$$

$${}^4\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}\left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}\right)$$

↳ for the bulk.

$$\begin{cases} {}^4\Gamma^t_{ab} = \frac{1}{2}\partial_t g_{ab} \\ {}^4\Gamma^a_{tb} = \frac{1}{2} g^{am}\,\partial_t g_{mb} \\ {}^4\Gamma^a_{bc} = \frac{1}{2} g^{am}\left(g_{mb,c} + g_{mc,b} - g_{bc,m}\right) \end{cases}$$

on $\Sigma\,(t=0)$

$${}^4\Gamma^t_{ab} = K_{ab} = \frac{1}{2}\partial_t g_{ab}(t=0)$$

$K_{ab} \Rightarrow$ "Extrinsic curvature"

↳ Its a 3-dimensional tensor of rank 2

$${}^4\Gamma^a_{tb} = h^{am} K_{mb} \equiv K^a{}_b$$

$${}^4\Gamma^a_{bc} = \Gamma^a_{bc}$$

K_{ab} scalar in bulk space time.
but tensor on hyper surface.

K_{ab} hyper surface tensor; (transform as such under $y^a \to y'^a$)
but space time scalar.

---

on Σ $(t=0)$

$${}^4R_{tatb} = -\frac{1}{2}\partial_t^2 g_{ab}(t=0) + K_{am}K^m{}_b$$

$${}^4R_{tabc} = D_c K_{ab} - D_b K_{ac}$$

$$\underbrace{{}^4R_{abcd}}_{\text{Bulk Curvation}} = \underbrace{R_{abcd}}_{\text{Intrinsic curvature}} + \underbrace{K_{ac}K_{bd} - K_{ad}K_{bc}}_{\text{Extrinsic curvature.}}$$

Tensorial equation.

Called Gauss - Codazzi equation.

Einstein tensor on Σ (at $t=0$)

$k = k^a{}_a = h^{ab}k_{ab}$

$${}^4G_{tt} = \frac{1}{2}\left({}^3R - K^{ab}K_{ab} + K^2\right)$$

$${}^4G_{ta} = D_b K^b{}_a - D_a K$$

$${}^4G_{ab} = G_{ab} + \frac{1}{2}\partial_t^2 g_{ab}(t=0) - \frac{1}{2}h_{ab}\left(h^{cd}\partial_t^2 g_{cb}(t=0)\right)$$
$$- 2K_{ac}K^c{}_b + \frac{3}{2}h_{ab}\left(K^{cd}K_{cd}\right)$$
$$+ K \cdot K_{ab} - \frac{1}{2}h_{ab}K^2$$

$$D_a A_b = \partial_a A_b - \Gamma^c_{ab} A_c$$

To turn all this into expression covariant under $x^\alpha \to x^{\alpha'}$

re-introduce the vectorial basis,

$$n^\alpha \text{ (normal) } \propto -\partial_\alpha \Phi$$

$$e^\alpha{}_a = \frac{\partial x^\alpha}{\partial y^a} \text{ (tangent vector)}$$

G.N.C. : $\Phi = t = 0$; $\Rightarrow n_\alpha \overset{*}{=} (-1, 0, 0, 0)$

(The star $*$ is ~~a~~ a reminder that the equality is in specific coordinate system)

$n^\alpha \overset{*}{=} (1, 0, 0, 0)$

$e^\alpha{}_1 \overset{*}{=} (0, 1, 0, 0)$; $e^\alpha{}_2 \overset{*}{=} (0, 0, 1, 0)$; $e^\alpha{}_3 \overset{*}{=} (0, 0, 0, 1)$

$\cancel{K_{ab} \overset{*}{=} \frac{1}{2} \partial_t g_{ab}(t=0)}$

Claim!! $K_{ab} \equiv n_{(\alpha;\beta)} e^\alpha{}_a e^\beta{}_b$ } completely covariant.

$\downarrow$

scalar w.r.t. $x^\alpha \to x'^\alpha$

Two sets of coordinate freedom

① $x^\alpha \to x'^\alpha$ (keeping y^a fixed)

② $y^a \to y'^a$ (keeping x^α fixed)

RHS in G.N.C.:

$$n_{\alpha;\beta} = n_{\alpha,\beta} - {}^4\Gamma^\gamma{}_{\alpha\beta} n_\gamma$$

$$\overset{*}{=} {}^4\Gamma^t{}_{\alpha\beta}$$

$\Rightarrow n_{(\alpha;\beta)} \overset{*}{=} {}^4\Gamma^t{}_{\alpha\beta}$

$\hookrightarrow$ we recognize it to be symmetric

$$n_{(\alpha;\beta)} e^\alpha{}_a e^\beta{}_b \overset{*}{=} {}^4\Gamma^t{}_{ab} = K_{ab} \rightarrow \text{Scalars in Spacetime}$$

(equality among scalars is true in all coordinate systems)

$\hookrightarrow$ scalars in spacetime

Equality of scalars in GNC ⇒ equality in all coordinates

$$ {}^4R_{\mu\alpha\beta\gamma}\, n^\mu e^\alpha{}_a e^\beta{}_b e^\gamma{}_c = D_c K_{ab} - D_b K_{ac} $$

$$ {}^4R_{\alpha\beta\gamma\delta}\, e^\alpha{}_a e^\beta{}_b e^\gamma{}_c e^\delta{}_d = R_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc} $$

$$ {}^4G_{\mu\nu}\, n^\mu n^\nu = \frac{1}{2}\left(R - K^{ab}K_{ab} + K^2\right) $$

$$ {}^4G_{\mu\alpha}\, n^\mu e^\alpha{}_a = D_b K^b{}_a - D_a K $$

→ Gauss - Codazzi equation in covariant form.

note// we cant give covariant form for double derivative's equation in g $\partial_t^2 g_{ab}(t=0)\ldots$

24/4/2020

Lec 12 : — Shoaib Akhtar

$$ h_{ab} = g_{\alpha\beta}\, e^\alpha{}_a e^\beta{}_b . $$

$$ K_{ab} = n_{(\alpha;\beta)}\, e^\alpha{}_a e^\beta{}_b . $$

Initial Value problem in G.R.

mechanics : $m\frac{d^2x}{dt^2} = F$; $x(0)$; $\dot{x}(0)$

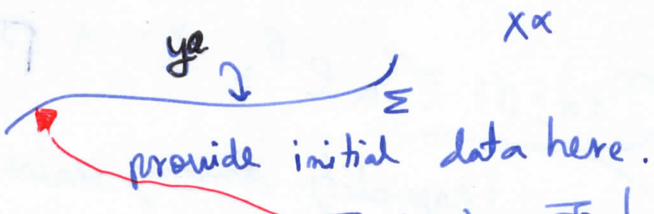
In the context of field theory;

$$ -\frac{\partial^2}{\partial t^2}\Phi + \nabla^2\Phi = \rho \Rightarrow \frac{\partial^2\Phi}{\partial t^2} = \nabla^2\Phi - \rho $$

$$ \Phi(t=0,\vec{x}),\ \partial_t\Phi(t=0,\vec{x}) $$

In curved spacetime

$$ g^{\alpha\beta}\nabla_\alpha\nabla_\beta\Phi = \rho $$

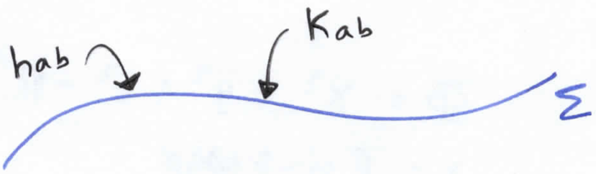


we need restriction of Φ on Σ : $\Phi(y^a) = \Phi|_\Sigma$

also: $n^\alpha \partial_\alpha \Phi(y^a)$ 

Then we have well posed I.V.P.

<u>G.R.</u> $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ (subset of this)



$\left.\begin{array}{l} g_{\alpha\beta}(t=0) \\ \partial_t g_{\alpha\beta}(t=0) \end{array}\right\} \Rightarrow$ over constraint. $\longrightarrow$ it contains information in a direction away from Σ.

$$K_{ab} = n_{(\alpha;\beta)} e^\alpha{}_a e^\beta{}_b$$
$$= \frac{1}{2} (\mathcal{L}_n g_{\alpha\beta}) e^\alpha{}_a e^\beta{}_b$$

Initial value problem in G.R. involves placing h_{ab} & K_{ab} on spacelike hypersurface Σ. $\longrightarrow$ time evolution with E.F.E. (or say subset of E.F.E.)

<u>Constraint equations</u>

$$\left.\begin{array}{ll} G_{tt} \rightarrow & R - K^{ab}K_{ab} + K^2 = 16\pi \underbrace{T_{\mu\nu} n^\mu n^\nu}_{\text{energy density on } \Sigma} \\ G_{ta} \rightarrow & D_b K^b{}_a - D_a K = 8\pi \underbrace{T_{\mu\alpha} n^\mu}_{\text{energy flux} \equiv \text{momentum density}} e^\alpha{}_a \end{array}\right\}$$

$\hookrightarrow$ restrictions on h_{ab}, K_{ab}.

$G_{ab} \rightarrow$ has flavour of $\partial_t K_{ab} = \ldots$

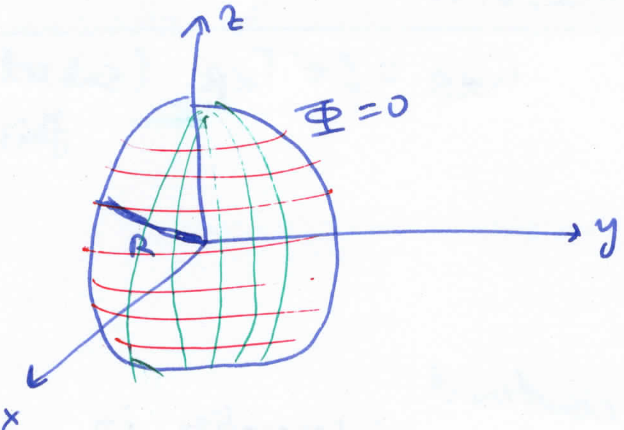
where; $\frac{1}{2}\partial_t h_{ab} = K_{ab}$

---

we dont have any restriction on choice of hypersurface Σ.

Example of K_{ab}

1. 2D sphere in 3D flat space



$x^\alpha = (x, y, z)$

$y^a = (\theta, \phi)$

$\Phi = x^2 + y^2 + z^2 - R^2 = 0$

$x = R \sin\theta \cos\phi$

$y = R \sin\theta \sin\phi$

$z = R\cos\theta$

from Φ we can get normal:

$$n_\alpha = (x/R,\ y/R,\ z/R)$$

$$e^\alpha{}_\theta = (R\cos\theta\cos\phi,\ R\cos\theta\sin\phi,\ -R\sin\theta)$$

$$e^\alpha{}_\phi = (-R\sin\theta\sin\phi,\ R\sin\theta\cos\phi,\ 0)$$

$$h_{ab} = g_{\alpha\beta}\, e^\alpha{}_a e^\beta{}_b \Rightarrow h_{\theta\theta} = R^2;\ h_{\theta\phi} = 0;\ h_{\phi\phi} = R^2\sin^2\theta$$

$$K_{ab} = n_{(\alpha;\beta)}\, e^\alpha{}_a e^\beta{}_b \Rightarrow K_{\theta\theta} = R;\ K_{\theta\phi} = 0;\ K_{\phi\phi} = R\sin^2\theta$$

$$\boxed{K_{ab} = \frac{1}{R} h_{ab}}$$

an outcome of spherical symmetry on S^2.

$K = \frac{3}{R}$

$$\boxed{R_{abcd} = \frac{1}{R^2}(h_{ac}h_{bd} - h_{ad}h_{bc})}$$

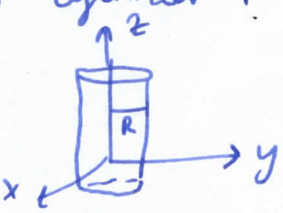
Gauss - Codazzi:

$$0 = {}^3R_{\alpha\beta\gamma\delta}\, e^\alpha{}_a e^\beta{}_b e^\gamma{}_c e^\delta{}_d$$

$$= R_{abcd} + K_{ad}K_{bc} - K_{ac}K_{bd}$$

$\hookrightarrow$ consistent

2. 2D cylinder in 3D flat space



$x^\alpha = (x, y, z)$

$y^a = (z, \phi)$

$\therefore\ \Phi = x^2 + y^2 - R^2$

$x = R\cos\phi$,
$y = R\sin\phi$,
$z = z$.

$n_\alpha = \left(\frac{x}{R}, \frac{y}{R}, 0\right)$

$e^\alpha{}_z = (0, 0, 1)$

$e^\alpha{}_\phi = (-R\sin\phi,\ R\cos\phi,\ 0)$

$$h_{ab}\, dy^a dy^b = dz^2 + R^2 d\phi^2 \equiv S^1 \times \mathbb{R}$$

$$K_{zz} = 0\ ,\ K_{z\phi} = 0\ ;\ K_{\phi\phi} = R\ ;\ \underline{K = 1/R}$$

* for cylinder there is curvature along circle not along all direction.

* In sphere; we have curvature along all direction; so; $K = 3/R$

This is basically turns 3 into 1.

$$\boxed{R_{abcd} = 0.}$$

Gauss-Codazzi also satisfied.

$\boxed{3}$ <u>F.R.W. (t = constant surface)</u>

metric of expanding universe

$$ds^2 = -dt^2 + a^2(t)\,(dx^2 + dy^2 + dz^2)$$

t = constant t_0

$\Phi = t$ - constant.

$x^\alpha = (t, x, y, z)$

$y^a = (x, y, z)$

$n_\alpha = (-1, 0, 0, 0)$

$e^\alpha{}_x = (0, 1, 0, 0)$

$e^\alpha{}_y = (0, 0, 1, 0)$

$e^\alpha{}_z = (0, 0, 0, 1)$

$$h_{ab}\, dy^a dy^b = a^2(t_0)\,(dx^2 + dy^2 + dz^2)$$

$K_{ab} = n_{(\alpha;\beta)} e^\alpha{}_a e^\beta{}_b \Rightarrow K_{xx} = a\dot{a} = K_{yy} = K_{zz}$ (pg 58)

$$\boxed{K_{ab} = \frac{\dot{a}}{a}\Big|_{t_0} h_{ab}} \qquad \boxed{K = \frac{3\dot{a}}{a}\Big|_{t_0}}$$

---

[4] Schwarzchild (t = constant)

$$ds^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2 d\Omega^2$$

$f = (1 - \frac{2M}{r})$; $x^\alpha = (t, r, \theta, \phi)$

$y^a = (r, \theta, \phi)$

$\Phi = t - t_0$: $n_\alpha = (-\sqrt{f}, 0, 0, 0)$

$e^\alpha{}_r = (0, 1, 0, 0)$

$e^\alpha{}_\theta = (0, 0, 1, 0)$

$e^\alpha{}_\phi = (0, 0, 0, 1)$

$h_{ab}\,dy^a dy^b = f^{-1} dr^2 + r^2 d\Omega^2$

$K_{ab} = n_{(\alpha;\beta)} e^\alpha{}_a e^\beta{}_b = 0$

$\hookrightarrow$ all components vanish; its not surprising because we know K_{ab} ~~was~~ ~~to~~ was supposed to calculate rate of change of metric w.r.t. time; & here we have static metric.

Ch 3 ends here

Advanced General Relativity — 24/11/2020 — Shoaib Akhtar

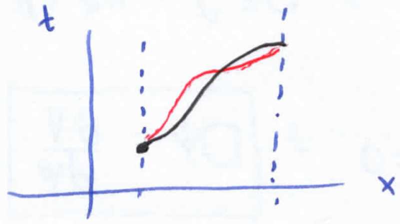
Lec 13: Ch4: Lagrangian and Hamiltonian formulations of general relativity.

H/W 3.13 : #3 4.5: #1, #2.

Lagrangian Formulation of G.R.

mechanics : $q(t)$, $\dot{q}(t)$, $L = L(q, \dot{q})$: $S[q] = \int_{t_1}^{t_2} L(q, \dot{q})\, dt$

E.O.M. of mechanics : $\delta S = 0$, $\delta q(t_1) = 0 = \delta q(t_2)$



$$\delta S = \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q}\cdot \delta q + \underbrace{\frac{\partial L}{\partial \dot{q}} \frac{d}{dt}\delta q}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\cdot\delta q\right) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q} \Big)\, dt$$

$$\Rightarrow \delta S = \frac{\partial L}{\partial \dot{q}}\delta q \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right)\delta q$$

$\delta S = 0 \Rightarrow$ Euler Lagrange Equation $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} - \dfrac{\partial L}{\partial q} = 0$

☆ Scalar field in Curved Spacetime

$\left\{ \begin{array}{l} \psi(x^\alpha) \\ \partial_\alpha \psi(x^\alpha) \end{array} \right.$: ∞ # of degree of freedom.

→ configuration variation.

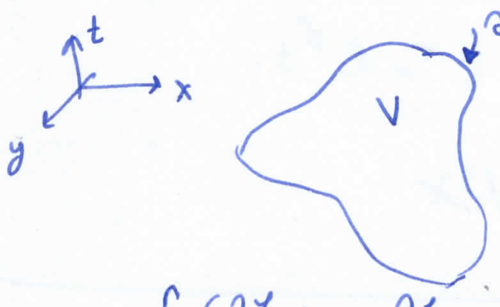
Lagrangian density : $\mathcal{L}(\psi, \partial_\alpha \psi)$ (a scalar function)

$$\boxed{S[\psi] = \int_V \mathcal{L}\cdot \sqrt{-g}\cdot d^4x}$$

→ invariant volume element.

V = fixed, ~~closed~~ finite 4D region in space time.

∂V = boundary closed 3 surface



$$\boxed{\delta\psi = 0 \quad \text{on } \partial V}$$

$$\partial_\alpha \delta\psi = \nabla_\alpha(\delta\psi)$$

metric is here assumed to be fixed.

$$0 = \delta S = \int_V \left(\frac{\partial \mathcal{L}}{\partial \psi}\delta\psi + \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\cdot \partial_\alpha \delta\psi \right)\sqrt{-g}\cdot d^4x$$

$$= \int_V \left\{ \frac{\partial \mathcal{L}}{\partial \psi}\delta\psi + \nabla_\alpha\left(\frac{\partial \mathcal{L}}{\partial \psi_{;\alpha}}\delta\psi\right) - \nabla_\alpha\left(\frac{\partial \mathcal{L}}{\partial \psi_{;\alpha}}\right)\cdot\delta\psi \right\}\sqrt{-g}\, d^4x$$

$$= \oint_{\partial V} \cancelto{0}{\left(\frac{\partial \mathcal{L}}{\partial \psi_{;\alpha}}\delta\psi\right)} d\Sigma_\alpha + \int_V \left(\frac{\partial \mathcal{L}}{\partial \psi} - \nabla_\alpha\left(\frac{\partial \mathcal{L}}{\partial \psi_{;\alpha}}\right)\right)\delta\psi\sqrt{-g}\cdot d^4x$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \psi} - \nabla_\alpha \frac{\partial \mathcal{L}}{\partial \psi_{;\alpha}} = 0 \Big\} \text{ Field Equation for field Theory.}$$

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu}\partial_\mu\psi\,\partial_\nu\psi - V(\psi)$$

$$\frac{\partial\mathcal{L}}{\partial\psi_{,\alpha}} = -\frac{1}{2} g^{\mu\nu}\left\{ \underbrace{\frac{\partial\psi_{,\mu}}{\partial\psi_{,\alpha}}}_{\delta^\alpha{}_\mu}\partial_\nu\psi + \partial_\mu\psi\cdot\underbrace{\frac{\partial_\nu\psi}{\partial_\alpha\psi}}_{\delta^\alpha{}_\nu} \right\}$$

$$= -\frac{1}{2}\left(g^{\alpha\nu}\partial_\nu\psi + g^{\mu\alpha}\partial_\mu\psi\right) = -g^{\alpha\mu}\partial_\mu\psi$$

$$D_\alpha\left(\frac{\partial\mathcal{L}}{\partial\psi_{,\alpha}}\right) = -g^{\alpha\mu}D_\alpha D_\mu\psi \equiv -\Box\psi \quad ; \quad \Box \equiv g^{\alpha\beta}D_\alpha D_\beta$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = -\frac{dV}{d\psi} = -V' \Rightarrow$$ Field equation.

$$-\Box\psi + \frac{dV}{d\psi} = 0 \Rightarrow \boxed{\Box\psi = \frac{dV}{d\psi}}$$

$\Rightarrow \cancel{\Box\psi = d\psi}$

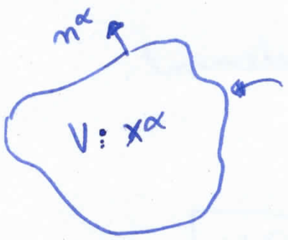
free massive field : $V = 0$

free massive " : $V = \frac{1}{2}m^2\psi^2$

Interacting field $\frac{1}{2}m^2\psi^2 + \frac{\lambda}{4!}\psi^4 = V$

G.R.|| $\quad S = S_G[g] + S_M[\psi, g]$

$$S_G = \frac{1}{16\pi}\int_V R\sqrt{-g}\cdot d^4x + \frac{1}{8\pi}\oint_{\partial V}\varepsilon\cdot K\,|h|^{1/2}d^3y$$



$\partial V: y^a$. $\quad \epsilon = \begin{cases} -1 & : \partial V \text{ spacelike} \\ +1 & : \partial V \text{ timelike}. \end{cases}$

∂V is almost nowhere null.

$$K_{ab} = n_{(\alpha;\beta)}e^\alpha{}_a e^\beta{}_b \qquad \text{(Extrinsic curvature)}$$

$$K = h^{ab}K_{ab} \quad ; \quad h_{ab} = g_{\alpha\beta}e^\alpha{}_a e^\beta{}_b.$$

$$S_M[\psi, g] = \int_V \mathcal{L}(\psi, \partial_\alpha\psi)\sqrt{-g}\,d^4x$$

Variation rules

$\delta g_{\alpha\beta} = 0$ on ∂V.

$\to \delta h_{ab} = 0$ (follows from $\delta g_{\alpha\beta} = 0$ on ∂V)

$\therefore\ \delta g_{\alpha\beta;\mu}e^\mu{}_a = 0 \qquad ((\delta g_{\alpha\beta})_{;\mu}e^\mu{}_a = 0)$

but $(\delta g_{\alpha\beta})_{;\mu}n^\mu \neq 0$ in general.

Instead of $\delta g_{\alpha\beta}$, we will work with $\delta g^{\alpha\beta}$.

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$$

$$\Rightarrow (\delta g^{\alpha\beta}) g_{\beta\gamma} + g^{\alpha\beta} (\delta g_{\beta\gamma}) = 0$$

$$\Rightarrow g_{\alpha\delta} g_{\beta\gamma} \delta g^{\alpha\beta} + \delta^{\beta}_{\delta} \delta g_{\beta\gamma} = 0$$

$$\Rightarrow \underline{\delta g_{\alpha\beta} = -g_{\alpha\nu} g_{\beta\mu} \delta g^{\nu\mu}}.$$

$$\boxed{\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} \cdot g_{\alpha\beta} \delta g^{\alpha\beta}}$$

$\Gamma^{\alpha}_{\beta\gamma}$ = not a tensor

$$\Gamma^{\alpha}_{\beta\gamma}[g] \xrightarrow{\delta g} \Gamma^{\alpha}_{\beta\gamma}[g+\delta g] = \Gamma^{\alpha}_{\beta\gamma}[g] + \delta\Gamma^{\alpha}_{\beta\gamma}$$

$$\delta\Gamma^{\alpha}_{\beta\gamma} = \Gamma[g+\delta g] - \Gamma[g]$$

$$= \text{Tensor}.$$

$$\Gamma^{\alpha'}_{\beta'\gamma'} = \frac{\partial}{\partial}\frac{\partial}{\partial}\frac{\partial}{\partial}\Gamma^{\alpha}_{\beta\gamma} + \frac{\partial^2}{\partial\gamma}(x_3).$$

& not a tensor.

<u>Bulk Term</u>

$$\delta\int_V R\sqrt{-g}\, d^4x = \delta\int_V R_{\alpha\beta} g^{\alpha\beta}\sqrt{-g} \cdot d^4x$$

$$= \int_V \left(\delta R_{\alpha\beta} g^{\alpha\beta}\sqrt{-g} + R_{\alpha\beta}\delta g^{\alpha\beta}\sqrt{-g} - \frac{1}{2}\sqrt{-g} \cdot g_{\alpha\beta}\delta g^{\alpha\beta} R\right) d^4x$$

$$= \int_V \delta R_{\alpha\beta} \cdot g^{\alpha\beta}\sqrt{-g}\, d^4x + \int_V \underbrace{\left(R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}\right)}_{G_{\alpha\beta}} \delta g^{\alpha\beta}\sqrt{-g}\, d^4x$$

$$= \int_V \delta R_{\alpha\beta} \cdot g^{\alpha\beta}\sqrt{-g}\, d^4x + \int_V G_{\alpha\beta}\,\delta g^{\alpha\beta} \cdot \sqrt{-g} \cdot d^4x$$

To calculate $\delta R_{\alpha\beta}$, work in local coordinate frame at P.

$$\Gamma^{\alpha}_{\beta\gamma}[g] \overset{*}{=} 0 \quad \text{at } P$$

$$\Gamma^{\alpha}_{\beta\gamma,\delta}[g] \neq 0$$

$$\delta\Gamma^{\alpha}_{\beta\gamma} \neq 0$$

$R = \partial\Gamma - \partial\Gamma + \Gamma^2 - \Gamma^2$

$\cancel{\delta R \sim \partial\delta}$

$R \sim \partial\delta\Gamma - \partial\delta\Gamma + 2\cancel{\Gamma}^{0}\delta\Gamma - 2\cancel{\Gamma}^{0}\delta\Gamma$

$\delta_{\alpha\beta}$

$$\delta R_{\alpha\beta} \overset{*}{=} \partial_\mu \delta\Gamma^\mu_{\alpha\beta} - \partial_\beta \delta\Gamma^\mu_{\mu\alpha}$$

covariant derivative defined w.r.t. ~~Backgr~~ Background metric

$$\boxed{\delta R_{\alpha\beta} = \nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - \nabla_\beta \delta\Gamma^\mu_{\mu\alpha}}$$

24/4/2020

Shamit Alchdhar

Lec 14

$$g^{\alpha\beta}\delta R_{\alpha\beta} = g^{\alpha\beta}\nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\beta}\nabla_\beta \delta\Gamma^\mu_{\mu\alpha}$$

$$= \nabla_\mu (g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta}) - \nabla_\beta (g^{\alpha\beta}\delta\Gamma^\mu_{\mu\alpha})$$

$$= \nabla_\mu (g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\nu_{\nu\alpha})$$

$$= \nabla_\mu (\delta V^\mu)$$

$$\therefore \quad \delta V^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\nu_{\nu\alpha}.$$

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \nabla_\mu (\delta V^\mu).$$

$$\int_V g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{-g}\, d^4x = \int_V \nabla_\mu(\delta V^\mu)\sqrt{-g}\, d^4x$$

$$= \oint_{\partial V} \delta V^\mu \underbrace{d\Sigma_\mu}_{\varepsilon n_\mu \sqrt{|h|}\, d^3y}$$

$$= \oint \varepsilon \cdot n_\mu \delta V^\mu \sqrt{|h|}\, d^3y$$

On ∂V: $\quad \delta g_{\alpha\beta} = 0 \;;\; \delta g^{\alpha\beta} = 0 \Longrightarrow \delta h_{ab} = 0$

$$g^{\alpha\beta} = \varepsilon n^\alpha n^\beta + h^{\alpha\beta} \;;\; h^{\alpha\beta} = h^{ab} e^\alpha_a e^\beta_b$$

$$\Gamma \sim \frac{1}{2} g^{-1} (\partial g) \Rightarrow \delta\Gamma \sim \frac{1}{2} \underbrace{(\delta g^{-1})}_{\text{zero}} \partial g + \frac{1}{2} g^{-1} \delta(\partial g)$$

zero because of working on boundary.

$$\delta\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} (\delta g_{\mu\beta,\gamma} + \delta g_{\mu\gamma,\beta} - \delta g_{\beta\gamma,\mu}) \quad \text{on } \partial V$$

$$g^{\beta\gamma} \delta\Gamma^{\mu}_{\beta\gamma} = \frac{1}{2} g^{\beta\gamma} g^{\mu\nu} (\delta g_{\nu\beta,\gamma} + \delta g_{\nu\gamma,\beta} - \delta g_{\beta\gamma,\nu})$$

$$\delta\Gamma^{\beta}_{\beta\gamma} = \frac{1}{2} g^{\beta\mu} (\delta g_{\mu\beta,\gamma} + \delta g_{\mu\gamma,\beta} - \delta g_{\beta\gamma,\mu})$$

$$= \frac{1}{2} g^{\beta\mu} \delta g_{\beta\mu,\gamma}$$

$$\delta V^{\mu} = \frac{1}{2} g^{\beta\gamma} g^{\mu\nu} (\delta g_{\nu\beta,\gamma} + \delta g_{\nu\gamma,\beta} - \delta g_{\beta\gamma,\nu}) - \frac{1}{2} g^{\mu\alpha} g^{\beta\nu} \delta g_{\beta\nu,\alpha}$$

$$= \frac{1}{2} g^{\beta\gamma} g^{\mu\nu} (2\delta g_{\nu\beta,\gamma} - \delta g_{\beta\gamma,\nu}) - \frac{1}{2} g^{\mu\nu} g^{\beta\gamma} \delta g_{\beta\gamma,\nu}$$

$$= g^{\beta\gamma} g^{\mu\nu} (\delta g_{\nu\beta,\gamma} - \delta g_{\beta\gamma,\nu})$$

$$\boxed{\delta V^{\mu} = g^{\alpha\beta} \cdot g^{\mu\nu} (\delta g_{\nu\alpha,\beta} - \delta g_{\alpha\beta,\nu})} \quad \text{on } \partial V$$

$$\boxed{\delta g_{\alpha\beta} = 0 \;\Downarrow\; \underbrace{\delta g_{\alpha\beta,\gamma} e^{\gamma}_{a}}_{\text{along tangential direction.}} = 0}$$

$$n_\mu \delta V^{\mu} = g^{\alpha\beta} \cdot n^{\mu} (\delta g_{\mu\alpha,\beta} - \delta g_{\alpha\beta,\mu})$$

$$= (\varepsilon n^{\alpha} n^{\beta} n^{\mu} + h^{\alpha\beta} n^{\mu}) (\delta g_{\mu\alpha,\beta} - \delta g_{\alpha\beta,\mu})$$

$$= h^{\alpha\beta} \underbrace{n^{\mu} \cdot (\delta g_{\mu\alpha,\beta}}_{\text{zero}} - \delta g_{\alpha\beta,\mu})$$

zero; gives component of derivative of δg along boundary.

$$\boxed{n_\mu \delta V^{\mu} = -h^{\alpha\beta} \cdot \underline{\delta g_{\alpha\beta,\mu} n^{\mu}}} \neq 0$$

This does not vanish. Because variation is zero on boundary; but not inward or outside boundary.

so; we finally have;

$$\int_V g^{\alpha\beta}\, \delta R_{\alpha\beta} \sqrt{-g}\, d^4x = -\oint_{\partial V} \varepsilon\, h^{\alpha\beta}\, \delta g_{\alpha\beta,\mu}\, n^\mu \sqrt{|h|}\, d^3y$$

$$\delta \int_V R\sqrt{-g}\, d^3x = \int_V G_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g}\, d^4x - \underbrace{\oint_{\partial V} \varepsilon\, h^{\alpha\beta}\, \delta g_{\alpha\beta,\mu}\, n^\mu \sqrt{|h|}\, d^3y}$$

we cannot set this to ~~be~~ be zero.

so; we need some other term in action to kill this

Boundary term: $K_{ab} = n_{(\alpha;\beta)}\, e^\alpha{}_a\, e^\beta{}_b$

$$K = (h^{ab} e^\alpha{}_a e^\beta{}_b)\, n_{(\alpha;\beta)}$$
$$= h^{\alpha\beta} .\, n_{\alpha;\beta}$$
$$= h^{\alpha\beta} .\, (n_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta}\, n_\gamma)$$

$$\delta K = -h^{\alpha\beta}(\delta\Gamma^\gamma_{\alpha\beta}) . n_\gamma$$
$$= -\frac{1}{2} h^{\alpha\beta} .\, g^{\gamma\mu} (\delta g_{\alpha\mu,\beta} + \delta g_{\beta\mu,\alpha} - \delta g_{\alpha\beta,\mu})\, n_\gamma$$
$$= -\frac{1}{2} h^{\alpha\beta} .\, (\underbrace{\delta g_{\alpha\mu,\beta}}_{\text{zero}} + \underbrace{\delta g_{\beta\mu,\alpha}}_{\text{zero}} - \delta g_{\alpha\beta,\mu})\, n^\mu$$

zero after $h^{\alpha\beta}(\ldots)$

$$\delta K = \frac{1}{2} h^{\alpha\beta} . \delta g_{\alpha\beta,\mu}\, n^\mu$$

$$\delta \oint_{\partial V} \varepsilon K |h|^{1/2} d^3y = \oint_{\partial V} \varepsilon\, \delta K\, |h|^{1/2} d^3y = \cancel{\tfrac{1}{2}\oint} \varepsilon h^{\alpha\beta}$$

$$= \frac{1}{2} \oint \varepsilon\, h^{\alpha\beta} .\, \delta g_{\alpha\beta,\mu}\, n^\mu\, |h|^{1/2} d^3y$$

~~so~~ $\cancel{\delta \int_V R\sqrt{-g}\, d^4x + 2\oint_{\partial V}}$

$$\boxed{\delta\left\{\int_V R\sqrt{-g}\,d^4x + 2\oint_{\partial V}\varepsilon K|h|^{1/2}d^3y\right\} = \int_V G_{\alpha\beta}\,\delta g^{\alpha\beta}\sqrt{-g}\,d^4x}$$

Works good.

Matter Term

$$S = S_G + S_M$$

$$S_G = \frac{1}{16\pi}\int_V R\sqrt{-g}\,d^4x + \frac{1}{8\pi}\oint_{\partial V}\varepsilon K\,|h|^{1/2}d^3y$$

$$\cancel{S_M = \int_V \mathcal{L}(\Psi, \partial_\alpha\Psi)\sqrt{-g}\,d^4x}$$

$$S_M = \int_V \mathcal{L}(\Psi, \partial_\alpha\Psi, g_{\alpha\beta})\sqrt{-g}\cdot d^4x$$

$$\underset{\text{w.r.t. } g_{\alpha\beta}}{\delta S_M} = \int_V\left(\frac{\partial\mathcal{L}}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta}\sqrt{-g} + \mathcal{L}\,(\underbrace{\delta\sqrt{-g}}_{-\frac{1}{2}\sqrt{-g}\,g_{\alpha\beta}\,\delta g^{\alpha\beta}})\right)d^4x$$

$$\delta S_M = \int_V \underbrace{\left[\frac{\partial\mathcal{L}}{\partial g^{\alpha\beta}} - \frac{1}{2}\mathcal{L}\cdot g_{\alpha\beta}\right]}_{\equiv -\frac{1}{2}T_{\alpha\beta}}\delta g^{\alpha\beta}\sqrt{-g}\cdot d^4x$$

$$\delta S_M = -\frac{1}{2}\int_V T_{\alpha\beta}\,\delta g^{\alpha\beta}\sqrt{-g}\cdot d^4x$$

where; $\boxed{T^{\alpha\beta} = -2\cdot\frac{\partial\mathcal{L}}{\partial g^{\alpha\beta}} + g_{\alpha\beta}\cdot\mathcal{L}}$

$$\delta S = \delta S_G + \delta S_M$$

$$= \int_V\left(\frac{1}{16\pi}G_{\alpha\beta} - \frac{1}{2}T_{\alpha\beta}\right)\delta g^{\alpha\beta}\sqrt{g}\,d^4x$$

so:- $\delta S = 0 \Rightarrow$ $\underline{G_{\alpha\beta} = 8\pi\,T_{\alpha\beta}}$

Einstein Field Equation.

<u>Scalar Field Theory</u> : $\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - V(\psi)$

$$\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} = -\frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial g^{\alpha\beta}} \cdot \partial_\mu \psi \partial_\nu \psi = -\frac{1}{2} \delta^\mu{}_\alpha \delta^\nu{}_\beta \partial_\mu \psi \partial_\nu \psi$$

$$= -\frac{1}{2} \partial_\alpha \psi \partial_\beta \psi$$

$$\boxed{T_{\alpha\beta} = \partial_\alpha \psi \partial_\beta \psi + g_{\alpha\beta}\left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - V\right)}$$

Energy momentum Tensor

$$T_{00} = (\partial_t \psi)^2 + \frac{1}{2}\left(-(\partial_t \psi)^2 + (\vec{\nabla}\psi \cdot \vec{\nabla}\psi)\right) + V)$$

$$= \frac{1}{2}\left((\partial_t \psi)^2 + (\vec{\nabla}^3 \psi)^2\right) + \frac{2}{2} V \quad > 0 .$$

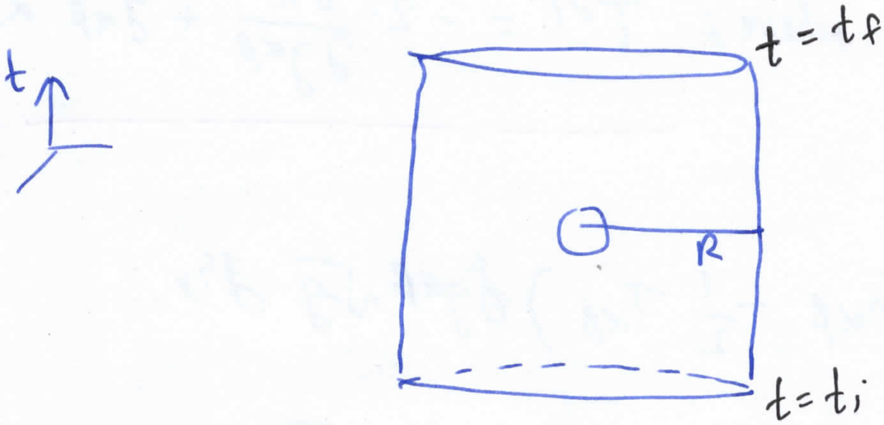
$$S_G = \frac{1}{16\pi} \int_V R \sqrt{-g}\, d^4x + \frac{2}{16\pi} \int_{\partial V} \varepsilon K |h|^{1/2} d^3y$$

lets evaluate S_G for asymptotically flat space time ; & take V to be whole spacetime.

Solutions to vaccum field equations. ~~then R = 0~~ . Evaluate ~~for~~ S_G for Schwarschild solution.

~~$S_G = \frac{1}{16\pi} \int_V$~~

~~to~~ become infinite for star or black holes.



let $R \to \infty$.

$S^2 \times \mathbb{R}$

~~first of all R = 0~~

$$\Phi = r - R = 0$$

$$y^a = (t, \theta, \phi)$$

$$h_{ab}\, dy^a dy^b = -f dt^2 + R^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

on rim of cylinder

$$f = (1 - 2M/R)$$

$$n_a = (0, f^{-1/2}, 0, 0)$$

$$K = \frac{f'}{2\sqrt{f}} + \frac{2\sqrt{f}}{R}$$

$$\sqrt{|h|} = \sqrt{f} \cdot R^2 \sin\theta$$

$$f = 1 - 2M/R$$

$$f' = \frac{2M}{R^2}$$

$$K = \frac{2}{\sqrt{f} R} \left\{ f + \frac{1}{4} R f' \right\}$$

$$= \frac{2}{R} \cdot \left(1 - \frac{M}{2R} + \dots \infty\right) \dots$$

here we see $S \to \infty$ for $R \to \infty$

so; we do renormalization.

∴ Action for flat spacetime also becomes ~~zero~~; ∞;
so; we set that to be 0 as reference
since that term comes in all action
for a reasonable solution

So, we will change defination of action.

$$\boxed{S_G = \frac{1}{16\pi} \int_V R \sqrt{-g} \cdot d^4x + \frac{1}{8\pi} \oint_{\partial V} \varepsilon (K - K_0) \sqrt{|h|}\, d^3y}$$

$K_0 \Rightarrow$ Extrinsic curvature of boundary as embedded in flat spacetime.

* S_G for flat spacetime is then zero now.

$K_0 = \frac{2}{R}$ for Schwarzschild.

so; we also tackled the problem for Schwarzschild.

S_{cr} (Swarchild) $\propto M(t_f - t_i)$

(check)

K_0 is not functional of $g_{\alpha\beta}$.

so; variation principle will be passive for K_0.

There will be no variation for K_0

so; we dont mess up action principle here.

Lec 15 :

24/3/2020

— Shomib Akhtar

Hamiltonian Formulation.

Mechanics; $L(q, \dot{q}) \rightarrow p = \frac{\partial L}{\partial \dot{q}}$; $H = p \cdot \dot{q} - L$.

* for field theory in flat spacetime

$\mathcal{L}(\psi, \partial_\alpha \psi) \longrightarrow \pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)}$

$\mathcal{H} = \pi \partial_t \psi - \mathcal{L} = T_{00}$

$H = \int \mathcal{H} d^3x$ = Total field energy.

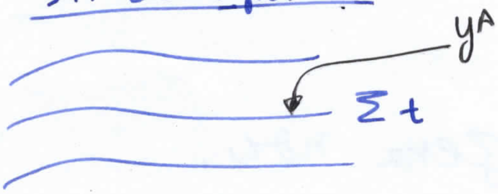
At the level of Hamiltonian; manifest lorentz invariant is lost.

Field Theory in Curved Spacetime: foliate spacetime with arbitrary spacelike hypersurface.

$\partial_t \psi \xrightarrow{\text{analogue}} n^\alpha \partial_\alpha \psi$

(normal derivatives)

3+1 decomposition

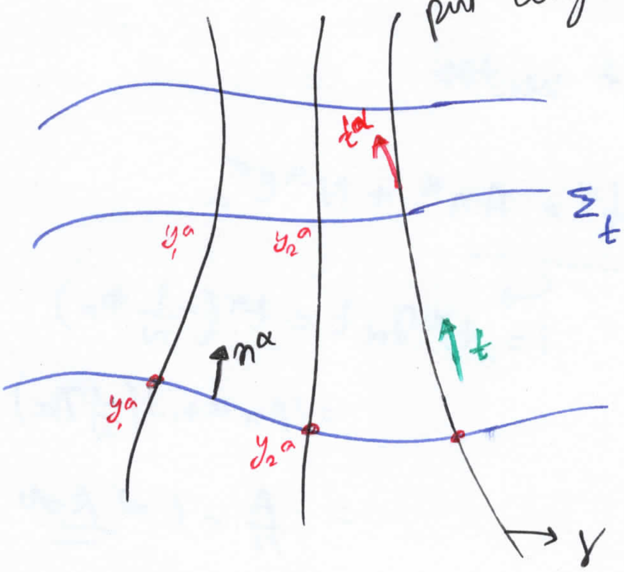


time function $t(x^\alpha)$; such that t = constant on each Σ_t

$n_\alpha \propto \partial_\alpha t$; $n_\alpha n^\alpha = -1$ (normalization)

We want to introduce relationship between y^A's coordinates on different Σ_t's. We need not do so ; in general it can be arbitrary ; but we will do this to simplify our lives.

put congruence (it can be arbitrary congruence ; ... need not be geodesics)



γ ; y^a = constant. ; parametrized by t.
, tangent vector t^α.

Displacement along curve ; $dx^\alpha = t^\alpha\, dt$

Cange in t will be given by $dt = \frac{\partial t}{\partial x^\alpha} dx^\alpha$

$$= \left(\frac{\partial t}{\partial x^\alpha} t^\alpha\right) dt$$

~~from~~ for along curve.

$$\boxed{t^\alpha \partial_\alpha t = 1}$$

Original coordinal system x^α
Alternative : (t, y^a)

Relation $x^\alpha = x^\alpha(t, y^a)$: parametric relations for γ.

$$\boxed{\left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} = t^\alpha} \qquad \boxed{\left(\frac{\partial x^\alpha}{\partial y^a}\right)_t = e^\alpha{}_a}$$

t^α → Tangent to congruence

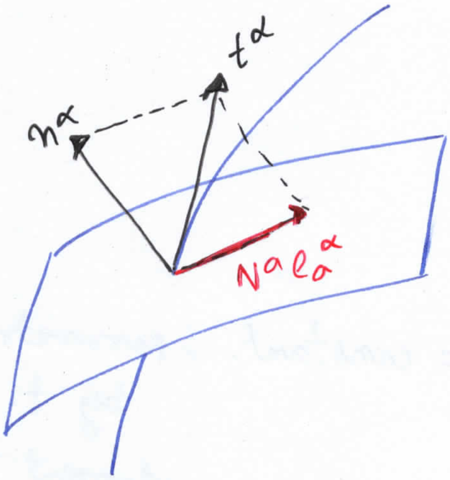
$e^\alpha{}_a$ → Tangent to Σ_t

$\boxed{\mathcal{L}_t \, e^\alpha{}_a = 0}$ ($n_\alpha \propto t_\alpha$ in general)

$\boxed{n_\alpha = -N \partial_\alpha t}$; $N \rightarrow$ lapse function.

$\boxed{n_\alpha e^\alpha{}_a = 0}$

$$\boxed{t^\alpha = \underbrace{N}_{\text{lapse}} n^\alpha + \underbrace{N^a}_{\text{shift vector}} e^\alpha{}_a}$$



$t^\alpha = A n^\alpha + N^a e^\alpha{}_a$

$1 = t^\alpha \partial_\alpha t = t^\alpha \left(-\frac{1}{N} n_\alpha\right)$

$= (A n^\alpha + \ldots)\left(-\frac{1}{N} n_\alpha\right)$

$= \frac{A}{N} = 1 \Rightarrow \underline{A = N}$

Metric in coordinates (t, y^a);

$dx^\alpha = t^\alpha dt + e^\alpha{}_a dy^a$

$= (N n^\alpha + N^a e^\alpha{}_a)\, dt \; e^\alpha{}_a dy^a$

$= (N dt) n^\alpha + (N^a dt + dy^a) e^\alpha{}_a$

$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$

$= g_{\alpha\beta} [N dt \, n^\alpha + (dy^a + N^a dt) e^\alpha{}_a]$
$[N dt \, n^\beta + (dy^b + N^b dt) e^\beta{}_b]$

$= -N^2 dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt)$

$\Rightarrow$ $\cancel{ds^2}$ $\boxed{h_{ab} = g_{\alpha\beta} e^\alpha{}_a e^\beta{}_a}$

$$\boxed{ds^2 = -N^2 dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt)}$$

3+1 decomposition of metric

For a displacement along t : $dy^a = 0$

$$ds^2 = -N^2 dt^2 + h_{ab} N^a N^b dt^2$$
$$= -(N^2 - h_{ab} N^a N^b) dt^2$$

$$\boxed{\sqrt{-g} = N\sqrt{h}}$$

Hamiltonian of a Field Theory

$\mathcal{L}(\psi, \partial_\alpha \psi)$

$\partial_t \psi \longrightarrow \underbrace{\dot{\psi} \equiv \mathcal{L}_t \psi}$ (derivative along t^α, not n^α)

$$\pi \equiv \frac{\partial}{\partial \dot{\psi}} (\sqrt{-g}\, \mathcal{L})$$

→ not a scalar (because of $\sqrt{g}$ here)

... called scalar density.

→ Canonical Momentum.

Hamiltonian density : $\boxed{\mathcal{H} = \pi \dot{\psi} - \sqrt{-g}\, \mathcal{L}}$

$$H = \int \mathcal{H}\, d^3y$$

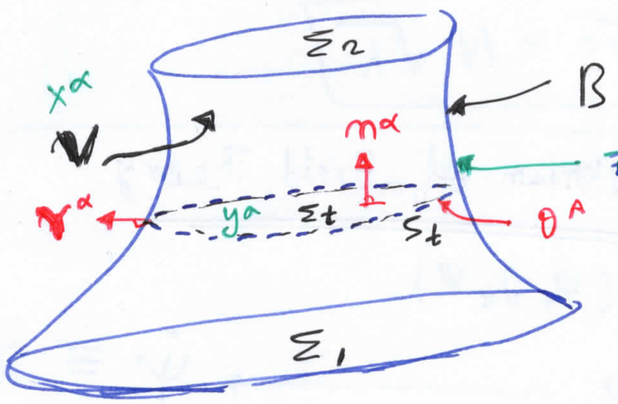
Hamiltonian.

→ we ~~get~~ dont put $\sqrt{|h|}$ here ~~????~~ because this already has $\sqrt{g}$ ... check & get ~~convinced~~ convinced.

H is something which lives on one of the hypersurface. $\therefore$ it is functional of fields on Σ.

A Hamiltonian formulation of GR.

$$16\pi S_G = \int_V {}^4R\sqrt{-g}\,d^4x + 2\oint_{\partial V} \varepsilon K\sqrt{|h|}\,d^3y$$



V volume; ; Boundary; $\partial V = B + \Sigma_1 + \Sigma_2$

V is foliated by Σ_t

Σ_t is bounded by S_t

B is foliated by S_t.

* θ^A instrinsic coordinates on S_t
* on B, we have another coordinate system. z^i which may not be related to t and θ^A.

24/4/2020
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Lec 15

Σ_t : $\begin{cases} t(x^\alpha) = \text{constant} \quad n_\alpha \\ x^\alpha = x^\alpha(y^a) \quad ; \quad e^\alpha{}_a = \partial x^\alpha/\partial y^a \\ h_{ab} = g_{\alpha\beta} e^\alpha{}_a e^\beta{}_b \quad ; \quad K_{ab} = n_{\alpha;\beta} e^\alpha{}_a e^\beta{}_b \\ g^{\alpha\beta} = -n^\alpha n^\beta + h^{\alpha\beta} e^\alpha{}_a e^\beta{}_b \quad \text{(completeness relationship)} \end{cases}$

S_t embedded in Σ_t: $\quad \Phi(y^a) = \theta \qquad r_a \propto \partial_a \Phi$

$\qquad y^a = y^a(\theta^A) \qquad e^a{}_A \propto \partial y^a/\partial\theta^A$

induced metric

$\sigma_{AB} = h_{ab} e^a{}_A e^b{}_B \quad ; \quad \cancel{k_{AB}} \; k_{AB} = r_{a|b} e^a{}_A e^b{}_B$

completeness relation $h^{ab} = r^a r^b + \sigma^{AB} e^a{}_A e^b{}_B$

plus sign; because that vector is spacelike; not timelike.

Embed S_t directly in spacetime.

S_t embedded in spacetime:

$\Psi(x^\alpha) = 0$; $x^\alpha = x^\alpha(\theta^A) = x^\alpha(y^a(\theta^A))$

$$\star \; e^\alpha{}_A = \frac{\partial x^\alpha}{\partial \theta^A} = \frac{\partial x^\alpha}{\partial y^a}\frac{\partial y^a}{\partial \theta^A} = e^\alpha{}_a e^a{}_A$$

note: for ~~normal~~ normal $r^\alpha = r^a e^\alpha{}_a$ → same normal vector.

$$\sigma_{AB} = (g_{\alpha\beta} e^\alpha{}_a e^\beta{}_b) e^a{}_A e^b{}_B$$

$$\sigma_{AB} = g_{\alpha\beta} e^\alpha{}_A e^\beta{}_B$$

$$\boxed{r_{a|b} = r_{\alpha;\beta} e^\alpha{}_a e^\beta{}_b}$$

$$K_{AB} = (r_{\alpha;\beta} \underbrace{e^\alpha{}_a e^\beta{}_b}_{\text{projected once}}) \underbrace{e^a{}_A e^b{}_B}_{\text{projected twice}}$$

$$= r_{\alpha;\beta} \underbrace{e^\alpha{}_A e^\beta{}_B}_{\text{full projected down from spacetime to 2D surface.}}$$

completeness relation

$$\boxed{g^{\alpha\beta} = -n^\alpha n^\beta + r^\alpha r^\beta + \sigma^{AB} e^\alpha{}_A e^\beta{}_B}$$

B embedded in spacetime ~~$\Xi(x^\alpha) = 0$~~

$\Xi(x^\alpha) = 0$ $\qquad r_\alpha \propto \partial_\alpha \Xi$ $\qquad : e^\alpha{}_i = \frac{\partial x^\alpha}{\partial z^i}$

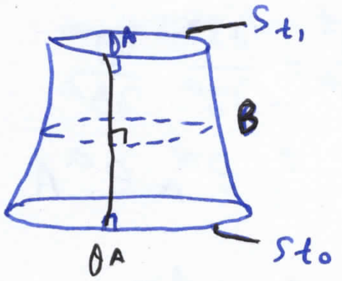
$x^\alpha = x^\alpha(z^i)$ $\qquad$ normal to $S_t \propto r_\alpha$

$$\gamma_{ij} = g_{\alpha\beta} e^\alpha{}_i e^\beta{}_j$$

$$\mathcal{K}_{ij} = r_{\alpha;\beta} e^\alpha{}_i e^\beta{}_j$$

$$g^{\alpha\beta} = r^\alpha r^\beta + \gamma^{ij} e^\alpha{}_i e^\beta{}_j$$

Foliation of $\mathcal{B}$ with S_t.



New congruence of curves on $\mathcal{B}$, along which θ^A is constant.

(∴ here we can restrict these congruence are orthogonal to S_t)

↪ to simplify.

Orthogonal to S_t within $\mathcal{B}$.

$$\left(\frac{\partial x^\alpha}{\partial t}\right)_{\theta^A} = N n^\alpha \quad (N^a \text{ zero}; \text{ zero shift})$$

$$n_\alpha = -N \partial_\alpha t$$

---

choose ; $z^i = (t, \theta^A)$

Displacement on $\mathcal{B}$:

$$dx^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{\theta^A} dt + \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_t d\theta^A$$

$$= N n^\alpha dt + e^\alpha{}_A d\theta^A$$

$$ds^2 = g_{\alpha\beta}(N n^\alpha dt + e^\alpha{}_A d\theta^A)(N n^\beta dt + e^\alpha{}_B d\theta^B)$$

$$= -N^2 dt^2 + \sigma_{AB} d\theta^A d\theta^B$$

$$\boxed{\gamma_{ij} dz^i dz^j = -N^2 dt^2 + \sigma_{AB} d\theta^A d\theta^B}$$

$$\sqrt{-\gamma} = N\sqrt{\sigma}$$

$$16\pi S_G = \int_V {}^4R \sqrt{g}\, d^4x - 2\int_{\Sigma_2} K\sqrt{h}\, d^3y + 2\int_{\Sigma_1} K \sqrt{h}\, d^3y + 2\int_{\mathcal{B}} \mathcal{K} \sqrt{-\gamma} \cdot d^3z$$

$${}^4R = {}^3R + K^{ab}K_{ab} - K^2 - 2(n^\alpha{}_{;\beta}n^\beta - n^\alpha n^\beta{}_{;\beta})_{;\alpha}$$

$$\sqrt{-g} = N\sqrt{h} \quad ; \quad d^4x = dt\, d^3y$$

$$\int {}^4R\sqrt{-g}\, d^4x = \int ({}^3R + K^{ab}K_{ab} - K^2)\, N\sqrt{h}\, dt\, d^3y$$
$$-2\oint_{\partial V} (n^\alpha{}_{;\beta}n^\beta - n^\alpha n^\beta{}_{;\beta})\, d\Sigma_\alpha$$

on Σ_2 : $d\Sigma_\alpha = -n_\alpha\sqrt{h}\, d^3y$

$$\longrightarrow 2\int_{\Sigma_2} (n^\alpha{}_{;\beta}n^\beta - n^\alpha n^\beta{}_{;\beta})\, n_\alpha \sqrt{|h|}\, d^3y$$

$$= 2\int_{\Sigma_2} (\underbrace{\cancel{n_\alpha n^\alpha{}_{;\beta}}}_{\frac{1}{2}(n_\alpha n^\alpha)_{;\beta} \to 0} + n^\beta{}_{;\beta})\sqrt{h}\, d^3y$$

$$g^{\alpha\beta}n_{\alpha;\beta} = (h^{\alpha\beta}n_{\alpha;\beta})$$

$$= 2\int_{\Sigma_2} K\sqrt{|h|}\, d^3y \qquad = h^{ab}n_{\alpha;\beta}e^\alpha{}_a e^\beta{}_b$$
$$= h^{ab}K_{ab} = K$$

on Σ_1 : $\longrightarrow -2\int_{\Sigma_1} K\sqrt{h}\, d^3y$

---

so only boundary terms ~~surf~~ survives is of $\mathcal{B}$.

on $\mathcal{B}$: $d\Sigma_\alpha = r_\alpha\sqrt{-\gamma}\, d^3z$

$$\rightarrow -2\int_{\mathcal{B}} (n^\alpha{}_{;\beta}n^\beta - n^\alpha \cancel{n^\beta{}_{;\beta}}^{\,0})\, r_\alpha\sqrt{-\gamma}\, d^3z$$

$$= -2\int_{\mathcal{B}} \underbrace{r_\alpha \cdot n^\alpha{}_{;\beta}n^\beta}_{(r_\alpha n^\alpha)_{;\beta}n^\beta - r_{\alpha;\beta}n^\alpha n^\beta} \sqrt{-\gamma}\, d^3z$$

$$= 2\int_{\mathcal{B}} r_{\alpha;\beta}n^\alpha n^\beta\sqrt{-\gamma}\, d^3x$$

$$16\pi S_G = \int_{t_1}^{t_2} dt \int_{\Sigma_t} N({}^3R + K^{ab}K_{ab} - K^2)\sqrt{h}\, d^3y$$
$$+ 2\int_{\mathcal{B}} (K + r_{\alpha;\beta}n^\alpha n^\beta)\sqrt{-\gamma}\, d^3z$$

$$K = \gamma^{ij} K_{ij} = \gamma^{ij}\, \gamma_{\alpha;\beta}\, e^{\alpha}{}_i e^{\beta}{}_j$$
$$= \gamma_{\alpha;\beta} (\gamma^{ij} e^{\alpha}{}_i e^{\beta}{}_j)$$
$$= \gamma_{\alpha;\beta} (g^{\alpha\beta} - \gamma^{\alpha}\gamma^{\beta})$$

$$(K + \gamma_{\alpha;\beta}\, n^{\alpha} n^{\beta}) = \gamma_{\alpha;\beta} (g^{\alpha\beta} + n^{\alpha} n^{\beta} - \gamma^{\alpha}\gamma^{\beta})$$
$$= \gamma_{\alpha;\beta}\, \sigma^{AB}\, e^{\alpha}{}_A e^{\beta}{}_B$$
$$= \sigma^{AB} . (k_{AB}) \equiv k$$

$$16\pi S_G = \int_{t_1}^{t_2} dt \left\{ \int_{\Sigma_t} N({}^3R + K^{ab}K_{ab} - K^2)\sqrt{h}\, d^3y + \oint_{S_t} N\, k \sqrt{\sigma}\, d^2\theta \right\}$$

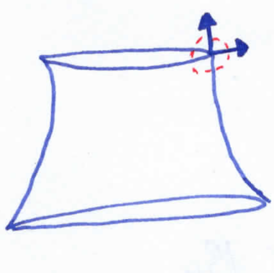
$$\Rightarrow 16\pi S_G = \int_{t_1}^{t_2} dt \left[\int_{\Sigma_t} N({}^3R + K^{ab}K_{ab} - K^2)\sqrt{h}\, d^3y + \oint_{S_t} N\, k \sqrt{\sigma}\, d^2\theta \right]$$

$$k \rightarrow (k - k_0)$$

2D extrinsic curvation on S_t embedded in Σ_t

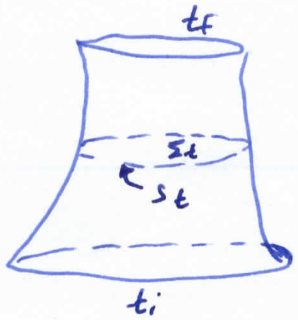
$$p^{ab} \sim \frac{\partial \mathcal{L}}{\partial \dot{h}_{ab}} \qquad \dot{h}_{ab} \sim \mathcal{L}_t h_{ab} \sim K_{ab}$$

$$\mathcal{H} \sim P_{ab}\dot{h}_{ab} - \mathcal{L}$$



$\Rightarrow$ singularity: but it will be of measure zero when integrated ~~over $\partial\Sigma$~~ over ∂V.

$$16\pi S_G = \int_{t_1}^{t_2} dt \left\{ \int_{\Sigma_t} N({}^3R + K^{ab}K_{ab} - K^2)\sqrt{h}\, d^3y + 2\oint_{S_t} N(k - k_0)\sqrt{\sigma}\, d^2\theta \right\}$$



K_{ab} = 3D extrinsic curvature of Σ_t.

K_{AB} = 2D " " of S_t embedded in Σ_t

K^0_{AB} = " " " " " " " flat space

$$\dot{h}_{ab} = \mathcal{L}_t h_{ab} = \mathcal{L}_t (g_{\alpha\beta} e^\alpha{}_a e^\beta{}_b)$$
$$= (\mathcal{L}_t g_{\alpha\beta}) e^\alpha{}_a e^\beta{}_b$$
$$= (t_{\alpha;\beta} + t_{\beta;\alpha}) e^\alpha{}_a e^\beta{}_b$$

$$t^\alpha = N n^\alpha + \underbrace{N^a e^\alpha{}_a}_{N^\alpha}$$

$$t_{\alpha;\beta} = N_{;\beta} n^\alpha + N n_{\alpha;\beta} + N_{\alpha;\beta}$$

$$t_{\alpha;\beta} e^\alpha{}_a e^\beta{}_b = N(n_{\alpha;\beta} e^\alpha{}_a e^\beta{}_b) + N_{\alpha;\beta} e^\alpha{}_a e^\beta{}_b$$
$$= N K_{ab} + N_{a|b}$$

$$\boxed{\dot{h}_{ab} = 2NK_{ab} + N_{a|b} + N_{b|a}}$$

$$p^{ab} = \frac{\partial}{\partial \dot{h}_{ab}} (\sqrt{-g}\, \mathcal{L}_{bulk})$$
$$= \frac{1}{2N} \frac{\partial}{\partial K_{ab}} \left(N\sqrt{h} \left({}^3R + K^{cd}K_{cd} - K^2 \right) \right)$$
$$= \frac{1}{2}\sqrt{h} \frac{\partial}{\partial K_{ab}} \left[\underbrace{(K_{cd}K_{ef})}_{\delta^a_c \delta^b_d K_{ef} + K_{cd}\delta^a_e \delta^b_f} (h^{ce}h^{df} - h^{cd}h^{ef}) \right]$$

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$$p^{ab} = \frac{1}{2}\sqrt{h}\left\{ K_{ef}\left(h^{ae}h^{bf} - h^{ab}h^{ef}\right) + K_{cd}\left(h^{ad}h^{bh} - h^{cd}h^{ab}\right)\right\}$$

$$\boxed{p^{ab} = \sqrt{h}\left(K^{ab} - K h^{ab}\right)}$$

$$\mathcal{H}_{Bulk} = p^{ab}\dot{h}_{ab} - \sqrt{-g}\,\mathcal{L}_{Bulk}$$

$$H = \int \mathcal{H}\, d^3y$$

$$\mathcal{H}_{Bulk} = \cancel{\sqrt{h}\left(K^{ab} - K h^{ab}\right)\left(2N K_{ab}\right.} + N_{a|b} + N_{b|a}\big)$$

$$- N\left({}^3R + \cancel{K^{ab}K_{ab} - K^2)\sqrt{h}}\right.$$

$$- 2\cancel{\oint_{S_t} N(k - k_0)\sqrt{\sigma}}\, d^2\theta$$

$$= \cancel{N\sqrt{h}\ \left(2K^{ab}K_{ab} - 2K^2\right.}$$

$$\mathcal{H}_{Bulk} = \left(2N K_{ab} + N_{a|b} + N_{b|a}\right)\sqrt{h}\left(K^{ab} - K h^{ab}\right) - N\sqrt{h}\left({}^3R + K^{ab}K_{ab} - K^2\right)$$

$$= N\sqrt{h}\left(2K_{ab}K^{ab} - 2K^2 - {}^3R - K^{ab}K_{ab} + K^2\right) + \sqrt{h}\left(K^{ab} - K h^{ab}\right)\left(2N_{a|b}\right)$$

$$= N\sqrt{h}\left(K_{ab}K^{ab} - K^2 - {}^3R\right) + 2\sqrt{h}\left[\left(K^{ab} - h^{ab}\right)N_a\right]_{|b} - 2\sqrt{h}\left(K^{ab} - K h^{ab}\right)_{|b} N_a$$

$$16\pi H = \int_{\Sigma_t} N\left(K_{ab}K^{ab} - K^2 - {}^3R\right)\sqrt{h}\,d^3y - 2\int_{\Sigma_t} N_a\left(K^{ab} - K h^{ab}\right)_{|b}\sqrt{h}\,d^3y + 2\oint_{S_t}\left(K^{ab} - h^{ab}K\right)N_a \underbrace{dS_b}_{r_b\sqrt{\sigma}\,d^2\theta} - 2\oint_{S_t} N(k - k_0)\sqrt{\sigma}\,d^2\theta$$

$$16\pi H_h = \int_{\Sigma_t}\left\{N\left(K_{ab}K^{ab} - K^2 - {}^3R\right) - 2N_a\left(K^{ab} - h^{ab}K\right)_{|b}\right\}\sqrt{h}\,d^3y + 2\oint_{S_t}\left\{-N(k - k_0) + N_a\left(K^{ab} - h^{ab}K\right)r_b\right\}\sqrt{\sigma}\,d^2\theta$$

Gravitational Hamiltonian.

$$(16\pi) H_G = \int_{\Sigma_t} \left[N (K^{ab} K_{ab} - K^2 - {}^3R) - 2 N_a (K^{ab} - K h^{ab})_{|b} \right] \sqrt{h}\, d^3y$$
$$- 2 \oint_{S_t} \left[N (k - k_0) - N_a (K^{ab} - K h^{ab}) r_b \right] \sqrt{\sigma}\, d^2\theta$$

Constraint equations in vacuum:

$$^3R - K_{ab} K^{ab} + K^2 = 0$$
$$(K^{ab} - h^{ab} K)_{|b} = 0$$

~~so, Bulk part of H_G vanishes; it's only surfacial term.~~

For a solution to vacuum field equation:

$$16\pi H_G = 2 \oint_{S_t} -N(k - k_0) + N_a (K^{ab} - h^{ab} K) r_b \, \} \sqrt{\sigma}\, d^2\theta$$

→ Pure Boundary term

Solution evaluated H_G

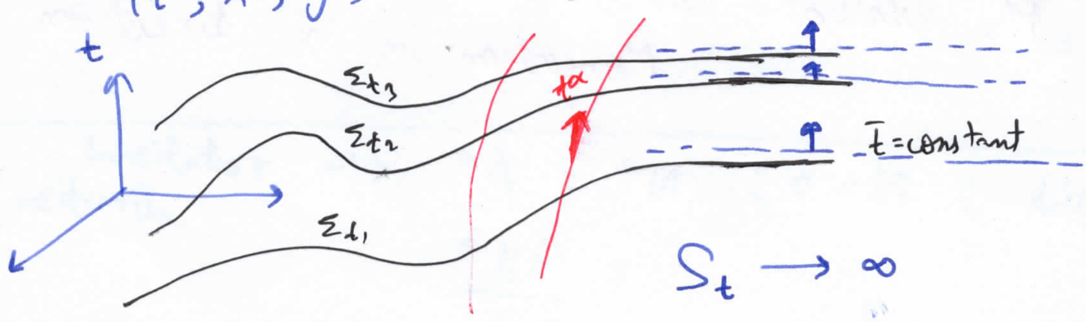
H_G depends on Σ_t and S_t; N, N_a choice of flow; t^α

~~we will~~

Application to asymptotically-flat spacetime

Restrict Σ_t such that it approaches a surface $\bar{t} = \text{constant}$ in asymptotic Minkowski spacetime

$(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ = Asymptotic Minkowski frame.



$S_t \to \infty$

flow is arbitrary: $t^\alpha = N n^\alpha + N^a e^\alpha{}_a$

$$= N \left(\frac{\partial x^\alpha}{\partial \bar{t}} \right) + N^a \left(\frac{\partial x^\alpha}{\partial \xi} \right)$$

H depends on flow through N, N^a.

Pick $N=1$, $N^a=0$. (so we pick flow to be asymptotic time ~~tran~~ time ~~transf~~ translation)

$\Rightarrow$ flow is asymptotic time translation.

$$H_G = -\frac{1}{8\pi}\oint_{S_t\to\infty} (k-k_0)\sqrt{\sigma}\, d^2\theta \equiv M \equiv \text{total mass-energy}$$

$\equiv$ ADM Mass

This hamiltonian is generated by Asymptotic time translation; ~~that~~ so we call it total energy of space time.

A way of extracting mass for total spacetime.

$$\boxed{H_G = -\frac{1}{8\pi}\oint_{S_t\to\infty} (k-k_0)\sqrt{\sigma}\, d^2\theta \equiv M}$$

if we pick $N=0$; $N^a = \frac{\partial y^a}{\partial \bar{x}}$ in some specific direction

Asymptotic translation along $\bar{x}$

so H_G evaluated with this be Total Linear Momentum in $\bar{x}$ direction

If we pick ; $N=0$; $N^a = \phi^a$ (in rotational direction)

$$= \frac{\partial y^a}{\partial \bar{\phi}}$$

$\bar{\phi} \Rightarrow$ rotational coordinate in asymptotically Minkowski frame

Then flow; describes rotation.

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$$H_G = +\frac{1}{8\pi} \oint_{S_t \to \infty} (k_{ab} - k h_{ab}) \phi^a r^b \sqrt{\sigma}\, d^2\theta$$

$$= -J$$

introduce minus for convention

Total Angular Momentum in ~~axis~~ the direction of rotational axis

We can mix them up; and consider Lorentz Transformations.

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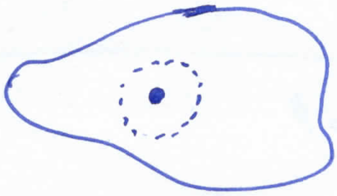
$$M = -\frac{1}{8\pi} \oint_{S_t \to \infty} (k - k_0) \sqrt{\sigma}\, d^2\theta$$

(similar to electromagnetism $Q \propto \oint \vec{E} \cdot d\vec{a}$)

$$J = -\frac{1}{8\pi} \oint_{S_t \to \infty} (k_{ab} - k h_{ab}) \phi^a r^b \sqrt{\sigma}\, d^2\theta$$

solution to linearized EFE in vacuum, ~~for~~ far away from a stationary body.



$$\cancel{ds^2 = -\left(1 - \frac{2m}{r} + \ldots\right) dt^2 + \left(1 + \frac{2m}{r} + \ldots\right) dr^2}$$

$$ds^2 = -\left(1 - \frac{2m}{r} + \ldots\right) dt^2 + \left(1 + \frac{2m}{r} + \ldots\right)(dr^2 + r^2 d\Omega^2)$$
$$- \left(\frac{4j \sin^2\theta}{r} + \ldots\right) dt\, d\phi$$

describes dragging.

$m, j \equiv$ operational defination of mass & angular momentum, in terms of precise physical process.

after calculation: $M = m$, $J = j$

$\xi^\alpha_{(t)} = (1,0,0,0)$

$\xi^\alpha_{(\phi)} = (0,0,0,1)$

$n^\alpha \propto (1,0,0,0)$

$r^\alpha \propto (0,1,0,0)$

for stationary, axially-symmetric spacetimes,

Killing vectors $\xi^\alpha_{(t)}$, $\xi^\alpha_{(\phi)}$

Alternative definations,

$$M = \frac{-1}{8\pi} \oint_{S_{t\to\infty}} \nabla^\alpha \xi^\beta_{(t)} \, dS_{\alpha\beta}$$

Komar formulas.

$$J = \frac{1}{16\pi} \oint_{S_{t\to\infty}} \nabla^\alpha \xi^\beta_{(\phi)} \, dS_{\alpha\beta}$$

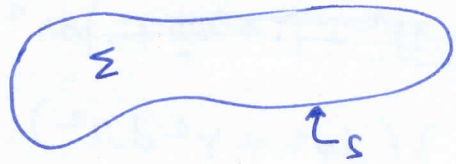
$$dS_{\alpha\beta} = -2 \, n_{[\alpha} r_{\beta]} \sqrt{\sigma} \, d^2\theta$$

by virtue of killing ~~vector~~ vector; $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$

Stokes Theorem

for an antisymmetric tensor field $B^{\alpha\beta}$,

$$\oint_S B^{\alpha\beta} \, dS_{\alpha\beta} = 2 \int_\Sigma B^{\alpha\beta}{}_{;\beta} \, d\Sigma_\alpha$$

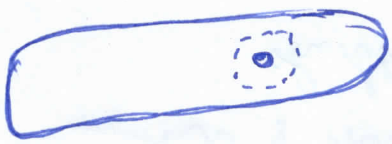


$$B^{\alpha\beta} = \nabla^\alpha \xi^\beta$$

$$B^{\alpha\beta}{}_{;\beta} - \nabla_\beta \nabla^\alpha \xi^\beta$$

$$= -\nabla_\beta \nabla^\beta \xi^\alpha = -\Box \xi^\alpha$$

$$= +R^\alpha{}_\beta \xi^\beta$$



$$= 8\pi \left(T^\alpha{}_\beta - \frac{1}{2} T \delta^\alpha{}_\beta \right) \xi^\beta$$

$$d\Sigma_\alpha =$$

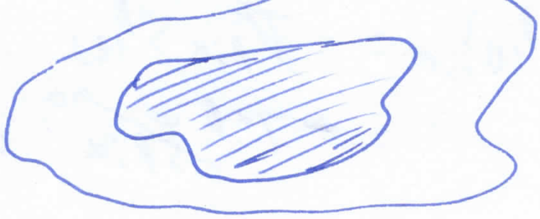
$$M = 2 \int_\Sigma \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) n^\alpha \xi^\beta_{(t)} \sqrt{h} \, d^3y + M_{int}$$

$$J = \int_\Sigma \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) n^\alpha \xi^\beta_{(\phi)} \sqrt{h} \, d^3y + J_{int}$$

mass density $= 2\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) n^\alpha \xi^\beta_{(t)}$

angular momentum density $= \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) n^\alpha \xi^\beta_{(\phi)}$

Matter should be changing energy with gravitational field } Not fully correct.. ... little ambiguous. ... and not well defined statement.



same answer... ~~unless yo~~ as far as your surface cover ~~whole~~ whole. mass distribution.

Mass and Angular Momentum Transfer

Stationary + Axis symmetric spacetime ; $\xi^\alpha_{(t)}$, $\xi^\alpha_{(\phi)}$

Test $T^{\mu\nu}$ represents a flow of matter.

$\Rightarrow$ transfer of ~~mass~~ M, J across hypersurface.

---

consider pressureless fluid : $T^{\alpha\beta} = \rho U^\alpha U^\beta$

$$0 = T^{\alpha\beta}{}_{;\beta} = \rho \underbrace{U^\alpha{}_{;\beta} U^\beta}_{a^\alpha} + (\rho U^\beta)_{;\beta} U^\alpha$$

a^α is orthogonal to U^α

so; in $0 = \rho a^\alpha + (\rho U^\beta)_{;\beta} U^\alpha$

both ~~should~~ should individually vanish

$\Rightarrow$ $\boxed{a^\alpha = 0}$ geodesic motion for fluid element

& $j^\alpha = \rho U^\alpha$; conserved $j^\alpha{}_{;\alpha} = 0$

= flux of rest mass

Conserved quantity: $\tilde{E} = -U_\alpha \xi^{\alpha}_{(t)} \equiv$ energy/mass

$\tilde{L} = U_\alpha \xi^{\alpha}_{(\phi)} \equiv$ Angular momentum/mass.

Energy flux vector = (mass flux vector) (energy/mass)

$$\boxed{\varepsilon^{\alpha} = \tilde{E}\,\rho\,U^{\alpha}}$$

$$\varepsilon^{\alpha} = -U_\beta \xi^{\beta}_{(t)} \rho U^{\alpha} = -T^{\alpha}{}_{\beta} \xi^{\beta}_{(t)}$$

We can verify $\varepsilon^{\alpha}{}_{;\alpha} = 0$

$$\varepsilon^{\alpha}{}_{;\alpha} = -(T^{\alpha}{}_{\beta}\xi^{\beta}_{(t)})_{;\alpha} = -\cancelto{0}{T^{\alpha}{}_{\beta;\alpha}}\,\xi^{\beta}_{(t)} + T^{\alpha\beta}\cancelto{0}{\xi_{\beta;\alpha}} = 0$$

Angular Momentum flux vector

$$\boxed{l^{\alpha} = \tilde{L}\,\rho\,U^{\alpha} = T^{\alpha}{}_{\beta}\xi^{\beta}_{(\phi)}}$$

$$\boxed{\begin{aligned}\varepsilon^{\alpha} &= -T^{\alpha}{}_{\beta}\xi^{\beta}_{(t)} \\ l^{\alpha} &= T^{\alpha}{}_{\beta}\xi^{\beta}_{(\phi)}\end{aligned}} \qquad \begin{aligned}\varepsilon^{\alpha}{}_{;\alpha} &= 0 \\ l^{\alpha}{}_{;\alpha} &= 0\end{aligned}$$

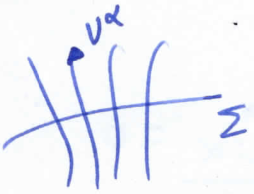
Transfer of mass-energy across hypersurface Σ

$$\Delta M = \int_{\Sigma} \varepsilon^{\alpha}\, d\Sigma_{\alpha} = -\int_{\Sigma} T^{\alpha}{}_{\beta}\xi^{\beta}_{(t)}\, d\Sigma_{\alpha}$$

Transfer of angular momentum.

$$\Delta J = \int_{\Sigma} l^{\alpha}\, d\Sigma_{\alpha} = \int_{\Sigma} T^{\alpha}{}_{\beta}\xi^{\beta}_{(\phi)}\, d\Sigma_{\alpha}$$

ex fluid ; Σ orthogonal to fluid worldlines



$$d\Sigma_{\alpha} = -U_{\alpha}\sqrt{h}\, d^3y$$

$$\Delta M = \int T^{\alpha}{}_{\beta}\xi^{\beta}_{(t)} U_{\alpha}\sqrt{h}\, d^3y$$

$$\Delta M = \int_{\Sigma} T^{\alpha\beta} \, V_\alpha \, \xi_{\beta(t)} \sqrt{h} \, d^3y = \int_{\Sigma} \rho \tilde{E} \sqrt{h} \, d^3y$$

$$\left. \begin{aligned} \Delta M &= 0 \\ \Delta J &= 0 \end{aligned} \right\} \text{ for closed } \Sigma .$$

Ch 4 completed here

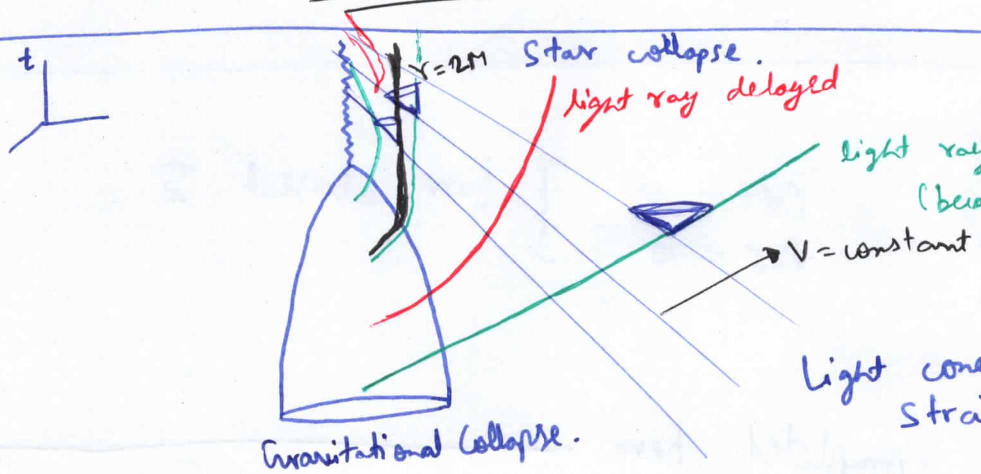
H/W 4.5 : #5
5.7 : #2 , #9

Advanced General Relativity

25/4/2020

Lec 19: Ch 5: Black Holes

– Shoaib Akhtar



Light cones are not straight.

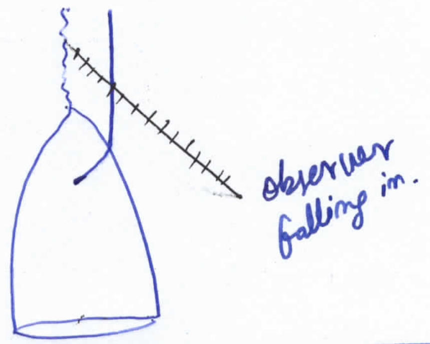
Non-rotating B.H. : Schwarzchild. $\quad ; \quad f = \left(1 - \frac{2M}{r}\right)$

$$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2$$

Birkoff's Theorem : vacuum + spherical symmetry $\Rightarrow$ Schwarschild.

Israel : Non-rotating + event horizon $\Rightarrow$ Schwarschild.

There is need to redefine t.



Eddington – Finkelstein

Design a new time coordinate by examining behavior of light rays.

Radial light rays : $\dot{\theta} = 0 = \dot{\phi}$

$$ds^2 = -f\cdot\underbrace{\left(dt - \frac{1}{f}dr\right)}_{du}\underbrace{\left(dt + \frac{1}{f}dr\right)}_{dV} = -f\,du\,dV$$

$\} \Rightarrow$ perfect differentials.

$$\boxed{u = t - \int\frac{dr}{f} \qquad V = t + \int\frac{dr}{f}}$$

null coordinates

$$\int\frac{dr}{f} = r - 2M\ln\left(\frac{r}{2M} - 1\right)$$

$u =$ constant $\Rightarrow$ outgoing light rays.

$V =$ " $\Rightarrow$ incoming " ".

$$V - u = 2\int\frac{dr}{f} = 2r - 4M\ln\left(\frac{r}{2M} - 1\right)$$

↳ to get rid of t

when $r \to 2M$: $V - u \to +\infty$

$v \to -\infty$ or $u \to +\infty$

as we approach event horizon.

Adapting v and r as coordinates;

$$dt = dv - \frac{1}{f} dr$$

$$ds^2 = -f\left(dv - \frac{1}{f}dr\right)^2 + f^{-1}dr^2 + r^2 d\Omega^2$$

$$= -f dv^2 - 2 dv\, dr - \frac{1}{f} dr^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2$$

$$\boxed{ds^2 = -f dv^2 + 2 dv\, dr + r^2 d\Omega^2}$$

change of sign at $r = 2M$ is not an issue.

r = constant observer (θ, ϕ constant)

$ds^2 = -f dv^2$ ← no problem $r \neq 2M$

if $r = 2M$; then; you are light ray; so you get $ds^2 = 0$ being consistent.

---

ex given vector field A^α in (t, r, θ, ϕ) coordinates.
Check regularity of vector field at $r = 2M$.
Adopt regular coordinate system (v, r, θ, ϕ)
; and then work out regularity conditions. (impose this)
at $r = 2M$.

$A^v, A^r, A^\theta, A^\phi$ all smooth

Translate; $$A^\alpha = A^{\mu'} \frac{\partial x^\alpha}{\partial x^{\mu'}}$$

$$A^t = A^v \underbrace{\frac{\partial t}{\partial v}}_{1} + A^r \underbrace{\frac{\partial t}{\partial r}}_{-1/f} = A^v - \frac{1}{f} A^r$$

$$\Rightarrow \boxed{A^t = A^v - \frac{1}{f} A^r}$$

Kruskal Coordinates

coordinates redefination $U = U(u)$, $V = V(v)$

$U = -e^{-ku}$

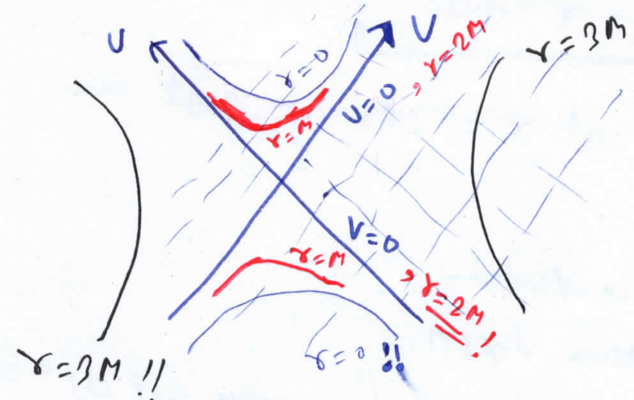
$V = +e^{kv}$ (to keep more symmetry)

$K = \frac{1}{4M}$ $\quad ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU dV + r^2 d\Omega^2$ (pg 88)

$$ds^2 = -\frac{32M^3}{r} \cdot e^{-r/2M} \, dU \, dV + r^2 d\Omega^2$$

$$\boxed{e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV}$$ (Hard to invert this !!)

when $r = 2M$; $\underline{UV = 0}$.



The construction gives two ~~sets~~ copies of solution.

∇ Maximally Extended Schwarzschild solution.

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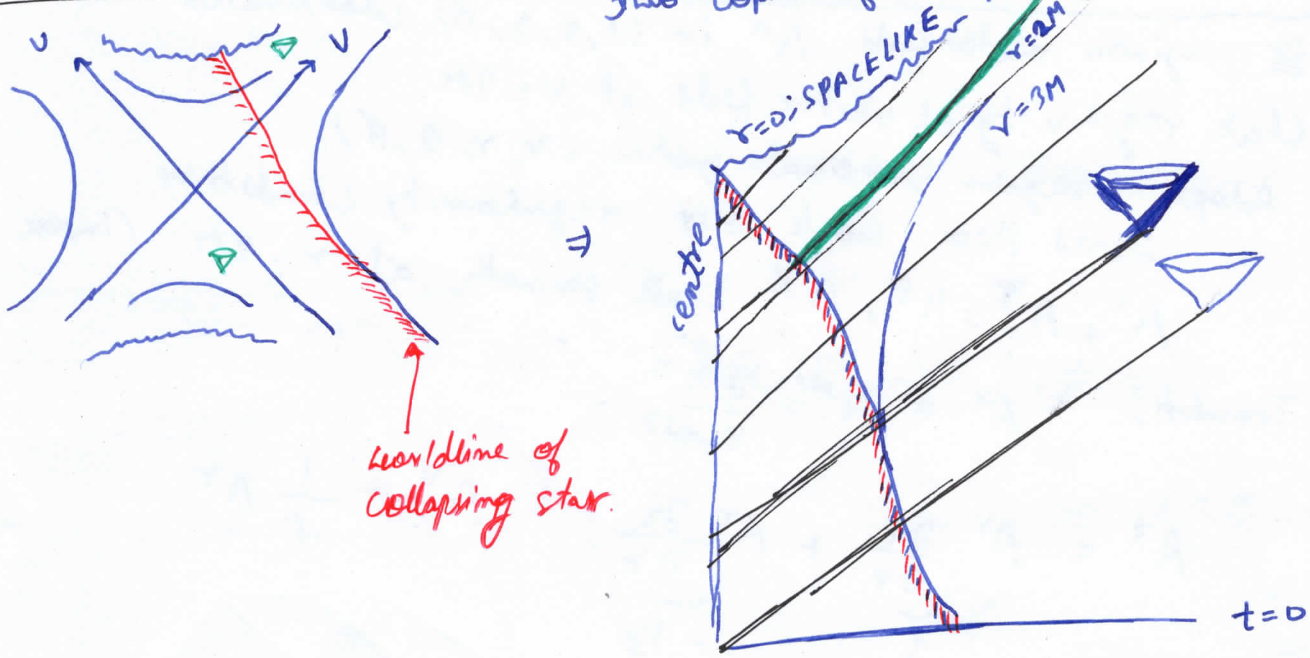
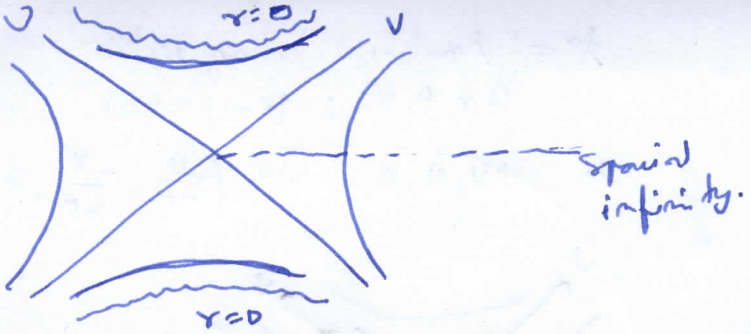
Lec 20

two copies of everything.



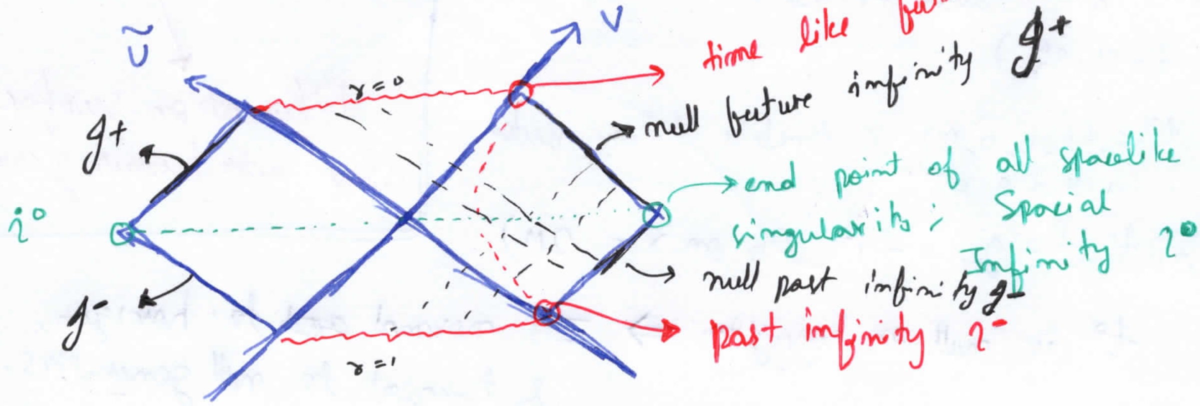
light ray moves at 45°.

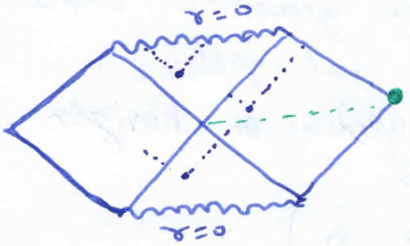


$$U = t - r^*$$
$$V = t + r^*$$
$$r^* = \int \frac{dr}{f} = r + 2M\ln\left(\frac{r}{2M} - 1\right)$$

Compactify the coordinates

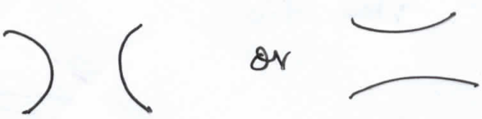
$$\tilde{U} = \arctan(U) \quad \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$\tilde{V} = \arctan(V) \quad \in \left(-\pi/2, \pi/2\right)$$

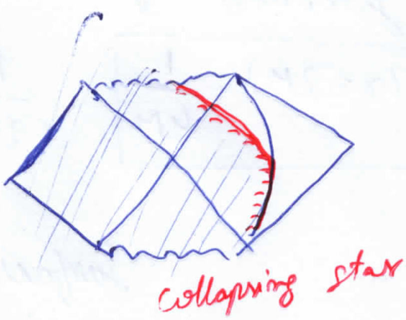




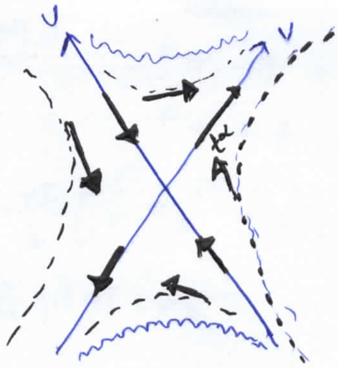
There are notions of infinity depending on how you get there

There will always be two regions for a given r in Schwarzschild space time





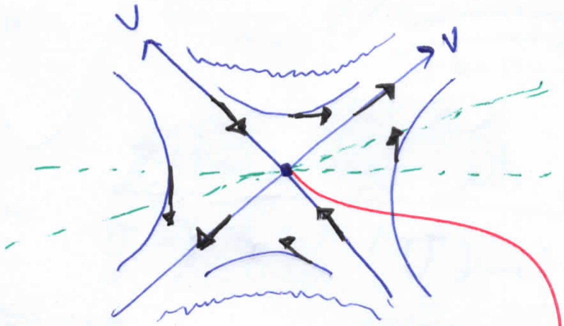
collapsing star



$t^\alpha \equiv$ "timelike" killing vector

(t, r, θ, ϕ) ; $t^\alpha = (1, 0, 0, 0)$

In (U, V, θ, ϕ) ; $t^\alpha = \left(\frac{-U}{4M}, \frac{-V}{4M}, 0, 0\right)$



bifurcation surface.
(killing vector becomes zero...)

$ds^2 = -f dv^2 + 2 dv dr + r^2 d\Omega^2$

$f = \left(1 - \frac{2M}{r}\right)$

$t^\alpha = (1, 0, 0, 0)$ = timelike killing vector
$\quad\;\; v \;\; r \;\; \theta \;\; \phi$

$g_{\alpha\beta} t^\alpha t^\beta = g_{vv} = -f$ (0 on $r = 2M$)

t^α is null on horizon $\Rightarrow$ It's normal ~~and~~ to horizon,
& tangent to null generators.

The killing vector will not satisfy geodesic inside and outside the horizon; because r = constant is not geodesic.
But it will satisfy geodesic on horizon.

$t^\alpha_{;\beta} t^\beta = \left(\frac{1}{2} f', \frac{1}{2} f f', 0, 0\right)$

when $f = 0$ $\Rightarrow \left(\frac{1}{2} f'(2M), 0, 0, 0\right)$

$\Rightarrow$ we get geodesic in ~~geo~~ generalized form

$\rightarrow \kappa t^\alpha$

$\boxed{; \kappa = \frac{1}{2} f'(r = 2M) = \frac{1}{4M}}$ $\frac{M}{(2M)^2}$ =

Surface gravity.

$\boxed{\text{On horizon; } t^\alpha{}_{;\beta} t^\beta = \kappa t^\alpha}$

generators parametrized by v

$dx^\alpha = t^\alpha dv$; its not an affine parameter.

RN Black Hole

mass M, charge Q

$$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2$$

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

$$f = \left(1 - \frac{2m(r)}{r}\right) \; ; \; m(r) = M - \frac{Q^2}{2r} \; ; \; m'(r) = \frac{Q^2}{2r^2}$$

total mass of spacetime including electrostatic energy

$$\rho = \frac{m'}{4\pi r^2} = \frac{Q^2}{8\pi r^4} = \frac{1}{8\pi}\left(\frac{Q}{r^2}\right)^2$$

$$\Rightarrow \boxed{\rho = \frac{E^2}{8\pi}}$$

$$F^{tr} = \frac{Q}{r^2}$$

Solution of Einstein Maxwell equation.

Horizon

$$r = r_\pm = M \pm \sqrt{M^2 - Q^2}$$

→ gives limit how much charge you can store inside black hole before you distort the black hole.

$$M > |Q|$$

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— Shoaib Akhtar

RN Black Hole II $ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2$; $f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$

$f = 1 - \frac{m(r)}{r}$; $m(r) = M - \frac{Q^2}{2r}$; we have two horizons $r_\pm$.

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

$$M \geq |Q|$$

Geodesic motion ; $L = \frac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \equiv$ lagrangian for geodesic motion.

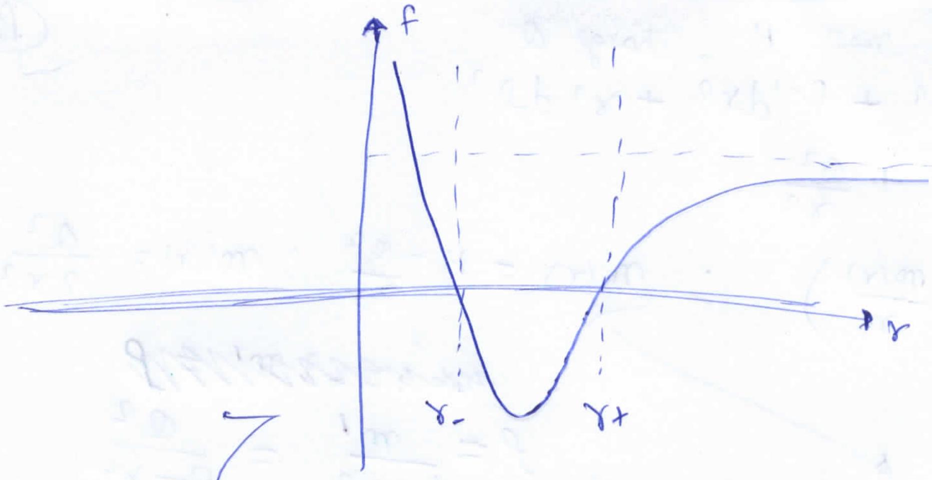
$\dot{\theta} = \dot{\phi} = 0$; $\theta = \frac{\pi}{2}$; $L = -\frac{1}{2} f \dot{t}^2 + \frac{1}{2} f^{-1}\dot{r}^2 = -\frac{1}{2}$ → normalizing velocity to unity.

$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{\partial L}{\partial \dot{t}} = \text{constant} = -f\dot{t} = -\tilde{E}$ = energy (rest mass)

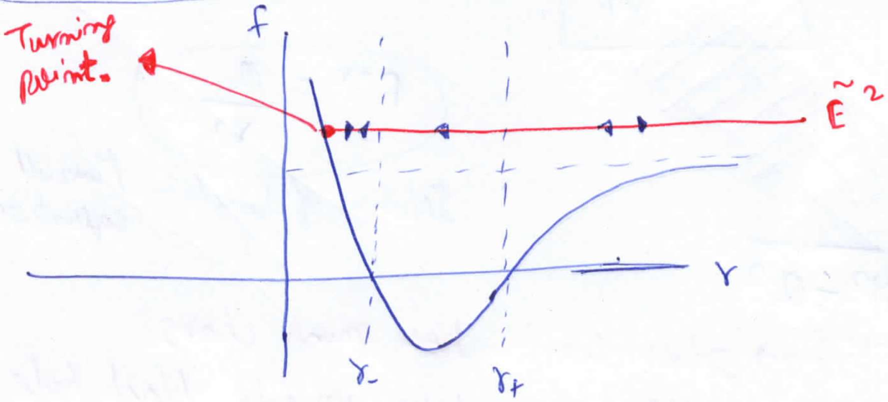
↳ we inserted minus here to get the result that for $\tilde{E} > 0$ time proceeds forward.

$$\dot{t} = +\frac{\tilde{E}}{f}$$

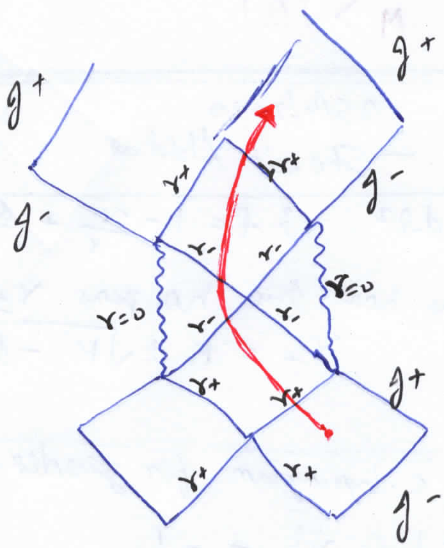
$$-1 = -f\frac{E^2}{f^2} + \frac{\dot{r}^2}{f} \Rightarrow \boxed{\dot{r}^2 + f^2 = E^2}$$



effectively this is potential energy.



Penrose diagram for RN ; repeat



~~Charge a schwarschild black hole ... & understand the dyn~~

Charge a schwarschild black hole & understand the dynamics.

→ origin of repulsion can ~~be~~ only be gravity.

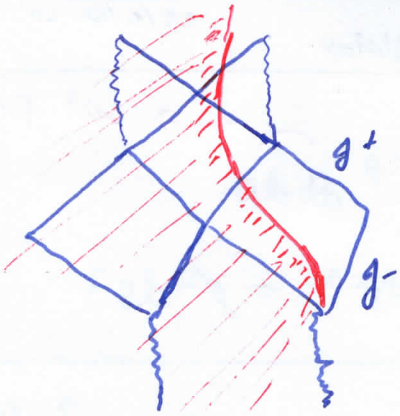
∴ (if you through neutral particle)

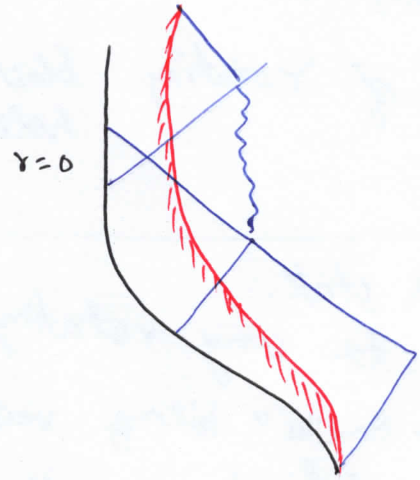
So. ~~m~~ Inside r_-, $m(r) < 0$

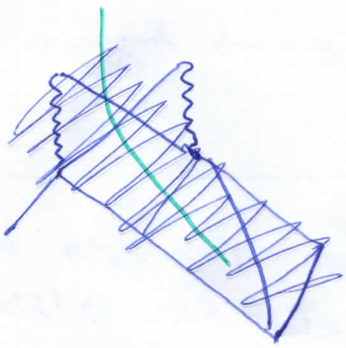
$\Rightarrow$ so; we have effective repulsion gravity

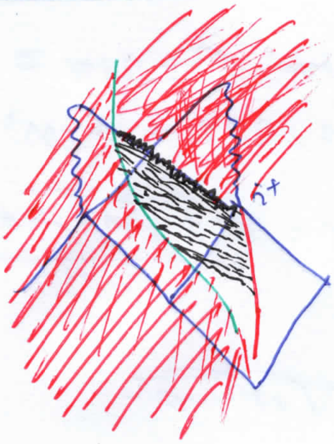
& gravity repels there!!!

with star









E finite.

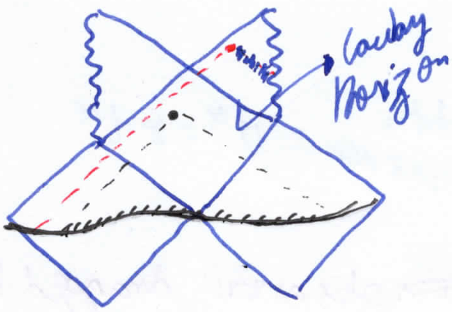
Energy density as measured by crossing observer $\to \infty$ as $r = r_-$

The Back Reaction $\to$ radical revision of internal structure.

~~Singul~~ Singularity ; $m(r) \to \infty$
$r \to$ finite

→ Mass inflation singularity.

Cauchy Horizon



Breakdown Cauchy Problem
predictibility.

Lec 22 — Shuaib Akhtar 23/4/2020 Pg 94

KERR BLACK HOLE

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}\,dt\,d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta\,d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

will produce dragging effect.

$\rho^2 = r^2 + a^2\cos^2\theta$

$\Delta = r^2 - 2Mr + a^2$; $\Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$

$a = \frac{J}{M}$ → Angular momentum of rotating black hole.

~~Kerr So~~

Kerr Solution is applied to black hole state.
It is not an exterior solution to any rotating star.

Killing vectors $t^\alpha = (1,0,0,0)$ "timelike" killing vector.
$\phi^a = (0,0,0,1)$ (rotational " ")

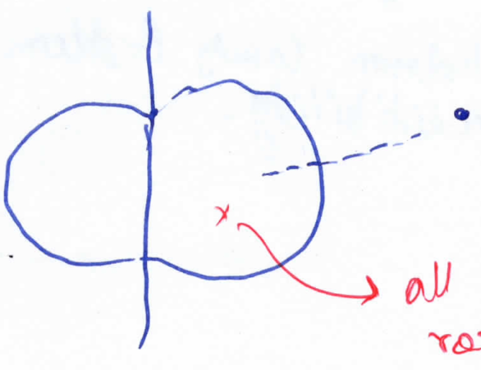
$g_{\alpha\beta}t^\alpha t^\beta = g_{tt} = 1 - \frac{2Mr}{\rho^2} = 0$ when $\rho^2 = 2Mr$
$\Rightarrow r^2 - 2Mr + a^2\cos^2\theta = 0$
$\Rightarrow r = \frac{2M \pm \sqrt{4M^2 - 4a^2\cos^2\theta}}{2}$

$\Rightarrow r = M + \sqrt{M^2 - a^2\cos^2\theta}$

not event horizon ; ~~$r = M + \sqrt{M^2 - a^2}$~~

Surface $r = M + \sqrt{M^2 - a^2\cos^2\theta}$ is not null.

↳ Static limit ⇒ ~~Ergos~~ Ergosphere ⇒ last place a static observer can exist.

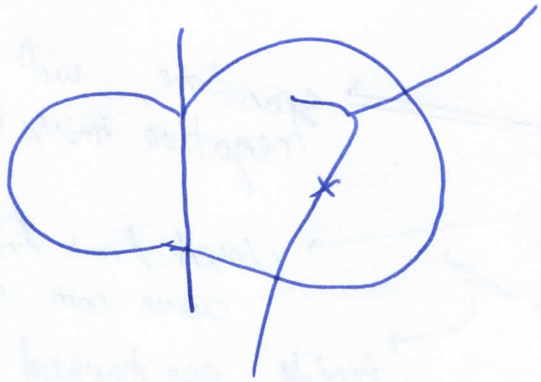


static observer $u^\alpha = \gamma t^\alpha$

all ~~observer~~ observers dragged by B.H. rotation.

$\tilde{E} = -U^\alpha t_\alpha = \begin{cases} +ve \\ -ve \end{cases}$

Penrose process.



As you extract energy from black hole

↪ you reduce J.

Stationary Observer: $U^\alpha = \gamma(t^\alpha + \Omega \phi^\alpha)$

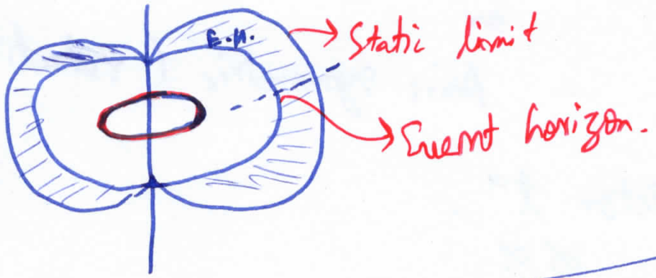
$\Omega = \frac{d\phi}{dt}$ = angular velocity.

for U^α timelike ; $\Omega_- < \Omega < \Omega_+$

↪ The interval ~~doesn~~ shrinks as r decreases ~~unti~~ untill

$\Omega_- = \Omega_+ = \Omega_H$

$= \frac{a}{r^2 + a^2}$



In that situation, $t^\alpha + \Omega_H \phi^\alpha$ = null

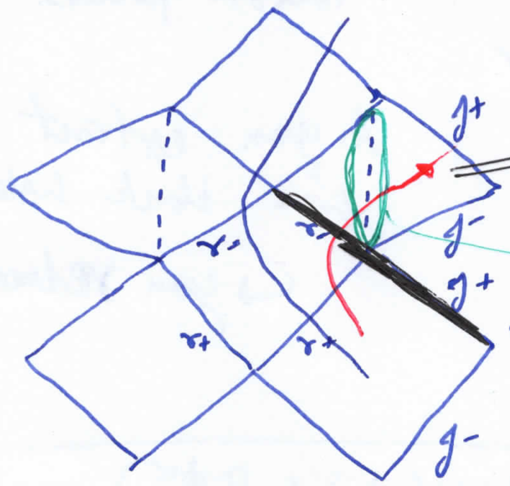
$$\Omega_- = \Omega_+ = \Omega_H = \frac{a}{r^2 + a^2}$$

$r = r_\pm M + \sqrt{M^2 - a^2}$

↪ location of event horizon ; null surface.

$a^2 < M^2 \Rightarrow \boxed{J \leq M^2}$

$a \leq M$



Spacetime with negative mass object.

closed time like curve can exist

inside perturbed solution should be applied.

Stationary B.H. (Kerr for vacuum; non Kerr with matter)

Hawking (1972): Stationary BH is either
— static (non-rotating)
or
— Axis symmetric (rotating)

$\Rightarrow$ Timelike Killing vector t^α.
Rotational " " ϕ^α.

$t^\alpha + \Omega_H \phi^\alpha = \xi^\alpha$ = null on Event Horizon. (E.H.)

If we are dealing with axis symmetric case, then we have this

$V^2 = g_{\alpha\beta}\xi^\alpha\xi^\beta = 0$ on E.H. } defining equation for E.H.

Killing Symmetry; $\mathcal{L}_\xi g_{\alpha\beta} = 0$

$\Rightarrow \mathcal{L}_\xi V^2 = 0$ ~~$\Rightarrow \xi^\alpha \partial_\alpha V = 0$~~

Normal to E.V. $= \partial_\alpha V^2 = \nabla_\alpha(g_{\alpha\beta}\xi^\alpha\xi^\beta)$

little wrong ... check it.

$= 2 g_{\mu\nu}\xi^\mu \nabla_\alpha \xi^\nu$

$= 2\xi_\mu \nabla_\alpha \xi^\nu$

$= -2\xi_\nu \nabla^\mu \xi_\alpha = -2 \ldots$

Stationary Blackhole

Stationary : no t-dependence ; existence of timelike killing vector.

Static : Time reversal invariance ; Killing vector hypersurface orthogonal.

Hawking (1972): Stationary BH is either static or axisymmetric.

t^α = "time like" killing vector
(puting " " because it is not time-like everywhere)

ϕ^a = rotational killing vector.

$\xi^\alpha = t^\alpha + \Omega_H \phi^a$ null on E.H.
$\Omega_H \to$ rotational velocity of E.H. ; (Ω_H = constant)

$\Omega_H = 0$ when B.H. is non-rotating

$\Phi^2 = g_{\alpha\beta}\,\xi^\alpha \xi^\beta = 0$ on E.H. $\Rightarrow$ null hypersurface traced by null generators to which ξ^α is tangent.

ξ^α is also normal to E.H.

ξ^α satisfies geodesic equation on E.H. only.

V = parameter.

$$\boxed{\xi^\alpha{}_{;\beta}\,\xi^\beta = k\,\xi^\alpha \quad \text{on E.H.}}$$

displacement on a generator : $dx^\alpha = \xi^\alpha dV$ $\qquad \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$

Proof //

$\Phi = 0$ on E.H.

$\partial_\alpha \Phi$ = normal to E.H. $= -2k\,\xi_\alpha$ (defines k)

$$\nabla_\alpha (g_{\mu\nu}\xi^\mu \xi^\nu) = 2\xi^\mu{}_{;\alpha}\xi_\mu = -2\xi_{\alpha;\mu}\xi^\mu \Rightarrow \boxed{\xi_{\alpha;\mu}\xi^\mu = k\,\xi_\alpha}$$

---

k will be shown to be constant on E.H.
$\to$ called "Surface Gravity"

Rayachaudhuri's Equation:

$$\frac{d\theta}{dV} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - R_{\alpha\beta}\xi^\alpha\xi^\beta + \cancel{\omega^{\alpha\beta}\omega_{\alpha\beta}} \to 0$$

$$\Rightarrow \boxed{\frac{d\theta}{dV} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta}$$

for stationary Horizon; $\theta = 0 \quad \therefore \frac{d\theta}{dV} = 0$

$$\Rightarrow \boxed{\sigma^{\alpha\beta}\sigma_{\alpha\beta} + 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta = 0}$$ $\Rightarrow$ for stationary Horizon.

with null energy condition; $8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta \geq 0$

$\therefore$ and $\sigma^{\alpha\beta}\sigma_{\alpha\beta} \geq 0$

so: $\underbrace{\sigma^{\alpha\beta}\sigma_{\alpha\beta} = 0}_{\Downarrow}$ & $\underbrace{8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta = 0}_{\Downarrow}$

$\sigma = 0$

(stationary Horizon has no shear and it cannot ~~cannot~~ have energy flux through it)

~~have energy~~

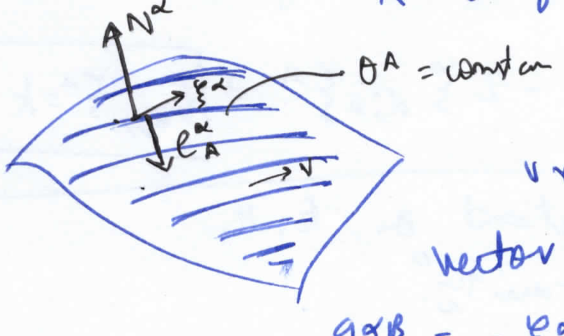
$\sigma_{\alpha\beta} = 0$

$T_{\alpha\beta}\xi^\alpha\xi^\beta = 0 \Rightarrow$ no flux

} Stationary Horizon.

Zeroth Law

for stationary BHs;
k = uniform on E.H.



$y^a = (v, \theta^A)$

v running on each generator.

vector basis; $N^\alpha, \xi^\alpha, e^\alpha_A$

$$g^{\alpha\beta} = -\xi^\alpha N^\beta - N^\alpha\xi^\beta + \sigma^{AB}e^\alpha_A e^\beta_B$$

$$\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial v}\right)_{\theta^A} \; ; \; e^\alpha{}_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_v$$

$$\sigma_{AB} = g_{\alpha\beta}\, e^\alpha{}_A e^\beta{}_B .$$

① $\left(\frac{\partial K}{\partial v}\right)_{\theta^A} = 0$. ②; $\left(\frac{\partial K}{\partial \theta^A}\right)_v = 0$. } need to prove this.

$$\xi_{\alpha;\beta} = a\xi_\alpha\xi_\beta + b\xi_\alpha N_\beta + c^A \xi_\alpha e^\alpha{}_A + d N_\alpha \xi_\beta + \cancel{e N_\alpha N_\beta} + \cancel{f^A} N_\alpha e_{A\beta} + g^A e_{A\alpha}\xi_\beta + \cancel{h^A e_{A\alpha} N_\beta} + \cancel{j^{AB} e_{A\alpha} e_{B\beta}}$$

$N_a N^a = 0$

$N_\alpha \xi^\alpha = -1$; $N_\alpha e^\alpha{}_A = 0$

a general decomposition.

(i)

ξ^α is null on EH : $0 = (\xi^\alpha\xi_\alpha)_{;\beta}\xi^\beta = 2\xi^\alpha\xi_{\alpha;\beta}\xi^\beta$

$0 = (\xi^\alpha\xi_\alpha)_{;\beta}e^\beta_A = 2\xi^\alpha\xi_{\alpha;\beta}e^\beta{}_A$

∴ so; $\xi^\alpha\xi_{\alpha;\beta} = -d\,\xi_\beta - eN_\beta - f^A e_{A\beta}$

$\xi^\alpha\xi_{\alpha;\beta}\xi^\beta = +e = 0$.

$0 = \xi^\alpha\xi_{\alpha;\beta}e^\beta{}_C = -f^A\sigma_{AC} \Rightarrow f^A = 0$ if $\sigma_{AC} \neq 0$

---

③ ξ^α is geodesic on E.H.

(ii) $\xi^\alpha{}_{;\beta}\xi^\beta = k\xi^\alpha = -b\xi^\alpha - h^A e^\alpha{}_A$

so; $\boxed{b = -k}$; $h^A = 0$

(iii) for stationary B.H.; $\theta = 0$: $\sigma_{\alpha\beta} = \omega_{\alpha\beta} = 0$

$\Rightarrow \tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu \xi_{\mu;\nu} = 0$

$\Rightarrow j^{AB} e_{A\alpha} e_{B\beta} \Rightarrow j^{AB} = 0$

(iv) Killing Vector

$\xi_{(\alpha;\beta)} = 0$: $\Rightarrow d = K$; $g^A = -c^A$

$$\xi_{\alpha;\beta} = -k\left(\xi_\alpha N_\beta - N_\alpha \xi_\beta\right) + c^A\left(\xi_\alpha e_{A\beta} - e_{A\alpha}\xi_\beta\right)$$

$$\cancel{-k = \xi_{\alpha;\beta} N^\alpha \xi^\beta} \qquad \boxed{-k = \xi_{\alpha;\beta} N^\alpha \xi^\beta}$$

$$\boxed{\xi_{\alpha;\beta} N^\alpha e^\beta_B = -c^A \sigma_{AB} \equiv -c_B}$$

$$\boxed{k = -\xi_{\alpha;\beta} N^\alpha \xi^\beta} \qquad \boxed{c_A = -\xi_{\alpha;\beta} N^\alpha e^\beta_A}$$

Killing Equations $\Rightarrow$ $\underline{\xi_{\alpha;\beta\nu} = R_{\alpha\beta\mu\nu}\xi^\nu}$

$$\left(\frac{\partial k}{\partial v}\right)_{\theta^A} = k_{;\mu}\xi^\mu$$

$$= -\xi_{\alpha;\beta\mu} N^\alpha \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha_{;\beta}\xi^\beta\xi^\mu$$

$$- \xi_{\alpha;\beta} N^\alpha \xi^\beta_{;\mu}\xi^\mu$$

$$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta \xi^\mu \xi^\nu \rightarrow 0$$

$$-k\,\xi_\alpha N^\alpha_{;\beta}\xi^\mu - k\,\xi_{\alpha;\beta} N^\alpha \xi^\beta$$

$$N_\alpha \xi^\alpha = -1 \Rightarrow N_{\alpha;\beta}\xi^\alpha + N_\alpha \xi^\alpha_{;\beta} = 0$$

so; $$\left(\frac{\partial k}{\partial v}\right)_{\theta^A} = k\left(\xi_{\alpha;\mu} N^\alpha \xi^\mu - k\,\xi_{\alpha;\beta} N^\alpha \xi^\beta\right) = 0.$$

$$\left(\frac{\partial k}{\partial \theta^A}\right)_v = k_{;\mu} e^\mu_A = (-\xi_{\alpha;\beta\mu} N^\alpha \xi^\beta e^\mu_A$$

$$- \xi_{\alpha;\beta} N^\alpha_{;\mu}\xi^\beta e^\mu_A - \xi_{\alpha;\beta} N^\alpha \xi^\beta_{;\mu} e^\mu_A$$

$$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu_A \xi^\nu - k\,\xi_\alpha N^\alpha_{;\mu} e^\mu_A - \xi_{\alpha;\beta} N^\alpha \xi^\beta_{;\mu} e^\mu_A$$

$$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu_A \xi^\nu + k\,\xi_{\alpha;\mu} N^\alpha e^\mu_A - N_\alpha \xi^\alpha_{;\beta}\xi^\beta_{;\mu} e^\mu_A$$

$$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu - k\, C_A + k\, C_A$$

$$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu$$

$$\Rightarrow \left(\frac{\partial K}{\partial \theta^A}\right)_V = -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu$$

$$g^{\mu\alpha} = -\xi^\mu N^\alpha - N^\mu \xi^\alpha + \sigma^{BC} e^\mu{}_B e^\alpha{}_C$$

$$-\xi N^\alpha = g^{\mu\alpha} + N^\mu \xi^\alpha - \sigma^{BC} e^\mu{}_B e^\alpha{}_C$$

so:

$$\left(\frac{\partial K}{\partial \theta^A}\right)_V = R_{\alpha\beta\mu\nu}\left(g^{\mu\alpha} + N^\mu \xi^\alpha - \sigma^{BC} e^\mu{}_B e^\alpha{}_C\right)\xi^\beta e^\mu{}_A$$

$$= -R_{\beta\nu} \xi^\beta e^\mu{}_A - \sigma^{BC} R_{\alpha\beta\mu\nu} \xi^\beta e^\alpha{}_C e^\nu{}_A e^\mu{}_B$$

$$\left(\frac{\partial K}{\partial \theta^A}\right)_V = -R_{\alpha\beta} \xi^\alpha e^\beta{}_A - \sigma^{BC} \cdot \underbrace{R_{\mu\nu\alpha\beta} e^\mu{}_A e^\nu{}_B e^\alpha{}_C \xi^\beta}_{=0}$$

we need dominated energy conditions to make this zero.

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$$\xi_{\alpha;\beta} e^\alpha{}_A e^\beta{}_B = 0.$$

$$(\xi_{\alpha;\beta} e^\alpha{}_A e^\beta{}_B)_{;\gamma} e^\gamma{}_C = 0$$

$$\Rightarrow \xi_{\alpha;\beta\gamma} e^\alpha{}_A e^\beta{}_B e^\gamma{}_C + \xi_{\alpha;\beta} e^\alpha{}_{A;\gamma} e^\gamma{}_C e^\beta{}_B + \xi_{\alpha;\beta} e^\alpha{}_A e^\beta{}_{B;\gamma} e^\gamma{}_C$$

$$= R_{\alpha\beta\gamma\delta} e^\alpha{}_A e^\beta{}_B e^\gamma{}_C \xi^\delta + (C_B \xi_\alpha) e^\alpha{}_{A;\gamma} e^\gamma{}_C - (C_A \xi_\beta e^\beta{}_{B;\gamma} e^\gamma{}_C)$$

$$= R_{\alpha\beta\gamma\delta} e^\alpha{}_A e^\beta{}_B e^\gamma{}_C \xi^\delta - C_B \underbrace{\xi_{\alpha;\gamma} e^\alpha{}_A e^\gamma{}_C}_{\to C_A \xi_\gamma} + C_A \underbrace{\xi_{\beta;c} e^\beta{}_B e^\gamma{}_C}_{-C_B \xi_\gamma e^\gamma{}_C}$$

$$= R_{\alpha\beta\gamma\delta} e^\alpha{}_A e^\beta{}_B e^\gamma{}_C \xi^\delta = 0.$$

$$\boxed{\left(\frac{\partial K}{\partial \theta^A}\right)_v = -R_{\alpha\beta}\xi^\alpha e^\beta_A}$$

$$= -8\pi\left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right)\xi^\alpha e^\beta_A$$

$$= -8\pi\, T_{\alpha\beta}\,\xi^\alpha e^\beta_A$$

$$= 8\pi\, j^\alpha e_{A\alpha} \quad ; \quad \boxed{j^\alpha = -T^\alpha{}_\beta \xi^\beta}$$

$$j^\alpha = a\xi^\alpha + bN^\alpha + j^A e^\alpha_A$$

$$0 = T_{\alpha\beta}\xi^\alpha\xi^\beta \Rightarrow \quad \boxed{b=0}$$

So; j^α has to be tangent to horizon

It has no component across horizon

$$j^\alpha = a\xi^\alpha + j^A e^\alpha_A$$

Dominant Energy Condition: $j^\alpha \equiv -T^\alpha{}_\beta\xi^\beta$ has to be future directed; timelike or null.

$\hookrightarrow$ so; $g_{\alpha\beta} j^\alpha j^\beta \leq 0$

$$g_{\alpha\beta}(a\xi^\alpha + j^A e^\alpha_A)(a\xi^\beta + j^B e^\beta_B)$$

$$= \sigma_{AB}\, j^A j^B \leq 0 \quad \Rightarrow j^A = 0$$

$\hookrightarrow$ positive definitive norm

$$\Rightarrow \sigma_{AB}\, j^A j^B \geq 0 \quad \Rightarrow \quad \sigma_{AB}\, j^A j^B = 0$$

$$\Rightarrow j^A = 0$$

$$\Rightarrow \boxed{j^\alpha = a\xi^\alpha}$$

$$\therefore j_\alpha e^\alpha_A = 0 \Rightarrow \quad a=0 \quad \left(\frac{\partial K}{\partial \theta^A}\right)_v = 0$$

under Dominant Energy condition.

First Law under a quasi-static process (infinitesimal change) that takes BH from one stationary state to other stationary state,

the B.H. parameters change according to

$$\delta M = \underbrace{\frac{K}{8\pi}\delta A}_{\text{heart term}} + \underbrace{\Omega_H \delta J}_{\text{work term}}$$

The analogy of K as surface gravity, and K as temperature like quantity;

The zeroth law of thermodynamics is statement that if you have system in equilibrium, stationary system; then temperature is uniform.

Here; we have stationary black hole for which surface gravity is uniform.

$$\frac{K}{8\pi}\delta A \cong T\, dS$$

$$dS \simeq \delta A$$

$$\frac{K}{8\pi} \simeq T$$

Infinitesimal $T^{\mu\nu}$

$$\delta M = -\int_H T^{\mu}{}_{\nu}\, t^{\nu} d\Sigma_{\mu}$$

$$\delta J = \int_H T^{\alpha}{}_{\beta}\, \phi^{\beta} d\Sigma_{\alpha}$$

first-order changes

$$d\Sigma_\alpha = -\xi_\alpha dS\, dV$$

$$dS = \sqrt{\sigma}\, d^2\theta$$

area of cross section

during transition, obviously there are no t^ν & ϕ^β killing vectors; but working to first order; we can use background killing vectors.

$$\delta M - \Omega_H \delta J = -\int_H T^\alpha{}_\beta (t^\beta + \Omega \phi^\beta) d\Sigma_\alpha$$

$$= \int_H T_{\alpha\beta} \xi^\alpha \xi^\beta \, dS dV$$ → integration over generators.

if our BH was remaining stationary during change; this will be zero; & we ~~will be talking~~ would not be ~~about~~ ~~about first law......~~ talking about first law...

∴ But; This being non-zero comes from the fact that $T_{\alpha\beta}\xi^\alpha\xi^\beta$ this component of $T_{\alpha\beta}$ is non-zero during transition.

Raychaudhuri's Equation $\quad \frac{d\theta}{dV} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta$

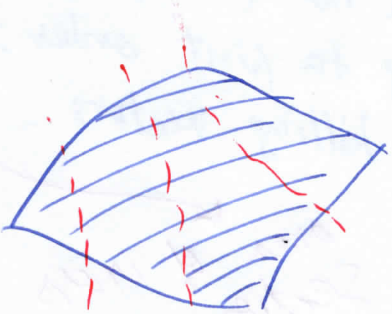
to zero order in perturbation theory $\theta = 0$ & $\sigma = 0$

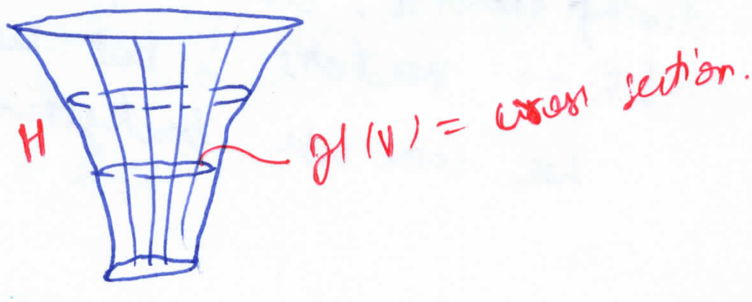
∴ to first order $\theta \neq 0$; $\sigma \neq 0$

σ^2 second order
θ^2 " "

upto first order $\quad 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta = k\theta - \frac{d\theta}{dV}$

so; $\delta M - \Omega_H \delta J = \frac{k}{8\pi}\int_H \theta \, ds dV - \frac{1}{8\pi}\int_H \frac{d\theta}{dV} \, ds dV$





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$\theta = \frac{1}{ds}\frac{d}{dV}ds$ = fractional rate of change of cross-sectional area.

$$\int_{H} \theta\, ds\, dV = \int_{V_1}^{V_2} dV \oint_{\mathcal{H}(V)} \frac{d}{dV} ds$$

$$= \int_{V_1}^{V_2} dV \frac{d}{dV} \oint_{\mathcal{H}(V)} ds$$

$$= \oint_{\mathcal{H}(V_i)} ds \Big]_{V_1}^{V_2}$$

$$= A(V_2) - A(V_1)$$

$$= \delta A$$

$\oint_{\mathcal{H}(V_i)} ds \Rightarrow$ Cross sectional area of BH at V_i $i=1,2$.

$$\int_{H} \frac{d\theta}{dV} ds\, dV = \int_{V_1}^{V_2} dV \oint_{\mathcal{H}(V)} \frac{d\theta}{dV} ds = \int_{V_1}^{V_2} dV \frac{d}{dV} \oint_{\mathcal{H}(V)} \theta\, ds$$

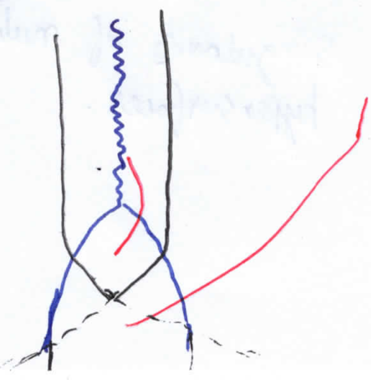
$$= \left[\oint_{H(V)} \theta\, ds\right]_{V_1}^{V_2} = \underline{\underline{0}}$$

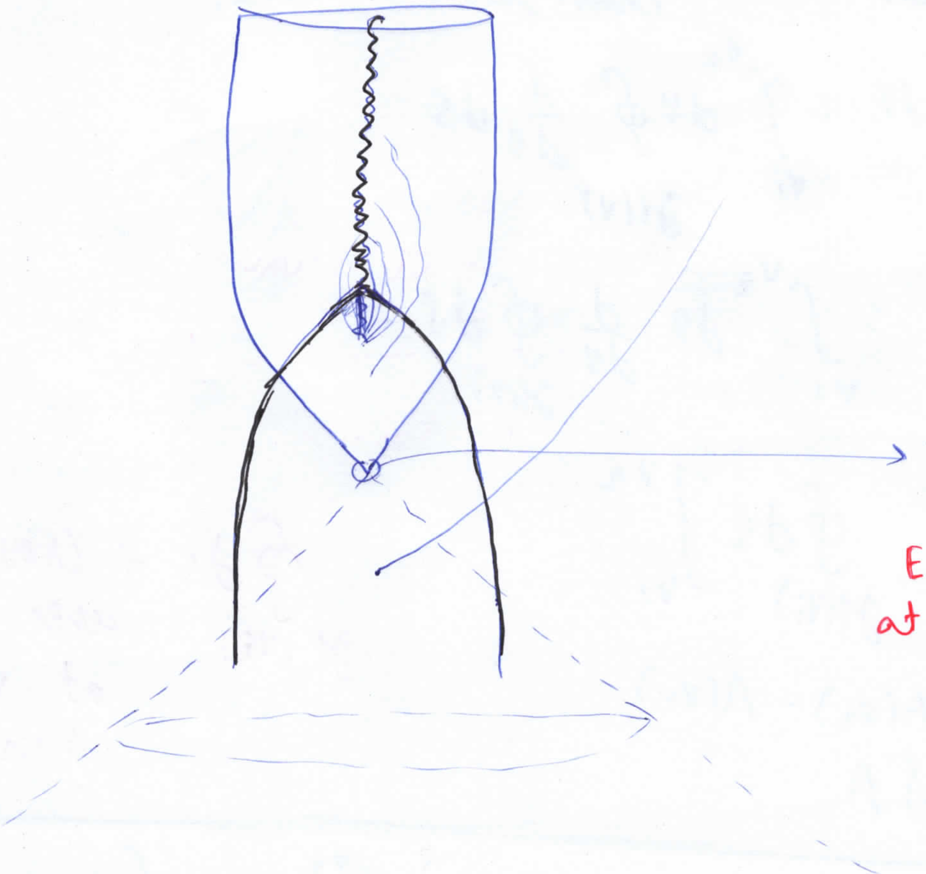
because we start (V_2) and finish (V_1) at stationary state; θ was zero at V_1 and V_2.

because $\theta = 0$ initially & finally.

$$\boxed{\delta M - \Omega_H \delta J = \kappa \delta A}$$ First law.

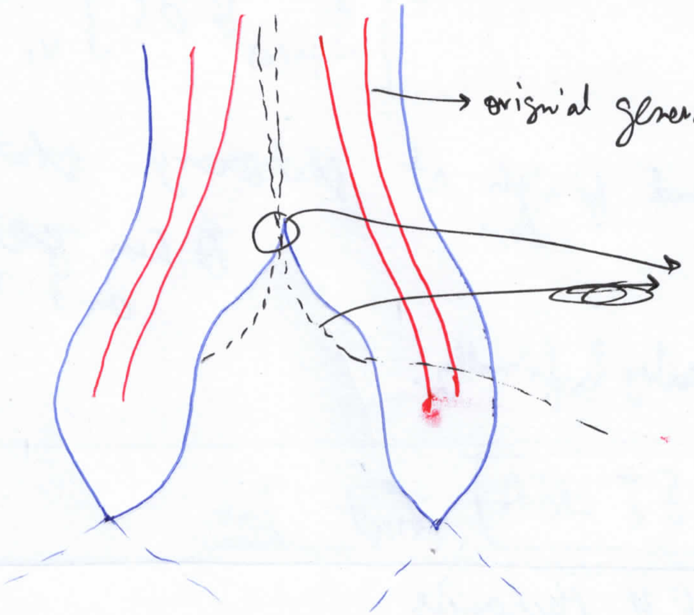
Second Law of B.H. Mechanics





Horizons has past : * Caustics are entry points in E.H.

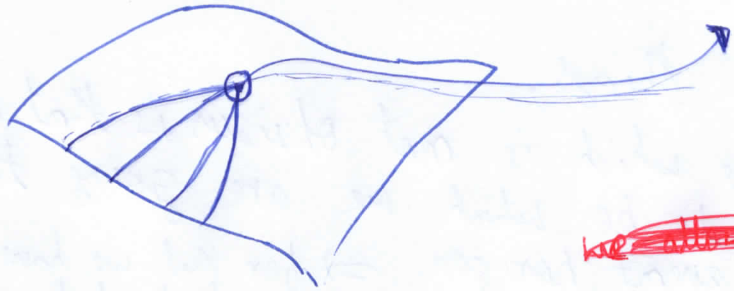
Merger of Two Black holes.



$\theta \to +\infty$

singularity of null hypersurfaces.

Future Caustics? Exit points?



~~we allow for fu~~

~~No exit point~~.
No exit point.

Violate integrety of E.H as Causal Boundary.

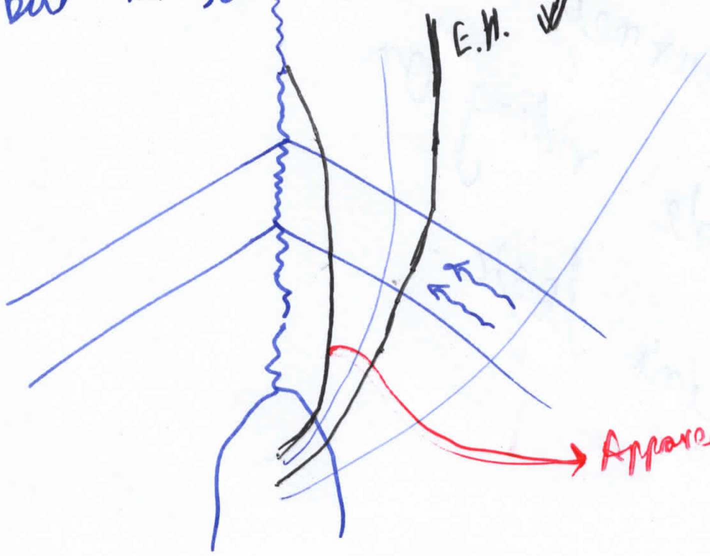
"No future caustic ⇒ congruence of generator cannot be contracting on E.H."

so; $\delta A \not< 0$ (becau ~~[illegible]~~ it will create future caustic) → and that will be end of Horizon.

so: $\boxed{\delta A \geq 0}$

~~Black holes at~~ * Black holes are not time reversal objects.

Event Horizon breakes time reversal ~~[illegible]~~ invariance; but in some sense it also breaks causality.



~~Act~~ Actual event horizon has to ~~ac~~ account for all matter falling in ~~future~~ ∞ future

so: E.H. satisfy bunch of differential equation; where you have to specify final condition instead of initial condition.
↪ In this sense they have non-causal behavior.

We have to wait for ∞ time for being sure that you are dealing with E.H.

All light rays do these things,
but the thing which is not obvious is that which light ray is going to be which we are going to declare generator of event horizon. → for that we have to ~~too~~ look at final condition.

↳ teleological property of E.H. (final causes)

End of Ch 5

End of the Notes

— Shoaib Akhtar

Advanced General Relativity.

A beautiful journey into the Black Hole riding on Einstein's logic.